Numerical Methods II

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SDE Path Simulation

Lectures 1-8 dealt with the case of European options for which the underlying SDE could be integrated exactly.

Lectures 9-16 address the more general case in which the solution to the SDE needs to be approximated because

- the option is path-dependent, and/or
- the SDE is not integrable

Euler-Maruyama method

The simplest approximation for the scalar SDE

$$\mathrm{d}S = a(S,t)\,\mathrm{d}t + b(S,t)\,\mathrm{d}W$$

is the forward Euler scheme, which is known as the Euler-Maruyama approximation when applied to SDEs:

$$\widehat{S}_{n+1} = \widehat{S}_n + a(\widehat{S}_n, t_n)\,h + b(\widehat{S}_n, t_n)\,\Delta W_n$$

Here h is the timestep, $\widehat{S}_n$ is the approximation to $S(nh)$ and the ΔW_n are i.i.d. $N(0,h)$ Brownian increments.

Euler-Maruyama method

For ODEs, the forward Euler method has $O(h)$ accuracy, and other more accurate methods would usually be preferred.

However, SDEs are very much harder to approximate so the Euler-Maruyama method is used widely in practice.

Numerical analysis is also very difficult and even the definition of “accuracy” is tricky.

Weak convergence

In finance applications, mostly concerned with **weak** errors, the error in the expected payoff. For a European payoff $f(S(T))$ this is

$$\mathbb{E}[f(S(T))] - \mathbb{E}[f(\widehat{S}_{T/h})]$$

and it is of order α if

$$\mathbb{E}[f(S(T))] - \mathbb{E}[f(\widehat{S}_{T/h})] = O(h^\alpha)$$

For a path-dependent option, the weak error is

$$\mathbb{E}[f(S)] - \mathbb{E}[\widehat{f}(\widehat{S})]$$

where $f(S)$ is a function of the entire path $S(t)$, and $\widehat{f}(\widehat{S})$ is a corresponding approximation.

Weak convergence

Key theoretical result (Bally and Talay, 1995):

If $p(S)$ is the p.d.f. for $S(T)$ and $\widehat{p}(S)$ is the p.d.f. for $\widehat{S}_{T/h}$ computed using the Euler-Maruyama approximation, then if $a(S,t)$ and $b(S,t)$ are Lipschitz w.r.t. S,t

$$\|p(S) - \widehat{p}(S)\|_1 = O(h)$$

and hence for bounded payoffs

$$\mathbb{E}[f(S(T))] - \mathbb{E}[f(\widehat{S}_{T/h})] = O(h)$$

(This holds even for digital options with discontinuous payoffs $f(S)$. Earlier theory covered only European options such as put and call options with Lipschitz payoffs.)

Weak convergence

Numerical demonstration: Geometric Brownian Motion

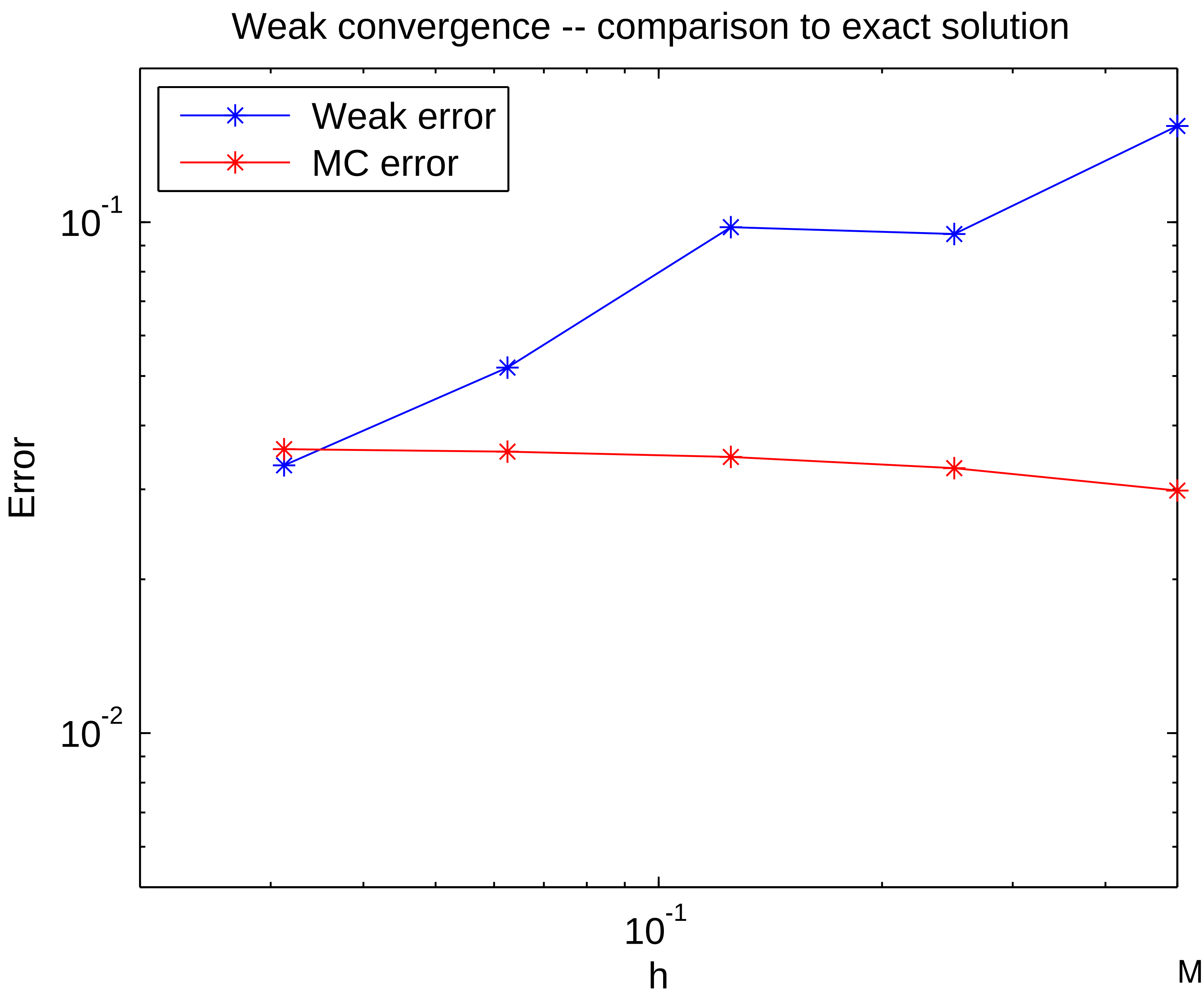
$$\mathrm{d}S = r\,S\,\mathrm{d}t + \sigma\,S\,\mathrm{d}W$$

$r = 0.05,\ \sigma = 0.5,\ T = 1$

European call: $S_0 = 100, K = 110$.

Plot shows weak error versus analytic expectation when using 10^8 paths, and also Monte Carlo error (3 standard deviations)

Weak convergence



Weak convergence

Previous plot showed difference between exact expectation and numerical approximation.

What if the exact solution is unknown? Compare approximations with timesteps h and $2h$.

If

$$\mathbb{E}[f(S(T))] - \mathbb{E}[f(\widehat{S}^{h}_{T/h})] \approx a\ h$$

then

$$\mathbb{E}[f(S(T))] - \mathbb{E}[f(\widehat{S}^{2h}_{T/2h})] \approx 2\,a\,h$$

and so

$$\mathbb{E}[f(\widehat{S}^{h}_{T/h})] - \mathbb{E}[f(\widehat{S}^{2h}_{T/2h})] \approx a\,h$$

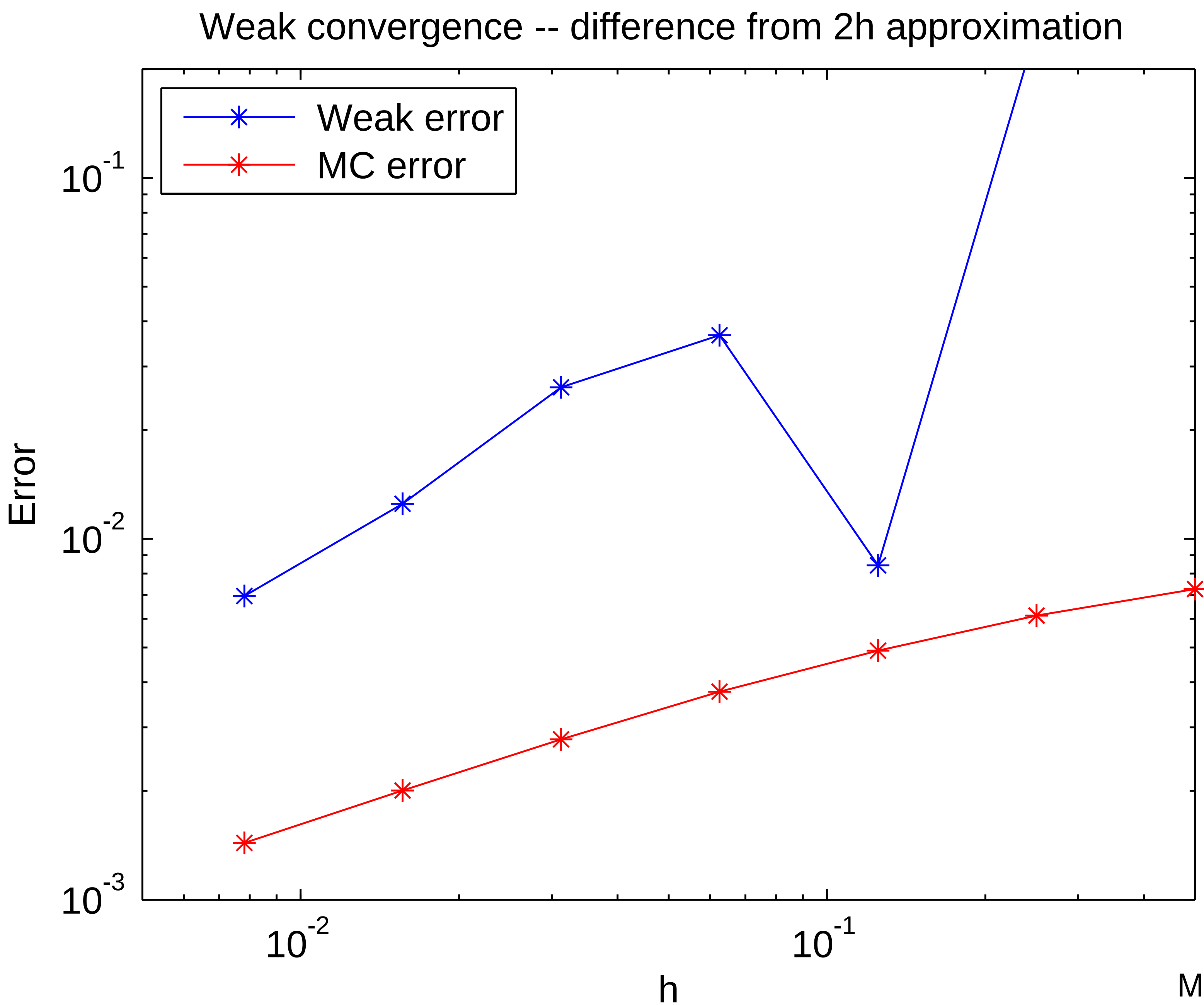
Weak convergence

To minimise the number of paths that need to be simulated, best to use **same** driving Brownian path when doing $2h$ and h approximations – i.e. take Brownian increments for h simulation and sum in pairs to get Brownian increments for $2h$ simulation.

This is like using the same driving Brownian paths for finite difference Greeks. The variance is lower because the h and $2h$ paths are close to each other (**strong** convergence).

In a later lecture, this forms the basis for the **Multilevel Monte Carlo** method (Giles, 2006)

Weak convergence



Strong convergence

Strong convergence looks instead at the average error in each individual path:

$$\left(\mathbb{E}\left[\left(S(T) - \widehat{S}_{T/h}\right)^2\right]\right)^{1/2} \quad \text{or} \quad \left(\mathbb{E}\left[\sup_{[0,T]}\left(S(t) - \widehat{S}_{t/h}\right)^2\right]\right)^{1/2}$$

It is of order β if it is $O(h^\beta)$ as $h \to 0$.

The main theoretical result (Kloeden & Platen 1992) is that for the Euler-Maruyama method if $a(S,t)$ and $b(S,t)$ are again Lipschitz then these are both $O(\sqrt{h})$.

Strong convergence

Thus, each approximate path deviates by $O(\sqrt{h})$ from its true path.

How can the weak error be $O(h)$? Because the error

$$S(T) - \widehat{S}_{T/h}$$

has mean $O(h)$ even though the r.m.s. is $O(\sqrt{h})$.

(In fact to leading order it is normally distributed with zero mean and variance $O(h)$.)

Strong convergence

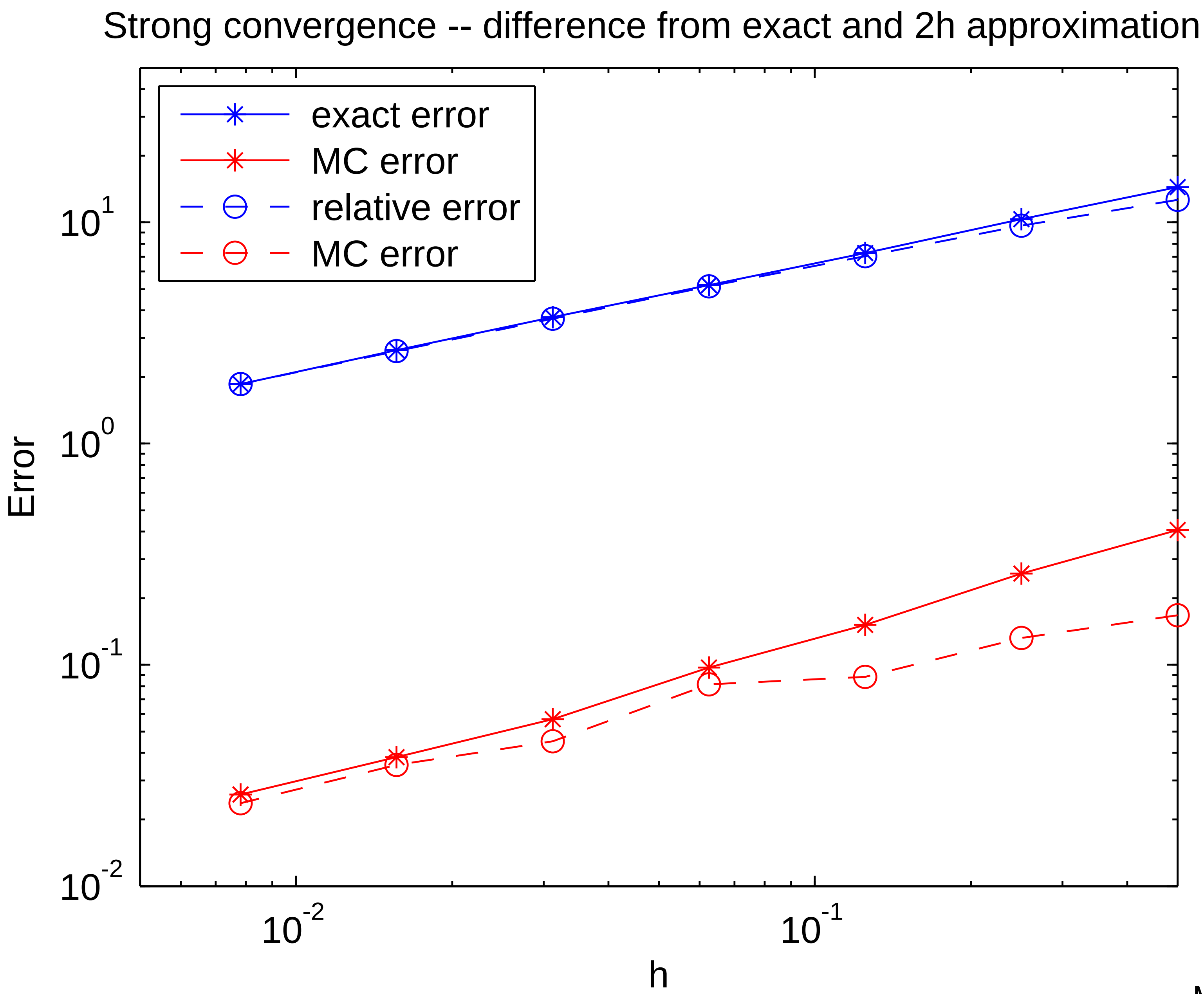
Numerical demonstration based on same Geometric Brownian Motion.

Plot shows two curves, one showing the difference from the true solution

$$S(T) = S_0 \, \exp\left((r - \tfrac{1}{2}\sigma^2)T + \sigma\, W(T) \right)$$

and the other showing the difference from the $2h$ approximation

Strong convergence



Mean Square Error

Finally, how to decide whether it is better to increase the number of timesteps (reducing the weak error) or the number of paths (reducing the Monte Carlo sampling error)?

If the true option value is $$V = \mathbb{E}[f]$$

and the discrete approximation is $$\widehat{V} = \mathbb{E}[\widehat{f}]$$

and the Monte Carlo estimate is $$\widehat{Y} = \frac{1}{N} \sum_{n=1}^{N} \widehat{f}^{(n)}$$

then ...

Mean Square Error

...the Mean Square Error is

$$
\begin{aligned}
\mathbb{E}\left[\left(\widehat{Y}-V\right)^2\right] &= \mathbb{E}\left[\left(\widehat{Y}-\mathbb{E}[\widehat{f}] \;+\; \mathbb{E}[\widehat{f}]-\mathbb{E}[f]\right)^2\right] \\
&= \mathbb{E}\left[(\widehat{Y}-\mathbb{E}[\widehat{f}])^2\right] + (\mathbb{E}[\widehat{f}]-\mathbb{E}[f])^2 \\
&= N^{-1}\mathbb{V}[\widehat{f}] + \left(\mathbb{E}[\widehat{f}]-\mathbb{E}[f]\right)^2
\end{aligned}
$$

- first term is due to the variance of estimator
- second term is square of bias due to weak error

Mean Square Error

If there are M timesteps, the computational cost is proportional to $C = NM$ and the MSE is approximately

$$a\,N^{-1} + b\,M^{-2} = a\,N^{-1} + b\,C^{-2}N^2.$$

For a fixed computational cost, this is a minimum when

$$N = \left(\frac{a\,C^2}{2\,b}\right)^{1/3}, \quad M = \left(\frac{2\,b\,C}{a}\right)^{1/3},$$

and hence

$$a\,N^{-1} = \left(\frac{2\,a^2 b}{C^2}\right)^{1/3}, \quad b\,M^{-2} = \left(\frac{a^2 b}{4\,C^2}\right)^{1/3},$$

so the MC term is twice as big as the bias term.

Final Words

- simple Euler-Maruyama method is basis for most Monte Carlo simulation in industry – $O(h)$ weak convergence and $O(\sqrt{h})$ strong convergence
- weak convergence is very important when estimating expectations
- strong convergence is usually not important – but is key for multilevel Monte Carlo method to be discussed later
- Mean-Square-Error is minimised by balancing bias due to weak error and Monte Carlo sampling error