A characterization of forward dynamic utilities

Mike Tehranchi (with Francois Berrier and Chris Rogers)

Statistical Laboratory
University of Cambridge

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If X_1 and X_2 are random variables modelling an agent's future endowment, then the classical von Neumann–Morgenstern theory asserts that the agent prefers X_1 to X_2 if and only if

$$\mathbb{E}[U(X_1)] \geq \mathbb{E}[U(X_2)]$$

for an increasing, concave utility function U.

An agent then tries to find the X^* in his set $\mathcal{X}$ of attainable future endowments so that

$$\mathbb{E}[U(X^*)] = \sup_{X \in \mathcal{X}} \mathbb{E}[U(X)]$$

Merton's problem: Suppose the agent can trade in a market $(S_t)_{t \geq 0}$.

Fix a time horizon $T > 0$ and an intial wealth $X_0 = x$ and let

$$\mathcal{X} = \{X_T(\pi) : X_t(\pi) > 0 \text{ a.s. for all } 0 \le t \le T\}$$

where

$$X_t(\pi) = x + \int_0^t \pi_s dS_s$$

for every adapted trading strategy π.

Given x, T and the function U, what is

$$\sup_{X \in \mathcal{X}} \mathbb{E}[U(X)]?$$

More importantly, what is the optimizing strategy π^* such that

$$X_T(\pi^*) = X^*?$$

Example: Suppose $r = 0$ and

$$dS_t = \mathrm{diag}(S_t)(\mu dt + \sigma dW_t)$$

for constants $\mu \in \mathbb{R}^d$ and $\sigma \in \mathbb{R}^{d \times d}$.
Let U have constant relative risk aversion, i.e.,

$$U(x) = \frac{x^\gamma}{\gamma}$$

for some $\gamma < 1, \gamma \neq 0$.

Merton showed

$$\sup_{X \in \mathcal{X}} \mathbb{E}[U(X)] = \frac{x^\gamma}{\gamma} \exp\left(\frac{\gamma}{1-\gamma} \frac{|\sigma^{-1}\mu|^2 T}{2} \right)$$

and the optimal strategy π^* is a scalar multiple of the mutual fund

$$(\sigma\sigma^{\mathrm{T}})^{-1}\mu.$$

- The maximum expected utility somehow represents the agent's utility from having an initial endowment of x.
- In general, however, the optimal strategy π^* may depend on the horizon T.
- Okay for an agent planning for retirement, but less helpful for a hedge fund manager.
- To address this issue, Musiela and Zariphopoulou indroduced forward dynamic utilities.
- Henderson and Hobson had previously developed the equivalent notion of horizon-unbiased utilities.

Let

$$\mathcal{X}(x, t, T) = \left\{ X_{x,t,T}(\pi) : X_{x,t,t'}(\pi) = x + \int_t^{t'} \pi_s dS_s \right\}$$

set of endowments attainable at time T starting with x at time t.

Definition

A function $U : \mathbb{R}_+ \times \mathbb{R}_+ \times \Omega \to \mathbb{R}$ is a forward dynamic utility if

- $x \mapsto U(x, t, \omega)$ is increasing and concave
- $U(x, t) = \sup_{X \in \mathcal{X}(x,t,T)} \mathbb{E}[U(X, T) | \mathcal{F}_t]$ for all $t \leq T$.

Question

Do forward dynamic utilities ever exist?

Answer

Yes. For instance, consider

$$dS_t = \mathrm{diag}(S_t)(\mu dt + \sigma dW_t).$$

Then

$$U(x,t) = \frac{x^\gamma}{\gamma} \exp\left(-\frac{\gamma}{1-\gamma} \frac{|\sigma^{-1}\mu|^2 t}{2} \right)$$

Musiela and Zariphopoulou have constructed a number of other examples.

Question

How can forward dynamic utilities be characterized?

Answer

Consider a general market model with

$$dS_t = \mathrm{diag}(S_t)(\mu_t dt + \sigma_t dW_t)$$

where $(\sigma_t^{-1})_{t\geq 0}$ *and* $(\mu_t)_{t\geq 0}$ *are bounded processes, adapted to a filtration* $(\mathcal{F}_t)_{t\geq 0}$ *which may be strictly larger than the filtration generated by* W*.*

Assume that

$$\int_0^\infty |\sigma_s^{-1}\mu_s|^2 ds = \infty$$

almost surely.

Theorem

The following are equivalent:

1. *U is a forward dynamic utility such that*
   - $x \mapsto U(x,t,\omega)$ *is twice continuously differentiable*
   - $t \mapsto U(x,t,\omega)$ *is continuously differentiable*
   - $U_x(0,t,\omega) = \infty$ *and* $U_x(\infty,t,\omega) = 0$
   - *there exists an* $X^* \in \mathcal{X}(x,t,T)$ *such that* $U(x,t) = \mathbb{E}[U(X^*,T)|\mathcal{F}_t]$
2. *There exists a finite measure ν such that*

$$I(y,t) = \int_{(0,\infty)} y^{-r} \exp\left(-\frac{r(r-1)}{2} \int_0^t |\sigma_s^{-1}\mu_s|^2 ds \right) \nu(dr)$$

where $I(\cdot,t) = U_x(\cdot,t)^{-1}$.

Corollary

If U is a forward dynamic utility satisfying the conditions of the theorem, then the optimal portfolio π_t^* is a scalar multiple of the mutual fund

$$(\sigma\sigma_t^{\mathrm{T}})^{-1}\mu_t.$$

Corollary

A function $u_0 : \mathbb{R}_+ \to \mathbb{R}$ *can be extended to a forward dynamic utility such that*

$$u_0(x) = U(x, 0)$$

if and only if $I_0 \circ \exp$ *is the Laplace transform of a finite measure, where* $I_0 = (u_0')^{-1}$.

Corollary

If U is a forward dynamic utility, then the convex dual function V defined by

$$V(y,t) = \sup_{x>0} U(x,t) - xy$$

satisfies

$$V(y,t) = \mathbb{E}\left[V\left(y\frac{Z_T}{Z_t}, T \right) \big| \mathcal{F}_t \right]$$

where

$$Z_t = \exp\left(-\frac{1}{2}\int_0^t |\sigma_s^{-1}\mu_s|^2 ds - \int_0^t \sigma_s^{-1}\mu_s dW_s \right)$$

is the density of the Foellmer–Schweizer minimal martingale measure.

Sketch of proof: If U is a forward dynamic utility function, then

$$(U(x,t))_{t\geq 0}$$

is a continuous semi-martingale, indexed by x. Hence the Itô-Wentzell formula holds

$$\begin{aligned} U(X_T,T) = U(X_t,t) \quad &+ \quad \int_t^T (\pi_s\cdot\sigma_s^{-1}\mu_s U_x + \frac{|\pi_s\cdot\sigma_s^{-1}|^2}{2}U_{xx})ds \\ &+ \quad \int_t^T \pi_s\cdot\sigma_s^{-1}\mu_s U_x dW_s \end{aligned}$$

Optimizing yields

$$U_t = \frac{U_x^2}{U_{xx}} \frac{|\sigma^{-1}\mu|^2}{2}.$$

Hence, the convex dual satisfies

$$V_t = y^2 V_{yy} \frac{|\sigma^{-1}\mu|^2}{2}.$$

But $y \mapsto V(y, t, \omega)$ is decreasing and convex. Together with Widder's characterization of positive space-time harmonic functions yields the direct implication.

Question

Is any of this true in a general semi-martingale market?