Numerical Methods II

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Euler-Maruyama method

For European options, Euler-Maruyama method has $O(h)$ weak convergence.

However, for some path-dependent options it can give only $O(\sqrt{h})$ weak convergence, unless the numerical payoff is constructed carefully.

Barrier option

A down-and-out call option has discounted payoff

$$\exp(-rT)\ (S(T)-K)^+ \mathbf{1}_{\min_t S(t)>B}$$

i.e. it is like a standard call option except that it pays nothing if the minimum value drops below the barrier B.

The natural numerical discretisation of this is

$$f = \exp(-rT)\ (\widehat{S}_{T/h}-K)^+ \mathbf{1}_{\min_n \widehat{S}_n>B}$$

Barrier option

Numerical demonstration: Geometric Brownian Motion

$$\mathrm{d}S = r\, S\, \mathrm{d}t + \sigma\, S\, \mathrm{d}W$$

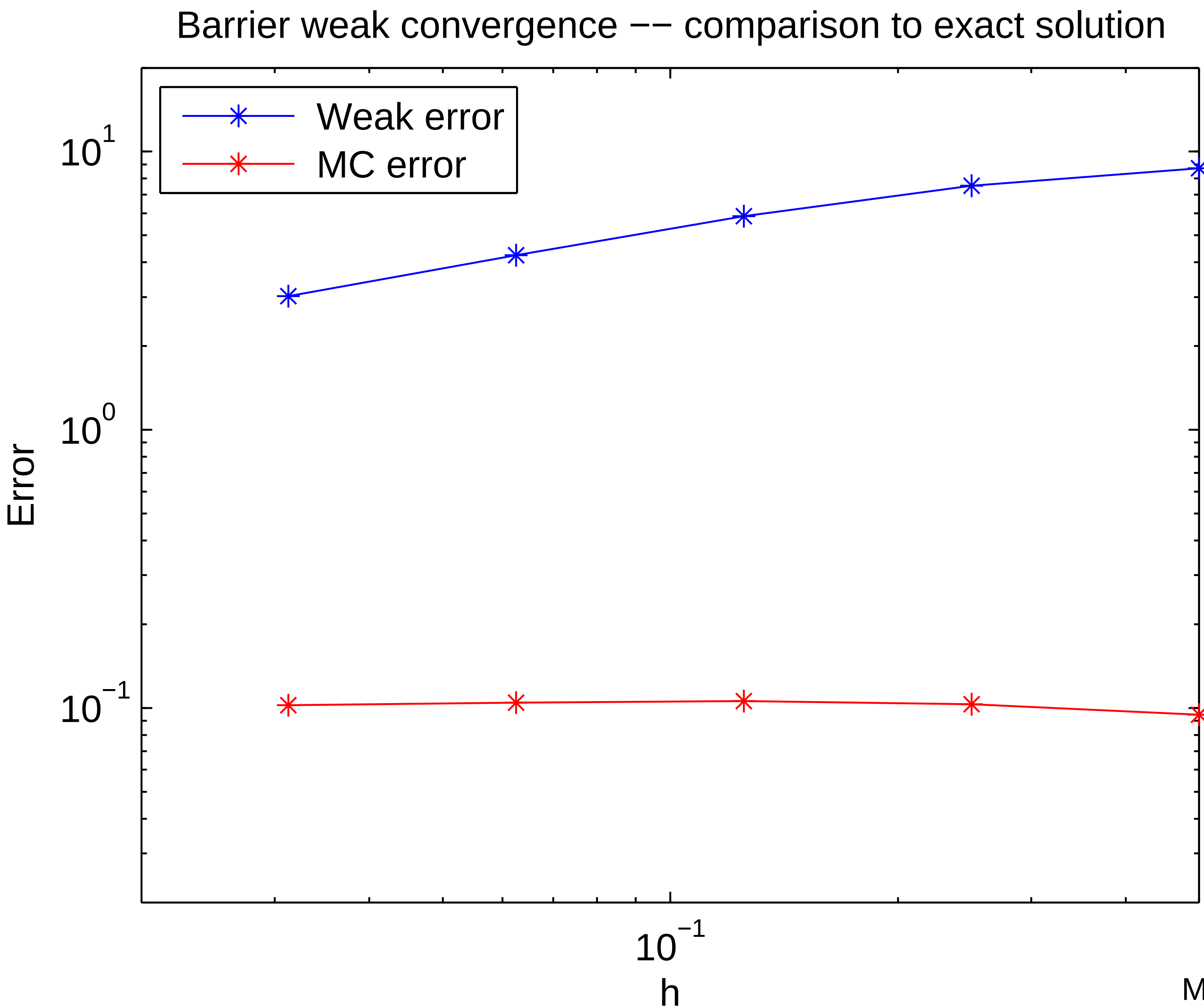
$r = 0.05,\ \sigma = 0.5,\ T = 1$

Down-and-out call: $S_0 = 100, K = 110, B = 90$.

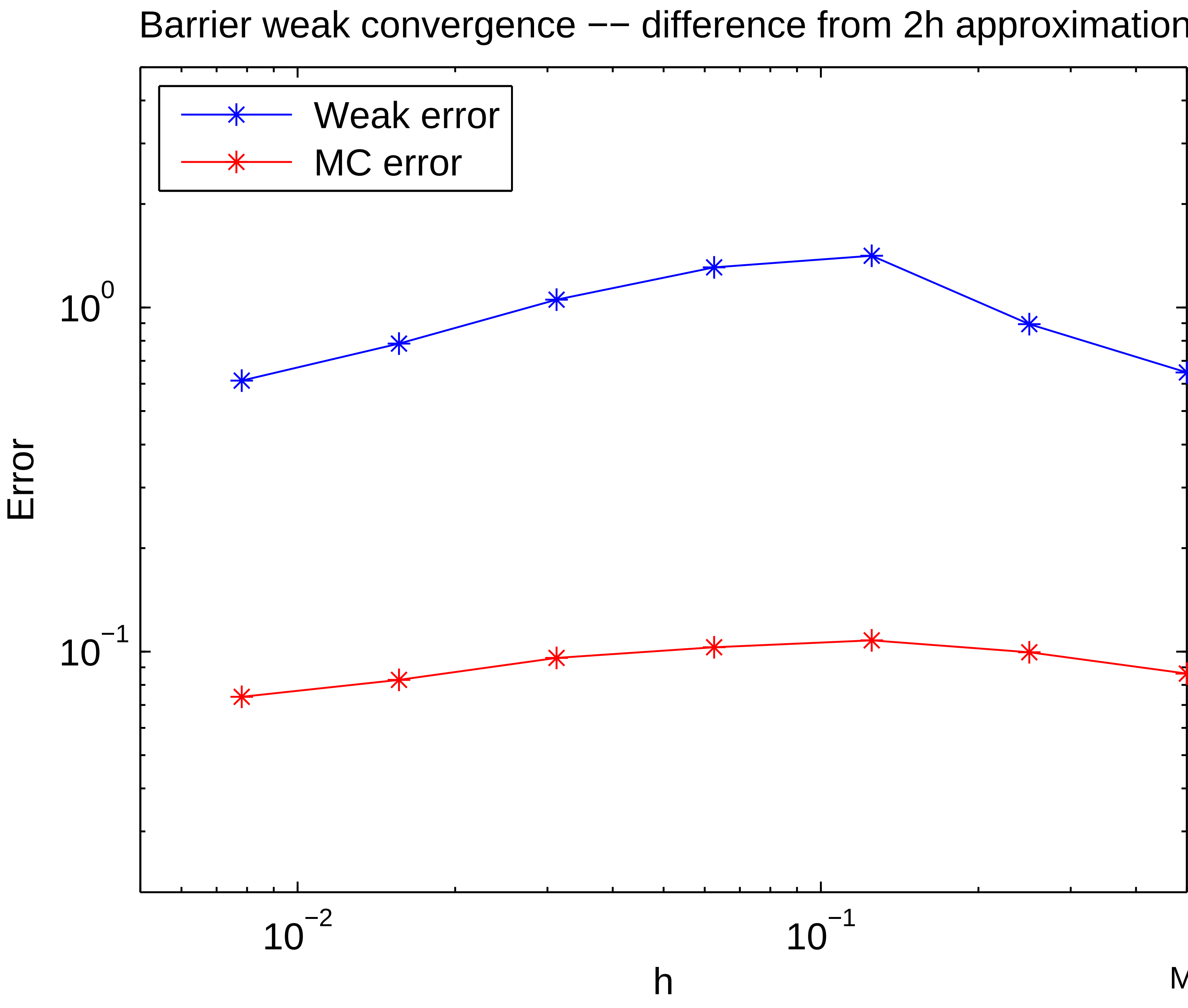
Plots shows weak error versus analytic expectation using 10^6 paths, and difference from $2h$ approximation using 10^5 paths.

(We don't need as many paths as in Lecture 9 because the weak errors are much larger in this case.)

Barrier option



Barrier option



Lookback option

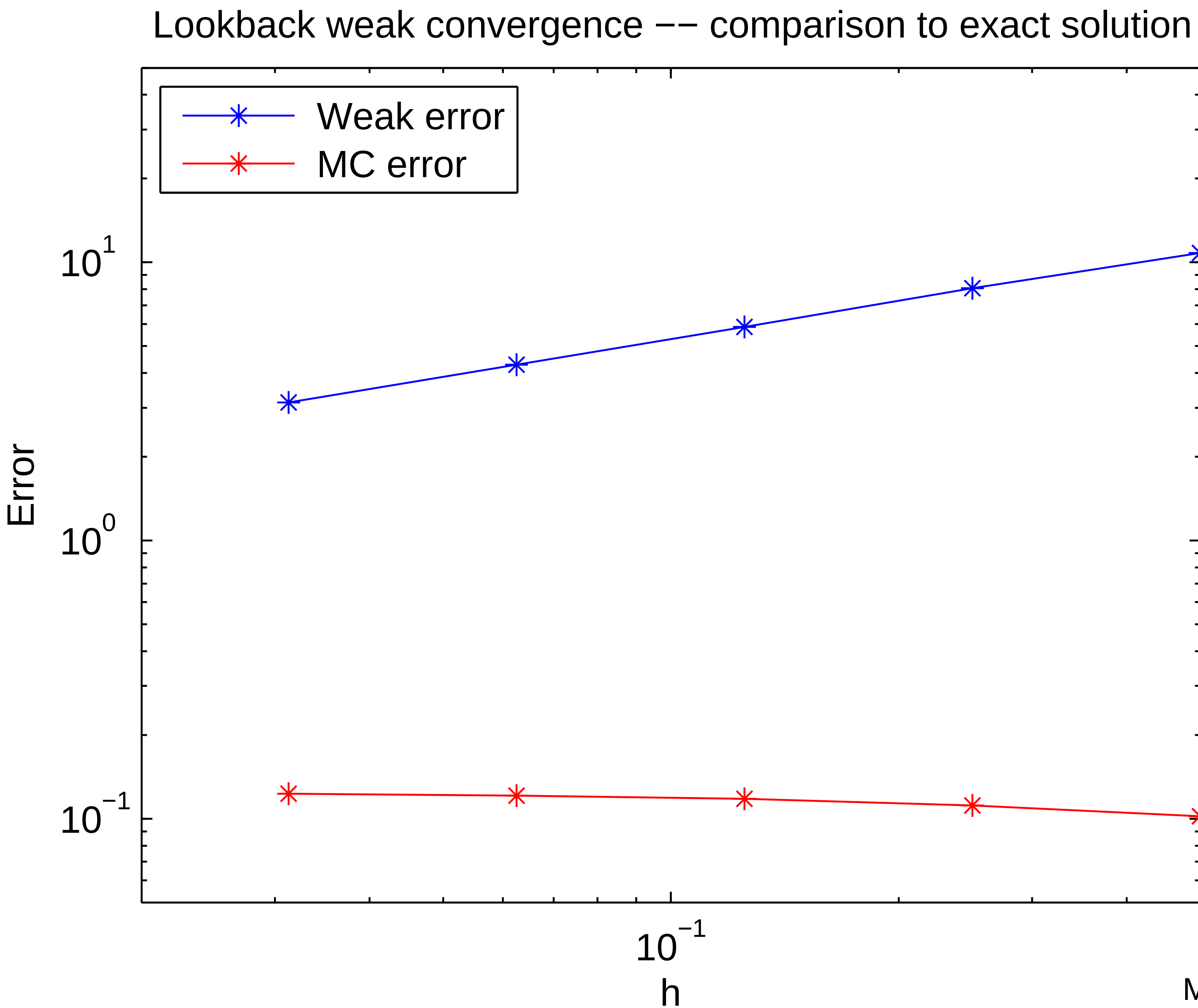
A floating-strike lookback call option has discounted payoff

$$\exp(-rT)\ \left(S(T) - \min_{[0,T]} S(t) \right)$$

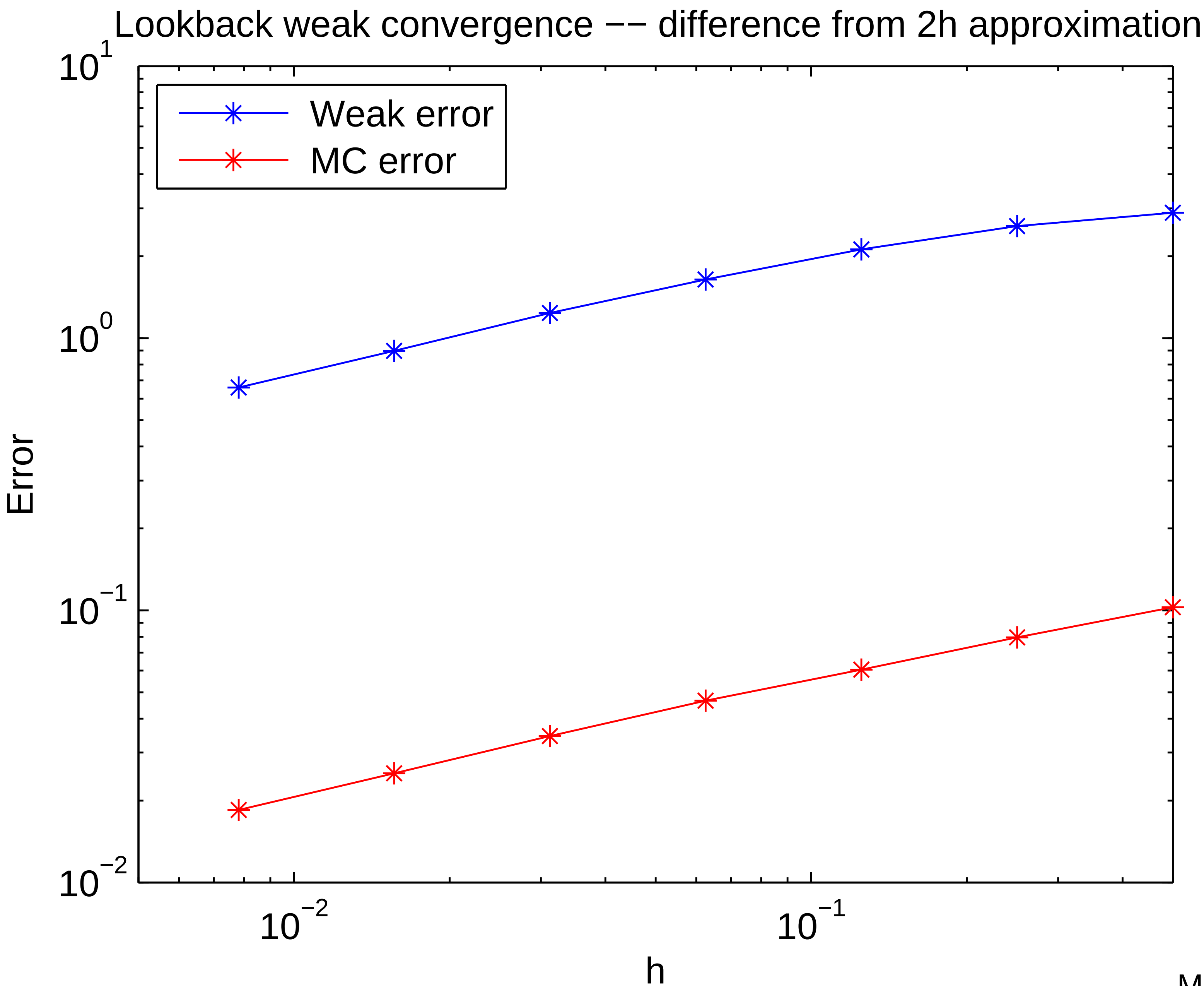
The natural numerical discretisation of this is

$$f = \exp(-rT)\ \left(\widehat{S}_{T/h} - \min_{n} \widehat{S}_n \right)$$

Lookback option



Lookback option



Brownian bridge

To recover $O(h)$ weak convergence we first need some theory.

Consider simple Brownian motion

$$\mathrm{d}S = a\,\mathrm{d}t + b\,\mathrm{d}W$$

with constant $a,\ b$ and initial data $S(0)\!=\!0$.

Question: given $S(T)$, what is conditional probability density for $S(T/2)$?

Conditional probability

With discrete probabilities,

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

Similarly, with probability density functions

$$p_1(x|y) = \frac{p_2(x,y)}{p_3(y)}$$

where

- $p_1(x|y)$ is the conditional p.d.f. for x, given y
- $p_2(x,y)$ is the joint probability density function for x, y
- $p_3(y)$ is the probability density function for y

Brownian bridge

In our case,

$$y \equiv S(T), \quad x \equiv S(T/2)$$

$$\begin{aligned}
p_2(x,y) &= \frac{1}{\sqrt{\pi T}\, b} \exp\left(-\frac{(x-aT/2)^2}{b^2\, T}\right) \\
&\times \frac{1}{\sqrt{\pi T}\, b} \exp\left(-\frac{(y-x-aT/2)^2}{b^2\, T}\right) \\
p_3(y) &= \frac{1}{\sqrt{2\pi T}\, b} \exp\left(-\frac{(y-aT)^2}{2\, b^2\, T}\right) \\
\Longrightarrow p_1(x|y) &= \frac{1}{\sqrt{\pi T/2}\, b} \exp\left(-\frac{(x-y/2)^2}{b^2\, T/2}\right)
\end{aligned}$$

Hence, x is Normally distributed with mean $y/2$ and variance $b^2T/4$.

Brownian bridge

Extending this to a particular timestep with endpoints $S(t_n)$ and $S(t_{n+1})$, conditional on these the mid-point is Normally distributed with mean

$$\frac{1}{2}\left(S(t_n) + S(t_{n+1})\right)$$

and variance $b^2 h/4$.

We can take a sample from this conditional p.d.f. and then repeat the process, recursively bisecting each interval to fill in more and more detail.

Note: the drift a is irrelevant, given the two endpoints. Because of this, we will take $a=0$ in the next bit of theory.

Barrier crossing

Consider zero drift Brownian motion with $S(0)\!>\!0$.

If the path $S(t)$ hits a barrier at 0, it is equally likely thereafter to go up or down. Hence, by symmetry, for $s>0$, the p.d.f. for paths with $S(T)=s$ after hitting the barrier is equal to the p.d.f. for paths with $S(T)=-s$.

Thus, for $S(T)>0$,

$$
\begin{aligned}
P(\text{hit barrier}|S(T)) &= \frac{\exp\left(-\frac{(-S(T)-S(0))^2}{2b^2T}\right)}{\exp\left(-\frac{(S(T)-S(0))^2}{2b^2T}\right)} \\
&= \exp\left(-\frac{2\,S(T)\,S(0)}{b^2T}\right)
\end{aligned}
$$

Barrier crossing

For a timestep $[t_n, t_{n+1}]$ and non-zero barrier B this generalises to

$$P(\text{hit barrier}|S_n, S_{n+1} > B) = \exp\left(- \frac{2\,(S_{n+1}\!-\!B)\,(S_n\!-\!B)}{b^2 h} \right)$$

This can also be viewed as the cumulative probability $P(S_{min} < B)$ where $S_{min} = \min\limits_{[t_n, t_{n+1}]} S(t)$.

Since this is uniformly distributed on $[0,1]$ we can equate this to a uniform $[0,1]$ random variable U_n and solve to get

$$S_{min} = \tfrac{1}{2}\left(S_{n+1} + S_n - \sqrt{(S_{n+1}\!-\!S_n)^2 - 2\,b^2 h\, \log U_n} \right)$$

Barrier crossing

For a barrier above, we have

$$P(\text{hit barrier}|S_n, S_{n+1} < B) = \exp\left(-\frac{2\,(B-S_{n+1})\,(B-S_n)}{b^2 h} \right)$$

and hence

$$S_{max} = \tfrac{1}{2}\left(S_{n+1} + S_n + \sqrt{(S_{n+1}-S_n)^2 - 2\,b^2 h\,\log U_n} \right)$$

where U_n is again a uniform $[0,1]$ random variable.

Barrier option

Returning now to the barrier option, how do we define the numerical payoff $\widehat{f}(\widehat{S})$?

First, calculate $\widehat{S}_n$ as usual using Euler-Maruyama method.

Second, two alternatives:

- use (approximate) probability of crossing the barrier
- directly sample (approximately) the minimum in each timestep

Barrier option

Alternative 1: treating the drift and volatility as being approximately constant within each timestep, the probability of having crossed the barrier within timestep n is

$$P_n = \exp\left(- \frac{2\,(\widehat{S}_{n+1} - B)^+\,(\widehat{S}_n - B)^+}{b^2(\widehat{S}_n, t_n)\ h} \right)$$

Probability at end of not having crossed barrier is $\displaystyle\prod_n (1-P_n)$ and so the payoff is

$$\widehat{f}(\widehat{S}) = \exp(-rT)\ (\widehat{S}_{T/h} - K)^+\ \prod_n (1-P_n).$$

I prefer this approach because it is differentiable – good for Greeks

Barrier option

Alternative 2: again treating the drift and volatility as being approximately constant within each timestep, define the minimum within timestep n as

$$\widehat{M}_n = \tfrac{1}{2}\left(\widehat{S}_{n+1} + \widehat{S}_n - \sqrt{(\widehat{S}_{n+1}-\widehat{S}_n)^2 - 2\, b^2(\widehat{S}_n, t_n)\, h\, \log U_n}\right)$$

where the U_n are i.i.d. uniform $[0,1]$ random variables.

The payoff is then

$$\widehat{f}(\widehat{S}) = \exp(-rT)\ (\widehat{S}_{T/h} - K)^+ \ \mathbf{1}_{\min_n \widehat{M}_n > B}$$

With this approach one can stop the path calculation as soon as one $\widehat{M}_n$ drops below B.

Lookback option

This is treated in a similar way to Alternative 2 for the barrier option.

We construct a minimum $\widehat{M}_n$ within each timestep and then the payoff is

$$\widehat{f}(\widehat{S}) = \exp(-rT) \left(\widehat{S}_{T/h} - \min_n \widehat{M}_n \right)$$

This is differentiable, so good for Greeks – unlike Alternative 2 for the barrier option.

Weak convergence

With these modification to the numerical payoff approximation, the weak convergence for both barrier and lookback options is improved from $O(\sqrt{h})$ to $O(h)$.

See practical 3 for numerical demonstration!

Final Words

- “natural” approximation of barrier and lookback options leads to poor $O(\sqrt{h})$ weak convergence
- this is an inevitable consequence of dependence on minimum/maximum and $O(\sqrt{h})$ path variation within each timestep
- improved treatment based on Brownian bridge theory approximates behaviour within timestep as simple Brownian motion with constant drift and volatility
  – gives $O(h)$ weak convergence