Stochastic Numerical Analysis

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Martingales

A martingale M_t has the property that for time $s > t$

$$\mathbb{E}[\, M_s \mid \mathcal{F}_t \,] = M_t$$

i.e. given complete knowledge at the current time t, the expected value of M at a future time s is equal to its current value M_t.

In this course, we are interested in both discrete and continuous-time martingales.

Martingales

Discrete martingales will usually be of the form:

$$M_n = \sum_{m=0}^{n-1} x_m \, \Delta W_m$$

Hence, for $p > n$,

$$M_p - M_n = \sum_{m=n}^{p-1} x_m \, \Delta W_m$$

and therefore

$$\mathbb{E}[\, M_p - M_n \mid \mathcal{F}_n] = 0$$

because ΔW_m is independent of x_m and $\mathbb{E}[\Delta W_m] = 0$.

Martingales

Continuous time:

$$M_t = \int_0^t x_u \, \mathrm{d}W_u$$

Hence

$$M_s - M_t = \int_t^s x_u \, \mathrm{d}W_u$$

and therefore

$$\mathbb{E}[\, M_s - M_t \mid \mathcal{F}_t \,] = 0$$

because $\mathbb{E}[\mathrm{d}W_u] = 0$.

Doob inequality

If M_t is a martingale with $M_0 = 0$, then for any $p > 1$

$$\mathbb{E}[\,|M_T|^p] \;\leq\; \mathbb{E}\left[\sup_{[0,T]} |M_t|^p\right] \;\leq\; \left(\frac{p}{p-1}\right)^p \mathbb{E}[\,|M_T|^p]$$

Why is this useful?

- a bound on $\mathbb{E}\left[\sup_{[0,T]} |M_t|^p\right]$ is a tight control on how big M_t can be
- however, usually easier to bound $\mathbb{E}[\,|M_t|^p]$
- we will use this to bound both the SDE solution, its discrete approximation, and the error

Quadratic variation

The quadratic variation of a discrete process X_n is defined as

$$[X]_n = \sum_{m=0}^{n-1} (X_{m+1} - X_m)^2$$

Note that if

$$X_{n+1} - X_n = a_n\, h + b_n\, \Delta W_n$$

with $\Delta W_n \sim N(0,h)$, then

$$(X_{n+1} - X_n)^2 = b_n^2 (\Delta W_n)^2 + o(h)$$

Quadratic variation

For a continuous time process X_t, the quadratic variation is defined similarly as

$$[X]_T = \lim_{N\to\infty} \sum_{n=0}^{N-1} \left(X_{(n+1)T/N} - X_{nT/N} \right)^2$$

which in the case of

$$X_t = X_0 + \int_0^t a(X_s, s)\ \mathrm{d}s + \int_0^t b(X_s, s)\ \mathrm{d}W_s$$

gives

$$[X]_t = \int_0^t b^2(X_s, s)\ \mathrm{d}s$$

BDG inequality

Like the Doob inequality, the Burkholder-Davis-Gundy inequality concerns martingales and provides bounds on the same quantity, but using the quadratic variation.

For any $p \geq 1$, there are constants c_p, C_p such that

- for a continuous-time martingale

$$c_p \, \mathbb{E}[\, [M]_T^{p/2}] \;\leq\; \mathbb{E}\left[\sup_{[0,T]} |M_t|^p\right] \;\leq\; C_p \, \mathbb{E}[\, [M]_T^{p/2}]$$

- for a discrete martingale,

$$c_p \, \mathbb{E}[\, [M]_N^{p/2}] \;\leq\; \mathbb{E}\left[\max_{0<n\leq N} |M_n|^p\right] \;\leq\; C_p \, \mathbb{E}[\, [M]_N^{p/2}]$$

BDG inequality

Note that if

$$M_t = \int_0^t b_u \, \mathrm{d}W_u$$

then

$$[M]_T = \int_0^T b_t^2 \, \mathrm{d}t.$$

It follows that when $p = 2$ we have

$$\mathbb{E}[\,[M]_T] \;=\; \int_0^T \mathbb{E}[b_t^2] \, \mathrm{d}t \;=\; \mathbb{E}[M_T^2]$$

by the Itô isometry, and therefore the Doob inequality implies the BDG inequality with $C_2 = 4$.

BDG inequality

Also, for general $p \geq 2$, Jensen's inequality gives us

$$\left(\int_0^T b_t^2 \, \mathrm{d}t \right)^{p/2} \;\leq\; T^{p/2-1} \int_0^T |b_t|^p \, \mathrm{d}t$$

and therefore

$$\mathbb{E}[\,[M]_T^{p/2}] \;\leq\; T^{p/2-1} \int_0^T \mathbb{E}[|b_t|^p] \, \mathrm{d}t$$

This is how we can bound $\mathbb{E}\left[\sup\limits_{[0,T]} |M_t|^p \right]$

SDEs

In Lecture 4 we saw the problems caused by superlinear growth in the drift so, following Kloeden & Platen's book, we will now restrict attention to SDEs

$$\mathrm{d}S = a(S_t, t)\ \mathrm{d}t + b(S_t, t)\ \mathrm{d}W$$

for which there exists a constant K such that

- Lipschitz continuity in space:

  $|a(x,t) - a(y,t)| + |b(x,t) - b(y,t)| \leq K\ |x-y|$

- linear growth bound:

  $|a(x,t)| + |b(x,t)|\ \leq\ K\ (1+|x|)$

- square-root continuity in time:

  $|a(x,s) - a(x,t)| + |b(x,s) - b(x,t)|\ \leq\ K\ (1+|x|)\sqrt{|s-t|}$

SDEs

Theorem: Under the standard conditions, starting from a fixed S_0, for any $T>0$ and any $p\geq 2$,

$$\mathbb{E}\left[\sup_{[0,T]} |S_t|^p\right] \ < \ \infty$$

i.e. all moments of $\sup_{s\in[0,t]} |S_s|$ are bounded for all time.

First we construct the SDE for S_t^n for $n\geq 2$. Itô 's formula gives

$$\begin{aligned} \mathrm{d}(S_t^n) \ &= \ n\, S_t^{n-1} \Big(a(S_t,t)\ \mathrm{d}t + b(S_t,t)\ \mathrm{d}W \Big) \\ &\quad + \tfrac{1}{2}\, n(n-1)\, S_t^{n-2}\ b^2(S_t,t)\ \mathrm{d}t \end{aligned}$$

SDEs

The integral form is $S_t^n = S_0^n + I_1(t) + I_2(t) + I_3(t)$ where

$$\begin{aligned} I_1(t) &= \int_0^t n\, S_s^{n-1} a(S_s, s)\ \mathrm{d}s \\ I_2(t) &= \int_0^t \tfrac{1}{2}\, n(n-1)\, S_s^{n-2}\ b^2(S_s, s)\ \mathrm{d}s \\ I_3(t) &= \int_0^t n\, S_s^{n-1}\, b(S_s, s)\ \mathrm{d}W_s \end{aligned}$$

and we have

$$\sup_{[0,t]} |S_s|^n \leq |S_0|^n + \sup_{[0,t]} |I_1(s)| + \sup_{[0,t]} |I_2(s)| + \sup_{[0,t]} |I_3(s)|$$

SDEs

By Jensen's inequality we have

$$\sup_{[0,t]} |S_s|^{2n} \le 4 \left(|S_0|^{2n} + \sup_{[0,t]} |I_1(s)|^2 + \sup_{[0,t]} |I_2(s)|^2 + \sup_{[0,t]} |I_3(s)|^2 \right)$$

Our objective is to establish an inequality of the form

$$\mathbb{E}[V_t^{2n}] \le \alpha + \beta \int_0^t \mathbb{E}[V_s^{2n}]\, \mathrm{d}s$$

for $0 \le t \le T$, where $V_t = \sup_{[0,t]} |S_s|$ and the constants α, β depend on T, n, K.

We can then use the Grönwall inequality to achieve our result with $p = 2n$.

SDEs

The first two integrals are fairly easy: using Jensen's inequality again, and $|S|^p \leq 1+\||S|^q$ for $1<p<q$, we have

$$\begin{aligned} I_1^2(t) &\leq \left(\int_0^t n\,|S_s|^{n-1} K(1+|S_s|)\;\mathrm{d}s\right)^2 \\ &\leq t \int_0^t \left(n\,|S_s|^{n-1} K(1+|S_s|)\right)^2\;\mathrm{d}s \\ &\leq T \int_0^t n^2\,K^2\,(3+4\,|S_s|^{2n})\;\mathrm{d}s \\ &\leq T \int_0^t n^2\,K^2\,(3+4\,V_s^{2n})\;\mathrm{d}s \end{aligned}$$

The final bound is also an upper bound for $I_1^2(s)$ for $s<t$, so it's an upper bound on $\sup_{[0,t]} I_1^2(s)$.

SDEs

Similarly,

$$
\begin{aligned}
I_2^2(t) &\leq \left(\int_0^t \tfrac{1}{2} n(n-1)\, |S_s|^{n-2} K^2 (1+|S_s|)^2 \mathrm{d}s \right)^2 \\
&\leq t \int_0^t \left(\tfrac{1}{2} n(n-1)\, |S_s|^{n-2} K^2 (1+|S_s|)^2 \right)^2 \mathrm{d}s \\
&\leq T \int_0^t \left(\tfrac{1}{2} n(n-1)\, K^2 \right)^2 \left(7 + 8\, |S_s|^{2n}\right) \mathrm{d}s \\
&\leq T \int_0^t \tfrac{1}{2} n(n-1)\, K^2 \left(7 + 8\, V_s^{2n}\right) \mathrm{d}s
\end{aligned}
$$

and the final bound is an upper bound on $\sup_{[0,t]} I_2^2(s)$.

SDEs

Finally, for the third we use the Doob inequality and Itô isometry to obtain

$$
\begin{aligned}
\mathbb{E}\left[\sup_{[0,t]} I_3^2(s)\right] &\leq 4\int_0^t \mathbb{E}\left[n^2\, S_s^{2n-2} K^2 (1+|S_s|)^2\right]\, \mathrm{d}s \\
&\leq 4\int_0^t \mathbb{E}\left[n^2\, K^2 (3+4|S_s|^{2n})\right]\, \mathrm{d}s \\
&\leq 4\int_0^t n^2\, K^2\, \left(3+4\,\mathbb{E}[V_s^{2n}]\right) \mathrm{d}s
\end{aligned}
$$

This concludes the proof for $p=2n$ and $n\geq 2$. For smaller values of p the result follows from the Hölder inequality, as discussed in lecture 4.