Module 2: Monte Carlo Methods
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Monte Carlo Practical

1. (a) Generate 10^6 uniform random variables using `rand`, then convert them into 10^6 unit Normal variables using `norminv`. Check that they have the expected mean and variance.

   (b) Given a covariance matrix

   $$\Sigma = \begin{pmatrix} 4 & 1 \\ 1 & 4 \end{pmatrix}$$

   perform a Cholesky factorisation (using Matlab function `chol(Sigma,'lower')`) to obtain a lower-triangular matrix L such that

   $$\Sigma = L\, L^T$$

   Use this matrix L to convert 2×10^6 independent unit Normals (generated using Matlab function `randn`) into 10^6 pairs of Normals with the desired covariance. Check that they have the expected mean and covariance.

2. The objective here is to estimate the price of a European call option with discounted payoff

   $$f(S) = \exp(-rT)\ (S-K)^+$$

   where the underlying is modelled by Geometric Brownian Motion so

   $$S_T = S_0\ \exp\left((r-\tfrac{1}{2}\sigma^2)T + \sigma W_T\right)$$

   where

   $$W_T = \sqrt{T}\ X$$

   and X is a unit Normal.

   Use the constants $r=0.05,\ \sigma=0.2,\ T=1,\ S_0=100,\ K=100.$

   (a) Using the Matlab `randn` function to generate unit Normals, write a Matlab program which computes

   $$Y_m = N^{-1} \sum_{n=1}^{N} f(S_T(W_T(X^{(m,n)}))), \quad m=1,\ldots 10000,\ n=1,\ldots 100,$$

   for 10000 different sets of 100 independent Normal variables $X^{(m,n)}$.

(b) Sort the Y_m into ascending order, and then plot $C_m = (m - 1/2)/10000$ versus Y_m – this is the numerical cumulative distribution function.

Superimpose on the same plot the cumulative distribution function you would expect from the Central Limit Theorem (using `normcdf` or `norminv`); you should find they match remarkably well.

The analytic value is given by the routine `european_call` available from my webpage; read its header to see how to call it. There is no need to compute the analytic variance; just use the unbiased estimator.

You may like to experiment by trying larger or smaller sets of points to improve your understanding of the asymptotic behaviour described by the CLT.

3. For the same European call as Q2, investigate the following forms of variance reduction:

   (a) First, try antithetic variables using $\frac{1}{2}\left(f(S_T(W_T)) + f(S_T(-W_T))\right)$ where W_T is the value of the underlying Brownian motion at maturity.

   What is the estimated correlation between $f(S_T(W_T))$ and $f(S_T(-W_T))$? How much variance reduction does this give?

   (b) Second, try using $\exp(-rT)\, S_T$ as a control variate, noting that its expected value is S_0.

   Again, how much variance reduction does this give?

4. For the case of Geometric Brownian Motion and a digital put option, with parameters, $r = 0.05,\ \sigma = 0.2,\ T = 1,\ S_0 = 100,\ K = 50$, investigate the use of importance sampling:

   (a) First, estimate the value without importance sampling. Check your results are correct by comparing to the analytic values given by `digital_put`.

   How many samples are needed to obtain a value which is correct to within 10%? (i.e. the 3 standard deviation confidence limit corresponds to ± 10%).

   (b) Second, try using importance sampling, adjusting the drift so that half of the samples are below the strike at maturity, and the other half are above.

   Now how many samples are required to get the value correct to within 10%?

5. For the same European call as Q2, use both the Likelihood Ratio Method and "pathwise" sensitivity analysis to compute delta and vega, the sensitivities to changes in the initial price and the volatility.

   Check your results are correct by comparing to the analytic values given by `european_call`.