110 Consider Riemann sphere S^2. ~~You~~

Go back to $Sp(2n)$ = subgp of $U(2n)$ ~~...~~ respecting

$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ $\quad g^t \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} g = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

$X = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $\quad \begin{bmatrix} a^t & c^t \\ b^t & d^t \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = 0$

$$\begin{bmatrix} -c^t & a^t \\ -d^t & b^t \end{bmatrix} + \begin{bmatrix} c & d \\ -a & -b \end{bmatrix} = 0 \qquad \boxed{\begin{matrix} b^t = b \\ c^t = c \\ d = -a^t \end{matrix}}$$

$\begin{bmatrix} a^* & c^* \\ b^* & d^* \end{bmatrix} + \begin{bmatrix} a & b \\ c & d \end{bmatrix} = 0$ $\quad c^* + b = 0$ $\quad a^* + a = 0$

$\bar{c} + b = 0, \quad d^* = a^t \Rightarrow \bar{d} = a$

$-a^t = d = -d^* \quad a^t = d^*$

$\bar{a} = d$

$\therefore X = \begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix} : \begin{matrix} a^* + a = 0 \\ b^t = b \end{matrix}$

Here's your Lie alg.

You have a candidate for Cartan subalg namely where $a = i\begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$. $\quad J = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$ $\quad [X, J] = 0 \iff b = 0.$

It seems that the diagonal a form comm. subalg

$J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ $\quad JX = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix} = \begin{bmatrix} -\bar{b} & \bar{a} \\ -a & -b \end{bmatrix}$

$XJ = \begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} -b & a \\ -\bar{a} & -\bar{b} \end{bmatrix}$ $\quad \begin{matrix} a = \bar{a} \\ b = \bar{b} \end{matrix}$

real

$X = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$ $\quad \begin{matrix} a^* + a = 0 \\ a^t + a = 0 \\ b^t = b \end{matrix}$ $\quad$ looks too hard

~~Cartan~~ rank of $Sp(2n, \mathbb{C})$ and $gl(2n, \mathbb{C})$ same

How does S^2 become a symmetric space? In more than one way? $SO(3)$ acts on S^2. Stabilizer of a point is $SO(1)$. Also $U(2)$ — also $SO(1)$ up to double covering.

(111) The rank of a sphere is probably 1.

Can you handle $G(m,n)$? ~~[illegible]~~

What is the Lie alg picture for a symmetric space? It's the tangent space to the ~~identity~~ origin acted on by ~~the~~ its isotropy group. Example $S^2 = SU(2)/U(1)$

$$LSU(2) = \left\{ \begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix} \middle| \ a + \bar{a} = 0 \right\}. \quad \text{Grass case} \quad \varepsilon = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \text{ on } \begin{bmatrix} V_+ \\ V_- \end{bmatrix}$$

$$\mathcal{L}U(V) = \left\{ \begin{bmatrix} a & b \\ -b^* & d \end{bmatrix} : \begin{array}{l} a^* + a = 0 \\ d^* + d = 0 \end{array} \right\}$$

Involution is conjugation by ε, ~~so fix~~ so

isot grp is $\begin{bmatrix} u_+ & 0 \\ 0 & u_- \end{bmatrix}$ tangent space is $b \in \mathrm{Hom}(V_-, V_+)$

so you have left & right mult by unitary matrices.

~~[illegible]~~ Maximal torus ~~an~~ analog for a symm space is a max abelian subspace of the ~~tangent space~~ infinitesimal ~~[illegible]~~ symm. space. It should be the diagonal b

e.g. $\begin{bmatrix} 0 & b \\ -\bar{b} & 0 \end{bmatrix}$ includes $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}$ which do not commute

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~~[illegible]~~ You want an understanding of symmetric spaces including the ones relevant for periodicity. How can you construct them, or how do they arise? The basic picture consists of a Lie group G equipped with an autom σ of order 2. The symmetric space is then the homogeneous space G/G^σ. Ex. $G = U(2n)$ σ is conjugation by

$\varepsilon = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ on $\begin{bmatrix} \mathbb{C}^n \\ \mathbb{C}^n \end{bmatrix}$ so $G^\sigma = U(n) \times U(n)$ and

G/G^σ = space of $F = F^* = F^{-1}$ = $\mathrm{Grass}(n, n)$

Infinitesimal symm space = -1 eigenspace in $\mathcal{L}G$ for conjugation by ε, equipped with the action of

(112) the isotropy group G^σ. (I guess you want to ignore differences between $\mathcal{L}G^\sigma = (\mathcal{L}G)^\sigma$ and G^σ.) Thus the action of G^σ and $\mathfrak{g}^\sigma$ should be considered equivalent.) The exponential map $X \mapsto e^X$ from $\mathfrak{g}$ to G, when restricted to $\mathfrak{g}^-$ should give some sort of picture for G/G^σ.

Example? Consider again $G = U(2n)$, σ = conj. by ε, $\mathfrak{g} = \{X : X^* = -X\}$, $X = \begin{bmatrix} a & b \\ -b^* & d \end{bmatrix}$ $\quad a^*+a=0$, $d^*+d=0$

$$\mathfrak{g}^+ = \left\{ \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} \text{skew-herm.} \right\} = \mathcal{L}U(n) \oplus \mathcal{L}U(n)$$

$$\mathfrak{g}^- = \left\{ \begin{bmatrix} 0 & b \\ -b^* & 0 \end{bmatrix}, \ b \in \mathrm{Hom}(V_-, V_+) \right\}.$$ The action of the isotropy group $U(n) \times U(n)$ is by left + right mult. So you're ~~looking~~ looking at linear operators $T = b : V_- \to V_+$ relative to pos herm forms given ~~on~~ $V_\pm$. Characteristic values = eigenvalues of $(T^*T)^{1/2}$ and perhaps also $(TT^*)^{1/2}$.

How is the preceding related to the idea that the symmetric space is the space of operators $F = F^* = F^{-1}$ on $\mathbb{C}^{2n}$? This space is disconnected. You have to choose a basepoint ε, and then the $U(2n)$ orbit will be the component containing ε.

~~Restrict~~ to case where $\varepsilon = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ with $n \times n$ blocks.

Let's begin again with the symmetric space $U(2n)/U(n) \times U(n)$, which is the conjugacy class of $\{F = F^* = F^{-1}\}$ self adjoint involutions with $\dim(F=1)$ and $\dim(F=-1)$ both n.

$\therefore$ You are looking at ~~self adjoint~~ n diml subspaces

(113) of $\mathbb{C}^{2n}$. Basepoint $\varepsilon = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ on $\begin{bmatrix} \mathbb{C}^n \\ \mathbb{C}^n \end{bmatrix}$

infinitesimal symm. space is described by $\mathcal{L}U(2n)$ under conjugation by ε.

$$\mathfrak{g} = \mathcal{L}U(2n) = \left\{ \begin{bmatrix} a & b \\ -b^* & d \end{bmatrix} : \begin{matrix} a^* + a = 0 \\ d^* + d = 0 \end{matrix} \right\}.$$

Centralizer of ε is $U(2n)$ is $\left[\begin{array}{c|c} U(n) & 0 \\ \hline 0 & U(n) \end{array}\right]$

$$\mathfrak{g}^- = \left\{ \begin{bmatrix} 0 & b \\ -b^* & 0 \end{bmatrix} : b \in M_n\mathbb{C} \right\}.$$

Action of $U(n) \times U(n)$ on $M_n\mathbb{C}$: ~~orbits~~ are described by the characteristic values of b. Let $X = \begin{bmatrix} 0 & b \\ -b^* & 0 \end{bmatrix} \in \mathfrak{g}^-$. Polar decomposition $-X^2 = \begin{bmatrix} +bb^* & 0 \\ 0 & +b^*b \end{bmatrix}$

$$|X| = (-X^2)^{1/2} = \begin{bmatrix} (bb^*)^{1/2} & 0 \\ 0 & (b^*b)^{1/2} \end{bmatrix}.$$

$$X|X|^{-1} = \begin{bmatrix} 0 & b \\ -b^* & 0 \end{bmatrix} \begin{bmatrix} (bb^*)^{-1/2} & 0 \\ 0 & (b^*b)^{-1/2} \end{bmatrix} = \begin{bmatrix} 0 & b(b^*b)^{-1/2} \\ -b^*(bb^*)^{-1/2} & 0 \end{bmatrix}$$

X skew-hermitian should imply that its phase $X|X|^{-1} = J$ satisfies $-J = J^* = J^{-1}$

$$\underbrace{b(b^*b)^{-1/2}}_{(bb^*)^{-1/2}\, b}(-b^*)(bb^*)^{-1/2} = -1$$

$$(bb^*)^{-1/2}\, b\,(-b^*)(bb^*)^{-1/2}$$

114 Digress to look again at $\mathcal{L}Sp(2n)$

$= \left\{ \begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix} : \begin{matrix} a^* + a = 0 \\ b^t = b \end{matrix} \right\}$. In the Morse theory for $Sp(2n)$, paths $1 \to -1$, you have a (non deg) critical submanifold of geodesics $e^{\theta J}$ $0 \leq \theta \leq \pi$, where J runs over $\{-J = J^* = J^{-1}\}$. Example $J = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$

centralizer of J is $\begin{bmatrix} a & 0 \\ 0 & \bar{a} \end{bmatrix}$, $a^* + a = 0$.

$$e^{\theta J} = \sum_{n \geq 0} \frac{\theta^n}{n!} J^n$$

$$= \sum_{m \geq 0} \frac{\theta^{2m}}{(2m)!} \overbrace{J^{2m}}^{(-1)^m} + \sum_{m \geq 0} \frac{\theta^{2m+1}}{(2m+1)!} (-1)^m J$$

$$= (\cos\theta) + (\sin\theta) J \qquad = J \quad \text{when } \theta = \frac{\pi}{2}$$

$$\frac{d}{d\theta}\left(\cos\theta + J(\sin\theta)\right) = -\sin\theta + J\cos\theta$$
$$= J(\cos\theta + J\sin\theta)$$

At some point you would like to know whether the subgroup $\left\{ \begin{bmatrix} g & 0 \\ 0 & \bar{g} \end{bmatrix} : g \in U(n) \right\} \subset Sp(2n)$ is related to $V \longmapsto \mathbb{H} \otimes V$.

This should be checkable at least on the Lie algebra level.

You have $U(n) \xhookrightarrow{\mathbb{H}\otimes -} Sp(2n) \subset U(2n)$

e.g. $U(1) \hookrightarrow Sp(2) \cong SU(2) \subset U(2)$

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functor $V \longmapsto \mathbb{H} \otimes_{\mathbb{C}} V$. Now tensor product of hermitean vector spaces should be hermitian. If so then you get a homom. $U(V) \to U(\mathbb{H} \otimes_{\mathbb{C}} V)$

(115) and this should be the homom. $g \mapsto \begin{bmatrix} g & 0 \\ 0 & \bar{g} \end{bmatrix}$

from $U(n)$ to $Sp(2n)$. Let's work on the Lie alg level.

$$\mathcal{L}U(2n) \longleftarrow \mathcal{L}Sp(2n) \longleftarrow \mathcal{L}U(n)$$

$$\left\{ \begin{bmatrix} a & b \\ -b^* & d \end{bmatrix}, \begin{matrix} a^*=-a \\ d^*=-d \end{matrix} \right\} \hookleftarrow \left\{ \begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix}, \begin{matrix} a^*=-a \\ b^t=b \end{matrix} \right\} \hookleftarrow \left\{ \begin{bmatrix} a & 0 \\ 0 & \bar{a} \end{bmatrix} : a^*=-a \right\}$$

$n=1$.

$$\left\{ \begin{bmatrix} a & b \\ -\bar{b} & d \end{bmatrix} \begin{matrix} \bar{a}+a=0 \\ \bar{d}+d=0 \end{matrix} \right\} \hookleftarrow \left\{ \begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix} \bar{a}+a=0 \right\}$$

At least for $n=1$, you can get $\mathcal{L}Sp(2)$ as trace zero elts in $\mathcal{L}U(2)$

You forgot the obvious thing $X^t \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} X = 0$

$$\begin{bmatrix} a^t & c^t \\ b^t & d^t \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} c & d \\ -a & -b \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} -d & c \\ b & -a \end{bmatrix}$$

$$g^t J g = J \qquad g^* = g^{-1} \qquad g^t = \bar{g}^{-1}$$

$$\bar{g}^{-1} J g = J \qquad \boxed{Jg = \bar{g} J} \qquad X^t J + JX = 0$$

$$X = JX^*J \qquad \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} a^* & c^* \\ b^* & d^* \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} b^* & d^* \\ -a^* & -c^* \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} -d^* & b^* \\ c^* & -a^* \end{bmatrix}$$

(116) There should be an involution σ on $U(2n)$ with fixpt subgroup $Sp(2n)$. $g \in Sp(2n)$ means g is unitary: $g^* = g^{-1}$ and that it preserves symplectic form: $g^t J g = J$ $\quad J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

now $g^t = \overline{g^*} = \bar{g}^{-1} \Rightarrow \bar{g}^{-1} J g = J$

or $Jg = \bar{g} J$

So you find $Sp(2n) = \{ g \in U(2n) \mid J g J^{-1} = \bar{g} \}$.

or the equiv. cond. $\boxed{g = J \bar{g} J^{-1}}$. The infinitesimal form is $X = J \bar{X} J^{-1}$ $\quad \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} \bar{a} & \bar{b} \\ \bar{c} & \bar{d} \end{bmatrix} \begin{bmatrix} 0 & -1 \\ +1 & 0 \end{bmatrix}$

$= \begin{bmatrix} \bar{c} & \bar{d} \\ -\bar{a} & -\bar{b} \end{bmatrix} \begin{bmatrix} 0 & -1 \\ +1 & 0 \end{bmatrix} = \begin{bmatrix} +\bar{d} & -\bar{c} \\ -\bar{b} & +\bar{a} \end{bmatrix}$ so $X = \begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix}$, but

also you want $X^* = -X \Leftrightarrow b^* = +\bar{b} \Leftrightarrow b^t = b$.

You are trying to find an involution on $\mathcal{L}U(n)$ whose fixed subalg is $Sp(2n)$.

$X = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$X = -X^*$

$X = J \bar{X} J^{-1}$

$U(2n) \longleftrightarrow Sp(2n) \longleftrightarrow U(n)$

Idea: $g \longmapsto J \bar{g} J^{-1}$ is a group auto from $U(n)$ to itself of order 2. $\quad J\, \overline{J \bar{g} J^{-1}}\, J^{-1} = \underbrace{J \bar{J}}\, g\, \underbrace{\bar{J}^{-1} J^{-1}}$

But $J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ $\quad \bar{J} = J$ and $\bar{J}^{-1} = J^{-1}$

(117) $g^* g = 1 \qquad g^t J g = J$ define $Sp(2n)$

$g^t \bar{g} = 1 \qquad (\bar{g})^{-1} J g = J \qquad Jg = \bar{g} J$

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} \bar{a} & \bar{b} \\ \bar{c} & \bar{d} \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \qquad \begin{aligned} \bar{a} &= d \\ \bar{b} &= -c \\ \bar{c} &= -b \\ \bar{d} &= -a \end{aligned} \qquad X = \begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix}$$

$$\begin{bmatrix} c & d \\ -a & -b \end{bmatrix} = \begin{bmatrix} -\bar{b} & \bar{a} \\ -\bar{d} & \bar{c} \end{bmatrix}$$

The hope is that such an X is an $\mathbb{H}$ linear endo.

$\mathbb{H} = \mathbb{R} + \mathbb{R}i + \mathbb{R}j + \mathbb{R}k = \mathbb{C} + \mathbb{C}j$

$$(a+bj)(x+yj) = (ax - b\bar{y}) + (ay + b\bar{x})j$$

$$\begin{bmatrix} ax - b\bar{y} & ay + b\bar{x} \end{bmatrix} = \begin{bmatrix} a & b \end{bmatrix}\begin{bmatrix} x & y \\ -\bar{y} & \bar{x} \end{bmatrix}$$

You want to left multiply by complex matrices.

First do $n = 1$. The Lie alg is set of $X = \begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix}$: $a + \bar{a} = 0$.

You want X to act on $\mathbb{C}^2$.

You get $M_2\mathbb{C}$ centralizer of mult. by $\mathbb{C}$

$\begin{bmatrix} \alpha \\ \beta \end{bmatrix}$. Consider $\mathbb{H}$

Start again with the ring of matrices $\begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix} \in M_2\mathbb{C}$. These are the X s.t. $JX = \bar{X} J$ where $J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

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$X = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$ $\quad \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} \overset{?}{=} \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

$$\begin{bmatrix} \bar a & \bar b \\ \bar c & \bar d \end{bmatrix}\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad \begin{bmatrix} 0 & -i \\ -i & 0 \end{bmatrix} - \begin{bmatrix} 0 & -i \\ -i & 0 \end{bmatrix}$$

$$\begin{bmatrix} i\bar a & -i\bar b \\ i\bar c & -i\bar d \end{bmatrix} = \begin{bmatrix} ia & ib \\ -ic & -id \end{bmatrix} \qquad \begin{matrix} a = \bar a & b = -\bar b \\ c = -\bar c & d = \bar d \end{matrix}$$

There should be a simple ~~relation between~~ interpretation of the condition $Jg = \bar g J$, ~~or~~ better ~~would be~~ $JX = \bar X J$ for $X \in M_{2n}\mathbb{C}$. $\quad J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

There should be a simple interpretation of the condition $JX = \bar X J$ for $X \in M_{2n}\mathbb{C}$. and $J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$.

Check first that such X form a ~~~~ real subalg

$$J(X_1X_2)J^{-1} = (JX_1J^{-1})(JX_2J^{-1}) = \bar X_1 \bar X_2 = \overline{X_1X_2}$$

Next ~~~~ use block picture $X = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and the condition $JX = \bar X J$ becomes $X = \begin{bmatrix} a & b \\ -\bar b & \bar a \end{bmatrix}$

Look at case $n=1$, ~~~~ where the real subalg. has basis. $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \underset{i}{\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}}, \underset{j}{\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}}, \underset{k}{\begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}}$

So you have the quaternions.

Now ~~form~~ let $\mathbb{H} \otimes I_n \simeq \mathbb{H}$ act on $\mathbb{C}^{2n}$ via the matrices for $1, i, j, k$ but as n scalar diagonal $n \times n$ matrices. Then $\mathbb{C}^{2n}$ is a vector space over $\mathbb{H}$. What are you

(119) really doing? Block matrices of total size $2n \times 2n$ with 4 $n \times n$ blocks are the same as elements of $M_2\mathbb{C} \otimes M_n\mathbb{C}$. | Review. $Sp(2n) = \{g \in U(2n) :$ $g^t J g = J\}$. $g \in U(2n) \iff g^* = g^{-1} \iff g^t = \bar{g}^{-1}$

$Sp(2n) = \{ g \in U(2n) \mid Jg = \bar{g} J \}$. ~~Point: If you started two~~

So you get two groups $Sp(2n, \mathbb{C})$ $GL(n, \mathbb{H})$??

Infinitesimally: $JX = \bar{X} J$ which leads (when $J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$) to $X = \begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix}$ $a, b \in gl(n, \mathbb{C})$.

real dim $= 2 \, 2n^2 = 4n^2$, & when you impose $X^* = -X$ you get $\begin{matrix} a^* + a = 0 \\ b^t = b \end{matrix}$) real dim $= n^2 + 2\frac{n(n+1)}{2} = 2n^2 + n$

What's important, useful? $\{ g \in GL(2n, \mathbb{C}) \mid Jg = \bar{g} J \}$

$= \{ X \in M_{2n}\mathbb{C} \mid JX = \bar{X}\bar{J} \}$. This is an alg/$\mathbb{R}$.

$= \left\{ X = \begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix} \subset M_{2n}\mathbb{C} \mid a, b \in M_n\mathbb{C} \right\}$

$n = 1$. $\left\{ \begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix} \right\}$ has basis $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}$

Suppose $X = \begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix}$ s.t. $X^* + X = 0$

$a^* + a = 0$

$\overline{a^*} + \bar{a} = 0$

$+\bar{b} = b^*$, $b^t = b$.

You learned that $(JX = \bar{X}J)$ & $(X^* = -X)$ yields $\mathcal{L}Sp(2n)$.

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3 conditions — Any two $\Rightarrow$ third

$X^* + X = 0$ — (1) $X^* = -X \Rightarrow X^t = -\bar{X} \Rightarrow JX \overset{(2)}{=} -X^t J \overset{(3)}{=} \bar{X} J$.

$X^t J + JX = 0$ — (2) $X^t J + JX \overset{(3)}{\Rightarrow} X^t J + \bar{X} J \Rightarrow X^t + \bar{X} = 0 \Rightarrow (1)$

$JX = \bar{X} J$ — (3) X

(120) What are you missing? The link between $JX = \bar{X}J$ and being a v.s. over $\mathbb{H}$.

What is the point? $\{X \in M_{2n}\mathbb{C} \mid JX = \bar{X}J\}$ is an algebra over $\mathbb{R}$. I think you want to translate the condition $JX = \bar{X}J$ to saying that X centralizes something

$n=1$. $\left\{X = \begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix} \in M_2\mathbb{C}\right\} = \mathcal{A}$ 4 diml alg/$\mathbb{R}$

$\mathcal{A}$ has $\mathbb{R}$-basis.
$$\underset{1}{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}, \underset{i}{\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}}, \underset{j}{\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}}, \underset{k}{\begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}}$$

So $\mathcal{A} = \mathbb{H}$ acting on $M_2\mathbb{C}$ in the way indicated

Ultimately you want ~~[illegible]~~ to identify $\{X \in M_{2n}\mathbb{C} \mid JX = \bar{X}J\} = \mathcal{A}_n$ with $M_n\mathbb{H}$

You've done this for $n=1$. What do you know about $M_n\mathbb{H}$? It's the ring of endos of the right vector space $\mathbb{H}^n$. Now $\mathbb{H}^n$ appears as ?

$n=1$. to identify ~~[illegible]~~ $\left\{X = \begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix} \in M_2\mathbb{C}\right\} = \mathcal{A}$ with the ring $\mathbb{H}$. That's done by the basis
$$\underset{1}{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}, \underset{i}{\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}}, \underset{j}{\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}}, \underset{k}{\begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix}}$$

Aim to identify $\mathcal{A}_n$ $\{X \in M_{2n}\mathbb{C} \mid JX = \bar{X}J\}$ with $M_n\mathbb{H}$. Now

$M_n\mathbb{H}$ = ring of endos of the right $\mathbb{H}$ vector space $\mathbb{H}^n$

What you need first is a right mult of $\mathbb{H}$ on $H(V)$

121 $\quad H(V) = \begin{bmatrix} V \\ \bar{V} \end{bmatrix}$ $\quad H(\mathbb{C}) = \begin{bmatrix} \mathbb{C} \\ \bar{\mathbb{C}} \end{bmatrix}$ not good notation. $H(\mathbb{C})$ is a complex vector space on which the real algebra $\left\{ \begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix} \right\}$ acts $\mathbb{C}$-linearly ?

All this is confused. What do you need to straighten things out? Go back to the 3 properties

$X^* = -X$, $\quad X^t J + J X = 0$, $\quad J X = \bar{X} J$

It might be easier to understand the algebra over $\mathbb{R}$ consisting of $2n \times 2n$ complex matrices X such that $JX = \bar{X} J$. $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \bar{a} & \bar{b} \\ \bar{c} & \bar{d} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

$= \begin{bmatrix} -\bar{c} & -\bar{d} \\ \bar{a} & \bar{b} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} \bar{d} & -\bar{c} \\ -\bar{b} & \bar{a} \end{bmatrix} \quad \therefore X = \begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix}$

IDEA. Choice of notation is poor, Normal is left multiplication by matrices on column vectors, so the columns of a matrix algebra are modules over the algebra. $\therefore$ Better should be $\begin{bmatrix} a & -\bar{b} \\ b & \bar{a} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax - \bar{b}y \\ bx + \bar{a}y \end{bmatrix}$

$(a + bj)(c + dj)$. ~~Clalgn~~

Goal. Symmetric space $Sp(2n)/U(n)$. ~~~~ infinitesimal picture is $\begin{bmatrix} 0 & -\bar{b} \\ b & 0 \end{bmatrix}$ $\quad$ isot. alg $\begin{bmatrix} a & 0 \\ 0 & \bar{a} \end{bmatrix}$

$\mathcal{L}\, Sp(2n) \quad X = \begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix}$. $\quad$ Action $\begin{bmatrix} u & 0 \\ 0 & \bar{u} \end{bmatrix} \begin{bmatrix} 0 & b \\ -\bar{b} & 0 \end{bmatrix} \begin{bmatrix} u^{-1} & \\ & \bar{u}^{-1} \end{bmatrix}$

$\bar{u}^{-1} = \overline{u^*} = u^t$

$u b \bar{u}^{-1} = u b u^t$. Now look at $\Omega\, Sp(2n)/U(n)$, Morse theory.

(122) $n=1$. $H(\mathbb{C}) = \begin{bmatrix}\mathbb{C}\\ \mathbb{C}\end{bmatrix}$, $\begin{bmatrix}x_1\\ y_1\end{bmatrix}^t \begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}\begin{bmatrix}x_2\\ y_2\end{bmatrix}$, $\begin{bmatrix}x_1\\ y_1\end{bmatrix}^* \begin{bmatrix}x_2\\ y_2\end{bmatrix}$

3 properties $X^tJ + JX = 0$, $X^* + X = 0$, $JX = \bar{X}J$. Now $JX = \bar{X}J \iff X = \begin{bmatrix}a & b\\ -\bar{b} & \bar{a}\end{bmatrix}$ these form a real alg $\mathcal{A}$. and $H(\mathbb{C})$ is a left $\mathcal{A}$ module.

I think the central problem is the meaning of the $\mathbb{R}$ algebra $\mathcal{A} = \{X \in M_{2n}\mathbb{C} \mid JX = \bar{X}J\}$. The algebra should be determined by its modules, NO only up to Morita equivalence.

$n=1$ again. $\mathcal{A} = \left\{\begin{bmatrix}a & b\\ -\bar{b} & \bar{a}\end{bmatrix} \in M_2\mathbb{C}\right\}$ $\quad X \mapsto J\bar{X}J^{-1}$

$$\begin{bmatrix}a & b\\ c & d\end{bmatrix} \longmapsto \begin{bmatrix}\bar{a} & \bar{b}\\ \bar{c} & \bar{d}\end{bmatrix} \longrightarrow \begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}\begin{bmatrix}\bar{a} & \bar{b}\\ \bar{c} & \bar{d}\end{bmatrix}\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}$$

$$= \begin{bmatrix}\bar{c} & \bar{d}\\ -\bar{a} & -\bar{b}\end{bmatrix}\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix} = \begin{bmatrix}\bar{d} & -\bar{c}\\ -\bar{b} & \bar{a}\end{bmatrix} = \begin{bmatrix}a & b\\ c & d\end{bmatrix}.$$

What can you do here? $\mathcal{A}$ has quaternionic basis over $\mathbb{R}$ $\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}, \begin{bmatrix}i & 0\\ 0 & -i\end{bmatrix}, \begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}, \begin{bmatrix}0 & i\\ i & 0\end{bmatrix}$

- An $\mathbb{H}$-module is a $\mathbb{C}$-module V equipped with an anti linear endomorphism σ ~~such that~~ satisfying $\sigma^2 = -1$.
- Tensor product

$$\sigma(v \otimes w) = \sigma(v) \otimes \sigma(w)$$
$$\sigma(z_1 v \otimes z_2 w) = \sigma(z_1 v) \otimes \sigma(z_2 w) = \bar{z}_1 \bar{z}_2\, \sigma(v) \otimes \sigma(w)$$

123 An $\mathbb{H}$-module structure on a complex vector space W is the same as an anti-linear map $\sigma: W \longrightarrow W$ (i.e. $\sigma(cw) = \bar{c}\,\sigma(w)$) satisfying $\sigma^2 = -1$. Why? $\mathbb{H} = \mathbb{C} + \mathbb{C}j$ where $jc = \bar{c}j$ and $j^2 = -1$. Given W, σ define left mult by $a + bj \in \mathbb{H}$ by $(a+bj)w = aw + b\sigma(w)$. Check this is a homomorphism $\mathbb{H} \longrightarrow \mathrm{Hom}_{\mathbb{R}}(W, W)$

$$(a+bj)\cdot(c+dj) \longrightarrow ac - b\bar{d} + (ad + b\bar{c})j$$

$$(a+b\sigma)\circ(c+d\sigma) = ac + b\sigma c \underbrace{+ ad\sigma + \underline{b\sigma d\sigma}}_{} $$
$$\underline{b\sigma d\sigma} = b\bar{d}(-1)$$
$$= (ac - b\bar{d}) + (ad + b\bar{c})\sigma$$

You really want $\mathbb{H}$ to act on $H(V)$, the hyperbolic space assoc. to V, where V is equipped with a positive hermitian form. Use the pos herm. form to get an anti-linear isom $\sigma: V \longrightarrow V$ such that $\sigma^2 = 1$. ??

Start again. $H(V) = \begin{bmatrix} V \\ V^\wedge \end{bmatrix}$ vector space over $\mathbb{C}$ equipped with the ~~obvious~~ canonical pairing $\varphi^t v = v^t \varphi$ for $v \in V$, $\varphi \in V^\wedge$. Get symmetric and skew symm. forms

$$\begin{bmatrix} v_1 \\ \varphi_1 \end{bmatrix}^t \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} v_2 \\ \varphi_2 \end{bmatrix}, \quad \begin{bmatrix} v_1 \\ \varphi_1 \end{bmatrix}^t \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} v_2 \\ \varphi_2 \end{bmatrix}$$

Important point I think is that if a pos. herm. form is given on V, then there is a canonical antilinear isom $\sigma: V \longrightarrow V^\wedge$ $v \mapsto \langle v|$. There should also be $\sigma: V^\wedge \longrightarrow V^{\wedge\wedge} = V$ similarly defined.

(124) V equipped with $\langle v_1 | v_2 \rangle$ pos. herm.
Then have canon anti linear isom $V \xrightarrow{\sim} V^\wedge$
$v_1 \longmapsto \langle v_1 |$. You want to get an anti linear isom of $W = \begin{bmatrix} V \\ V^\wedge \end{bmatrix}$ with itself having square 1.

Start with V a complex v.s. equipped with pos herm form. Then you get an anti linear isom $V \longrightarrow V^\wedge$, call this σ. You want another half namely $\sigma : V^\wedge \longrightarrow V$. The obvious thing to do is to make $\sigma^2 = 1$. See what this means.

Take $V = \mathbb{C}$ $\langle x | y \rangle = \bar{x} y$
Identify $V^\wedge$ with $\mathbb{C}$ via product in $\mathbb{C}$.

Then
$$\begin{array}{ccc} V & \longrightarrow & V^\wedge \\ \Psi & & \\ y & & \bar{y} \end{array} \qquad \begin{array}{ccc} V & & V^\wedge \\ z & \overset{\sigma}{\longmapsto} & \bar{z} \end{array}$$

So on $W = \begin{bmatrix} V \\ V^\wedge \end{bmatrix}$ you have W

$$\begin{array}{ccc} W & \longrightarrow & W^\wedge \\ \shortparallel & & \\ \begin{bmatrix} V \\ V^\wedge \end{bmatrix} & & \begin{bmatrix} \quad \end{bmatrix} \end{array}$$

$$\begin{tikzcd} W \arrow[r, hook] \arrow[d] & \begin{bmatrix} V \\ V^\wedge \end{bmatrix} \arrow[d, "\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}"] \\ W^\wedge & \begin{bmatrix} V^\wedge \\ V \end{bmatrix} \arrow[l] \end{tikzcd}$$

$$\begin{bmatrix} v_2 \\ \varphi_2 \end{bmatrix}^t \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ \varphi_1 \end{bmatrix}$$

125 V complex vector space + pos herm. form

$\simeq$ ~~[illegible]~~ Euclidean space with ~~ϕ~~ $-I = I^* = I^{-1}$

There's too much confusion. Let's start again with a complex vector space V equipped with pos. herm. form. Let $V^\wedge$ be the $\mathbb{C}$-linear dual of V. Representation thm for linear fnls says that the map

$V \xrightarrow{\theta} V^\wedge$, $\quad x \longmapsto h(x,-)$ is an antilinear isom.

It should also satisfy a symmetry condition arising from the identity $h(x,y) = \overline{h(y,x)}$. (If you were dealing with Euclidean spaces, then the condition is

$$V \underset{\text{canon}}{=} \hat{\hat{V}} \xrightarrow{\theta^t} \hat{V} \qquad \theta = \theta^t \circ (\text{canon}). \qquad \begin{aligned} V &\longrightarrow (V^\wedge)^\wedge \\ x &\longmapsto (\xi \mapsto \xi(x)) \end{aligned}$$

what is $\theta^t : (V^\wedge)^\wedge \longrightarrow V^\wedge$? $\qquad \theta^t(\varphi) = \varphi\theta$

~~[illegible]~~ Start again with Euclidean space V.

and $\theta : V \longrightarrow V^\wedge$, $x \longmapsto \theta x = (y \mapsto x \cdot y)$

$$\begin{aligned} V \times V &\longrightarrow \mathbb{R} \\ (x, y) &\longmapsto x \cdot y \end{aligned} \qquad\qquad \begin{aligned} V &\xrightarrow{\theta} V^\wedge \\ x &\longmapsto (y \mapsto x\cdot y) \end{aligned}$$

Want θ^t where $\theta : V \longrightarrow V^\wedge$, $\theta x = (y \mapsto x \cdot y)$

θ^t is a map $(V^\wedge)^\wedge \longrightarrow V^\wedge$ defined by

$$\theta^t(\varphi) = \varphi\theta = (x$$

The point must be that $\varphi \in (V^\wedge)^\wedge$ is equiv to an $x \in V$.

126 First let V, W be (fin dim'l) v.s.

Then bilinear form $V \times W \xrightarrow{B} F$ $\quad v \mapsto v^t B$

is the same as $\quad V \longrightarrow W^\wedge$, $\quad v \mapsto (w \mapsto B(v,w))$

also $\quad W \longrightarrow V^\wedge \quad w \mapsto Bw$

matrix notation $\quad B(v,w) = v^t B w = w^t B^t v$

$B(v,w)$ bilinear $\quad V \times W \longrightarrow F$

$V \xrightarrow{T} W^\wedge$

What is the problem? Given V, W and $B: V \times W \to F$ bilinear. ~~Then~~ You should get two linear maps $V \xrightarrow{T} W^\wedge$ and $W \xrightarrow{T^t} V^\wedge$ which are mutually transpose. Define $Tv = (w \mapsto B(v,w))$

$T: V \longrightarrow W$ induces $T^t: W^\wedge \longrightarrow V^\wedge$ defined by $(T^t\lambda)(v) = \lambda(Tv)$.

$$V \xrightarrow{T} W$$

$$V^\wedge \xleftarrow{T^t} W^\wedge$$

$$(V^\wedge)^\wedge \xrightarrow{(T^t)^t} (W^\wedge)^\wedge$$

Start again. The problem is to understand the symplectic or orthogonal structure on $H(V) = \begin{bmatrix} V \\ V^\wedge \end{bmatrix}$ when V is a $\mathbb{C}$-vector space equipped with positive herm. form. This should be easy. Why? Because the pos herm. form ~~should~~ gives an anti-linear isom $V \longrightarrow V^\wedge$ which should yield an antilinear isom $\begin{bmatrix} V \\ V^\wedge \end{bmatrix} \longrightarrow \begin{bmatrix} V^\wedge \\ V \end{bmatrix}$ of order 2.

How to make this feel "real". Take $V = \mathbb{C}^n$, $n = 1$. Then $\sigma \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} \bar{z}_2 \\ \bar{z}_1 \end{bmatrix}$

so $\sigma^2 \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \sigma \begin{bmatrix} \bar{z}_2 \\ \bar{z}_1 \end{bmatrix} = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$. The other thing is to make $\sigma^2 = -1$.

$$\sigma \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} \bar{z}_1 \\ \bar{z}_2 \end{bmatrix} = \begin{bmatrix} \bar{z}_2 \\ -\bar{z}_1 \end{bmatrix}$$

then $\sigma \begin{bmatrix} \bar{z}_2 \\ -\bar{z}_1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} z_2 \\ -z_1 \end{bmatrix} = \begin{bmatrix} -z_1 \\ -z_2 \end{bmatrix} = - \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$

Maybe its useful to note that to give an anti linear operator (transformation) σ of square -1 on a complex vector space V does not seem to make V either a left or right $\mathbb{H}$-module, unless there is some other choice made.

More precisely consider what structure you put on V: a real v.s. structure, an operator i of square -1, and an operator σ of square -1 anti commuting with i.

(128) IDEA: Your description of $\mathbb{H}$-module as a $\mathbb{C}$-module V equipped with an anti-linear transf. $\sigma : V \to V$ satisfying $\sigma^2 = -1$ seems fishy, ~~because~~ because you expect a difference between left and right $\mathbb{H}$-modules. Is there an isomorphism between $\mathbb{H}$ and $\mathbb{H}^{op}$? a canonical anti-isomorphism of $\mathbb{H}$ with itself. $i \mapsto -i$ and similarly for j, k?

Point: $\mathbb{H}$ = real Clifford alg $C(\mathbb{R}^2)$ generated by i, j ~~anticommuting~~ satisfying $i^2 = j^2 = -1$ $ij + ji = 0$. An $\mathbb{H}$-module is a $\mathbb{R}$ vector space equipped with operators i, j satisfying these relations.

$$\overset{1}{\mathbb{R}} \quad \overset{2}{\mathbb{C}} \quad \overset{4}{\mathbb{H}}$$

Recall what the real Clifford algs are.
Clifford alg is like Weyl alg. Generated by a quadratic space V relations are that

$$c(v)^2 = -(v, v)$$

Still no real progress. Let's try creation and annihilation operators as the models for the hyperbolic quadratic and symplectic forms, especially displaying the positive hermitian form.

Start with $n=1$. $[a, a^*] = 1$, $[a, a] = 0$, $[a^*, a^*] = 0$

(129) $W = \mathbb{C}a \oplus \mathbb{C}a^*$ How do you get the pos hermitian form? Obvious thing is to make a, a^* orthonormal basis.

For arb n. $[a_i, a_j^*] = \delta_{ij}$, $[a_i, a_j] = 0$, $[a_i^*, a_j^*] = 0$

$$\left[\sum z_i a_i, \sum \zeta_j a_j^*\right] = \sum_i z_i \zeta_i$$

$$\begin{bmatrix} z \\ \zeta \end{bmatrix}^t \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} \tilde{z} \\ \tilde{\zeta} \end{bmatrix} = z^t \tilde{\zeta} - \zeta^t \tilde{z}$$

This is a very clear correspondence between W with basis a_i, a_j^* and the two structure $\begin{cases} \text{symp} \\ \text{pos herm.} \end{cases}$

$$g^t J g = J, \quad g^* g = 1, \quad J g = \bar{g} J. \qquad g \in GL(2n, \mathbb{C})$$

$\overset{t}{g}\bar{g} = 1$, $\; = \bar{g}^{-1}$

What missing is the link with the quaternions, the meaning of $J g J^{-1} = \bar{g}$ which tells you that the g satisfying this condition ~~form an algebra~~ are the invertible elements of a certain real algebra;

$$\mathcal{A} = \{X \in gl(2n, \mathbb{C}) \mid JX = \bar{X} J\}.$$

Try $n = 1$. $\quad \mathcal{A} = \left\{X = \begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix} \in M_2\mathbb{C}\right\}$

$\mathcal{A}$ has basis $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}$

Obvious conjecture is that $\mathcal{A} = M_n\mathbb{H}$

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(130) $n=1$. $\quad \mathcal{A} = \left\{ X = \begin{bmatrix} a & b \\ -\bar b & \bar a \end{bmatrix} \in M_2\mathbb{C} \right\}$

so $\mathcal{A} = \mathbb{H}$ acts on $\begin{bmatrix} \mathbb{C} \\ \mathbb{C} \end{bmatrix}$ in $\mathbb{C}$ linear fashion.

Suppose you consider a non $\mathbb{C}$-linear ~~action~~ transformation on $\mathbb{C}^2$ given by $\sigma\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} \bar x \\ \bar y \end{bmatrix} = \begin{bmatrix} \bar y \\ -x \end{bmatrix}$.

Then compare

$$\begin{bmatrix} a & b \\ -\bar b & \bar a \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax+by \\ -\bar b x + \bar a y \end{bmatrix} \xmapsto{\sigma} \begin{bmatrix} -b\bar x + a\bar y \\ -\bar a \bar x - \bar b \bar y \end{bmatrix} = \begin{bmatrix} a & b \\ -\bar b & \bar a \end{bmatrix}\begin{bmatrix} \bar y \\ -\bar x \end{bmatrix}$$

$$\therefore \quad \sigma\left(\begin{bmatrix} a & b \\ -\bar b & \bar a \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} a & b \\ -\bar b & \bar a \end{bmatrix} \sigma \begin{bmatrix} x \\ y \end{bmatrix}$$

So it seems that σ on $\begin{bmatrix} \mathbb{C} \\ \mathbb{C} \end{bmatrix}$ which is antilinear and $\sigma^2 = -1$ commutes with the action of $\mathcal{A}$. At this point it should be clear that this action of σ amounts to a right action of $\mathbb{H}$, etc.

It should be easy to see that $\mathbb{C}[\sigma]$ acting on $\mathbb{C}^2$ has commutant $\mathbb{H}$. Still want to see that $\begin{bmatrix} \mathbb{C}^n \\ \mathbb{C}^n \end{bmatrix}$ leads to $M_n\mathbb{H}$.

131 Recall calculation from yesterday where you finally got the quaternions to act on $H(V)$. Use the idea that an $\mathbb{H}$-module is a $\mathbb{C}$-module equipped with an anti-linear transformation σ such that $\sigma^2 = -1$.

Let $V = \mathbb{C}^n$, let $H(V) = \begin{bmatrix} V \\ V \end{bmatrix}$, with $\mathbb{C}$ acting, let σ be the antilinear transformation on $H(V)$ given by $\sigma\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} \bar{x} \\ \bar{y} \end{bmatrix} = \begin{bmatrix} \bar{y} \\ -\bar{x} \end{bmatrix}$, $\sigma^2 = -1$.

This defines an action of $\mathbb{H}$ on $H(V)$.

Next you want the commutant, i.e. the $X \in M_{2n}\mathbb{C}$, $X = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ such that

$\sigma X \sigma^{-1} = X$, note X already commutes with $\mathbb{C}$.

$$\sigma\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} \bar{x} \\ \bar{y} \end{bmatrix} = \begin{bmatrix} \bar{y} \\ -\bar{x} \end{bmatrix} \qquad \sigma^{-1}\begin{bmatrix} x \\ y \end{bmatrix} = -\sigma\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -\bar{y} \\ \bar{x} \end{bmatrix}$$

$$\sigma\begin{bmatrix} \bar{y} \\ -\bar{x} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} y \\ -x \end{bmatrix} = \begin{bmatrix} -x \\ -y \end{bmatrix} \qquad ??$$

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$$\sigma\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}0&1\\-1&0\end{bmatrix}\begin{bmatrix}\bar{x}\\\bar{y}\end{bmatrix} = \begin{bmatrix}\bar{y}\\-\bar{x}\end{bmatrix}$$

$\sigma^2=-1 \Leftrightarrow \sigma^{-1}=-\sigma$

$$\sigma^{-1}\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}-\bar{y}\\\bar{x}\end{bmatrix}$$

$$\begin{bmatrix}a&b\\c&d\end{bmatrix}\begin{bmatrix}-\bar{y}\\\bar{x}\end{bmatrix} = \begin{bmatrix}-a\bar{y}+b\bar{x}\\-c\bar{y}+d\bar{x}\end{bmatrix}$$

$$\sigma\begin{bmatrix}-a\bar{y}+b\bar{x}\\-c\bar{y}+d\bar{x}\end{bmatrix} = \begin{bmatrix}-\bar{c}y+\bar{d}x\\\bar{a}y-\bar{b}x\end{bmatrix} = \underbrace{\begin{bmatrix}\bar{d}&-\bar{c}\\-\bar{b}&\bar{a}\end{bmatrix}}\begin{bmatrix}x\\y\end{bmatrix}$$

$$= \begin{bmatrix}a&b\\c&d\end{bmatrix}$$ YES.

Review various ideas.

V complex vector space with pos hermitian form. $V \xrightarrow{\nu} V^{\wedge}$ antilinear

$v \longmapsto v^{+}$

$$H(V) = \begin{bmatrix}V\\V\end{bmatrix}$$

3 structures

$$\begin{bmatrix}v_1\\w_1\end{bmatrix}^t\begin{bmatrix}0&1\\1&0\end{bmatrix}\begin{bmatrix}v_2\\w_2\end{bmatrix}, \quad \begin{bmatrix}v_1\\w_1\end{bmatrix}^*\begin{bmatrix}1&0\\0&1\end{bmatrix}\begin{bmatrix}v_2\\w_2\end{bmatrix}$$

$$\sigma\begin{bmatrix}v\\w\end{bmatrix} = \begin{bmatrix}0&1\\1&0\end{bmatrix}\begin{bmatrix}\bar{v}\\\bar{w}\end{bmatrix} = \begin{bmatrix}\bar{w}\\\bar{v}\end{bmatrix}$$

doing O case

$$\left\{ X \in M_{2n}\mathbb{C} \;\middle|\; X^* + X = 0 \right\}$$

$$\left\{ \text{———} \;\middle|\; X^t s + s X = 0 \right\}$$

$$\left\{ \text{———} \;\middle|\; \sigma X = \bar{X}\sigma \right\}$$

$$\sigma\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}\bar{y}\\\bar{x}\end{bmatrix}$$

133 $\quad \sigma = s \circ \bar{\ }$ | Suppose $X^* + X = 0$

$\Leftrightarrow -X^t = \overline{X} \Leftrightarrow$

~~assume~~ $X^t s + sX = 0$ $\Leftrightarrow$ $\overline{X} s = -X^t s$

$\Rightarrow \overline{X} s = sX \Rightarrow Xs = s\overline{X}$

$X^* + X = 0, \quad X^t s + sX = 0$

$X^t + \overline{X} = 0 \qquad X = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$$\begin{bmatrix} a^t & c^t \\ b^t & d^t \end{bmatrix} = -s \begin{bmatrix} a & b \\ c & d \end{bmatrix} s = -\begin{bmatrix} d & c \\ b & a \end{bmatrix}$$

$d = -a^t, \quad b^t + b = c^t + c = 0$

$X^* + X = 0 \qquad \begin{bmatrix} a^* & c^* \\ b^* & d^* \end{bmatrix} + \begin{bmatrix} a & b \\ c & d \end{bmatrix} = 0.$

$a^* + a = 0$

$d^* + d = 0$

$c = -b^* = +\overline{b}$

$\overline{d} = -a^* = a$

$d = -a^t$

$X = \begin{bmatrix} a & b \\ +\overline{b} & \overline{a} \end{bmatrix} \quad a^* + a = 0, \ b^t + b = 0$

You want to understand $sXs = \overline{X} \qquad s = \overline{s}$

point Ass. $\begin{cases} X^* + X = 0 \\ X^t s + sX = 0 \end{cases} \qquad X^t = -\overline{X}, \quad -\overline{X} s + sX = 0 \qquad \boxed{sXs = \overline{X}}$

$$s \begin{bmatrix} a & b \\ c & d \end{bmatrix} s = \begin{bmatrix} d & c \\ b & a \end{bmatrix} = \begin{bmatrix} \overline{a} & \overline{b} \\ \overline{c} & \overline{d} \end{bmatrix} \qquad X = \begin{bmatrix} a & b \\ \overline{b} & \overline{a} \end{bmatrix}$$

134 Work a little on the Cayley Transform. This ~~[illegible]~~ might be useful in understanding the symmetric spaces.

At some point you should make a list of topics which are relevant to your program and not yet understood properly. e.g.

- Clifford algebras
- Cayley Transform.
- A-S periodicity proof using the $\exp \pi$ map and Kuiper's theorem.

Let's first look at the real Clifford algebras. $C_n = \mathrm{Cliff}(\mathbb{R}^n)$. In general given a vector space V with quadratic form $q(x)$, the corresp. Clifford alg is the alg gen. by V subject to the relations $x^2 = q(x)$, whence $xy + yx = (x+y)^2 - x^2 - y^2 = q(x+y) - q(x) - q(y) = (x,y) + (y,x)$, where (x,y) is the associated symmetric bilinear form.

A Clifford alg is $\mathbb{Z}/2$ graded naturally - by the length mod 2 of words in the generator. There's also an increasing filtration $F_0 \subset F_1 \subset \cdots$ and $gr = \Lambda V$. Orthogonal direct sum of quadratic spaces yields super algebra tensor product.

In the interesting case for periodicity one takes $\mathbb{R}^n$ with ~~the~~ relations $x^2 = -|x|^2$ i.e. the negative norm^2 of the vector x. $C_0 = \mathbb{R}$, $C_1 = \mathbb{C}$.

(135) Let $e_1, \dots, e_n$ be orth. basis for $\mathbb{R}^n$, and use same notation for generator $e_i \in C_n$

Thus

$$e_i^2 = -1 \quad \text{and} \quad e_i e_j + e_j e_i = 0, \quad i \neq j.$$

$\therefore$ $C_2 = \mathbb{H}$ with $e_1 = i$ $e_2 = j$

basis for C_2 is $1, e_1, e_2, e_1 e_2$

Maybe next look at complex Clifford algebras with the same generators and relations

$$e_i^2 = -1 \qquad e_i e_j + e_j e_i = 0 \qquad i \neq j.$$

What's nice about the Clifford alg. C_n is that when graded C_n-module $M = M_+ \oplus M_-$ is pulled up to $\mathbb{R}^n$:

$$\begin{array}{c} \mathbb{R}^n \times M \\ \downarrow \\ \mathbb{R}^n \end{array}$$

there is a canonical isom between $M_+ \simeq M_-$ over the unit sphere. So you get a KR class on $\mathbb{R}^n \cup \infty$. Apparently this is the way to get the $\widetilde{KR}$ classes of the spheres, (generators for $\widetilde{KR}(\mathbb{R}^n \cup \infty)$ which are cyclic groups. $\mathbb{Z}, \mathbb{Z}/2, \mathbb{Z}/2, 0, \mathbb{Z}, 0, 0, 0, \mathbb{Z}$

$\pi_0 = \mathbb{Z}$ $\quad \mathbb{Z}/2 \quad \mathbb{Z}/2$

$\mathbb{Z} \times BO$, O, SO/U

Let's return to the Morse theory. Review $\Omega SO(2n)$. Go back to orthogonal group details

136 orthogonal group. $O(2n)$

$$\{ g \in GL(2n,\mathbb{R}) \mid g^t g = 1 \}$$

Lie alg $= \{ X \in gl(2n,\mathbb{R}) \mid X^t + X = 0 \}$.

$$X = \begin{bmatrix} a & b \\ -b^t & d \end{bmatrix} \qquad \begin{array}{l} a^t + a = 0 \\ d^t + d = 0 \end{array}$$

To continue you probably want the Cartan subalg where b is diagonal. Maybe introduce $J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$?
Situation not clear.

$$O(2n,\mathbb{C}) = \{ g \in GL(2n,\mathbb{C}) \mid g^t S g = S \} \qquad S = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\text{Lie } O(2n,\mathbb{C}) = \{ X \in gl(2n,\mathbb{C}) \mid \boxed{X^t S + S X = 0} \}.$$

$$\begin{bmatrix} a^t & c^t \\ b^t & d^t \end{bmatrix} = -S \begin{bmatrix} a & b \\ c & d \end{bmatrix} S = \begin{bmatrix} -d & -c \\ -b & -a \end{bmatrix} \qquad \begin{array}{l} d = -a^t \\ b^t = -b \\ c^t = -c \end{array}$$

combine with $\boxed{X^* + X = 0}$

$$\bar{d} = -a^* = a$$

$$\begin{bmatrix} a^* & c^* \\ b^* & d^* \end{bmatrix} + \begin{bmatrix} a & b \\ c & d \end{bmatrix} = 0 \qquad \begin{array}{l} a^* + a = 0 \\ b^* + c = 0 \\ d + d^* = 0 \end{array} \qquad \begin{array}{l} \overline{b^t} = -\bar{b} \\ \shortparallel \\ b^* = -c \end{array}$$

$$X = \begin{bmatrix} a & b \\ \bar{b} & \bar{a} \end{bmatrix} \qquad \begin{array}{l} a^* + a = 0 \\ b^t + b = 0 \end{array}$$

$$\boxed{S X S = \bar{X}} \qquad \bar{X} = -X^t$$

$$S X S = -X^t S$$

$\sigma X = S\bar{X}$ so σ antilinear

$$\sigma(\sigma X) = \quad \sigma(S\bar{X}) = S\overline{S\bar{X}} = X$$

137 The aim is to finish the Morse theory.

Consider $O(2n) = \{g \in U(2n) : g^t S g = S\}$

$$\mathcal{L}O(2n) = \{X \in \mathcal{L}U(2n) : X^t S + SX = 0\}$$

$$= \left\{ X = \begin{bmatrix} a & b \\ \bar{b} & \bar{a} \end{bmatrix} : \begin{matrix} a^* + a = 0 \\ b^t + b = 0 \end{matrix} \right\}$$

$$\{X \in gl(2n, \mathbb{C}) \mid SXS = \bar{X}\}$$

$$\begin{bmatrix} \bar{a} & \bar{b} \\ \bar{c} & \bar{d} \end{bmatrix} = \begin{bmatrix} d & c \\ b & a \end{bmatrix} \qquad X = \begin{bmatrix} a & b \\ \bar{b} & \bar{a} \end{bmatrix}$$

~~Other point is that if $a^* + a = 0$, $b^t + b = 0$ then $a + ib$ ... $a^* - ib^* = -a - i(b^t) = -a - i(-b)$ ... $a^* = -a$ ... $b^* = \bar{b}$ ... $b^t = -b$~~

Other point is that you have this real description of $\mathcal{L}ie(O(2n))$. The point maybe is that $SXS = \bar{X}$ (the reality condition) says $X = \begin{bmatrix} a & b \\ +\bar{b} & \bar{a} \end{bmatrix}$. But the other conditions on X are $X^t S + SX = 0$ and $\boxed{X^* + X = 0}$

$\boxed{a^* + a = 0}$

$\bar{b} + b^* = 0$

same as $\boxed{b + b^t = 0}$

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[illegible]

$-b^* \stackrel{?}{=} \bar{b}$
$-b^t = b$

Try to use the model for $\mathcal{L}\,O(2n) = \left\{ \begin{bmatrix} a & b \\ \bar{b} & \bar{a} \end{bmatrix} : \begin{matrix} a^* + a = 0 \\ b^t + b = 0 \end{matrix} \right\}$

You want geodesics from 1 to -1.

It seems that you want X so that $\exp(\pi X) = -1$

Take $b = 1$. $X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = S$ No.

You want ~~X~~ all ~~X~~ $X = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}$

~~n=1~~ What is $O(2)$? $g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \ni$

$$\begin{bmatrix} a & c \\ b & d \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad \text{if } \det = +1.$$

$$\begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} a & -c \\ -b & d \end{bmatrix} \qquad \therefore b = c = 0, \; ad = 1.$$

If $\det = -1$, then $a = d = 0$. $\begin{bmatrix} 0 & b^{-1} \\ b & 0 \end{bmatrix}$?

What should be interesting is the action of $U(n)$. The symmetric spaces should be $SO(2n)/U(n)$ and $Sp(2n)/U(n)$. Have $U(n)$ acting on b which is skew symm.

So what have you learned? $-b^* \stackrel{?}{=} +\bar{b}$
$\Downarrow$
$-b^t = b$

You've found $\operatorname{Lie} O(2n) = \left\{ \begin{bmatrix} a & b \\ \bar{b} & \bar{a} \end{bmatrix} : \begin{matrix} a^* + a = 0 \\ b^t + b = 0 \end{matrix} \right\}$

$n^2 + 2\dfrac{n(n-1)}{2} = 2n^2 - n$ The point now is that a acts on b. You have $U(n)$ acting on skew symmetric complex matrices.

(139) $\begin{bmatrix} a & 0 \\ 0 & \bar a \end{bmatrix} \begin{bmatrix} 0 & b \\ \bar b & 0 \end{bmatrix} - \begin{bmatrix} 0 & b \\ \bar b & 0 \end{bmatrix} \begin{bmatrix} a & 0 \\ 0 & \bar a \end{bmatrix}$

$$= \begin{bmatrix} 0 & ab \\ \bar a \bar b & 0 \end{bmatrix} - \begin{bmatrix} 0 & b\bar a \\ \bar b a & 0 \end{bmatrix} = \begin{bmatrix} 0 & ab - b\bar a \\ \bar a \bar b - \bar b a & 0 \end{bmatrix}.$$

~~So that~~ So if $g \in U(n)$, the action on b complex skew-symm should be $g b \bar g^{-1} = g b g^t$ $\quad (\bar g)^{-1} = \overline{(g^{-1})} = \overline{g^*} = g^t$

Check $(gbg^t)^t = g b^t g^t = g(-b)g^t = -gbg^t$. Also

$\delta(gbg^t) = ab + ba^t = ab - b\bar a$. Seems OK.

Try $\mathcal{L}\,Sp(2n) = \left\{ X \in \mathcal{L}U(2n) \,\middle|\, X = \begin{bmatrix} a & b \\ -\bar b & \bar a \end{bmatrix} : \begin{array}{l} a^* + a = 0 \\ b^t = b \end{array} \right\}$

$$\begin{bmatrix} a & 0 \\ 0 & \bar a \end{bmatrix} \begin{bmatrix} 0 & b \\ -\bar b & 0 \end{bmatrix} - \begin{bmatrix} 0 & b \\ -\bar b & 0 \end{bmatrix} \begin{bmatrix} a & 0 \\ 0 & \bar a \end{bmatrix}$$

$$\begin{bmatrix} 0 & ab \\ -\bar a \bar b & 0 \end{bmatrix} - \begin{bmatrix} 0 & b\bar a \\ -\bar b a & 0 \end{bmatrix} = \begin{bmatrix} 0 & ab - b\bar a \\ -\bar a \bar b - \bar b a & 0 \end{bmatrix}$$

$a * b = ab - b\bar a$ $\qquad a^* = (\bar a)^t = -a \qquad \bar a = -a^t$

$= ab + ba^t$

$= \delta(gbg^t)$ $\qquad (g_1 g_2) b (g_1 g_2)^t = g_1 (g_2 b g_2^t) g_1^t$

$$(ab - b\bar a)^t = b^t a^t - \bar a^t b^t = -b^t \bar a + ab^t = -b\bar a + ab$$

So what's happening? You now should be able to handle the Morse theory for the symmetric spaces $O(2n)/U(n)$ and $Sp(2n)/U(n)$

(140) You now have the Lie algebras of $SO(2n)$ and $Sp(2n)$ understood well enough to see the symmetric spaces $SO(2n)/U(n)$, $Sp(2n)/U(n)$ on the infinitesimal level

$$\mathcal{L}\,SO(2n) = \left\{ X = \begin{bmatrix} a & b \\ \bar b & \bar a \end{bmatrix} : \begin{matrix} a^* + a = 0 \\ b^t + b = 0 \end{matrix} \right\}.$$

$\mathcal{L}U(n)$ is the Lie subalg $\begin{bmatrix} a & 0 \\ 0 & \bar a \end{bmatrix}$ of $\mathcal{L}SO(2n)$ and it acts by bracket on $\begin{bmatrix} 0 & b \\ \bar b & 0 \end{bmatrix}$:

$$\begin{bmatrix} a & 0 \\ 0 & \bar a \end{bmatrix}\begin{bmatrix} 0 & b \\ \bar b & 0 \end{bmatrix} - \begin{bmatrix} 0 & b \\ \bar b & 0 \end{bmatrix}\begin{bmatrix} a & 0 \\ 0 & \bar a \end{bmatrix}$$

$$= \begin{bmatrix} 0 & ab \\ \bar a \bar b & 0 \end{bmatrix} - \begin{bmatrix} 0 & b\bar a \\ \bar b a & 0 \end{bmatrix} = \begin{bmatrix} 0 & ab - b\bar a \\ \bar a \bar b - \bar b a & 0 \end{bmatrix}$$

So $a * b = ab - b\bar a$ should be an inf action of $U(n)$ on complex symmetric matrices. Why?

$(g,b) \mapsto gb(g^t)$ is an action on symm. matrices, rest. to unitary $= ub(u)^t$ $\quad u^* = u^{-1},\ u^t = \bar u^{-1}$

to get $(u,b) \longmapsto ub(\bar u)^{-1}$, inf'lly $\delta(ub(\bar u)) = ab - b\bar a$.

It seems that the action $(u,b) \longmapsto ub\bar u^{-1}$ of unitaries on complex skew-symmetric matrices has nice properties such as a description of the orbits by diagonal matrices.

$n = 1 \qquad U(1) = \{e^{i\theta}\} = \mathbb{T}$ acting trivial on $b = 0$

What is this classification of skew-symm complex b under $b \mapsto ub\bar u^{-1}$? $\qquad u^t = \bar u^{-1} = \overline{u^{-1}} = \overline{u^*}$

$(ub\bar u^{-1})^t = ub^t u^t$. The action is just the usual one on symm. or skewsym form, but rest to unitary

(14 I) So consider a skew symmetric form on a complex vector space V equipped with pos herm form. Do the symmetric case first.

V has pos herm. form given. Simple cases first. $\dim(V) = 2$, V has complex symplectic form given also. Now there's a unique symplectic form on V up to an element of $\mathbb{C}^\times$. The symp form is a nonzero map $\Lambda^2 V \xrightarrow{S} \mathbb{C}$. Choose an orthonormal basis v_1, v_2 for V wrt pos. herm form. Then you get $S v_1 \wedge v_2$. How unique is this number? Another orth basis ~~is linked~~ is linked by a determinant to v_1, v_2 and the det of unitary ~~can~~ be any elt $\in \mathbb{T}$. So it's clear you get a positive real number invariant.

Next consider $\dim V = 1$ equipped with a nonzero symmetric bilinear form $S^2 V \longrightarrow \mathbb{C}$. Try to find an invariant of such a form using the pos herm form. Orth basis of V is a unit vector, two such related by elt of $\mathbb{T}$, the effect of $e^{i\theta}$ on V becomes $e^{2i\theta}$ on $S^2 V$. So again you get a positive real number invariant. $\boxed{u^t = \overline{u^*} = \bar{u}^{-1}}$

Try $\dim V = 2$ with ~~a~~ nondegenerate symmetric form. Looking at $\{b \text{ complex symm.}\}$ with $U(n)$ action $u b u^t$ $\qquad b^t = b$ $\qquad n(n+1) \sim n^2$

Try polar decomp $(ubu^t)^*(ubu^t) = \bar{u} b^* \cancel{u^*} u b u^t$ $\quad \bar{u}^{-1}$

(142) Polar decomp. $(ubu^t)(ubu^t)^* = ub\cancel{u^t\bar{u}}b^*u^*$
$= u(bb^*)u^*$. Also $(ubu^t)^*(ubu^t) = \bar{u}b^*\cancel{u^*u}bu^t = \bar{u}(b^*b)u^t$

$\boxed{(ubu^t)(ubu^t)^* = (ub)(ub)^*}$ $\boxed{(ubu^t)^*(ubu^t) = (bu^t)^*(bu^t)}$

So you have these nice ≥ 0 hermitian operators $u(bb^*)u^*$ and $\bar{u}(b^*b)u^t$ whose eigenvalues are invariants for the U-action of complex symmetric matrices b.

Let b be complex symm $n\times n$ matrix. You want the polar decomposition of b. Suppose b invertible.
You want to find a unitary u and positive diagonal matrix d such that
$ubu^t = d$. If so then one has
$d^2 = (ubu^t)(ubu^t)^* = ubu^t\bar{u}b^*u^* = ubb^*u^*$.

Start again with b complex symmetric. Form bb^* which is >0, diagonalize bb^*; i.e. find u unitary such that $ubb^*u^* = d^2$ where d is a positive diagonal matrix. Then $bb^* = u^*ddu$.

$(bb^*)^t = (b^*)^t b^t = \bar{b}b^t = \bar{b}b^t$??
$(b^*b)^t = b^t(b^*)^t = b^t\bar{b} = b^t\bar{b}$

You would to go from $b \mapsto b^*b$?

You start with the action $u * b = ubu^t$
You want any b to have the form $u\lambda u^t$ where $\lambda > 0$ & diagonal. If $b = u\lambda u^t$, then

$bb^* = u\lambda u^t\bar{u}\lambda u^*$
$= u\lambda^2 u^*$

$(u^*b)(b^*u) = \lambda^2$??

(143) You don't understand yet the eigenvalue theory. Meaning? You ~~consider~~ are studying the symm spaces $SO(2n)/U(n)$ and $Sp(2n)/U(n)$. You have [redacted] an understanding of the Lie theory, namely, [redacted] the Lie algebras $\mathcal{L}SO(2n)$, $\mathcal{L}Sp(2n)$ and their $\pm$ splitting into $\mathcal{L}U(n)$, the isotropic Lie alg, and the complements resp. $\begin{bmatrix} 0 & b \\ \bar{b} & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & b \\ -\bar{b} & 0 \end{bmatrix}$ where $b^t = -b$ in the former, $b^t = b$ — latter.

Symmetry ~~space~~ theory should tell you that there's ~~an abelian~~ a maximal abelian Lie subalg of $\mathcal{L}SO(2n)$ resp $\mathcal{L}Sp(2n)$ invariant under the $\pm$ splitting, that the $+$ part exponentiates to the max torus of $U(n)$, + that the $-$ part exponentiates to the analog of the maximal torus in the symmetric space, which is a [redacted] flat submanifold reversed by the involution. Also [redacted] the conjugates of the $-$ torus under the isotropy group cover the symmetric space. This conjugacy theorem for the $-$Cartan subalg symmetric space is what you call the eigenvalue picture.

How to get started? [redacted] Maybe you need to go back to Killing form. Some sort of minimization. [redacted]

Let's try next reflecting. Go back to $V = \begin{bmatrix} V_+ \\ V_- \end{bmatrix}$, $\varepsilon = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ and a $W \subset V$, let $F = \begin{matrix} +1 & \text{on } W \\ -1 & \text{on } W^\perp \end{matrix}$

Let's discuss where to go. Look at $Sp(2n)/U(n)$

(144) Fix attention on $O(2n,\mathbb{C}) =$

$\{ g \in GL(2n,\mathbb{C}) \mid g^t S g = S \}$, where $S = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

Behind this is the hyperbolic space $\begin{bmatrix} \mathbb{C}^n \\ \mathbb{C}^n \end{bmatrix}$, $\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}^t \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}$

So $O(2n,\mathbb{C})$ has been defined are the group of autos of the hyperbolic space $H(\mathbb{C}^n)$. Next ~~introduce~~ put pos hermitian form on $H(\mathbb{C}^n)$ $\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}^* \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}$

$\boxed{g^*g = 1 \;\&\; g^t S g = S}$

$O(2n) = \{ g \in U(2n) \cap O(2n,\mathbb{C}) \}$. You ~~look~~ at Lie version. $\mathcal{L}O(2n) = \{ X \in gl(2n,\mathbb{C}) \mid \underline{X^* + X = 0,\; X^t S + SX = 0} \}$

$X = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $\quad \begin{bmatrix} a^t & c^t \\ b^t & d^t \end{bmatrix} + \begin{bmatrix} d & c \\ b & a \end{bmatrix} = 0$ $\quad \begin{matrix} b^t + b = 0 \\ c^t + c = 0 \\ d = -a^t \quad a = -d^t \end{matrix}$

$X = \begin{bmatrix} a & b \\ c & -a^t \end{bmatrix}$: $\begin{matrix} b^t + b = 0 \\ c^t + c = 0 \end{matrix}$ $\quad \begin{bmatrix} a^* & c^* \\ b^* & d^* \end{bmatrix} + \begin{bmatrix} a & b \\ c & d \end{bmatrix} = 0$ $\quad \begin{matrix} a^* + a = 0 \\ c^* + b = 0 \\ c + b^* = 0 \\ d^* + d = 0 \end{matrix}$

$\boxed{\begin{matrix} b^t + b = 0 & b^* + c = 0 \\ c^t + c = 0 & b + c^* = 0 \end{matrix}}$

$\bar{b} = c$ $\quad \begin{matrix} d + a^t = 0 \\ d + d^* = 0 \end{matrix}$ $\quad \begin{matrix} a^* + a = 0 \\ a^t + d = 0 \end{matrix}$ $\quad a = \bar{d}$

$\left\{ X = \begin{bmatrix} a & b \\ \bar{b} & \bar{a} \end{bmatrix} : \begin{matrix} a^* + a = 0 \\ b^t + b = 0 \end{matrix} \right\} = \mathcal{L}O(2n)$

<u>You</u> should emphasize the operators on $\begin{bmatrix} x \\ y \end{bmatrix} \in \begin{bmatrix} \mathbb{C}^n \\ \mathbb{C}^n \end{bmatrix}$

What do you want? You would like ~~the~~ a version of the F, ε formalism in the Grassmannian situation. Work near the origin

145 $SO(2n)/U(n)$ = space of orthogonal operators of square $-I$. Try for a Lie picture. First point is that a Lie element is a skew symmetric complex matrix b. Go back to $\left[\begin{array}{c|c} 0 & b \\ \hline \bar{b} & 0 \end{array}\right]$

Can you find the Cartan subalg of $\mathcal{L}\, O(2n)$. Obvious choice is $\left[\begin{array}{c|c} 0 & d \\ \hline & \end{array}\right]$

Try the eigenvalue approach.

$u \in U(n)$ acts on b complex skew symm.

by $u \cdot b = u b u^t$ $\qquad (ubu^t)^t = u b^t u^t = -ubu^t$

$$\begin{bmatrix} u & 0 \\ 0 & \bar{u} \end{bmatrix}\begin{bmatrix} 0 & b \\ \bar{b} & 0 \end{bmatrix}\begin{bmatrix} u^{-1} & 0 \\ 0 & \bar{u}^{-1} \end{bmatrix} = \begin{bmatrix} 0 & ub \\ \bar{u}\bar{b} & 0 \end{bmatrix}\begin{bmatrix} u^{-1} & 0 \\ 0 & \bar{u}^{-1} \end{bmatrix} = \begin{bmatrix} 0 & ub\bar{u}^{-1} \\ \bar{u}\bar{b}u^{-1} & 0 \end{bmatrix}$$

~~[illegible]~~ So you have this nice action of $U(n)$ on comp skew matrices.

Let V be complex vector space equipped with a skew symmetric bilinear form. Let's start with a symplectic form.

So you have a complex v.s. with two structures pos herm + symplectic. Take dim = 2. Choose a line $L \subset V$ $\qquad L \hookrightarrow V \longrightarrow V/L$ orth comp for herm. form. So what's significant $\Lambda^2 V \longrightarrow \mathbb{C}$

V complex vs equipped with $\begin{cases} \text{pos herm form} \\ \text{skew-symmetric } \mathbb{C} \text{ linear form } \omega \end{cases}$

Let $\dim(V) = 2$ so that $\Lambda^2 V$ is 1-diml, and $\omega : \Lambda^2 V \to \mathbb{C}$

Then the pos herm form determines a pos. herm form in $\Lambda^2 V$, namely choose orthonormal basis v_1, v_2 put $\|v_1 \wedge v_2\| = 1$. A different orth basis

(146) changes $v_1 \wedge v_2$ by det of $U(2)$ matrix which $\in \mathbb{T}$. Better: Choose v_1, v_2 orth basis look at $|\omega(v_1, v_2)|$, ind of choice.

[redacted] Interesting "confusion" unitary on b skew-symm belongs to $SO(2n)/U(n)$, unitary on b symm belongs to $Sp(2n)/U(n)$.

IDEA. Conjugacy in the Lie algebra can be proved by minimizing the distance from a generic point in the Cartan subalgebra to the orbit.

Let's examine this idea in simple cases e.g. to [redacted] do spectral thm. for hermitian operators. Distance fn. $tr(A^2)$ diagonal matrix Λ distinct λ_i

You start with A hermitian, then let $u \in U(n)$ act: uAu^{-1} and you want to minimize the distance to this generic Λ.

$$tr(A - u\Lambda u^{-1})^2$$

$$0 = \delta\, tr(u\Lambda u^{-1} - A)^2 = 2tr\left((u\Lambda u^{-1} - A)[\delta u, \Lambda]\right)$$

$$\delta(u\Lambda u^{-1} - A) = (\delta u)\Lambda u^{-1} + u\Lambda(-u^{-1}\delta u\, u^{-1})$$

$$\delta\, tr(\Lambda - u^{-1}Au)^2 = 2tr(\Lambda - u^{-1}Au)\,\delta(\Lambda - u^{-1}Au)$$

$$\delta(u^{-1}Au) = -u^{-1}\delta u\, u^{-1}A + u^{-1}A\delta u$$

147 Consider A hermitian, let Λ be a diagonal matrix with distinct real entries. Let u be a variable unitary matrix. We seek to minimize the distance^2 between A and $u\Lambda u^{-1}$. Let u be a stationary point. Then

$$0 = \delta \tfrac{1}{2} \operatorname{tr} (A - u\Lambda u^{-1})^2 = \operatorname{tr} (A - u\Lambda u^{-1}) \delta (u\Lambda u^{-1})$$

Consider an infinitesimal variation $u + \delta u = u(1+X)$ where $X^* = -X$. Then $\delta u = uX$ and $\delta(u^{-1}) = -u^{-1}\delta u\, u^{-1} =$ $= -Xu^{-1}$, so $\delta(u\Lambda u^{-1}) = uX\Lambda u^{-1} - u\Lambda X u^{-1} = u[X,\Lambda]u^{-1}$.
Since u stationary one has

$$0 = \operatorname{tr}(A - u\Lambda u^{-1})\, u[X,\Lambda]u^{-1} = \operatorname{tr}(u^{-1}Au - \Lambda)[X,\Lambda]$$

for all skewhermitian X.

$$\| \qquad \operatorname{tr} [\Lambda, u^{-1}Au - \Lambda] X$$

Therefore $u^{-1}Au$ centralizes Λ i.e. it's diagonal.

Identity.
$$\operatorname{tr} X[Y,Z] = \operatorname{tr} XYZ - \operatorname{tr} XZY$$
$$\operatorname{tr} [X,Y]Z = \operatorname{tr} XYZ - \operatorname{tr} YXZ$$

Interesting question is the significance of the "eigenvalues" for a ~~symmetric~~ skew symmetric bilinear form under the action of $U(n)$. This should be related to "frequencies."

Let's consider V 2dim over $\mathbb{C}$ w. pos herm. form and with symmetric bilinear form over $\mathbb{C}$. You have 3 complex coefficients for the latter form

(148) Siegel U.H.P. [redacted] generalizes $z = x + iy$ with $y > 0$. Complex symmetric matrices with positive imaginary part

$n^2 + 2\frac{n(n+1)}{2}$

$$\mathcal{L}\, Sp(2n) \cong \left\{ X = \begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix} : \begin{array}{l} a^* + a = 0 \\ b^t = b \end{array} \right\}.$$

Symmetric space (inf picture) is $u \in U(n)$ acting on $b = b^t$ via $u \cdot b = ubu^t$. Why? inf action is

$$\begin{bmatrix} a & 0 \\ 0 & \bar{a} \end{bmatrix}\begin{bmatrix} 0 & b \\ -\bar{b} & 0 \end{bmatrix} - \begin{bmatrix} 0 & b \\ -\bar{b} & 0 \end{bmatrix}\begin{bmatrix} a & 0 \\ 0 & \bar{a} \end{bmatrix}$$

$$\begin{bmatrix} 0 & ab \\ -\bar{a}\bar{b} & 0 \end{bmatrix} - \begin{bmatrix} 0 & +b\bar{a} \\ -\bar{b}a & 0 \end{bmatrix} = \begin{bmatrix} 0 & ab - b\bar{a} \\ -\bar{a}\bar{b} + \bar{b}a & 0 \end{bmatrix}$$

$a \cdot b = ab - b\bar{a}$ $\qquad$ $u \cdot b = ubu^t \Rightarrow$ for $u = 1 + a$

$(1+a)b(1+a^t) - b = ab + ba^t = ab - b\bar{a}$ $\quad \Big| \quad a^t = \overline{a^*} = -\bar{a}$

You keep thinking that the orbit structure of $U(n)$ acting on complex symmetric matrices via $u \cdot b = ubu^t = ub(\bar{u})^{-1}$, should be simple to derive by some induction, something like choosing an orthonormal basis. Here are other methods to try:

1) [redacted] Find the symmetric space analog of the [redacted] the conjugacy theorem for the Lie alg.

2) [redacted] Some version of polar decomposition.

In the latter note that $(ubu^t)(ubu^t)^* = u(bb^*)u^*$ and $(ubu^t)^*(ubu^t) = \bar{u}(b^*b)(\bar{u})^{-1}$, so the spectrum of bb^* and b^*b are invariant of the U orbit of b. Also one has $b^*b = \bar{b}b$ because $b^* = \overline{b^t} = \bar{b}$.

[redacted] So $b^*b = \bar{b}b$ and $bb^* = b\bar{b}$

These are fixed under $*$, and under $-$ and t they go into each other.

Let's now try to understand the conjugacy thm for the symmetric space $Sp(2n)/U(n)$.

IDEA: Don't forget the creation & annihilation operator picture of the hyperbolic spaces (symplectic and quadratic). Simple boson case: $\mathcal{L}\,Sp(2)$ has basis a^*a, $\frac{a^2}{2}$, $\frac{a^{*2}}{2}$.

Look at conjugacy theorem in $\mathcal{L}Sp(2n) = \left\{ X = \begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix} : \begin{matrix} a^* + a = 0 \\ b^t = b \end{matrix} \right\}$ [redacted]

What is the Cartan subalg? It should be a maximal abelian subspace. Take $n = 1$.

Consider $Sp(2)/U(1) = SU(2)/U(1) =$ Riemann sphere

The involution you want is conjugation by ε

$$\varepsilon \begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix} \varepsilon^{-1} = \begin{bmatrix} a & -b \\ +\bar{b} & \bar{a} \end{bmatrix}$$

, the fixed [redacted] sub Lie alg is $\begin{bmatrix} a & 0 \\ 0 & \bar{a} \end{bmatrix}$ (the image of $\mathcal{L}U(1)$).

This situation should be close to $U(2)/U(1) \times U(1)$, so that maybe you can see the Cayley Transform. [redacted] It seems you want $X = \begin{bmatrix} 0 & b \\ -\bar{b} & 0 \end{bmatrix} = \begin{bmatrix} 0 & b \\ -b^* & 0 \end{bmatrix}$, because $-\bar{b} = -\bar{b^t} = -b^*$.

So it seems you should get $g = \frac{1+X}{1-X} \in Sp(2n)$. The reason $g \in Sp(2n)$ is because $g \in U(2n)$ and X and hence g lie in the real subalgebra of $X \in M(2n, \mathbb{C})$ satisfying $\sigma X = \bar{X} \sigma$??

 Review $Sp(2n)$ Objects + Results

$H(\mathbb{C}^n)$, symplectic form, positive herm. form.

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}^t \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} \qquad \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}^* \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}$$ [redacted]

$$Sp(2n) = U(2n) \cap Sp(2n,\mathbb{C}) \subset GL(2n,\mathbb{C}).$$
$$= \{ g \in GL(2n,\mathbb{C}) \mid g^t J g = J,\ g^* g = 1 \}.$$

$$\mathrm{Lie}\, Sp(2n) = \{ X \in M_{2n}\mathbb{C} \mid X^t J + JX = 0,\ X^* + X = 0 \}$$

$$\mathrm{Lie}\, Sp(2n,\mathbb{C}) = \left\{ X = \begin{bmatrix} a & b \\ c & d \end{bmatrix} : \begin{array}{l} d = -a^t \\ b^t = b,\ c^t = c \end{array} \right\}$$

$$\mathrm{Lie}\, Sp(2n) = \left\{ X = \begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix} : \begin{array}{l} a^* = -a \\ b^t = b \end{array} \right\}$$

You've left out the [redacted] $\mathbb{H}$ module structure on $H(\mathbb{C}^n)$

$$\sigma \begin{bmatrix} x \\ y \end{bmatrix} = J \begin{bmatrix} \bar{x} \\ \bar{y} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} \bar{x} \\ \bar{y} \end{bmatrix} = \begin{bmatrix} \bar{y} \\ -\bar{x} \end{bmatrix}$$

$$\sigma \begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \sigma \begin{bmatrix} ax + by \\ -\bar{b}x + \bar{a}y \end{bmatrix} = \begin{bmatrix} -b\bar{x} + a\bar{y} \\ -\bar{a}\bar{x} - \bar{b}\bar{y} \end{bmatrix} = \begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix} \begin{bmatrix} \bar{y} \\ -\bar{x} \end{bmatrix}$$

Better $$\sigma \begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix} \sigma^{-1} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} \bar{a} & \bar{b} \\ -b & a \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -b & a \\ -\bar{a} & -\bar{b} \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix}$$

Let's work on the Cayley transform for $Sp(2n)/U(n)$, [redacted] whatever this means. ?? You mean the conjugacy theorem for the action of $U(n)$ on the space of complex symmetric matrices b given by $u \cdot b = ubu^t$. There should be some sort of "characteristic values" for b.

Look at the case $n=1$ where you have the flag manifold $SU(2)/U(1) = U(2)/U(1)\times U(1) =$ Riemann sphere

[redacted] What's going on?

(151) Discuss what to do. You want to understand the conjugacy thm. for $U(n)$ acting on complex symm matrices via $u \cdot b = ubu^t$.

Alternatively given a complex v.s. with pos. herm. form and complex bilinear symmetric form you want to construct an orthonormal basis which diagonalizes the symm bilinear form.

Let's look at complex bilinear skew symmetric ~~symplectic~~ case first where lines are automatically isotropic. No

Let's look at the case of a 2 diml $V/\mathbb{C}$ with pos herm form and a symmetric $\mathbb{C}$ bilinear form. There are 3 parameters describing symmetric $\mathbb{C}$ bilinear forms. These are $\mathbb{C}$ linear fnls. $S^2V \longrightarrow \mathbb{C}$, ~~and they~~ equiv. elts of $S^2(V^\wedge)$. You can factor them into linear factors.

Let V have coords x, y $\quad V = \begin{bmatrix} x \\ y \end{bmatrix}$

$V = \mathbb{C}^2$

$$\begin{bmatrix} x \\ y \end{bmatrix}^t \begin{bmatrix} a & b \\ b & c \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = ax^2 + 2bxy + cy^2$$

~~So you have this quadra~~ [redacted] ~~form~~ Obvious invariant is the two null lines.

Symm bilinear form $\longmapsto$ quadratic function of $\frac{x}{y} \in$ Riemann sphere.

Have divisor of degree 2 on $\mathbb{P}\mathbb{C}$. Now let

$SU(2)$? ? $\quad \begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix}$ $\quad a^* + a = 0$, $b^t = b$ $\quad n^2 + n(n+1) = 10$

$\dim Sp(4)/U(2) = 6$

Consider $Sp(4)/U(2)$

letting $U(2)$ act on $\begin{bmatrix} 0 & b \\ -\bar{b} & 0 \end{bmatrix}$ via $u \cdot b = ubu^t$

(152) Consider the symmetric space $Sp(2n)/U(n)$.
Maybe begin with $Sp(2n)$ whose Lie algebra is
$\left\{ X = \begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix} : \begin{matrix} a^* + a = 0 \\ b^t = b \end{matrix} \right\}$. On this is an involution given by conjugation with $\varepsilon = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.

First look at Lie $Sp(2n)$. What is the rank, + Cartan subalg?

$$SU(2) = \left\{ \begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix} \in GL(2, \mathbb{C}) \;\middle|\; |a|^2 + |b|^2 = 1 \right\}$$

$$\mathcal{L}\, SU(2) = \text{——— gl ———} \quad a + \bar{a} = 0.$$

Symmetric space picture is clear. Apparently the max torus is in the isotropy group.

Start again with $SU(2) = \left\{ \begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix} \;\middle|\; |a|^2 + |b|^2 = 1 \right\}$

Involution: conjugation with ε
$U(1) =$

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} a & b \\ \bar{b} & -\bar{a} \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} a & -b \\ \bar{b} & \bar{a} \end{bmatrix}$$

It fixes $\left\{ \begin{bmatrix} a & 0 \\ 0 & \bar{a} \end{bmatrix} : |a|^2 = 1 \right\}$ and sends b to $-b$.

You now want the C.T. Let's use $SU(2)/U(1) = U(2)/U(1)\times U(1)$ because you understand a Grassmannian.

Review. For a general Gras. $U(k+l)/U(k)\times U(l)$ the subgroup $U(k)\times U(l)$ is the centralizer of $\varepsilon = \begin{bmatrix} 1_k & 0 \\ 0 & -1_l \end{bmatrix}$, so you know an point F also has k +'s and l −'s.

What is the symmetric space idea? Something to do with midpoint.

$$\varepsilon \longrightarrow F$$

If $g = F\varepsilon$, then

$$g^{1/2}\,\varepsilon\, g^{-1/2} = g^{1/2} g^{+1/2}\varepsilon = g\varepsilon = F.$$ Not clear

Basic property is that each point of the symmetric space has a reflection through the point which is an isometry (or inversion)

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$SU(2) = \left\{ \begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix} : |a|^2+|b|^2=1 \right\} = S^3$

Lie $SU(2) = \underbrace{\qquad\qquad}_{a+\bar{a}=0}$

Involution conj by Σ

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} a & b \\ \bar{b} & -\bar{a} \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} a & -b \\ \bar{b} & \bar{a} \end{bmatrix}$$

Fixes $U(1)$, flips b to $-b$. Get symmetric space $SU(2)/U(1)$ which we can identify with the space of field homomorphisms $\mathbb{C} \to \mathbb{H}$, equivalently $\sqrt{-1}$'s in $\mathbb{H}$, i.e. $xi + yj + zk$ $\quad x^2+y^2+z^2=1$.

$$\begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix}\begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix} = \begin{bmatrix} a^2-|b|^2 & ab+b\bar{a} \\ -(\bar{b}a+\bar{a}\bar{b}) & \bar{a}^2-|b|^2 \end{bmatrix}$$ if $= \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ then

$b(a+\bar{a})=0$. If $b=0$ then $a^2=-1$, $\bar{a}^2=-1 \Rightarrow a \pm i$

If $b \neq 0$, then $a+\bar{a}=0$ so $a=i|a|$, $\bar{a}=-i|a|$, and get

$$\begin{bmatrix} -|a|^2-|b|^2 & 0 \\ 0 & -|a|^2-|b|^2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

So you have a picture of the symmetric space $SU(2)/U(1)$ = Riemann sphere consisting of the elements in $\mathbb{H}$ of square $= -1$.

You would like to generalize this to $Sp(2n)/U(n)$ and perhaps also understand the C.T.

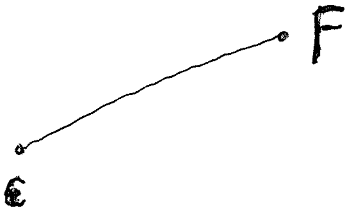
But first finish $SU(2)/U(1) = S^2$. Symmetric space theory $\underbrace{\text{embeds}}_{}$ the space into the group of isometries. By defn for each point of the symmetric space you have an inversion through the point.

Do $U(k+\ell)/U(k)\times U(\ell)$ $\qquad$ conj. by $\Sigma = \begin{bmatrix} 1_k & 0 \\ 0 & -1_\ell \end{bmatrix}$

This is the symmetric space, you have picture using involution $F^2=1$. You want what?

(154) ~~[illegible]~~. Review the Grass case.

ε fixed involution "the origin", F is another inv.
You want to show F and ε are conjugate,
Simplest



$g = F\varepsilon$ assume $g^{1/2}$ defd.
then $g^{1/2}\varepsilon g^{-1/2} = g\varepsilon = F$.

Let's see how this looks for $SU(2)/U(1)$.
Lie picture is $\left\{ \begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix} : a + \bar{a} = 0 \right\}$.

Consider the tangent vector $\begin{bmatrix} 0 & b \\ -\bar{b} & 0 \end{bmatrix}$ at the origin

form the corresp. geodesic, assume given by 1 param. s/gp

$\exp t \begin{bmatrix} 0 & b \\ -\bar{b} & 0 \end{bmatrix}$ You can conjugate by $U(1)$ to assume b real > 0.

and you get $\begin{pmatrix} \cos(bt) & \sin(bt) \\ -\sin(bt) & \cos(bt) \end{pmatrix}$. Alternative is C.T.

Normalization: $t = \pi$, $|b| = 1$ and you get $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$.

C.T. Take $X = \begin{bmatrix} 0 & b \\ -\bar{b} & 0 \end{bmatrix}$ and form $g = \dfrac{1+X}{1-X}$

$$g^{1/2} = \frac{1+X}{\sqrt{1-X^2}} = \begin{bmatrix} 1 & b \\ -\bar{b} & 1 \end{bmatrix} \frac{1}{\sqrt{1+|b|^2}}$$

$$g^{1/2}\varepsilon g^{-1/2} = \begin{bmatrix} 1 & b \\ -\bar{b} & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & -b \\ \bar{b} & 1 \end{bmatrix} \frac{1}{1+|b|^2}$$

$$\begin{bmatrix} 1 & -b \\ -\bar{b} & -1 \end{bmatrix} \begin{bmatrix} 1 & -b \\ \bar{b} & 1 \end{bmatrix} = \begin{bmatrix} 1-|b|^2 & -2b \\ -2\bar{b} & -1+|b|^2 \end{bmatrix}$$

(155) Symm. space $U(2)/U(1)\times U(1) = SU(2)/U(1)$

Involution on $U(2)$ ~~is~~ conj by $\varepsilon = \begin{bmatrix}1&0\\0&-1\end{bmatrix}$.

$$\begin{bmatrix}a&b\\-\bar b&\bar a\end{bmatrix} \longmapsto \begin{bmatrix}a&-b\\\bar b&\bar a\end{bmatrix}.$$ fixpt subgp

is $U(1)$, $u \mapsto \begin{bmatrix}u&0\\0&\bar u\end{bmatrix}$, the centralizer of ε. ~~Tangent space at origin~~

Points of $SU(2)/U(1)$ are the elements of $U(2)$ having square $= -1$.

This is confused. Start with $SU(2)$ and the involution $\begin{bmatrix}a&b\\-\bar b&\bar a\end{bmatrix} \longmapsto \begin{bmatrix}a&-b\\\bar b&\bar a\end{bmatrix}$. The fixed subgroup for the involution is $U(1) = \left\{\begin{bmatrix}a&0\\0&\bar a\end{bmatrix} \;|a|^2=1\right\}$

Note that $\begin{bmatrix}i&0\\0&-i\end{bmatrix} = \begin{bmatrix}i&0\\0&i\end{bmatrix}\begin{bmatrix}1&0\\0&-1\end{bmatrix}$, so that $\begin{bmatrix}i&0\\0&-i\end{bmatrix}$, which $\in SU(2)$, has same centralizer as ε. Therefore

$SU(2)/U(1) = \{g\in SU(2) \mid g^2=-1\}$.

$$\begin{bmatrix}a&b\\-\bar b&\bar a\end{bmatrix}\begin{bmatrix}a&b\\-\bar b&\bar a\end{bmatrix} = \begin{bmatrix}a^2-|b|^2 & \underbrace{(a+\bar a)}_{0}b\\ -\underbrace{(a+\bar a)}_{0}\bar b & \bar a^2-|b|^2\end{bmatrix}$$

$a+\bar a=0$
$a = i|a|$
$\bar a = -i|a|$
$a^2=\bar a^2 = -|a|^2$

It's not clear that $\{g\in SU(2) \mid g^2=-1\}$ is a good description of points of the ~~symm~~ symmetric space $SU(2)/U(1)$.

$\mathrm{Lie}\, SU(2) = \left\{\begin{bmatrix}a&b\\-\bar b&\bar a\end{bmatrix},\ a+\bar a=0\right\}$.

$$\begin{bmatrix}a&b\\-\bar b&\bar a\end{bmatrix} \longrightarrow a+b\sigma$$

Link with $\mathbb{H}$.

$$\sigma\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}0&1\\-1&0\end{bmatrix}\begin{bmatrix}\bar x\\\bar y\end{bmatrix} = \begin{bmatrix}\bar y\\-\bar x\end{bmatrix}$$

$$\sigma\begin{bmatrix}a&b\\-\bar b&\bar a\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}a&b\\-\bar b&\bar a\end{bmatrix}\sigma\begin{bmatrix}x\\y\end{bmatrix}$$

156 The ~~[illegible]~~ good way to do this

$$\sigma\begin{bmatrix}a&b\\-\bar b&\bar a\end{bmatrix}\sigma^{-1} = \begin{bmatrix}0&1\\-1&0\end{bmatrix}(c)\begin{bmatrix}a&b\\-\bar b&\bar a\end{bmatrix}(c)\begin{bmatrix}0&-1\\1&0\end{bmatrix}$$

$$=\begin{bmatrix}0&1\\-1&0\end{bmatrix}\begin{bmatrix}\bar a&\bar b\\-b&a\end{bmatrix}\begin{bmatrix}0&-1\\1&0\end{bmatrix} = \begin{bmatrix}-b&a\\-\bar a&-\bar b\end{bmatrix}\begin{bmatrix}0&-1\\1&0\end{bmatrix} = \begin{bmatrix}a&b\\-\bar b&\bar a\end{bmatrix}$$

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finish everything for $SU(2)/U(2)$. Let $g \in SU(2)$, $g^2 = -1$. Eigenvalues of g are roots $\lambda^2+1=0$, $\lambda = \pm i$. product of eigenvalues is $+1$. $i, -i$ corresponding eigenspace are orthogonal lines in $\mathbb{C}^2$, which can be moved by a unitary u to the coord lines $\begin{bmatrix}\mathbb{C}\\0\end{bmatrix}, \begin{bmatrix}0\\\mathbb{C}\end{bmatrix}$. Then $g = u\begin{bmatrix}i&0\\0&-i\end{bmatrix}u^{-1}$. $\therefore \{g \in SU(2) : g^2 = -1\}$ is the orbit of $\begin{bmatrix}i&0\\0&-i\end{bmatrix}$, orbit = conjugacy class, & centralizer of $\begin{bmatrix}i&0\\0&-i\end{bmatrix}$ is the image of $U(1) \to SU(2)$, $a \mapsto \begin{bmatrix}a&0\\0&\bar a\end{bmatrix}$, $|a|=1$. $\therefore SU(2)/U(1) \cong \{J \in SU(2) \mid J^2 = -1\}$

Why is $SU(2)/U(1)$ a symm. space?

$J^* = J^{-1}$ J unitary $J = -J^* = -J^{-1}$

$J^{-1} = -J$ $J^2 = -1$ $\boxed{-J = J^{-1} = J^*}$

(157) Why is $\underbrace{SU(2)/U(1)}_{= \text{Riemann sphere } S^2}$ a symmetric space?

This means that at each point of S^2 the reflection through the point (defined using the exp. map) is an isometry (whatever this means).

$$S^2 = \{ J \in M_2(\mathbb{C}) \mid -J = J^* = J^{-1} \}$$

Why are you stuck?

J skew adjoint $\iff$ $J = \begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix}$: $a + \bar{a} = 0$

$J \in SU(2) \iff$ " : $|a|^2 + |b|^2 = 1$

Point; $\begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix}$ has $\mathbb{R}$-basis

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix},$$

$$\mathbb{H} = \mathbb{R}1 + \mathbb{R}\hat{i} + \mathbb{R}\hat{j} + \mathbb{R}\hat{k}$$

Somehow you should make clear that $S^2 =$ embeddings of the field $\mathbb{C}$ into the sfield $\mathbb{H}$.

$$q = t + x\hat{i} + y\hat{j} + z\hat{k}, \quad q^* = t - x\hat{i} - y\hat{j} - z\hat{k}$$

$$q q^* = [t + (\quad)][t - (\quad)] = t^2 + x^2 + y^2 + z^2$$

$$q^* q = [t - (\quad)][t + (\quad)]$$

Other point is that

$$\mathcal{L}SU(2) = \left\{ \begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix} : a + \bar{a} = 0 \right\} = \{ x\hat{i} + y\hat{j} + z\hat{k} \}.$$

Maybe you should avoid $\mathbb{H}$.

(158) $\mathcal{L}\, SU(2) = \left\{ \begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix} : a + \bar{a} = 0 \right\}$

$$= k\left\{ = \begin{bmatrix} a & 0 \\ 0 & \bar{a} \end{bmatrix} ; a + \bar{a} = 0 \right\} + p\left\{ \begin{bmatrix} 0 & b \\ -\bar{b} & 0 \end{bmatrix} \right\}$$

Your aim is C.T. $\qquad u \cdot b = u b u^t$

$$\mathcal{L}\, Sp(2n) = \left\{ \begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix} : \begin{matrix} a^* + a = 0 \\ b^t = b \end{matrix} \right\} = k \oplus p$$

eigensp of invo. $a \leftrightarrow a$, $b \leftrightarrow -b$

Today you must clearly understand the C.T.

$$X = \begin{bmatrix} 0 & -b^* \\ b & 0 \end{bmatrix} \in \mathcal{L}(U(2n)) \qquad (+\text{ extra } b^t = b)$$

Is it true that characteristic values decomp. for Grassmannian should be compatible with the $b^t = b$ condition?

So go back to the C.T. formalism.

$\mathrm{Grass}(n, n) = U(2n) / U(n) \times U(n)$. So you have

$$X = \begin{bmatrix} 0 & -b^* \\ b & 0 \end{bmatrix}$$

$b^* = \bar{b}^t = \bar{b}$ extra condition

$$g = \frac{1+X}{1-X} = \left[\frac{1+X}{(1-X^2)^{1/2}} \right]^2 \qquad g^{1/2} = \frac{1+X}{(1-X^2)^{1/2}}$$

Aim? Symmetric space picture with reflections.

$\varepsilon \xrightarrow{\;g^{1/2}\varepsilon\;} F$ $\qquad g = F\varepsilon \qquad (F\varepsilon)\varepsilon = F$

origin $\qquad$ arb. pt.

???

159 You need conjugation?

General Picture. G group, $\sigma: G \to G$ autom s.t. $\sigma^2 = 1$. Let $G^\sigma = \{y \in G \mid \sigma y = y\}$.

map $x \longmapsto x(\sigma x)^{-1}$ from G to itself.

$y \in G^\sigma \Rightarrow xy \longmapsto xy(\sigma(xy))^{-1} = xy(\sigma y)^{-1}(\sigma x)^{-1} = x(\sigma x)^{-1}$

Thus we have a map of G-sets.

$$G/G^\sigma \longrightarrow G \qquad xG^\sigma \longmapsto x(\sigma x)^{-1}$$

In fact if we define G action on itself by $g \cdot x = gx(\sigma g)^{-1}$, then the stabilizer of $x = 1$ is G^σ so $G/G^\sigma \hookrightarrow G$ is the orbit of 1 under this action.

Back to symm. space stuff. G, $\sigma: G \to G$, $\sigma^2 = 1$.

Define $g \cdot x = gx(\sigma g)^{-1}$ action of G on the set G.

Stabilizer of $x = 1$ is $\{g \mid g = \sigma g\} = G^\sigma$, so you have $G/G^\sigma \xrightarrow{\sim} G\text{-orbit of } 1$
$= \{g(\sigma g)^{-1} \in G,\ g \in G\}$.

Let's see what this means for $G = SU(2)$ and $\sigma =$ conjugation by $\varepsilon = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, which is the same as conjugation by $i\varepsilon = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$. $\sigma\left(\begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix}\right) = \begin{bmatrix} a & -b \\ +\bar{b} & \bar{a} \end{bmatrix}$

where $|a|^2 + |b|^2 = 1$. Here $G^\sigma = \left\{\begin{bmatrix} u & 0 \\ 0 & \bar{u} \end{bmatrix}, u \in \mathbb{T}\right\} \simeq U(1)$

You've seen that $G/G^\sigma =$ conjugacy class of $\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} \doteq S^2$

(160) What is $g(\sigma g)^{-1}$ for $g = \begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix}$ $|a|^2+|b|^2=1$. $\sigma g = \begin{bmatrix} a & -b \\ \bar{b} & \bar{a} \end{bmatrix}$. Since $\det(\sigma g) = |a|^2+|b|^2=1$, $(\sigma g)^{-1} = \begin{bmatrix} \bar{a} & b \\ -\bar{b} & a \end{bmatrix}$, so

$$g(\sigma g)^{-1} = \begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix} \begin{bmatrix} \bar{a} & b \\ -\bar{b} & a \end{bmatrix} = \begin{bmatrix} |a|^2-|b|^2 & 2ab \\ -2\bar{a}\bar{b} & |a|^2-|b|^2 \end{bmatrix}.$$

This has the form $\begin{bmatrix} \cos 2\theta & \zeta \sin 2\theta \\ -\bar{\zeta} \sin 2\theta & \cos 2\theta \end{bmatrix}$ with ζ = phase of ab.

I don't see any significance for $g(\sigma g)^{-1}$.

Let's try to improve understanding of the Cayley transform. Especially you want to understand the involution at each point of $SU(2)/U(1) \simeq S^2$. Such a point is a 2×2 complex matrix J such that $-J = J^* = J^{-1}$.

You have the symm. space G/G^σ which is the G orbit of $x=1$ for the action $g \cdot x = g x (\sigma g)^{-1}$. Maybe what you want to do is to form the semi-direct product $\langle \sigma \rangle \ltimes G$. This abstract stuff is too hard

---

Go back to ε, F put $g = F\varepsilon$. Try carefully to understand the symmetric space picture at least with simple examples.

Actually the situation above is familiar, where you tried to find a canonical for a repn of $\langle F, \varepsilon \rangle$

(161) IDEA Earlier stuff about phases occurring with representations of $\langle F, \varepsilon \rangle$. IDEA
This is to be found in the binder from α to ω'''.

You still need a better picture for a symmetric space. Consider the Grassmannian again $V = \begin{bmatrix} V_+ \\ V_- \end{bmatrix}$, $\varepsilon = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$W \hookrightarrow V$, let $F = +1$ on W, -1 on $W^\perp$.

Get $\langle F, \varepsilon \rangle$ acting on V.

Picture of the symmetric space $=$ $U(V)$-orbit of $\varepsilon \simeq U(V) / U(V_+) \times U(V_-)$

What do you want? The reflection at each point of the symmetric space. The reflection $\in U(V)$?

You still need a picture for a symmetric space

Grass example $U(V)/U(V_+) \times U(V_-) \xrightarrow{\sim}$ conjugacy class in $U(V)$ consisting of all $\{F = F^* = F^{-1}\}$ where mult of ± 1 eigenvalue of F $= \dim(V_\pm)$.

First model for Grass symm. space is

$$U(V) / U(V_+) \times U(V_-)$$

2nd model is $\{ g \varepsilon g^{-1} \mid g \in U(V) \} = \{ F \in U(V) \mid F = F^* = F^{-1}$, mult of ± 1 eigen v. same as ε $\}$

next: tangent space to Grass at ε is

$$\mathrm{Hom}_{\mathbb{C}}(V_+, V_-) = \left\{ \begin{bmatrix} 0 & -T^* \\ T & 0 \end{bmatrix} : T : V_+ \to V_- \right\}$$

Same true for any F using $F = \pm 1$ eigenspaces

Summary. You begin with Grass $= \{F\}$. which is a homog. space over $U(V)$, action by gFg^{-1}. Fix basept ε. Pass to Lie picture. Tangent space to Grass at ε is

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Then $\left\{ X = \left[\begin{array}{c|c} 0 & -T^* \\ \hline T & 0 \end{array}\right] \right\}$. You should have a reflection map through ε defined by either

$$\exp X \longmapsto \exp(-X) \quad \text{or by} \quad \frac{1+X}{1-X} \longmapsto \frac{1-X}{1+X}.$$

The actual reflection map should be conjugation by ε restricted to $\exp \mathfrak{p}$

Go back to $SU(2)/U(1)$. The symm. space is the conjugacy class of $\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$, i.e. all $g \in SU(2)$ sat $g^* = g^{-1} = -g$ (i.e. unitaries s.t. $g^2 = -1$)

The basepoint you choose is $i\varepsilon = \begin{bmatrix} i & \\ & -i \end{bmatrix}$.

$$\text{Lie } SU(2) = \left\{ X = \begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix},\ \bar{a} + a = 0 \right\}.$$

$$= \left\{ x \underbrace{\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}}_{\hat{i}} + y \underbrace{\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}}_{\hat{j}} + z \underbrace{\begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}}_{\hat{k}} \right\} \subset \mathbb{H}$$

Summary: In the Grass case you identify the symm. space with the conjugacy class of unitary involutions F of the right "signature". You choose a basepoint ε and consider the tangent space $T_\varepsilon Gr$ with action of the isotropy grp. of ε. $T_\varepsilon Gr = -1$ eigenspace for conjugation by ε.

Next you want to extend the structure on the tangent space at the origin to the symmetric space at least locally.

(163) $\sigma : G \longrightarrow G$ $\quad$ σ homom. $\quad \sigma^2 = 1 \Rightarrow \sigma$ autom.

Let G act on the set G by $g \circ x = g x \sigma(g)^{-1}$.
Then fixpt subgp G^σ = stabilizer of $1 \in G$ so you have
an injection $G/G^\sigma \hookrightarrow G$

$$g G^\sigma \longmapsto g\sigma(g)^{-1}$$

Example: $G = SU(2) = \left\{ \begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix}, |a|^2+|b|^2 = 1 \right\}$. Let σ = conjugation with $\iota\varepsilon = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$: $\sigma \begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix} = \underbrace{\begin{bmatrix} a & -b \\ \bar{b} & \bar{a} \end{bmatrix}}_{\sigma(g)}$. det $= +1$.

So $g\sigma(g)^{-1}$ for $g = \begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix}$ is

$$g\sigma(g)^{-1} = \begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix} \begin{bmatrix} \bar{a} & b \\ -\bar{b} & a \end{bmatrix} = \begin{bmatrix} |a|^2 - |b|^2 & 2ab \\ -2\bar{a}\bar{b} & |a|^2 - |b|^2 \end{bmatrix} = \begin{bmatrix} \cos 2\theta & \xi \sin 2\theta \\ -\bar{\xi} \sin 2\theta & \cos 2\theta \end{bmatrix}$$

where ξ is the phase of ab.

Is there any connection between $g\sigma(g)^{-1}$ and a complex structure: J unitary, $J^2 = -1$?

Return to the Grassmannian of all self adjoint involutions on V having the same multiplicity p for the eigenvalue $+1$. In other words you consider the manifold of subspaces W of dim $= p$ in V. Fix a basepoint ε corresp to $V = \begin{bmatrix} V_+ \\ V_- \end{bmatrix}$.

Organize!! which means maybe distinguishing action versus conjugation and such things. The Grass is a conjugacy class of involutions, so that for two involutions which are conjugate there are isomorphic $+1$ and isomorphic -1 subspaces. This suggests that you might look at twisting somehow so as to allow the glueing of V_+ to V_-.

(164) Let $\mathcal{G}_p$ be the Grassmannian of p-diml subspaces W of V; assume V has pos herm. form. A point of $\mathcal{G}_p$ can be identified with a splitting $V = \begin{bmatrix} W \\ W^\perp \end{bmatrix}$, equivalently the unitary operator F such $F^2 = 1$, $F = \begin{cases} +1 & \text{on } W \\ -1 & \text{on } W^\perp \end{cases}$. $\dim(W) = p$

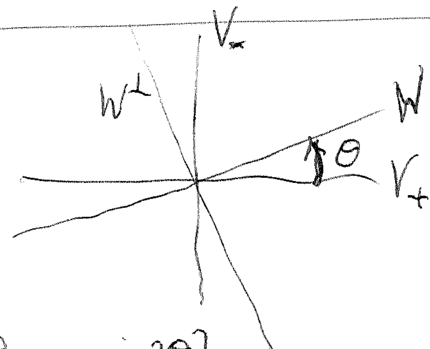
$U(V)$ acts transitively on $\mathcal{G}_p$.

Pick basepoint $V = \begin{bmatrix} V_+ \\ V_- \end{bmatrix}$ $\quad \varepsilon = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.

ε, F two points of $\mathcal{G}_p$. You want to find a unitary conjugating ε to F. Look at $g = F\varepsilon$ which is the difference (mult.)

$\varepsilon g \varepsilon^{-1} = \varepsilon F = g^{-1}$ $\qquad \varepsilon g^{1/2} \varepsilon^{-1} = g^{-1/2}$ assuming some sort of uniqueness.

What did you learn last fall?



$$\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta & -\sin\theta \\ -\sin\theta & -\cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} \cos 2\theta & -\sin 2\theta \\ -\sin 2\theta & -\cos 2\theta \end{bmatrix}$$

ε corresponds to $V = \begin{bmatrix} V_+ \\ V_- \end{bmatrix}$, when you conjugate by $\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ = rotation thru θ you get the splitting $V = \begin{bmatrix} W \\ W^\perp \end{bmatrix}$

Check, let $\rho_\theta = \begin{bmatrix} \cos\theta & +\sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ = rotation through θ.

Then $\rho_\theta \, \varepsilon \, \rho_{-\theta}$ is $= 1$ on $W = \rho_\theta V_+$

(165) Consider a unitary representation on V of the dihedral group $\langle F, \varepsilon \rangle$ such that $\frac{1}{2}(g+g^{-1}) = \frac{1}{2}(F\varepsilon + \varepsilon F)$ is a scalar operator λ.

~~g unitary ⇒ eigenvalues of g are on unit circle.~~

~~If g has eigenvalue $e^{i\theta}$ then $g^{-1} = \varepsilon g \varepsilon^{-1}$ has eigenvalue $e^{i\theta}$, so g~~

g unitary ⇒ eigenvalues of g have form $e^{i\theta}$

& $\frac{1}{2}(g+g^{-1}) = \lambda \implies \lambda = \cos(\theta)$.

$$g^2 - 2\lambda g + 1 = (g-e^{i\theta})(g-e^{-i\theta})$$

$$J = \begin{bmatrix} i\alpha & -\bar{b} \\ b & -i\alpha \end{bmatrix}$$

Recall $J = \begin{bmatrix} a & -\bar{b} \\ b & \bar{a} \end{bmatrix}$ s.t. $J^2 = -1$ when $a + \bar{a} = 0$, $|a|^2 + |b|^2 = 1$

so $a = i\alpha$, $\alpha \in \mathbb{R}$

$$\underbrace{\begin{bmatrix} i\alpha & -\bar{b} \\ b & -i\alpha \end{bmatrix} \begin{bmatrix} 1+\alpha & -i\bar{b} \\ -ib & 1+\alpha \end{bmatrix}}_{\parallel} = \underbrace{\begin{bmatrix} 1+\alpha & -i\bar{b} \\ -ib & 1+\alpha \end{bmatrix} \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}}_{\parallel}$$

$$\begin{bmatrix} i\alpha + i\alpha^2 + i|b|^2 & \cancel{\alpha\bar{b}} - \bar{b} - \cancel{\bar{b}\alpha} \\ b + \cancel{b\alpha} - \cancel{\alpha b} & -i|b|^2 - i\alpha - i\alpha^2 \end{bmatrix} = \begin{bmatrix} i(1+\alpha) & -\bar{b} \\ b & -i(1+\alpha) \end{bmatrix}$$

(166) $SU(2)$ $\quad J = \begin{bmatrix} i\alpha & -\bar{b} \\ b & -i\alpha \end{bmatrix}$: α real, $\alpha^2 + |b|^2 = 1.$

J should be equivalent to ~~an ordered pair of antipodal points~~ a point on S^2 i.e. a line in $\mathbb{C}^2$, it's the eigenspace for $\lambda = i$

$$\begin{bmatrix} i(\alpha-1) & -\bar{b} \\ b & -i(\alpha+1) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0$$

$$(-i)(1-\alpha)x = \bar{b}y$$

$$bx = i(1+\alpha)y$$

$$\begin{matrix} x & i(1+\alpha) \\ y & b \end{matrix} \qquad \begin{bmatrix} 1+\alpha \\ -ib \end{bmatrix}$$

$$\cancel{\frac{y}{x} = \frac{b}{i(1+\alpha)} = \frac{b\bar{b}}{i(1+\alpha)} = \frac{1-\alpha^2}{i(1+\alpha)} \neq \frac{1-\alpha}{i}}$$

$$bx = i(\alpha+1)y \qquad \frac{x}{y} = \frac{i(1+\alpha)}{b} = \frac{1+\alpha}{-ib}$$

$$\frac{(1+\alpha)\bar{b}}{(-ib)\bar{b}} = \frac{i(1+\alpha)\bar{b}}{1-\alpha^2} = \frac{i\bar{b}}{1-\alpha}$$

$$\boxed{\frac{x}{y} = \frac{1+\alpha}{-ib} = \frac{i\bar{b}}{1-\alpha}}$$

~~What else~~

$\frac{1}{2}$ angle. $\quad 1 = \alpha^2 + |b|^2 \qquad \alpha = \cos\theta \qquad b = \zeta \sin\theta$

$$\frac{1+\cos\theta}{\sin\theta} = \frac{2\cos^2\theta/2}{2\sin\theta/2\,\cos\theta/2} = \frac{\cos\theta/2}{\sin\theta/2}$$

(167) $SU(2)$ $\quad J = \begin{bmatrix} i\alpha & -\bar{b} \\ b & -i\alpha \end{bmatrix}$ $\quad \alpha^2 + |b|^2 = 1$, α real

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$S^2 = U(2)/U(1)\times U(1)$ $\qquad \varepsilon = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

$L = \begin{bmatrix} 1 \\ z \end{bmatrix}\mathbb{C} \qquad L^{\perp} = \begin{bmatrix} 1 \\ -\bar{z}^{-1} \end{bmatrix}\mathbb{C} = \begin{bmatrix} -\bar{z} \\ 1 \end{bmatrix}\mathbb{C}$

$$F \begin{bmatrix} 1 & -\bar{z} \\ z & 1 \end{bmatrix} = \begin{bmatrix} 1 & -\bar{z} \\ z & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$F = \begin{bmatrix} 1 & \bar{z} \\ z & -1 \end{bmatrix} \frac{1}{1+|z|^2} \begin{bmatrix} 1 & \bar{z} \\ -z & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1-|z|^2 & 2\bar{z} \\ 2z & -1+|z|^2 \end{bmatrix} \frac{1}{1+|z|^2}$$

Review.

$H(\mathbb{C}^n) = \begin{bmatrix} \mathbb{C}^n \\ \mathbb{C}^n \end{bmatrix} \qquad \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}^* \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}, \quad \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}^t \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}$

pos herm. $\qquad$ symp.

$Sp(2n) \qquad U(2n) \qquad Sp(2n,\mathbb{C})$

$\mathcal{L}\,Sp(2n) = \qquad \mathcal{L}U(2n) = \left\{ \begin{bmatrix} a & b \\ -b^* & d \end{bmatrix} \; \begin{matrix} a + a^* = 0 \\ d + d^* = 0 \end{matrix} \right\}$

$Sp(2n) = \left\{ g \in GL(2n,\mathbb{C}) \;\middle|\; g^* g = 1,\; g^t \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} g = \right.$

$\mathcal{L}\,Sp(2n) \quad \{ X \in M_{2n}\mathbb{C} \mid X^* + X = 0, \quad X^t J + J X = 0 \}$

$= \left\{ \begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix} : \; \begin{matrix} a + a^* = 0 \\ b^t = b \end{matrix} \right\}$

(168) Diversion: Sym case $H(\mathbb{C}^n) = \begin{bmatrix}\mathbb{C}^n\\ \mathbb{C}^n\end{bmatrix} \to \begin{bmatrix}x\\ y\end{bmatrix}$

pos herm. form $\begin{bmatrix}x_1\\ y_1\end{bmatrix}^* \begin{bmatrix}x_2\\ y_2\end{bmatrix}$, $\begin{bmatrix}x_1\\ y_1\end{bmatrix}^t \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\begin{bmatrix}x_2\\ y_2\end{bmatrix}$

$$\sigma\begin{bmatrix}x\\ y\end{bmatrix} = \underbrace{\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}}_{S}\begin{bmatrix}\bar{x}\\ \bar{y}\end{bmatrix} = \begin{bmatrix}\bar{y}\\ \bar{x}\end{bmatrix}$$

$$\mathcal{L}\, O(2n) = \left\{ X = \begin{bmatrix}a & b\\ c & d\end{bmatrix} : X^t S + S X = 0,\ X^* + X = 0 \right\}$$

$$XS = S\bar{X}$$

$$\sigma(X\xi) = S\bar{X}\bar{\xi} \qquad XS = S\bar{X}$$

$$X\sigma\xi = XS\bar{\xi} \qquad \begin{bmatrix}a & b\\ c & d\end{bmatrix} = \begin{bmatrix}\bar{d} & \bar{c}\\ \bar{b} & \bar{a}\end{bmatrix}$$

$$X^t = -SXS \qquad X = \begin{bmatrix}a & b\\ \bar{b} & \bar{a}\end{bmatrix} \qquad \begin{matrix} a + a^* = 0 \\ \bar{b} = -b^*,\ b = -b^t\end{matrix}$$

$$\mathcal{L}\, O(2n) = \left\{ X = \begin{bmatrix}a & b\\ \bar{b} & \bar{a}\end{bmatrix} : \begin{matrix} a + a^* = 0 \\ b^t = -b \end{matrix} \right\} \qquad \begin{matrix} n^2 \\ 2\frac{n(n-1)}{2} \end{matrix}$$

$$a \mapsto -a^t = -(-a^*)^t = \bar{a}$$

Basic problem is how to understand group elements.

$$J = \begin{bmatrix} i\alpha & -\bar{b}\\ b & -i\alpha\end{bmatrix} \qquad \varepsilon J = \begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\begin{bmatrix} i\alpha & -\bar{b}\\ b & -i\alpha\end{bmatrix} = \begin{bmatrix} i\alpha & -\bar{b}\\ -b & i\alpha\end{bmatrix}$$

$$J\varepsilon = \begin{bmatrix} i\alpha & -\bar{b}\\ b & -i\alpha\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix} = \begin{bmatrix} i\alpha & \bar{b}\\ b & i\alpha\end{bmatrix}$$

$$\frac{1}{2}(\varepsilon J + J\varepsilon) = \begin{bmatrix} i\alpha & 0\\ 0 & i\alpha\end{bmatrix}$$

Question: Does it make sense to have a $J \in \mathcal{L}Sp(2n)$ s.t. $-J = J^* = J^{-1}$?

$$J^* = \begin{bmatrix} a^* & -b\\ \bar{b}^* & a^t\end{bmatrix} = -\begin{bmatrix} a & b\\ -\bar{b} & \bar{a}\end{bmatrix}$$

$$J = \begin{bmatrix} a & b\\ -\bar{b} & \bar{a}\end{bmatrix} \qquad \begin{matrix} a + a^* = 0 \\ b^t = b\end{matrix}$$

$a^* = -a \implies a^* + a = 0$

$a^t = -\bar{a} \implies a^* = -a$

this is automatically skew adjoint

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$$i\varepsilon = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} \qquad J = \begin{bmatrix} i\alpha & -\bar{b} \\ b & -i\alpha \end{bmatrix}$$

$$i\varepsilon J = \begin{bmatrix} -\alpha & -i\bar{b} \\ -ib & -\alpha \end{bmatrix}$$

$$J i\varepsilon = \begin{bmatrix} i\alpha & -\bar{b} \\ b & -i\alpha \end{bmatrix}\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} = \begin{bmatrix} -\alpha & i\bar{b} \\ ib & -\alpha \end{bmatrix}$$

$$\therefore \tfrac{1}{2}\left[(i\varepsilon)J + J(i\varepsilon)\right] = \begin{bmatrix} -\alpha & 0 \\ 0 & -\alpha \end{bmatrix}$$

Recap. Studying $SU(2)/U(1)$ the symmetric space given by the Riemann sphere. Given $L \subset \mathbb{C}^2$ pick $L = \begin{bmatrix} i\alpha \\ b \end{bmatrix}\mathbb{C}$ $\quad \alpha^2 + |b|^2 = 1$, α real.

Then $J = \begin{bmatrix} i\alpha & -\bar{b} \\ b & -i\alpha \end{bmatrix}$ satisf. $-J = J^* = J^{-1}$

$J \in \mathcal{L}\, SU(2) \qquad J \in SU(2)$.

$$J^* = \begin{bmatrix} -i\alpha & \bar{b} \\ -b & i\alpha \end{bmatrix} \qquad J^{-1} = \begin{bmatrix} -i\alpha & \bar{b} \\ -b & i\alpha \end{bmatrix}\frac{1}{\alpha^2+|b|^2}$$

Other approaches: Use $\mathcal{L}$ Symp. is S^2V

$$X^t\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}X = 0 \qquad X^t\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \overset{?}{=} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}X$$

$$X^t = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}X\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} a^t & c^t \\ b^t & d^t \end{bmatrix} = \begin{bmatrix} c & d \\ -a & -b \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} -d & c \\ b & -a \end{bmatrix}$$

$c^t = c \qquad d = -a^t$
$b^t = b$

$$\left(X^t\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\right)^t = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}X$$

(170) $\dot{p} = -\dfrac{\partial H}{\partial q} \qquad \dot{q} = \dfrac{\partial H}{\partial p}$

$$H = \tfrac{1}{2} p^t m^{-1} p + \tfrac{1}{2} q^t k q = \tfrac{1}{2}\begin{bmatrix} p \\ q \end{bmatrix}^t \begin{bmatrix} m^{-1} & 0 \\ 0 & k \end{bmatrix}\begin{bmatrix} p \\ q \end{bmatrix}$$

$$\dot{p} = -kq \qquad \dot{q} = m^{-1}p$$

$$X\begin{bmatrix} p \\ q \end{bmatrix} = \begin{bmatrix} -kq \\ m^{-1}p \end{bmatrix} = \begin{bmatrix} 0 & -k \\ m^{-1} & 0 \end{bmatrix}\begin{bmatrix} p \\ q \end{bmatrix}$$

Symplectic form is

$$\begin{bmatrix} p_1 \\ q_1 \end{bmatrix}^t \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} p_2 \\ q_2 \end{bmatrix}$$

First you need ~~the~~ multiplicative X as a "difference" between H A.

Let V be phase space $\begin{bmatrix} p \\ q \end{bmatrix} \in V$. $\qquad X : V \longrightarrow V$

$H, A : V \longrightarrow V^*$

$$\boxed{AX = H}$$

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 0 & -k \\ m^{-1} & 0 \end{bmatrix} = \begin{bmatrix} m^{-1} & 0 \\ 0 & k \end{bmatrix}$$

$$X = \underset{A^{-1}}{\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}}\underset{H}{\begin{bmatrix} m^{-1} & 0 \\ 0 & k \end{bmatrix}} = \begin{bmatrix} 0 & -k \\ m^{-1} & 0 \end{bmatrix}.$$

Checks? X resps. A, H.

$X^t A + AX = 0$?

$X^t H + HX = 0$?

$X^t A X + A X X.$

$AX = H = H^t \qquad H = X^t A^t = X^t A$

$X^t A = H \qquad AX = H$