Numerical Methods II

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Estimating Greeks

Finite differences can again be used to estimate Greeks, with all of the advantages/disadvantages discussed in lectures 7 & 8.

We will now look at the extensions of

- Likelihood Ratio Method (LRM)
- pathwise sensitivity method

for path simulations.

To understand details and efficiency, will compare how each is used to estimate Vega for Geometric Brownian Motion with European payoff.

Likelihood Ratio Method

Reminder of lecture 8: defining $p(S)$ to be the p.d.f. for the final state $S(T)$, then

$$V = \mathbb{E}[f(S(T))] = \int f(S)\, p(S)\, \mathrm{d}S,$$

Dependence on input parameters (e.g. σ) comes in through p.d.f. $p(S)$ and so

$$\Longrightarrow \quad \frac{\partial V}{\partial \theta} = \int f\, \frac{\partial p}{\partial \theta}\, \mathrm{d}S = \int f\, \frac{\partial (\log p)}{\partial \theta}\, p\, \mathrm{d}S = \mathbb{E}\left[f\, \frac{\partial (\log p)}{\partial \theta} \right]$$

Likelihood Ratio Method

For Geometric Brownian Motion,

$$\log p = -\log S - \log \sigma - \tfrac{1}{2}\log(2\pi T) - \tfrac{1}{2}\,\frac{\left(\log(S/S_0) - (r-\tfrac{1}{2}\sigma^2)T\right)^2}{\sigma^2\, T}$$

$$\Longrightarrow \quad \frac{\partial(\log p)}{\partial \sigma} = -\,\frac{1}{\sigma} - \frac{\log(S/S_0) - (r-\tfrac{1}{2}\sigma^2)T}{\sigma} + \frac{\left(\log(S/S_0) - (r-\tfrac{1}{2}\sigma^2)T\right)^2}{\sigma^3\, T}$$

$$= \frac{Z^2-1}{\sigma} - \sqrt{T}\, Z$$

where Z is the unit Normal defined by

$$\log(S/S_0) - (r-\tfrac{1}{2}\sigma^2)T = \sigma\,\sqrt{T}\,Z$$

Likelihood Ratio Method

Hence,

$$\text{Vega} = \mathbb{E}\left[\left(\frac{Z^2-1}{\sigma} - \sqrt{T}\, Z \right) f(S(T)) \right]$$

Note that this correctly gives zero for $f(S) \equiv 1$

- useful check when using LRM
- could also use $\dfrac{Z^2-1}{\sigma} - \sqrt{T}\, Z$ as a control variate

Likelihood Ratio Method

Extending this to a SDE path simulation with M timesteps, with the payoff a function purely of the discrete states $\widehat{S}_n$, we have the M-dimensional integral

$$V = \mathbb{E}[\widehat{f}(\widehat{S})] = \int \widehat{f}(\widehat{S})\, p(\widehat{S})\, \mathrm{d}\widehat{S},$$

where $\mathrm{d}\widehat{S} \equiv \mathrm{d}\widehat{S}_1\ \mathrm{d}\widehat{S}_2\ \mathrm{d}\widehat{S}_3\ \ldots\ \mathrm{d}\widehat{S}_M$

and $p(\widehat{S})$ is the product of the p.d.f.s for each timestep

$$\begin{aligned} p(\widehat{S}) &= \prod_n p_n(\widehat{S}_{n+1}|\widehat{S}_n) \\ \log p(\widehat{S}) &= \sum_n \log p_n(\widehat{S}_{n+1}|\widehat{S}_n) \end{aligned}$$

Likelihood Ratio Method

For the Euler-Maruyama approximation of Geometric Brownian Motion,

$$\log p_n = -\log \widehat{S}_n - \log \sigma - \tfrac{1}{2}\log(2\pi h) - \tfrac{1}{2}\,\frac{\left(\widehat{S}_{n+1} - \widehat{S}_n(1{+}r\,h)\right)^2}{\sigma^2\,\widehat{S}_n^2\,h}$$

$$\Longrightarrow \quad \frac{\partial(\log p_n)}{\partial\sigma} \;=\; -\frac{1}{\sigma} + \frac{\left(\widehat{S}_{n+1} - \widehat{S}_n(1{+}r\,h)\right)^2}{\sigma^3\,\widehat{S}_n^2\,h}$$

$$\;=\; \frac{Z_n^2 - 1}{\sigma}$$

where Z_n is the unit Normal defined by

$$\widehat{S}_{n+1} - \widehat{S}_n(1{+}r\,h) = \sigma\,\widehat{S}_n\,\sqrt{h}\,Z_n$$

Likelihood Ratio Method

Hence, the approximation of Vega is

$$\frac{\partial}{\partial \sigma}\, \mathbb{E}[f(\widehat{S}_M)] = \mathbb{E}\left[\left(\sum_n \frac{Z_n^2 - 1}{\sigma} \right) f(\widehat{S}_M) \right]$$

Note that again this gives zero for $f(S) \equiv 1$.

Note also that $\mathbb{V}[Z_n^2 - 1] = 2$ and therefore

$$\mathbb{V}\left[\left(\sum_n \frac{Z_n^2 - 1}{\sigma} \right) f(\widehat{S}_M) \right] \;=\; O(M) \;=\; O(T/h)$$

This $O(h^{-1})$ blow-up is the great drawback of the LRM.

Pathwise Sensitivity

Reminder of lecture 8: defining $p(W)$ to be the p.d.f. for the driving Brownian motion $W(T)$, then

$$V = \mathbb{E}[f(S(T))] = \int f(S(T))\, p(W)\, \mathrm{d}W$$

and so differentiating gives

$$\frac{\partial V}{\partial \theta} = \int \frac{\partial f}{\partial S}\, \frac{\partial S(T)}{\partial \theta}\, p\, \mathrm{d}W = \mathbb{E}\left[\frac{\partial f}{\partial S}\, \frac{\partial S(T)}{\partial \theta}\right]$$

with $\partial S(T)/\partial \theta$ being evaluated at fixed W.

Pathwise Sensitivity

To allow for possibility of calculating sensitivity to changes in correlation, better to start with integral with respect to unit Normal Z:

$$V = \mathbb{E}[f(S(T))] = \int f(S(T))\ \phi(Z)\ \mathrm{d}Z$$

where $\phi(Z)$ is unit Normal p.d.f.

Differentiation then gives

$$\frac{\partial V}{\partial \theta} = \mathbb{E}\left[\frac{\partial f}{\partial S}\ \frac{\partial S(T)}{\partial \theta}\right]$$

with $\partial S(T)/\partial\theta$ being evaluated at fixed Z.

Pathwise Sensitivity

In the multiple dimensional GBM case,

$$S_i(T) = S_i(0)\exp((r - \tfrac{1}{2}\sigma_i^2)T + \sigma_i\sqrt{T}\,(LZ)_i)$$

where $L\,L^T$ is the correlation matrix for $\mathrm{d}W$, and the components of Z are i.i.d. unit Normals.

Hence for vega, we have

$$\left.\frac{\partial S_i}{\partial \sigma_i}\right|_Z = S_i(T)\left(-\,\sigma_i T + \sqrt{T}\,(LZ)_i\right)$$

Pathwise Sensitivity

The extension to SDE path simulations is quite natural, with

$$V = \mathbb{E}[\widehat{f}(\widehat{S})] = \int \widehat{f}(\widehat{S}(Z))\, \phi(Z)\, \mathrm{d}Z$$

where $\mathrm{d}Z \equiv \mathrm{d}Z_0\, \mathrm{d}Z_1\, \mathrm{d}Z_2\, \ldots\, \mathrm{d}Z_{M-1}$ and $\phi(Z)$ is the product of the unit Normal p.d.f.'s $\phi(Z) = \prod_n \phi(Z_n)$

Differentiation then gives

$$\frac{\partial V}{\partial \theta} = \mathbb{E}\left[\frac{\partial \widehat{f}}{\partial \widehat{S}}\, \frac{\partial \widehat{S}}{\partial \theta}\right]$$

with $\partial \widehat{S}/\partial\theta$ being evaluated at fixed Z.

Pathwise Sensitivity

For a scalar GBM, defining $\widehat{s}_n \equiv \dfrac{\partial \widehat{S}_n}{\partial \sigma}$ then differentiating the initial data $\widehat{S}_0 = S(0)$ gives $\widehat{s}_0 = 0$, and differentiating

$$\widehat{S}_{n+1} = \widehat{S}_n \, (1 + r\, h + \sigma \sqrt{h}\, Z_n)$$

gives

$$\widehat{s}_{n+1} = \widehat{s}_n \, (1 + r\, h + \sigma \sqrt{h}\, Z_n) + \widehat{S}_n \sqrt{h}\, Z_n$$

and then

$$\text{Vega} = \mathbb{E}\left[\frac{\partial \widehat{f}}{\partial \widehat{S}_M} \, \widehat{s}_M \right]$$

Pathwise Sensitivity

As $h \to 0$,

$$\widehat{s}_M \to \frac{\partial S(T)}{\partial \sigma}$$

so the approximate path sensitivity tends to the true value, and hence both the expectation and variance of

$$\frac{\partial \widehat{f}}{\partial \widehat{S}_M}\, \widehat{s}_M$$

converge to the expectation and variance of

$$\frac{\partial f}{\partial S}\, \frac{\partial S(T)}{\partial \sigma}$$

Thus, there is no variance “blow-up”.

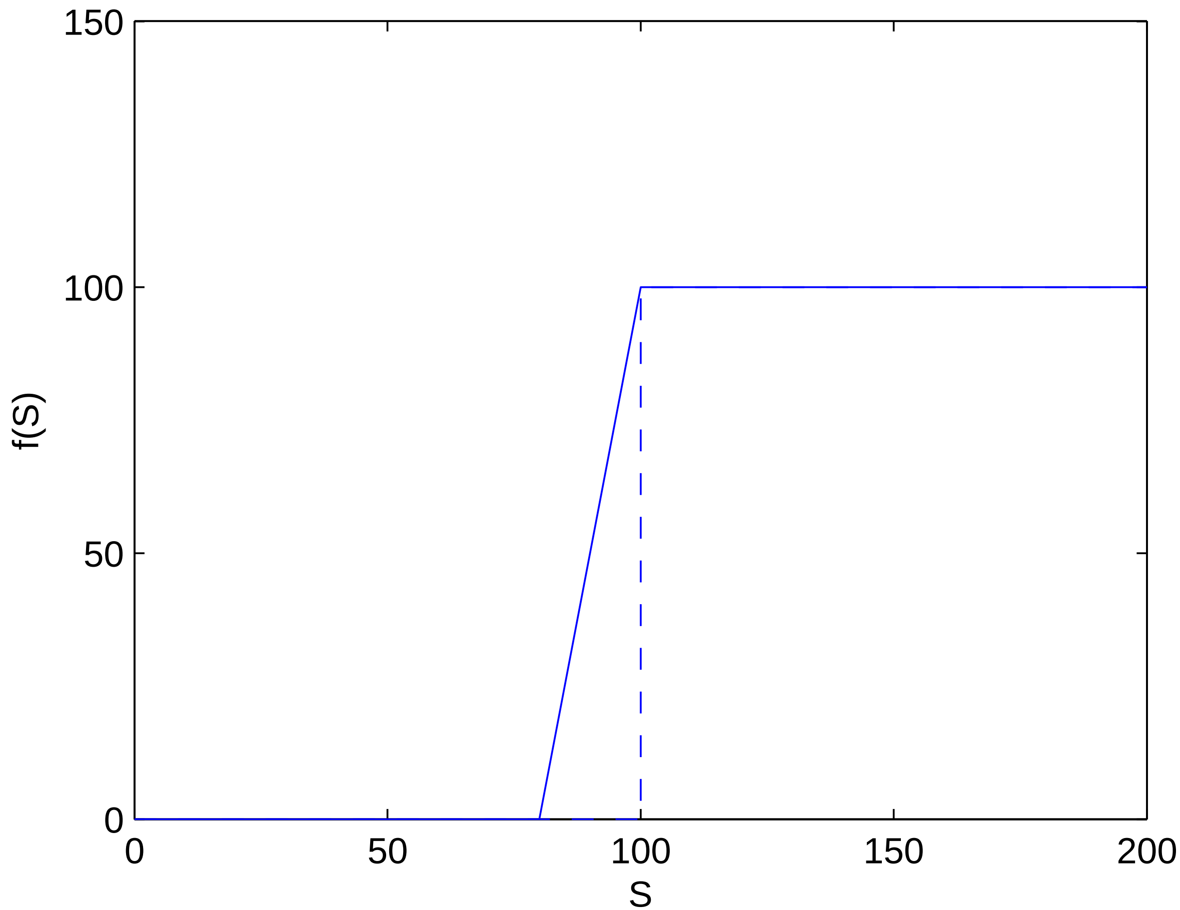
Pathwise Sensitivity

The big limitation of the pathwise sensitivity approach is that it requires that the payoff is continuous and piecewise differentiable.

One practical “solution” is to use a continuous piecewise linear approximation:

- a pair of put or call options, one long and one short
- under-estimate or over-estimate, depending whether buying or selling the option
- also limits the magnitude of Delta close to maturity, which limits the transaction costs

Pathwise Sensitivity



Call with strike $K = 80$, minus call with strike $K = 100$ gives over-estimate of digital option with strike $K = 100$

Final words

- LRM and pathwise sensitivity approaches both extend to SDE path simulations
- weakness of LRM is the blow-up in the variance as $h \to 0$
- weakness of pathwise sensitivity approach is the requirement that the payoff be continuous
- payoff smoothing is often used in practice
- alternatively, there is research on hybrid methods which combine the strengths of both approaches (Malliavin calculus, “vibrato” Monte Carlo)
- for computational efficiency, can use adjoint implementation of pathwise sensitivity to get all first order Greeks for same cost as original simulation