Numerical Methods II
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Problem sheet 2

1. In stratified sampling, we considered one sample per stratum, and each sample was uniformly distributed on its stratum, independent of the others.

   Suppose instead that we have N samples defined by

   $$x_j = \frac{j+U}{N}, \quad j = 0, 1, 2, \ldots, N-1,$$

   where U is a single random variable, uniformly distributed on $[0,1]$.

   We can then estimate the value of $\mathbb{E}[f(x)]$ by using

   $$N^{-1} \sum_{j=0}^{N-1} f(x_j).$$

   This is a form of 1D Quasi-Monte Carlo, using a regular lattice j/N with a random offset, U.

   Prove that the estimate is unbiased, and find the asymptotic form of the variance when $N \gg 1$ (assuming that $f(x)$ has a bounded second derivative).

2. Suppose that X_t satisfies the SDE

   $$\mathrm{d}X_t = (a - bX_t)\ \mathrm{d}t + c\ \mathrm{d}W_t, \quad X_0 = x_0.$$

   Determine the probability distribution for X_T.

   (Hint: you might find it helpful to start by deriving the SDE for

   $$Y_t \equiv \exp(b\,t)\ (X_t - d)$$

   for a particular choice of constant d.)

   Derive the score functions in the likelihood ratio method estimates for option derivatives with respect to the initial value x_0 and the volatility parameter c.

3. **[2013 exam question]**

Suppose that we have a European option based on 5 underlying assets with log-normal distributions so that

$$S_i(T) = S_i(0)\ \exp\left((r - \tfrac{1}{2}\sigma_i^2)T + \sigma_i W_i(T)\right), \quad i = 1, 2, \ldots, 5$$

where $W(T)$ has a multivariate Normal distribution with covariance matrix Σ.

(i) Explain how it is possible to simulate the $W_i(T)$ by generating independent uniform random variates U_i on the interval $(0, 1)$, and then defining

$$W(T) = L\ \Phi^{-1}(U)$$

where $\Phi(x)$ is the Normal cumulative distribution function, and L is a matrix which you should define.

(ii) One variance reduction technique for general payoff functions is the use of quasi-random numbers. Outline the ideas behind this.

In the best case, how does the error decrease with the number of sampling points, compared to the standard Monte Carlo method?

Outline how you can obtain a confidence interval through the introduction of randomisation. (There is no need to explain the details of how the randomisation is performed.)

(iii) Another variance reduction technique is the use of control variates. Explain how a weighted average of the $S_i(T)$ can be used as a control variate, and derive an expression for the optimal weighting.

(iv) Suppose that it is known that the European payoff function is non-zero only when each of the uniform random variates is within a small range,

$$l_i < U_i < u_i.$$

Explain how importance sampling can be used to reduce the variance of the standard Monte Carlo estimator by uniformly sampling U within this range. In particular, derive an expression for the variance to prove that it has been reduced.

How could we generate uniform samples of U within this range?

4. **[2013 exam question]**

A simple model for an asset S undergoing a single period jump process is

$$S = S_0 \exp(X)$$

where X is a random variable with the double-tail exponential distribution with density $\frac{\lambda}{2}\exp(-\lambda\,|x|)$ where $\lambda > 0$.

(i) Determine the cumulative distribution function for X, and explain how this can can be used to generate samples $X^{(n)}$ given a random number generator which produces uniformly distributed random numbers on the open interval $(0, 1)$.

(ii) Explain how you could use Monte Carlo simulation to estimate $\mathbb{E}[P]$, where $P = \max(S - K, 0)$. Explain how the Central Limit Theorem enables you to determine a 99% confidence interval for your estimate.

(iii) Explain how you would use "pathwise sensitivity" analysis to determine the sensitivity of the expected value to changes in a) the initial asset price S_0 and b) the jump parameter λ.

(iv) Determine the probability density function for S.

(v) Explain why you cannot use "pathwise sensitivity" analysis to determine the sensitivity of the expected value of a digital payoff $P = \mathbf{1}_{S>K}$ to changes in S_0. Explain how you could instead use the Likelihood Ratio Method.