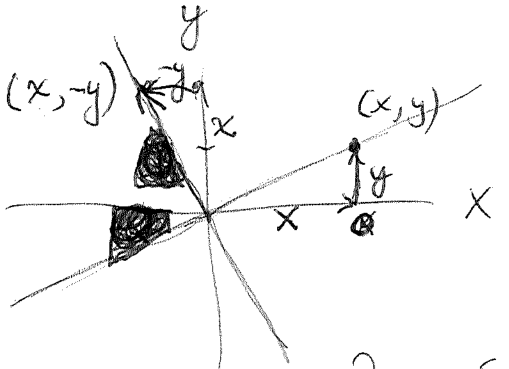
171 Put some effort into the Cayley transf. You have get straight what the C.T. is. The simplest version maybe occurs in the case of the orthogonal group. $SO(n) = \{g \in GL(n, \mathbb{R}) \mid g^t g = 1\}$

Lie $SO(n) = \{X \in M_n(\mathbb{R}) \mid X^t + X = 0\}$. In this case you have a map from Lie $SO(n)$ to $SO(n)$ sending X to $\frac{1+X}{1-X} = g$

$$g^{1/2} = \frac{1+X}{(1-X^2)^{1/2}} \qquad g = \frac{(1+X)^2}{(1+X)(1-X)} = \frac{1+X}{1-X}.$$

Notice that so far everything is taking place in $M_n\mathbb{R}$ in which you have the appropriate functional calcullus, both $\exp(tX)$ and $\frac{1+tX}{1-tX}$. Since a skew symmetric op X is a direct of infinitesimal 2 dim'l rotations: $X = \omega\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, It should be possible to understand what's happenning.

$$\exp\left(\theta\begin{bmatrix} 0 & -1 \\ +1 & 0 \end{bmatrix}\right) =$$



$$\begin{bmatrix} 0 & -1 \\ +1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -y \\ x \end{bmatrix}$$

$$\exp\left(\theta\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\right) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \xrightarrow{\partial_\theta} \begin{bmatrix} -\sin\theta & -\cos\theta \\ \cos\theta & -\sin\theta \end{bmatrix}$$

$$\shortparallel$$

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} -\sin\theta & -\cos\theta \\ \cos\theta & -\sin\theta \end{bmatrix}$$

$$\int_0^\infty e^{-st}\dot{u}\,dt = \int_0^\infty \left\{\frac{d}{dt}\left[e^{-st}u\right] + se^{-st}u\right\}dt = -u(0) + s\hat{u}(s)$$

(172) [redacted] Looking at $O(n)$ or should it be $SO(2n)$? The aim is to use the C.T. Lie $O(n) \longrightarrow SO(n)$, $X \mapsto \frac{1+X}{1-X}$ to understand group elements.

[redacted]

You had an idea linking Cayley tranform of X to frequencies for a harmonic oscillator. Involves inverse Cayley transform I think.

Given an orthogonal transformation g you want X skew-symmetric s.t. $\frac{1+X}{1-X} = g$. Then X is the infinitesimal generator of the time evolution for a harmonic oscillator. This is the "odd" ungraded case.

But there's also an even case, where $X = \begin{bmatrix} 0 & -T^* \\ T & 0 \end{bmatrix}$

$$\begin{tikzcd}
\tilde{C}^0 \arrow[r, "{\begin{bmatrix}1\\T\end{bmatrix}}"] & \begin{bmatrix}C^1_C\\C^1_L\end{bmatrix} \arrow[r, "{\begin{bmatrix}-T & 1\end{bmatrix}}"] & H^1 \\
\bar{C}_0 & \begin{bmatrix}C_1^C\\C_1^L\end{bmatrix} \arrow[l, "{\begin{bmatrix}1 & T^*\end{bmatrix}}"'] \arrow[u, "{\begin{bmatrix}s^{-1} & \\ & s\end{bmatrix}}"'] & H_1 \arrow[l, "{\begin{bmatrix}-T^*\\1\end{bmatrix}}"']
\end{tikzcd}$$

[redacted] [redacted]

Important varbles

V_C I_L dominant

I_C V_L weak

[redacted] $\dot{V}_C = I_C$

[redacted] $\dot{I}_L =$ [redacted] V_L

$\hat{I}_C = \widehat{\text{[redacted]}\dot{V}_C} = \text{[redacted]} s\hat{V}_C - V_C(0)$

$\hat{V}_L = \widehat{\text{[redacted]}\dot{I}_L} = \text{[redacted]} s\hat{I}_L - I_L(0)$

$-T\hat{V}_C + \hat{V}_L = 0$

$\hat{I}_C + T^*\hat{I}_L = 0$

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$$V_C(0) = s\hat{V}_C - \hat{I}_C = s\hat{V}_C + T^*\hat{I}_L$$

$$I_L(0) = s\hat{I}_L - \hat{V}_L = -T\hat{V}_C + s\hat{I}_L$$

$$\underbrace{\begin{bmatrix} s & +T^* \\ -T & s \end{bmatrix}}_{s - \begin{bmatrix} 0 & -T^* \\ T & 0 \end{bmatrix}} \begin{bmatrix} \hat{V}_C \\ \hat{I}_L \end{bmatrix} = \begin{bmatrix} V_C(0) \\ I_L(0) \end{bmatrix}$$

[illegible]

[illegible] Harmonic oscillator

Ham eq $\quad \dot{p} = -kq, \quad \dot{q} = m^{-1}p \qquad X\begin{bmatrix} p \\ q \end{bmatrix} = \begin{bmatrix} \dot{p} \\ \dot{q} \end{bmatrix} = \begin{bmatrix} 0 & -k \\ m^{-1} & 0 \end{bmatrix}\begin{bmatrix} p \\ q \end{bmatrix}$ ✓

$$H = \begin{bmatrix} m^{-1} & 0 \\ 0 & k \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} 0 & -k \\ m^{-1} & 0 \end{bmatrix}}_{X}$$

$X^tA + AX = 0$ because [illegible]

$AX = H \qquad (AX)^t = H^t \Longleftrightarrow -X^tA = H \qquad \therefore AX = -X^tA$

$X^t\underbrace{AX}_{H} + \underbrace{AX}_{H}X = 0$. Can you get anywhere?

~~What is the point?~~ [illegible]

$$X^2 = \begin{bmatrix} 0 & -k \\ m^{-1} & 0 \end{bmatrix}\begin{bmatrix} 0 & -k \\ m^{-1} & 0 \end{bmatrix} = \begin{bmatrix} -km^{-1} & 0 \\ 0 & -m^{-1}k \end{bmatrix} = -\begin{bmatrix} km^{-1} & 0 \\ 0 & m^{-1}k \end{bmatrix}.$$

$$-X^2 = \begin{bmatrix} km^{-1} & 0 \\ 0 & m^{-1}k \end{bmatrix}$$

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Go to binder about harm osc., C.T. and look for results.

$$H \simeq \begin{bmatrix} m^{-1} & 0 \\ 0 & k \end{bmatrix} : \begin{bmatrix} V_+ \\ V_- \end{bmatrix} \longrightarrow \begin{bmatrix} V_+^{\wedge} \\ V_-^{\wedge} \end{bmatrix}$$

---

do the "mechanical" harmonic oscillator $H = \frac{1}{2} p^t m^{-1} p + \frac{1}{2} q^t k q$

$$\dot{p} = -\frac{\partial H}{\partial q} = -kq \quad , \quad \dot{q} = \frac{\partial H}{\partial p} = m^{-1} p$$

$$\begin{bmatrix} \dot{p} \\ \dot{q} \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & -k \\ m^{-1} & 0 \end{bmatrix}}_{X} \begin{bmatrix} p \\ q \end{bmatrix}$$

$$AX = H$$

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -k \\ m^{-1} & 0 \end{bmatrix} = \begin{bmatrix} m^{-1} & 0 \\ 0 & k \end{bmatrix}$$

what's going on here? You have a $2n$ dimensional phase space of $\begin{bmatrix} p \\ q \end{bmatrix}$ with symplectic form $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = A$ and pos symm form $\begin{bmatrix} m^{-1} & 0 \\ 0 & k \end{bmatrix} = H$.

Question. How do you get from this picture to the situation [redacted] of $Sp(2n)$?

Take $n = 1$.

$$X = A^{-1} H = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} m^{-1} & 0 \\ 0 & k \end{bmatrix} = \begin{bmatrix} 0 & -k \\ m^{-1} & 0 \end{bmatrix}$$

$$-X^2 = \begin{bmatrix} 0 & -k \\ m^{-1} & 0 \end{bmatrix} \begin{bmatrix} 0 & k \\ -m^{-1} & 0 \end{bmatrix} = \begin{bmatrix} km^{-1} & 0 \\ 0 & m^{-1}k \end{bmatrix}$$

[redacted] Another idea you've forgotten [redacted]

$$L = \frac{1}{2} \dot{q}^t m \dot{q} - \frac{1}{2} q^t k q \, , \quad p = \frac{\partial L}{\partial \dot{q}} = m \dot{q} \qquad \frac{\partial L}{\partial q} = -kq$$

$m \ddot{q} + kq = 0$

frequencies $m\omega^2 + k = 0$

$m^{-1}k$ has real ≥ 0 eigenvalues

WHY?

~~[illegible]~~ $x^t k x > 0$ fa $x \neq 0$

$x^t m(m^{-1}k)x > 0$. Other way to proceed is via $m^{1/2}$? m pos symmetric has a positive symm. $m^{1/2}$. The point is that $m^{-1}k$ is symmetric wrt the pos. inner prod $x^t m x$

~~$(m^{-1}kx)^t m (m^{-1}k)x = x^t k m^{-1} m m^{-1} k x$~~

~~$(m^{-1}kx)^t m x$~~ ~~$x^t m m^{-1} k x$~~

$$(m^{-1}kx)^t m x \overset{?}{=} x^t \cancel{m}(\cancel{m}^{-1}kx)$$

$$x^t (m^{-1}k)^t m x = x^t k m^{-1} m x = x^t k x$$

The problem: Apparently there is a simple way to get the frequencies, namely the eigenvalues of the operator $m^{-1}k$ or km^{-1}. ~~[illegible]~~ ~~normal modes~~ None of this looks symplectic

$$X = \begin{bmatrix} 0 & -k \\ m^{-1} & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} m^{-1} & 0 \\ 0 & k \end{bmatrix}$$

$$-X^2 = -\begin{bmatrix} 0 & -k \\ m^{-1} & 0 \end{bmatrix} \begin{bmatrix} 0 & -k \\ m^{-1} & 0 \end{bmatrix} = \begin{bmatrix} +km^{-1} & 0 \\ 0 & +m^{-1}k \end{bmatrix} = \begin{bmatrix} km^{-1} & 0 \\ 0 & m^{-1}k \end{bmatrix}$$

$$\frac{1+X}{(1-X^2)^{1/2}}$$

$$1+X = \begin{bmatrix} 1 & -k \\ +m^{-1} & 1 \end{bmatrix} \begin{bmatrix} (1+km^{-1})^{-1/2} & 0 \\ 0 & (1+m^{-1}k)^{-1/2} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{(1+km^{-1})^{1/2}} & -k\frac{1}{(1+m^{-1}k)^{1/2}} \\ m^{-1}(1+km^{-1})^{-1/2} & \frac{1}{(1+m^{-1}k)^{1/2}} \end{bmatrix}$$

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$$X\begin{bmatrix}p\\ q\end{bmatrix} = \begin{bmatrix}\dot p\\ \dot q\end{bmatrix} = \begin{bmatrix}-kq\\ m^{-1}p\end{bmatrix} = \cancel{\begin{bmatrix}0 & m^{-1}\\ -k & 0\end{bmatrix}\begin{bmatrix}q\\ p\end{bmatrix}}$$

$$X = \begin{bmatrix}0 & -k\\ m^{-1} & 0\end{bmatrix} = A^{-1}H = \begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}\begin{bmatrix}m^{-1} & 0\\ 0 & k\end{bmatrix} = \begin{bmatrix}0 & -k\\ m^{-1} & 0\end{bmatrix}\begin{bmatrix}p\\ q\end{bmatrix}$$

Can you combine A, H to get the eigenvalues? $\begin{bmatrix}0 & -k\\ m^{-1} & 0\end{bmatrix}$

$$\begin{bmatrix}m^{-1} & \lambda\\ -\lambda & k\end{bmatrix} = \lambda A + H = A[\lambda + A^{-1}H] = A[\lambda + X]$$

Yes

$$X = A^{-1}H$$

$$\therefore \lambda - X = \lambda - A^{-1}H = A^{-1}(\lambda A - H).$$

Go back to the LC network idea that the network gives rise to a rep of $\langle F, \varepsilon\rangle$ and the dynamics of the network are obtained via L.T. from $(s - X)^{-1}$, where X is the I.C.T. of $F\varepsilon = g$.

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p. ω″ Introduce orth. coords. At the moment you have $A = \begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}$ $\begin{bmatrix}p_1\\ q_1\end{bmatrix}^t\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}\begin{bmatrix}p_2\\ q_2\end{bmatrix} = p_1^t q_2 - q_1^t p_2$

$$H = \begin{bmatrix}m^{-1} & 0\\ 0 & k\end{bmatrix} \qquad X = A^{-1}H = \begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}\begin{bmatrix}m^{-1} & 0\\ 0 & k\end{bmatrix} = \begin{bmatrix}0 & -k\\ m^{-1} & 0\end{bmatrix}$$

Use Graham Schmidt $k = K^tK$ $q^tkq = \overbrace{q^tK^tKq}^{= (Kq)^t(Kq)}$

k is a symmetric pos. def matrix

pp. ω″ thru ν‴ need clarification about the mechanical harmonic oscillator.

(177) $$SU(2) = \left\{ \begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix} : |a|^2 + |b|^2 = 1 \right\}$$

$$\mathcal{L}\, SU(2) = \left\{ X = \begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix} : \begin{matrix} a + \bar{a} = 0 \\ a = i\alpha \end{matrix} \right\}. \quad \alpha \text{ real}$$

$$\mathcal{L}\, SU(2) = \left\{ X = \begin{bmatrix} i\alpha & b \\ -\bar{b} & -i\alpha \end{bmatrix} : \alpha \text{ real} \right\}$$

$$X^2 = \begin{bmatrix} i\alpha & b \\ -\bar{b} & -i\alpha \end{bmatrix} \begin{bmatrix} i\alpha & b \\ -\bar{b} & -i\alpha \end{bmatrix} = \begin{bmatrix} -\alpha^2 - |b|^2 & 0 \\ 0 & -\alpha^2 - |b|^2 \end{bmatrix} = -1$$

[redacted] $$J = \begin{bmatrix} i\alpha & b \\ -\bar{b} & -i\alpha \end{bmatrix} \in \mathcal{L}\, SU(2)$$

and $J \in SU(2)$ when $\alpha^2 + |b|^2 = 1$.

$$\mathcal{J} = \{ -J = J^* = J^{-1} \} = S^2$$

So $J^2 = -1 \Longrightarrow$ [redacted] $$e^{\theta J} = \sum_{n=0}^{\infty} \frac{\theta^{2n} \overset{(-1)^n}{J^{2n}}}{(2n)!} + \sum_{n=0}^{\infty} \frac{\theta^{2n+1} \overset{(-1)^n J}{J^{2n+1}}}{(2n+1)!}$$

$\underline{e^{\theta J} =}$ $\cos\theta + (\sin\theta) J$ so $e^{\pi J} = -1$, $e^{\frac{\pi}{2} J} = J$

basepoint $i\varepsilon = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$

$$\left.\begin{aligned} \varepsilon J + \underbrace{(-J)\varepsilon}_{(\varepsilon J)^*} &= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} i\alpha & b \\ -\bar{b} & -i\alpha \end{bmatrix} = \begin{bmatrix} i\alpha & b \\ \bar{b} & i\alpha \end{bmatrix} \\ &+ \begin{bmatrix} i\alpha & b \\ -\bar{b} & -i\alpha \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \underline{\begin{bmatrix} i\alpha & -b \\ -\bar{b} & i\alpha \end{bmatrix}} \end{aligned}\right\} = 2 \begin{bmatrix} i\alpha & 0 \\ 0 & i\alpha \end{bmatrix}$$

$$\frac{i\varepsilon J - Ji\varepsilon}{2} = \alpha \qquad (i\varepsilon)J + (-J)(-i\varepsilon) = i(\varepsilon J + J\varepsilon)$$

Try again $\quad \varepsilon J + (\varepsilon J)^* = \varepsilon J + (-J)\varepsilon = 2 \begin{bmatrix} i\alpha & 0 \\ 0 & i\alpha \end{bmatrix}$

$I, J \in \mathcal{J}$ $\qquad IJ + JI$ should be herm.

$$(IJ)^* = (-J)(-I) = JI$$

Try exponentiating $\begin{bmatrix} 0 & i\beta \\ -i\beta & 0 \end{bmatrix}$. This may be wrong. But perhaps you can diagonalize. $b, \bar{b}$

Recall $J^2 = \begin{bmatrix} 0 & b \\ -\bar{b} & 0 \end{bmatrix}^2 = \begin{bmatrix} -b\bar{b} & 0 \\ 0 & -\bar{b}b \end{bmatrix} = -1$

so $b\bar{b} = \bar{b}b = 1$. Also $b = b^t$ is assumed so that $\bar{b} = b^*$. Thus $b^{-1} = b^*$ (so $b \in U(n)$) and $b = b^t$ so that b is symmetric. The idea now is to diagonalize b. using the action of $U(n)$ given by $u \cdot b = ubu^t = ub\bar{u}^{-1}$.

Perhaps you should try to follow the conjugacy theorem in the Lie algebra. This proceeds by minimizing the distance squared between a the $U(n)$ orbit of b and a generic diagonal element.

List of steps. The basic action is $u \cdot b = ubu^t = ub\bar{u}^{-1}$ of $U(n)$ on complex symmetric matrices of size $n \times n$. Suppose $b = \mathrm{diag}(\lambda_1, \cdots, \lambda_n)$. What is the isotropy group? $n = 1$. Then $u \cdot b = ubu^t = u^2 b$, here $u = e^{i\theta}$. If $b \neq 0$ the isotropy group is $\{u = \pm 1\}$. Next take $n = 2$, where $b_1 \neq b_2$ are distinct $\neq 0$ complex numbers.

say $u = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}$

$$\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} \alpha & \gamma \\ \beta & \delta \end{bmatrix}$$

It seems that you ought to be able to use $SU(2)$.

R81

$$\begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} a & -\bar{b} \\ b & \bar{a} \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ -b & \bar{a} \end{bmatrix} \begin{bmatrix} \lambda_1 & \\ & \lambda_2 \end{bmatrix} = \begin{bmatrix} \lambda_1 & \\ & \lambda_2 \end{bmatrix} \begin{bmatrix} \bar{a} & \bar{b} \\ -b & a \end{bmatrix}$$

$$\begin{bmatrix} \lambda_1 a & \lambda_2 b \\ -\lambda_1 b & \lambda_2 \bar{a} \end{bmatrix} = \begin{bmatrix} \lambda_1 \bar{a} & \lambda_1 \bar{b} \\ -\lambda_2 b & \lambda_2 a \end{bmatrix} .$$

$$\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} \alpha & \gamma \\ \beta & \delta \end{bmatrix}$$

$$\begin{bmatrix} \lambda_1 \alpha & \lambda_2 \beta \\ \lambda_1 \gamma & \lambda_2 \delta \end{bmatrix} \begin{bmatrix} \alpha & \gamma \\ \beta & \delta \end{bmatrix} = \begin{bmatrix} \lambda_1 \alpha^2 + \lambda_2 \beta^2 & \lambda_1 \alpha\gamma + \lambda_2 \beta\delta \\ \lambda_1 \gamma\alpha + \lambda_2 \delta\beta & \lambda_1 \gamma^2 + \lambda_2 \delta^2 \end{bmatrix}$$

$\lambda_1 \gamma\alpha + \lambda_2 \delta\beta = 0$ $\quad |\lambda_1| \cancel{|\gamma|} |\alpha| = |\lambda_2| |\delta| \cancel{|\beta|}.$

But you know $|\alpha|^2 + |\beta|^2 = 1$ $\quad \therefore \quad |\beta| = |\gamma|$

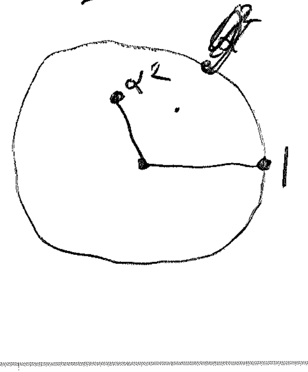
$|\alpha|^2 + |\gamma|^2 = 1$

Finally $\lambda_1 = \lambda_1 \alpha^2 + \lambda_2 \beta^2$ $\quad \lambda_1 (1 - \alpha^2) = \lambda_2 \beta^2$

$\lambda_2 = \lambda_1 \gamma^2 + \lambda_2 \delta^2$ $\quad \lambda_2 (1 - \delta^2) = \lambda_1 \gamma^2$

suppose $\lambda_1 > \lambda_2 > 0$ $\quad \lambda_1 |1 - \alpha^2| = \lambda_2 |\beta|^2$

$= \lambda_2 (1 - |\alpha|^2)$



(182) Diagonalization ~~[illegible]~~ for self-adjoint matrices A. Pick a diagonal matrix D with distinct real eigenvalues. Define a functional $\frac{1}{2}\operatorname{tr}\left(D - uAu^{-1}\right)^2 = \Phi(u)$. Because $U(n)$ is compact $\exists$ critical point u_0. Then consider a tangent vector $u_0 X$ at u_0

$$\Phi(u_0 + \varepsilon u_0 X) - \Phi(u_0)$$

$$= \frac{1}{2}\operatorname{tr}\left(D - u_0(1+\varepsilon X)A(1-\varepsilon X)u_0\right) \quad ?$$

$$\Phi(u_0 + \delta u) - \Phi(u_0) =$$

$$\delta \frac{1}{2}\operatorname{tr}\left(uAu^{-1} - D\right)^2 = \operatorname{tr}\left[(uAu^{-1}-D)\cdot\delta(uAu^{-1})\right]$$

$$\delta(uAu^{-1}) = uXAu^{-1} - uAu^{-1}uXu^{-1} = u[X,A]u^{-1}$$

$$\operatorname{tr}\left((uAu^{-1}-D)\cdot u[X,A]u^{-1}\right)$$

$$= \operatorname{tr}\left((A - u^{-1}Du)\cdot[X,A]\right) = \operatorname{tr}\left([A, \cancel{A} - u^{-1}Du]X\right)$$

A, D hermitian, D diagonal with distinct eigenvalues minimize $\frac{1}{2}\operatorname{tr}(uAu^{-1} - D)^2$. to prove $\exists u$ such that $[uAu^{-1}, D] = 0$. $\exists\, u_0$ critical point of $\frac{1}{2}\operatorname{tr}(u_0Au_0^{-1} - D)^2$ Replace A by $u_0^{-1}Au_0$?? Logic?

Let $u = u_0$ be a critical point of $\frac{1}{2}\operatorname{tr}(uAu^{-1} - D)^2$ $= \frac{1}{2}\operatorname{tr}(A - u^{-1}Du)^2$. A, D hermitian. Claim $\frac{1}{2}\operatorname{tr}(uAu^{-1} - D)^2$ has a critical point at $u = 1$ $\Longleftrightarrow [A, D] = 0$. $\frac{1}{2}\operatorname{tr}\left(\underbrace{(1+X)A(1-X)}_{A+[X,A]} - D\right)^2$

103 $\delta\ \frac{1}{2}\operatorname{tr}(uAu^{-1}-D)^2 = \operatorname{tr}(\cancel{u}A\cancel{u^{-1}}-D)([X,A])$

$= 0\ \forall X.$ $\qquad \frac{1}{2}\operatorname{tr}\left(((1+X)A(1-X))\cdot(A-D)\right) = 0$

$\qquad\qquad A+[X,A]$

$$\delta\ \frac{1}{2}\operatorname{tr}(uAu^{-1}-D)^2 = \underbrace{\operatorname{tr}(\delta(uAu^{-1})\cdot(uAu^{-1}-D))}_{\operatorname{tr}([X,A]\cdot(A-D))}$$

$u+\delta u = 1+X$

$$\operatorname{tr}([X,A]\cdot(A-D)) = \operatorname{tr}(X\cdot[A, A-D])$$

$$\operatorname{tr}([X,Y]Z) = \operatorname{tr}(X[Y,Z]) = \operatorname{tr}([Z,X], Y)$$

$XYZ - YXZ \qquad XYZ - XZY$

$\mathcal{L}\, Sp(2n) = \left\{ X = \begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix} : \begin{matrix} a^* = -a \\ b^t = b \end{matrix} \right\}.$

Cartan subalg is where $a = \begin{bmatrix} s_1 & & \\ & \ddots & \\ & & s_n \end{bmatrix}$ $\qquad \bar{s}_i = -s_i$

Action $\quad a \# b = ab - b\bar{a} = ab + ba^t$

Let $\hat{a}_{ij}$ denote the matrix with $\begin{cases} 1 & \text{in position } i,j \\ 0 & \text{otherwise} \end{cases}$

$s \# \hat{a}_{ij} = (s_i - s_j)\hat{a}_{ij}$. Let $\hat{b}_{ij} = \begin{cases} 1 & \text{position } i,j \\ 0 & \text{otherwise} \end{cases}$

$s \# \hat{b}_{ij} = s\hat{b}_{ij} - \hat{b}_{ij}\bar{s} = s_i\hat{b}_{ij} - \hat{b}_{ij}\bar{s}_j = (s_i + s_j)\hat{b}_{ij}$

You've confused real + complex for the b root space. When you write $a\#b = ab - b\bar{a}$? You have to straighten out the

(184) Go back to $\mathcal{L}Sp(2n) = \left\{ X = \begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix} : \begin{matrix} a^* = -a \\ b^t = b \end{matrix} \right\}$

Have embedding $U(n) \hookrightarrow Sp(2n)$, $u \longmapsto \begin{bmatrix} u & 0 \\ 0 & \bar{u} \end{bmatrix} = \begin{bmatrix} u & 0 \\ 0 & (u^t)^{-1} \end{bmatrix}$

$U \# B = \begin{bmatrix} u & 0 \\ 0 & \bar{u} \end{bmatrix} \begin{bmatrix} 0 & b \\ -\bar{b} & 0 \end{bmatrix} \begin{bmatrix} u^* & 0 \\ 0 & u^t \end{bmatrix} = \begin{bmatrix} 0 & ubu^t \\ -\bar{u}\bar{b}u^* & 0 \end{bmatrix}$. U Your aim is a conjugacy thm: that $\forall B \ \exists U$ such that $U \# B = UBU^*$ centralizes some B_0.

You are working in the space ~~[illegible]~~

$\mathcal{B} = \left\{ \begin{bmatrix} 0 & b \\ -\bar{b} & 0 \end{bmatrix} \middle| \ b^t = b \right\}$. Functional $\frac{1}{2}\mathrm{tr}(UBU^* - B_0)^2$

Let U be a critical point, ~~[illegible]~~

Functional stationary for $\overset{\text{1st order}}{\text{variation}}$ $U + \delta U = (1 + \varepsilon X)U$

$$\delta \ \tfrac{1}{2}\mathrm{tr}(UBU^* - B_0)^2 =$$

~~$= -\frac{1}{2}\mathrm{tr}(UBU^* - B_0)^2 + \frac{1}{2}\mathrm{tr}((U+\delta U)B(U+\delta U)^* - B_0)^2$~~

~~$UBU^* + UB(\delta U)^* + (\delta U)BU$~~

$$= \delta\left\{ \tfrac{1}{2}\mathrm{tr}\Big(\overbrace{\cancel{(UBU^*)^2}}^{UB\cancel{U^*}UBU^*} - UBU^*B_0 - B_0UBU^* + \cancel{B_0^2}\Big)\right\}$$

$$= -\tfrac{1}{2}\mathrm{tr}\,\delta\left\{ 2UBU^*B_0 \right\} = -\mathrm{tr}\left\{ \delta(UBU^*B_0) \right\}$$

$$= -\mathrm{tr}\left\{ \underbrace{(1+\varepsilon X)B(1-\varepsilon X)}_{\varepsilon[X,B]} B_0 \right\}$$

~~[illegible]~~

(185) Look at $\mathcal{L}O(2n) = \left\{ X = \begin{bmatrix} a & b \\ \bar{b} & \bar{a} \end{bmatrix} : \begin{array}{l} a^* = -a \\ b^t = -b \end{array} \right\}$

$K = \left\{ \begin{bmatrix} u & 0 \\ 0 & \bar{u} \end{bmatrix} : u \in \mathcal{U}(n) \right\}$ $\qquad \mathfrak{p} =$ [illegible] $= \left\{ p = \begin{bmatrix} 0 & b \\ \bar{b} & 0 \end{bmatrix} : b^t = -b \right\}$

$$\underbrace{\begin{bmatrix} u & 0 \\ 0 & \bar{u} \end{bmatrix}}_{\kappa} \underbrace{\begin{bmatrix} 0 & b \\ \bar{b} & 0 \end{bmatrix}}_{p} \underbrace{\begin{bmatrix} u^* & 0 \\ 0 & u^t \end{bmatrix}}_{\kappa^{-1}} = \begin{bmatrix} 0 & ubu^* \\ \bar{u}\bar{b}u^t & 0 \end{bmatrix}$$

$$\Phi(\kappa) = \frac{1}{2} \mathrm{tr} \left(\kappa p \kappa^{-1} - p_0 \right)^2 \qquad \kappa = \begin{bmatrix} u & 0 \\ 0 & \bar{u} \end{bmatrix}$$

~~Let u_0 be a stationary point: $\Phi(u_0 + \delta u) = \Phi(u_0)$~~

~~can take $\delta u = a u_0$~~ Let $\kappa_0 = \begin{bmatrix} u_0 & 0 \\ 0 & \bar{u}_0 \end{bmatrix}$ be a stationary point. $\kappa + \delta\kappa = \kappa_0 + \varepsilon X \kappa_0$ $\qquad X = \begin{bmatrix} a & 0 \\ 0 & \bar{a} \end{bmatrix}$

$$\underline{\left((1+\varepsilon X) \kappa p \kappa^{-1} (1 - \varepsilon X) - p_0 \right)^2}$$

$$\frac{1}{2} \mathrm{tr} \left(\kappa p \kappa^{-1} - p_0 \right)^2 = \frac{1}{2} \mathrm{tr} \left[(\kappa p \kappa^{-1})^2 - \kappa p \kappa^{-1} p_0 - p_0 \kappa p \kappa^{-1} + p_0^2 \right]$$

$$= \mathrm{tr}(p^2 + p_0^2) - \mathrm{tr}(\kappa p \kappa^{-1} p_0)$$

Assuming κ_0 stationary point for $\mathrm{tr}(\kappa_0 p \kappa_0^{-1} p_0)$ take $\kappa_0 + \delta\kappa = (1+\varepsilon X)\kappa_0$ $\qquad X = \begin{bmatrix} a & 0 \\ 0 & \bar{a} \end{bmatrix}$

$$\mathrm{tr}\left((1+\varepsilon X) \kappa_0 p \kappa_0^{-1} (1 - \varepsilon X) p_0 \right)$$

$$= \mathrm{tr}(\kappa_0 p \kappa_0^{-1} p_0) + \varepsilon \underbrace{\mathrm{tr}\left([X, \kappa_0 p \kappa_0^{-1}] p_0 \right)}$$

$$\pm \varepsilon \, \mathrm{tr}\left(X [\kappa_0 p \kappa_0^{-1}, p_0] \right)$$

(186) Centralizer of a generic p_0

$$p_0 = \begin{bmatrix} 0 & b_0 \\ \bar{b}_0 & 0 \end{bmatrix} \qquad \begin{bmatrix} u & 0 \\ 0 & \bar{u} \end{bmatrix}\begin{bmatrix} 0 & b \\ \bar{b} & 0 \end{bmatrix}\begin{bmatrix} u^* & 0 \\ 0 & u^t \end{bmatrix} = \begin{bmatrix} 0 & ubu^t \\ \bar{u}\bar{b}u^* & 0 \end{bmatrix}$$

inf action $\left[\begin{bmatrix} a & 0 \\ 0 & \bar{a} \end{bmatrix}, \begin{bmatrix} 0 & b \\ \bar{b} & 0 \end{bmatrix}\right] = \begin{bmatrix} 0 & ab + ba^t \\ \bar{a}\bar{b} + \bar{b}a^* & 0 \end{bmatrix}$

$b = 1.$ $\begin{bmatrix} u & 0 \\ 0 & \bar{u} \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} u^* & 0 \\ 0 & u^t \end{bmatrix} = \begin{bmatrix} uu^t & 0 \\ 0 & \bar{u}u^* \end{bmatrix} \overset{?}{=} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

iff $uu^t = 1$ $\qquad \bar{u} = u$

$\bar{u}u^* = 1 \Rightarrow \bar{u}\cancel{u^*}u = u$

$$\begin{bmatrix} 0 & b \\ \bar{b} & 0 \end{bmatrix} = \left[\begin{array}{c|c} & \lambda_1 \cdots \lambda_n \\ \hline \bar{\lambda}_1 \cdots \bar{\lambda}_n & \end{array}\right]$$

$$\begin{bmatrix} a & 0 \\ 0 & \bar{a} \end{bmatrix}, \left[\begin{array}{c|c} 0 & \Lambda \\ \hline \bar{\Lambda} & 0 \end{array}\right] = \begin{bmatrix} 0 & a\Lambda \\ \bar{a}\bar{\Lambda} & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & \Lambda \\ \bar{\Lambda} & 0 \end{bmatrix} \cdot \begin{bmatrix} a & 0 \\ 0 & \bar{a} \end{bmatrix} = \begin{bmatrix} 0 & \Lambda\bar{a} \\ \bar{\Lambda}a & 0 \end{bmatrix}$$

$$\begin{bmatrix} & a\Lambda - \Lambda\bar{a} \\ \bar{a}\bar{\Lambda} - \bar{\Lambda}a & \end{bmatrix}$$

$\Lambda = 1$, then $\begin{bmatrix} a & 0 \\ 0 & \bar{a} \end{bmatrix}$ centralizes

$\Lambda \Leftrightarrow a = \bar{a}$. Next compute

$$[a_{ij}]\begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} - \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}[\bar{a}_{ij}] = 0$$

$$[a_{ij}]\begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} + \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}[a_{ji}] = 0 \qquad ?$$

$a_{ij}\lambda_j \; +$

(187) take $b = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$ $P = \left|\begin{array}{c|c} 0 & \begin{matrix} \lambda_1 & \\ & \lambda_2 \end{matrix} \\ \hline \begin{matrix} \bar\lambda_1 & \\ & \bar\lambda_2 \end{matrix} & 0 \end{array}\right|$

$X = \left[\begin{array}{c|c} \begin{matrix} a & b \\ c & d \end{matrix} & \\ \hline & \begin{matrix} \bar a & \bar b \\ \bar c & \bar d \end{matrix} \end{array}\right]$

$\bar a = -a$
$\bar d = -d$
$\bar b = -c$

$$XP = \left|\begin{array}{c|c} \begin{matrix} a & b \\ c & d \end{matrix} & 0 \\ \hline 0 & \begin{matrix} \bar a & \bar b \\ \bar c & \bar d \end{matrix} \end{array}\right| \left|\begin{array}{c|c} 0 & \begin{matrix} \lambda_1 & \\ & \lambda_2 \end{matrix} \\ \hline \begin{matrix} \bar\lambda_1 & \\ & \bar\lambda_2 \end{matrix} & 0 \end{array}\right| = \left[\begin{array}{c|c} 0 & \begin{matrix} a\lambda_1 & b\lambda_2 \\ c\lambda_1 & d\lambda_2 \end{matrix} \\ \hline \begin{matrix} \bar a\bar\lambda_1 & \bar b\bar\lambda_2 \\ \bar c\bar\lambda_1 & \bar d\bar\lambda_2 \end{matrix} & 0 \end{array}\right]$$

$$\cancel{PX = \left|\begin{array}{c|c} 0 & \begin{matrix} \lambda_1 & \\ & \lambda_2 \end{matrix} \\ \hline \begin{matrix} \bar\lambda_1 & \\ & \bar\lambda_2 \end{matrix} & 0 \end{array}\right| \left|\begin{array}{c|c} 0 & \begin{matrix} a & b \\ c & d \end{matrix} \\ \hline \begin{matrix} \bar a & \bar b \\ \bar c & \bar d \end{matrix} & 0 \end{array}\right| = \left|\begin{array}{c|c} \begin{matrix} \lambda_1\bar a & \lambda_1\bar b \\ \lambda_2\bar c & \lambda_2\bar d \end{matrix} & \\ \hline & \begin{matrix} \bar\lambda_1 a & \bar\lambda_1 b \\ \bar\lambda_2 c & \bar\lambda_2 d \end{matrix} \end{array}\right|}$$

$$PX = \left[\begin{array}{c|c} & \begin{matrix} \lambda_1 & \\ & \lambda_2 \end{matrix} \\ \hline \begin{matrix} \bar\lambda_1 & \\ & \bar\lambda_2 \end{matrix} & \end{array}\right] \left[\begin{array}{c|c} \begin{matrix} a & b \\ c & d \end{matrix} & \\ \hline & \begin{matrix} \bar a & \bar b \\ \bar c & \bar d \end{matrix} \end{array}\right] = \left[\begin{array}{c|c} 0 & \begin{matrix} \lambda_1\bar a & \lambda_1\bar b \\ \lambda_2\bar c & \lambda_2\bar d \end{matrix} \\ \hline \begin{matrix} \bar\lambda_1 a & \bar\lambda_1 b \\ \bar\lambda_2 c & \bar\lambda_2 d \end{matrix} & 0 \end{array}\right]$$

$a\lambda_1 = \lambda_1\bar a$ $\quad b\lambda_2 = \lambda_1\bar b$

$c\lambda_1 = \lambda_2\bar c$ $\quad d\lambda_2 = \lambda_2\bar d$

If $\lambda_1 \neq 0$, then
$a = \bar a \Longrightarrow a = 0$
If $\lambda_2 \neq 0$, then
$d = \bar d \Longrightarrow d = 0$

If $|\lambda_1| \neq |\lambda_2|$ then $|c| = 0$

$a\lambda_1 = \lambda_1(-a)$ $\quad b\lambda_2 = \lambda_1(-c)$

$c\lambda_1 = \lambda_2(-\bar b)$ $\quad d\lambda_2 = \lambda_2(-d)$

Ass $\lambda_1, \lambda_2 \neq 0$
then $a = d = 0$.

(188) Suppose $\begin{bmatrix}\lambda_1 & 0 \\ 0 & \lambda_2\end{bmatrix} = \begin{bmatrix}1 & 0 \\ 0 & 1\end{bmatrix}$. Then you seem to get $a=\bar{a}$, $b=\bar{b}$, $c=\bar{c}$, $d=\bar{d}$. $\therefore$ $\begin{bmatrix}a & b \\ c & d\end{bmatrix}$ is real and ~~skew symmetric~~ skew adjoint, therefore real skew symmetric $a=d=0$ $b=-c$. $\begin{bmatrix}0 & b \\ -b & 0\end{bmatrix}$.

Assume $\lambda_1, \lambda_2 \neq 0$. then $a=d=0$.

$$XP - PX = \left[\begin{array}{c|c} & \\ \hline & \end{array}\right]$$

$$\begin{bmatrix} a\lambda_1 - \lambda_1\bar{a} & b\lambda_2 - \lambda_1\bar{b} \\ c\lambda_1 - \lambda_2\bar{c} & d\lambda_2 - \lambda_2\bar{d} \end{bmatrix} = \begin{bmatrix} \lambda_1\overbrace{(a-\bar{a})}^{2a} & \\ & \lambda_2\underbrace{(d-\bar{d})}_{2d} \end{bmatrix}$$

$c = -\bar{b}$ $\qquad$ $b\lambda_2 + \lambda_1 c$

You see that $c\lambda_1 - \lambda_2\bar{c} = c\lambda_1 + \lambda_2 b$ $\|$

$b\lambda_2 - \lambda_1\bar{b} = b\lambda_2 + \lambda_1 c$ $\qquad$?

Take $P_0 = \begin{bmatrix}0 & b \\ \bar{b} & 0\end{bmatrix}$ $\qquad$ $b^t = -b$

$$\mathrm{Lie}\ O(2n) = \left\{ X = \begin{bmatrix}a & b \\ \bar{b} & \bar{a}\end{bmatrix} : \begin{array}{l} a^* = -a \\ b^t = -b \end{array} \right\}$$

$K = \begin{bmatrix} a & 0 \\ 0 & \bar{a} \end{bmatrix}$, $P = \begin{bmatrix} 0 & b \\ -\bar{b} & 0 \end{bmatrix}$ b diag

$a = [a_{ij}]$ $a_{ij} = \bar{a}_{ji}$ $b = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$ $b_{ij} = \lambda_i \delta_{ij}$ no summ conv.

$$(ab)_{ik} = \sum_j a_{ij} \lambda_j \delta_{jk} = a_{ik}\lambda_k, \quad (b\bar{a})_{ik} = \sum_j \lambda_i \delta_{ij} \bar{a}_{jk} = \lambda_i \bar{a}_{ik}$$

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} a_{11}\lambda_1 & a_{12}\lambda_2 \\ a_{21}\lambda_1 & a_{22}\lambda_2 \end{bmatrix}$$

action

$$a \# b = ab + ba^t = ab - b\bar{a}$$

$$\begin{bmatrix} \lambda_1 & \\ & \lambda_2 \end{bmatrix}\begin{bmatrix} \bar{a}_{11} & \bar{a}_{12} \\ \bar{a}_{21} & \bar{a}_{22} \end{bmatrix} = \begin{bmatrix} \lambda_1\bar{a}_{11} & \lambda_1\bar{a}_{12} \\ \lambda_2\bar{a}_{21} & \lambda_2\bar{a}_{22} \end{bmatrix} = -\begin{bmatrix} \lambda_1 a_{11} & \lambda_1 a_{21} \\ \lambda_2 a_{12} & \lambda_2 a_{22} \end{bmatrix}$$

~~$a_{11}\lambda_1 = \lambda_1 \bar{a}_{11}$ $a_{22}\lambda_2 = \lambda_2 \bar{a}_{22}$~~

$a_{11}\lambda_1 = -a_{11}\lambda_1$ $a_{22}\lambda_2 = -\lambda_2 a_{22}$ $\therefore a_{11} = a_{22} = 0$

$a_{12}\lambda_2 = -a_{21}\lambda_1$ $a_{21}\lambda_1 = -a_{12}\lambda_2$

plus single relation $\underbrace{a_{12}}_{+\lambda_1}\lambda_2 + \underbrace{a_{21}}_{-\lambda_2}\lambda_1 = 0$. $\therefore$ λ_2

$a \in \begin{bmatrix} 0 & \lambda_1 \\ -\lambda_2 & 0 \end{bmatrix}\mathbb{C}$ still need $a^* = -a$. $\overline{(\lambda_1 z)} = -\lambda_2 z$

$\bar{\lambda}_1 \bar{z} = -\lambda_2 z \Longrightarrow |\lambda_1| = |\lambda_2|$ unless $z = 0$. ??

Go to $\mathcal{L}O(2n)$. You can't have a diagonal b which is skew-symm. You probably made a bad choice

$b^* = \bar{b}^t = -\bar{b}$

$$\mathcal{L}O(2n) = \left\{ X = \begin{bmatrix} a & b \\ \bar{b} & \bar{a} \end{bmatrix} : \begin{matrix} a^* = -a \\ b^t = -b \end{matrix} \right\}$$

If $n = 2$, then $b = \begin{bmatrix} 0 & \lambda \\ -\lambda & 0 \end{bmatrix}$

(190) $\mathcal{L}\, O(2n) = \left\{ X = \begin{bmatrix} a & b \\ \bar b & \bar a \end{bmatrix} : \begin{matrix} a^* = -a \\ b^t = -b \end{matrix} \right\}.$

$n = 1$ have only $\left\{ \begin{bmatrix} a & 0 \\ 0 & \bar a \end{bmatrix} : a \in i\mathbb{R} \right\}.$

$n = 2$ have $\left\{ \begin{bmatrix} a & b \\ \bar b & \bar a \end{bmatrix} : a \in \mathcal{L}U(2), \quad b = \begin{bmatrix} 0 & \lambda \\ -\lambda & 0 \end{bmatrix} \right\}$

have action $a \# b = ab - b\bar a$

$$\overset{a}{\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} \bar\alpha & \bar\beta \\ \bar\gamma & \bar\delta \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} \bar\gamma & \bar\delta \\ -\bar\alpha & -\bar\beta \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} \bar\delta & -\bar\gamma \\ -\bar\beta & \bar\alpha \end{bmatrix}$$

$\bar\alpha = \delta$
$\gamma = -\bar\beta$ $\quad \therefore \quad a = \begin{bmatrix} \alpha & \beta \\ -\bar\beta & \bar\alpha \end{bmatrix} \qquad a^* = \begin{bmatrix} \bar\alpha & -\beta \\ \bar\beta & \alpha \end{bmatrix}$

$\bar\alpha = -\alpha.$

$\therefore \quad a = \begin{bmatrix} \alpha & \beta \\ -\bar\beta & \bar\alpha \end{bmatrix}$ where $\alpha + \bar\alpha = 0 \quad \therefore a \in \mathcal{L}\,SU(2)$

$\mathcal{L}\, Sp(2n) = \left\{ X = \begin{bmatrix} a & b \\ -\bar b & \bar a \end{bmatrix} : \begin{matrix} a^* = -a \\ b^t = b \end{matrix} \right\}.$

again the action is $a \# b = ab - b\bar a = ab + ba^t$

What's a generic b?

General theory. You get a Cartan subalgebra as centralizer of a generic element. Rank of $O(2n)$ should be n. Cartan subalg is $\oplus$ of n inf rotations $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\omega$

Look for max abelian subspace of $\mathfrak{f}$.

$$\begin{bmatrix}0 & b_1\\ \bar b_1 & 0\end{bmatrix}\begin{bmatrix}0 & b_2\\ \bar b_2 & 0\end{bmatrix} - \begin{bmatrix}0 & b_2\\ \bar b_2 & 0\end{bmatrix}\begin{bmatrix}0 & b_1\\ \bar b_1 & 0\end{bmatrix}$$ [illegible]

$$\begin{bmatrix}b_1\bar b_2 & 0\\ 0 & \bar b_1 b_2\end{bmatrix} - \begin{bmatrix}b_2\bar b_1 & 0\\ 0 & \bar b_2 b_1\end{bmatrix} = \begin{bmatrix}b_1\bar b_2 - b_2\bar b_1 & 0\\ 0 & \bar b_1 b_2 - \bar b_2 b_1\end{bmatrix}$$

~~$(b_1\bar b_2 - b_2\bar b_1)^* = (b_1 b_2^* - b_2 b_1^*)^*$~~ [illegible]

$$\underbrace{(b\bar c - c\bar b)}^{*} + (b\bar c - c\bar b) = 0$$

$$(\bar b c - \bar c b)^t = (-c)(-\bar b) - (-b)(-\bar c) = -b\bar c + c\bar b$$

want pairing $\langle b, c\rangle = b\bar c - c\bar b$

~~[illegible]~~ Let's stop this calculation

Go to Poisson brackets?

Let try to understand the picture from the creation and annihilation operator viewpoint.

H has basis $a_1, \dots, a_n, a_1^*, \dots, a_n^*$ get complex linear space of operators somewhere. Also have symplectic relation $\{a_i, a_j^*\} = \delta_{ij}$ $\{a_i, a_j\} = 0 = \{a_i^*, a_j^*\}$

Next you can enlarge this symplectic space by semidirect prod with S^2H. How does this look? So far you have been using this picture $H = \left\{\begin{bmatrix}x\\y\end{bmatrix}\right\}$, $\begin{bmatrix}x_1\\y_1\end{bmatrix}^*\begin{bmatrix}x_2\\y_2\end{bmatrix}$, $\begin{bmatrix}x_1\\y_1\end{bmatrix}^t\begin{bmatrix}0&1\\-1&0\end{bmatrix}\begin{bmatrix}x_2\\y_2\end{bmatrix}$

S^2H generated by [redacted] a quadratic map

$$\begin{bmatrix}u\\v\end{bmatrix} \longmapsto \qquad S^2\begin{bmatrix}V\\V^\wedge\end{bmatrix} = S^2V \oplus V\otimes V^\wedge \otimes S^2V^\wedge$$

$$\mathcal{L}Sp(2n) = \left\{\begin{bmatrix}a&b\\-\bar{b}&\bar{a}\end{bmatrix} : \begin{matrix}a^*=-a\\b^t=b\end{matrix}\right\}$$

$$X=\begin{bmatrix}a&b\\c&d\end{bmatrix}\in gl(2n,\mathbb{C}) \qquad X^t\begin{bmatrix}0&1\\-1&0\end{bmatrix}+\begin{bmatrix}0&1\\-1&0\end{bmatrix}X=0$$

$$X^t = -JXJ^{-1} = JXJ$$

$$\begin{bmatrix}a^t&c^t\\b^t&d^t\end{bmatrix} = \begin{bmatrix}0&1\\-1&0\end{bmatrix}\begin{bmatrix}a&b\\c&d\end{bmatrix}\begin{bmatrix}0&1\\-1&0\end{bmatrix}$$

$$= \begin{bmatrix}c&d\\-a&-b\end{bmatrix}\begin{bmatrix}0&1\\-1&0\end{bmatrix} = \begin{bmatrix}-d&c\\b&-a\end{bmatrix}$$

$\therefore\ d=-a^t,\ b^t=b,\ c^t=c$ So you see $\underset{b}{S^2V} \oplus \underbrace{V\otimes V^\wedge}_{a} \oplus \underset{c}{S^2V^\wedge}$

suppose you replace a, b by bilinear expressions

What does this mean? $b = b_{ij}$ where $b_{ij}=b_{ji}$

$$a_{ij} = \bar{a}_{ji}$$

(193) problems. Max commutative subspace of $\mathfrak{p}$.

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p} = \left\{ \begin{bmatrix} a & 0 \\ 0 & \bar{a} \end{bmatrix} : a^* = -a \right\} + \left\{ \begin{bmatrix} 0 & b \\ -\bar{b} & 0 \end{bmatrix} : b^t = b \right\}$$

$$\begin{bmatrix} 0 & b \\ -\bar{b} & 0 \end{bmatrix} \begin{bmatrix} 0 & c \\ -\bar{c} & 0 \end{bmatrix} = \begin{bmatrix} -b\bar{c} & 0 \\ 0 & -\bar{b}c \end{bmatrix}$$

$$\begin{bmatrix} 0 & c \\ -\bar{c} & 0 \end{bmatrix} \begin{bmatrix} 0 & b \\ -\bar{b} & 0 \end{bmatrix} = \begin{bmatrix} -c\bar{b} & 0 \\ 0 & -\bar{c}b \end{bmatrix}$$

$$\begin{bmatrix} -b\bar{c} + c\bar{b} & \\ & -\bar{b}c + \bar{c}b \end{bmatrix}$$

$$SU(2) = Sp(2). \qquad \mathcal{L}Sp(2) = \left\{ \begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix} : \begin{matrix} a^* + a = 0 \\ b = b^t \end{matrix} \checkmark \right\}$$

means $a = i\alpha$

$$\mathcal{L}SU(2) = \left\{ \begin{bmatrix} i\alpha & b \\ -\bar{b} & -i\alpha \end{bmatrix} \right\} = \left\{ x\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} + y\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} + z\begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix} \right\}$$

$$x\hat{i} + y\hat{j} + z\hat{k}$$

So a maximal abelian subspace of $\mathcal{L}SU(2)$ is 1-diml. Any real line.

$$\begin{bmatrix} e^{i\alpha} & 0 \\ 0 & e^{-i\alpha} \end{bmatrix} \begin{bmatrix} 0 & b \\ -\bar{b} & 0 \end{bmatrix} \begin{bmatrix} e^{-i\alpha} & 0 \\ 0 & e^{+i\alpha} \end{bmatrix} = \begin{bmatrix} 0 & e^{2i\alpha}b \\ -\bar{b}e^{-2i\alpha} & 0 \end{bmatrix}$$

Clear you have conjugacy for the maximal abelian subspaces.

Consider next example $\mathcal{L}SO(2n)$ $n=2$

$$\mathcal{L}SO(2n) = \mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p} = \left\{ \begin{bmatrix} a & 0 \\ 0 & \bar{a} \end{bmatrix} : a^* = -a \right\} \oplus \left\{ \begin{bmatrix} 0 & b \\ \bar{b} & 0 \end{bmatrix} : b^t = -b \right\}$$

$$\begin{bmatrix} a & 0 \\ 0 & \bar{a} \end{bmatrix} \begin{bmatrix} 0 & b \\ \bar{b} & 0 \end{bmatrix} - \begin{bmatrix} 0 & b \\ \bar{b} & 0 \end{bmatrix} \begin{bmatrix} a & 0 \\ 0 & \bar{a} \end{bmatrix} = \begin{bmatrix} 0 & ab \\ \bar{a}\bar{b} & 0 \end{bmatrix} - \begin{bmatrix} 0 & b\bar{a} \\ \bar{b}a & 0 \end{bmatrix} = \begin{bmatrix} 0 & ab - b\bar{a} \\ \bar{a}\bar{b} - \bar{b}a & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & b \\ \bar{b} & 0 \end{bmatrix} \begin{bmatrix} 0 & c \\ \bar{c} & 0 \end{bmatrix} - \begin{bmatrix} 0 & c \\ \bar{c} & 0 \end{bmatrix} \begin{bmatrix} 0 & b \\ \bar{b} & 0 \end{bmatrix} = \begin{bmatrix} b\bar{c} & 0 \\ 0 & \bar{b}c \end{bmatrix} - \begin{bmatrix} c\bar{b} & 0 \\ 0 & \bar{c}b \end{bmatrix} = \begin{bmatrix} b\bar{c} - c\bar{b} & 0 \\ 0 & \bar{b}c - \bar{c}b \end{bmatrix}$$

(194) Check $(b\bar{c} - c\bar{b})^t =$ [redacted] $\bar{c}^t b^t - \bar{b}^t c^t = \bar{c}b - \bar{b}c$

$(b\bar{c} - c\bar{b})^* = c\bar{b} - b\bar{c}$ ✓

$$\begin{bmatrix} 0 & b \\ \bar{b} & 0 \end{bmatrix}^* = \begin{bmatrix} 0 & \bar{b}^* \\ b^* & 0 \end{bmatrix} = \begin{bmatrix} 0 & b^t \\ \bar{b}^t & 0 \end{bmatrix} = \begin{bmatrix} 0 & -b \\ -\bar{b} & 0 \end{bmatrix} = -\begin{bmatrix} 0 & b \\ \bar{b} & 0 \end{bmatrix}$$

So now you want $n=2$ $\quad b = \begin{bmatrix} 0 & \lambda \\ -\lambda & 0 \end{bmatrix}$ $\quad c = \begin{bmatrix} 0 & \mu \\ -\mu & 0 \end{bmatrix}$

[redacted]

$$b\bar{c} - c\bar{b} = \begin{bmatrix} 0 & \lambda \\ -\lambda & 0 \end{bmatrix}\begin{bmatrix} 0 & \bar{\mu} \\ -\bar{\mu} & 0 \end{bmatrix} - \begin{bmatrix} 0 & \mu \\ -\mu & 0 \end{bmatrix}\begin{bmatrix} 0 & \bar{\lambda} \\ -\bar{\lambda} & 0 \end{bmatrix}$$

$$= \begin{bmatrix} -\lambda\bar{\mu} & 0 \\ 0 & -\lambda\bar{\mu} \end{bmatrix} - \begin{bmatrix} -\mu\bar{\lambda} & 0 \\ 0 & -\mu\bar{\lambda} \end{bmatrix}$$

$$= \begin{bmatrix} -\lambda\bar{\mu} + \mu\bar{\lambda} & 0 \\ 0 & -\lambda\bar{\mu} + \mu\bar{\lambda} \end{bmatrix} = \overbrace{(-\lambda\bar{\mu} + \mu\bar{\lambda})}^{-\begin{vmatrix} \lambda & \bar{\lambda} \\ \mu & \bar{\mu} \end{vmatrix}} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

So the symm space has rank 1: [redacted] any maximal abelian subspace in $\mathfrak{p}$ is 1 diml.

Count dims. $\dim \mathfrak{k} = 4$, $\dim \mathfrak{p} = 2$ have $\mathfrak{k}$ acting on $\mathfrak{p}$: $a \# b = ab - b\bar{a}$. There is a structure here that you don't understand. 4 diml compact Lie group acting on a complex line.

[redacted]

$$\left[\begin{bmatrix} a & 0 \\ 0 & \bar{a} \end{bmatrix}, \begin{bmatrix} 0 & b \\ \bar{b} & 0 \end{bmatrix}\right] = \begin{bmatrix} 0 & ab - b\bar{a} \\ \bar{a}\bar{b} - \bar{b}a & 0 \end{bmatrix}$$

$a \# b = ab - b\bar{a}$

Take $b = \begin{bmatrix} 0 & \lambda \\ -\lambda & 0 \end{bmatrix}$

$$\underbrace{\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}}_{a}\underbrace{\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}}_{b} - \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} \bar{\alpha} & \bar{\beta} \\ \bar{\gamma} & \bar{\delta} \end{bmatrix} = \begin{bmatrix} -\beta & \alpha \\ -\delta & \gamma \end{bmatrix} - \begin{bmatrix} \bar{\gamma} & \bar{\delta} \\ -\bar{\alpha} & -\bar{\beta} \end{bmatrix}$$

So get: $a \# b = 0 \iff \bar{\gamma} = -\beta$, $\delta = \bar{\alpha}$

$a^* + a = 0$ means $\alpha + \bar{\alpha} = 0$, $\gamma = -\bar{\beta}$; $\delta + \bar{\delta} = 0$. Conclude that $a = \begin{bmatrix} it & \beta \\ -\bar{\beta} & -it \end{bmatrix}$ $\underbrace{}_{t \in \mathbb{R}}$ i.e. $a \in \mathcal{L}SU(2)$.

(195) A better way to see this is to use $u * b = u b u^t = u b \bar{u}^{-1}$ If $u b u^t = b$ then $(\det u)^2 \det b = \det b$ $\therefore$ $\det(u) = \pm 1$. So infinitesimally $tr(a) = 0$. Better: $ab = b\bar{a} \Rightarrow$ $tr(ab) = tr(b\bar{a}) = tr(\bar{a}b) \Rightarrow tr((a-\bar{a})b) = 0$

$a - \bar{a}$ = purely imag skew adj. $\therefore$ $a - \bar{a} = ih$, h real symm.

h symm. $b = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ skew symm. $\therefore$ $tr(hb) = 0$

So in the case of $\mathcal{L}SO(2n)$, $n=2$ You get $U(2)$ acting on $\Lambda^2\mathbb{C}^2$, the kernel must be $SU(2)$, & the action on this line is via the determinant.

Next case is $\mathcal{L}Sp(2n)$, $n=2$. $X = \begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix}$: $a^* = -a$, $b^t = +b$

Action $\begin{bmatrix} u & 0 \\ 0 & \bar{u} \end{bmatrix}\begin{bmatrix} 0 & b \\ -\bar{b} & 0 \end{bmatrix}\begin{bmatrix} u^* & 0 \\ 0 & u^t \end{bmatrix} = \begin{bmatrix} 0 & ubu^t \\ -\bar{u}\bar{b}u^* & 0 \end{bmatrix}$

Inf. action $a \# b = ab - b\bar{a} = ab + ba^t$. You have the action of $u \in U(n)$ on $\{b : b^t = b\}$ = symmetric bilinear forms, usual action $u \# b = ubu^t$

Now if $n=2$, then b should be the same as a quadratic form on $\mathbb{C}^n$, the space is 3 diml/$\mathbb{C}$ and the corresponding projective space should be the Riemann sphere S^2. You want to identify the complex lines in $S^2(\mathbb{C})$ with degree 2 positive divisors on S^2, and $U(2)$ should be acting by orthogonal rotations on S^2. Count dims $\mathcal{L}Sp(4)$ has real dim $2^2 + 2(3) = 10$.

$U(2)$ acting on $S^2\mathbb{C}^2$. It looks like the scalars matrices in $U(2)$ map onto the scalar operators on $S^2\mathbb{C}^2$. It seems that the interesting action is

(196) [illegible] SU(2) rotations on degree 2 divisors on S^2. ~~[illegible]~~ There's one invariant here, namely, $\cos\theta$ where θ is the $<$ between the 2 pts of the divisor.

Puzzle: $U(2)$ acting on $S^2\mathbb{C}^2 \simeq \mathbb{C}^3$, yield $SU(2)$ acting on $\mathbb{P}(S^2\mathbb{C}^2) \simeq S^2$

Situation V complex v.s. with pos herm. form say $V = \mathbb{C}^n$ with $\langle x, y\rangle = x^* y$. Suppose also given $b^t = b$. Then you have a quadratic function $\frac{1}{2}x^t b x$. The obvious thing to do is to maximize $\frac{1}{2}x^t b x$ subject to $x^* x = 1$

$$F(x) = \frac{1}{2}x^t b x + \lambda\left(\frac{1}{2}x^* x - 1\right)$$

except $\frac{1}{2}x^t b x$ is not real valued

~~See what is made~~ Go back to

$$F(u) = \frac{1}{2}\mathrm{tr}\,\text{[illegible]}\left(gpg^{-1} - p_0\right)^2$$

$$= \frac{1}{2}\mathrm{tr}(p^2) + \frac{1}{2}\mathrm{tr}(p_0^2) \overset{-}{\text{[illegible]}} \mathrm{tr}(gpg^{-1}p_0)$$

assume $g = 1$ is critical point for F.

$$0 \overset{\forall X}{=} \mathrm{tr}([X, p]\, p_0) = \mathrm{tr}(X[p, p_0]) \Longrightarrow [p, p_0] = 0$$

$Sp(2n)$ $\quad g = \begin{bmatrix} u & 0 \\ 0 & \bar{u} \end{bmatrix} \quad p = \begin{bmatrix} 0 & b \\ -\bar{b} & 0 \end{bmatrix} \quad b^t = b.$

Want p_0 chosen so that it's centralizer is max ab.

~~[illegible]~~. $n = 2$ $\quad \mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$

$$\begin{bmatrix} a & 0 \\ 0 & \bar{a} \end{bmatrix} \quad \left[\begin{array}{c|c} 0 & b \\ -\bar{b} & 0 \end{array}\right]$$

(197) Simple calculations. $P_0 = \left[\begin{array}{c|c} 0 & b \\ \hline +\bar{b} & 0 \end{array}\right]$ $b^t = -b$

centralizer of p_0 where b is something easy. First case $b = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

$$\left[\left[\begin{array}{c|c} a & \\ \hline & \bar{a} \end{array}\right], \left[\begin{array}{c|c} & b \\ \hline +\bar{b} & \end{array}\right]\right] = \left[\begin{array}{c|c} 0 & \underset{\sim}{ab - b\bar{a}} \\ \hline +\bar{a}\bar{b} - \bar{b}a & 0 \end{array}\right]$$

$$\underset{a}{\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}}\underset{b}{\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}} - \underset{b}{\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}}\underset{\bar{a}}{\begin{bmatrix} \bar{\alpha} & \bar{\beta} \\ \bar{\gamma} & \bar{\delta} \end{bmatrix}} = \begin{bmatrix} -\beta & \alpha \\ -\delta & \gamma \end{bmatrix} - \begin{bmatrix} \bar{\gamma} & \bar{\delta} \\ -\bar{\alpha} & -\bar{\beta} \end{bmatrix}$$

$\delta = \bar{\alpha}$, $\alpha = \bar{\delta}$

$\gamma = -\bar{\beta}$, $\beta = -\bar{\gamma}$

$a^* = -a \Longrightarrow \gamma = -\bar{\beta}$,

$\bar{\alpha} = -\alpha$, $\bar{\delta} = -\delta$

$\alpha = it$ $\delta = -it$

$a = \begin{bmatrix} it & \beta \\ -\bar{\beta} & -it \end{bmatrix}$ trace 0 ~~But now is that~~

~~What~~

$$\left[\left[\begin{array}{c|c} 0 & b \\ \hline \bar{b} & 0 \end{array}\right], \left[\begin{array}{c|c} 0 & c \\ \hline \bar{c} & 0 \end{array}\right]\right] = \begin{bmatrix} b\bar{c} - c\bar{b} & 0 \\ 0 & -\bar{c}b + \bar{b}c \end{bmatrix}$$

Perhaps try $\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$, $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}$

Ask about centralizer. What you want is a $P_0 = \begin{bmatrix} 0 & b \\ +\bar{b} & 0 \end{bmatrix}$ $b^t = -b$ with a small centralizer.

Basic observation is that you need 2 planes.

Basically you would like to start with ~~a finite dim'l Hilbert space~~ a complex space V equipped with pos. herm. form, then add a symmetric $\mathbb{C}$ linear form.

$n = 1$. Given $\mathbb{C}$ with x^*x and a symm form ax^2. Obvious invariant, namely $|a| = |ax^2|$ when $|x| = 1$.

$n = 2$. $\mathbb{C}^2$ idea is to ~~compare invariants~~ look at all $L \subset \mathbb{C}^2$

Morse theory for $\mathbb{C}P^n$. Something like height

~~complex~~

$$\sum \lambda_i z_i \ ?$$

Use method you know, which involves $U(n)$ acting by conjugation on hermitian matrices.

$$gPg^* -$$

Point of $\mathbb{C}P^{n-1}$ is a hermitian projection of rank 1. Then take any hermitian op. Λ and use the trace. So the Morse function amounts to contracting $\iota^*\Lambda\iota$, $\iota : L \hookrightarrow \mathbb{C}^n$

Method: Let A be a hermitian operator on $\mathbb{C}^n$, Then x^*Ax for $|x| = 1$ descends to a real function on $P\mathbb{C}^n$. Assume $x\mathbb{C}$ is a stationary point and let δx be any variation preserving $x^*x = 1$ to first order: $\delta x^* x = 0$. Then

$$0 = (\delta x)^* A x + x^* A \delta x = 2(\delta x)^* A x$$

But $(\delta x)^* A x = 0$ for all $\delta x \perp x$ implies $Ax = \lambda x$; $\lambda \in \mathbb{C}$; then $\lambda \in \mathbb{R}$ because A hermitian.

Next you want to handle $\mathcal{L}Sp(2n)$ with $n=2$. $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$

$$n^2 + 2\left(\frac{n(n+1)}{2}\right) \qquad \left\{\begin{bmatrix} a & 0 \\ 0 & \bar a \end{bmatrix}\right\} \left\{\begin{bmatrix} 0 & b \\ -\bar b & 0 \end{bmatrix}\right\} \qquad \begin{matrix} a^* = -a \\ b^t = b. \end{matrix}$$

$$a \# b = ab + ba^t = ab - b\bar a.$$

Why look first at this case? Instead take $\mathcal{L}O(2n)$ with $n=2$.

$$\mathfrak{g} = \left\{\begin{bmatrix} a & 0 \\ 0 & \bar a \end{bmatrix} : a^* + a = 0\right\} \oplus \left\{\begin{bmatrix} 0 & b \\ \bar b & 0 \end{bmatrix} : b^t = -b\right\}$$

$n^2 + n(n-1)$

$b = \lambda \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, $\lambda \in \mathbb{C}$ Let $a = \begin{bmatrix} is & \beta \\ -\bar\beta & it \end{bmatrix}$ $s,t \in \mathbb{R}$, $\beta \in \mathbb{C}$

$$\begin{bmatrix} is & \beta \\ -\bar\beta & it \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} -\beta & is \\ -it & -\bar\beta \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} -is & \bar\beta \\ -\beta & -it \end{bmatrix} = \begin{bmatrix} -\beta & -it \\ is & -\bar\beta \end{bmatrix}$$

$$\begin{bmatrix} 0 & i(s+t) \\ -i(s+t) & 0 \end{bmatrix}$$

$u \in U(2)$

$u \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} u^t \overset{?}{=} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

$u = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \in U(2)$

$u J \overset{u^t}{\boxed{\bar u^{-1}}} = J \qquad uJ = J\bar u \qquad u = -J\bar u J$

$$\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} \bar\alpha & \bar\beta \\ \bar\gamma & \bar\delta \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} -\bar\gamma & -\bar\delta \\ \bar\alpha & \bar\beta \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} \bar\delta & -\bar\gamma \\ -\bar\beta & \bar\alpha \end{bmatrix}$$

$u = \begin{bmatrix} \alpha & \beta \\ -\bar\beta & \bar\alpha \end{bmatrix}$

$|\alpha|^2 + |\beta|^2 = 1$

$\det(u)^2 = 1$ $\therefore$

What have you learned? Just that the action $u \# b = ubu^t$ on $b = J$ seems to involve only the determinant of u.

Go over this again. You have $b = \lambda\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}$ $\lambda \in \mathbb{C}$.
acted on by $u \in U(2)$ via $u * b = ubu^t = ub\bar{u}^{-1}$.
The action is $\mathbb{C}$ linear on the $\mathbb{C}$-line $\{b\}$, so it's via a character of $U(2)$. Let $u = \begin{bmatrix}\alpha & \beta\\ \gamma & \delta\end{bmatrix} \in U(2)$.
You [redacted] want

$$uJu^t = \begin{bmatrix}\alpha & \beta\\ \gamma & \delta\end{bmatrix}\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}\begin{bmatrix}\alpha & \gamma\\ \beta & \delta\end{bmatrix} = \begin{bmatrix}-\beta & \alpha\\ -\delta & \gamma\end{bmatrix}\begin{bmatrix}\alpha & \gamma\\ \beta & \delta\end{bmatrix}$$

$$= \begin{bmatrix}0 & \alpha\delta - \beta\gamma\\ -\alpha\delta + \beta\gamma & 0\end{bmatrix} = \det(u)\, J$$

$\therefore$ The centralizer of J in $U(2)$ is $SU(2)$.
This handles the case of $k \oplus p$ for $\mathcal{L}SO(4)$.

Next case is $\mathcal{L}Sp(4) = \left\{\begin{bmatrix}a & 0\\ 0 & \bar{a}\end{bmatrix} : a^* = -a\right\} \oplus \left\{\begin{bmatrix}0 & b\\ -\bar{b} & 0\end{bmatrix} : b^t = b\right\}$

In this case k has real dim 4, p has real dim 6.

Recall $\mathcal{L}Sp(2) = \begin{bmatrix}it & \beta\\ -\bar{\beta} & -it\end{bmatrix}$ $t \in \mathbb{R}$, $\beta \in \mathbb{C}$. Then action of [redacted] $u = e^{it}$ on β is $e^{it} \# \beta = e^{2it}\beta$.

Viewpoint: You want to classify symmetric bilinear forms on a complex vector space equipped with a positive hermitian form. What approaches to use?

1) Polar decomposition
2) Find a maximal abelian subspace of p, which probably is the centralizer of a generic element.

201 $\quad n=2 \qquad b = \begin{bmatrix} x & y \\ y & z \end{bmatrix}$. You want a

[illegible] a maximal abelian subspace of $\mathfrak{p} = \left\{ \begin{bmatrix} 0 & b \\ -\bar b & 0 \end{bmatrix} : b^t = b \right\}$.

$$\begin{bmatrix} 0 & b \\ -\bar b & 0 \end{bmatrix} \begin{bmatrix} 0 & c \\ -\bar c & 0 \end{bmatrix} - \begin{bmatrix} 0 & c \\ -\bar c & 0 \end{bmatrix} \begin{bmatrix} 0 & b \\ -\bar b & 0 \end{bmatrix}$$

$$\begin{bmatrix} -b\bar c & 0 \\ 0 & -\bar b c \end{bmatrix} - \begin{bmatrix} -c\bar b & 0 \\ 0 & -\bar c b \end{bmatrix} = \begin{bmatrix} -b\bar c + c\bar b & 0 \\ 0 & -\bar b c + \bar c b \end{bmatrix}$$

Check this lies in k. $\quad a = -b\bar c + c\bar b$

$$a^* = -\bar c^* b^* + \bar b^* c^* = -c\bar b + b\bar c = -a$$

This calculation works for any n, e.g. $n=1$.

where you are getting $-b\bar c + c\bar b = 2i\,\mathrm{Im}(c\bar b)$?

$$\mathrm{Im}(c\bar b) = \mathrm{Im}((c_1 + ic_2)(b_1 - ib_2)) = c_2 b_1 - c_1 b_2$$

$$\shortparallel$$

$$\frac{c\bar b - \bar c b}{2i}$$

You want something like $\quad b = \left[\begin{array}{c|c} & \\ \hline & \end{array}\right]$

$b = \begin{bmatrix} x & y \\ y & z \end{bmatrix} \qquad c = \begin{bmatrix} \xi & \eta \\ \eta & \zeta \end{bmatrix} \qquad$ Find $-b\bar c + c\bar b$

$$-\begin{bmatrix} x & y \\ y & z \end{bmatrix} \begin{bmatrix} \bar\xi & \bar\eta \\ \bar\eta & \bar\zeta \end{bmatrix} = -\begin{bmatrix} x\bar\xi + y\bar\eta & x\bar\eta + y\bar\zeta \\ y\bar\xi + z\bar\eta & y\bar\eta + z\bar\zeta \end{bmatrix}$$

$$+\begin{bmatrix} \xi & \eta \\ \eta & \zeta \end{bmatrix} \begin{bmatrix} \bar x & \bar y \\ \bar y & \bar z \end{bmatrix} \qquad + \begin{bmatrix} \xi\bar x + \eta\bar y & \xi\bar y + \eta\bar z \\ \eta\bar x + \zeta\bar y & \eta\bar y + \zeta\bar z \end{bmatrix}$$

$b = \begin{bmatrix} x & 0 \\ 0 & 0 \end{bmatrix}$ $c = \begin{bmatrix} \xi & \eta \\ \eta & \zeta \end{bmatrix}$

$$-b\bar{c} + c\bar{b} = -\begin{bmatrix} x\bar{\xi} & x\bar{\eta} \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} \xi\bar{x} & 0 \\ \eta\bar{x} & 0 \end{bmatrix} = \left[\begin{array}{c|c} -x\bar{\xi}+\xi\bar{x} & -x\bar{\eta} \\ \hline \eta\bar{x} & 0 \end{array}\right]$$

Therefore the centralizer of $\begin{bmatrix} x & 0 \\ 0 & 0 \end{bmatrix}$ $(x \neq 0)$ is $\begin{bmatrix} \mathbb{R} & 0 \\ 0 & \mathbb{C} \end{bmatrix}$

Next $b = \begin{bmatrix} 0 & 0 \\ 0 & z \end{bmatrix}$

$$-bc + c\bar{b} = -\begin{bmatrix} 0 & 0 \\ z\bar{\eta} & z\bar{\xi} \end{bmatrix} + \begin{bmatrix} 0 & \eta\bar{z} \\ 0 & \zeta\bar{z} \end{bmatrix} = \begin{bmatrix} 0 & \eta\bar{z} \\ -z\bar{\eta} & -z\bar{\zeta}+\zeta\bar{z} \end{bmatrix}$$

Thus centralizer of $\begin{bmatrix} 0 & 0 \\ 0 & z \end{bmatrix}$ is $\begin{bmatrix} \mathbb{C} & 0 \\ 0 & \mathbb{R}z \end{bmatrix}$

cent of $\begin{bmatrix} 0 & y \\ y & 0 \end{bmatrix}$

$$-\begin{bmatrix} y\bar{\eta} & y\bar{\zeta} \\ y\bar{\xi} & y\bar{\eta} \end{bmatrix} + \begin{bmatrix} \eta\bar{y} & \xi\bar{y} \\ \zeta\bar{y} & \eta\bar{y} \end{bmatrix}$$

$$= \begin{bmatrix} -y\bar{\eta}+\eta\bar{y} & -y\bar{\zeta}+\xi\bar{y} \\ -y\bar{\xi}+\zeta\bar{y} & -y\bar{\eta}+\eta\bar{y} \end{bmatrix} = 0$$ when $\eta \in \mathbb{R}y$ and $y\bar{\xi} = \zeta\bar{y}$

What does $y\bar{\xi} = \zeta\bar{y}$ mean? Say y real $\neq 0$, says $\zeta = \bar{\xi}$. So centralizer is [illegible] $c = \begin{bmatrix} \xi & t \\ t & \bar{\xi} \end{bmatrix}$ where $\xi \in \mathbb{C}$ and $t \in \mathbb{R}$.

There should be better ways to calculate the centralizer. What do these b matrices mean?

	$n=1$	$n=2$	$n=3$
$SO(2n)$	0	2	6
$Sp(2n)$	2	6	

table gives real dim of space of b matrices

~~So what you need this~~ [illegible] What you want to understand is the $U(n)$ action on the space of [illegible] symmetric $\mathbb{C}$ bilinear forms.

(203) Do calculations in simple cases

$b = \begin{bmatrix}1&0\\0&0\end{bmatrix}$ $\quad c = \begin{bmatrix}\xi&\eta\\\eta&\zeta\end{bmatrix}$ [redacted]

$-b\bar{c} + c\bar{b} = cb - b\bar{c} = \begin{bmatrix}\xi&0\\\eta&0\end{bmatrix} - \begin{bmatrix}\bar\xi&\bar\eta\\0&0\end{bmatrix} = \begin{bmatrix}\xi-\bar\xi&-\bar\eta\\\eta&0\end{bmatrix}$

$\therefore \xi = \bar\xi,\ \eta = 0,\ \zeta \in \mathbb{C}.$ $\quad c = \begin{bmatrix}\xi&0\\0&\zeta\end{bmatrix} : \xi \in \mathbb{R}, \zeta \in \mathbb{C}$

[redacted]

$b = \begin{bmatrix}0&1\\1&0\end{bmatrix}$ $\quad c\bar{b} - b\bar{c} = \begin{bmatrix}\xi&\eta\\\eta&\zeta\end{bmatrix}\begin{bmatrix}0&1\\1&0\end{bmatrix} - \begin{bmatrix}0&1\\1&0\end{bmatrix}\begin{bmatrix}\bar\xi&\bar\eta\\\bar\eta&\bar\zeta\end{bmatrix}$

$= \begin{bmatrix}\eta&\xi\\\zeta&\eta\end{bmatrix} - \begin{bmatrix}\bar\eta&\bar\zeta\\\bar\xi&\bar\eta\end{bmatrix} = \begin{bmatrix}\eta-\bar\eta&\xi-\bar\zeta\\\zeta-\bar\xi&\eta-\bar\eta\end{bmatrix}$ whence

[redacted] b centralizes c iff $\eta \in \mathbb{R},\ \zeta = \bar\xi,\ \xi \in \mathbb{C}$

$c = \begin{bmatrix}\xi&t\\t&\bar\xi\end{bmatrix}$ $\quad t \in \mathbb{R}, \xi \in \mathbb{C}.$

$b = \begin{bmatrix}0&0\\0&1\end{bmatrix}$, $\begin{bmatrix}\xi&\eta\\\eta&\zeta\end{bmatrix}\begin{bmatrix}0&0\\0&1\end{bmatrix} - \begin{bmatrix}0&0\\0&1\end{bmatrix}\begin{bmatrix}\bar\xi&\bar\eta\\\bar\eta&\bar\zeta\end{bmatrix} = \begin{bmatrix}0&\eta\\-\bar\eta&\zeta-\bar\zeta\end{bmatrix}$

c centralizes b $\iff \eta = 0,\ \zeta \in \mathbb{R}$ i.e. $c = \begin{bmatrix}\xi&0\\0&t\end{bmatrix}$ $\quad \xi \in \mathbb{C},\ t \in \mathbb{R}$

You should need now a root calculation only. Go back to $\mathcal{L}$ [redacted] $SO(2n)$ [redacted] $= \left\{\begin{bmatrix}a&0\\0&\bar a\end{bmatrix} : a^* = -a\right\} \oplus \left\{\begin{bmatrix}0&b\\\bar b&0\end{bmatrix}, b^t = -b\right\}$

Take $n = 3$. $\quad b = \begin{bmatrix}0&x&y\\-x&0&z\\-y&-z&0\end{bmatrix}$

Let's review the conjugacy argument. You have $p, p_0 \in \mathfrak{p}$

You want a maximal abelian subspace of $\mathfrak{p}$. You have $b = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ [redacted] commuting with $c = \begin{bmatrix} \xi & 0 \\ 0 & \zeta \end{bmatrix}$ $\forall \xi \in \mathbb{R}$, $\zeta \in \mathbb{C}$.

It seems that $\mathfrak{a} = \left\{ b = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} : \lambda_1, \lambda_2 \in \mathbb{R} \right\}$ is a maximal abelian subspace of $\mathfrak{p} = \left\{ \begin{bmatrix} 0 & b \\ -\bar{b} & 0 \end{bmatrix} : b^t = b \right\}$, because the centralizer [redacted] in $\mathfrak{p}$ of $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ is $\begin{bmatrix} \mathbb{R} & 0 \\ 0 & \mathbb{C} \end{bmatrix}$

$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ is $\begin{bmatrix} \mathbb{C} & 0 \\ 0 & \mathbb{R} \end{bmatrix}$

Try extending to $Sp(2n)$, $n = 3$.

$$b = \begin{bmatrix} 1 & & \\ & 0 & \\ & & 0 \end{bmatrix} \qquad c = \begin{bmatrix} \text{[redacted]}\, p & q & r \\ q & r & s \\ r & s & t \end{bmatrix}$$

$$cb - b\bar{c} = \begin{bmatrix} p & 0 & 0 \\ q & 0 & 0 \\ r & 0 & 0 \end{bmatrix} - \begin{bmatrix} \bar{p} & \bar{q} & \bar{r} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} p - \bar{p} & -\bar{q} & -\bar{r} \\ q & 0 & 0 \\ r & 0 & 0 \end{bmatrix}$$

$$\mathrm{Cent} \begin{bmatrix} 1 & & \\ & 0 & \\ & & 0 \end{bmatrix} = \left\{ \begin{bmatrix} \mathbb{R} & 0 & 0 \\ 0 & r & s \\ 0 & s & t \end{bmatrix} : r, s, t \in \mathbb{C} \right\}$$

obviously works for general n

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back to $SO(2n)$, $n=1$, [redacted]

$$\mathfrak{g} = \mathcal{L}SO(2n) = \left\{ \begin{bmatrix} a & 0 \\ 0 & \bar{a} \end{bmatrix} : \bar{a} = -a \right\} \oplus ~~\left\{ \begin{bmatrix} 0 & b \\ \bar{b} & 0 \end{bmatrix} : b^t = -b \right\}~~$$

$n=2$

$$\mathfrak{g} = \left\{ \left[\begin{array}{c|c} a & \\ \hline & \bar{a} \end{array} \right] : a^* = -a \right\} \oplus \left\{ \left[\begin{array}{c|c} 0 & b \\ \hline \bar{b} & 0 \end{array} \right] : b^t = -b \right\}.$$

$$b = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \qquad c = \text{[redacted]} \begin{bmatrix} 0 & q \\ -q & 0 \end{bmatrix}$$

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$$\begin{bmatrix} 0 & b \\ \bar{b} & 0 \end{bmatrix}\begin{bmatrix} 0 & c \\ \bar{c} & 0 \end{bmatrix} = \begin{bmatrix} b\bar{c} & 0 \\ 0 & \bar{b}c \end{bmatrix}$$

$$\begin{bmatrix} 0 & c \\ \bar{c} & 0 \end{bmatrix}\begin{bmatrix} 0 & b \\ \bar{b} & 0 \end{bmatrix} = \begin{bmatrix} c\bar{b} & 0 \\ 0 & \bar{c}b \end{bmatrix}$$

$$\begin{aligned}(b\bar{c}-c\bar{b})^* &= \bar{c}^*b^* - \bar{b}^*c^* \\ &= c^t\bar{b}^t - b^t\bar{c}^t \\ &= c\bar{b} - b\bar{c}\end{aligned}$$

$$\left[\begin{bmatrix} 0 & b \\ \bar{b} & 0 \end{bmatrix}, \begin{bmatrix} 0 & c \\ \bar{c} & 0 \end{bmatrix}\right] = \begin{bmatrix} b\bar{c}-c\bar{b} & 0 \\ 0 & \bar{b}c-\bar{c}b \end{bmatrix}$$

$b\bar{c} - c\bar{b}$ when $b = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ $c = \begin{bmatrix} 0 & q \\ -q & 0 \end{bmatrix}$

$$b\bar{c} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 0 & \bar{q} \\ -\bar{q} & 0 \end{bmatrix} = \begin{bmatrix} -\bar{q} & 0 \\ 0 & -\bar{q} \end{bmatrix}$$

$-\bar{q}+q$

$-\bar{q}+q$

$$c\bar{b} = \begin{bmatrix} 0 & q \\ -q & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} -q & 0 \\ 0 & -q \end{bmatrix}$$

$$b\bar{c} - c\bar{b} = -\begin{bmatrix} q+\bar{q} & 0 \\ 0 & +q+\bar{q} \end{bmatrix}$$

$b\bar{c} - \bar{c}b = 0 \iff q+\bar{q} = 0$

i.e. q real

~~What's happening for $\mathcal{L}SO(4)$? [illegible]~~

It seems that if $q+\bar{q}=0$, then

$$\begin{bmatrix} 0 & b \\ \bar{b} & 0 \end{bmatrix} = \left[\begin{array}{cc|cc} & & 0 & 1 \\ & & -1 & 0 \\ \hline 0 & 1 & & \\ -1 & 0 & & \end{array}\right] \quad \text{and} \quad \begin{bmatrix} 0 & c \\ \bar{c} & 0 \end{bmatrix} = \left[\begin{array}{cc|cc} & & 0 & q \\ & & -q & 0 \\ \hline 0 & \bar{q} & & \\ -\bar{q} & 0 & & \end{array}\right]$$

Commute.

Repeat: $\mathcal{L}SO(2n)$ $\begin{bmatrix} a & b \\ \bar{b} & \bar{a} \end{bmatrix}$: $a^* = -a$, $b^t = -b$ $n=2. \Rightarrow b = \lambda\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

Take $b = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, $c = \begin{bmatrix} 0 & q \\ -q & 0 \end{bmatrix}$ where $q \in \mathbb{C}$.

$$\left[\begin{bmatrix} 0 & b \\ \bar{b} & 0 \end{bmatrix}, \begin{bmatrix} 0 & c \\ \bar{c} & 0 \end{bmatrix}\right] = \begin{bmatrix} b\bar{c}-\bar{b}c & 0 \\ 0 & \bar{b}c-\bar{c}b \end{bmatrix}$$

(206) need $b\bar{c} - \bar{b}c = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 0 & \bar{q} \\ -\bar{q} & 0 \end{bmatrix}$

$b\bar{c} - \bar{b}c = b(\bar{c} - c)$ $\qquad \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 0 & q \\ -q & 0 \end{bmatrix}$

$= \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 0 & \bar{q} - q \\ -q + \bar{q} & 0 \end{bmatrix}$

$= \begin{bmatrix} -q + \bar{q} & 0 \\ 0 & q - \bar{q} \end{bmatrix}$ so $\begin{bmatrix} 0 & b \\ \bar{b} & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & c \\ \bar{c} & 0 \end{bmatrix}$ commute iff $q = \bar{q}$ i.e. $q \in \mathbb{R}$.

Lets return to conjugacy thm. Consider $\mathcal{L}Sp(2n) = \left\{ \begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix} : \begin{matrix} a^* = -a \\ b^t = b \end{matrix} \right\}$ and consider

$b = \begin{bmatrix} 1 & & & \\ & 0 & & \\ & & \ddots & \\ & & & 0 \end{bmatrix}$ better: $\left[\begin{array}{c|c} & 1 \\ \hline -1 & \\ & \end{array}\right] = p_0$

You want to find all $p = \left[\begin{array}{c|c} 0 & c \\ \hline -\bar{c} & 0 \end{array}\right]$

commuting with p_0. This means finding all $c = c^t$ such that

$$\left[\begin{array}{c|c} & b \\ \hline -\bar{b} & \end{array}\right]\left[\begin{array}{c|c} & c \\ \hline -\bar{c} & \end{array}\right] = \left[\begin{array}{c|c} -b\bar{c} & \\ \hline & -\bar{b}c \end{array}\right]$$

$$\left[\begin{array}{c|c} & c \\ \hline -\bar{c} & \end{array}\right]\left[\begin{array}{c|c} & b \\ \hline -\bar{b} & \end{array}\right] = \left[\begin{array}{c|c} -c\bar{b} & \\ \hline & -\bar{c}b \end{array}\right]$$

$$\left[\begin{array}{c|c} -b\bar{c} + c\bar{b} & \\ \hline & -\bar{b}c + \bar{c}b \end{array}\right]$$

(207) $b\bar{c} = c\bar{b}$ or $b\bar{c} = cb$

$$\begin{bmatrix} 1 & & & \\ & 0 & & \\ & & \ddots & \\ & & & 0 \end{bmatrix} \begin{bmatrix} \bar{c}_{ij} \end{bmatrix} = \begin{bmatrix} c_{ij} \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 0 & & \\ & & \ddots & \\ & & & 0 \end{bmatrix}$$

$$\begin{bmatrix} \bar{c}_{11} & \bar{c}_{12} & \cdots & \bar{c}_{1n} \\ & & & \\ & & & \end{bmatrix} = \begin{bmatrix} c_{11} & 0 & 0 \\ c_{21} & & \\ \vdots & & 0 \\ c_{n1} & & \end{bmatrix}$$

$c_{11} = \bar{c}_{11}$

$c_{21} = \cdots = c_{n1} = 0$

So you found that the centralizer of [illegible]

$$p_0 = \left[\begin{array}{c|c} & 1 \\ \hline 1 & \end{array}\right]$$ is [illegible]

$$p_0 = \left[\begin{array}{c|c} & 1 \\ \hline -1 & \end{array}\right] \qquad \left[\begin{array}{cc|cc} & & c_{11} & 0 \\ & & 0 & \\ \hline -c_{11} & 0 & & \\ 0 & & & \end{array}\right]$$

complex symm $(n-1)\times(n-1)$

You would like to do this in a non computational way. Somehow there should be a way to take a symmetric $\mathbb{C}$-bilinear form on $\mathbb{C}^n$ and to construct an ~~flag~~ orthonormal basis for the pos. herm fm which also diagonalizes the symm form. It should involve looking at complex lines in $\mathbb{C}^n$, + some sort of functional on these lines.

Given symm. $[c_{ij}]$. Take $L \subset \mathbb{C}^n$.

Can restrict. $L = z_1 : z_2 : \cdots : z_n \quad \frac{1}{2}\sum_{i,j} c_{ij} z_i z_j$

~~Restrict~~ to $\sum^n |z_i|^2 = 1$

Consider $\mathcal{L}SO(2n) = \mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p} = \left\{ \begin{bmatrix} a & b \\ \bar{b} & \bar{a} \end{bmatrix} : \begin{matrix} a^* = -a \\ b^t = -b \end{matrix} \right\}$

$$b_0 = \left[\begin{array}{c|c} \begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix} & \\ \hline & \end{array}\right] \qquad p_0 = \begin{bmatrix} 0 & b_0 \\ b_0 & 0 \end{bmatrix} \qquad q = \begin{bmatrix} 0 & c \\ \bar{c} & 0 \end{bmatrix}$$

~~$\left[\begin{bmatrix} 0 & b \\ \bar{b} & 0 \end{bmatrix}, \begin{bmatrix} 0 & c \\ \bar{c} & 0 \end{bmatrix} \right] = \begin{bmatrix} b\bar{c} \\ \bar{b}c \end{bmatrix}$~~

$$\begin{bmatrix} 0 & c \\ \bar{c} & 0 \end{bmatrix} \left[\begin{bmatrix} 0 & b \\ \bar{b} & 0 \end{bmatrix}, \begin{bmatrix} 0 & c \\ \bar{c} & 0 \end{bmatrix} \right] = \begin{bmatrix} b\bar{c} - c\bar{b} & 0 \\ 0 & \bar{b}c - \bar{c}b \end{bmatrix}$$

want $b_0\bar{c} - c\bar{b}_0 = b_0\bar{c} - cb_0$

[illegible]

$$K = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

$$b_0\bar{c} - cb_0 \quad \left[\begin{array}{c|c} K & 0 \\ \hline 0 & 0 \end{array}\right] \left[\begin{array}{c|c} \bar{A} & \bar{B} \\ \hline \bar{C} & \bar{D} \end{array}\right] - \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \left[\begin{array}{c|c} K & 0 \\ \hline 0 & 0 \end{array}\right]$$

$$= \begin{bmatrix} K\bar{A} & K\bar{B} \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} AK & 0 \\ CK & 0 \end{bmatrix}$$

$$= \begin{bmatrix} K\bar{A} - AK & K\bar{B} \\ -CK & 0 \end{bmatrix}$$

You want $b_0\bar{c} - cb_0 = 0$

This means $B = 0$, $C = 0$

and that $K\bar{A} = AK$

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Recall that $c = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ sat $c^t = -c$

so $A^t = -A$, $D^t = -D$ $\quad A = \begin{bmatrix} 0 & x \\ -x & 0 \end{bmatrix}$

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 0 & \bar{x} \\ -\bar{x} & 0 \end{bmatrix} - \begin{bmatrix} 0 & x \\ -x & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} -\bar{x} & 0 \\ 0 & -\bar{x} \end{bmatrix} - \begin{bmatrix} -x & 0 \\ 0 & -x \end{bmatrix} = \begin{bmatrix} x-\bar{x} & 0 \\ 0 & x-\bar{x} \end{bmatrix} \qquad \text{i.e. } x \text{ real}$$

Repeat this. $\mathcal{L}Sp(2n) = \left\{ \begin{bmatrix} a & b \\ \bar{b} & \bar{a} \end{bmatrix} : \begin{matrix} a^* = -a \\ b^t = -b \end{matrix} \right\}$.

Let $b_0 = \begin{bmatrix} \begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix} & 0 \\ 0 & 0 \end{bmatrix}$ $\quad p_0 = \begin{bmatrix} & b_0 \\ +\bar{b}_0 & \end{bmatrix}$ $\quad p = \begin{bmatrix} & c \\ \bar{c} & \end{bmatrix}$

$\overset{\longleftarrow 2 \longrightarrow}{} \overset{\longleftarrow n-2 \longrightarrow}{}$

$c^t = -c$

$q^t = -q$, $\ell^t = -\ell$

$r^t = -s$

$c = \begin{bmatrix} q & r \\ s & \ell \end{bmatrix}$ $\qquad \therefore q = \begin{bmatrix} 0 & \lambda \\ -\lambda & 0 \end{bmatrix}$

When do p and p_0 commute?

$$\begin{bmatrix} & b_0 \\ \bar{b}_0 & \end{bmatrix}\begin{bmatrix} & c \\ \bar{c} & \end{bmatrix}\begin{bmatrix} & b_0 \\ \bar{b}_0 & \end{bmatrix}$$

$$[p_0, p] = \begin{bmatrix} b_0\bar{c} & \\ & \bar{b}_0 c \end{bmatrix} - \begin{bmatrix} c\bar{b}_0 & \\ & \bar{c}b_0 \end{bmatrix} = \begin{bmatrix} b_0\bar{c} - c\bar{b}_0 & 0 \\ 0 & \bar{b}_0 c - \bar{c}b_0 \end{bmatrix}$$

Ans. when $b_0\bar{c} = c b_0$

$$b_0 = \left[\begin{array}{c|c} J & 0 \\ \hline 0 & 0 \end{array}\right] \qquad c = \left[\begin{array}{c|c} q & r \\ \hline s & \ell \end{array}\right]$$

$$b_0\bar{c} = \left[\begin{array}{c|c} J\bar{q} & J\bar{r} \\ \hline 0 & 0 \end{array}\right] \qquad c\bar{b}_0 = \left[\begin{array}{c|c} qJ & 0 \\ \hline sJ & 0 \end{array}\right]$$

so $$b_0\bar{c} - c\bar{b}_0 = \left[\begin{array}{c|c} J\bar{q} - qJ & J\bar{r} \\ \hline -sJ & 0 \end{array}\right]$$

$$\overset{J}{\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}} \overset{\bar{q}}{\begin{bmatrix} 0 & \lambda \\ -\lambda & 0 \end{bmatrix}} = \begin{bmatrix} -\lambda & 0 \\ 0 & -\lambda \end{bmatrix}$$

$$\begin{bmatrix} 0 & \lambda \\ -\lambda & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} -\lambda & 0 \\ 0 & -\lambda \end{bmatrix}$$

$$\therefore \quad [p_0, p] = 0 \iff r = 0,\ s = 0,\ \underbrace{q = \begin{bmatrix} 0 & \lambda \\ -\lambda & 0 \end{bmatrix} : \lambda \in \mathbb{R}}$$

[illegible] $$\{p : [p_0, p] = 0\} = \left\{ \left[\begin{array}{c|c} & c \\ \hline \bar{c} & \end{array}\right] : \quad c = \begin{bmatrix} q & 0 \\ 0 & \ell \end{bmatrix} \right. \quad \ell^t = -\ell$$

$$\underset{2}{\longleftrightarrow}\ \underset{n-2}{\longleftrightarrow}$$

You would like next to really understand how the above calculation leads [illegible] inductively to eigenvalues. You start with $U(n)$ acting on complex skew-symmetric $n \times n$ matrices: $u \# b = ubu^t$, but maybe better notation:

$$\begin{bmatrix} u & 0 \\ 0 & \bar{u} \end{bmatrix}\begin{bmatrix} 0 & b \\ \bar{b} & 0 \end{bmatrix}\begin{bmatrix} u^* & 0 \\ 0 & u^t \end{bmatrix} = \begin{bmatrix} 0 & ubu^t \\ \bar{u}\bar{b}u^* & 0 \end{bmatrix}$$

Your aim is to find ~~an orthonormal basis of $\mathbb{C}^n$~~ ~~such that~~ $u \in U(n)$ such that ubu^t is diagonal with $\lambda \geq 0$.

(211) [redacted] Return to $\mathcal{L}Sp(2n) = k \oplus p$

$k = \left\{ \begin{bmatrix} a & 0 \\ 0 & \bar{a} \end{bmatrix} : a^* = -a \right\}$; $p = \left\{ \begin{bmatrix} 0 & b \\ -\bar{b} & 0 \end{bmatrix} : b^t = b \right\}$

[redacted] $$b_0 = \begin{array}{c} \left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & 0 \end{array}\right] \\ \xrightarrow{} 1 \xleftarrow{} \xleftrightarrow{n-1} \end{array}$$

$$\begin{bmatrix} 0 & b \\ -\bar{b} & 0 \end{bmatrix}\Big], \Big[\begin{bmatrix} 0 & c \\ -\bar{c} & 0 \end{bmatrix}\begin{bmatrix} 0 & b \\ -\bar{b} & 0 \end{bmatrix}\Big] = \begin{bmatrix} -c\bar{b}+b\bar{c} & \\ & \bar{b}c-\bar{c}b \end{bmatrix}$$

Find when [redacted] $p = \begin{bmatrix} 0 & c \\ -\bar{c} & 0 \end{bmatrix}$ centralizes $p_0 = \begin{bmatrix} 0 & b_0 \\ -b_0 & 0 \end{bmatrix}$

$$c = \left[\begin{array}{c|c} q & r \\ \hline s & l \end{array}\right] \qquad b\bar{c} - c\bar{b} = \left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & 0 \end{array}\right]\left[\begin{array}{c|c} \bar{q} & \bar{r} \\ \hline \bar{s} & \bar{l} \end{array}\right] = \left[\begin{array}{c|c} \bar{q} & \bar{r} \\ \hline 0 & 0 \end{array}\right]$$

$$- \left[\begin{array}{c|c} q & r \\ \hline s & l \end{array}\right]\left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & 0 \end{array}\right] = -\left[\begin{array}{c|c} q & 0 \\ \hline s & 0 \end{array}\right]$$

$$b_0\bar{c} - cb_0 = \left[\begin{array}{c|c} \bar{q}-q & \bar{r} \\ \hline -s & 0 \end{array}\right] = 0 \iff q \in \mathbb{R},\ r=0,\ s=0.$$

centralizer of p_0 $= \left\{ c = \begin{bmatrix} \mathbb{R} & 0 \\ 0 & \underset{\text{symm}}{n-1 \times n-1} \end{bmatrix} \right\}$

Discuss the situation. You have made some progress on the idea of splitting off one line. Start earlier. You are studying a complex v.s. V with pos herm. form together with a symmetric form, [redacted] equivalently you want to study the action of $u \in U(n)$ on complex symmetric matrices b given by $u \# b = ubu^t$. Special case $b = \begin{bmatrix} 1 & & & \\ & 0 & & \\ & & 0 & \\ & & & 0 \end{bmatrix}$ [redacted]

$$b = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}[1\ 0\ 0\ 0] \qquad ubu^t = u\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}[1\ 0\ 0\ 0]u^t$$

In this case ubu^t is vv^t where v is a unit vector

$$= \begin{bmatrix} u_{11} \\ \vdots \\ u_{1n} \end{bmatrix}[u_{11} \cdots u_{1n}]$$

(212) Repeat: You have $u \in U(n)$ acting on complex symm. b by $u \# b = ubu^t$. Better you have a complex v.s. V equipped with a pos herm. form and a $\mathbb{C}$ bilinear symm. form.

You want to find a line $l \subset V$ such that l and $l^\perp$ are orthogonal with respect to the symmetric bilinear form.

l is generated by a unit vector v, unique up to mult. by $\mathbb{T}$.

Idea today, Apr 9 is to go back to a complex vector space equipped with a symplectic structure and also a pos hermitian form. You recall trying to construct a canonical form for such a situation.

Begin with $n = 1$ i.e. $\dim_{\mathbb{C}} V = 2$. Pick a line l, then you get a $\perp$ line $l^\perp$, and the symplectic form gives you canon isom $l \otimes l^\perp \xrightarrow{\sim} \mathbb{C}$. There should be an invariant ≥ 0 here. Choose unit vectors in l and $l^\perp$

$\dim_{\mathbb{C}} V = 2$. V equipped with $\left(\begin{array}{l} \text{pos herm form} \\ \text{symplectic "} \end{array} \right.$

l $\mathbb{C}$line in V, $l^\perp$ $\quad l \hookrightarrow V \twoheadrightarrow l^\perp$

have canon isom: $\Lambda^2 V \simeq l \otimes l^\perp$

have $\Lambda^2 V \xrightarrow{\sim} \mathbb{C}$ given by the symplectic form.

Choose unit vectors v, w in l and $l^\perp$. Then $\omega(v, w) \in \mathbb{C}^\times$. Arbitrariness of phases in v, w yields positive $\lambda_1 > 0$ as invariant

(213) Use components. $V = \mathbb{C}^2$ let first + second unit vectors be $\begin{bmatrix} p \\ q \end{bmatrix}, \begin{bmatrix} r \\ s \end{bmatrix}$ so that one has $|p|^2+|q|^2 = 1 = |r|^2+|s|^2$, $\bar p \overset{(-\bar q)}{r} + \bar q \overset{(\bar p)}{s} = 0$.

$$\begin{bmatrix} p & -\bar q \\ q & +\bar p \end{bmatrix} = \begin{bmatrix} p & r \\ -\bar r & \bar p \end{bmatrix}$$

Review. V 2diml/C with pos herm. form and symplectic form $\Lambda^2 V \xrightarrow{\sim} \mathbb{C}$. You pick a line $l \subset V$, let $l^\perp \subset V$ be the orthogonal line, then get $l \otimes l^+ \xrightarrow{\sim} \Lambda^2 V$ via $\wedge$ followed by $\Lambda^2 V \xrightarrow{\sim} \mathbb{C}$. If you pick unit vectors $x \in l$, $y \in l^\perp$ then $x \wedge y \mapsto \omega(x,y)$ is determined up to $\mathbb{T}$ factors ∴ get inv. $|\omega(x,y)|$

Example $V = \mathbb{C}^2$ with standard pos. herm. $x^* y$ and symp. form $\lambda \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} = \lambda \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}^t \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$

There's confusion about the problem. This arose because you want a "flag" approach to symplectic structures under unitary equivalence, and you think there's [redacted] a similarity with with the orbit structure of $U(n)$ acting on complex symmetric $n \times n$ matrices via [redacted] $u * b = ubu^t$. This seems reasonable but you have to be careful. First check that $\begin{bmatrix} 0 & b \\ -\bar b & 0 \end{bmatrix}$ is skew hermitian $b^* = \overline{(b^t)} = \bar b$

[redacted] $\begin{bmatrix} 0 & b \\ c & 0 \end{bmatrix}^* = \begin{bmatrix} 0 & c^* \\ b^* & 0 \end{bmatrix} = \begin{bmatrix} 0 & \bar c \\ \bar b & 0 \end{bmatrix}$ If $c = -\bar b$, then $\bar c = -b$ so OK

$$\begin{bmatrix} 0 & b \\ -\bar{b} & 0 \end{bmatrix}^* = \begin{bmatrix} 0 & -(\bar{b})^* \\ b^* & 0 \end{bmatrix} = \begin{bmatrix} 0 & -b^t \\ \bar{b} & 0 \end{bmatrix} = \begin{bmatrix} 0 & -b \\ \bar{b} & 0 \end{bmatrix}$$

Similarly if $b^t = -b$, then

$$\begin{bmatrix} 0 & b \\ +\bar{b} & 0 \end{bmatrix}^* = \begin{bmatrix} 0 & \bar{b}^* \\ b^* & 0 \end{bmatrix} = \begin{bmatrix} 0 & b^t \\ \overline{b^t} & 0 \end{bmatrix} = \begin{bmatrix} 0 & -b \\ -\bar{b} & 0 \end{bmatrix}$$

~~[illegible] Let's check [illegible]~~

Repeat: Problem. Given a complex vector space V

with | pos herm form
| symplectic form. To classify these.

$\dim_{\mathbb{C}} V = 2$. Pick l get $l \oplus l^\perp = V$

and $l \otimes l^\perp = \Lambda^2 V \xrightarrow{\omega} \mathbb{C}$

Better: Pick a unit vector v

~~dim~~ $V = 2$ inner product $x^* y$, symplectic form $x^t \begin{bmatrix} 0 & \lambda \\ -\lambda & 0 \end{bmatrix} y$ change orthonormal bases $U \begin{bmatrix} 0 & \lambda \\ -\lambda & 0 \end{bmatrix} U^t$

$$u = \begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix} \begin{bmatrix} e^{i\theta} & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} a & -\bar{b} \\ b & \bar{a} \end{bmatrix}$$

$$= \begin{bmatrix} -b & a \\ -\bar{a} & -\bar{b} \end{bmatrix} \begin{bmatrix} a & -\bar{b} \\ b & \bar{a} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & |a|^2 + |b|^2 \\ -|a|^2 - |b|^2 & 0 \end{bmatrix}$$

$$\begin{bmatrix} e^{i\theta} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} e^{+i\theta} & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & e^{2i\theta} \\ -e^{2i\theta} & 0 \end{bmatrix}$$

$$g^t \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} g = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

~~If~~ $\det g = 1$, no condition on g:

$$\begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \begin{bmatrix} 0 & -1 \\ +1 & 0 \end{bmatrix} = \begin{bmatrix} -c & a \\ -d & b \end{bmatrix} \begin{bmatrix} 0 & -1 \\ +1 & 0 \end{bmatrix} = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$$

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original approach: Pick unit vector $\begin{bmatrix} x \\ y \end{bmatrix}$ then $\begin{bmatrix} -\bar{y} \\ \bar{x} \end{bmatrix}$ is a ⊥ unit vector so $\begin{bmatrix} x & -\bar{y} \\ y & \bar{x} \end{bmatrix} \in SU(2)$. nice choice of orth basis

215 Symplectic form applied to $\begin{bmatrix} x \\ y \end{bmatrix}$ $\begin{bmatrix} -\bar{y} \\ x \end{bmatrix}$

is $\begin{bmatrix} x \\ y \end{bmatrix}^t \begin{bmatrix} 0 & \lambda \\ -\lambda & 0 \end{bmatrix} \begin{bmatrix} -\bar{y} \\ x \end{bmatrix} = \lambda(|x|^2 + |y|^2) = \lambda$

Aim? $V = \mathbb{C}^2$ with $\|v\|^2$ and $v_1 \wedge v_2$

Your aim is to classify ~~symmetric bilinear~~ symplectic forms under unitary equivalence. When $n=2$ the symp forms are $\left\{ \lambda \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} : \lambda \in \mathbb{C}^{\times} \right\}$. A general unitary transf. is the product of an $\begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix}$ $|a|^2 + |b|^2 = 1$ and $\begin{bmatrix} e^{i\theta} & 0 \\ 0 & 1 \end{bmatrix}$

Since $\left\{ \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \mathbb{C} \right\}$ is a 1-dim'l repn of $U(2)$, $SU(2)$ must act trivially.

$$\begin{bmatrix} e^{i\theta} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & \lambda \\ -\lambda & 0 \end{bmatrix} \begin{bmatrix} e^{i\theta} & 0 \\ 0 & 1 \end{bmatrix} = e^{2i\theta}\lambda \begin{bmatrix} & 1 \\ -1 & \end{bmatrix}$$

So the classification is clear i.e. $|\lambda| \geq 0$ is the invariant. NEXT you want $\dim V = 4$, that is, to classify symplectic forms on $\mathbb{C}^4 = V$ under the action of $U(4)$. Try choosing a symplectic flag. Choose a line l in V and a line orthogonal to the annihilator of l

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$n=1$. Choose unit vector $\begin{bmatrix} x \\ y \end{bmatrix}$ then extend to $\begin{bmatrix} x & -\bar{y} \\ y & \bar{x} \end{bmatrix} \in SU(2)$, then apply symp. form

which should be $\begin{bmatrix} x \\ y \end{bmatrix}^t \begin{bmatrix} 0 & \lambda \\ -\lambda & 0 \end{bmatrix} \begin{bmatrix} -\bar{y} \\ x \end{bmatrix} =$

$$\lambda [x \;\; y] \begin{bmatrix} \bar{x} \\ \bar{y} \end{bmatrix} = \lambda (|x|^2 + |y|^2) = \lambda$$

$$\begin{bmatrix} x & -\bar{y} \\ y & \bar{x} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x & y \\ -\bar{y} & \bar{x} \end{bmatrix} = \begin{bmatrix} x & -\bar{y} \\ y & \bar{x} \end{bmatrix} \begin{bmatrix} -\bar{y} & \bar{x} \\ -x & -y \end{bmatrix} \simeq \begin{bmatrix} 0 & |x|^2+|y|^2 \\ -|y|^2-|x|^2 & 0 \end{bmatrix}$$

216 So now what to do? $n=2$. You want to try something similar, namely, you choose a line (which is isotropic) l, then choose a line l' $\perp$ annihilator of l, ~~and~~ choose unit vectors v, v' in l, l' resp, and adjust the phase of v' say so that $\omega(v \wedge v') > 0$.

What do you know? Ignoring the inner product you can study a symplectic form by choosing a "symplectic flag" namely, you choose a line $F_1 \subset V$, pass to the symp quot. F_1^0/F_1, then choose a line $F_2/F_1 \subset F_1^0/F_1$ pass to the symp quot F_2^0/F_2 etc. It seems you get $0 < F_1 < F_2 < \cdots < F_n = F_n^0 < F_{n-1}^0 < \cdots < F_1^0 < V$

$n=2$. $0 < F_1 < F_2 = F_2^0 < F_1^0 < V$

Let's start with $\mathbb{C}^4 = V$

Go back over what you can do in symmetric space situation: where $U(n)$ acts on a complex symmetric and skew symmetric matrices. Take symm. matrices, the case of $\mathcal{L}\, Sp(2n) = \left\{ \begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix} : \begin{matrix} a^* = -a \\ b^t = +b \end{matrix} \right\}$.

$$X = \begin{bmatrix} a & 0 \\ 0 & \bar{a} \end{bmatrix} \qquad Y = \begin{bmatrix} 0 & b \\ -\bar{b} & 0 \end{bmatrix} \qquad Z = \begin{bmatrix} 0 & c \\ -\bar{c} & 0 \end{bmatrix} \qquad b, c \text{ symm.}$$

$$[Y, Z] = \begin{bmatrix} 0 & b \\ -\bar{b} & 0 \end{bmatrix} \begin{bmatrix} 0 & c \\ -\bar{c} & 0 \end{bmatrix} - \begin{bmatrix} 0 & c \\ -\bar{c} & 0 \end{bmatrix} \begin{bmatrix} 0 & b \\ -\bar{b} & 0 \end{bmatrix}$$

$$= \begin{bmatrix} -b\bar{c} & 0 \\ 0 & -\bar{b}c \end{bmatrix} - \begin{bmatrix} -c\bar{b} & 0 \\ 0 & -\bar{c}b \end{bmatrix} = \begin{bmatrix} -b\bar{c} + c\bar{b} & 0 \\ 0 & -\bar{b}c + \bar{c}b \end{bmatrix}$$

$$(-b\bar{c} + c\bar{b})^* = b^t \bar{c} - c^t b^* = b\bar{c} - c\bar{b}$$

$b = b_0 = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & 0 & \\ 0 & & & \end{bmatrix}$ $\quad c = [c_{ij}]$ (symm).

$$-b_0\bar{c} + c\bar{b}_0 = -\left[\begin{array}{ccc} \bar{c}_{11} & \cdots & \bar{c}_{1n} \\ \hline & 0 & \end{array}\right] + \begin{bmatrix} c_{11} & \\ \vdots & 0 \\ c_{n1} & \end{bmatrix}$$ ok

$$= \left[\begin{array}{c|ccc} c_{11} - \bar{c}_{11} & -\bar{c}_{12} & \cdots & -\bar{c}_{1n} \\ \hline c_{21} & & & \\ \vdots & & 0 & \\ c_{n1} & & & \end{array}\right]$$

vanishes when $c_{11} = \bar{c}_{11}$
$c_{21}, \ldots, c_{n1} = 0$
$\bar{c}_{12}, \ldots, \bar{c}_{1n} = 0$.

This calculation is used to get

centralizer of b_0 is $\left\{ C = \left[\begin{array}{c|c} \mathbb{R} & 0 \\ \hline 0 & \begin{array}{c}(n-1)\times(n-1) \\ \text{symmetric}\end{array} \end{array}\right] \right\}$

Somehow this is the right ~~[illegible]~~ calculation

Recall existence of ~~[illegible]~~ eigenvectors for hermitian matrices.

$V = \mathbb{C}^n$ let $x \in V$ $\|x\| = 1$. then

$x x^*$ $\quad \frac{1}{2} tr(xx^* A) = \frac{1}{2} x^* A x$ real fn.

on $P\mathbb{C}^n$ $\quad x^* \delta x = 0$ $\quad$ i.e. $\delta x \perp x$

fn. ~~[illegible]~~ $x\mathbb{C}$ stationary $\Longrightarrow \frac{1}{2}(\delta x^* A x + x^* A \delta x)$
$= (\delta x)^* A x = 0$

for all $\delta x \perp x$. $\Longrightarrow Ax \in \mathbb{C}x$

Let's go over the calculations until they become clear. Your starting point should be the conjugacy theorem for $U(n)$ acting on complex symmetric matrices, eventually skew symm. ones.

So where next? You want ~~[illegible]~~ to handle U(n) and complex skew symmetric bilinear forms..

~~[illegible]~~ $\mathcal{L}\, SO(2n) = \mathfrak{k} \oplus \mathfrak{p}$ $\quad \begin{bmatrix} a & \\ & \bar{a} \end{bmatrix}, \begin{bmatrix} 0 & b \\ \bar{b} & 0 \end{bmatrix} = p$

You want the conjugacy thm. $K = \left\{ \begin{bmatrix} u & 0 \\ 0 & \bar{u} \end{bmatrix} : u \in U(n) \right\}.$

$$\frac{1}{2} tr\left(kpk^{-1} - p_0\right)^2 = \frac{1}{2} tr\left(p^2 + p_0^2\right) - tr\left(kpk^{-1}p_0\right)$$

$$k + \delta k = (1+X)k \qquad X = \begin{bmatrix} a & 0 \\ 0 & \bar{a} \end{bmatrix}$$

$$\underbrace{\delta\, tr\left((1+X)kpk^{-1}(1-X)p_0 - kpk^{-1}p_0\right)}$$

$$0 = tr\left([X, kpk^{-1}]p_0\right) = tr\left(X, [kpk^{-1}, p_0]\right)$$

$$p = \begin{bmatrix} 0 & b \\ \bar{b} & 0 \end{bmatrix} \qquad b^t = -b \qquad p_0 = \begin{bmatrix} 0 & b_0 \\ \bar{b}_0 & 0 \end{bmatrix}$$

$$b_0 = \begin{bmatrix} J & 0 \\ 0 & 0 \end{bmatrix} \qquad J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \qquad p_0 = \begin{bmatrix} & J \\ J & \end{bmatrix}$$

→ 2 ← ← n−2 →

~~So~~ Your linear functional is $tr(pp_0)$

$$\delta\, tr(pp_0) = tr([X,p]p_0) = tr(X[p,p_0])$$

~~[illegible]~~

$$\left[\begin{bmatrix} 0 & b \\ \bar{b} & 0 \end{bmatrix}, \begin{bmatrix} 0 & c \\ \bar{c} & 0 \end{bmatrix} \right] = \begin{bmatrix} b\bar{c} - c\bar{b} & 0 \\ 0 & \bar{b}c - \bar{c}b \end{bmatrix}$$

$$tr\left(\begin{bmatrix} 0 & b \\ \bar{b} & 0 \end{bmatrix} \begin{bmatrix} 0 & c \\ \bar{c} & 0 \end{bmatrix} \right) = tr(b\bar{c} + \bar{b}c)$$

(219) Repeat with the aim of making the inductive step clear. It should be possible to do everything in half the degree.

Let's begin with V, ~~equipped with~~ a finite dim'l complex Hilbert space equipped with symmetric bilinear form. Your model should be $u \in U(n)$ acting on b symmetric via ubu^t. You have this simple picture. The obvious thing to try for? At the moment you have a specific conjugacy class in $\mathcal{L}Sp(2n)$ (or $\mathcal{L}SO(2n)$)

$$\begin{bmatrix} 0 & b \\ -\bar{b} & 0 \end{bmatrix} : b^t = b \qquad\qquad \begin{bmatrix} 0 & b \\ \bar{b} & 0 \end{bmatrix} : b^t = -b$$

How to tell: you want $\begin{bmatrix} 0 & b \\ \bar{b} & 0 \end{bmatrix}$ to be skew-symm.

i.e. $-b^* = \bar{b}$ i.e. $-b^t = b$. You can also keep them straight because $\mathcal{L}Sp(2n)$ has larger dim.

What might be **signifant about** $b_0 = \left[\begin{array}{c|c} 1 & r \\ \hline q & s \end{array}\right]$

This is supposed to be symmetric: $s^t = s$, $q^t = r$

$$tr\left(\begin{bmatrix} 0 & b \\ -\bar{b} & 0 \end{bmatrix}\begin{bmatrix} 0 & c \\ -\bar{c} & 0 \end{bmatrix}\right) = tr(-b\bar{c} - \bar{b}c)$$

Another idea is to use $\begin{bmatrix} 1 \\ q \end{bmatrix}\begin{bmatrix} 1 & q^t \end{bmatrix}$. What is the significance of $p_0 = \begin{bmatrix} 0 & b_0 \\ -\bar{b}_0 & 0 \end{bmatrix}$ as an operator on V

Start again V complex v.s. with pos herm. form and also a complex bilinear symmetric form. You want an eigenvector picture for these. ~~Simplest method~~ This means an orthonormal basis for V which diagonalizes the symmetric form.

(220) Ideas: difference (mult. sense) between the pos. hermitian form and the symmetric bilinear form

$$\begin{tikzcd} V \arrow[r, "b"] & V^{\wedge} & V \arrow[l, "\sim"'] \\ & x^{*} & x \arrow[l, mapsto] \end{tikzcd}$$

So it seems that you get an antilinear endo of V. What sort of symmetry?

Choose $V = \mathbb{C}^n$ orthonormal basis.

$$\begin{tikzcd} \mathbb{C}^n & \mathbb{C}^n \arrow[l] \arrow[r, "b_{ij}"] & \mathbb{C}^n & \mathbb{C}^n \arrow[l] \\ {[\bar{y}_1 \cdots \bar{y}_n]} & \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} \arrow[l, mapsto] & {[\bar{x}_1 \cdots \bar{x}_n]} & \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \arrow[l, mapsto] \end{tikzcd}$$

Let's be more precise.

Begin with inner product $\langle v_1 | v_2 \rangle$ and the symm form $S(v_1, v_2)$. $S : V \longrightarrow V^{\wedge}$

$$\begin{tikzcd} V_J & \hat{V}_{-J} \arrow[l, "\sim"'] \arrow[r, leftrightarrow, "S"] & V_{-J} & \hat{V}_J \arrow[l, "\sim"'] \end{tikzcd}$$

Apparently this doesn't work because you get an anti-linear map. You can try forgetting the complex structure.

So you assume V is a Euclidean space equipped with a complex structure J. $\hat{V}$ is V equipped with complex structure $-J$. Can you show there's a canonical isom. $V \xrightarrow{\sim} \hat{V}$. This should be just the scalar product from the Euclidean structure

J^t

V Euclidean space, there is a canonical isomorphism $\varphi : V \xrightarrow{\sim} \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{R})$, J complex structure in V, J is orthogonal of square -1.

$$\varphi(v)(v_1) = (v, v_1) \qquad (Jv, Jv_1) = (J^tJv, v_1)$$

$$\overset{\parallel}{(v, v_1)} \quad \therefore J^tJ = 1$$

$$\begin{tikzcd}
v \in V \arrow[r, "\varphi"] \arrow[d, "J"'] & \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{R}) \\
Jv \in V & \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{R}) \arrow[u, "J^t"']
\end{tikzcd}$$

$$Jv \longmapsto (v_1 \longmapsto (Jv, v_1)) = \varphi_{Jv}$$

V $\mathbb{R}$-vector space with (v_1, v_2) scalar product.

W ——————— (w_1, w_2) ———————

$$T : V \longrightarrow W \quad \text{induces} \quad T^t : W^\wedge \longrightarrow V^\wedge$$

$$\begin{matrix} \overset{\psi}{g} & \longmapsto & gT \\ \parallel & & \parallel \\ \varphi_w & \longmapsto & \varphi_w T \end{matrix}$$

But you have $\varphi : W \xrightarrow{\sim} W^\wedge$

$\varphi_w(w') = (w, w')$

$$(\varphi_w T)(v) = \varphi_w(Tv) = (w, Tv)$$

V with (v, v') $\qquad \varphi : V \longrightarrow V^\wedge$, $v \longmapsto \varphi_v$ $\qquad \varphi_v(v') = (v, v')$

W with (w, w') $\qquad \varphi : W \longrightarrow W^\wedge$ $\qquad \varphi_w(w') = (w, w')$

Given $T : V \longrightarrow W$ get $T^\wedge : W^\wedge \longrightarrow V^\wedge$ $\qquad T^\wedge g = gT$

$$\begin{tikzcd}
W^\wedge \arrow[r] & V^\wedge \\
W \arrow[u, "S"'] \arrow[r] & V \arrow[u, "S"'] \\
\varphi_w \arrow[r, mapsto] & Tw'
\end{tikzcd}$$

?

$$(T^\wedge \varphi_w)(w') = \varphi_w(Tw') = (w, Tw')$$

(222) V with (v, v') $\quad V \xrightarrow{\sim} V^\wedge$

sim W with (w, w') $\quad v \longmapsto \varphi_v(v') = (v, v')$

Given $T : V \longrightarrow W$, get $T^\wedge : W^\wedge \longrightarrow V^\wedge$, $\quad T^\wedge g = gT$

$$(T^\wedge \varphi_w)(v') = \varphi_w(Tv') = (w, Tv')$$

So $T^\wedge \varphi_w \in V^\wedge$ ~~is given by~~ is φ_v

where $\varphi_v(v') = (w, Tv')$. Define $T^* w = v$

Given $T : V \longrightarrow W$, get $T^\wedge : W^\wedge \longrightarrow V^\wedge$

$$\begin{tikzcd}
W^\wedge \arrow[r] & V^\wedge \\
W \arrow[u, "s"] \arrow[r, "T^*"] & V \arrow[u, "s"]
\end{tikzcd}$$

i.e. $(T^* w, v') = (w, Tv')$

$$\begin{tikzcd}
V \arrow[r, "T"] \arrow[d, "s", "\varphi_V"'] & W \arrow[d, "s"', "\varphi_W"] \\
V^\wedge & W^\wedge \arrow[l, "T^\wedge"']
\end{tikzcd}$$

$$\boxed{T^* = \varphi_V^{-1} T^\wedge \varphi_W}$$

$$\varphi_V T^* = T^\wedge \varphi_W = \varphi_W T$$

$$(T^* w, v') = (w, Tv')$$

The next step is when $W = V$ and V is equipped with a complex structure J.

$$\begin{tikzcd}
V \arrow[r, "J"] \arrow[d, "s"'] & V \arrow[d, "S"] \\
V^\wedge & V^\wedge \arrow[l, "J^\wedge"']
\end{tikzcd}$$

$$J^t = \varphi_V^{-1} J \varphi_V \quad ?$$

The point is that by defn. a J is orthogonal: $J^t = J^{-1}$ & has $J^2 = -1$.

Use Euclidean structure to identify V with $\mathrm{Hom}_{\mathbb{R}}(V, \mathbb{R}) = \hat{V}$, and then you get $T^t = \varphi_V^{-1} \hat{T} \varphi_W$

Now J on V satisfies $J^t = J^{-1} = -J$

V Euclidean with $J^t = J^{-1} = -J$. Now you also have have $S : V \longrightarrow$ [illegible] $\operatorname{Hom}_{\mathbb{C}}(V, \mathbb{C})$ symmetric. $\operatorname{Hom}_{\mathbb{C}}(V, \mathbb{C}) = \operatorname{Hom}_{\mathbb{R}}(V, \mathbb{R})$

[illegible] $\operatorname{Hom}_{\mathbb{C}}(V, \operatorname{Hom}_{\mathbb{R}}(\mathbb{C}, \mathbb{R})) = \operatorname{Hom}_{\mathbb{R}}(\mathbb{C} \otimes_{\mathbb{C}} V, \mathbb{R})$

V has symmetric bilinear form.

You [illegible] can try to reduce to V Euclidean. So far you've reduced V & herm inner product to a Euclidean ~~space~~ with J. Next you want to include ~~a map~~ a $\mathbb{C}$-linear map $S^2 V \longrightarrow \mathbb{C}$. Such a thing is the same as [illegible] an $\mathbb{R}$-linear $S^2 V \longrightarrow \mathbb{R}$

$$V \xrightarrow{S} \hat{V}$$

There should be a good way to handle a symmetric form. As a kind of operator [illegible] on V? Not ~~[illegible]~~ obvious because the operator goes from V to $\hat{V}$.

What's your aim? The idea was to start with the symmetric form $S : V \longrightarrow \hat{V}$. [illegible] [illegible] Here $\hat{V}$ = $\mathbb{R}$-dual of V, [illegible] which you can identify with V, but equipped with the appropriate $\mathbb{C}$-structure. S should amount to a pairing $V \otimes_{\mathbb{R}} V \longrightarrow \mathbb{R}$ with ~~a [illegible] J can~~ appropriate symmetry and J conditions. What could they be?

$f : V \longrightarrow \mathbb{R}$ $\mathbb{R}$-linear

$\tilde{f}(v) = f(v) - i f(Jv)$

$$\tilde{f}(Jv) = f(Jv) + i f(v)$$
$$= i[f(v) - i f(Jv)]$$
$$= i \tilde{f}(v)$$

Now look at a bilinear form $V \otimes_{\mathbb{C}} V \to \mathbb{R}$.

$B(v_1, v_2)$. So you want $\underset{J}{V} \xrightarrow{B} \hat{V} \xleftarrow{\sim} V$

$$B : V \otimes_{\mathbb{C}} W \longrightarrow \mathbb{C}$$

$$B(c_1 v_1 + c_2 v_2 \, , \, d_1 w_1 + d_2 w_2)$$

$$= \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}^t \begin{bmatrix} B(v_1, w_1) & B(v_1, w_2) \\ B(v_2, w_1) & B(v_2, w_2) \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}$$

$$B(cv, dw) = cd\, B(v, w).$$

$$B((c_1 + c_2 J)v, (d_1 + d_2 J)w)$$

$$= c_1 d_1 B(v, w) + c_1 d_2 B(v, Jw)$$

$$c_2 d_1 B(Jv, w) + c_2 d_2 B(Jv, Jw)$$

$$= c_1 d_1 B(v, w) + c_1 d_2 i B(v, w)$$

$$+ c_2 d_1 i B(v, w) + c_2 d_2 i^2 B(v, w)$$

$$= (c_1 d_1 + (c_1 d_2 + c_2 d_1) i - c_2 d_2] B(v, w)$$

So $\mathbb{C}$-bilinear means $B(\lambda v, \mu w) = \lambda \mu B(v, w)$

Reduces conditions $B(v, Jw) = B(Jv, w) = i B(v, w)$

$B(Jv, Jw) = - B(v, w)$

(225) V real Euclidean dim $2n$ with J

have canon. isom $V \xrightarrow{\sim} \mathrm{Hom}_{\mathbb{C}}(V,\mathbb{C}) \simeq \mathrm{Hom}_{\mathbb{R}}(V,\mathbb{R})$.

Suppose given $S : V \xrightarrow{\sim} \hat{V}$ symmetric non deg and you combine it with $\hat{V} \xrightarrow[\sim]{A^{-1}} \bar{V}$ assoc. to the hermitian inner product. Then you'll get an anti-linear map $A^{-1}S : V \longrightarrow \bar{V}$ also non deg.

Point eigenvectors makes sense; they are the lines in V fixed under $A^{-1}S$. It should be clear that an anti-linear map f on V induces a map on $\mathbb{P}V$. $f(cv) = \bar{c} f(v)$.

Can you say anything about an eigenvalue for an eigenline? This is a 1-diml matter

First define an eigenline of T to be a line $l \subset V$ such that $T(l) \subset l$, whence $T(l) = l$. What kind of eigenvalue. Pick $0 \neq v_0 \in l$. Then $T(v_0) = c v_0$ for some $c \neq 0$. A different choice of generator $v_1 = d v_0$ yields

$$\frac{T(v_1)}{v_1} = \frac{\bar{d}\, T(v_0)}{d\, v_0} = \frac{\bar{d}}{d} c .$$ ~~So the modulus~~

so $\left|\frac{T(v_0)}{v_0}\right|$ is independent of the choice of gen.

~~Looks like eigenvalues are~~

Next look at T^2 which is a nondeg $\mathbb{C}$ linear transf on V. Assume $T(v_0) = c v_0$, then $T(T(v_0)) = T(c v_0) = \bar{c}\, T(v_0) = \bar{c} c\, v_0$, so the

(226) eigenvalues of T^2 should be >0. Now you need the existence.

Let T be an antilinear transf on V. $T(cv) = \bar{c}\,T(v)$.
Then T^2 is linear: $T(T(cv)) = T(\bar{c}\,T(v)) = c\,T(T(v))$.

Let μ be an eigenvalue of T^2, put $V_\mu = \mathrm{Ker}(T^2 - \mu)$. $T^2 v = \mu v$

$\Longrightarrow$

$v \in V_\mu = \mathrm{Ker}(T^2 - \mu)$ $\quad T^2 v = \mu v$

$Tv \in V_{\bar{\mu}} = \mathrm{Ker}(T^2 - \mu)$ $\quad T^3 v = \bar{\mu}\, Tv$

T antilinear transf on V. Notion of eigenvector: $v \neq 0$ s.t. $Tv = \lambda v$ λ scalar.

$T^2 v = T\lambda v = \bar{\lambda}\, Tv = |\lambda|^2 v$.

Question: Do the eigenvectors span V?

T antilinear on V, notion of eigenvector: $v \neq 0 : Tv = \lambda v$

Then $T(Tv) = T(\lambda v) = \bar{\lambda}\, Tv$ so Tv is also an eigenvector. Another point is that the eigenline is the interesting thing. and if $c \neq 0$ then $T(cv) = \bar{c}\, Tv = \bar{c}\lambda v$, so $\frac{T(cv)}{cv} = \frac{\bar{c}}{c}\lambda = \frac{\bar{c}}{c}\frac{Tv}{v}$

The point is that only $|\lambda|$ is determined by the eigenline.

Now look at T^2 which is $\mathbb{C}$ linear.

Really you should be doing this with Euclidean spaces and complex structures. What does this mean? Ratio of symmetric and hermitian forms?

$$\begin{tikzcd} V \arrow[r, "H"] & \hat{V} \arrow[r, "A^{-1}"] & V \end{tikzcd}$$

V complex vector space equipped with $\langle x|y\rangle$ pos herm. form. This should be the same as a Euclidean space together with an operator J satisfying $J^t = J^{-1} = -J$ (that is a complex structure)

Problem: Given a symmetric complex bilinear form S on V (this should be the same as a symmetric real bilinear form on V satisfying some compatibility with the operator J) you want a "time evolution" associated to S and the pos. herm form.

Ideas. A complex Hilbert space (= Euclidean space + J), from the real viewpoint, has both symmetric^S and skew-symmetric forms^A, whose "ratio" should be J. Ratio might be understood as the eigenvalues (better: spectrum) of $A^{-1}S$.

First task. Make sense of time evolution! There should be someway to do this on the real level.

(228) To understand the situation in real terms.

The situation: An n-diml Hilb space V equipped with a symmetric bilinear form S.
Then we have $V \xrightarrow{S} \hat{V} \xleftrightarrow[\sim]{A^{-1}} \bar{V}$ which gives an anti-linear $A^{-1}S$ map on V. What's the significance of S being symmetric? Let's do this by introducing coords. $V = \mathbb{C}^n$ space of column vectors.

$S : V \longrightarrow \hat{V}$ is $\sum_j s_{ji} v_j$

$$\hat{V} \longrightarrow \bar{V}$$

$$[y_1, \dots, y_n] \longmapsto \begin{bmatrix} \bar{y}_1 \\ \vdots \\ \bar{y}_n \end{bmatrix}$$

~~Combination~~ composition is

~~$\sum_j s_{ij} v_j \longmapsto \sum_j \bar{v}_j \bar{s}_{ij}$~~

$$y_i = \sum_i s_{ji} v_j \longmapsto \bar{y}_i = \sum_j \bar{s}_{ji} \bar{v}_j = \sum_j \bar{s}_{ij} \bar{v}_j$$

It seems that is

$$\begin{tikzcd} \mathbb{C}^n \arrow[r, dash, "S"] & \widehat{\mathbb{C}^n} \arrow[r, "*"] & \mathbb{C}^n \end{tikzcd}$$

$$\begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} \longmapsto \Big[\sum_j s_{ji} v_j\Big] \longmapsto \sum_j \bar{s}_{ji} v_j$$

$$\sum_j s_{ij} v_j \xleftrightarrow{*} \sum_j \bar{s}_{ij} \bar{v}_j$$

So the anti linear map is it seems equal to

$$[v_j] \longmapsto \sum_j \bar{s}_{ij} \bar{v}_j$$

~~$v \mapsto \bar{s}\bar{v} \mapsto \bar{s}\,\overline{\bar{s}\bar{v}}$~~

~~$v_k \mapsto s_{jk} \bar{v}_k$~~

and its square is

$$v \xmapsto{T} \bar{s}\bar{v} \xmapsto{T} \bar{s}(\overline{\bar{s}\bar{v}})$$

$$(\bar{s}s)v$$

$$(\bar{s}s)^* = s^* \bar{s}^* = \overline{(s^t)}\, s^t = \bar{s}s$$