MLMC for Parabolic PDEs

Mike Giles, Francisco Bernal

Mathematical Institute, University of Oxford, UK

Instituto Superior Tecnico, Portugal

Slightly expanded version of talk given at

IMACS Monte Carlo Methods 2015

July 6, 2015

Outline

- Feynman-Kac formula
- prior work – Gobet & Menozzi
- multilevel Monte Carlo
- prior work – Higham *et al*
- new idea – approximating a conditional expectation
- outline numerical analysis

Feynman-Kac formula

Suppose that $u(x,t)$ satisfies the parabolic PDE

$$\frac{\partial u}{\partial t} + \sum_j a_j \frac{\partial u}{\partial x_j} + \tfrac{1}{2} \sum_{j,k,l} b_{jl} b_{kl} \frac{\partial^2 u}{\partial x_j \partial x_k} - V\, u + f = 0$$

in bounded domain D, subject to $u(x,t) = g(x,t)$ on the boundary ∂D.

It will be assumed that $f(x,t), V(x,t), a(x,t), b(x,t)$ are all Lipschitz continuous, and $g(x,t)$ is continuously twice-differentiable.

Feynman-Kac formula

Feynman and Kac proved that $u(x,t)$ can also be expressed as

$$u(x,t) = \mathbb{E}\left[\int_t^\tau E(t,s)\, f(X_s,s)\,\mathrm{d}s + E(t,\tau)\, g(X_\tau,\tau) \mid X_t = x \right]$$

where X_t satisfies the SDE

$$\mathrm{d}X_t = a(X_t,t)\,\mathrm{d}t + b(X_t,t)\,\mathrm{d}W_t,$$

with W_t being a Brownian motion with independent components,
τ is the first time at which X_t leaves D, and

$$E(t_0,t_1) = \exp\left(-\int_{t_0}^{t_1} V(X_t,t)\,\mathrm{d}t \right).$$

Note: in the special case in which $f(x,t)\!=\!0$, $g(x,t)\!=\!t$, $V(x,t)\!=\!0$
$u_{exit}(x,t)$ is the expected exit time.

Feynman-Kac formula

Proof: suppose that $u(x,t)$ satisfies the PDE, and X_s satisfies the SDE starting from $X_t = x$. Then, for $t \leq s \leq \tau$ define

$$Y(s) = E(t,s)\, u(X_s, s) + \int_t^s E(t,r)\, f(X_r, r)\, \mathrm{d}r$$

$\mathrm{d}E = -V(X_s, s)\, E\, \mathrm{d}s$, and therefore Itô calculus gives

$$\begin{aligned}
\mathrm{d}Y &= E \left\{ -V\, u\, \mathrm{d}s + \sum_j \frac{\partial u}{\partial x_j} \left(a_j\, \mathrm{d}s + \sum_k b_{jk} \mathrm{d}W_k \right) + \frac{\partial u}{\partial t}\, \mathrm{d}s \right. \\
&\qquad \left. + \ \tfrac{1}{2} \sum_{j,k,l} b_{jl}\, b_{kl}\, \frac{\partial^2 u}{\partial x_j \partial x_k}\, \mathrm{d}s + f\, \mathrm{d}s \right\} \\
&= E \sum_{j,k} \frac{\partial u}{\partial x_j} b_{jk} \mathrm{d}W_k.
\end{aligned}$$

Feynman-Kac formula

Hence, by integrating we get

$$Y(\tau) = Y(t) + \int_t^\tau E(t,\tau) \sum_{j,k} \frac{\partial u}{\partial x_j}\, b_{jk}\, \mathrm{d}W_k$$

Taking expectations, and noting that $u(X_\tau, \tau) = g(X_\tau, \tau)$ and $X_t = x$, we then get

$$\begin{aligned} \mathbb{E}\left[E(t,\tau)\, g(X_\tau,\tau) + \int_t^\tau E(t,r)\, f(X_r, r)\, \mathrm{d}r \right] &= \mathbb{E}[Y(\tau)] \\ &= \mathbb{E}[Y(t)] \\ &= u(x,t) \end{aligned}$$

Feynman-Kac formula

Why is this Feynman-Kac formula useful?

In high dimensions, approximating the parabolic PDE can be expensive because the cost increases exponentially – *curse of dimensionality*

The cost of Monte Carlo simulation for the SDE scales linearly (or possibly quadratically) with dimension

Numerical approximation

An Euler-Maruyama discretisation with uniform timestep h gives

$$\widehat{X}_{t_{n+1}} = \widehat{X}_{t_n} + a(\widehat{X}_{t_n}, t_n)\, h + b(\widehat{X}_{t_n}, t_n)\, \Delta W_n,$$

with initial data $\widehat{X}_0 = x$ at time t.

If $\widehat{X}(t)$ is the piecewise-constant interpolant, we then have

$$\widehat{u}(x,t) = \mathbb{E}\left[\int_t^{\widehat{\tau}} \widehat{E}(t,s)\, f(\widehat{X}(s), s)\, \mathrm{d}s + \widehat{E}(t,\widehat{\tau})\, g(\widehat{X}(\tau), \widehat{\tau}) \right].$$

with $\widehat{\tau}$ being the exit time, and

$$\widehat{E}(t_0, t_1) = \exp\left(- \int_{t_0}^{t_1} V(\widehat{X}_t, t)\, \mathrm{d}t \right).$$

Prior work – Gobet & Menozzi

The Euler-Maruyama method has strong accuracy

$$\left(\mathbb{E}\left[\sup_{[0,\min(\tau,\widehat{\tau})]} \|X_t - \widehat{X}(t)\|^2 \right] \right)^{1/2} = O(h^{1/2} |\log h|^{1/2}),$$

and Gobet & Menozzi (2007) proved it has weak error

$$u(x,t) - \widehat{u}(x,t) = O(h^{1/2}).$$

For standard Monte Carlo method, ε RMS accuracy needs $O(\varepsilon^{-2})$ paths, each with $h = O(\varepsilon^2)$, so total cost is $O(\varepsilon^{-4})$

Gobet & Menozzi (2010) reduced this to $O(\varepsilon^{-3})$ by shifting the boundary by $O(h^{1/2})$ to improve the weak error to $O(h)$.

Multilevel Monte Carlo

MLMC is based on the telescoping sum

$$\mathbb{E}[\widehat{P}_L] = \mathbb{E}[\widehat{P}_0] + \sum_{\ell=1}^{L} \mathbb{E}[\widehat{P}_\ell - \widehat{P}_{\ell-1}]$$

where $\widehat{P}_\ell$ represents an approximation using timestep $h_\ell = 2^{-\ell}\, h_0$, with weak convergence

$$\mathbb{E}[\widehat{P}_\ell - P] = O(2^{-\alpha\,\ell}).$$

If $\widehat{Y}_\ell$ is an unbiased estimator for $\mathbb{E}[\widehat{P}_\ell - \widehat{P}_{\ell-1}]$, based on N_ℓ samples, with variance

$$\mathbb{V}[\widehat{Y}_\ell] = O(N_\ell^{-1}\, 2^{-\beta\,\ell})$$

and expected cost

$$\mathbb{E}[C_\ell] = O(N_\ell\; 2^{\gamma\,\ell}), \quad \ldots$$

Multilevel Monte Carlo

... then the finest level L and the number of samples N_ℓ on each level can be chosen to achieve an RMS error of ε at an expected cost

$$C = \begin{cases} O\left(\varepsilon^{-2}\right), & \beta > \gamma, \\ O\left(\varepsilon^{-2}(\log \varepsilon)^2\right), & \beta = \gamma, \\ O\left(\varepsilon^{-2-(\gamma-\beta)/\alpha}\right), & 0 < \beta < \gamma. \end{cases}$$

Prior work – Higham

Higham *et al* (2013) developed a MLMC treatment of the exit time problem:

- Euler-Maruyama discretisation
- $O(h_\ell^{1/2})$ weak convergence $\Longrightarrow\ \alpha = 1/2$
- $\mathbb{V}[\widehat{P}_\ell - \widehat{P}_{\ell-1}] = O(h_\ell^{1/2} |\log h_\ell|^{1/2}) \Longrightarrow\ \beta \approx 1/2$
- $O(h_\ell^{-1})$ cost per path $\Longrightarrow\ \gamma = 1$

Hence, overall cost is approximately $O(\varepsilon^{-3})$.

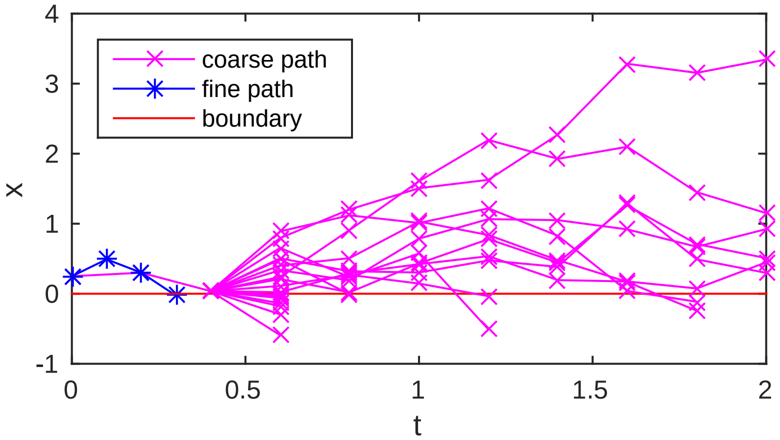
Gobet & Menozzi's boundary treatment would improve this to $O(\varepsilon^{-2.5})$.

G & Primozic (2011) developed $O(\varepsilon^{-2})$ treatment using Milstein discretisation for SDEs with special commutativity property.

MLMC challenge

When coarse or fine path exits the domain, the other is within $O(h^{1/2})$.

However, there is a $O(h^{1/2})$ probability that it will not exit the domain until much later $\Longrightarrow V_\ell \approx O(h^{1/2})$.



MLMC challenge

How can we do better?

Similar to previous work on digital options, and also used by Dickmann & Schweizer, split second path into multiple copies, and average their outputs to approximate the conditional expectation.

$O(h^{1/2})$ expected time to exit for second path, so can afford to use $O(h^{-1/2})$ copies of second path.

This gives an approximation to the conditional expectation resulting in $\widehat{P}_\ell - \widehat{P}_{\ell-1} \approx O(h^{1/2})$, so $V_\ell \approx O(h)$.

This gives $\alpha = 1/2,\ \beta \approx 1,\ \gamma \approx 1$ and the complexity is $O(\varepsilon^{-2}\,|\log \varepsilon|^3)$.

Numerical Analysis

Assumption 1: There is a Lipschitz constant L_f such that

$$|f(x,t) - f(y,s)| \leq L_f \left(\|x-y\|_2 + |t-s| \right), \quad \forall (x,t),(y,s) \in D,$$

and there are similar Lipschitz constants L_g, L_V, L_a, L_b, L_u, L_{exit} for g, V, a, b, u, u_{exit}. In addition, $g \in C^{2,1}(D)$, with a bounded Hessian $H_g \equiv \partial^2 g / \partial x^2$.

Comment: assumption about L_u, L_{exit} may require the boundary ∂D to be smooth, or at least not have re-entrant corners.

Assumption 2: There is a unit computational cost for each timestep, and in determining whether or not $\widehat{X}_{t_{n+1}} \in D$.

Assumption 3: There exist constants C_u and C_{exit} s.t. for all $(x,t) \in D$

$$\begin{aligned} |u(x,t) - \widehat{u}(x,t)| &\leq C_u \; h^{1/2} \\ |u_{exit}(x,t) - \widehat{u}_{exit}(x,t)| &\leq C_{exit} \; h^{1/2} \end{aligned}$$

Numerical Analysis

Defining the output functional

$$P_{t_0} = \int_{t_0}^{\tau} E(t_0, s)\, f(X_s, s)\, \mathrm{d}s + E(t_0, \tau)\, g(X_\tau, \tau),$$

we get

Lemma

Given Assumption 1, there exists C such that for any $(x_0, t_0) \in D$

$$\mathbb{V}[P_{t_0} \,|\, X_{t_0} = x_0] \leq C\ \mathbb{E}[\tau - t_0 \,|\, X_{t_0} = x_0].$$

By Itô calculus,

$$\mathrm{d}\left(E(t_0, s)\, g(X_s, s)\right) =$$

$$E(t_0, s)\left(\left(-V\, g + \dot{g} + (\nabla g)^T a + \tfrac{1}{2}\,\mathrm{trace}(b^T H_g\, b)\right) \mathrm{d}s + (\nabla g)^T\, b\, \mathrm{d}W_s\right)$$

with a, b, g, $\dot{g} \equiv \partial g / \partial t$, ∇g, H_g, all evaluated at (X_s, s).

Numerical Analysis

Hence, $P_{t_0} - g(x_0, t_0) = p^{(1)} + p^{(2)}$, where

$$
\begin{aligned}
p^{(1)} &= \int_{t_0}^{\tau} E(t_0, s) \left(f - V\, g + \dot{g} + (\nabla g)^T a + \tfrac{1}{2}\, \mathrm{trace}(b^T H_g\, b) \right) \mathrm{d}s, \\
p^{(2)} &= \int_{t_0}^{\tau} E(t_0, s)\, (\nabla g)^T\, b\, \mathrm{d}W_s.
\end{aligned}
$$

Considering the second term, since $E(t_0, s) \leq \exp(T \|V\|_\infty)$, we have

$$
\begin{aligned}
\mathbb{E}[(p^{(2)})^2] &= \mathbb{E}\left[\int_{t_0}^{\tau} (E(t_0, s))^2\ \|(\nabla g)^T b\|_2^2\ \mathrm{d}s \right] \\
&\leq \exp(2T\|V\|_\infty)\ \|\nabla g\|_{2,\infty}^2 \|b\|_{2,\infty}^2\ \mathbb{E}[\tau - t_0 \,|\, X_{t_0} = x_0],
\end{aligned}
$$

where $\|b\|_{2,\infty}$, $\|\nabla g\|_{2,\infty}$ are the maximum values of $\|b\|_2$, $\|\nabla g\|_2$ over D.

The first term is handled similarly to complete the proof.

Numerical Analysis

The following is a standard result:

Lemma

If W and Z are independent random variables, then

$$Y = M^{-1} \sum_{m=1}^{M} f(W, Z^{(m)})$$

with independent samples W and $Z^{(m)}$ is an unbiased estimator for $\mathbb{E}\left[f(W,Z)\right]$ and its variance is

$$\mathbb{V}[Y] = \mathbb{V}\left[\mathbb{E}[f(W,Z) \mid W]\right] + M^{-1}\, \mathbb{E}\left[\mathbb{V}[f(W,Z) \mid W]\right].$$

Numerical Analysis

Proof: For a given W, Y has expected value $\mathbb{E}[f(W,Z)\,|\,W]$ and variance $M^{-1}\,\mathbb{V}[f(W,Z)\,|\,W]$, and therefore

$$\mathbb{E}\left[Y^2\,|\,W\right] = \Big(\mathbb{E}[f(W,Z)\,|\,W]\Big)^2 + M^{-1}\,\mathbb{V}[f(W,Z)\,|\,W]$$

Taking the expectation over W then gives

$$\begin{aligned}\mathbb{E}[Y] &= \mathbb{E}\Big[\mathbb{E}[f(W,Z)\,|\,W]\Big] = \mathbb{E}[f(W,Z)],\\ \mathbb{E}\left[Y^2\right] &= \mathbb{E}\Big[(\mathbb{E}[f(W,Z)\,|\,W])^2\Big] + M^{-1}\,\mathbb{E}\Big[\mathbb{V}[f(W,Z)\,|\,W]\Big],\end{aligned}$$

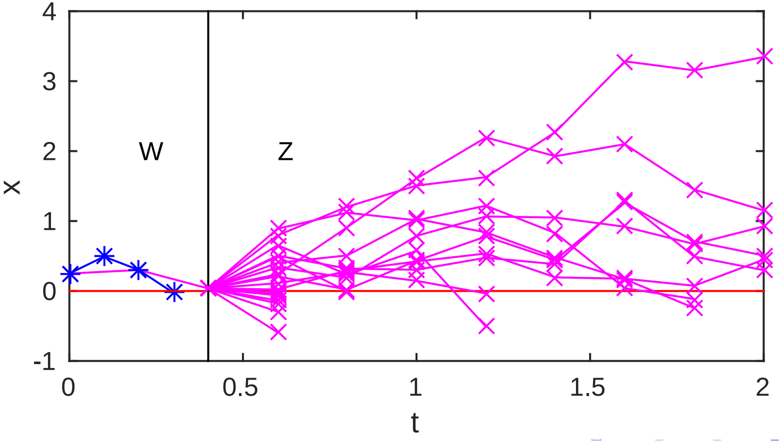
from which it follows that

$$\begin{aligned}\mathbb{V}[Y] &= \mathbb{E}\left[Y^2\right] - \Big(\mathbb{E}[Y]\Big)^2\\ &= \mathbb{V}\Big[\mathbb{E}[f(W,Z)\,|\,W]\Big] + M^{-1}\,\mathbb{E}\Big[\mathbb{V}[f(W,Z)\,|\,W]\Big]\end{aligned}$$

Numerical Analysis

Let $\underline{\tau}$ be the exit time of the first of a pair of coarse/fine paths, and $\overline{\tau}$ be $\underline{\tau}$ rounded up to the end of a coarse timestep.

In our application W represents the Brownian path up to $\overline{\tau}$, and Z is the Brownian path therafter.



Numerical Analysis

Lemma

Given Assumptions 1 and 3, we have

$$\begin{aligned} \mathbb{E}[\sup_{[0,\underline{\tau}]} \|\widehat{X}_{\ell,t} - \widehat{X}_{\ell-1,t}\|^2\,]^{1/2} &= O(h_{\ell-1}^{1/2}\,|\log h_{\ell-1}|^{1/2}) \\ \mathbb{E}[\,\|\widehat{X}_{\ell,\underline{\tau}} - \widehat{X}_{\ell-1,\overline{\tau}}\|^2\,]^{1/2} &= O(h_{\ell-1}^{1/2}\,|\log h_{\ell-1}|^{1/2}) \\ \Longrightarrow \quad \mathbb{V}\left[\,\mathbb{E}[\widehat{P}_\ell - \widehat{P}_{\ell-1}\,|\,W]\,\right] &= O(h_{\ell-1}|\log h_{\ell-1}|) \end{aligned}$$

The key here is that if $0 \leq t \leq \tau$ then

$$P_0 = E(0,t)\left\{\int_t^\tau E(t,s)\,f(X_s,s)\,\mathrm{d}s + E(t_0,\tau)\,g(X_\tau,\tau)\right\}$$

$$\Longrightarrow \quad \mathbb{E}[P_0\,|\,\mathcal{F}_t] \;=\; E(0,t)\;\mathbb{E}[P_t\,|\,\mathcal{F}_t] \;=\; E(0,t)\,u(X_t,t)$$

Something similar for the discrete approximation yields

$$\mathbb{E}[\widehat{P}_\ell - \widehat{P}_{\ell-1}\,|\,W] = O(h_{\ell-1}^{1/2}\,|\log h_{\ell-1}|^{1/2})$$

Numerical Analysis

Lemma

Given Assumptions 1 and 3,

$$\mathbb{E}\left[\,\mathbb{V}[\widehat{P}_\ell - \widehat{P}_{\ell-1} \,|\, W]\,\right] = O(h_{\ell-1}^{1/2} |\log h_{\ell-1}|^{1/2})$$

The key here is that, similar to the SDE analysis, there exists C such that

$$\begin{aligned}\mathbb{V}[\widehat{P}_\ell - \widehat{P}_{\ell-1} \,|\, W] &\leq C\ \mathbb{E}\,[\,|\widehat{\tau}_\ell - \widehat{\tau}_{\ell-1}| \,|\, W]) \\ &= O(h_{\ell-1}^{1/2}\,|\log h_{\ell-1}|^{1/2})\end{aligned}$$

Numerical Analysis

Corollary

Under the given assumptions, an RMS error of ε can be achieved with an $O(\varepsilon^{-2}|\log \varepsilon|^3)$ expected computational cost.

The proof is slightly non-standard because of log terms.

- $h_\ell = 4^{-\ell} h_0$
- $M_\ell = \lceil 2^\ell / \ell^{1/2} \rceil$ paths in the splitting estimator
- expected cost is $O(h_\ell^{-1})$
- variance $V_\ell \; = O(h_\ell \, |\log h_\ell|) = O(h_\ell \, \ell)$.

This eventually gives the desired cost bound.

Conclusions

- conditional expectation / splitting is a useful technique in MLMC estimation
- in Feynmac-Kac application it improves the MLMC variance from approximately $O(h^{1/2})$ to $O(h)$, reducing the complexity from $O(\varepsilon^{-3})$ to $O(\varepsilon^{-2}|\log\varepsilon|^3)$
- numerical analysis is now complete but relies on key assumption of uniform $O(h^{1/2})$ weak convergence – an open problem

Webpages:

`people.maths.ox.ac.uk/gilesm/mlmc.html`

`people.maths.ox.ac.uk/gilesm/mlmc_community.html`

`people.maths.ox.ac.uk/gilesm/acta/`