Crank-Nicolson time-marching

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Abstract

This entry describes the Crank-Nicolson time-marching discretisation and its numerical properties. It also presents the Rannacher startup procedure which is required to achieve second order accuracy for the option value and its first and second derivatives, and discusses extensions to nonlinear and multi-factor applications.

Keywords: time-marching, discretisation, numerical stability, iterative solution, Black-Scholes, Crank-Nicolson, Rannacher, θ-scheme

1   Algorithm and key features

In a paper published in 1947 [2], John Crank and Phyllis Nicolson presented a numerical method for the approximation of diffusion equations. Starting from the simplest example

$$\frac{\partial V}{\partial t} = \frac{\partial^2 V}{\partial x^2},$$

a spatial approximation on a uniform grid with spacing h leads to the semi-discrete equations

$$\frac{\mathrm{d}V_j}{\mathrm{d}t} = h^{-2}\,\delta_j^2 V_j,$$

where $V_j(t)$ is an approximation to $V(x_j, t)$, and $\delta_j^2 V_j \equiv V_{j+1} - 2V_j + V_{j-1}$ is a central second difference. Crank-Nicolson time-marching discretises this in time with a uniform timestep k using the approximation

$$k^{-1}(V_j^{n+1} - V_j^n) = \tfrac{1}{2}h^{-2}\left(\delta_j^2 V_j^{n+1} + \delta_j^2 V_j^n\right),$$

which can be re-arranged to give

$$\left(1 - \tfrac{1}{2}kh^{-2}\delta_j^2\right)V_j^{n+1} = \left(1 + \tfrac{1}{2}kh^{-2}\delta_j^2\right)V_j^n.$$

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This can be viewed as the $\theta = \frac{1}{2}$ case of the more general θ-scheme

$$\left(1 - \theta \, kh^{-2}\delta_j^2\right) V_j^{n+1} = \left(1 + (1-\theta) \, kh^{-2}\delta_j^2\right) V_j^n.$$

$\theta = 0$ corresponds to explicit Euler time-marching, while $\theta = 1$ corresponds to fully-implicit Euler time-marching. For $\theta > 0$, the θ-scheme defines a tri-diagonal system of simultaneous equations which can be solved very efficiently using the Thomas algorithm to obtain the values for V_j^{n+1}. The scheme is unconditionally stable in the L_2 norm, meaning that the L_2 norm of the solution does not increase for any value of k, provided $\theta \geq \frac{1}{2}$. The Crank-Nicolson scheme is thus on the boundary of unconditional stability. It is also special in having a numerical error for smooth initial data which is $O(h^2, k^2)$, whereas the error is $O(h^2, k)$ for other values of θ. The unconditional L_2 stability means that one can choose to make k proportional to h, and together with the second order accuracy this makes the scheme both accurate and efficient, and hence a very popular choice for approximating parabolic PDEs.

2 Application to Black-Scholes equation

The Crank-Nicolson method is used extensively in mathematical finance for approximating parabolic PDEs such as the Black-Scholes equation (EQF12/003) which can be written in reversed-time form (with $\tau \equiv T{-}t$ being the time to maturity T) as

$$\frac{\partial V}{\partial \tau} = -rV + rS\,\frac{\partial V}{\partial S} + \tfrac{1}{2}\,\sigma^2 S^2\,\frac{\partial^2 V}{\partial S^2}.$$

Switching to the new coordinate $x \equiv \log S$ gives the transformed equation

$$\frac{\partial V}{\partial \tau} = -rV + \left(r - \tfrac{1}{2}\sigma^2\right)\frac{\partial V}{\partial x} + \tfrac{1}{2}\,\sigma^2\,\frac{\partial^2 V}{\partial x^2},$$

and its Crank-Nicolson discretisation on a grid with uniform timestep k and uniform grid spacing h is

$$(I + \tfrac{1}{2}D)\,V_j^{n+1} = (I - \tfrac{1}{2}D)\,V_j^n$$

where the discrete operator D is defined by

$$D = r\,k - \tfrac{1}{2}kh^{-1}\left(r - \tfrac{1}{2}\sigma^2\right)\delta_{2j} - \tfrac{1}{2}kh^{-2}\,\sigma^2\,\delta_j^2,$$

with the central first difference operator δ_{2j} defined by $\delta_{2j}V_j \equiv V_{j+1} - V_{j-1}$.

For a European call option with strike K, the initial data at maturity is $V(S,0) = \max(S - K, 0)$. The top-left plot in Figure 1 shows the numerical solution $V(S,2)$ for parameter values $r = 0.05$, $\sigma = 0.2$, $K = 1$, and timestep/spacing ratio $\lambda \equiv k/h = 10$. The agreement between

the numerical solution and the known analytic solution appears quite good, but the other two left-hand plots show much poorer agreement for the approximations to $\Delta \equiv \partial V/\partial S$ and $\Gamma \equiv \partial^2 V/\partial S^2$ (EQF07/017, EQF07/018) obtained by central differencing of the numerical solution V_j^n. In particular, note that the maximum error in the computed value for Γ occurs at $S\!=\!1$, which is the location of the discontinuity in the first derivative of the initial data. The left-hand plots in Figure 2 show the behaviour of the maximum error as the computational grid is refined, keeping fixed the ratio $\lambda \equiv k/h$. It can be seen that for largest value of λ the numerical solution V_j exhibits first order convergence, while the discrete approximation to Δ does not converge, and the approximation to Γ diverges. For smaller values of λ it appears the convergence is better, but in fact the asymptotic behaviour is exactly the same except that it becomes evident only on much finer grids.

At first sight, this is a little surprising as textbooks almost always describe the Crank-Nicolson method as unconditionally stable and second order accurate. The key is that it is only unconditionally stable in the L_2 norm, and this only ensures convergence in the L_2 norm for initial data which has a finite L_2 norm [9]. Furthermore, the order of convergence may be less than second order for initial data which is not smooth; for example, the L_2 order of convergence for discontinuous initial data is $\frac{1}{2}$. With the European call Black-Scholes application, the initial data for V lies in L_2, as does its first derivative, but the second derivative is the Dirac delta function which does not lie in L_2. This is the root cause of the observed failure to converge as the grid is refined. Furthermore, it is the maximum error, the L_∞ error, which is most relevant in financial applications.

One solution to this problem is to use an alternative second order backward difference method, but these methods require special startup procedures because they require more than one previous time level, and they are usually less accurate than the Crank-Nicolson method for the same number of timesteps. Better alternatives are higher order backward difference methods [5] or the Rannacher startup procedure described next.

3   Rannacher startup procedure

Rannacher analysed this problem of poor L_2 convergence of convection-diffusion approximations with discontinuous initial data [8], and recovered second order convergence by replacing the Crank-Nicolson approximation for the very first timestep by two half-timesteps of implicit Euler time integration, and using a finite element projection of the discontinuous initial data onto the computational grid. This technique, often referred to as Rannacher timestepping, has been used with success in approximations of the Black-Scholes equations [6, 7], with the

half-timestep implicit Euler discretisation given by

$$(I + \tfrac{1}{2}D)\, V_j^{n+1/2} = V_j^n.$$

The problem has been further investigated by Giles and Carter [4] who analysed the maximum errors in finance applications and proved it is necessary to go further and replace the first two Crank-Nicolson timesteps by four half-timesteps of implicit Euler to achieve second order accuracy in the L_∞ norm for V, Δ and Γ for put, call and digital options. The improved accuracy is demonstrated by the right-hand plots in Figures 1–2.

4   Nonlinear and multi-factor extensions

The use of a nonlinear penalty function in approximating American options (EQF12/002) leads to a nonlinear discretisation of the form [3]

$$(I + \tfrac{1}{2}D)\, V_j^{n+1} = (I - \tfrac{1}{2}D)\, V_j^n + P(V_j^{n+1}),$$

where the nonlinear penalty term $P(V_j^{n+1})$ is negligible in the region where the option is not exercised, and elsewhere ensures that V_j^{n+1} is approximately equal to the exercise value.

This nonlinear system of equations can be solved using a Newton iteration, starting with $V_j^{n+1,0} = V_j^n$ and defining the $m+1^{th}$ iterate to be $V_j^{n+1,m+1} = V_j^{n+1,m} + \Delta V_j$ with the correction ΔV_j given by the linear equations

$$\left(I + \tfrac{1}{2}D - \frac{\partial P}{\partial V}\right)\Delta V_j$$
$$= -(I + \tfrac{1}{2}D)\, V_j^{n+1,m} + (I - \tfrac{1}{2}D)\, V_j^n + P(V_j^{n+1,m}),$$

Alternatively, one can use just one step of the Newton iteration, in which case one has $V_j^{n+1} = V_j^n + \Delta V_j$ with the change ΔV_j given by

$$\left(I + \tfrac{1}{2}D - \frac{\partial P}{\partial V}\right)\Delta V_j = -D\, V_j^n + P(V_j^n).$$

In one dimension, the linear equations are a tridiagonal system which can be solved very efficiently. In higher dimensions the direct solution cost is much greater and alternative approaches are usually adopted. One is to use an ADI (alternative direction implicit) approximate factorisation into a product of operators, each of which involves differences in only one direction [9]. To maintain second order accuracy it is necessary to use the Craig-Sneyd treatment for any cross-derivative term [1]. Another approach is to use a preconditioned iterative solver such as BiCGStab with ILU preconditioning.

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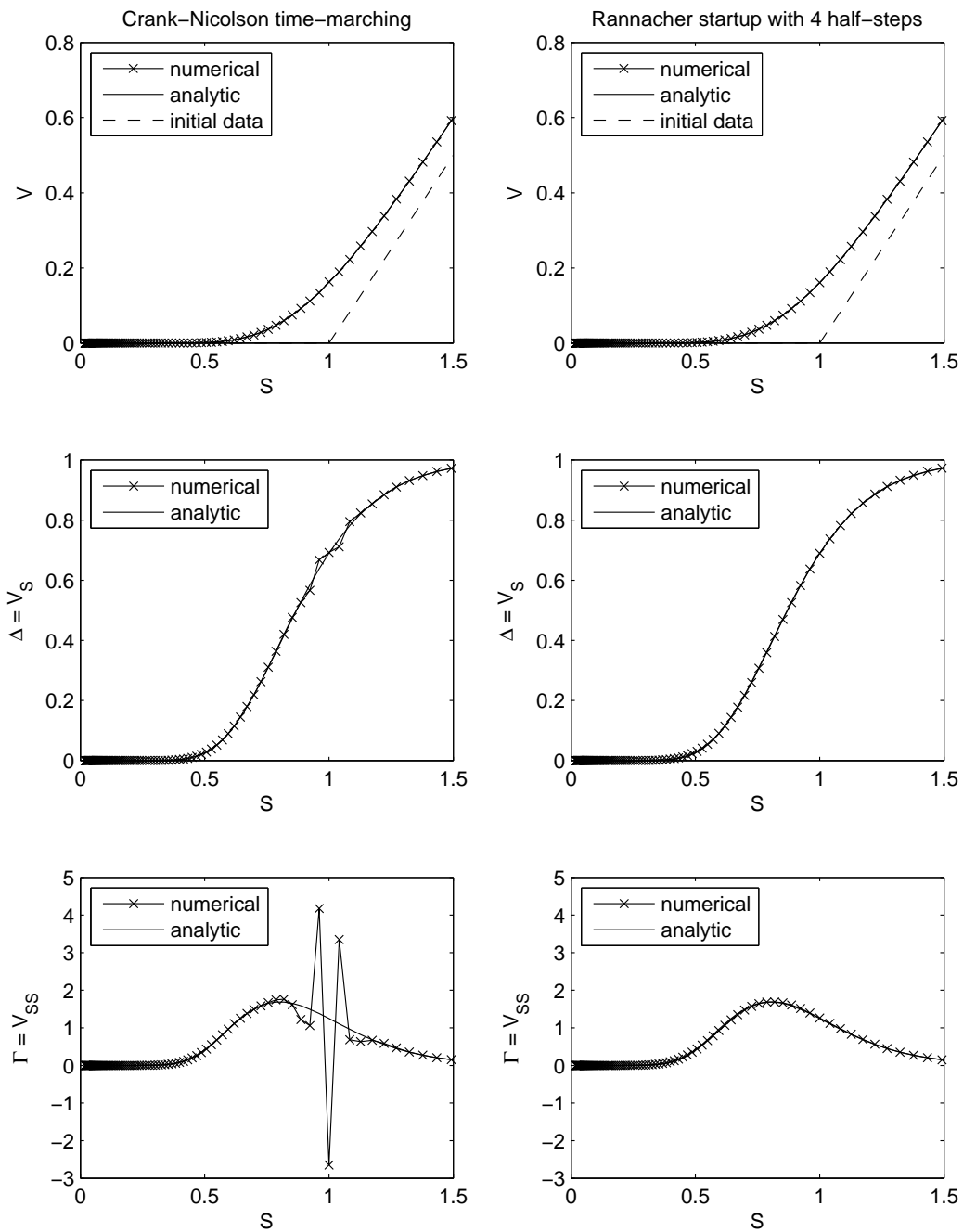
Figure 1: V, Δ and Γ for a European call option, with $\lambda \equiv k/h = 10$
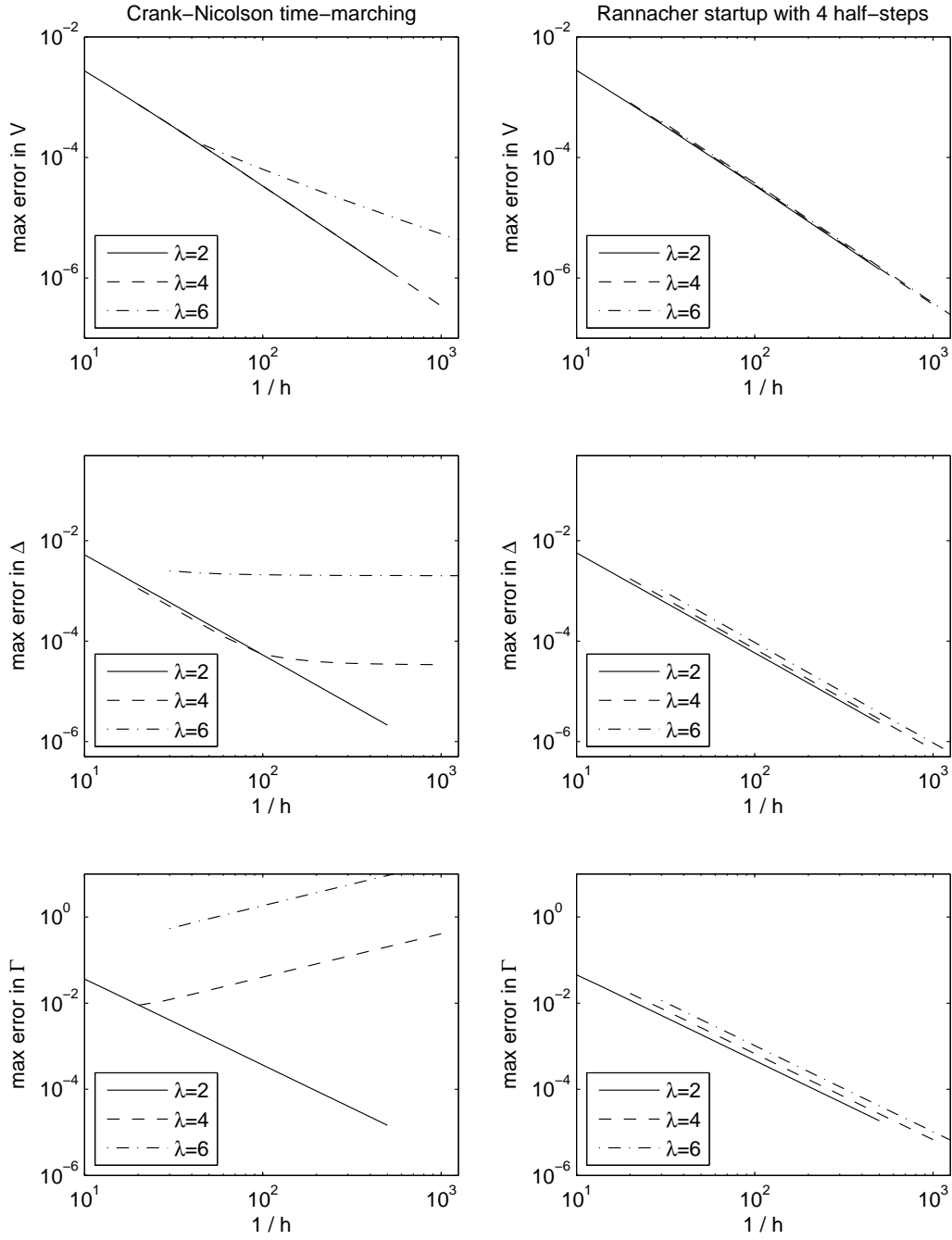
Figure 2: Grid convergence for a European call option, with fixed $\lambda \equiv k/h$