Stochastic Numerical Analysis

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Mean-square stability analysis

Today we are looking at the numerical analysis in

D.J. Higham. 'Mean-square and asymptotic stability of the stochastic theta method'. *SIAM Journal of Numerical Analysis*, 38(3):753-769, 2000

M.B. Giles, C. Reisinger. 'Stochastic finite differences and multilevel Monte Carlo for a class of SPDEs in finance'. *SIAM Journal of Financial Mathematics*, 3(1):572-592, 2012.

What's new?

- new numerical methods
- new definition and analysis of numerical stability

Stochastic theta method

The stochastic theta method for approximating the SDE

$$\mathrm{d}X_t = f(X_t)\,\mathrm{d}t + g(X_t)\,\mathrm{d}W_t$$

is

$$\widehat{X}_{n+1} = \widehat{X}_n + (1-\theta) f(\widehat{X}_n)\,h + \theta\, f(\widehat{X}_{n+1})\,h + g(\widehat{X}_n)\,\Delta W_n$$

- $\theta = 0$ is Euler-Maruyama method
- $\theta = 1$ is drift-implicit method
- $\theta = 1/2$ is stochastic equivalent of Crank-Nicholson method

ODE stability analysis

The linear ODE:

$$\mathrm{d}X_t = -\lambda\, X_t\; \mathrm{d}t$$

has a solution which decays exponentially if $\lambda > 0$.

The numerical approximation

$$\widehat{X}_{n+1} = \widehat{X}_n - (1-\theta)\lambda\, h\, \widehat{X}_n - \theta\, \lambda\, h\, \widehat{X}_{n+1}$$

also decays exponentially if

$$\left| \frac{1-(1-\theta)\lambda\, h}{1+\theta\lambda\, h} \right| < 1$$

so need either $\theta \geq 1/2$, or $\lambda\, h < 2/(1-2\theta)$.

SDE stability analysis

What about the corresponding SDE?

$$\mathrm{d}X_t = -\lambda\, X_t\, \mathrm{d}t + \mu\, X_t\, \mathrm{d}W_t$$

First of all, how does the analytic solution behave?

Itô calculus gives us

$$\mathrm{d}X_t^2 = -2(\lambda - \tfrac{1}{2}\mu^2)\, X_t^2\, \mathrm{d}t + 2\mu\, X_t^2\, \mathrm{d}W_t$$

and hence

$$\mathrm{d}\left(\mathbb{E}[X_t^2]\right) = -2\,(\lambda - \tfrac{1}{2}\mu^2)\; \mathbb{E}[X_t^2]\; \mathrm{d}t$$

so $\mathbb{E}[X_t^2]$ decays exponentially if $\lambda > \frac{1}{2}\mu^2$

SDE stability analysis

The numerical approximation is

$$\widehat{X}_{n+1} = \widehat{X}_n - (1-\theta)\lambda\, h\, \widehat{X}_n - \theta\, \lambda\, h\, \widehat{X}_{n+1} + \mu\, \widehat{X}_n\, \Delta W_n$$

Setting $\Delta W_n = h^{1/2} Z_n$ where Z_n is a standard Normal r.v., we can re-arrange to get

$$\widehat{X}_{n+1} = (a + b\, Z_n)\ \widehat{X}_n$$

where

$$a = \frac{1 - (1-\theta)\lambda\, h}{1 + \theta\lambda\, h}, \qquad b = \frac{\mu\, h^{1/2}}{1 + \theta\lambda\, h}$$

SDE stability analysis

Since

$$\widehat{X}_{n+1}^2 = (a + b\, Z_n)^2\; \widehat{X}_n^2$$

it follows that

$$\mathbb{E}[\widehat{X}_{n+1}^2] = (a^2 + b^2)\; \mathbb{E}[\widehat{X}_n^2]$$

so $\mathbb{E}[\widehat{X}_n^2]$ decays exponentially iff $a^2 + b^2 < 1$, which corresponds to

$$(1-2\,\theta)\,\lambda^2\,h < 2\,(\lambda - \tfrac{1}{2}\mu^2)$$

If $\lambda - \frac{1}{2}\mu^2 > 0$, then it's unconditionally stable for $\theta \geq 1/2$, while for $\theta < 1/2$ the timestep stability limit is

$$h < \frac{2\,(\lambda - \frac{1}{2}\mu^2)}{(1-2\,\theta)\,\lambda^2}$$

SDE stability analysis

Generalisation to vector systems – the algorithm

$$\widehat{X}_{n+1} = (A + Z_n\, B)\; \widehat{X}_n$$

with scalar Z_n, vector $\widehat{X}_n$ and matrices A, B, leads to

$$\widehat{X}_{n+1}^T \widehat{X}_{n+1} = \widehat{X}_n^T (A + Z_n\, B)^T (A + Z_n\, B)\; \widehat{X}_n$$

and hence

$$\mathbb{E}\left[\widehat{X}_{n+1}^T \widehat{X}_{n+1}\right] = \mathbb{E}\left[\widehat{X}_n^T (A^T A + B^T B) \widehat{X}_n\right]$$

$A^T A + B^T B$ is symmetric and positive (semi-)definite, so a sufficient condition for mean-square stability is that the largest eigenvalue is less than 1.

SDE stability analysis

It can be further generalised to

$$\widehat{X}_{n+1} = (A + Z_n\, B + W_n\, C)\ \widehat{X}_n$$

where W_n is an additional, independent Normal r.v.

This leads to

$$\mathbb{E}\left[\widehat{X}_{n+1}^T \widehat{X}_{n+1}\right] = \mathbb{E}\left[\widehat{X}_n^T (A^T A + B^T B + C^T C) \widehat{X}_n\right]$$

with a similar test for mean-square stability.

Parabolic SPDE

Unusual parabolic SPDE arises in CDO modelling (Bush, Hambly, Haworth & Reisinger)

$$\mathrm{d}p = -\mu\, \frac{\partial p}{\partial x}\, \mathrm{d}t + \frac{1}{2}\, \frac{\partial^2 p}{\partial x^2}\, \mathrm{d}t + \sqrt{\rho}\, \frac{\partial p}{\partial x}\, \mathrm{d}W$$

with absorbing boundary $p(0,t) = 0$

- derived in limit as number of firms $\longrightarrow \infty$
- x is distance to default
- $p(x,t)$ is probability density function
- $\mathrm{d}W$ term corresponds to systemic risk
- $\partial^2 p / \partial x^2$ comes from idiosyncratic risk

Parabolic SPDE

Milstein and central difference discretisation leads to

$$\begin{aligned} p_j^{n+1} \; &= \; p_j^n \; - \; \frac{\mu\, k + \sqrt{\rho\, k}\, Z_n}{2h} \left(p_{j+1}^n - p_{j-1}^n\right) \\ &\quad + \; \frac{(1-\rho)\, k + \rho\, k\, Z_n^2}{2h^2} \left(p_{j+1}^n - 2p_j^n + p_{j-1}^n\right) \end{aligned}$$

where k is the timestep, h is the uniform grid spacing, and $Z_n \sim N(0,1)$,

Considering a Fourier mode

$$p_j^n = g_n \exp(ij\theta), \quad |\theta| \leq \pi$$

leads to . . .

Parabolic SPDE

$$g_{n+1} = \left(a(\theta) + b(\theta)\, Z_n + c(\theta)\, Z_n^2 \right)\; g_n,$$

where

$$\begin{aligned}
a(\theta) &= 1 - \frac{i\,\mu\,k}{h}\;\sin\theta - \frac{2\,(1-\rho)\,k}{h^2}\sin^2\tfrac{\theta}{2},\\
b(\theta) &= -\frac{i\sqrt{\rho\,k}}{h}\;\sin\theta,\\
c(\theta) &= -\frac{2\,\rho\,k}{h^2}\sin^2\tfrac{\theta}{2}.
\end{aligned}$$

Parabolic SPDE

Following Higham's mean-square stability analysis approach,

$$\begin{aligned}\mathbb{E}[\,|g_{n+1}|^2] &= \mathbb{E}\left[(a + b\,Z_n + c\,Z_n^2)(a^* + b^* Z_n + c^* Z_n^2)\ |g_n|^2\right] \\ &= \left(\,|a{+}c|^2 + |b|^2 + 2|c|^2\,\right)\ \mathbb{E}\left[\,|g_n|^2\right]\end{aligned}$$

so stability requires $|a{+}c|^2 + |b|^2 + 2|c|^2 \ \leq\ 1$ for all θ, which leads to a timestep stability limit:

$$\begin{aligned}\mu^2 k &\leq 1 - \rho, \\ \frac{k}{h^2} &\leq (1 + 2\rho^2)^{-1}.\end{aligned}$$

Parabolic SPDE

This can be extended to finite domains using matrix stability analysis, writing the discrete equations as

$$P_{n+1} = (A + B\, Z_n + C\, Z_n^2)\; P_n, \quad \text{where}$$

$$A = I - \frac{\mu\, k}{2h}\, D_1 + \frac{(1-\rho)\, k}{2h^2}\, D_2, \quad B = -\,\frac{\sqrt{\rho\, k}}{2h}\, D_1, \quad C = \frac{\rho\, k}{2h^2}\, D_2,$$

and D_1 and D_2 look like

$$D_1 = \begin{pmatrix} 0 & 1 & & \\ -1 & 0 & 1 & \\ & -1 & 0 & 1 \\ & & -1 & 0 \end{pmatrix}, \quad D_2 = \begin{pmatrix} -2 & 1 & & \\ 1 & -2 & 1 & \\ & 1 & -2 & 1 \\ & & 1 & -2 \end{pmatrix}.$$

Parabolic SPDE

$$\mathbb{E}[\, P_{n+1}^T P_{n+1}] \;=\; \mathbb{E}\left[P_n^T (A^T\!+\!B^T\, Z_n\!+\!C^T\, Z_n^2)(A\!+\!B\, Z_n\!+\!C\, Z_n^2)\; P_n\right]$$

$$=\; \mathbb{E}\left[P_n^T \left((A\!+\!C)^T(A\!+\!C) + B^T B + 2\, C^T C\,\right) P_n\right]$$

D_1 is anti-symmetric and D_2 is symmetric, and

$$D_1 D_2 - D_2 D_1 = E_1 - E_2, \quad D_1^2 = D_3 + E_1 + E_2$$

where D_3 looks like

$$D_3 = \begin{pmatrix} -3 & 0 & 1 & \\ 0 & -2 & 0 & 1 \\ 1 & 0 & -2 & 0 \\ & 1 & 0 & -3 \end{pmatrix},$$

Parabolic SPDE

and E_1 and E_2 are zero apart from one corner element,

$$E_1 = \begin{pmatrix} 2 & & & \\ & & & \\ & & & \\ & & & \end{pmatrix}, \qquad E_2 = \begin{pmatrix} & & & \\ & & & \\ & & & \\ & & & 2 \end{pmatrix}$$

This leads to

$$\begin{aligned} & \mathbb{E}\left[V_n^T \left((A{+}C)^T (A{+}C) + B^T B + 2\, C^T C\,\right) V_n\right] \\ = \;\; & \mathbb{E}\left[V_n^T M V_n\right] - (e_1 + e_2)\, \mathbb{E}[(v_1^n)^2] - (e_1 - e_2)\, \mathbb{E}[(v_{J-1}^n)^2], \end{aligned}$$

where e_1 and e_2 are scalars and

$$M = I - \frac{k}{h^2}\, D_2 + \frac{k^2}{4\,h^4}\, D_2^2 - \left(\frac{\rho k}{4\,h^2} + \frac{\mu^2 k^2}{4\,h^2}\right) D_3.$$

Parabolic SPDE

It can be verified that the m^{th} eigenvector of M is a Fourier mode and the associated eigenvalue is

$$|a(\theta_m)+c(\theta_m)|^2 + |b(\theta_m)|^2 + 2|c(\theta_m)|^2$$

where $a(\theta), b(\theta), c(\theta)$ are the same functions as before.

In the limit $h, k/h \to 0$, $e_1 \pm e_2 > 0$, and therefore the Fourier stability condition

$$\sup_\theta \left\{ |a(\theta)+c(\theta)|^2 + |b(\theta)|^2 + 2|c(\theta)|^2 \right\} \leq 1$$

is also a sufficient condition for mean-square matrix stability.

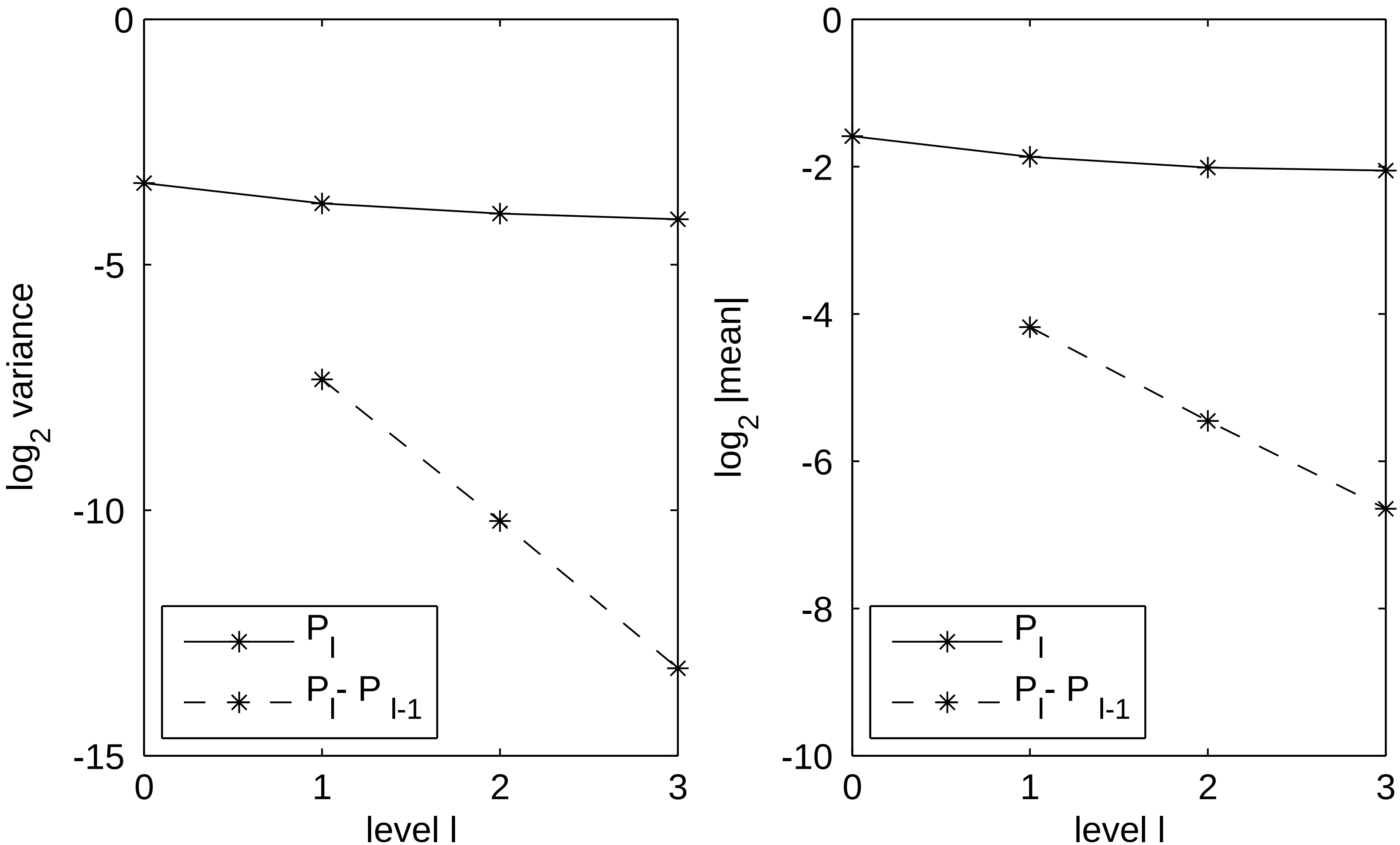
Parabolic SPDE

This turns out to be a good application for multilevel MC:

- coarsest level of approximation uses 1 timestep per quarter, and 10 spatial points
- each finer level uses four times as many timesteps, and twice as many spatial points – ratio is due to numerical stability constraints
- computational cost $C_\ell \propto 8^\ell$
- numerical results suggest variance $V_\ell \propto 8^{-\ell}$
- can prove $V_\ell \propto 16^{-\ell}$ when no absorbing boundary

Parabolic SPDE

Fractional loss on equity tranche of a 5-year CDO:



Parabolic SPDE

Fractional loss on equity tranche of a 5-year CDO: