$$g = \begin{pmatrix}0&1\\-1&1\end{pmatrix} \cdot \begin{pmatrix}0&1\\-1&1\end{pmatrix} = \begin{pmatrix}-1&+1\\-1&0\end{pmatrix} \cdot \begin{pmatrix}0&1\\-1&1\end{pmatrix} = \begin{pmatrix}-1&0\\0&-1\end{pmatrix}$$

$$g = \begin{pmatrix}0&-1\\1&-1\end{pmatrix} \cdot \begin{pmatrix}0&-1\\1&-1\end{pmatrix} = \begin{pmatrix}-1&1\\-1&0\end{pmatrix} \cdot \begin{pmatrix}0&-1\\1&-1\end{pmatrix} = \begin{pmatrix}1&0\\0&1\end{pmatrix}$$

$$g^* x = -y \qquad g^* y = x-y$$

$$g^*(x\,dy) - x\,dy = -y\,d(x-y) - x\,dy = -y\,dx - x\,dy + y\,dy = d\left(-xy + \frac{y^2}{2}\right)$$

$Q = 2\pi i\left(-xy + \frac{y(y-1)}{2}\right)$. Thus

$$\psi \in \mathcal{L} \Longrightarrow e^{2\pi i xy - \frac{y(y-1)}{2}}\, \psi(-y, x-y) \in \mathcal{L}$$

$$Q' = \frac{1}{2\pi i} Q = -xy + \frac{y^2}{2} - \frac{y}{2} \qquad \Big| \quad -xy + \frac{y^2}{2} \quad \Big| \quad -\frac{y}{2}$$

$$g^*Q' = +(+y)(x-y) + \frac{(x-y)^2}{2} - \frac{x-y}{2}$$

$$= \cancel{xy} + y^2 + \frac{x^2}{2} - \cancel{xy} + \frac{y^2}{2} - \frac{x}{2} + \frac{y}{2}$$

$$= -\frac{y^2}{2} + \frac{x^2}{2} - \frac{x}{2} + \frac{y}{2} \qquad \Big| \quad -\frac{y^2}{2} + \frac{x^2}{2} \quad \Big| \quad -\frac{x}{2} + \frac{y}{2}$$

$$g^*g^*Q' = -\frac{(x-y)^2}{2} + \frac{(-y)^2}{2} + \frac{y}{2} + \frac{x-y}{2}$$

$$= -\frac{x^2}{2} + xy - \frac{y^2}{2} + \frac{y^2}{2} + \frac{x}{2}$$

$$= -\frac{x^2}{2} + xy + \frac{x}{2} \qquad \Big| \quad -\frac{x^2}{2} + xy + \frac{y^2}{2} \quad \Big| \quad \frac{x}{2}$$

Does $e^Q g^*$ $\quad g = \begin{pmatrix}0&-1\\1&-1\end{pmatrix}$ $\quad Q = 2\pi i\left(-xy + \frac{y(y-1)}{2}\right)$

commute with the central element ε^*: $\psi(x,y) \longmapsto \psi(-x,-y)$.

$$\varepsilon = \begin{pmatrix}-1&0\\0&-1\end{pmatrix}, \quad \begin{pmatrix}ac&bc\\bc&bd\end{pmatrix} = \begin{pmatrix}0&0\\0&0\end{pmatrix}$$

$$\varepsilon^* Q' = \varepsilon^*\left(-xy + \frac{y(y-1)}{2}\right) = -(-x)(-y) + \frac{(-y)(-y-1)}{2}$$

$$= -xy + \frac{y^2+y}{2}$$

answer No. $\quad \begin{pmatrix}0&-1\\1&-1\end{pmatrix}\begin{pmatrix}0&-1\\1&-1\end{pmatrix} = \begin{pmatrix}-1&1\\-1&0\end{pmatrix} \qquad \begin{pmatrix}1&-1\\-1&0\end{pmatrix}^{ca}$

$$\psi(x,y) \longmapsto e^{2\pi i\left(-xy + \frac{x^2-x}{2}\right)}\, \psi(x-y, -x)$$

$$\tau = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}(\tau) = \frac{-1}{\tau - 1} \Rightarrow \tau^2 - \tau + 1 = 0 \Rightarrow \tau = \frac{-1 \pm i\sqrt{3}}{2} \text{ if } \tau \in \text{UHP}$$

stabilizer of $\frac{-1+i\sqrt{3}}{2}$ is $\left\{ g = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}, \pm I, g^2 = \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}, -g, -g^2 \right\}$,
in fact it's the cyclic group of order 6 generated by
$-g = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}$.

Let's go back to the problem of identifying the principal $\mathbb{T}$-bundle P associated to L with the Heisenberg group H. What is H? It's the central group extension with cross-section

Notation P, L refer to absence of automorphy condition

$$\mathbb{T} \longrightarrow H \longrightarrow \mathbb{R}^2$$

associated to the bilinear cocycle $c((x_1,y_1),(x_2,y_2)) = e^{2\pi i y_1 x_2}$.
But you should start with the operators on $\mathcal{L}$. Recall

$D_x = \partial_x$, $D_y = \partial_y + 2\pi i x$.

$$e^{aD_x} e^{bD_y} \psi(x,y) = e^{a\partial_x} e^{b 2\pi i x} e^{b\partial_y} \psi(x,y)$$
$$= e^{2\pi i b x} e^{a\partial_x} e^{b\partial_y} \psi(x,y) = e^{2\pi i b x} \psi(x+a, y+b)$$

$$e^{a_1 D_x} e^{b_1 D_y} e^{a_2 D_x} e^{b_2 D_y} = e^{a_1 D_x} e^{a_2 D_x} e^{[b_1 D_y, a_2 D_x]} e^{b_1 D_y} e^{b_2 D_y}$$
$$= e^{2\pi i b_1 a_2} e^{(a_1+a_2) D_x} e^{(b_1+b_2) D_y}$$

Let's begin with H described as $\mathbb{T} \times \mathbb{R}^2$ with mult $(e^{i\varphi_1}, a_1, b_1)\cdot(e^{i\varphi_2}, a_2, b_2) = (e^{i(\varphi_1+\varphi_2+2\pi b_1 a_2)}, a_1+a_2, b_1+b_2)$

You want the infinitesimal left (resp right) translations

$$\delta\psi(e^{i\varphi}, a, b) = \partial_z \psi(e^{i\varphi}, a, b)\, i e^{i\varphi}\delta\varphi$$
$$+ \partial_x \psi(e^{i\varphi}, a, b)\,\delta a + \partial_y \psi(e^{i\varphi}, a, b)\,\delta b$$

$\delta e^{i\varphi} = i e^{i\varphi}\delta\varphi$

$\frac{\delta z}{iz} = \delta\varphi$

Try $(e^{i\varphi}, x, y)\cdot(1+i\delta\varphi, \delta x, \delta y)$

$$= (\underbrace{e^{i\varphi}(1+i\delta\varphi)\,e^{iy\delta x}}, x+\delta x, y+\delta y)$$

$$\psi\left(e^{i\varphi}(1+i\delta\varphi+iy\delta x), x+\delta x, y+\delta y\right)$$

$$= \psi(e^{i\varphi}, x, y) + z\partial_z\psi(e^{i\varphi}, x, y)\, i(\delta\varphi + y\delta x) + \partial_x\psi\,\delta x + \partial_y\psi\,\delta y$$

So the variation of ψ under the infinitesimal right translation by $(e^{i\delta\varphi}, \delta x, \delta y)$ should be

$$iz\partial_z\psi\;\delta\varphi + (\partial_x\psi + iy)\,\delta x + (\partial_y\psi)\,\delta y$$

You need to put in the 2π

$$\psi(e^{2\pi i\varphi}, x, y)$$

$$\psi + \delta\psi = \psi\left((e^{2\pi i\varphi}, x, y)\cdot(e^{2\pi i\delta\varphi}, \delta x, \delta y)\right)$$

$$= \psi(\underbrace{e^{2\pi i(\varphi+\delta\varphi+y\delta x)}}_{e^{2\pi i\varphi}(1+2\pi i\delta\varphi+2\pi i y\delta x)}, x+\delta x, y+\delta y)$$

$$= \psi + z\partial_z\psi\; 2\pi i(\delta\varphi + y\delta x) + \partial_x\psi\,\delta x + \partial_y\psi\,\delta y$$

$$\delta\psi = (2\pi i z\partial_z\psi)\,\delta\varphi + (2\pi i z\partial_z\psi)\,y\,\delta x + \partial_x\psi\,\delta x + \partial_y\psi\,\delta y$$

$$= (2\pi i z\partial_z\psi)\,\delta\varphi + (\partial_x\psi + 2\pi i y\, z\partial_z\psi)\,\delta x + (\partial_y\psi)\,\delta y$$

So this inf right mult yields the vector fields

$$\boxed{2\pi i z\partial_z\,,\quad \nabla_x = \partial_x + 2\pi i y z\partial_z\,,\quad \nabla_y = \partial_y}$$

Next look at inf left translations

$$\psi + \delta\psi = \psi\left((e^{2\pi i\delta\varphi}, \delta x, \delta y)\cdot(e^{2\pi i\varphi}, x, y)\right)$$

$$= \psi\left(e^{2\pi i(\delta\varphi+\varphi)}\,e^{2\pi i(\delta y)x}, \delta x + x, \delta y + y\right)$$

$$= \psi\left(e^{2\pi i\varphi}(1+2\pi i\,\delta\varphi+2\pi i x\,\delta y),\ x+\delta x,\ y+\delta y\right)$$

$$= \psi\left(e^{2\pi i\varphi}, x, y\right) + \partial_z\psi\left(e^{2\pi i\varphi}, x, y\right)e^{2\pi i\varphi}\,2\pi i(\delta\varphi + x\,\delta y) + \partial_x\psi\,\delta x + \partial_y\psi\,\delta y$$

$$= \psi\left(e^{2\pi i\varphi}, x, y\right) + 2\pi i z\,\partial_z\psi(\ )(\delta\varphi + x\,\delta y) + \partial_x\psi(\ )\,\delta x + \partial_y\psi(\ )\,\delta y$$

$$\delta\psi = 2\pi i z\,\partial_z\psi\,\delta\varphi + \partial_x\psi\,\delta x + \left(\partial_y + 2\pi i x z\,\partial_z\right)\delta y$$

So inf left mult. yields the vector fields

$$\boxed{2\pi i z\,\partial_z\ ,\quad D_x = \partial_x\ ,\quad D_y = \partial_y + 2\pi i x\, z\,\partial_z}$$

$z = e^{2\pi i\varphi}$

$dz = 2\pi i z\, d\varphi$

$\partial_z = \dfrac{\partial}{\partial z} = \dfrac{1}{2\pi i z}\dfrac{\partial}{\partial\varphi}$

$\dfrac{\partial}{\partial\varphi} = 2\pi i z\,\partial_z$

Note that if we use $z = e^{2\pi i\varphi}$, then

$\partial_\varphi = 2\pi i z\,\partial_z$ because $\varphi = \dfrac{1}{2\pi i}\log z$

$d\varphi = \dfrac{1}{2\pi i}\dfrac{dz}{z}$ $\quad \partial_\varphi = 2\pi i z\,\partial_z$

also $[D_x, D_y] = D_\varphi$

Let $\psi \in \mathcal{L}$: $\psi(x,y)$ periodic in y but !! $\psi(x+m, y) = e^{-2\pi i m y}\psi(x,y)$, the phase is in some sense linear in x. You want perhaps to describe the situation using tools from your determinant line bundle papers. On the determinant line bundle you put a hermitian metric, you calculated the curvature, then you introduce an exponential factor to cancel the curvature, the resulting connection is flat yielding the desired determinant up to scalar factor.

Where do you start? With $\mathcal{L}$ and the autom. condition

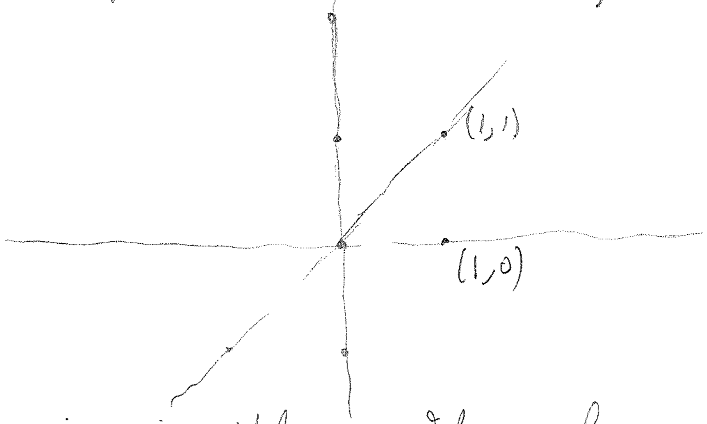
$$\psi(x+m, y+n) = e^{-2\pi i m y}\psi(x,y) \quad \text{or} \quad \psi = e^{m(\partial_x + 2\pi i y)}e^{n\partial_y}\psi$$

periodic in y; in the vertical direction; in the x direction the phase is linear in m.

Let's pay attention to the periodic line; line means summand of $\mathbb{Z}^2$ of rank 1. There's the real version also. Given a line l, i.e. point of $P_1\mathbb{Q}$, what else do you need to obtain an automorphic condition?

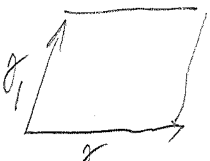
Given a $\mathbb{Z}$-line l, i.e. a subgroup $l \subset \mathbb{Z}^2$ of rank 1 such that $\mathbb{Z}^2/l$ is free, what is needed to obtain an automorphic condition having periodicity l.

Consider the case $l = \{(0,n) \mid n \in \mathbb{Z}\} \subset \mathbb{Z}^2$; thus $l = \mathbb{Z}(0,1)$.
Possible complements are $\mathbb{Z}(1,n) \subset \mathbb{Z}^2$ for any $n \in \mathbb{Z}$.



Discuss invariantly. You have a free abelian group Γ of rank 2, the real plane $\Gamma \otimes_{\mathbb{Z}} \mathbb{R}$ it generates and a basis γ_1, γ_2 for Γ. Equip $\Gamma_{\mathbb{R}}$ with the volume such that $\gamma_1 \wedge \gamma_2 \mapsto 1$. You want to understand?

Start again with the lattice Γ in the real plane $\Gamma_{\mathbb{R}} = \Gamma \otimes_{\mathbb{Z}} \mathbb{R}$ such that volume is 1 on $\Gamma_{\mathbb{R}}/\Gamma$. Pick a unimodular basis for Γ, say γ_1, γ_2 so that the ▱



has vol $= 1$

Is there a natural autom. condition associated to this basis? You want periodicity in the γ_1 direction, and also some

Try a different viewpoint, namely, consider a connection on the plane, rather, consider a line bundle L over the plane equipped with a connection whose curvature is a translation invariant purely imaginary non degenerate 2 form. There's an associated Heisenberg group action on sections of this line bundle, because you are given a connection on L. Suppose now that the plane comes from of lattice: $\Gamma_{\mathbb{R}} = \Gamma_{\mathbb{Z}} \otimes \mathbb{R}$, and the "volume" of $\Gamma_{\mathbb{R}}/\Gamma_{\mathbb{Z}}$ is integral

6 Let's look at the symmetries of L over $\mathbb{R}^2$. 1180

Review the classical mechanics, Hamilton Jacobi theory.

You have a symplectic manifold $(\mathbb{R}^2, dxdy)$, this is phase space. Recall that phase space is ordinarily the cotangent bundle of configuration space. There is a canonical 1-form pdq with $d(pdq) = dpdq$, the symp. form.

Contact transformations: $PdQ - pdq = dF$. θ

In the case of the plane an

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Your aim is to understand "geometric quantization" in the special case of 1 degree of freedom. Various things come to mind such as Feynmann's path integral picture.

$\boxed{\text{Idea: Role of convexity}}$ You know this appears in the Legendre transform (see book by Hörmander), convexity of the moment mapping (Atiyah, Guilleman-Sternberg, Kirwan).

Q: Is there any link between convexity of this sort and partitions of unity?

Go back to phase space for 1 degree of freedom: the (q,p)-plane equipped with the 1-form pdq giving the action leading to Hamilton's principle saying the dynamics are paths of stationary phase for the action $\int p\dot{q}\,dt$.

What are the symmetries of phase space? Because you've given the 1-form pdq, you should have a canonical complex line bundle equipped with connection having constant curvature, nondegenerate purely imaginary. The line bundle is unique up to scalar multiplication.

What are the symmetries of phase space? Say configuration space is an affine line, then phase space should be the cotangent bundle of this affine line; if x is the coordinate on the line, then the canonical 1-form is $y\,dx = p\,dq$.

Still not clear! Apparently you want to start with the real plane equipped with translation invariant symplectic structure. Choose a 1-form η with $d\eta = \omega$; you want to restrict to "bilinear" forms.

April 20, 02. Consider phase space for 1 degree of freedom, in other words the cotangent bundle (T^*X) of an affine line X.

Choose a coord $x : X \xrightarrow{\sim} \mathbb{R}$; a point of T^*X over x is linear functional $y\,dx$ on the tangent space $(TX)_x$ at x. $(x,y) : T^*X \xrightarrow{\sim} \mathbb{R}^2$. There's a canonical 1-form, the contact form $y\,dx$ on T^*X, whose diffl $dy\,dx = -dx\,dy$ is a symplectic form on T^*X.

To study symplectic diffeos of phase space arising from affine linear transfs, which must be in $SL(2,\mathbb{R}) \ltimes \mathbb{R}^2$ in order to preserve the symp. form. Your aim is to understand the automorphic conditions under ~~such~~ symplectic diffeos. respecting ~~the~~ the behavior of integral structure.

The first thing to understand is whether, and if so, how there is a link between the 1-form and the automorphic condition.

Take $d +_x^{2\pi i} y\,dx = dx(\partial_x + 2\pi i y) + dy(\partial_y)$

$$d + 2\pi i\, x\,dy = dx(\partial_x) + dy(\partial_y + 2\pi i x)$$

Recall usual form

$$e^{m(\partial_x + 2\pi i y)} e^{n(\partial_y)} \psi(x,y) = \boxed{e^{2\pi i m y} \psi(x+m, y+n) = \psi(x,y)}$$

What does this mean? ψ inv under $y \mapsto y+n$

What is special about the 1-forms that occur? They have the same curvature as $y\,dx$, hence differ from $y\,dx$ by an exact 1-form, the differential of a form of degree ≤ 2 in x, y.

You ought to review symplectic stuff, especially the link between Lagrangian subspaces and quadratic forms.

Let V be a vector space, $W = V \oplus V^*$ equipped with the anti symm. form $A(v \oplus \lambda, v' \oplus \lambda') = \lambda(v') - \lambda'(v)$. Let $\Gamma_T = \{v \oplus Tv\}_{v \in V}$ be isotropic i.e. $0 = A(v \oplus Tv, v' \oplus Tv') = (Tv)(v') - (Tv')(v)$ equivalently $T : V \longrightarrow V^*$ satisf $T = T^*$. Note: $T^*v \in V^*$ is defined by $(T^*v)(v') = (Tv')(v)$. Thus Γ_T is a Lagrangian subspace $\Longleftrightarrow$ $T = T^*$.

How is this related to something like $y\,dx$? You can interpret $y\,dx$ as a bilinear form in 2 variables

$$\begin{pmatrix} x \\ y \end{pmatrix}^* \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} d \begin{pmatrix} x \\ y \end{pmatrix}$$

which skew-symmetrizes to $\begin{pmatrix} dx \\ dy \end{pmatrix}^* \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} dx \\ dy \end{pmatrix} = dy\,dx - dx\,dy$.

Answer: Consider two symplectic vector spaces W_1, W_2 and a symplectic isom. $U : W_1 \longrightarrow W_2$. Form s.d.s symp. d.s. $W_1 \oplus W_2$ with symp. form ω_1 on W_1 and $-\omega_2$ on W_2. The graph of U should be Lagrangian, which should translate into a quadratic form.

But the situation might be more interesting when you introduce translation

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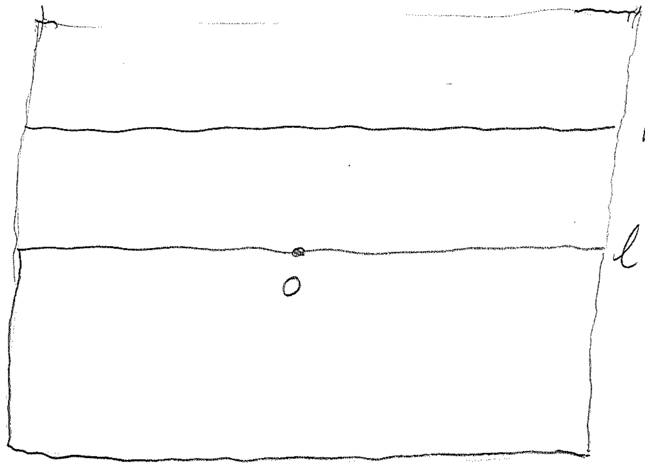
Idea: Novikov in his talk about metals mentioned quasi-momenta, points in a 3 torus, which contains the Fermi surface, on which there is some kind of dynamics.

Phase space is just $\mathbb{R}^2$ equipped with the 1-form $y\,dx$. Here $\mathbb{R}^2$ is viewed as affine space. A better way to put this might be to use the affine plane $z = 1$ in $\mathbb{R}^3$. When does $g \in GL(n+1, \mathbb{R})$ preserve the last coordinate: $(0 \cdots 0\ 1) \left(\begin{array}{c|c} * & \begin{matrix} * \\ * \\ * \end{matrix} \\ \hline 0 & 1 \end{array}\right)$

April 21, 2002

1 degree freedom, configuration space $\simeq$ real affine line, a torsor L for the Lie group $(\mathbb{R}, +)$. $\mathbb{R}^\times = GL(1, \mathbb{R})$ is the group of symmetries of this Lie group. Then $\mathbb{R}^\times$ and $(\mathbb{R}, +)$ combined suitably (semi direct product $\mathbb{R}^\times \ltimes (\mathbb{R}, +)$) yield the symmetries of the affine line. Next comes phase space, the cotangent bundle $T^*\mathbb{R}$.

Problem. Intrinsic definition of an affine line L as a nonzero coset of a 1-diml subspace l of a 2 diml vector space V. Any coset of l is a torsor for l. Can you reconstruct V from l and L?



The idea here is that a standard way to represent an affine space L is via a linear fnl $f: V \longrightarrow \mathbb{R}$ as $L = f^{-1}(1)$

At some point you need to understand Suslin's theorem on excision for K-theory of h-unital rings.

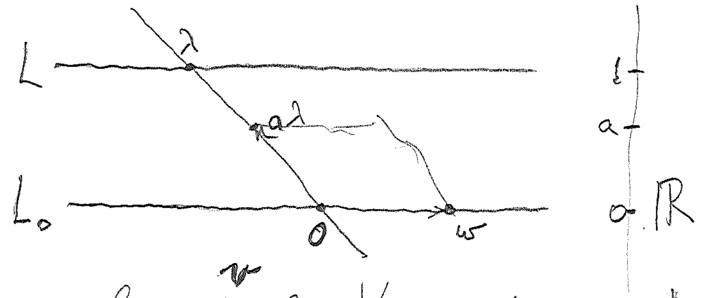
Go back to affine line $\mathbb{R}$. Do affine spaces form a category like vector spaces?

April 22, 02 Discuss the category of affine spaces. Hopefully these are related to partitions of 1. The first step should be to identify an affine space L of dimension n with a pair (V, f), where V is v.s. of dim $n+1$ and $f: V \twoheadrightarrow \mathbb{R}$ is a nonzero linear fnl. Then $L = f^{-1}(1)$. These $f: V \twoheadrightarrow \mathbb{R}$ form a category.

Another definition of an affine space is a torsor L for a vector space L_0, i.e. an action $L \times L_0 \xrightarrow{\mu} L$ such that $L \times L_0 \xrightarrow[\sim]{(pr_1, \mu)} L \times L$.

Can you construct $f: V \longrightarrow \mathbb{R}$ with $f^{-1}(1) = L$?

Given a torsor L for the vector space W you want to construct $V \xrightarrow{f} \mathbb{R}$ such that $L = f^{-1}(1)$, $W = f^{-1}(0)$.



It seems that V should be described by generators and relations, something like $a\lambda + w$. Note that every element v of V not in W is uniquely $v = a\lambda$ with $\lambda \in L$ and $a \in \mathbb{R}^\times$. You should only need to get elements of W. So you choose $\lambda \in L$ and consider $a\lambda + w$ with $a \in \mathbb{R}$, $w \in W$. These form a vector space $\simeq \mathbb{R} \times W$.

Exact sequence $0 \longrightarrow W \longrightarrow V \longrightarrow \mathbb{R} \longrightarrow 0$

Legendre transform in 1-dim = poor man's F.T.

$$\int e^{-x\xi} e^{F(x)}\, dx \sim e^{-(x\xi - F)}$$

where $\frac{d}{dx}(x\xi - F) = 0$

i.e. $\xi = \frac{dF}{dx}$

Use $\xi = \frac{dF}{dx}$ and Implicit Fn Thm. to make $G = x\xi - F$ to regard G as a function of ξ. Then

$$\frac{dG}{d\xi} = x + \cancel{\frac{dx}{d\xi}\xi - \frac{dF}{dx}\frac{dx}{d\xi}} = x.$$

Apr 23, 02

$$\{\text{Affine spaces}\} \xleftarrow{\sim} \{V \xrightarrow{f} \mathbb{R}\}.$$

$$f^{-1}(1) \longleftarrow\!\shortmid \ (V \xrightarrow{f} \mathbb{R})$$

Here define affine space L as a torsor under the $\overbrace{\text{group}}^{\text{additive}}$ of a vector space W. Thus there is an action $L \times W \xrightarrow{\mu} L$ such that $L \times W \xrightarrow{(pr_1, \mu)} L \times L$ is an isom. But W also has scalar multiplication by elts $a \in \mathbb{R}$ yielding

$$\begin{array}{ccccccc} L \times L & \xleftarrow{(pr_1,\mu)} & L \times W & \xrightarrow{1 \times a} & L \times W & \xrightarrow{\mu} & L \\ (\lambda_0, \lambda_1) & & (\lambda_0, \lambda_1 - \lambda_0) & & (\lambda_0, a(\lambda_1 - \lambda_0)) & & \underbrace{\lambda_0 + a(\lambda_1 - \lambda_0)}_{(1-a)\lambda_0 + a\lambda_1} \end{array}$$

Problem: Axioms for this operations. $\underbrace{\underline{\text{Barycentric}} \text{ linear combinations}}_{\text{partitions of } 1}$.

IDEA Convexity $\overset{\text{link}}{\longleftrightarrow}$ positive partitions of 1. Can you weaken positivity to some kind of semi-boundedness? From Poisson Sum Fmla work you get an x affine line with $\mathbb{Z}$ translation, a y-circle (1-torus) of quasi-momenta.

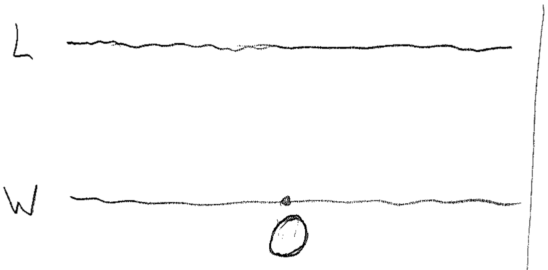
Novikov's metal talk suggests looking at a ~~surface~~ crystalline with $\mathbb{Z}^2$ translational symmetry, 2-torus of quasi-momenta, Fermi curves. Is this Fermi curve related to Novikov's version of Morse theory where the Morse function is $\mathbb{T}$-valued?

In your situation you have a complex line bundle over T^2

$$\begin{array}{ccc} L \times W & \xrightarrow{\sim} & L \times L \\ (\lambda, \omega) & \longmapsto & (\lambda, \lambda + \omega) \end{array}$$

Somehow you should be able to construct V from this.

Maybe V is a quotient of $\underset{n}{L} \times \underset{n}{W} \times \underset{1}{\mathbb{R}}$ by something n dimens.



$(\lambda, \omega, a) \longmapsto$

Go back to $W \xrightarrow{i} V \xrightarrow{f} \mathbb{R}$, $L = f^{-1}(1)$
$W = f^{-1}(0)$.

You want to obtain V as a quotient of W and L in some way, but L is not a vector space; moreover you need products $a\lambda$ for $a \in \mathbb{R}$, $\lambda \in L$. Note that $\mathbb{R} \times L$ is also not a vector space; you need $\mathbb{R}[L]$. There is a canonical map $\mathbb{R}[L] \longrightarrow V$ sending $\sum_{\lambda} a_\lambda [\lambda]$ to $\sum a_\lambda \lambda$ in V. Is this map onto? You need to show that W is in the image. $f(\sum_{\lambda \in L} a_\lambda \lambda) = \sum_{\lambda} a_\lambda = 0$, so it should be clear, namely given w choose $\lambda_0 - \lambda_1 = w$ and then take $a_{\lambda_0} = 1$, $a_{\lambda_1} = -1$.

So V is the vector space generated by the elements $[\lambda]$ for $\lambda \in L$ subject to the relations: $[\lambda + w] = [\lambda] + iw$ for all $\lambda \in L$ and $w \in W$. If $\lambda_0 \in L$ is chosen, then one has $\forall \lambda$: $[\lambda] = [\lambda - \lambda_0 + \lambda_0] = [\lambda_0] + i(\lambda - \lambda_0)$, so that $V = \mathbb{R}[\lambda_0] + iW$, etc.

Now you should be able to write up an account of affine spaces. These form a category equivalent to the category of v.s. V equipped with a surj $f : V \twoheadrightarrow \mathbb{R}$. There is a final object $\mathbb{R} \xrightarrow{1} \mathbb{R}$, but no initial object.

It should be true that free a" map of affine spaces is a set map respecting barycentric linear combinations. So the join of two affine spaces should be the category direct sum.

Given $W \longrightarrow V \xrightarrow{f} \mathbb{R}$, $W' \longrightarrow V' \xrightarrow{f'} \mathbb{R}$ get

$$\begin{tikzcd} W \times W' \arrow[r] & V \times V' \arrow[r] & \mathbb{R} \times \mathbb{R} \\ & \cup & \\ W \times W' \arrow[r] & V \times_{\mathbb{R}} V' \arrow[r] & \mathbb{R} \arrow[uu, "\Delta"'] \end{tikzcd}$$

$L \times L' = (f \times f')^{-1}(1,1)$ should give the product affine space $L \times L'$.

Let's now look at the cotangent bundle of the affine line $\mathbb{R}$. The tangent space of an affine space L at any point is the vector space W, so the tangent bundle is $L \times W$, and the cotangent bundle should be $L \times W^*$.

What's happening? You have the affine group operating on the affine space L and also on its cotangent bundle $L \times W^*$. Somewhere here should be Galilean symmetry, maybe?

You are looking at symmetries of the affine line first, this is configuration space, and then symmetries of phase space. You know the affine symmetries of the line $\mathbb{R}$ are $x \longmapsto ax + b \quad a \neq 0$. What are the natural affine symmetries of phase space?

Start with contact transformations: 4 variables P, Q, p, q: such that $P\,dQ - p\,dq = dS$. Example:

Hamilton's principle $\quad \delta \int_a^b L(q, \dot{q}, t)\, dt = 0$

$$\delta \int_a^b L(q,\dot{q},t)\,dt = \int_a^b \left(\frac{\partial L}{\partial q}\delta q + \frac{\partial L}{\partial \dot{q}} \underbrace{\delta \dot{q}}_{\frac{d}{dt}\delta q} \right) dt$$

$$= \int_a^b \left[\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\delta q\right) - \left(\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q}\right)\delta q \right] dt$$

$$= \left[\frac{\partial L}{\partial \dot{q}}\delta q\right]_a^b - \int \left(\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q}\right)\delta q\, dt$$

This vanishes for δq vanishing at a, b. $\Longrightarrow$ $\boxed{\text{Lagrange eqn of motion} \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = 0}$

Solve eqn of motion to get $q(t)$, then integrate

Apr 24, 02

Solve Lagrange eqn to get a family of paths $q(t)$ for $a \le t \le b$ which relate the initial conditions at a to the ones at b. Integrating L over these paths yields the action S satisfying

$$\delta S = \left[p\, \delta q \right]_a^b \qquad \text{where } p = \frac{\partial L}{\partial \dot q}$$

whence a contact transf.

Suppose P, Q, p, q are variables such that $P\,dQ = p\,dq$, for example $P = a^{-1}p$, $Q = aq$. Another example: $P = p$ and $Q = q + b$. Combining these two types yields

$$Q = aq + b \qquad\qquad a^{-1}p\, d(aq+b) = p\,dq$$
$$P = a^{-1}p$$

This should be the symmetry of phase space induced by the affine symmetry $q \longmapsto aq + b$ of $\mathbb{R}$. Converse: Is any diffeomorphism of $\mathbb{R}^2$ preserving $p\,dq$ necessarily affine? NO because any diffeo of the line induces a diffeom. of its phase space preserving the canonical 1-form. Note that $p = 0$ is where the 1-form $p\,dq$ vanishes, and that vertical lines $q =$ constant are the null curves for $p\,dq$.

A diffeomorphism g_P of phase space preserving $p\,dq$ has to induce a diffeom g_ℓ on the line $p = 0$. It seems likely that latter induces the former, for consider the "difference" between g_P on phase space and g_ℓ^* the induced diffeo on phase space. This g_d should be the identity on the line $p = 0$, and one has

$$p\,dq = g_d^*(p\,dq) = g_d^*(p)\, d g_d^*(q) = g_d^*(p)\, dq$$

so $g_d^*(p) = p$.

IDEA. It may be important to use scaling transformations because of <u>wavelet</u> theory, although there only $a = 2$ appears.

April 25, 02.

Consider phase space $\mathbb{R}^2$ with the canonical 1-form $\eta = p\,dq$. Define a <u>contact transformation</u> to be a pair (f, S_f) consisting of a diffeomorphism f of $\mathbb{R}^2$ and a function S_f on $\mathbb{R}^2$ satisfying $f^*\eta - \eta = dS_f$. Consider two contact transfos (f, S_f) and (g, S_g). Then

$$f^*\eta - \eta = dS_f\,,\quad g^*f^*\eta - g^*\eta = d(g^*S_f)\,,\quad g^*\eta - \eta = dS_g$$

$\Longrightarrow (fg)^*\eta - \eta = d(g^*S_f + S_g)$, so composition can be defined for contact transformations by:

$$(f, S_f)(g, S_g) = (fg,\ g^*S_f + S_g)$$

This seems to be the semi direct product of the symplectic diffeomorphism group operating the additive group of functions on phase space.

Now $f^*\eta - \eta = dS_f \Longrightarrow f^*(d\eta) = d\eta$ i.e. f is symplectic. Conversely if f is a symplectic diffeo, then $f^*\eta - \eta$ is closed, hence exact $= dS_f$ because phase space is contractible. S_f is determined up to an additive constant. So it seems that the group of contact transfs. is a central extension by $\mathbb{R}$ of the group of symplectic diffeos.

Next consider the infinitesimal situation. An infinitesimal contact transformation on phase space should be (X, φ_X), where X is a vector field and φ_X a function satisfying

$$\mathcal{L}_X\eta = d\varphi_X \qquad (\Longrightarrow \mathcal{L}_X(d\eta) = 0 \text{ so } X \text{ is a Hamiltonian v.f.})$$

Conversely if $0 = \mathcal{L}_X(d\eta) = d\iota_X d\eta$, then $\iota_X d\eta = d\varphi_X$, where φ_X is a function determined up to an additive constant.

Let's make inf contact transf. into a Lie algebra. Let $\mathcal{L}_X \eta = d\varphi_X$ and $\mathcal{L}_Y \eta = d\varphi_Y$. Then

$$\mathcal{L}_{[X,Y]}\eta = [\mathcal{L}_X, \mathcal{L}_Y]\eta = \mathcal{L}_X d\varphi_Y - \mathcal{L}_Y d\varphi_X = d(X\varphi_Y - Y\varphi_X)$$

so $[(X,\varphi_X),(Y,\varphi_Y)]$ should be $([X,Y], X\varphi_Y - Y\varphi_X)$. One can probably check easily the Jacobi identity, but instead let's exhibit a faithful representation of the inf contact transfs. on functions

Try the connection $\psi \mapsto (d+\eta)\psi$ on the trivial line bundle.

$$\mathcal{L}_X(d\psi + \eta\psi) = d(X\psi) + (d\varphi_X)\psi + \eta X\psi$$

$$\mathcal{L}_X(d+\eta)\psi = (d+\eta)X\psi + d\varphi_X\psi$$

$$(d+\eta)\varphi_X\psi = (d\varphi_X)\psi + \varphi_X d\psi + \varphi_X\eta\psi$$

$$\left.\begin{aligned}(d+\eta)\varphi_X &= d\varphi_X + \varphi_X(d+\eta)\\ (d+\eta)\mathcal{L}_X &= -d\varphi_X + \mathcal{L}_X(d+\eta)\end{aligned}\right\} \Rightarrow [\mathcal{L}_X+\varphi_X, d+\eta] = 0$$

Check: $[\mathcal{L}_X + \varphi_X, \mathcal{L}_Y + \varphi_Y] = \mathcal{L}_{[X,Y]} + X\varphi_Y - Y\varphi_X$. This should make clear the Jacobi identity for the bracket on line 4 above, provided you can recover from the operator $\mathcal{L}_X + \varphi_X$ the vector field X and function φ_X. Clear because $(\mathcal{L}_X+\varphi_X)1 = \varphi_X$ and because X is the symbol of the first order operator $\mathcal{L}_X+\varphi_X$.

So what to do about quantization. You want to quantize the real affine plane equipped with a translation invariant symplectic structure (i.e. volume elt. + orientation). Notion of polarization - two intersecting lines

April 28, 02

Viewpoint: configuration space = the affine line $\mathbb{R}$, that is, $\mathbb{R}$ equipped with translational symmetries, in other words a torsor for the Lie group $(\mathbb{R},+)$.

phase space = cotangent bundle of configuration space.
The important point is that phase space has more symmetries than configuration space, and these are relevant for physics.

Look at the cotangent bundle of the affine line $\mathbb{R}$. Let q be position coord, then the tangent bundle is $\mathbb{R}^2$ with coords $(q, \dot{q})$, the cotangent bundle is $\mathbb{R}^2$ with coords (q,p), the pairing between tangent + cotangent spaces is $p, \dot{q} \longmapsto p\dot{q}$, so that the contact form on phase space is $\eta = p\,dq$. Phase space is an affine plane, however, its translation group does not preserve the contact form, η, although it does preserve the symplectic form $d\eta$.

$$\mathcal{L}_{\partial_q}(p\,dq) = 0 \quad , \quad \mathcal{L}_{\partial_p}(p\,dq) = dq$$

$$\varphi_{\partial_q} = 0 \qquad\qquad \varphi_{\partial_p} = q$$

$$\left[\left(\mathcal{L}_{\partial_p} + q\right), \left(d + \eta\right)\right] = \mathcal{L}_{\partial_p}(\eta) + [q, d] = dq - [d, q] = 0$$

$$\left[\mathcal{L}_{\partial_q}, d + \eta\right] = 0.$$

Some geometry associated to the Poisson ~~sum~~ ~~[illegible]~~ summation formula

important result linking Fourier series and Fourier integrals

leads naturally to a complex line bundle over the 2-diml torus ~~[illegible]~~ non trivial

discuss ~~[illegible]~~ structure

hermitian scalar product

connection, curvature

fundamental ~~[illegible]~~ representation of the Heisenberg group

~~[illegible]~~ ~~Inter~~

abstract

The Poisson summation formula is a basic result linking Fourier series and Fourier integrals, which has many important applications. This formula leads naturally to a complex line bundle over the 2-dimensional torus whose geometric structure will be discussed. Sections of the line bundle provide a realization of the basic representation of the Heisenberg commutation relations different from the familiar ones from quantum mechanics. I will also discuss the connection with "aliasing" and "sampling" in electrical engineering.