Numerical Methods II
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Problem sheet 3

1. Applying the Euler-Maruyama method with timestep h and initial data to the Geometric Brownian motion SDE

$$\mathrm{d}S = r\, S\, \mathrm{d}t + \sigma\, S\, \mathrm{d}W$$

gives

$$\widehat{S}_{n+1} = \widehat{S}_n \left(1 + r\, h + \sigma\, \Delta W_n\right)$$

If the initial data is $S(0)=1$, and we change variables to $\widehat{X}_n = \log \widehat{S}_n$, then

$$\widehat{X}_{T/h} = \sum_n \log(1 + r\, h + \sigma\, \Delta W).$$

By performing an asymptotic expansion of the log terms, deduce that the leading order error in $\widehat{X}_{T/h}$ is

$$\sum_n \tfrac{1}{2}\, \sigma^2 \left((\Delta W_n)^2 - h \right)$$

and prove that in the limit as $h \to 0$ this is Normally distributed with zero mean and $O(h)$ variance.

2. One simple second-order method for approximating the ODE

$$\mathrm{d}S = a(S,t)\ \mathrm{d}t$$

is the predictor-corrector scheme

$$\begin{aligned}
\widehat{S}^{(p)}_{n+1} &= \widehat{S}_n + a(\widehat{S}_n, t_n)\, h \\
\widehat{S}_{n+1} &= \widehat{S}_n + \tfrac{1}{2} \left(a(\widehat{S}_n, t_n) + a(\widehat{S}^{(p)}_{n+1}, t_{n+1}) \right)\ h
\end{aligned}$$

Trying to apply this scheme to the SDE

$$\mathrm{d}S = a(S,t)\ \mathrm{d}t + b(S,t)\, \mathrm{d}W$$

one might think of trying

$$\begin{aligned}
\widehat{S}^{(p)}_{n+1} &= \widehat{S}_n + a(\widehat{S}_n, t_n)\, h + b(\widehat{S}_n, t_n)\, \Delta W_n \\
\widehat{S}_{n+1} &= \widehat{S}_n + \tfrac{1}{2} \left(a(\widehat{S}_n, t_n) + a(\widehat{S}^{(p)}_{n+1}, t_{n+1}) \right)\ h + \tfrac{1}{2} \left(b(\widehat{S}_n, t_n) + b(\widehat{S}^{(p)}_{n+1}, t_{n+1}) \right)\ \Delta W_n
\end{aligned}$$

Show that if this scheme is applied to the geometric Brownian motion in the previous question this approximation will not even converge to the true solution as $h \to 0$.

3. Determine the discrete equations one gets by applying the Milstein method to the following SDEs:

   (a) Ornstein-Uhlenbeck process

   $$\mathrm{d}S = \kappa\,(\theta - S)\,\mathrm{d}t + \sigma\,\mathrm{d}W$$

   (b) CIR process

   $$\mathrm{d}S = \kappa\,(\theta - S)\,\mathrm{d}t + \sigma\,\sqrt{S}\,\mathrm{d}W$$

4. Write out in full detail (so that you can implement it in the next Practical!) how one would use the pathwise sensitivity method to estimate the Delta and Vega for the following two cases with the standard Geometric Brownian Motion

   $$\mathrm{d}S = r\,S\,\mathrm{d}t + \sigma\,S\,\mathrm{d}W$$

   as the underlying SDE:

   - down-and-out barrier option using Alternative 1 from lecture 10 to achieve $O(h)$ weak convergence
   - floating-strike lookback call option using the approximate sampling of the minimum of each timestep to achieve $O(h)$ weak convergence