# Numerical Methods II

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# Expectation and Integration

If $x$ is a random variable uniformly distributed on $[0,1]$ then the expectation of a function $f(x)$ is equal to its integral:

$$\overline{f} = \mathbb{E}[f(x)] = I[f] = \int_0^1 f(x)\,\mathrm{d}x.$$

The generalisation to a $d$-dimensional "cube" $I^d = [0,1]^d$, is

$$\overline{f} = \mathbb{E}[f(x)] = I[f] = \int_{I^d} f(x)\,\mathrm{d}x.$$

Thus the problem of finding expectations in finance is directly connected to the problem of numerical quadrature (integration), often in very large dimensions.

# Expectation and Integration

Suppose we have a sequence $x_n$ of independent samples from the uniform distribution.

An approximation to the expectation/integral is given by

$$I_N[f] = N^{-1} \sum_{n=1}^{N} f(x_n).$$

Two key features:

- Unbiased: $\mathbb{E}\left[I_N[f]\right] = I[f]$
- Convergent: $\lim_{N\to\infty} I_N[f] = I[f]$

# Expectation and Integration

In general, define

- error $\varepsilon_N(f) = I[f] - I_N[f]$
- bias $= \mathbb{E}[\varepsilon_N(f)]$
- RMSE, “root-mean-square-error” $= \sqrt{\mathbb{E}[(\varepsilon_N(f))^2]}$

The Central Limit Theorem proves that for large $N$

$$\varepsilon_N(f) \sim \sigma\, N^{-1/2}\, Z$$

with $Z$ a $N(0,1)$ random variable and $\sigma^2$ the variance of $f$:

$$\sigma^2 = \mathbb{E}[(f-\overline{f})^2] = \int_{I^d} \left(f(x) - \overline{f}\right)^2 \, \mathrm{d}x.$$

# Expectation and Integration

More precisely, provided $\sigma$ is finite, then as $N \longrightarrow \infty$,

$$\text{CDF}(N^{1/2}\sigma^{-1}\varepsilon_N) \longrightarrow \text{CDF}(Z)$$

so that

$$\mathbb{P}\left[N^{1/2}\sigma^{-1}\varepsilon_N < s\right] \;\longrightarrow\; \mathbb{P}\left[Z < s\right] \;=\; \Phi(s)$$

and

$$\mathbb{P}\left[\left|N^{1/2}\sigma^{-1}\varepsilon_N\right| > s\right] \;\longrightarrow\; \mathbb{P}\left[|Z| > s\right] \;=\; 2\,\Phi(-s)$$

$$\mathbb{P}\left[\left|N^{1/2}\sigma^{-1}\varepsilon_N\right| < s\right] \;\longrightarrow\; \mathbb{P}\left[|Z| < s\right] \;=\; 1 - 2\,\Phi(-s)$$

# Expectation and Integration

Given $N$ samples, the empirical variance is

$$\widetilde{\sigma}^2 = N^{-1} \sum_{n=1}^{N} \left(f(x_n) - I_N\right)^2 = I_N^{(2)} - (I_N)^2$$

where

$$I_N = N^{-1} \sum_{n=1}^{N} f(x_n), \qquad I_N^{(2)} = N^{-1} \sum_{n=1}^{N} \left(f(x_n)\right)^2$$

$\widetilde{\sigma}^2$ is a slightly biased estimator for $\sigma^2$; an unbiased estimator is

$$\widehat{\sigma}^2 = (N-1)^{-1} \sum_{n=1}^{N} \left(f(x_n) - I_N\right)^2 = \frac{N}{N-1} \left( I_N^{(2)} - (I_N)^2 \right)$$

# Expectation and Integration

Objective: want an accuracy of $\overline{\varepsilon}$ with confidence $c$.
i.e. $|\varepsilon| < \overline{\varepsilon}$ with probability $c$.

How many samples do we need to use?

Recall,

$$\mathbb{P}\left[N^{1/2}\sigma^{-1}|\varepsilon| < s\right] \approx 1 - 2\,\Phi(-s),$$

so define function $s(c)$ such that

$$1 - 2\,\Phi(-s) = c \iff s = -\Phi^{-1}((1-c)/2)$$

# Expectation and Integration

| $c$ | 0.683 | 0.9545 | 0.9973 | 0.99994 |
|---|---|---|---|---|
| $s$ | 1.0 | 2.0 | 3.0 | 4.0 |

Then $|\varepsilon| < N^{-1/2}\,\sigma\,s(c)$ with probability $c$, so to get $|\varepsilon| < \overline{\varepsilon}$ we can put

$$N^{-1/2}\,\widehat{\sigma}\,s(c) = \overline{\varepsilon} \quad\Longrightarrow\quad N = \left(\frac{\widehat{\sigma}\,s(c)}{\overline{\varepsilon}}\right)^2.$$

Note: twice as much accuracy requires 4 times as many samples.

# Expectation and Integration

How does Monte Carlo integration compare to grid based methods for $d$-dimensional integration?

MC error is proportional to $N^{-1/2}$ independent of the dimension.

If the integrand is sufficiently smooth, trapezoidal integration with $M = N^{1/d}$ points in each direction has

$$\text{Error} \ \propto \ M^{-2} \ = \ N^{-2/d}$$

This scales better than MC for $d < 4$, but worse for $d > 4$.
i.e. MC is better at handling high dimensional problems.

# Applications

Geometric Brownian motion for single asset:

$$S(T) = S_0 \ \exp\left((r - \tfrac{1}{2}\sigma^2)T + \sigma\, W(T)\right)$$

$W(T)$ has a Normal distribution with mean 0, variance $T$; from this we will calculate the risk-neutral expectation for

$$V = \mathbb{E}\left[f(S(T))\right]$$

# Applications

We can put

$$W(T) = \sqrt{T}\,Y = \sqrt{T}\,\Phi^{-1}(U)$$

where $Y$ is a $N(0,1)$ random variable, and $U$ is uniformly distributed on $[0,1]$.

Thus

$$V = \mathbb{E}\left[f(S(T))\right] = \int_0^1 f(S(T))\,\mathrm{d}U,$$

with

$$\begin{aligned} S(T) &= S_0\,\exp\left((r-\tfrac{1}{2}\sigma^2)T + \sigma\sqrt{T}\,Y\right) \\ &= S_0\,\exp\left((r-\tfrac{1}{2}\sigma^2)T + \sigma\sqrt{T}\,\Phi^{-1}(U)\right) \end{aligned}$$

# Applications

For the European call option,

$$f(S) = \exp(-rT)\ (S-K)^+$$

while for the European put option

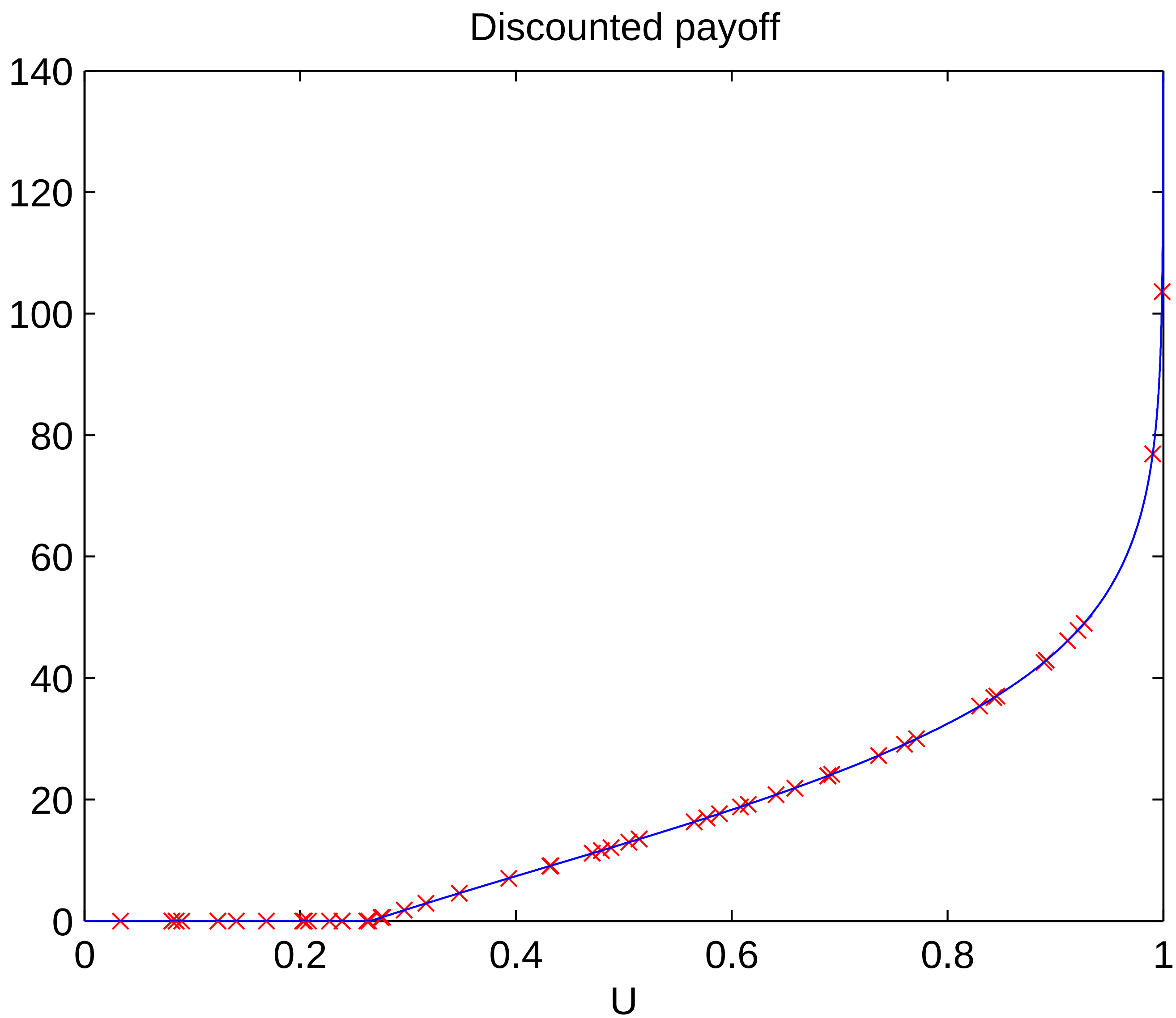
$$f(S) = \exp(-rT)\ (K-S)^+$$

where $K$ is the strike price.

For numerical experiments we will consider a European call with $\quad r=0.05, \quad \sigma = 0.2, \quad T=1, \quad S_0=110, \quad K=100.$

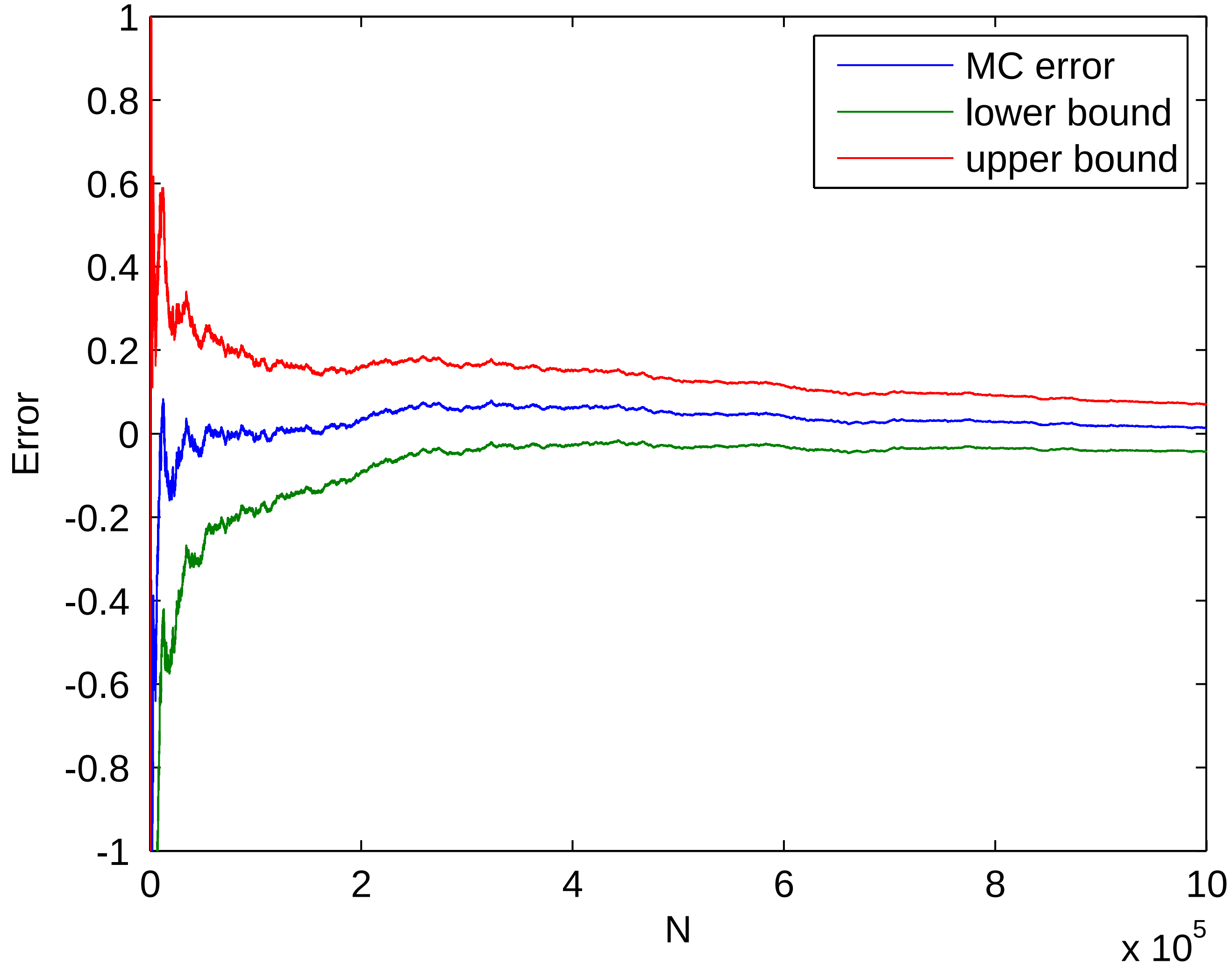
The analytic value is known for comparison.

# Applications



# Applications

MC calculation with up to $10^6$ paths; true value = 17.663



# Applications

The upper and lower bounds are given by

$$\text{Mean} \pm \frac{3\,\widetilde{\sigma}}{\sqrt{N}},$$

so more than a 99.7% probability that the true value lies within these bounds.

# Applications

MATLAB code:

```
r=0.05;  sig=0.2;  T=1;  S0=110;  K=100;
N = 1:1000000;
U = rand(1,max(N));     % uniform random variable
Y = norminv(U);         % inverts Normal cum. fn.
S = S0*exp((r-sig^2/2)*T + sig*sqrt(T)*Y);
F = exp(-r*T)*max(0,S-K);

sum1 = cumsum(F);       % cumulative summation of
sum2 = cumsum(F.^2);    % payoff and its square
val  = sum1./N;
rms  = sqrt(sum2./N - val.^2);
```

# Applications

```
err = european_call(r,sig,T,S0,K,'value') - val;

plot(N,err,                     ...
     N,err-3*rms./sqrt(N),      ...
     N,err+3*rms./sqrt(N))
axis([0 length(N) -1 1])
xlabel('N'); ylabel('Error')
legend('MC error','lower bound','upper bound')
```

# Applications

New application: basket option

European call for arithmetic average of M stocks which are correlated so that

$$\mathrm{d}S_i = r\, S_i\, \mathrm{d}t + \sigma_i S_i \mathrm{d}W_i$$

with the different $\mathrm{d}W_i$ not independent.

As before, get

$$S_i(T) = S_i(0)\ \exp\left( (r - \tfrac{1}{2}\sigma_i^2) T + \sigma_i W_i(T) \right)$$

# Applications

If $\sigma_i W_i(T)$ have covariance matrix $\Sigma$, then use Cholesky factorisation $LL^T = \Sigma$ to get

$$S_i(T) = S_i(0)\ \exp\left( (r - \tfrac{1}{2}\sigma_i^2)T + \sum_j L_{ij} Y_j \right)$$

where $Y_j$ are independent $N(0,1)$ random variables.

Each $Y_i$ can in turn be expressed as $\Phi^{-1}(U_i)$ where the $U_i$ are uniformly, and independently, distributed on $[0,1]$.

# Applications

The payoff is

$$f = \exp(-rT) \left( \frac{1}{M} \sum_i S_i - K \right)^+$$

and so the expectation can be written as the $M$-dimensional integral

$$\int_{I^M} f(U) \, \mathrm{d}U.$$

This is a good example for Monte Carlo simulation – cost scales linearly with the number of stocks, whereas it would be exponential for grid-based numerical integration.

# Final Words

- Monte Carlo quadrature is straightforward and robust
- Confidence bounds can be obtained as part of the calculation
- Can calculate the number of samples $N$ needed for chosen accuracy
- Much more efficient than grid-based methods for high dimensions
- Accuracy = $O(N^{-1/2})$, CPU time = $O(N)$

$\Longrightarrow$ accuracy = $O(\text{CPU time}^{-1/2})$

$\Longrightarrow$ CPU time = $O(\text{accuracy}^{-2})$