① Review

$$\underbrace{\begin{bmatrix}\bar{C}^0\\ H_1\end{bmatrix}}_{K} \underset{\begin{bmatrix}1\\1\end{bmatrix}}{\overset{\begin{bmatrix}1\\-1\end{bmatrix}}{\hookrightarrow}} \underbrace{\begin{bmatrix}C^1\\ C_1\end{bmatrix}}_{S} \underset{\begin{bmatrix}1 & -1\end{bmatrix}}{\overset{\begin{bmatrix}1 & 1\end{bmatrix}}{\twoheadrightarrow}} \underbrace{\begin{bmatrix}H^1\\ \bar{C}_0\end{bmatrix}}_{S/K = K^*}$$

$$\begin{tikzcd} & S \\ & \Omega \arrow[u] \arrow[r, "\sim"] & S/K \end{tikzcd}$$

$$\Omega \ni \begin{bmatrix}R & 0\\ 0 & 0\\ 1 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}I_R\\ I_e\end{bmatrix} = \begin{bmatrix}RI_R\\ 0\\ I_R\\ I_e\end{bmatrix}$$ typical elt of Ω.

goes to

$$\begin{bmatrix}1 & 1 & 0 & 0\\ 0 & 0 & 1 & -1\end{bmatrix}\begin{bmatrix}RI_R\\ 0\\ I_R\\ I_e\end{bmatrix} = \begin{bmatrix}RI_R\\ I_R - I_e\end{bmatrix} \in \begin{bmatrix}H^1\\ \bar{C}_0\end{bmatrix}$$

You have isom $\quad \Omega \xrightarrow{\sim} \begin{bmatrix}H^1\\ \bar{C}_0\end{bmatrix}$

$$\begin{bmatrix}RI_R\\ 0\\ I_R\\ I_e\end{bmatrix} \longmapsto \begin{bmatrix}RI_R\\ I_R - I_e\end{bmatrix} = \begin{bmatrix}U\\ J\end{bmatrix} \in \begin{bmatrix}H^1\\ \bar{C}_0\end{bmatrix}$$

What is the inverse map.

$$\begin{bmatrix}1 & 1 & 0 & 0\\ 0 & 0 & 1 & -1\end{bmatrix}\begin{bmatrix}R & 0\\ 0 & 0\\ 1 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}I_R\\ I_e\end{bmatrix} = \begin{bmatrix}R & 0\\ 1 & -1\end{bmatrix}\begin{bmatrix}I_R\\ I_e\end{bmatrix} = \begin{bmatrix}U\\ J\end{bmatrix}$$

Then $\quad \begin{bmatrix}I_R\\ I_e\end{bmatrix} = \frac{1}{+R}\begin{bmatrix}+1 & 0\\ +1 & -R\end{bmatrix}\begin{bmatrix}U\\ J\end{bmatrix}$

(2)

$$\begin{bmatrix} R & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \frac{1}{R} \begin{bmatrix} 1 & 0 \\ \text{[redacted]} & -R \end{bmatrix} \begin{bmatrix} u \\ J \end{bmatrix} = \begin{bmatrix} R & 0 \\ 0 & 0 \\ 1 & 0 \\ 1 & -R \end{bmatrix} \frac{1}{R} \begin{bmatrix} u \\ J \end{bmatrix}$$

Go over what you want. You have this sub-space [redacted] $\Omega \hookrightarrow \begin{bmatrix} C^1 \\ C_1 \end{bmatrix}$. [redacted]

$$\begin{tikzcd} \Omega \arrow[r, hook] \arrow[dr, "\sim"'] & \begin{bmatrix} C^1 \\ C_1 \end{bmatrix} \arrow[d] \\ & \begin{bmatrix} H^1 \\ \bar{C}_0 \end{bmatrix} \end{tikzcd}$$

So Ω splits into $\begin{bmatrix} R \\ 0 \\ 1 \\ 1 \end{bmatrix} \frac{1}{R}$ and $\begin{bmatrix} 0 \\ 0 \\ 0 \\ -1 \end{bmatrix}$

You guess that $\begin{bmatrix} u & u' \\ v' & v \end{bmatrix} : \begin{bmatrix} \bar{C}^0 \\ H_1 \end{bmatrix} \longleftarrow \begin{bmatrix} H^1 \\ \bar{C}_0 \end{bmatrix}$

[redacted]

$$\Omega \ni \begin{bmatrix} R & 0 \\ 0 & 0 \\ 1 & 0 \\ 1 & -R \end{bmatrix} \frac{1}{R} \begin{bmatrix} u \\ J \end{bmatrix} \longleftarrow\!\shortmid \begin{bmatrix} u \\ J \end{bmatrix} \in \begin{bmatrix} H^1 \\ \bar{C}_0 \end{bmatrix}$$

$$\begin{tikzcd} \bar{C}^0 \arrow[r, hook] & C^1 \arrow[r, two heads] & H^1 \\ & \Omega \arrow[u] \arrow[d] \arrow[r, "\sim"] & \begin{bmatrix} H^1 \\ \bar{C}_0 \end{bmatrix} \arrow[u] \arrow[d] \\ H_1 \arrow[r, hook] & C_1 \arrow[r, two heads] & \bar{C}_0 \end{tikzcd}$$

(3) apparently you now have the two splittings for the voltage + current s.e.s.

$$\bar{C}^0 \xrightarrow{\begin{bmatrix}1\\-1\end{bmatrix}} C^1 \xleftarrow{\begin{bmatrix}1\\0\end{bmatrix}} H^1$$

$$H_1 \xrightarrow{\begin{bmatrix}1\\1\end{bmatrix}} C_1 \xleftarrow{\begin{bmatrix}0\\-1\end{bmatrix}} \bar{C}_0$$

$$\begin{bmatrix}R & 0\\ 0 & 0\\ 1 & 0\\ 0 & 1\end{bmatrix}\frac{1}{R}\begin{bmatrix}1 & 0\\ 1 & -R\end{bmatrix} = \begin{bmatrix}R & 0\\ 0 & 0\\ 1 & 0\\ 1 & -R\end{bmatrix}\frac{1}{R}\begin{bmatrix}u\\ J\end{bmatrix} \longleftarrow\!\shortmid \begin{bmatrix}u\\ J\end{bmatrix}$$

$$\Omega \xleftarrow{\sim} \begin{bmatrix}H^1\\ \bar{C}_0\end{bmatrix}$$

$$\begin{bmatrix}u & u'\\ v' & v\end{bmatrix}$$ it looks like $v' = \frac{1}{R} : H^1 \to H_1$

There should be a better way to do all this. Today [redacted] you want a better approach

Principles: Importance of Hodge decomposition

Idea of adjoining a zero resistance branch [redacted] in order to handle an external node voltage. ~~Ifor~~ ~~[illegible]~~ Basic object seems to be a dual pair of voltage-current s.e.s. together with ~~an Ω on the tot~~ a Lagrangian subspace ~~transverse~~ complementary to the Kirchhoff space

~~[illegible]~~

④ Start with $A \hookrightarrow B \twoheadrightarrow C$

$C^* \hookrightarrow B^* \twoheadrightarrow A^*$

You need something on $\begin{bmatrix} A \\ A^* \end{bmatrix}$

Review augmenting your graph.

$$\begin{tikzcd}
\bar{C}^0 K \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\text{cocart}"] & C^1 K \arrow[r, two heads] \arrow[d] & H^1 K \arrow[d, equal] \\
V' \arrow[r, hook] & V \arrow[r] & H^1 K
\end{tikzcd}$$

Discuss general case with external node

$$R \twoheadleftarrow \bar{C}^0 \hookrightarrow C^1 \twoheadrightarrow H^1$$

$$R \hookrightarrow \bar{C}_0 \twoheadleftarrow C_1 \hookleftarrow H_1$$

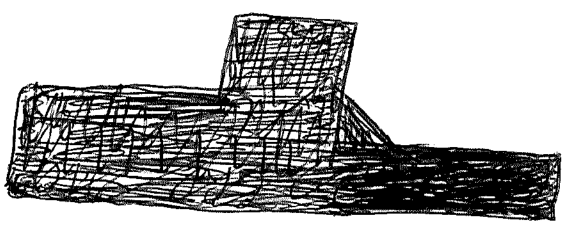
From this you should get

$$\bar{C}^0 \hookrightarrow \begin{bmatrix} C^1 \\ \mathbb{R} \end{bmatrix} \twoheadrightarrow \tilde{H}^1$$

$$\bar{C}_0 \twoheadleftarrow \begin{bmatrix} C_1 \\ \mathbb{R} \end{bmatrix} \hookleftarrow \tilde{H}_1$$

You want

$\tilde{\Omega} = -\Omega$



(5) Look at K-theory aspects. Consider quadratic ~~forms~~ or symplectic spaces direct sum operation.

First point should be that you are concerned with retracts of hyperbolic spaces.

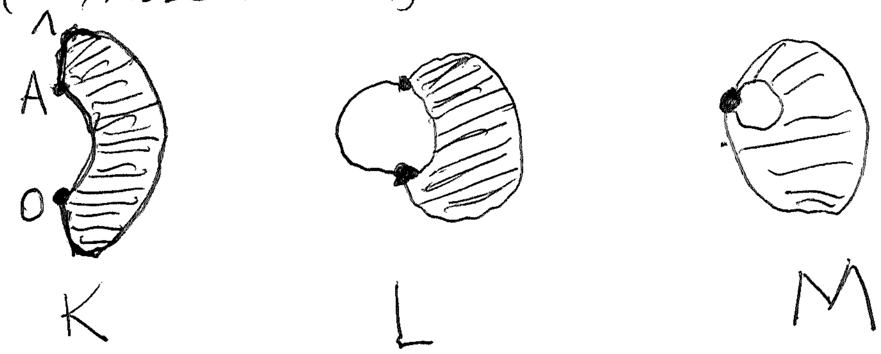
Begin with symmetric case: a v.s. V equipped with a symm. map $T: V \longrightarrow V^*$ which is non degenerate i.e. invertible. Too abstract. Begin with real case i.e. invertible symmetric matrix

In the case of $\mathbb{R}$ you get the $\boxed{\text{signature!}}$

$$\begin{bmatrix} x \\ y \end{bmatrix}^t \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = x^2 - y^2 = (x+y)(x-y)$$

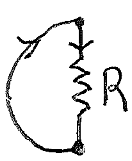
to show this equivalent to $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

Let's return to external node forcing. This means you have a connected R-network and ~~too~~ two nodes A, O



In the case of a single resistance

You've handled



which is the graph L.

(6) Program: to understand hermitian K-theory. This is a kind of algebraic K-theory, it turns out. It's related to surgery obstructions on non-simply-conn manifolds. [redacted]

Examples? Start with real vector spaces (f.d) with nondegenerate quadratic form: $V \xrightarrow{T} V^*$ $\quad T = T^t$
T invertible, basic invariant: the signature, define by choosing a pos. def. scalar product on V, so T becomes a symmetric operator which is invertible, then $V = V_+ \oplus V_-$ given by the phase of T: $\varepsilon = T(T^2)^{-1/2}$. ~~Then you have~~ [redacted]

What is the alg. theory interpretation of the preceding? This means some sort of stabilization or Grothendieck group. You have notions of $\oplus$, and retract, and isomorphism. These should lead to a Grothendieck group, which is iso to $\mathbb{Z}$ via the signature.

What should be true?
On one hand you have real quadratic spaces (V,T) V splits canonically into $V = V_+ \oplus V_-$, $T = T_+ \oplus T_-$ where $T_+ > 0$, $T_- < 0$. Over $\mathbb{R}$ you have pos. square roots. Over $\mathbb{Q}$ not so. But you have a nice embedding thm:

$$(V,T) \oplus (V,-T) \simeq H(V).$$

So if you invert hyperbolic quadratic spaces you get a Grothendieck group. Wait

(7) Discuss $\mathbb{R}$-vector spaces V with nondeg symm. bilinear form. Have $\oplus$ defined such that the two summands are orthogonal. Have isom. notion.

Choose pos. def. scalar product, quad form becomes symmetric operator. Use polar decomp to make operator an involution. Iso classes are pairs $(m_+, m_-) \in \mathbb{N} \times \mathbb{N}$. ~~Next you kill hyperbolic~~

Next: hyperbolic quadratic forms

$$\begin{bmatrix} V \\ V^* \end{bmatrix} \text{ with } \begin{bmatrix} v_1 \\ \varphi_1 \end{bmatrix}^t \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} v_2 \\ \varphi_2 \end{bmatrix} = v_1^t \varphi_2 + \varphi_1^t v_2$$

If $T : V \longrightarrow V^*$ sk symmetric.

$$0 = \begin{bmatrix} v_1 \\ T v_1 \end{bmatrix}^t \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} v_2 \\ \varphi_2 \end{bmatrix} = v_1^t \begin{bmatrix} 1 & T^t \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} v_2 \\ \varphi_2 \end{bmatrix}$$

$$\forall v_1 \quad 0 = \begin{bmatrix} 1 & T^t \end{bmatrix} \begin{bmatrix} \varphi_2 \\ v_2 \end{bmatrix} = \varphi_2 + T^t v_2$$

$$\therefore \left(\begin{bmatrix} 1 \\ T \end{bmatrix} V \right)^\perp = \left\{ \begin{bmatrix} v_2 \\ \varphi_2 \end{bmatrix} \,\middle|\, \varphi_2 = -T^t v_2 \right\}$$

$$= \left\{ \begin{bmatrix} v_2 \\ -T^t v_2 \end{bmatrix} \,\middle|\, v_2 \in V \right\} = \begin{bmatrix} 1 \\ -T^t \end{bmatrix} V$$

get $\begin{bmatrix} 1 \\ T \end{bmatrix} V$ isotropic $\Leftrightarrow$ $T = -T^t$

Basic statement should say $(V, T) \oplus (V, -T)$ is hyperbolic.

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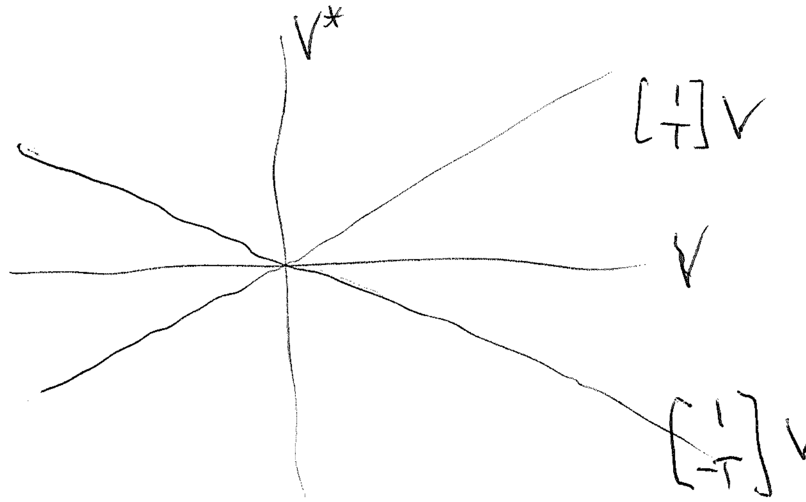

the quad form on $\begin{bmatrix}1\\T\end{bmatrix}V$ is isom. to $2T$ on V, the quad form on $\begin{bmatrix}1\\-T\end{bmatrix}V$ is $-2T$ and $\begin{bmatrix}1\\T\end{bmatrix}V$, $\begin{bmatrix}1\\-T\end{bmatrix}V$ are $\perp$, when T sym

$$\begin{bmatrix}1\\T\end{bmatrix}^t \begin{bmatrix}0&1\\1&0\end{bmatrix}\begin{bmatrix}1\\-T\end{bmatrix} = \begin{bmatrix}1 & T^t\end{bmatrix}\begin{bmatrix}-T\\1\end{bmatrix} = -T+T^t = 0$$

$$\begin{bmatrix}1\\T\end{bmatrix}^t \begin{bmatrix}0&1\\1&0\end{bmatrix}\begin{bmatrix}1\\T\end{bmatrix} = T+T^t = 2T \quad \text{invertible}$$

$$\therefore \begin{bmatrix}1\\T\end{bmatrix}V \oplus \begin{bmatrix}1\\-T\end{bmatrix}V \cong \text{hyperbolic form}$$

Look at K-theory: real vector bundles E over X equipped with a family of non deg quadratic forms on fibres. You seem to get (up to isom.) a v.b. with splitting $E_+ \oplus E_-$. Hyperbolic E should be $E = \begin{bmatrix}V\\V\end{bmatrix}$ where $E_+ = E_- = V$. If you kill hyperbolic E, this should be the same as adding the same bundle to E_+, E_-. $\therefore$ K-gp. is $KR(X)$.

sesquilinear form $f(v_1, v_2)$ Conjugate linear in v_1 and $\mathbb{C}$ linear in v_2. $v_2 \longmapsto f(\cancel{v}, v_2)$

$V \longrightarrow \bar{V}^* = V^\dagger$. Alternatively it's a real bilinear form s.t. $f(Jv_1, v_2) = -J f(v_1, v_2)$

$f(v_1, Jv_2) = f(v_1, v_2)\, J$

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where to go? You might try Clifford algs.

Cliff$_n$ Cliff$(\mathbb{R}^n)$ generated by

$x_1, \dots, x_n$ $x_i^2 = -1$ $x_i x_j + x_j x_i = 0$ $i \neq j$.

$C_0 = \mathbb{R}$, $C_1 = \mathbb{C}$, $C_2 = \mathbb{H}$?

complex case $\mathbb{C}$ $\mathbb{C} \times \mathbb{C}$ $M_2 \mathbb{C}$

You feel that there's something significant in hermitian K-theory that is different from ordinary K-theory. It involves a different kind of stabilizing, a different kind of Grothendieck group.

Let's study an example.

Let's first get the hermitian theory straight. You want to consider a ~~comp~~ $\mathbb{C}$-vector space W together with a sesquilinear form $f(w_1, w_2)$ on W, which means

$$f(c_1 w_1, c_2 w_2) = \bar{c}_1 f(w_1, w_2)\, c_2 \quad \therefore \text{ conj linear in } w_1, \text{ linear in } w_2$$

For example a pos herm. scalar product $\langle w_1, w_2 \rangle$

$f : \bar{W} \otimes W \longrightarrow \mathbb{C}$ same as $\bar{W} \longrightarrow W^*$

$$w_1 \longmapsto (w_2 \longmapsto f(w_1, w_2))$$

$f : \overline{W} \otimes W \to \mathbb{C}$ same as $W \longrightarrow (\overline{W})^* = W^\dagger$
$w_2 \longmapsto (w_1 \longmapsto f(w_1, w_2))$

So if $p : \overline{W} \otimes W \to \mathbb{C}$ is pos. def. herm symm form

$p : W \longrightarrow W^\dagger$

$p^t : \overline{W} \longrightarrow W^*$ $\qquad \overline{p^t} : W \longrightarrow W^\dagger$

herm symm means $p^\dagger = \overline{p^t} = p$. So if you have

$f : W \longrightarrow W^\dagger$ and $p : W \xrightarrow{\sim} W^\dagger \xleftarrow{f} W$

get $p^{-1}f : W \longleftarrow W$ $\qquad (p^{-1}f)^\dagger = f^\dagger p^{-1}$??

Back to the idea that there is something special about herm. K-theory, namely, that stabilization seems to be done via the hyperbolic functor.

hermitian symmetric form on a $\mathbb{C}$-vector space W.
<u>Pick basis</u> w_j. Then $h(w_i, w_j)$ is a herm. matrix.
$\overline{h(w_i, w_j)} = h(w_j, w_i)$. The basis w_i yield a positive definite scalar product $\langle w_i | w_j \rangle = \delta_{ij}$
So now you basically understand the hermitian vector space category. Next require h_{ij} nonsing. + you get polarization

Having done hermitian stuff over a pt. you might look at what happens over a connected X. Get a polarized v.s.

(11) Complex K-theory, spaces U, $\mathbb{Z}\times BU$, Fredholm operators. Key steps like identifying unitaries $\equiv -1$ mod K with s.a. Fred essential spectrum ± 1. Even version? $U = \Omega(\mathbb{Z}\times BU)$. This should be easy because of some geometric fibration, but it's trickier, involves Calkin algebra. Fred instead of $\mathbb{Z}\times BU$, the restricted Grass.

Begin with a simple example: the Witt group for real quadratic spaces. Def: Take abelian monoid M of iso classes of real quadratic spaces. Look at monoid homomorphisms $M \longrightarrow A$, A abelian group, such that all hyperbolic classes in M go to 0, and take universal A. $W(\mathbb{R})$ to be the. $W(\mathbb{R})$ should be some quotient of M. So the question is when are two quad spaces V_1, V_2 the same in W? Probably means that $V_1 \oplus (-V_2)$ is stably hyperbolic: $V_1 \oplus (-V_2) \oplus H(P_1) \simeq H(P_2)$ for some v.s. P_1, P_2. Simpler might be that

$$[V_1] = [V_2] \iff \exists\, P_1, P_2 \text{ s.t. } V_1 \oplus H(P_1) \simeq V_2 \oplus H(P_2).$$

Obviously the same.

So now you see some sort of infinite orthogonal group arising. Take $P = \mathbb{R}^n$, form $H(\mathbb{R}^n)$, and take its automorphisms. Say $n=1$. $H(\mathbb{R}) = \begin{bmatrix} \mathbb{R} \\ \mathbb{R} \end{bmatrix}$ with $\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}^t \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}$

$= x_1 y_2 - y_1 x_2$. There should be some connection

(12) with special relativity, because you have signature $1,-1$ (Lorentzian) on $\mathbb{R}^2$. Want $g \in GL_2(\mathbb{R})$ to satisfy $g^t \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} g = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Clearly $\det(g) = \pm 1$, so restrict to $+1$, let $g = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$$g^t = \begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} g^{-1} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} -b & d \\ a & -c \end{bmatrix}$$

If $\det = -1$ then $a = d = 0$

$\boxed{bc = \pm 1}$

$$= \begin{bmatrix} a & -c \\ -b & d \end{bmatrix} \quad \therefore \; b = c = 0, \quad ad = 1$$

$O(1,1)$

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can you describe orthogonal matrices on $H(\mathbb{R}^n)$ by some version of $g^t g = 1$?

$$g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} d & c \\ b & a \end{bmatrix}$$

want

$$g^t \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} g \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} a^t & c^t \\ b^t & d^t \end{bmatrix} \begin{bmatrix} d & c \\ b & a \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ [redacted]}$$

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} b & a \\ d & c \end{bmatrix} = \begin{bmatrix} d & c \\ b & a \end{bmatrix}$$

[redacted] Look at Lie algebra (keeping C.T. in mind)

$$X^t \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} X = 0$$

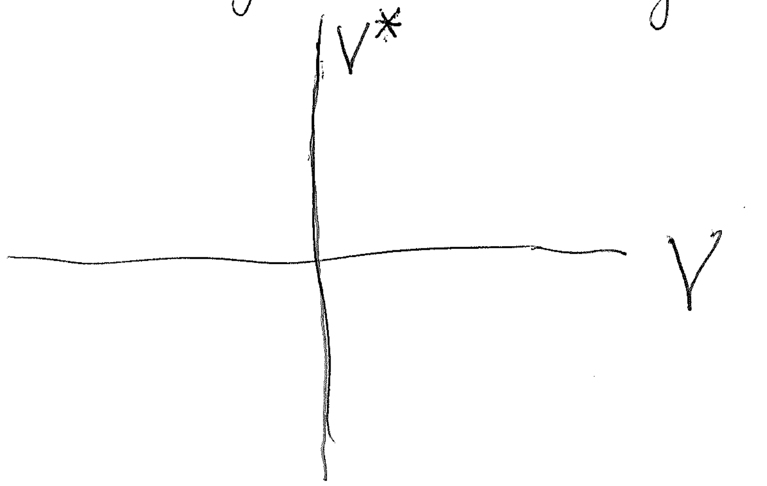
(13)
$$\begin{bmatrix}\alpha&\beta\\\gamma&\delta\end{bmatrix}^t\begin{bmatrix}0&1\\1&0\end{bmatrix}+\begin{bmatrix}0&1\\1&0\end{bmatrix}\begin{bmatrix}\alpha&\beta\\\gamma&\delta\end{bmatrix}=0$$

$$\underbrace{\begin{bmatrix}\alpha^t&\gamma^t\\\beta^t&\delta^t\end{bmatrix}\begin{bmatrix}0&1\\1&0\end{bmatrix}}$$

$$\begin{bmatrix}\gamma^t&\alpha^t\\\delta^t&\beta^t\end{bmatrix}+\begin{bmatrix}\gamma&\delta\\\alpha&\beta\end{bmatrix}=0$$

$\gamma^t+\gamma=0$, $\beta^t+\beta=0$, $\alpha+\delta^t=0$, $\alpha^t+\delta=0$.

you encountered this as a kind of skew-symmetric matrix condition. You're looking at it the wrong way.



What viewpoint? $GL_{2n}\mathbb{R}$ = inv. elts. of $M_{2n}\mathbb{R}$ $= M_2(M_n\mathbb{R})$. At the moment you want to understand the condition $g^t\begin{bmatrix}0&1\\1&0\end{bmatrix}g=\begin{bmatrix}0&1\\1&0\end{bmatrix}$, and its inf. version: $X^t\begin{bmatrix}0&1\\1&0\end{bmatrix}+\begin{bmatrix}0&1\\1&0\end{bmatrix}X=0$. This is a kind of skew symmetry condition. Compare with $g^tg=1$ and $X^t+X=0$.

Maybe you should look at $Sp(2n,\mathbb{R})$. Simplest is $n=1$ where you should get $SL(2,\mathbb{R})$. Say $g\in M_2\mathbb{R}$ $g=\begin{bmatrix}a&b\\c&d\end{bmatrix}$. $g\in Sp(2,\mathbb{R})$ means

(14) $g^t\begin{pmatrix}0&1\\-1&0\end{pmatrix}g=\begin{bmatrix}0&1\\-1&0\end{bmatrix}$, $\begin{bmatrix}a&c\\b&d\end{bmatrix}\begin{bmatrix}0&1\\-1&0\end{bmatrix}\begin{bmatrix}a&b\\c&d\end{bmatrix}=$

$=\begin{bmatrix}a&c\\b&d\end{bmatrix}\begin{bmatrix}c&d\\-a&-b\end{bmatrix}=$ [redacted] $\begin{bmatrix}0&\det(g)\\-\det(g)&0\end{bmatrix}\overset{?}{=}\begin{bmatrix}0&1\\-1&0\end{bmatrix}$

so it's clear. Lie alg: $X^t\begin{bmatrix}0&1\\-1&0\end{bmatrix}+\begin{bmatrix}0&1\\-1&0\end{bmatrix}X=0$

$$\begin{bmatrix}\alpha^t&\gamma^t\\\beta^t&\delta^t\end{bmatrix}\begin{bmatrix}0&1\\-1&0\end{bmatrix}+\begin{bmatrix}0&1\\-1&0\end{bmatrix}\begin{bmatrix}\alpha&\beta\\\gamma&\delta\end{bmatrix}=$$

$$=\begin{bmatrix}-\gamma^t&\alpha^t\\-\delta^t&\beta^t\end{bmatrix}+\begin{bmatrix}\gamma&\delta\\-\alpha&-\beta\end{bmatrix}=0\iff \gamma=\gamma^t,\ \beta^t=\beta$$
$$\alpha^t=-\delta \quad \alpha=-\delta^t$$

[redacted] This is the same as [redacted] $\begin{bmatrix}0&1\\-1&0\end{bmatrix}\begin{bmatrix}\alpha&\beta\\\gamma&\delta\end{bmatrix}=\begin{bmatrix}\gamma&\delta\\-\alpha&-\beta\end{bmatrix}$ being symmetric

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IDEA: Karoubi's $\lambda+\bar\lambda=1$ condition suggests [redacted] (or is, or should be) a partition of unity condition. Could this be linked to embedding a quadratic space in a trivial (i.e. hyper[redacted]bolic) quadratic space? If you embed a quadratic space into a hyperbolic one, are there interesting families of such embeddings?

Recall that to embed a vector bundle E as a retract of a trivial vector bundle is the same as writing, expressing 1_E as a nuclear map $\left(E\otimes_{\mathcal{O}}E^\vee\longrightarrow \mathrm{Hom}_{\mathcal{O}}(E,E)\right)$

More precisely, choosing $\xi_i\in\Gamma(E)$, $\eta_i\in\Gamma(E^\vee)$, $i=1,\dots,N$ such that $\sum_i \xi_i\,\eta_i = id_E$.

(15) Q: Is there a variant of this for quadratic [redacted] spaces, or ~~vector~~ quadratic bundles, which involves the hyperbolic functor? You need to understand the embedding-as-a-retract process. What do you know? If you are given a quadratic space $(V, T : V \to V^*)$, you know there's a direct embedding [redacted] of (V,T) into $H(V)$. Review $H(V) = \begin{bmatrix} V \\ V^* \end{bmatrix}$, $\begin{bmatrix} v_1 \\ \varphi_1 \end{bmatrix}^t \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} v_2 \\ \varphi_2 \end{bmatrix} = v_1^t \varphi_2 + \varphi_1^t v_2$

Then $\begin{bmatrix} V \\ V^* \end{bmatrix} \xleftarrow{\begin{bmatrix} 1 & 1 \\ T & -T \end{bmatrix}} \begin{bmatrix} V \\ V \end{bmatrix}$ [redacted] is invertible when T is

$$\begin{bmatrix} 1 & 1 \\ T & -T \end{bmatrix}^t \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ T & -T \end{bmatrix} = \begin{bmatrix} 1 & T^t \\ 1 & -T^t \end{bmatrix} \begin{bmatrix} T & -T \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2T & 0 \\ 0 & -2T \end{bmatrix}$$

This means that $\begin{bmatrix} 1 \\ T \end{bmatrix} V$ is a quadratic subspace of $H(V)$

also $\begin{bmatrix} 1 \\ -T \end{bmatrix} V$, and these are orthogonal complements.

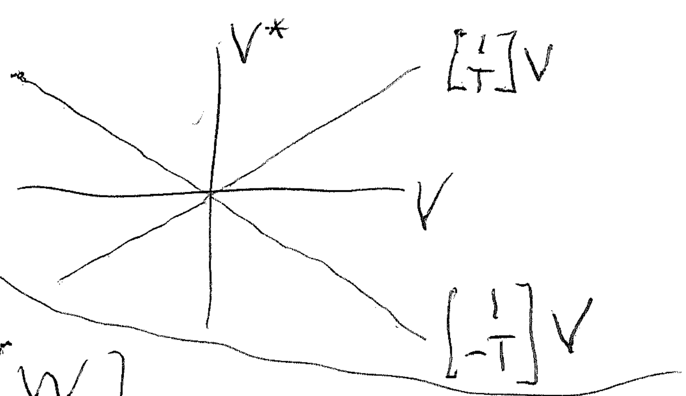
Check: $0 = \begin{bmatrix} 1 \\ T \end{bmatrix}^t \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} v \\ \varphi \end{bmatrix} = \begin{bmatrix} 1 & T^t \end{bmatrix} \begin{bmatrix} \varphi \\ v \end{bmatrix} = \varphi + T^t v$

$\varphi + T v = 0$ means $\begin{bmatrix} v \\ \varphi \end{bmatrix} = \begin{bmatrix} 1 \\ -T \end{bmatrix} v$

need $\begin{bmatrix} 1 & 1 \\ T & -T \end{bmatrix}^{-1} = \frac{1}{-2T} \begin{bmatrix} -T & -1 \\ -T & 1 \end{bmatrix} = \frac{1}{2T} \begin{bmatrix} T & 1 \\ T & -1 \end{bmatrix}$

$$\begin{bmatrix} 1 & 1 \\ T & -T \end{bmatrix} \begin{bmatrix} T & 1 \\ T & -1 \end{bmatrix} = \text{[redacted]} \begin{bmatrix} 2T & 0 \\ 0 & 2T \end{bmatrix}$$

(16) Picture



Question. Given a quadratic space (V,T), can you describe possible embeddings of V into $H(W)$. You want

$$\begin{tikzcd}
V \arrow[r, "{\left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right]}"] \arrow[d, "T"'] & \begin{bmatrix}W\\ W^*\end{bmatrix} \arrow[d, "{\left[\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right]}"] \\
V^* & \begin{bmatrix}W^*\\ W\end{bmatrix} \arrow[l, "{[\alpha^t\ \beta^t]}"']
\end{tikzcd}$$

$$T = [\beta^t\ \alpha^t]\begin{bmatrix}0&1\\1&0\end{bmatrix}\begin{bmatrix}\alpha\\ \beta\end{bmatrix}$$

$$= [\alpha^t\ \beta^t]\begin{bmatrix}\beta\\ \alpha\end{bmatrix} = \alpha^t\beta + \beta^t\alpha$$

Claim then is that an embedding of a quadratic space (V,T) into $H(W)$ is the same as a pair of maps $\alpha : V \to W$, $\beta : V \to W^*$ such that $\begin{bmatrix}\alpha\\ \beta\end{bmatrix}^t\begin{bmatrix}0&1\\1&0\end{bmatrix}\begin{bmatrix}\alpha\\ \beta\end{bmatrix} = \alpha^t\beta + \beta^t\alpha = T.$

So it seems that to embed (V,T) in a hyperbolic space, you first choose $h : V \to V^*$ such that $h + h^t = T$. Then choose

$$\begin{tikzcd}
V \arrow[rr, "h"] \arrow[dr, "\alpha"'] & & V^* \\
& W \arrow[ur, "\beta^t"'] &
\end{tikzcd}$$

Then factor h into $V \xrightarrow{\alpha} W \xrightarrow{\beta^t} V^*$. Seems amazingly simple.

Now $h = \frac{1}{2}(T + X)$ where X is skew symm.

so you want to factor

$$\begin{tikzcd}
V \arrow[dr, "\alpha"] \arrow[dd, "h"'] & \\
& W \arrow[dl, "\beta^t"] \\
V^* &
\end{tikzcd}$$

and then you get an embedding of V into $H(W)$.

(17) Review Bott periodicity, maybe the ABShapiro paper, or the AS paper where Kuiper's thm. is used. First look at the Clifford algs.

Sequence of Clifford algs. with mult. properties yielding Thom class. In the complex case, V a $\mathbb{C}$ vector space, form ΛV $\forall v \in V$ get a complex by exterior multiplication, acyclic $v \neq 0$. Over $V - \{0\}$ as a top space you have a K-class.

Consider complex vectors bundles E over X compact equipped with a nonsingular hermitian (symmetric) form. If you choose a pos herm. form on E, the first form becomes a hermitian operator on E, nonsingular, so there's a canonical splitting $E = E_+ \oplus E_-$. The bundle theory you have is pairs of v.b. or polarized v.b.

It should be true that the set of iso classes of these hermitian vector bundles is $K_0(X) \oplus K_0(X)$. NO for X connected you get $\coprod_{p,q \geq 0} Vect_p(X) \times Vect_q(X)$, where $Vect_p(X)$ mean iso classes of vb of rank p. Next want action of hyperbolic bundles.

There's a hyperbolic functor from vb + isos to nonsing. herm. v.b. + isos. Look at this from the viewpoint of $\varinjlim_n O(H(\mathbb{C}^n))$. Also there's Novikov's Hamiltonian formalism which perhaps is the appropriate tool for handling orthog + symp autos.

At this point I would like to understand better how to handle external voltage + current (?) sources from Thevenin's picture.

(18) Recall idea that the hyperbolic functor might give rise to a sort of triple. The "free" quadratic spaces should be the hyperbolic ones. So yesterday you learned about embedding quadratic spaces as retracts of hyperbolic ones.

$$\begin{tikzcd} V \arrow[r, "\begin{bmatrix}\alpha\\ \beta\end{bmatrix}"] \arrow[d, "T"'] & \begin{bmatrix} W \\ W^* \end{bmatrix} \arrow[d, "\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}"] \\ V^* & \begin{bmatrix} W^* \\ W \end{bmatrix} \arrow[l, "{[\alpha^t\ \beta^t]}"'] \end{tikzcd}$$

$T = \alpha^t\beta + \beta^t\alpha$. So to get (W, α, β) you need an operator h s.t. $h + h^t = T$, which you factor:

$$\begin{tikzcd} V \arrow[r, "\alpha"] \arrow[d, "h"'] & W \arrow[dl, "\beta^t"] \\ V^* & \end{tikzcd}$$

There's a minimal factorization which is unique up to canon. isom. h itself has the form $h = \frac{1}{2}(T+X)$ where $X = -X^t : V \longrightarrow V^*$ can be arbitrary. Obvious embedding $h = \frac{1}{2}T$ and $\begin{bmatrix}\alpha \\ \beta\end{bmatrix} = \begin{bmatrix} 1 \\ \frac{1}{2}T \end{bmatrix} \Rightarrow \beta^t\alpha = \frac{1}{2}T = h.$

Review triple stuff: F free, left adj; G forget, right adj.

$$\mathrm{Hom}(FX, Y) = \mathrm{Hom}(X, GY)$$

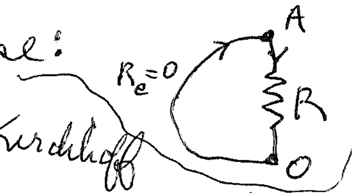
$\alpha : FGY \to Y$

$\beta : X \longrightarrow GFX$

$$\left.\begin{array}{l} GY \xrightarrow{\beta \cdot G} GFGY \xrightarrow{G \cdot \alpha} GY \\ FX \xrightarrow{F \cdot \beta} FGFX \xrightarrow{\alpha \cdot F} FX \end{array}\right\} = \text{identity}$$

Is it possible to take $F : W \longmapsto H(W)$ and for G to take (V, T) to V?

(19) Back to circuit theory, aim to handle external nodes. First review the calculations, get the Hodge decomposition in the case:



conditions:
$V_R + V_e = 0$
$I_R - I_e = 0$

4 vbles V_R, I_R, V_e, I_e subject to 2 Kirchhoff

2 Ohm conditions $V_R = R I_R$, $V_e = 0$. These 4 conditions have only 0 solution $\Longrightarrow$ $S = K \oplus \Omega$ where recall

$$\underset{K}{\underset{\parallel}{\begin{bmatrix} \bar{C}^0 \\ H_1 \end{bmatrix}}} \hookrightarrow \underset{S}{\underset{\parallel}{\begin{bmatrix} C^1 \\ C_1 \end{bmatrix}}} \twoheadrightarrow \underset{S/K \cong K^*}{\underset{\parallel}{\begin{bmatrix} H^1 \\ \bar{C}_0 \end{bmatrix}}}$$

$$\begin{tikzcd} K \arrow[r, hook] & S \arrow[r, "\pi"] & S/K \\ & \Omega \arrow[u, phantom, "\cup"] \arrow[ur, "\int"] & \end{tikzcd}$$

A state, i.e. point of S is given by a 4 vector $\begin{bmatrix} V_R \\ V_e \\ I_R \\ I_e \end{bmatrix} \in \begin{bmatrix} C^1 \\ \\ C_1 \end{bmatrix}$

A point of S/K is a 2-vector $\begin{bmatrix} U \\ J \end{bmatrix}$. A point of K is a 2-vector $\begin{bmatrix} \varphi \\ I \end{bmatrix}$. The voltage and current s.e.s.'s are

$$\varphi \longmapsto \begin{bmatrix} 1 \\ -1 \end{bmatrix} \varphi$$

$$\bar{C}^0 \xhookrightarrow[\begin{bmatrix} 1 \\ -1 \end{bmatrix}]{} C^1 \xrightarrow{[1\ 1]} H^1 \qquad H_1 \xhookrightarrow{\begin{bmatrix} 1 \\ 1 \end{bmatrix}} C_1 \xrightarrow{[1\ -1]} \bar{C}_0$$

The surjection π gives the values for any state of the Kirchhoff constraints.

$$\begin{bmatrix} V_R \\ V_e \\ I_R \\ I_e \end{bmatrix} \xmapsto{\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} V_R \\ V_e \\ I_R \\ I_e \end{bmatrix} = \begin{bmatrix} V_R + V_e \\ I_R - I_e \end{bmatrix} = \begin{bmatrix} U \\ J \end{bmatrix}$$

Ω is the graph of $\begin{bmatrix} I_R \\ I_e \end{bmatrix} \longmapsto \begin{bmatrix} R & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} I_R \\ I_e \end{bmatrix} = \begin{bmatrix} R I_R \\ 0 \end{bmatrix}$

$$\therefore \Omega = \left\{ \begin{bmatrix} R & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} I_R \\ I_e \end{bmatrix}, \ \forall \begin{bmatrix} I_R \\ I_e \end{bmatrix} \right\}$$

(20) π applied to an elt of Ω is

$$\begin{bmatrix}1&1&0&0\\0&0&1&-1\end{bmatrix}\begin{bmatrix}R&0\\0&0\\1&0\\0&1\end{bmatrix}\begin{bmatrix}I_R\\I_e\end{bmatrix} = \begin{bmatrix}R&0\\1&-1\end{bmatrix}\begin{bmatrix}I_R\\I_e\end{bmatrix} = \begin{bmatrix}U\\J\end{bmatrix}$$

The inverse of $\begin{bmatrix}I_R\\I_e\end{bmatrix} \longmapsto \begin{bmatrix}U\\J\end{bmatrix}$ is $\begin{bmatrix}I_R\\I_e\end{bmatrix} = \frac{1}{+R}\begin{bmatrix}+1&0\\+1&-R\end{bmatrix}\begin{bmatrix}U\\J\end{bmatrix}$

The projection of S onto Ω with kernel K is

$$\frac{1}{R}\begin{bmatrix}R&0\\0&0\\1&0\\0&1\end{bmatrix}\begin{bmatrix} &0\\ &-R\end{bmatrix}\begin{bmatrix}1&1&0&0\\0&0&1&-1\end{bmatrix}$$

$$\frac{1}{R}\begin{bmatrix}R&0\\0&0\\1&0\\0&1\end{bmatrix}\begin{bmatrix}1&1&0&0\\1&1&-R&R\end{bmatrix} = \frac{1}{R}\begin{bmatrix}R&R&0&0\\0&0&0&0\\1&1&0&0\\1&1&-R&R\end{bmatrix}$$

$$= \begin{bmatrix}1&1&0&0\\0&0&0&0\\R^{-1}&R^{-1}&0&0\\R^{-1}&R^{-1}&-1&1\end{bmatrix}$$

Thus $\begin{bmatrix}0&-1&0&0\\0&1&0&0\\-R^{-1}&-R^{-1}&1&0\\-R^{-1}&-R^{-1}&1&0\end{bmatrix}$ is the proj of S onto K with kernel Ω

The sum of these two matrices is I, so they commute, and their product is zero.

Now what's the response to the forcing term given by $V_e = -\mathcal{E}$ and $V_R, I_R, I_e = 0$.

$$\begin{bmatrix}0&-1&0&0\\0&1&0&0\\-R^{-1}&-R^{-1}&1&0\\-R^{-1}&-R^{-1}&1&0\end{bmatrix}\begin{bmatrix}0\\-\mathcal{E}\\0\\0\end{bmatrix} = \begin{bmatrix}\mathcal{E}\\-\mathcal{E}\\R^{-1}\mathcal{E}\\R^{-1}\mathcal{E}\end{bmatrix} = \begin{bmatrix}V_R\\V_e\\I_R\\I_e\end{bmatrix}$$

This is what you expect.

$$\begin{bmatrix}\bar{C}^0\\ H_1\end{bmatrix} \hookrightarrow \begin{bmatrix}C^1\\ C_1\end{bmatrix} \twoheadrightarrow \begin{bmatrix}H^1\\ \bar{C}_0\end{bmatrix}$$

$$K \hookrightarrow S \twoheadrightarrow S/K = K^*$$

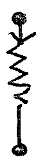
Symplectic analogy of a short exact sequence

Where to start:

Consider a connected R-network with one external node pair. picture

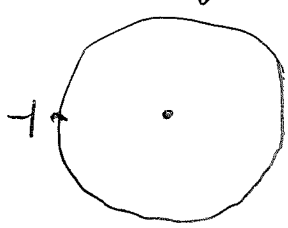
$$\begin{tikzcd}
\mathbb{R} & \bar{C}^0 \arrow[l, two heads] \arrow[r, hook] & C^1 \arrow[r, two heads] & H^1 \\
\mathbb{R} \arrow[r, hook] & \bar{C}_0 & C_1 \arrow[l, two heads] \arrow[u, "S"', "R"] & H_1 \arrow[l, hook']
\end{tikzcd}$$

Is there a surgery viewpoint you might develop?

The graph is  R $\quad \bar{C}^0, C^1$ dim 1, $H^1 = 0$

Review Clifford modules + K-theory, and Bott periodicity proof by AS using Kuiper's theorem + quasi-fibration arguments.

Complex case first. Ungraded case. There are 2 spaces of type $U = \varinjlim U(n)$. First there is the space of unitary operators on Hilb space H which are $\equiv -1$ mod $\mathcal{K}$. $\quad U(H) \cap (-1 + \mathcal{K})$ unitary ops with ess sp $\{-1\}$.



2nd self adjoint contractions $-1 \leq A \leq 1$ s.t. essential spectrum $= \{\pm 1\}$.

Have wrapping map $[-1, 1] \longrightarrow S^1 \quad t \mapsto e^{i\pi t}$

Get map $\{A = A^* \mid -1 \leq A \leq 1,\ A^2 - 1 \in \mathcal{K}\} \to \{U \mid u^* = u^{-1},\ U + 1 \in \mathcal{K}\}$

Filtration by mult of eigenvalue $+1$ for U, 0 for A

(22) $\mathcal{F}_k$ where mult $= k$. Fibre of $A \mapsto U$ over a point of $\mathcal{F}_k$ should be same for $\mathcal{A}$ + $\mathcal{U}$?

Take $k=0$ i.e. you look at A's invertible and $U \ni U+1$ inv. You can always deform the nonzero spectrum to $\{\pm 1\}$ [redacted] for A and -1 for U. Such an A corresponds to a polarization of H, and these [redacted] form a contractible space by Kuiper.

Next you want to bring in Clifford algebras, or really Clifford modules. In the complex case, the basic object is a $\mathbb{Z}/2$ graded vector space $V_+ \oplus V_-$ equipped with n anticommuting odd involutions.

Discuss: You get a Clifford algebra from a v.s. V equipped with a symmetric bilinear form (v,v'). $C(V)$ is the alg gen by V module the relation $vv = (v,v)$

$$(v+v', v+v') = (v,v) + (v',v) + (v,v') + (v',v')$$

$$(v+v')(v+v') = v^2 + v'v + vv' + (v')^2$$

$$\therefore \quad v'v + vv' = 2(v,v') \qquad \text{so } v \perp v' \Rightarrow vv' + v'v = 0.$$

Other facts. [redacted] $C(V)$ is $\mathbb{Z}/2$-graded with degree v odd. Tautological action of $C(V)$ on ΛV $\quad v \mapsto e_v + \iota_v$ where ι_v is interior product wrt $v' \mapsto \iota_v v' = (v,v')$.

$$(e_v + \iota_v)^2 = e_v\iota_v + \iota_v e_v = (v,v)$$

$$(e_v\iota_v + \iota_v e_v)(\omega) = \cancel{e_v \iota_v \omega} + \underbrace{\iota_v(v\wedge\omega)}_{(\iota_v v)\omega \,-\, \cancel{v\wedge \iota_v\omega}}$$

$$= (v,v)\,\omega.$$

There is an increasing filtration (like for Weyl alg, + universal env. algs.) [redacted] of $C(V)$ such [redacted] that there is a canonical alg surj

$$\Lambda V \twoheadrightarrow \operatorname{gr} C(V)$$

If I remember OK, one can show this alg surj is an isomorphism using the action of $C(V)$ on ΛV.

Next look at Clifford modules. C_n is the Clifford alg assoc to the n dim'l space $\mathbb{C}^n$ equipped the diagonal bilinear form $(v, v') = \sum_{i=1}^{n} v_i v_i'$. If ξ_i is the standard basis for $\mathbb{C}^n$ and s_i = the operator in C_n corresp. to $\xi_i \in \mathbb{C}^n$, then $s_i^2 = 1$ and $s_i s_j + s_j s_i = 0$ $i \neq j$.

So a graded C_n-module E is a $\mathbb{Z}/2$ graded vector space equipped with n anti commuting odd involutions.

Clearly a graded C_n-module = an ungraded C_{n+1}-module, where the n+1 st involution is the grading involution ε.

Next you want to link Clifford modules and periodicity in the setting of AS. Where to start? You have reviewed the ungraded case identifying (up to heq) self-adjoint Fredholm contractions and unitaries $\equiv -1 \bmod \mathcal{K}$. Try to understand the graded case.

The basic object is a Fredholm operator $V_+ \xrightarrow{F} V_-$ between Hilbert spaces. Better would be an odd operator F on a graded Hilbert space, such that $F^2 - 1 \in \mathcal{K}$. (This is analogous to the condition $A^2 - 1 \in \mathcal{K}$ treated above.)

It looks like you want $V_+ \oplus V_-$ to be a C_n-Clifford module, and the F to be some lifting which will present obstructions.

Consider the typical Fredholm operator case, where you have a graded C_0-module Hilbert space: $H = H_+ \oplus H_-$ and an odd self-adjoint operator F on H such that $F^2 - I \in \mathcal{K}$. You ultimately want F to be an odd self-adjoint contraction: $F = \begin{bmatrix} 0 & \alpha^* \\ \alpha & 0 \end{bmatrix}$ where $\alpha : H_+ \longrightarrow H_-$ satisfies $\alpha^*\alpha, \alpha\alpha^* \leq I$, so that $I - F^2 \geq 0$ and $\in \mathcal{K}$. Then you have the

(24) graded case of an $F=F^*$, [redacted] $1-F^2 \geq 0$ and $\in \mathcal{K}$.

Discuss $n=1$: Consider a graded C_1-module Hilbert space; this should be the same as a single Hilbert space H.

[redacted] i.e. $\begin{bmatrix} H \\ H \end{bmatrix}$ with $s_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $\varepsilon = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

Next you need F to satisfy $F = F^*$, $0 \leq 1 - F^2 \in \mathcal{K}$, which should be the same as F being a self-adjoint contraction with essential spectrum $\{\pm 1\}$.

Points arising: Everything should be set up to exploit Clifford periodicity, which should be a kind of Morita equivalence. Contractible components "left to the reader." Grassmannian example.

Begin with graded C_0 modules $\begin{bmatrix} V_+ \\ V_- \end{bmatrix}$

The Groth group is $\mathbb{Z} \oplus \mathbb{Z}$.

graded C_1-modules same as modules over the alg with generators s, ε $\quad s^2 = \varepsilon^2 = 1 \quad s\varepsilon + \varepsilon s = 0$

$$\underset{s}{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}} \underset{\varepsilon}{\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \quad \Bigg| \quad \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \underset{\varepsilon}{\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}} \underset{s}{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}$$

$\therefore \; s\varepsilon = -\varepsilon s$ So a graded C_1 module is a graded module $\begin{bmatrix} V_+ \\ V_- \end{bmatrix}$, $\varepsilon = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ together with an odd operator $s \ni s^2 = 1$. $\therefore$ mult by s gives an iso $V_+ \cong V_-$. So graded C_1 modules up to isom. are $\begin{bmatrix} V \\ V \end{bmatrix}$ with $s = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $\varepsilon = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. K grp C_1 is $\mathbb{Z}$

(25) Another point is using the super $\otimes$ for superalgebras: $C_1 \tilde{\otimes} C_1 = C_2$?

$$\boxed{\cancel{\begin{aligned}(s\otimes 1)(1\otimes s_2) &= s_1\otimes s_2 \\ (1\otimes s_2)(s_1\otimes 1) &= -s_1\otimes s_2\end{aligned}}} \qquad \begin{aligned} s_1 &= s\otimes 1 \\ s_2 &= 1\otimes s\end{aligned}$$

$$s_1 s_2 = (s\otimes 1)(1\otimes s) = s\otimes s, \quad s_2 s_1 = (1\otimes s)(s\otimes 1) = -s\otimes s.$$

[redacted] So C_2 has basis $1, s_1, s_2, s_1 s_2$ with relations $s_1^2 = 1$, $s_2^2 = 1$, $s_1 s_2 = -s_2 s_1$

You need to get over this Clifford obstruction. A place to begin is with the [redacted] model for $\mathbb{Z}\times BU$ given by the graded version of $\{F \mid F = F^*, [redacted] 0 \le 1 - F^2 \in \mathcal{K}\}$. That is, you have a graded Hilbert space $H = H_+ \oplus H_-$, $\varepsilon = \begin{bmatrix} 1 & 0 \\ 0 & -1\end{bmatrix}$ and $F = \begin{bmatrix} 0 & \alpha^* \\ \alpha & 0\end{bmatrix}$ is an odd self-adj contraction, because

[redacted] $F^2 = \begin{bmatrix} \alpha^*\alpha & 0 \\ 0 & \alpha\alpha^*\end{bmatrix} \le \begin{bmatrix} 1 & 0 \\ 0 & 1\end{bmatrix}$. Also $\begin{matrix} 1-\alpha^*\alpha \\ 1-\alpha\alpha^*\end{matrix} \in \mathcal{K}$.

so you see that α is unitary modulo $\mathcal{K}$. What is the spectral picture of such an F?

$$\begin{bmatrix} 0 & \alpha^* \\ \alpha & 0\end{bmatrix}\begin{bmatrix} \xi \\ \eta\end{bmatrix} = \lambda\begin{bmatrix} \xi \\ \eta\end{bmatrix} \qquad \begin{aligned} \alpha^*\eta &= \lambda\xi \\ \alpha\xi &= \lambda\eta \end{aligned} \qquad \begin{aligned}\alpha^*\alpha\xi &= \lambda\alpha^*\eta \\ &= \lambda^2\xi\end{aligned}$$

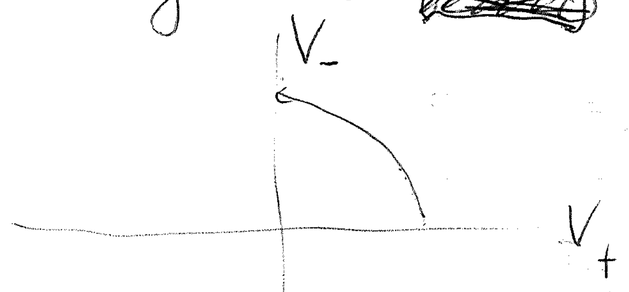
You [redacted] need a good picture. graded C_0 module with [redacted] an F, $F = F^*$, $0 \le I - F^2 \in \mathcal{K}$

Another approach to Clifford algebras + Bott periodicity might be [redacted] Bott's Morse theory [redacted] method. [redacted] Let's try to recall this. [redacted]

You want Bott's basic map from a Grassmannian into the loop space of some unitary group. There's

(26) the idea of nondegenerate critical submanifolds.
Let's try to recall the map Bott uses. Consider $SU(2n)$ and the geodesics going from $+I$ to $-I$. Such a geodesic should be $\exp(tX)$ essentially, i.e. a 1-parameter subgroup, so $0 \leq t \leq 1$, and so X satisfies $\exp(X) = -I$ and X skew adjoint. X skew adjoint means it can be diagonalized, eigenvalues are $i\omega$, ω real. Then you want $e^{i\omega} = -1$ $\omega \in \pi + 2\pi\mathbb{Z}$. You expect the eigenvalues for a minimal geodesic to be $\pm i\pi$, then for the determinant of $\exp(tX)$ to be 1, same as $tr(X) = 0$, so you have n $+\pi$'s and n $-\pi$'s. Then $U(2n)$ acts transitively on these X, and the stabilizer of a pt is $U(n) \times U(n)$. Then the Morse theory says that $G(n,n) = U(2n)/U(n)\times U(n) \hookrightarrow \Omega SU(2n)$ is a k-equivalence where k increases as $n \to \infty$.

Next consider some Grassmannian. This is a symmetric space, so the Morse theory arguments should work. You want to look for a nondegenerate critical submanifold of geodesics. want V_+, V_- same dimension. These should be like $+I$ and $-I$. Then a unitary isom $V_+ \; V_-$ will give a geodesic path from V_+ to V_-



Unfortunately this seems to be far from Clifford algebras.

(27) Let's do periodicity in the real case.

Start with $\Omega\, SO(2n)$, geodesics from $+I$ to $-I$

e^{tX} $\quad X^t = -X.$ $\quad$ Want $0 \le t \le 1.$ $\quad e^X = 1$

X direct sum of $\omega \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ $\quad \omega$ real, in this case

$$e^{tX} = \cos \omega t + \sin \omega t \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} \cos(\omega t) & \sin(\omega t) \\ -\sin(\omega t) & \cos(\omega t) \end{bmatrix}$$

$$e^{t\omega J} = \sum_{n \ge 0} \frac{t^{2n}\omega^{2n}}{(2n)!} \underbrace{J^{2n}}_{(-1)^n} + \sum_{n \ge 0} \frac{(t\omega)^{2n+1}}{(2n+1)!} \underbrace{J^{2n+1}}_{(-1)^n J}$$

You want least ω so that $\begin{bmatrix} \cos \omega & \sin \omega \\ -\sin \omega & \cos \omega \end{bmatrix} = -I$

i.e. $\omega = \pm \pi$

$SO(2n)$ $\quad$ Lie alg. = skew symm. matrices.

$n = 1.$

$$\exp t\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \exp tJ \qquad J^2 = -1.$$

$$= \sum_{n \ge 0} \frac{1}{(2n)!} t^{2n}(-1)^n + \sum_{n \ge 0} \frac{1}{(2n+1)!} t^{2n+1}(-1)^n J$$

$$= (\cos t) I + (\sin t) J = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}$$

$$\frac{d}{dt}\begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} = \begin{pmatrix} -\sin t & \cos t \\ -\cos t & -\sin t \end{pmatrix} \overset{?}{=} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}$$

First conj pt. You have geod. $\exp tX$ joining I to a point of $SO(2n)$ fixed under ~~conjugate~~ adj action.

28 $t = \frac{\pi}{2}$ $\exp \frac{\pi}{2} J = J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ centralizer of this is $SO(2n)$. You want the homotopy type of $\Omega SO(2n)$ for n large. Use Morse theory for a non degenerate critical sub manifold, where the index (dimension of negative subbundle) becomes large as $n \to \infty$.

Consider the exponential map $X \mapsto e^X$ at the identity in $SO(2n)$. What's important is to find a geodesic, really a tangent vector X at the origin of the symmetric space, such that the stabilizer of e^X is larger that the stabilizer of X.

Actually you want the stabilizer of e^{tX} for $0<t<1$ call this K_t, to be G = Isotropy group of origin at $t=0$, then to be constant $K_t = H$ for $0<t<1$ and then jump to $K_1 = K$ at $t=1$. Then you get $G \supset H \subset K$.

Recall another point: You choose a maximal torus in the case of a group, really a maximal abelian subspace $\mathfrak{a}$ of the Lie algebra. There's an analog of or for a symmetric space. Geodesics are lines in $\mathfrak{a}$. One has an affine root system, an $\mathfrak{a}$ where the hyperplanes intersecting the geodesic line contribute to the nullity of the geodesic.

Go back to the case $\Omega SU(2n)$, Lie alg of $SU(2n)$ is the space of hermitian matrices of trace 0. $\mathfrak{a}$ is the subspace of diagonal matrices in $SU(2n)$.

(29) Symplectic group - compact form, this should [redacted] arise from $Sp(2n, \mathbb{C})$ together with a "Cartan" involution. Other ideas - finite diml $\mathbb{H}$-module with suitable scalar product, which should make $Sp(2n)$ a subgroup of $U(2n)$. Start with $U(2n)$ acting on $\mathbb{C}^{2n}$ in the usual way by left mult. View $\mathbb{C}^{2n}$ as $\mathbb{H}^n$ with $\mathbb{H}$ acting by right mult. [redacted] Then $Sp(2n) \subset U(2n)$ is the [redacted] centralizer of right mult by j

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Go back [redacted] to $SO(2n)$, $\mathfrak{so}(2n)$ = space of real skew-symm $2n \times 2n$ matrices. [redacted] $\mathfrak{a} \subset \mathfrak{so}(2n)$ is the subspace [redacted] of $X = \omega_1 J \oplus \omega_2 J \oplus \cdots \oplus \omega_n J$

where $J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ $\omega_1, \ldots, \omega_n \in \mathbb{R}$.

Consider the [redacted] geodesic e^{tX} $0 \leq t \leq 1$. You want the ω_i such that [redacted] the endpoint e^X has a high degree of symmetry (isotropy gp is big). Also you want the path tX for $0 < t < 1$ to avoid the affine hyperplanes. [redacted] Otherwise you get a critical point with bad index.

$$e^{t\omega \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}} = \begin{bmatrix} \cos(t\omega) & \sin(t\omega) \\ -\sin(t\omega) & \cos(t\omega) \end{bmatrix}$$

[redacted] Let's work in $SO(2)$, replace $t\omega$ by θ. Then $e^{\theta J} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ is [redacted] rotation through the angle θ. The centralizer of $e^{\theta J}$??

(30) Go back to $\Omega SU(2n)$. $G = SU(2n)$ acting on itself by conjugation, better might be acted on by $U(2n)$ via conjugation. $X \in \text{Lie } SU(2n) = su(2n)$ = space of skew ~~adjoint~~ hermitian matrices of trace 0.

Take $X = \text{diag}(i\theta_1, \ldots, i\theta_{2n})$

$e^{tX} = \text{diag}(e^{it\theta_1}, \ldots, e^{it\theta_{2n}})$. e^{tX} for $0<t<1$ should avoid the affine hyperplanes if $0 < |\theta_j| \leq \frac{\pi}{2}$ $\forall j$.

Other condition about $e^X = \text{diag}(e^{i\theta_1}, \ldots, e^{i\theta_{2n}})$ having high symmetry ??

Repeat. $G = SU(2n)$, $T = \text{Ker}\{\mathbb{T}^{2n} \xrightarrow{\Sigma} \mathbb{T}\}$

$n=1$ $\quad G = SU(2)$, $T = \{\text{diag}(e^{i\theta}, e^{-i\theta}) \mid e^{i\theta} \in \mathbb{T}\}$

For $G = SU(2n)$, $T = \{\text{diag}(\zeta_1, \ldots, \zeta_{2n}) \mid \prod_1^{2n} \zeta_j = 1\}$

What do you want? A specific geodesic going from I to $-I$ with only trivial Jacobi fields ??

$$X = \pi i \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_{2n} \end{bmatrix} \qquad \lambda_j \in \mathbb{R} \qquad \sum_{j=1}^{2n} \lambda_j = 0$$

You want $e^X = -1$

$e^{\pi i \lambda_j} = -1 \quad \forall j \implies \lambda_j$ odd integer. There's some condition which forces $\lambda_j = \pm 1$.

But $\text{tr}(X) = 0$ so $\# +1\text{'s} = \# -1\text{'s}$.

(31) Next $G = SO(2n)$

$$X = \begin{bmatrix} \boxed{\lambda_1 J} & & \\ & \boxed{\lambda_2 J} & \\ & & \ddots \\ & & & \boxed{\lambda_n J} \end{bmatrix}$$

$\lambda \in \mathbb{R}$

$$e^{\lambda J} = \begin{bmatrix} \cos(\lambda) & \sin\lambda \\ -\sin\lambda & \cos\lambda \end{bmatrix}$$

If you look at $n=1$ case $G = SO(2)$ = rotations in $\mathbb{R}^2$

If $\lambda = 0 \implies e^{\lambda J} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\lambda = \frac{\pi}{2} \implies e^{\lambda J} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

$\lambda_j = \pm \frac{\pi}{2}$?

It seems you want to take all $\lambda_j = +\frac{\pi}{2}$ and then $e^{tX} = e^{t\frac{\pi}{2}J} = J$ at $t=1$, so the geodesic goes from I to J, and the stabilizer of J should be $U(n)$.

$\therefore \; SO/U \sim \Omega SO$

Next try $G = Sp(2n)$. Start with $\mathbb{C}^{2n} = \mathbb{H}^n$.

Take $n=1$. $\mathbb{C}^2 = \mathbb{H} = \mathbb{C}1 + \mathbb{C}j$ where $j^2 = -1$, $ij + ji = 0$.

You think $Sp(2n)$ is the subgroup of $U(2n)$ of operators commuting with right mult by j. Take $n=1$.

$U(2n) \simeq$ unitary 2×2 matrices

$$\mathbb{H} = \mathbb{C}1 + \mathbb{C}j = \begin{bmatrix} \mathbb{C} & \mathbb{C} \end{bmatrix} \begin{bmatrix} 1 \\ j \end{bmatrix}$$

$$\mathbb{H} = \{ z_1 1 + z_2 j \}$$

(32) $\mathbb{H} = \mathbb{C}1 \oplus \mathbb{C}j$

$\mathbb{H}^n = \mathbb{C}^{2n}$ because $\mathbb{H} = \mathbb{C}^2$

you want $Sp(2n)$ to be a subgroup of $U(2n)$.

$Sp(2) = SU(2)$ expected, where

$$SU(2) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \;\middle|\; \begin{array}{l} |a|^2+|c|^2=1 \\ a\bar{b}+c\bar{d}=0 \\ |b|^2+|d|^2=1 \end{array} \quad ad-bc=1 \right\}$$

$g \in GL(n)$ is unitary when $g^{\dagger} g = I$, i.e.

$$g^{\dagger} = g^{-1} \iff \begin{bmatrix} \bar{a} & \bar{c} \\ \bar{b} & \bar{d} \end{bmatrix} = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \qquad \begin{array}{l} \bar{a} = d \\ \bar{b} = -c \end{array}$$

$$g = \begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix} \qquad |a|^2+|b|^2 = 1.$$

$$\boxed{SU(2) = \left\{ \begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix} \;\middle|\; |a|^2+|b|^2=1 \right\}}$$

Next want to generalize.

$\cancel{(r+sj)(\lambda+\mu j) = r\lambda + r\mu j + sj\lambda - s\mu i}$

$\cancel{(r\lambda - s\mu) + (r\mu + s\lambda)j}$

$\cancel{(a+bj)(\lambda+\mu j) = a\lambda + bj\lambda + a\mu j + bj\mu j}$

$\cancel{= a\lambda + b\bar{\lambda} j + a\mu j - b\bar{\mu} i}$

$\cancel{\lambda+\mu j \longmapsto (a\lambda - b\bar{\mu} i) + (b\bar{\lambda} + a\mu)j}$

(33) Try a different approach namely identify the ring of quarternions $\mathbb{H}$ with a ring of 2×2 complex matrices. ~~Use the basis Calculate~~

$$SU(2) = \left\{ g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \;\middle|\; g^\dagger g = 1,\ \det(g) = 1 \right\}.$$

$$g^\dagger = g^{-1} \qquad \begin{bmatrix} \bar a & \bar c \\ \bar b & \bar d \end{bmatrix} = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \qquad \bar a = d,\ \bar b = -c$$

$$\therefore\ g = \begin{bmatrix} a & b \\ -\bar b & \bar a \end{bmatrix}$$

So it should be clear that $\mathbb{H} = \left\{ \begin{bmatrix} a & b \\ -\bar b & \bar a \end{bmatrix} \;\middle|\; a, b \in \mathbb{C} \right\}$

$$\begin{bmatrix} a & b \\ -\bar b & \bar a \end{bmatrix}\begin{bmatrix} c & d \\ -\bar d & \bar c \end{bmatrix} = \begin{bmatrix} ac - b\bar d & ad + b\bar c \\ -\bar b c - \bar a \bar d & -\bar b d + \bar a \bar c \end{bmatrix}$$

[redacted]

Obvious question is your idea that $Sp(2n)$ is the subgroup of $U(2n)$ commuting with J somehow.

Discuss. You start with $Sp(2) = SU(2)$

$Sp(2) \subset$ [redacted] $\mathbb{H}^\times \longrightarrow \mathbb{R}^\times_{>0}$. Try to understand the left action of $\begin{bmatrix} a & b \\ -\bar b & \bar a \end{bmatrix}$ on $\begin{bmatrix} \\ \end{bmatrix} = \mathbb{C}^2$. In fact you have an $\mathbb{R}$-alg isomorphism from $\mathbb{H}$ to M_2 NO

You have $\mathbb{H}$ acting on $\mathbb{C}^2$ tautologically, $\mathbb{H}$ is a division alg over $\mathbb{R}$, dim = 4. ~~There~~

4 diml

Q: $\mathbb{H} \otimes_{\mathbb{C}} \mathbb{H} = M_2\mathbb{C}$?

NO $\mathbb{H}$ is not a $\mathbb{C}$-algebra.

Start again. You have the repn $\mathbb{H} \longrightarrow M_2\mathbb{C}$, probably a $*$ repn. $a(z_1 + j z_2) = a z_1 + \bar a z_2$

$j b (z_1 + j z_2) =$

(34) Repeat. $Sp(2) = SU(2)$, why? because

$$SU(2) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} = g \;\middle|\; \begin{matrix} g^{\dagger} g = 1 \\ \det(g) = 1 \end{matrix} \right\} \qquad g^{\dagger} = g^{-1} \quad \begin{bmatrix} \bar a & \bar c \\ \bar b & \bar d \end{bmatrix} = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$= \left\{ \begin{bmatrix} a & b \\ -\bar b & \bar a \end{bmatrix} \;\middle|\; a, b \in \mathbb{C} \text{ and } \underbrace{|a|^2 + |b|^2 = 1}_{} \right\}.$$

now relax the determinant $=1$ condition. This yields a real 4 dim'l subalgebra of $M_2\mathbb{C}$

$$\begin{bmatrix} a & b \\ -\bar b & +\bar a \end{bmatrix} \begin{bmatrix} c & d \\ -\bar d & \bar c \end{bmatrix} = \begin{bmatrix} ac - b\bar d & ad + b\bar c \\ -\bar b c - \bar a \bar d & -\bar b d + \bar a \bar c \end{bmatrix}$$

Let's find the obvious real basis

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \underset{I}{\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}} \quad \underset{J}{\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}} \quad \underset{K}{\begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}}$$

$$K^2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$I^2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \qquad J^2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$IJ = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix} = K$$

$$IK = I^2 J = -J$$

$$IJ = K$$
$$IJK = -1 \qquad JK = I$$
$$\Rightarrow KIJ = -1 \qquad IJ = K$$
$$\Rightarrow JKI = -1 \qquad \cancel{JK = I}$$
$$IJ = K.$$

(35) You're still missing something important. At the moment you have this embedding

$$\mathbb{H} \hookrightarrow M_2\mathbb{C}, \quad \mathbb{H} = \left\{ \begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix} \;\middle|\; a, b \in \mathbb{C} \right\}$$

You want to be able to handle $Sp(2n)$, which you believe is a quaternionic version of $U(2n)$. You want a natural embedding of $Sp(2n)$ into $U(2n)$ related to ~~the centralizer of an involution~~ centralizing a "J" operator.

$$M_n\mathbb{H} = M_n\mathbb{R} \otimes_{\mathbb{R}} \mathbb{H} \hookrightarrow M_n\mathbb{R} \otimes_{\mathbb{R}} M_2\mathbb{C} = M_{2n}\mathbb{C}$$

Inside $M_n\mathbb{H}$ might be $Sp(2n)$.

At some point you ought to be able to handle the maximal torus of $Sp(2n)$, which should have rank n, because $Sp(2) = SU(2)$ has rank 1.

What is the Lie alg of $Sp(2n)$? This should be the space of symmetric bilinear forms in the case of $Sp(2n, \mathbb{C})$.

Actually maybe you should do the complex case and find the Cartan involution, i.e. the compact form of $Sp(2n, \mathbb{C})$. Begin with the hyperbolic symplectic space $\begin{bmatrix} V \\ V^* \end{bmatrix}$ with symp. form

$$\begin{bmatrix} v_1 \\ \varphi_1 \end{bmatrix}^t \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} v_2 \\ \varphi_2 \end{bmatrix} = v_1^t \varphi_2 - \varphi_1^t v_2$$

36 What should be the maximal [redacted] compact version? First try choosing a positive definite hermitian form on V, at which point V becomes the same as V^*, and the symplectic form becomes the skew hermitian operator $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ on $\begin{bmatrix} V \\ V \end{bmatrix}$.

Look at $n=1$ for $Sp(2n,\mathbb{C})$. Then $Sp(2n,\mathbb{C}) = SL(2,\mathbb{C})$

Check $Sp(2,\mathbb{C}) = \left\{ g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_2\mathbb{C} \;\middle|\; g^t \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} g = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right\}$

$$\begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \begin{bmatrix} 0 & -1 \\ +1 & 0 \end{bmatrix} \quad \text{if } \det(g) = 1.$$

$$= \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} -b & -d \\ +a & +c \end{bmatrix} = \begin{bmatrix} +a & +c \\ +b & +d \end{bmatrix}$$

no condition at all for $g \in SL(2,\mathbb{C})$.

If $g^t \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} g = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, then $+\det(g)^2 = +1$.

If $\det(g) = -1$, then you get the ~~condition~~ that $g^t = -g^t$ so $g^t = 0$. $\therefore Sp(2,\mathbb{C}) = SL(2,\mathbb{C})$

~~[illegible]~~ You want the compact form of $Sp(2n,\mathbb{C})$, so you need some sort of $\dagger$ operator. You start with $\begin{bmatrix} V \\ V^* \end{bmatrix}$ where ~~[illegible]~~ $V = V^*$ via pos. def. herm. form on V.

I think it's true that the symplectic form amounts to $\begin{bmatrix} V \\ V \end{bmatrix}$ being the complexification ~~of~~ $\mathbb{C} \otimes V$

What you need now is ~~—~~ to handle the Lie algebra. This is ~~[illegible]~~ easy for $sp(2n,\mathbb{C})$ and probably not too hard for $sp(2n)$, the compact form

37 symplectic group. You can understand this via CCR. Your symplectic space has basis a_i, a_i^* $\quad 1 \le i \le n$ satisfying

$$[a_i, a_j] = [a_i^*, a_j^*] = 0 \quad \forall i,j, \quad [a_i, a_j^*] = \delta_{ij}.$$

Lie of $Sp(2n, \mathbb{C})$ is $S_2(V \oplus V^*) = S_2 V \oplus V \otimes V^* \oplus S_2 V^*$

Point: You have a description of Lie $Sp(2n)$. The problem is now to find a maximal abelian or, or Cartan subalgebra. Guess these spanned by $a_i^* a_i$ $\quad i = 1, \dots, n$.

Start with $V = \mathbb{C}^n$, form the hyperbolic symplectic space $H(V) = \begin{bmatrix} V \\ V^* \end{bmatrix}$ with nondeg skew-form

$$\begin{bmatrix} v_1 \\ \varphi_1 \end{bmatrix}^t \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} v_2 \\ \varphi_2 \end{bmatrix} = v_1^t \varphi_2 - \varphi_1^t v_2 .$$

$Sp(2n, \mathbb{C})$ is the group of autos of $H(V)$ preserving the symplectic form: $\{ g \in \underbrace{GL\left(\left[\begin{smallmatrix} V \\ V^* \end{smallmatrix}\right]\right)}_{GL(2n,\mathbb{C})} \mid g^t \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} g = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \}$. Lie $Sp(2n, \mathbb{C})$

$$= \{ X \in M_{2n}\mathbb{C} \mid X^t \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} X = 0 \}, \text{ so}$$

if $X = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $\begin{bmatrix} a^t & c^t \\ b^t & d^t \end{bmatrix} = -\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

$$= \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} b & -a \\ d & -c \end{bmatrix} = \begin{bmatrix} -d & +c \\ b & -a \end{bmatrix} \qquad a^t = -d, \quad b = b^t, \quad c = c^t$$

$$\therefore \quad X = \begin{bmatrix} a & b \\ c & -a^t \end{bmatrix}$$

where b, c are symmetric

$a \in M_n\mathbb{C} = V \otimes V^*$

(38) A better viewpoint is perhaps to use the Weyl algebra associated to the symplectic space $H(V)$.

Look at rank 1 situations. First do $Sp(2) = SU(2)$

Look at $\begin{bmatrix} V \\ V^* \end{bmatrix} = H(V)$, ~~mess~~ choose a pos. def. scalar product on V, then have $\begin{bmatrix} V \\ V \end{bmatrix}$ equipped with ?

$$\begin{bmatrix} v_1 \\ \varphi_1 \end{bmatrix}^{\dagger} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} v_2 \\ \varphi_2 \end{bmatrix} ?$$

Start again. V complex vector space with positive hermitean scalar product, that is, ~~an iso~~ a hermitian symmetric iso $V \xrightarrow[\sim]{T} V^{\dagger} = \bar{V}^*$ = space of antilinear functionals on V. T herm. symm means

$$V^* \xleftarrow{T^t} \bar{V} \quad \text{same as} \quad \bar{V}^* \xleftarrow{\bar{T}^t} V$$

Seems like ??

Look at $Sp(2) = SU(2)$ real forms of
$Sp(2,\mathbb{C}) = SL(2,\mathbb{C})$

Point of departure is a canonical group map $U(n) \hookrightarrow Sp(2n)$ which is the natural action of the unitary group of V on the hyperbolic symplectic space $H(V) = \begin{bmatrix} V \\ V^* \end{bmatrix}$.

From Lie alg viewpoint you have

$$\text{Lie } Sp(2n,\mathbb{C}) = S_2(V \oplus V^*) = S_2 V \oplus \underbrace{V \otimes V^*}_{\text{End}(V)} \oplus S_2 V^*$$

You are talking about sending $X \in \text{End}(V)$ to $\begin{bmatrix} X & 0 \\ 0 & -X^t \end{bmatrix} \in \text{End} \begin{bmatrix} V \\ V^* \end{bmatrix}$ $\quad \text{End } V \longrightarrow \underbrace{\text{End } H(V)}_{S_2(V \oplus V^*)}$

(39) First point $Sp(2n,\mathbb{C}) = \{ g \in GL(2n,\mathbb{C}) \mid g^t \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} g = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, Lie case: $X^t J + JX = 0$

So if $X = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{End} \begin{bmatrix} V \\ V^* \end{bmatrix}$, then $X^t = \begin{bmatrix} a^t & c^t \\ b^t & d^t \end{bmatrix} =$

$$= JXJ = J \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} -b & a \\ -d & c \end{bmatrix} = \begin{bmatrix} -d & c \\ b & -a \end{bmatrix}$$

$\iff \begin{matrix} b = b^t \\ c = c^t \end{matrix}$ and $d = -a^t$ $\therefore X = \begin{bmatrix} a & b \\ c & -a^t \end{bmatrix}$ b, c symm.

to get nice embedding $U(n) \hookrightarrow Sp(2n)$, at least for the complex groups: $GL(n,\mathbb{C}) \hookrightarrow Sp(2n,\mathbb{C})$. At some point you must clarify the Cartan involution.

Take $n=1$. You have $U(1) \hookrightarrow SU(2)$, 1-parameter subgroup given by $e^{i\theta} \mapsto \begin{bmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{bmatrix}$. This [redacted] should be a maximal ~~torus~~ of $SU(2)$. [redacted]

At this point you should find the Weyl group, roots, hyperplanes, etc. The Weyl group of $SU(2)$ is [redacted] $\mathbb{Z}/2$, [redacted] so in the case of $\mathbb{T}^n \hookrightarrow U(n) \hookrightarrow Sp(2n)$, it seems the Weyl group should be [redacted] $\Sigma_n \ltimes (\mathbb{Z}/2)^n$.

Let's try to guess what [redacted] (better: Morse theory) Bott periodicity says about $\Omega Sp(2n)$. It seems that you want the geodesic in $Sp(2n)$ going from I to $-I$, which is the direct sum of n maps $\theta \longmapsto \begin{bmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{bmatrix}$ for $0 \leq \theta \leq \pi$.

When is $e^{i\theta} = e^{-i\theta}$? $\iff e^{2i\theta} = 1 \iff$ [redacted] $\theta \in \mathbb{Z}\pi$.

Next, the (you need) stabilizer of the [redacted] geodesic.

40

Atiyah "Real" K-theory, $KR(X)$ defined for X a compact space with $\mathbb{Z}/2$ action. A "Real" v.b. over X is a complex vector bundle E over X equipped with a $\mathbb{Z}/2$ action on E ??

Try again.

What you want should be clear if you work with an algebraic variety Y defined over $\mathbb{R}$. The maximal ideals have residue field $\mathbb{R}$ or $\mathbb{C}$. Complexify the variety Y defined $\mathbb{R}$ to get $\mathbb{C}$-variety X. For each max ideal in Y with residue field $\mathbb{R}$ you get one point of X which is fixed under conjugation. Other case gives 2 pts.

There should be a compact space version. Take a compact space X with $\mathbb{Z}/2$ action, Consider continuous maps $X \xrightarrow{f} \mathbb{C}$ which are equivariant wrt the given $\mathbb{Z}/2$ action on X, and the conjugation action on $\mathbb{C}$.

$$f(\bar{x}) = \overline{f(x)} \qquad \forall x \in X.$$

What are vector bundles in this framework?

$\mathbb{R}[x]/(x^2+1)$ is a 2 diml field over $\mathbb{R}$

$$\mathbb{C}[x]/(x^2+1) = \mathbb{C} \times \mathbb{C}$$

$$\mathbb{R}[x,y]/(x^2+y^2+1)$$

41 X compact space with $\mathbb{Z}/2$ action, then let ~~define~~

$$C(X) \simeq \{ \text{cont. } f : X \to \mathbb{C} \mid f(\bar{x}) = \overline{f(x)} \}$$

Recall Atiyah introduces $\mathbb{R}^{p,q}$ ~~which conjugation~~ a representation of $\mathbb{Z}/2$ where trivial on $\mathbb{R}^p$ and -1 on $\mathbb{R}^q$. $KR^{p,q}(X) = KR(\mathbb{R}^{p,q} \times X)$

Periodicity thm (Elementary proof) says that

$KR(\mathbb{C} \times X) = KR(X)$ $\qquad \mathbb{C} = \mathbb{R}^{1,1}$

$KR^{1,1}(X) = KR(X)$. You need next to get ~~values for~~ simple cases.

Suppose $\forall x \in X \quad \bar{x} = x$. Then a "Real" v.b. should be just a usual $\mathbb{R}$-bdle

$$K^{p,0}(X) = KO(\mathbb{R}^p \times X)$$

$$K^{0,1}(X) = KO(\underset{\uparrow}{\mathbb{R}} \times X) \qquad [-1, 1]$$

$\mathbb{Z}/2$ acts.

$$\{-1, 1\} \times X \hookrightarrow [-1, 1] \times X \longrightarrow \mathbb{R} \times X$$

$$K\cancel{O}^0(X) \longleftarrow KO^0(X) \longleftarrow K^{0,1}(X)$$

~~History~~ $\qquad KR^{p,q}(X) = KR(\mathbb{R}^p_+ \times \mathbb{R}^q_- \times X)$

and it depends only on $p-q$. Problem is to relate KR to KO, KU, KSp. Long exact sequences arising from

$$S^0 \hookrightarrow D^1 \longrightarrow S^1$$

$$S^1 \hookrightarrow D^2 \longrightarrow S^2$$

You need correct indices for $H^k(\mathbb{R}^p_+ \times X)$

(42) $H^k(\mathbb{R}^p_\bullet \wedge X) = H^k(S^pX) = H^{k-p}(X)$. ~~...~~ Besides there a canonical class in $H^p(\mathbb{R}^p)$

$$
\begin{array}{ccccc}
\{\pm 1\}\times X & \hookrightarrow & [-1,1]\times X & & \overset{(-1,1)}{\mathbb{R}_-}\times X \\
KR(\{\pm 1\}_-\times X) & \leftarrow & KR([-1,1]_-\times X) & \leftarrow & KR^{0,1}(X) \\
\| & & & & \\
KU(X) & & KR^{0,0}(X) & \xleftarrow{?} & KR^{0,1}(X) \\
KR(S^1_-\times X) & & KR(D^2_-\times X) & & KR^{0,2}(X)
\end{array}
$$

---

need a lot of review; first look at path space. $\Omega SU(2n)$, inside this, more precisely paths from I to $-I$, you have the Grassmannian $U(2n)/U(n)\times U(n)$.

In the case $G = SO(2n)$ you consider paths from I to the conjugacy class of $J = \begin{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} & & \\ & \ddots & \\ & & \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \end{bmatrix}$

What is the homotopy type of this path space?

Paths in G ~~starting~~ at I and ending on

Point: ~~...~~ The theory concerns symmetric spaces. A compact conn. Lie group G is a symmetric space with $G\times G$ acting by left and right mult. $(g_1,g_2)*x = g_1 x g_2^{-1}$ The stabilizer of $I\in G$ is $K = \Delta G$

(43) So the K orbits are conjugacy classes. You to understand the space of paths going from the identity conj class to the conjugacy class of complex structures on $\mathbb{R}^{2n}$.

Point: Paths in G are closely related to connections. ~~A path starting at the origin~~ A path g_t, $0 \le t \le 1$, starting at the origin is essentially equivalent to a loop A_t in the Lie algebra $\mathfrak{g}$.

$$\frac{d}{dt} g_t = A_t g_t$$

gives rise to an equivalence between paths $[0,1] \to G$ and G-connections over the circle S^1. The monodromy of the connection is the conjugacy class of the endpoint g_1 of the path.

Point: To consider the space of paths going from the origin to a conjugacy class in G is very natural, so there should be a nice picture of the homotopy type.

Consider $SO(2n)$ with maximal torus the direct sum of 2×2 blocks $e^{\theta J}$, where $J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, and $e^{\theta J} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$, so the maximal torus T consists of

$$\begin{bmatrix} e^{\theta_1 J} & & & \\ & e^{\theta_2 J} & & \\ & & \ddots & \\ & & & e^{\theta_n J} \end{bmatrix}$$

$$e^{\frac{\pi}{2} J} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = J$$

Now ~~let~~ $\theta_i = \theta \quad \forall i$

As θ runs: $0 \le \theta \le \frac{\pi}{2}$ the corresp elt of T is $e^{\theta (J^{\oplus n})}$. This is a geodesic going from the identity I to $e^{\frac{\pi}{2} J^{\oplus n}} = J^{\oplus n}$

(44) $SO(2n) = \{ g \in M_{2n}\mathbb{R} \mid g^t = g^{-1} \text{ and } \det(g) = 1 \}$.

$\mathrm{Lie}\, SO(2n) = \{ X \in M_{2n}\mathbb{R} \mid X^t = -X \}$

If X invertible, then get polar decomp.

$|X| = (X^t X)^{1/2} = (-X^2)^{1/2}$, then $\frac{X}{|X|} = J$ satisfies

$$J J = \frac{+X}{|X|}\frac{X}{|X|} = + \frac{X^2}{|X|^2} = \frac{X^2}{-X^2} = -1$$

So you get $X = |X| J$ where $J^t = -J$, $J^2 = -1$, and $|X|^t = |X| > 0$. Then split $\mathbb{R}^{2n}$ into eigenspaces for $|X|$. You get $0 < \lambda_1 \le \lambda_2 \le \dots \le \lambda_n$

$$X = \begin{bmatrix} \lambda_1 J_2 & & & \\ & \lambda_2 J_2 & & \\ & & \ddots & \\ & & & \lambda_n J_2 \end{bmatrix}$$

What went wrong: Look at $SO(2)$ and the obvious geodesic (1-par. subgp)

$$\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} = e^{\theta \overbrace{\left[\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right]}^{J}}$$

What is the centralizer of $e^{\theta J}$? The eigenvalues are $e^{i\theta}, e^{-i\theta}$. One has $e^{i\theta} = e^{-i\theta} \iff e^{2i\theta} = 1 \iff \theta \in \mathbb{Z}\pi$. If $\theta = \frac{\pi}{2}$ then $e^{\theta J} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = J$

But the centralizer doesn't jump. (nondeg critical)

So it seems that you want the submanifold of geodesics to go from 1 to -1. ~~The basis~~ Take the tangent vector to be $J_n = \begin{bmatrix} J & & \\ & J & \\ & & \ddots \end{bmatrix}$ n times

$SO(2n)$ acts via conjugation

centralizer is $U(n)$ $\therefore$ nondeg critical submanifold should be $SO(2n)/U(n)$.

(45) Next $\Omega Sp(2n)$. $Sp(2n)$ You start with $Sp(2) = SU(2)$. ?? ~~You have a canonical map~~

Maybe start with $Sp(2n, \mathbb{C})$, the group of autos of hyperbolic symplectic space $\underset{H(V)}{}\begin{bmatrix} V \\ V^* \end{bmatrix}$ $V = \mathbb{C}^n$

There is an obvious action of $GL(n, \mathbb{C})$ on $H(V)$.

$$\text{Lie } Sp(2n, \mathbb{C}) = S_2(V \oplus V^*) = S_2 V \oplus \underbrace{(V \otimes V^*)}_{gl(n,\mathbb{C})} \otimes S_2 V^*$$

$$Sp(2, \mathbb{C}) = \{ g \in GL(2, \mathbb{C}) \mid \underbrace{g^t \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} g = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}}_{\det(g)^2 = 1} \}.$$

$$\text{Lie } Sp(2n, \mathbb{C}) = \{ X \in gl(2n, \mathbb{C}) \mid X^t J + JX = 0 \}$$

$$\begin{bmatrix} a^t & c^t \\ b^t & d^t \end{bmatrix} = +JXJ = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} -b & a \\ -d & c \end{bmatrix} = \begin{bmatrix} -d & c \\ b & -a \end{bmatrix}$$

$-a^t = +d$

$b = b^t, c = c^t$

If $\det(g) = 1$ then

$$\begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} -b & -d \\ a & +c \end{bmatrix} = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$$

If $\det(g) = -1$, then $g = 0$.

This is still incomplete about the compact $Sp(2n)$.

(46) You know the maximal torus for $Sp(2) = SU(2)$ is $\left\{ \begin{bmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{bmatrix} \right\}$. The max torus for $Sp(2n)$ seems to be the direct sum

$$\begin{bmatrix} e^{i\theta_1} & 0 \\ 0 & e^{-i\theta_1} \end{bmatrix} \oplus \begin{bmatrix} e^{i\theta_2} & 0 \\ 0 & e^{-i\theta_2} \end{bmatrix} \oplus \cdots \oplus \begin{bmatrix} e^{i\theta_n} & 0 \\ 0 & e^{-i\theta_n} \end{bmatrix}$$

Weyl group should be $\Sigma_n \ltimes (\mathbb{Z}/2)^n$

Look at the centralizer of $\begin{bmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{bmatrix}$ in $SU(2)$

It jumps where $e^{i\theta} = e^{-i\theta}$, i.e. $\theta \in \mathbb{Z}\pi$. So it looks like the nondeg critical submanifold of geodesics to use go from 1 to -1. You act ~~on~~ by $Sp(2n)$ conjugation on the geodesic $\bigoplus_1^n \begin{bmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{bmatrix}$ $\quad 0 \leq \theta \leq \pi$

The tangent vector to this geodesic is $\bigoplus_1^n \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$.

~~So it seems you want the conjugacy cl~~

Weyl group allows you to move i and $-i$ together, to maybe get a polarization. There should be some natural subgroup of $Sp(2n)$?

(47) It's time to understand $Sp(2n)$ better.
$Sp(2n)$ is a compact form of $Sp(2n, \mathbb{C})$ the Lie group of autos of the hyperbolic symplectic space $V = \mathbb{C}^n$

$$H(V) = \begin{bmatrix} V \\ V^* \end{bmatrix}, \quad \begin{bmatrix} v_1 \\ \varphi_1 \end{bmatrix}^t \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} v_2 \\ \varphi_2 \end{bmatrix} = v_1^t \varphi_2 - \varphi_1^t v_2$$

Your idea is to find a compact form for $H(V)$ and use it to get $Sp(2n)$. The compact form for $H(V)$ should arise from a pos. def. hermitian form on V. Then $U(n) = U(V)$ naturally acts on $H(V)$, this is the usual action of $U(n)$ on V direct sum with the contragredient repn. on V^*.

Let's work on the real form for $H(V)$.

To get started begin with $Sp(2)$ which you know is $SU(2)$. $\quad Sp(2) = SU(2) \subset SL(2, \mathbb{C})$

Another point of departure would be creation & annihilation operators. $\begin{bmatrix} V \\ V^* \end{bmatrix}$

Look for a conjugation on $H(V)$.
Choose pos herm. scalar product on V $\quad \langle v_1 | v_2 \rangle$

$$\langle v_1 \lambda | v_2 \mu \rangle = \bar{\lambda} \langle v_1 | v_2 \rangle \mu \qquad \bar{V} \longrightarrow V^*$$
$$v_1 \longmapsto \langle v_1 |$$

herm. symm means $\quad \overline{\langle v_1 | v_2 \rangle} = \langle v_2 | v_1 \rangle$

What to do? $\quad$ pos herm. scalar product $T : V \xrightarrow{\sim} V^t$
such that $T = T^t : V^* \longleftarrow \bar{V}$

(48) herm form $h(v_1, v_2) = v_1^t h v_2$

perhaps all you need to do is to replace the pairing $V \times V^*$ $(v, \varphi) \longmapsto v^t\varphi = \varphi^t v$

$\varphi \in V^*$ i.e. $\varphi : \mathbb{C} \longrightarrow V^*$, $\varphi^t : \mathbb{C} \longleftarrow V$

$v \in V$ i.e. $v : \mathbb{C} \longrightarrow V$, $v^t : \mathbb{C} \longleftarrow V^*$

Check this. $\mathbb{C} \xrightarrow{v} V \xrightarrow{\varphi} \mathbb{C}$

Maybe you start w. V, $V^* = \mathrm{Hom}(V, \mathbb{C})$

$$V \xrightarrow{u} W \Rightarrow W^* \xrightarrow{u^t} V^* \Rightarrow \begin{array}{c} V^{**} \xrightarrow{(u^t)^t} W^{**} \\ \| \\ V \xrightarrow{u} W \end{array}$$

Aim: to interpret the pairing (v, φ) using maps.

$v \rightsquigarrow \mathbb{C} \xrightarrow{v} V$, $c \mapsto$ [redacted] vc $\qquad \varphi \rightsquigarrow \mathbb{C} \longrightarrow V^*$, $c \mapsto \varphi c$

transpose

$\mathbb{C} \xleftarrow{v^t} V^*$

$(v, \varphi) \longleftarrow\!\shortmid\ \varphi$

Claim: View $v \in V$, $\varphi \in V^*$ as maps $\mathbb{C} \xrightarrow{\hat{v}} V$, $\mathbb{C} \xrightarrow{\hat{\varphi}} V^*$ then $\langle v, \varphi \rangle = \hat{\varphi}^t \circ \hat{v}$ $= \hat{v}^t \circ \hat{\varphi}$

$\langle v, \varphi \rangle = v^t\varphi = \varphi^t v$ provided you interpret v, φ as maps.

You want to do the same in the hermitian case. What does this mean? Mainly you replace the dual V^* by the antidual $V^\dagger$. A sesquilinear form ~~should~~ be a $\mathbb{C}$-linear map $V \xrightarrow{T} V^\dagger$

~~$(Tv_1)v_2$~~ $\qquad v_2 \, T v_1 \qquad$ What do you need?

(49) If $v_1 \in V$, then $Tv_1 \in V^\dagger$, which means (v_2, Tv_1) is [redacted] anti linear in v_2. Formulate using maps. $Tv_1 \in V^\dagger$, so $Tv_1 : \mathbb{C} \to V^\dagger$, $v_2 \in V$ is a map $v_2 : \mathbb{C} \to V$, apply $\dagger$ to get $V^\dagger \xrightarrow{v_2^\dagger} \mathbb{C}^\dagger$, can compose $v_2^\dagger T v_1 : \mathbb{C} \longrightarrow V^\dagger \longrightarrow \mathbb{C}^\dagger$. It seems you need to identify $\mathbb{C}$ $\mathbb{C}^\dagger$ probably using 1.

Next given V with pos herm. $T : V \xrightarrow{\sim} V^\dagger$

~~Need notation $v^\dagger$ is the~~

Goal: Real form for $H(V) = \begin{bmatrix} V \\ V^* \end{bmatrix}$ $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

$Sp(H(V))$ = intrinsic version of $Sp(2n, \mathbb{C})$. You want a real form for $H(V)$, ~~which is preserved~~ ~~which yields the~~ whose autos yield the compact form of $Sp(H(V))$.

You want a [redacted] conjugation on $\begin{bmatrix} V \\ V^* \end{bmatrix}$, probably odd wrt the grading. [redacted] So you want

[redacted] $\begin{bmatrix} & 1 \\ T & \end{bmatrix} : \begin{bmatrix} V \\ V^* \end{bmatrix} \longrightarrow \begin{bmatrix} V \\ V^* \end{bmatrix}^* = \begin{bmatrix} V^* \\ V \end{bmatrix}$

$$H(V) = \begin{bmatrix} V \\ V^* \end{bmatrix} \qquad \begin{bmatrix} v_1 \\ \varphi_1 \end{bmatrix}^t \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} v_2 \\ \varphi_2 \end{bmatrix}$$

$$\Big\downarrow \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

$$H(V)^* = \begin{bmatrix} V^* \\ V \end{bmatrix}$$

You want to combine this with conjugation

$H(V)$ is a complex symplectic space [redacted] for which you seek a real form with nice properties. A

(50) real form should be an antilinear isomorphism. Look at $SL(2,\mathbb{C}) = Sp(2,\mathbb{C})$. Two real forms are $SU(2)$ and $SL(2,\mathbb{R})$. What is the involution on $SL(2,\mathbb{C})$? It's what yields unitary matrices. $g^\dagger g = 1$. Polar decomp for $SL(2,\mathbb{C})$

$(g^* g)^{1/2} = |g| \qquad g|g|^{-1} = u$

Look at the Lie algebra $\quad \mathrm{Lie}\, SL(2,\mathbb{C}) = \left\{ \begin{bmatrix} a & b \\ c & -a \end{bmatrix} \right\}$

$$SU(2) = \left\{ \begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix} \middle| \, |a|^2 + |b|^2 = 1 \right\} \qquad \mathrm{Lie}\, SU(2) = \left\{ \begin{bmatrix} ia & b \\ -\bar{b} & -ia \end{bmatrix} \right\}$$

a real

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functions on phase space, linear + quadratic form a Lie alg under Poisson bracket. ~~Should~~ be basis is $a, a^*, \frac{a^2}{2}, \frac{a^{*2}}{2}, aa^*$ For this there's an obvious conjugation $*$. Check it out for $H(V)$, $V = \mathbb{C}^n$

$H(V)$ has basis $\underbrace{a_1, \dots, a_n}_{V} \quad \underbrace{a_1^*, \dots, a_n^*}_{V^*}$

$$[a_i, a_j^*] = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$$

It should be true that $\sigma a_i = a_i^* \qquad \sigma(a_i^*) = a_i$

preserves the symp. form.

V complex vector space of dim n $\qquad \varphi_2(v_1) - \varphi_1(v_2)$

$$H(V) = \begin{bmatrix} V \\ V^* \end{bmatrix} \quad \text{with} \quad \begin{bmatrix} v_1 \\ \varphi_1 \end{bmatrix}^t \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} v_2 \\ \varphi_2 \end{bmatrix} = v_1^t \varphi_2 - \varphi_1^t v_2$$

$n = 1$. $\quad H(V) = \left\{ \begin{bmatrix} v \\ \varphi \end{bmatrix} \right\}$ with $\begin{bmatrix} v_1 \\ \varphi_1 \end{bmatrix}^t \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} v_2 \\ \varphi_2 \end{bmatrix}$

$= v_1 \varphi_2 - v_2 \varphi_1$

$H(V)$ complex 2 diml space equipped with symplectic form

You want a real form for $H(V)$.

(51)

$V = \mathbb{C}^n = \{(a_1, \dots, a_n)\}$, $V^* = \{(b_1, \dots, b_n)\}$.

ab^t duality.

$ad-bc$

$$H(\mathbb{C}) = \left\{ \begin{bmatrix} a \\ b \end{bmatrix} \right\}, \quad \begin{bmatrix} a \\ b \end{bmatrix} \wedge \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}^t \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} c \\ d \end{bmatrix}$$

positive herm form on V is $|a|^2 = \bar{a} a = a^\dagger 1 a$

$a \longmapsto b = \bar{a}$ $\qquad a \mapsto a^t$ = linear fnl

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Start again: $\quad H(\mathbb{C}) = \begin{bmatrix} \mathbb{C} \\ \mathbb{C} \end{bmatrix} \quad \begin{bmatrix} a_1 \\ b_1 \end{bmatrix} \wedge \begin{bmatrix} a_2 \\ b_2 \end{bmatrix} = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$

Identify V^* with $V^\dagger$ $\qquad V^\dagger \longrightarrow V^*$

$\qquad a^\dagger \quad a^t$

~~Start~~ again. $V = \mathbb{C}^n = \left\{ \begin{bmatrix} a_1 \\ a_n \end{bmatrix} \right\}$, $\cancel{V^* = \{[b_1 \cdots b_n]\}}$

$V^* = \left\{ \begin{bmatrix} b_1 \\ \\ b_n \end{bmatrix} \right\}$ $\quad (\underline{b}, \underline{a}) = \underline{b}^t \underline{a}$ Also you have the hermitian symmetric form $\langle a' | a \rangle = a'^\dagger a$

$V = \{a \in \mathbb{C}^n\}$ $\quad a = \begin{bmatrix} a_1 \\ a_n \end{bmatrix}$ $\quad$ ~~$\bar{V}$ = V with λ~~

~~$\lambda \bar{b} =$~~

$V^* = \{b^t \mid b \in \mathbb{C}^n\}$. $\quad$ duality pairing $b^t a$

$\bar{V} = \{\bar{a} \mid a \in \mathbb{C}^n\}$ $\quad \lambda \bar{a} = \overline{\lambda a}$

$V^\dagger = \{b^\dagger = \bar{b}^t \mid b \in \mathbb{C}^n\}$.

not clean yet

have iso. $V \simeq V^\dagger$

$a \longmapsto a^\dagger$

(52) $V = \mathbb{C}^n = \left\{ a = \begin{bmatrix} a_1 \\ a_n \end{bmatrix},\ a_i \in \mathbb{C} \right\}$

$V^* = \mathrm{Hom}(V, \mathbb{C})$ can be ident w. $\left\{ b = \begin{bmatrix} b_1 \\ b_n \end{bmatrix} \in \mathbb{C}^n \right\}$

via the pairing $b^t a = \sum b_i a_i$

You also have the hermitian positive form

$$\langle b | a \rangle = \bar{b}^t a = \sum_i \bar{b}_i a_i$$

~~Other~~ Representation of linear functional says

$\bar{V} \longrightarrow V^*$???

Intrinsic stuff first. V f.d. complex v.s.

$V^* = \mathrm{Hom}(V, \mathbb{C})$ $\qquad V^* \times V \longrightarrow \mathbb{C}$

sesquilinear form $\bar{V} \times V \longrightarrow \mathbb{C}$ same as $V \longrightarrow \bar{V}^* = V^\dagger$

$f(\lambda v_1, \mu v_2) = \bar{\lambda} f(v_1, v_2) \mu$ $\qquad \overline{f(v_1, v_2)} = f(v_2, v_1)$ herm symm.

$v_2 \longmapsto (v_1 \longmapsto f(v_1, v_2))$ $\qquad V \to V^\dagger$

V fin diml v.s. over $\mathbb{C}$. $\qquad V^* = \mathrm{Hom}(V, \mathbb{C})$

canon pairing $V^* \times V \longrightarrow \mathbb{C}$

$(\varphi, v) \longmapsto \varphi(v)$

$B(v_1, v_2)$ sesquilinear $\qquad B(\lambda v_1, \mu v_2) = \bar{\lambda} B(v_1, v_2) \mu$

the same as $v_1 \longmapsto \underbrace{(v_2 \longmapsto B(v_1, v_2))}_{\in V^*}$

as a map $\bar{V} \longrightarrow V^*$, which is the same a

a $\mathbb{C}$-linear map $V \longrightarrow \bar{V}^* = V^\dagger$

$\overline{B(v_1, v_2)} = B(v_2, v_1)$ herm. symmetry

better

$v_1 \longmapsto (v_2 \longmapsto \overline{B(v_1, v_2)})$, $\bar{V} \longrightarrow \bar{V}^*$

$v_1 \longmapsto (v_2 \longmapsto B(v_2, v_1))$, $V \longrightarrow V^\dagger$

Repeat $\overline{B(v_1, v_2)} = B(v_2, v_1)$.

$$v_1 \longmapsto (v_2 \longmapsto \overline{B(v_1, v_2)}) \qquad V \longrightarrow V^\dagger$$

$$v_1 \longmapsto (v_2 \longmapsto B(v_2, v_1)) \qquad V \longrightarrow V^\dagger$$

So hermitian symmetry of $B(v_1, v_2)$ is the same as the two possible maps $V \longrightarrow V^\dagger$ for a sesquilinear form coinciding.

~~Let's go back to matrices, vectors~~ Choose a positive hermitian form on V, and use it to identify V and V^* ?? Not possible, but you can identify $\bar{V}$ and V^*. So return to $H(V) = \begin{bmatrix} V \\ V^* \end{bmatrix} = \begin{bmatrix} V \\ \bar{V} \end{bmatrix}$ which might be $\mathbb{H} \otimes_{\mathbb{C}} V$. Seems OK because you have combined complex conjugation with an involution interchanging V and V^*.

What about the old idea about a finite diml Hilbert space being phase space, where the real and imaginary parts of the hermitian scalar product are the Hamiltonian and symplectic form? e.g. $\mathbb{C}$

$$\overline{(x_1 + iy_1)}(x_2 + iy_2) = (x_1x_2 + y_1y_2) + i(x_1y_2 - y_1x_2) = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}^{t} \begin{bmatrix} 1 & i \\ -i & 1 \end{bmatrix} \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}$$

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$W = \begin{bmatrix} \mathbb{C} \\ \mathbb{C} \end{bmatrix}$; $\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}^t \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = x_1 y_2 + y_1 x_2$

You want to define a conjugation on W respecting the symmetric bilinear form. Simplest case is

$\sigma \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \bar{x} \\ \bar{y} \end{bmatrix}$. Then $W^\sigma = \begin{bmatrix} \mathbb{R} \\ \mathbb{R} \end{bmatrix}$ with same hyperbolic form, but over $\mathbb{R}$.

Try $\sigma \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \bar{y} \\ \bar{x} \end{bmatrix}$ $\quad \sigma(x_1 y_2 + y_1 x_2) = \bar{x}_1 \bar{y}_2 + \bar{y}_1 \bar{x}_2$

$\cancel{\begin{bmatrix} \bar{x}_1 \\ \bar{y}_1 \end{bmatrix}^t \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \bar{x}_2 \\ \bar{y}_2 \end{bmatrix} = \bar{x}_1 \bar{y}_2 + \bar{y}_1 \bar{x}_2}$

$\sigma \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} \bar{y}_1 \\ \bar{x}_1 \end{bmatrix}$ $\quad \begin{bmatrix} \bar{y}_1 \\ \bar{x}_1 \end{bmatrix}^t \begin{bmatrix} 0 & +1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \bar{y}_2 \\ \bar{x}_2 \end{bmatrix} = \bar{y}_1 \bar{x}_2 + \bar{x}_1 \bar{y}_2$

Go back to something ~~[illegible]~~ concrete, namely $\mathbb{H} \otimes_{\mathbb{C}} V = \begin{bmatrix} 1 \otimes V \\ j \otimes V \end{bmatrix}$. this is a vector space over $\mathbb{H}$, there should be some scalar product? Suppose V equipped with pos. def herm. form. An orthonormal basis of V ~~should [illegible]~~ yields $\mathbb{H} \otimes_{\mathbb{C}} V \xrightarrow{\sim} \mathbb{H}^n$. Given $\mathbb{H}^n$ the orthogonal $\oplus$ of the "inner product" on $\mathbb{H}$.

Your problem is to find a positive definite structure on $\mathbb{H} \otimes_{\mathbb{C}} V$ This should be obvious by tensor product. $V = \mathbb{C}^n \quad \mathbb{H} \otimes_{\mathbb{C}} V \simeq \mathbb{C}^{2n}$

What distinguishes $\mathbb{H} \otimes_{\mathbb{C}} V$ from $\mathbb{C}^{2n}$ is left mult by j. Focus on this point: An $\mathbb{H}$ vector space is a $\mathbb{C}$ vector space W equipped with a special operator

55 j such that $j \lambda w = \bar{\lambda} j w \quad \lambda \in \mathbb{C}$

$j^2 = -1$

What happens if $j^2 = 1$. Given a complex vector space W and such a $\underbrace{j}_{\text{call it } \varepsilon}$ you can split $W = W_+ \oplus W_-$ $\quad \varepsilon = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ on $\begin{bmatrix} W_+ \\ W_- \end{bmatrix}$

$\varepsilon \lambda = \bar{\lambda} \varepsilon \quad \forall \lambda \in \mathbb{C}. \quad \Rightarrow \varepsilon i = -i \varepsilon$

Notice that ε is not $\mathbb{C}$ linear

So where next? V complex vector space n dim

$H(V) = \begin{bmatrix} V \\ V^* \end{bmatrix}$ hyperbolic, either $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ or $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

symm. $\quad$ skewsymm.

~~Pick a pos herm symm. form on V, i.e. an isom $\bar{V} \xrightarrow{\sim} V^*$ $\quad v_1 \longmapsto (v_2 \longmapsto B(v_1, v_2))$~~

~~herm. symm means $\overline{B(v_1, v_2)} = B(v_2, v_1)$~~

~~$\bar{V} \to V^*$~~

~~$v_1 \longmapsto (v_2 \longmapsto B(v_1, v_2))$ $\quad$ want transpose $V^{**} = V \longrightarrow \bar{V}^*$?~~

~~$v_1 \longmapsto$~~

---

pairing $B(v_1, v_2)$ of type $(-1, 1)$

B is a pairing $\bar{V} \times V \longrightarrow \mathbb{C}$

$H(V) = \begin{bmatrix} V \\ V^* \end{bmatrix}$ hyperbolic either $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ or $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

Choose pos. herm. $B(v_1, v_2)$ get $\bar{V} \xrightarrow{\sim} V^*$

$v_1 \longmapsto (v_2 \mapsto B(v_1, v_2))$

Then $H(V) = \begin{bmatrix} V \\ \bar{V} \end{bmatrix}$ complex vector space

You have $\sigma : V \longrightarrow \bar{V}$

V n-diml v.s. with $\langle v_1 | v_2 \rangle = v_1^\dagger v_2$

$\Vert$

$\mathbb{C}^n$ equipped with $v_1^\dagger v_2$

$\mathbb{C}^n$ hermitian scalar product $v^\dagger w = \sum \bar{v}_i w_i$

$$\overline{\mathbb{C}}^n \xrightarrow{\sim} (\mathbb{C}^n)^\wedge$$

$$v \longmapsto (w \mapsto v^\dagger w)$$

$$\begin{bmatrix} \mathbb{C}^n \\ \overline{\mathbb{C}}^n \end{bmatrix} \qquad \sigma \begin{bmatrix} v \\ w \end{bmatrix} = \qquad v^\dagger$$

V complex vector space of dim n equipped with pos. herm. inner product.

$$H(V) = \begin{bmatrix} V \\ V^\wedge \end{bmatrix} \quad \text{equipped with} \quad \begin{bmatrix} a \\ \alpha \end{bmatrix}^t \begin{bmatrix} 0 & 1 \\ +1 & 0 \end{bmatrix} \begin{bmatrix} b \\ \beta \end{bmatrix} = \begin{matrix} a^t\beta \\ + \\ \alpha^t b \end{matrix}$$

To define a conjugation σ on $H(V)$. $\sigma\lambda = \bar{\lambda}\sigma$, $\sigma^2 = 1$

If $a \in V$ let $\sigma \begin{bmatrix} a \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ a^\dagger \end{bmatrix}$

If $\alpha = \,?$ Note since $\overline{V} = V^\wedge$ any elt of $H(V)$ is a pair $\begin{bmatrix} a \\ b^\dagger \end{bmatrix} \in \begin{bmatrix} V \\ \overline{V} \end{bmatrix}$. Now

define $\sigma \begin{bmatrix} a \\ b^\dagger \end{bmatrix} = \begin{bmatrix} b \\ a^\dagger \end{bmatrix}$ $\qquad \sigma^2 \begin{bmatrix} a \\ b^\dagger \end{bmatrix} = \sigma \begin{bmatrix} b \\ a^\dagger \end{bmatrix} = \begin{bmatrix} a^{\dagger\dagger} \\ b^\dagger \end{bmatrix} = \begin{bmatrix} a \\ b^\dagger \end{bmatrix}$

$$\sigma i \begin{bmatrix} a \\ b^\dagger \end{bmatrix} = \sigma \begin{bmatrix} ia \\ ib^\dagger \end{bmatrix} = \begin{bmatrix} (ib^\dagger)^\dagger \\ (ia)^\dagger \end{bmatrix} = \begin{bmatrix} -ib \\ -ia^\dagger \end{bmatrix} = -i\sigma \begin{bmatrix} a \\ b^\dagger \end{bmatrix}$$

What elements are fixed under σ?

$$\sigma \begin{bmatrix} a \\ b^\dagger \end{bmatrix} = \begin{bmatrix} b \\ a^\dagger \end{bmatrix} \overset{?}{=} \begin{bmatrix} a \\ b^\dagger \end{bmatrix} \iff a = b.$$
