Mar 20, 02

You have two cats. $\mathcal{W}$ consists of $W = \begin{pmatrix} W_+ \\ W_- \end{pmatrix}$, a $\mathbb{Z}/2$-graded v.s., equipped with an odd operator X: $\varepsilon X \varepsilon = -X$

$\mathcal{V}$ consists of a v.s. V equipped with 2 projections $p_\pm$. You want to understand to what extent these categories are equivalent. Both are unital module categories, but you don't get a get a M.eq between these unital rings, rather you need to replace $\mathcal{V}$ by reduced modules: $V = p_+V + p_-V$, $\operatorname{Ker}(p_+, p_-) = 0$.

$$B = \mathcal{E}_\Gamma \rtimes \Gamma = \mathbb{C}[X] \rtimes \Gamma \qquad \varepsilon X \varepsilon = -X.$$

also putting $h = \frac{1}{2}(1+X)$, $\varepsilon h \varepsilon = \frac{1}{2}(1-X)$

gives $h + \varepsilon h \varepsilon = 1$

B is a nice unital ring. You want a Morita context joining B to the nonunital free product $A = \mathbb{C}p_+ * \mathbb{C}p_-$. Recall how a M context linking a unital ring B to an idempotent ring A appears.

$$\begin{pmatrix} A & Q \\ P & B \end{pmatrix}$$

the fact that B has 1_B implies P, Q f.g. proj over A^{op}, A resp.

$$Q \otimes P \longrightarrow A \qquad q \otimes p \longmapsto \langle q | p \rangle$$

$$B = P \otimes_A Q \longrightarrow \operatorname{Hom}_A(Q, A) \otimes_A Q \longrightarrow \operatorname{Hom}_A(Q, Q)$$

$$p \otimes q \qquad (q' \mapsto \langle q' | p \rangle) \otimes q \qquad (q' \mapsto \langle q' | p \rangle q)$$

$$B = P \otimes_A Q \longrightarrow P \otimes_A \operatorname{Hom}_{A^{op}}(P, A) \longrightarrow \operatorname{Hom}_{A^{op}}(P, P)$$

$$p \otimes q \longmapsto p \otimes (p' \mapsto \langle q | p' \rangle) \longmapsto (p' \mapsto p \langle q | p' \rangle)$$

You expect the M. cont. $\begin{pmatrix} hBh & hB \\ Bh & B \end{pmatrix}$ where the product is defined pretending h idemp. Important point is that $BhB = B$, and that should.

$\Lambda = \mathbb{C} \times \mathbb{C} = \mathbb{C}e_+ \oplus \mathbb{C}e_-$, $1 = e_+ + e_-$, $e_+e_- = e_-e_+ = 0$.

A Λ-module is a v.s. W equipped with a splitting $W = W_+ \oplus W_-$, $W_\pm = \{ w \in W \mid e_\pm w = w \}$.

Ex. $\Lambda \otimes V$ the free Λ module gen. by V.

$$\Lambda \otimes V = \begin{pmatrix} e_+ \otimes V \\ e_- \otimes V \end{pmatrix} \simeq \begin{pmatrix} V \\ V \end{pmatrix} \qquad e_+ V = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$e_- = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

Let W be a Λ-mod retract of $\Lambda \otimes V$

$$W \xleftarrow{\beta} \begin{pmatrix} V \\ V \end{pmatrix} \xleftarrow{\alpha} W \qquad \beta\alpha = 1$$

$$\alpha\beta = p \in \mathrm{End}_\Lambda(\Lambda \otimes V)$$

$W \; \shortparallel$

Get

$$W_+ \xleftarrow{\beta_+} V \xleftarrow{\alpha_+} W_+$$

$$W_- \xleftarrow{\beta_-} V \xleftarrow{\alpha_-} W_-$$

$$p = \begin{pmatrix} \alpha_+\beta_+ & \\ & \alpha_-\beta_- \end{pmatrix} = \begin{pmatrix} p_+ & 0 \\ 0 & p_- \end{pmatrix}$$

What you find: Given Λ-module $\begin{pmatrix} V \\ V \end{pmatrix}$ with $e_+ = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $e_- = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

$$\mathrm{End}_\Lambda \begin{pmatrix} V \\ V \end{pmatrix} = \begin{pmatrix} \mathcal{L}(V) & 0 \\ 0 & \mathcal{L}(V) \end{pmatrix}$$ so $p \in \mathrm{End}_\Lambda(\Lambda \otimes V) = \begin{pmatrix} p_+ & 0 \\ 0 & p_- \end{pmatrix}$

where $p_\pm$ are projections. Here is a critical point.

Puzzle: (1) A Λ-mod retract (W, β, α) of $\Lambda \otimes V$ is the same as a projection $p \in \mathrm{End}_\Lambda(\Lambda \otimes V) \simeq \begin{pmatrix} \mathcal{L}(V) & 0 \\ 0 & \mathcal{L}(V) \end{pmatrix}$

(2) p is the same as a pair of proj. $(p_+, p_-) \in \mathcal{L}(V)$

The puzzle is how to use the odd operator

$\Lambda = \mathbb{C}e_+ \oplus \mathbb{C}e_-$, $\quad 1 = e_+ + e_-$, $\quad e_+e_- = e_-e_+ = 0$.

initial Λ module is a v.s. W equipped with a splitting

$W = W_+ \oplus W_-$, $\quad W_\pm = \{ w \in W \mid e_\pm w = w \}$.

free Λ-module $\quad \Lambda \otimes V = \begin{pmatrix} e_+ \otimes V \\ e_- \otimes V \end{pmatrix} \simeq \begin{pmatrix} V \\ V \end{pmatrix} \quad e_+ = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad e_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

A retract W of the Λ-module $\Lambda \otimes V$ is the same as a proj $p = p^2 \in \mathrm{End}_\Lambda(\Lambda \otimes V) = \begin{pmatrix} \mathcal{L}(V) & 0 \\ 0 & \mathcal{L}(V) \end{pmatrix}$ i.e.

$p = \begin{pmatrix} p_+ & 0 \\ 0 & p_- \end{pmatrix}$ with $p_\pm = p_\pm^2$ in $\mathcal{L}(V)$.

So at the moment you are considering (V, W, α, β) and have deleted W, α, β keeping $(V, p_\pm)$. Next delete V keep $W = \begin{pmatrix} W_+ \\ W_- \end{pmatrix}$ and $X = \begin{pmatrix} \beta_+ & 0 \\ 0 & \beta_- \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \alpha_+ & 0 \\ 0 & \alpha_- \end{pmatrix} = \begin{pmatrix} 0 & \beta_+\alpha_- \\ \beta_-\alpha_+ & 0 \end{pmatrix}$

$$\begin{pmatrix} W_+ \\ W_- \end{pmatrix} \xleftarrow{\begin{pmatrix} \beta_+ & \\ & \beta_- \end{pmatrix}} \begin{pmatrix} V \\ V \end{pmatrix} \xleftarrow{\begin{pmatrix} \alpha_+ & 0 \\ 0 & \alpha_- \end{pmatrix}} \begin{pmatrix} W_+ \\ W_- \end{pmatrix}$$

Repeat. Given $\quad W \xleftarrow{\beta} \begin{pmatrix} V \\ V \end{pmatrix} \xleftarrow{\alpha} W \quad \beta\alpha = 1$. you get two projs $p_\pm = \alpha_\pm \beta_\pm$ on V. Conversely given $p_\pm$ on V you get $W = \begin{pmatrix} W_+ \\ W_- \end{pmatrix}$ together with $X = \begin{pmatrix} 0 & \beta_+\alpha_- \\ \beta_-\alpha_+ & 0 \end{pmatrix}$.

Again: Given projs $p_+, p_- \in \mathcal{L}(V)$ you define the retracts

$$\begin{array}{ll} W_+ \xleftarrow{\beta_+} V \xleftarrow{\alpha_+} W_+ & \qquad \beta_\pm \alpha_\pm = 1 \\ W_- \xleftarrow{\beta_-} V \xleftarrow{\alpha_-} W_- & \qquad \alpha_\pm \beta_\pm = p_\pm \end{array}$$

equipped with $X = \begin{pmatrix} 0 & \beta_+\alpha_- \\ \beta_-\alpha_+ & 0 \end{pmatrix}$

Conversely given $W = \begin{pmatrix} W_+ \\ W_- \end{pmatrix}$ + odd op X you want to get a (V, p_+, p_-). $\quad h = \frac{1}{2}(1 + X) \qquad h + \varepsilon h \varepsilon$

Look first at the case where $\frac{1}{2}(1+X)$ is invertible on W. Then V is isomorphic to W

$$\underbrace{V \xleftarrow{(\alpha_+\ \alpha_-)} \begin{pmatrix} W_+ \\ W_- \end{pmatrix}}_{1-X} \xleftarrow{\begin{pmatrix}\beta_+ \\ \beta_-\end{pmatrix}} \underbrace{V \xleftarrow{(\alpha_+\ \alpha_-)} \begin{pmatrix} W_+ \\ W_- \end{pmatrix}}_{1+X}$$

$$(1+X)\varepsilon(1+X)^{-1} = (1+X)(1-X)^{-1}\varepsilon = g\varepsilon = F$$

$$F(1+X) = g\varepsilon(1+X) = g(1-X)\varepsilon = (1+X)\varepsilon$$

$$1+X = \begin{pmatrix} 1 & -T^* \\ T & 1 \end{pmatrix} \quad \text{in Hilbert space case}$$

Assume V reduced whence $\quad V = \text{Image of } 1+X$

$$\begin{pmatrix} W_+ \\ W_- \end{pmatrix} \qquad V$$

$(\alpha_+\ \alpha_-)$ surj

$\begin{pmatrix}\beta_+ \\ \beta_-\end{pmatrix}$ inj.

Assume $1+X$ is an isom on W. Clearly you get two ~~splittings~~ of V.

$$\begin{pmatrix} W_+ \\ W_- \end{pmatrix} \xleftarrow[\sim]{\begin{pmatrix}\beta_+ \\ \beta_-\end{pmatrix} = \tilde\beta} V \xleftarrow[\sim]{\overset{\tilde\alpha}{(\alpha_+\ \alpha_-)}} \begin{pmatrix} W_+ \\ W_- \end{pmatrix}$$

$$\tilde\beta^{-1}\varepsilon\tilde\beta \qquad \tilde\alpha\,\varepsilon\,\tilde\alpha^{-1} \qquad \text{two involutions on } V$$

$$\tilde\beta^{-1}\varepsilon\underbrace{\tilde\beta\,\tilde\alpha}_{1+X}\varepsilon\,\tilde\alpha^{-1} = \tilde\beta^{-1}(1-X)\tilde\alpha^{-1}$$

You learned today that

$$(1+X)\varepsilon(1+X)^{-1} = (1+X)(1-X)^{-1}\varepsilon = g\varepsilon = F$$

so actually $1+X$ and not just $g^{1/2} = \frac{1+X}{(1-X^2)^{1/2}}$ conjugates ε to F.

Look again at

$$\begin{pmatrix} W_+ \\ W_- \end{pmatrix} \xleftarrow{\begin{pmatrix} \beta_+ & 0 \\ 0 & \beta_- \end{pmatrix}} \begin{pmatrix} V \\ V \end{pmatrix} \xleftarrow{\begin{pmatrix} \alpha_+ & 0 \\ 0 & \alpha_- \end{pmatrix}} \begin{pmatrix} W_+ \\ W_- \end{pmatrix} \xleftarrow{\begin{pmatrix} \beta_+ & 0 \\ 0 & \beta_- \end{pmatrix}} \begin{pmatrix} V \\ V \end{pmatrix} \qquad \beta_\pm \alpha_\pm = 1_{W_\pm}$$

Replace the identity op on $\begin{pmatrix} V \\ V \end{pmatrix}$ by $\frac{1}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \end{pmatrix}$

or by $\frac{1}{2}\begin{pmatrix} 1 \\ -1 \end{pmatrix}\begin{pmatrix} 1 & -1 \end{pmatrix}$

March 21, 02 $\quad \Lambda = \mathbb{C}e_+ \oplus \mathbb{C}e_-$ $\qquad e_+ = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $e_- = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

Identify the free Λ module $\Lambda \otimes V$ with $\begin{pmatrix} V \\ V \end{pmatrix}$ and $\mathrm{End}_\Lambda(\Lambda \otimes V)$ with $\begin{pmatrix} \mathcal{L}(V) & 0 \\ 0 & \mathcal{L}(V) \end{pmatrix}$.

retract of $\Lambda \otimes V$ = 2 projection (p_+, p_-) on V

$$\boxed{\begin{aligned} \beta_\pm \alpha_\pm &= 1_{W_\pm} \\ p_\pm &= \alpha_\pm \beta_\pm \end{aligned}}$$

$$\begin{pmatrix} W_+ \\ W_- \end{pmatrix} \xleftarrow{\begin{pmatrix} \beta_+ & 0 \\ 0 & \beta_- \end{pmatrix}} \begin{pmatrix} V \\ V \end{pmatrix} \xleftarrow{\begin{pmatrix} \alpha_+ & \\ & \alpha_- \end{pmatrix}} \begin{pmatrix} W_+ \\ W_- \end{pmatrix}$$

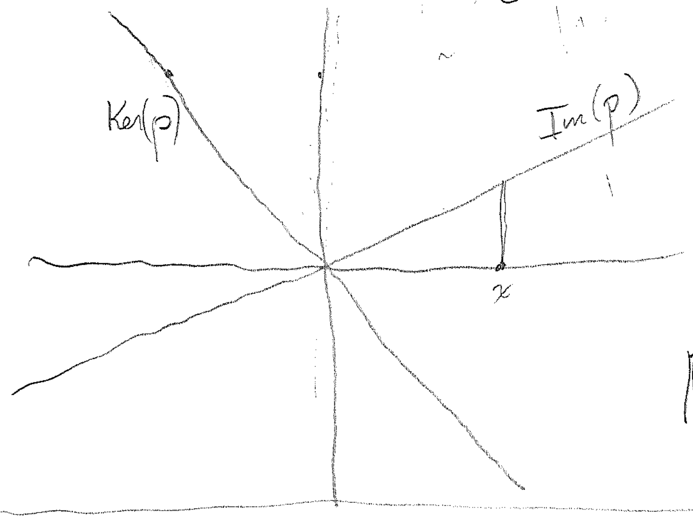
on W you have the grading together with $X = \begin{pmatrix} 0 & \beta_+\alpha_- \\ \beta_-\alpha_+ & 0 \end{pmatrix}$

Problem: Assume V reduced, whence

$$\begin{tikzcd}
\begin{pmatrix} W_+ \\ W_- \end{pmatrix} \arrow[d, "\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}"'] & V \arrow[l, "\begin{pmatrix} \beta_+ \\ \beta_- \end{pmatrix}"'] \arrow[d, equal] & \begin{pmatrix} W_+ \\ W_- \end{pmatrix} \arrow[l, "\begin{pmatrix} \alpha_+ & \alpha_- \end{pmatrix}"'] \\
\begin{pmatrix} W_+ \\ W_- \end{pmatrix} & V \arrow[l, "\begin{pmatrix} \beta_+ \\ -\beta_- \end{pmatrix}"] & \begin{pmatrix} W_+ \\ W_- \end{pmatrix} \arrow[l, "\begin{pmatrix} \alpha_+ & -\alpha_- \end{pmatrix}"] \arrow[u, "\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}"']
\end{tikzcd}$$

Mar 21, 02

It appears that you still lack an understanding of the image factorization.



$$\begin{pmatrix}1 & T'\\ T & 1\end{pmatrix} = 1+X$$

$$p\begin{pmatrix}1 & T'\\ T & 1\end{pmatrix} = \begin{pmatrix}1 & 0\\ T & 0\end{pmatrix} = \begin{pmatrix}1 & T'\\ T & T\end{pmatrix}\underbrace{\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}}_{e}$$

Start over again $\Lambda = \mathbb{C}e_+ \oplus \mathbb{C}e_-$ $\quad 1 = e_+ + e_-$, $0 = e_+e_- = e_-e_+$

W set of $\Lambda \otimes V \implies$ W has following structure: $W = \begin{pmatrix}W_+\\ W_-\end{pmatrix}$ + odd operator $X = \begin{pmatrix}0 & \beta_+\alpha_-\\ \beta_-\alpha_+ & 0\end{pmatrix}$. V structure: (p_+, p_-) on V

V reduced means that V is the image of $1+X$

$$\begin{pmatrix}W_+\\ W_-\end{pmatrix} \xleftarrow{\begin{pmatrix}\beta_+\\ \beta_-\end{pmatrix}} V \xleftarrow{(\alpha_+\ \alpha_-)} \begin{pmatrix}W_+\\ W_-\end{pmatrix}$$

The problem: Given $\begin{pmatrix}W_+\\ W_-\end{pmatrix}$ with $(1+X)$, let V be the image of $1+X$, show that there are two projections p_+, p_- on V

$$1+X = \begin{pmatrix}1 & T'\\ T & 1\end{pmatrix} \qquad T = \beta_-\alpha_+ ,\ T' = \beta_+\alpha_-$$

Case where $1+X$ is an isom. seems clear. You get one involution on V from $\begin{pmatrix}\beta_+\\ \beta_-\end{pmatrix} = b$ and another involution on V from $(\alpha_+\ \alpha_-) = a$. These involutions are resp. $b^{-1}\varepsilon b$ and $a\varepsilon a^{-1}$, their product is $b^{-1}\varepsilon \underbrace{ba}_{1+X} \varepsilon a^{-1}$

$= b^{-1}(1-X)b$ $\qquad \boxed{ba = 1+X}$

Try
$$a\varepsilon a^{-1}b^{-1}\varepsilon b^{-1} = a\varepsilon(1+X)^{-1}\varepsilon b = a(1-X)^{-1}b$$

transport this invertible op on V to left side to get
$$ba(1-X)^{-1}b\,b^{-1} = (1+X)(1-X)^{-1}$$

Review: A retract W of $\Lambda \otimes V$ has following structure: grading $W = \begin{pmatrix} W_+ \\ W_- \end{pmatrix}$, odd operator $X = \left(\begin{array}{c|c} 0 & * \\ \hline * & 0 \end{array}\right)$

The projection on $\Lambda \otimes V$ corresp to the retract is the same as a pair (p_+, p_-) of proj on V. Assume V reduced:

$$\begin{tikzcd} \begin{pmatrix} W_+ \\ W_- \end{pmatrix} & V \arrow[l, "\begin{pmatrix} \beta_+ \\ \beta_- \end{pmatrix}"'] & \begin{pmatrix} W_+ \\ W_- \end{pmatrix} \arrow[l, two heads, "(\alpha_+ \ \alpha_-)"'] \arrow[ll, bend left] \end{tikzcd}$$

$$1+X = \begin{pmatrix} 1 & \beta_+\alpha_- \\ \beta_-\alpha_+ & 1 \end{pmatrix}$$

Thus V is canonically isom to the image of $1+X$. Now the problem is to find the two projections $p_\pm$ on V.

First case $1+X$ invertible whence you have two isos.

$$\begin{pmatrix} W_+ \\ W_- \end{pmatrix} \xleftarrow[b]{\sim} V \xleftarrow[a]{\sim} \begin{pmatrix} W_+ \\ W_- \end{pmatrix}$$

with composition $ba = 1+X$.

Thus you have two involutions $b^{-1}\varepsilon b$, $a\varepsilon a^{-1}$ on V

It might be simpler to say you have isom.

$$\begin{pmatrix} W_+ \\ W_- \end{pmatrix} \xleftarrow[\sim]{1+X} \begin{pmatrix} W_+ \\ W_- \end{pmatrix} \qquad (1+X)\,\varepsilon\,(1+X)^{-1} = \overbrace{\frac{1+X}{1-X}\,\varepsilon}^{F}$$

$$F(1+X) = \frac{1+X}{1-X}\,\varepsilon(1+X) = \frac{1+X}{1-X}(1-X)\varepsilon = (1+X)\varepsilon$$

This is the special case where $1+X$ is invertible. Perhaps the same sort of argument can be used. You want to show how the image of $1+X$ carries two involutions

So you have W with splitting and $h = \frac{1}{2}(1+X)$, so automatically things should work. $V = hW$

$$W \xleftarrow{i} V \twoheadleftarrow W$$

so on V you end up with

$p(1) = ji$; $p(\varepsilon) = j\varepsilon i$

$$(p(1) \pm p(\varepsilon))^2 = jiji \pm jij\varepsilon i \pm j\varepsilon iji + j\varepsilon ij\varepsilon i$$
$$= \underbrace{j(h + \varepsilon h \varepsilon)i}_{ji} \pm \underbrace{j(\varepsilon h + h\varepsilon)i}_{j\varepsilon i}$$

Seems to work but why?

$$\begin{pmatrix} W_+ \\ W_- \end{pmatrix} \xleftarrow{\iota = \begin{pmatrix}\beta_+ \\ \beta_-\end{pmatrix}} V \twoheadleftarrow{} \!\!\xleftarrow{j = (\alpha_+ \, \alpha_-)} \begin{pmatrix} W_+ \\ W_- \end{pmatrix}$$

Question: Does the grading on W induce a grading on V via ι, and another grading via j.

$$\begin{tikzcd}
\begin{pmatrix} W_+ \\ W_- \end{pmatrix} & & \begin{pmatrix} W_+ \\ W_- \end{pmatrix} \arrow[ll, "\begin{pmatrix} 1 & T' \\ T & 1 \end{pmatrix}"'] \\
\begin{pmatrix} W_+ \\ W_- \end{pmatrix} \arrow[u, "\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}"] & & \begin{pmatrix} W_+ \\ W_- \end{pmatrix} \arrow[ll, "\begin{pmatrix} 1 & -T' \\ -T & 1 \end{pmatrix}"'] \arrow[u, "\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}"']
\end{tikzcd}$$

$$\varepsilon(1+X) = (1+X)\varepsilon$$

$$W_+ \xleftarrow{\beta_+} V \xleftarrow{\alpha_+} W_+$$

$$W_- \xleftarrow{\beta_-} V \xleftarrow{\alpha_-} W_-$$

Two splittings $\quad \Lambda \otimes V \simeq \begin{pmatrix} V \\ V \end{pmatrix}$

$$1 = \iota_1 j_1 + \varepsilon \iota_1 j_1 \varepsilon \qquad j_1 \varepsilon \iota_1 = 0$$

$$1 = \iota_+ j_+ + \iota_- j_- \qquad j_- \iota_+ = j_+ \iota_- = 0$$

$$\begin{tikzcd}
& V \arrow[d, shift right] & \\
W & \Lambda \otimes V \arrow[l, "\beta"'] \arrow[u, shift right] \arrow[d, shift left] & W \arrow[l, "\alpha"'] \\
& V \arrow[u, shift left] &
\end{tikzcd}$$

$$1_W = \underbrace{\beta \iota_1 j_1 \alpha}_{h} + \underbrace{\varepsilon \beta \iota_1 j_1 \alpha \varepsilon}_{\varepsilon h \varepsilon}$$

$$\begin{pmatrix} V \\ V \end{pmatrix} \xleftarrow{\frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}} \begin{pmatrix} V \\ V \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

You want to express $e_1 = \iota_1 j_1$ $\overbrace{\text{and } e_2 = \varepsilon \iota_1 j_1 \varepsilon}$ in terms of e_+, e_-

$$e_+ = e_1 + \varepsilon e_1 \varepsilon$$
$$e_- = e_1 - \varepsilon e_1 \varepsilon$$

$$(e_1 + \varepsilon e_1 \varepsilon)(e_1 - \varepsilon e_1 \varepsilon)$$
$$= e_1 + \varepsilon e_1 \varepsilon e_1 - e_1 \varepsilon e_1 \varepsilon + \varepsilon e_1 \cancel{\varepsilon^2} e_1 \varepsilon \qquad ?$$
$$= e_1 - \varepsilon e_1 \varepsilon$$

$$\begin{pmatrix} e_+ \otimes V \\ e_- \otimes V \end{pmatrix} \xrightarrow[\sim]{1+\varepsilon} \Lambda \otimes V \longleftarrow \begin{pmatrix} 1 \otimes V \\ \varepsilon \otimes V \end{pmatrix}$$

$$\begin{pmatrix} e_+ \\ e_- \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ \varepsilon \end{pmatrix} \qquad \begin{pmatrix} 1 \\ \varepsilon \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} e_+ \\ e_- \end{pmatrix}$$

Situation Given $\begin{pmatrix} W_+ \\ W_- \end{pmatrix}$ and an odd operator $X = \begin{pmatrix} 0 & T' \\ T & 0 \end{pmatrix}$ you want to embed W as a (retract / summand) of a free Λ module $\Lambda \otimes V \simeq \begin{pmatrix} V \\ V \end{pmatrix}$ with $\varepsilon = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$.

OK the problem seems to be about what is a free module

Let W be a retract of $\Lambda \otimes V = e_+ \otimes V \oplus e_- \otimes V \simeq \begin{pmatrix} V \\ V \end{pmatrix}$ with $e_+ = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $e_- = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. One has retracts

$$\begin{pmatrix} W_+ \\ W_- \end{pmatrix} \xleftarrow{\begin{pmatrix} \beta_+ & \\ & \beta_- \end{pmatrix}} \begin{pmatrix} V \\ V \end{pmatrix} \xleftarrow{\begin{pmatrix} \alpha_+ & \\ & \alpha_- \end{pmatrix}} \begin{pmatrix} W_+ \\ W_- \end{pmatrix} \qquad \beta_\pm \alpha_\pm = 1_{W_\pm} \qquad \rho_\pm = \alpha_\pm \beta_\pm$$

and the odd operator $X = \beta \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \alpha$

Point: If W ret of $\Lambda \otimes V$ which we ident with $\begin{pmatrix} V \\ V \end{pmatrix}$ then W is a

Mar 23, 02

$\Lambda = \mathbb{C}e_+ \oplus \mathbb{C}e_-$, $e_+ + e_- = 1$, $e_+e_- = e_-e_+ = 0$.

Λ-module = v.s. W equipped with a splitting $W = \begin{pmatrix} W_+ \\ W_- \end{pmatrix}$

Ex. e.g. $\Lambda\otimes V = e_+\otimes V \oplus e_-\otimes V \simeq \begin{pmatrix} V\\ V\end{pmatrix}$

Let W be a retract of the Λ-module $\Lambda\otimes V$:

$$W \xleftarrow{\beta} \Lambda\otimes V \xleftarrow{\alpha} W \qquad \beta\alpha = 1_W$$

Same as a projection p on $\Lambda\otimes V$.

$$\begin{pmatrix}W_+\\ W_-\end{pmatrix} \xleftarrow{\begin{pmatrix}\beta_+&0\\ 0&\beta_-\end{pmatrix}} \begin{pmatrix}V\\ V\end{pmatrix} \xleftarrow{\begin{pmatrix}\alpha_+&0\\ 0&\alpha_-\end{pmatrix}} \begin{pmatrix}W_+\\ W_-\end{pmatrix}$$

Same as a pair $p_\pm = \alpha_\pm\beta_\pm$ of projections on V.

"odd" operator $X = \beta\begin{pmatrix}0&1\\ 1&0\end{pmatrix}\alpha = \begin{pmatrix}0&\beta_+\alpha_-\\ \beta_-\alpha_+ & 0\end{pmatrix}$

Conclude a retract W of $\Lambda\otimes V$ inherits the structure of a $\mathbb{Z}/2$ graded v.s. together with an odd operator X.

Conversely, given $\begin{pmatrix}W_+\\ W_-\end{pmatrix}$ with $X = \begin{pmatrix}0&T'\\ T&0\end{pmatrix}$ choose a factorization of

$$\begin{tikzcd} V & \begin{pmatrix}W_+\\ W_-\end{pmatrix} & V \arrow[l, "\begin{pmatrix}\beta_+\\ \beta_-\end{pmatrix}"'] & \begin{pmatrix}W_+\\ W_-\end{pmatrix} \arrow[l, "(\alpha_+\ \alpha_-)"'] \arrow[ll, bend left, "1+X"] & V \arrow[l, dashed, "\begin{pmatrix}\beta_+\\ \beta_-\end{pmatrix}"'] \arrow[ll, bend right, "p_+ + p_-"'] \end{tikzcd}$$

of $1+X$. Then $\beta_\pm\alpha_\pm = 1_{W_\pm}$ so that you have two proj $p_\pm = \alpha_\pm\beta_\pm$ on V, such that $W_\pm$ are corresp. retracts of V; also get back X as $\begin{pmatrix}0&\beta_+\alpha_-\\ \beta_-\alpha_+&0\end{pmatrix}$.

Question: On V you two projections $p_\pm$ which you convert to involutions $2p_+-1$, $2p_- -1$ then you should have an invertible $g = (2p_+-1)(2p_--1)$.

$$(1+X)\varepsilon = \varepsilon(1-X)$$

You want an invertible operator on V somehow induced by $1+X$

Consider some factorization of $1+X$

$$\begin{pmatrix} W_+ \\ W_- \end{pmatrix} \xleftarrow[\sim]{\begin{pmatrix}\beta_+\\ \beta_-\end{pmatrix}} V \xleftarrow[\sim]{(\alpha_+ \ \alpha_-)} \begin{pmatrix} W_+ \\ W_- \end{pmatrix} \xleftarrow[\sim]{\begin{pmatrix}\beta_+\\ \beta_-\end{pmatrix}} V$$

$1+X$

operators on V arising are $p_+ = \alpha_+\beta_+$; $p_- = \alpha_-\beta_-$ projections

Assume $b = \begin{pmatrix}\beta_+\\ \beta_-\end{pmatrix}$ $a = (\alpha_+ \ \alpha_-)$ are bijectives.

on V you have two projections and what they generate

$$p_+ = \alpha_+\beta_+ = (\alpha_+ \ \alpha_-)\begin{pmatrix} 1_{W_+} & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix}\beta_+\\ \beta_-\end{pmatrix} \qquad F_+ = 2p_+ - 1$$

$$p_- = \alpha_-\beta_- = (\alpha_+ \ \alpha_-)\begin{pmatrix} 0 & 0 \\ 0 & 1_{W_-} \end{pmatrix}\begin{pmatrix}\beta_+\\ \beta_-\end{pmatrix} \qquad F_- = 2p_- - 1$$

$$F_+ = 2p_+ - 1 = a\begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} b - 1_V$$

$$F_- = 2p_- - 1 = a\begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix} b - 1_V$$

$$p_+^2 = a\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} b\, a\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} b = a\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & T' \\ T & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} b$$

$$= a\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} b = p_+$$

$$p_-^2 = a\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} b\, a\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} b = a\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & T' \\ T & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} b$$

$$= a\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} b = \alpha_-\beta_- = p_-$$

$$F = (1+X)\varepsilon(1+X)^{-1} = \frac{1+X}{1-X}\varepsilon$$

$$(1+X)^{-1}\varepsilon(1+X) = \frac{1-X}{1+X}\varepsilon$$

$$\begin{pmatrix} W_+ \\ W_- \end{pmatrix} \xleftarrow{b = \begin{pmatrix}\beta_+ \\ \beta_-\end{pmatrix}} V \xleftarrow{a = (\alpha_+ \ \alpha_-)} \begin{pmatrix} W_+ \\ W_- \end{pmatrix}$$

$$ba = 1 + X \qquad \beta_\pm \alpha_\pm = 1_{W_\pm}.$$

Assume b, a bijectives. Then you get two involutions on V, namely $a\varepsilon a^{-1}$, $b^{-1}\varepsilon b$ whose product is $a\varepsilon a^{-1} b^{-1}\varepsilon b = a\varepsilon (ba)^{-1}\varepsilon b = a\varepsilon(1+X)^{-1}\varepsilon b =$ $a(1-X)^{-1}b = a(1-X)^{-1}(1+X)a^{-1}$, which is the C.T. of X transported to V via a.

Now you have two projections $p_\pm = \alpha_\pm \beta_\pm$ on V. How are they related to $a\frac{1+\varepsilon}{2}a^{-1}$ and $b^{-1}\frac{1+\varepsilon}{2}b$

$$a\frac{1+\varepsilon}{2}a^{-1} = a\frac{1+\varepsilon}{2}(1+X)^{-1}b, \qquad b^{-1}\frac{1+\varepsilon}{2}b = a(1+X)^{-1}\frac{1+\varepsilon}{2}b$$

Clearly $a\frac{1+\varepsilon}{2}a^{-1}$, $b^{-1}\frac{1+\varepsilon}{2}b$ are projections on V.

Question: Are they the same as $p_\pm = \alpha_\pm \beta_\pm$?

$$\begin{pmatrix} W_+ \\ W_- \end{pmatrix} \underset{\sim}{\xleftarrow{\begin{pmatrix}\beta_+ \\ \beta_-\end{pmatrix}}} V \underset{\sim}{\xleftarrow{(\alpha_+ \ \alpha_-)}} \begin{pmatrix} W_+ \\ W_- \end{pmatrix}$$

$$\begin{pmatrix}\beta_+ \\ \beta_-\end{pmatrix}(\alpha_+ \ \alpha_-) = \begin{pmatrix} 1 & T' \\ T & 1 \end{pmatrix}$$

Let's find the projection on V given by e_+ on W. Given $v \in V$ write $v = \alpha_+ w_+ + \alpha_- w_- = (\alpha_+ \ \alpha_-)\begin{pmatrix} w_+ \\ w_- \end{pmatrix}$ then apply e_+ to $\begin{pmatrix} w_+ \\ w_1 \end{pmatrix}$ and then $(\alpha_+ \ \alpha_-)$.

$$v \longmapsto \underbrace{(\alpha_+ \ \alpha_-)\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}}_{(\alpha_+ \ 0)} \underbrace{(\alpha_+ \ \alpha_-)^{-1} v}_{(1+X)^{-1}\begin{pmatrix}\beta_+ \\ \beta_-\end{pmatrix} v}$$

$$v \longmapsto (\alpha_+ \ 0)(\alpha_+ \ \alpha_-)^{-1} v$$

$$\begin{pmatrix} W_+ \\ W_- \end{pmatrix} \xleftarrow[i]{\begin{pmatrix}\beta_+ \\ \beta_-\end{pmatrix}} V \xleftarrow[j]{(\alpha_+ \ \alpha_-)} \begin{pmatrix} W_+ \\ W_- \end{pmatrix} \xleftarrow{\begin{pmatrix}\beta_+ \\ \beta_-\end{pmatrix}} V$$

Define a projection on V by

$$p_1 v = (\alpha_+ \ \alpha_-)\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}(\alpha_+ \ \alpha_-)^{-1} v \longleftarrow v$$

Given $v \in V$, choose $\begin{pmatrix} w_+ \\ w_- \end{pmatrix} \in W$: $v = \alpha_+ w_+ + \alpha_- w_-$, then apply $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ to $\begin{pmatrix} w_+ \\ w_- \end{pmatrix}$ to get $\begin{pmatrix} w_+ \\ 0 \end{pmatrix}$, then apply $(\alpha_+ \ \alpha_-)$ to get $\alpha_+ w_+$. This is obviously a projection operator on V when $(\alpha_+ \ \alpha_-)$ is bijective. So all you have to do now is to check it's well-defined. So suppose $\alpha_+ w_+ + \alpha_- w_- = 0$. You wish to show that $\alpha_+ w_+ = 0$. So apply $\begin{pmatrix}\beta_+ \\ \beta_-\end{pmatrix}$

$$0 = \begin{pmatrix}\beta_+ \\ \beta_-\end{pmatrix}(\alpha_+ \ \alpha_-)\begin{pmatrix} w_+ \\ w_- \end{pmatrix} = \begin{pmatrix} \beta_+\alpha_+ & \beta_+\alpha_- \\ \beta_-\alpha_+ & \beta_-\alpha_- \end{pmatrix}\begin{pmatrix} w_+ \\ w_- \end{pmatrix} \quad ?$$

Let's work on the Morita context.

data $W = \begin{pmatrix} W_+ \\ W_- \end{pmatrix}$ and X, V with p_+, p_-

$$\begin{pmatrix} W_+ \\ W_- \end{pmatrix} \xleftarrow{\begin{pmatrix}\beta_+ \\ \beta_-\end{pmatrix}} V \xleftarrow{(\alpha_+ \ \alpha_-)} \begin{pmatrix} W_+ \\ W_- \end{pmatrix}$$

fact of $1+X$

$W = \begin{pmatrix} W_+ \\ W_- \end{pmatrix}$ and odd operator X.

$$\begin{pmatrix} W_+ \\ W_- \end{pmatrix} \xleftarrow{\begin{pmatrix}\beta_+ & \\ & \beta_-\end{pmatrix}} \begin{pmatrix} V \\ V \end{pmatrix} \xleftarrow{\begin{pmatrix}\alpha_+ & \\ & \alpha_-\end{pmatrix}} \begin{pmatrix} W_+ \\ W_- \end{pmatrix} \xleftarrow{\begin{pmatrix}\beta_+ & \\ & \beta_-\end{pmatrix}} \begin{pmatrix} V \\ V \end{pmatrix}$$

To set up the Morita context you need? relations

$$\left(\begin{array}{cc|c} 1 & T' & \beta_+ \\ T & 1 & \beta_- \\ \hline \alpha_+ & \alpha_- & \end{array}\right) \begin{pmatrix} W_+ \\ W_- \\ V \end{pmatrix} \qquad \beta_\pm \alpha_\pm = 1_{W_\pm}$$

Let's go back to Γ case, Γ finite

$\mathcal{W}$: obj is Γ-module W plus $h \in \mathrm{Hom}_{\mathbb{C}}(W,W)$

satisf $\sum_{s\in\Gamma} s h s^{-1} = 1_W$

$$\begin{tikzcd} W \arrow[r, "h"] \arrow[d] & W \arrow[d] \\ W' \arrow[r, "h"] & W' \end{tikzcd}$$

$\mathcal{U}$: obj is $(W, V, \beta_1, \alpha_1)$ where W is a Γ-module, V is a v.s. and

$W \xleftarrow{\beta_1} V \overset{\alpha_1}{\twoheadleftarrow} W$ are $\mathbb{C}$-linear maps

satisfying $\sum_{s\in\Gamma} s\beta_1\alpha_1 s^{-1} = 1_W$ $\quad$ β_1 inj, α_1 surj.

$\mathcal{V}$: obj is a v.s. V equipped with $p(s) \in \mathrm{Hom}_{\mathbb{C}}(V,V)$, $s\in\Gamma$

satisf. $\sum_{u=st} p(s)p(t) = p(u)$. $\quad$ $V = \sum_s p(s)V$

$\bigcap_s \mathrm{Ker}(p(s)) = 0$

$\mathcal{U}$ $\qquad\qquad$ $\mathcal{V}$

$(W, V, \beta_1, \alpha_1) \longmapsto V, \ p(s) = \alpha_1 s\beta_1$

$$\sum_s (\alpha_1 s\beta_1)(\alpha_1 s^{-1}t\beta_1) = \alpha_1 t\beta_1$$

$\sum p(s)V = \sum_s \alpha_1 s\beta_1 V = \alpha_1 \sum_s s\beta_1 V$ $\qquad$ $\sum s\beta_1 V \supset s\underbrace{\beta_1\alpha_1 s^{-1}W}_{W}$

$(\forall s)\ \alpha_1 s\beta_1 v = 0 \implies \sum_s s^{-1}\beta_1\alpha_1 s\beta_1 v = 0 \implies \beta_1 v = 0 \implies v = 0.$

$\Lambda = \mathbb{C}[\Gamma]$ $\qquad$ $\Lambda\otimes V = \bigoplus_t t\otimes V = \left\{\sum_t t\otimes f(t) \,\middle|\, f:\Gamma\to V\right\}$

$p = \sum_u u\otimes p(u) \in \Lambda\otimes\mathcal{L}(V)$ $\qquad$ $p^2 = p$. $\quad$ $ps = sp$

p acts on $\Lambda\otimes V$ via $p(t\otimes v) = \sum_u tu^{-1}\otimes p(u)v$

$$p\left(\sum_t t\otimes f(t)\right) = \sum_{t,u} tu^{-1}\otimes p(u)f(t) \qquad s = tu^{-1}, \ u = s^{-1}t$$

$$= \sum_s s\otimes \sum_t p(s^{-1}t)f(t)$$

Define W to be the retract (summand) corresp. to p

$$\begin{tikzcd} W & \Lambda\otimes V \arrow[l, two heads, "\beta"'] \arrow[d, shift right, "j_1"'] & W \arrow[l, tail, "\alpha"'] \arrow[dl, "\alpha_1 = j_1\alpha"] & \Lambda\otimes V \arrow[l, two heads, "\beta"'] \\ & V \arrow[ul, "\beta_1=\beta\iota_1"] \arrow[u, shift right, "\iota_1"'] & & \end{tikzcd}$$

$\beta\alpha = 1_W$

$p = \alpha\beta$

$\iota_1 v = 1\otimes v$

$j_1(t\otimes v) = 0 \quad t\neq 1$
$ = v \quad t=1$

$$\beta\ 1_{\Lambda\otimes V}\,\alpha = \sum_s s\,\iota_1 j_1 s^{-1} \implies 1_W = \sum_s s\beta_1\alpha_1 s^{-1}$$

$\mathcal{W}$ = cat of unital modules over the unital ring

$$\mathcal{E}_\Gamma \rtimes \Gamma = \bigoplus_s \mathcal{E}_\Gamma \otimes s \qquad \mathcal{E}_\Gamma = \mathbb{C}\langle h_s,\ s\in\Gamma\rangle \Big/ \sum_s h_s = 1$$

$$t_*(h_s) = h_{ts}$$

$\mathcal{V}$ = cat of reduced modules over $\mathcal{P}_\Gamma$ generators

$$p(s),\ s\in\Gamma, \quad \sum_{u=st} p(s)p(t) = p(u).$$

$$B = \mathcal{E}_\Gamma \rtimes \Gamma \qquad \begin{pmatrix} hBh & hB \\ Bh & B \end{pmatrix}$$

is a Morita context with mult defined pretending $h^2 = h$.

$$\mathcal{P}_\Gamma \longrightarrow hBh$$
$$p(s) \longmapsto hsh$$

$$\sum_t p(st^{-1})p(t) = \sum_t hs\cancel{t^{-1}}h\cancel{t}h = hsh = p(s)$$

Problem is to compare

$$hb_1\otimes w \longmapsto (b_2h \mapsto b_2hb_1w)$$

$$\mathrm{Im}\left\{ hB\otimes_B W \longrightarrow \mathrm{Hom}_B(Bh, W) \right\}$$

$$\begin{tikzcd} hB\otimes_B W \arrow[dr, two heads] & & \mathrm{Hom}_B(Bh,W) \\ & hW \arrow[ur] & \end{tikzcd}$$

$$hb_1\otimes w \longmapsto hb_1w \qquad hw' \longmapsto (b_2h \mapsto b_2hw')$$

$hw' \mapsto 0$ means

$B.hw' = 0$

$\implies \underbrace{BhB}_{\text{contains } \sum shs^{-1}}hw' = 0 \implies hw' = 0$

Next compare $b_1 h \otimes v \longmapsto (h b_2 \mapsto (h b_2 b_1 h) v)$

$$\mathrm{Im}\left\{ Bh \otimes_A V \longrightarrow \mathrm{Hom}_A(hB, V) \right\}?$$

NOT CLEAR

$b_1 h \otimes v$

$h = \beta_1 \alpha_1$

$$\begin{tikzcd}
Bh \otimes_A V \arrow[r] & W \arrow[r] \arrow[dr, "\alpha"] & \mathrm{Hom}_A(hB, V) \arrow[d] \\
\Lambda \otimes V \arrow[u] \arrow[ur, "\beta"] & & \Lambda \otimes V
\end{tikzcd}$$

$t \otimes v \longmapsto t\beta_1 v$

you missed $V = hW$

You need a clearer approach than "pretending" h is idempotent. The idea might be that if the Morita context is defined by generators and relations, then the important maps, the ones needed to yield the image are easily handled.

For example. Instead of $Bh \otimes_A V$ you use $\Lambda \otimes V$.

$$Bh = \Lambda \otimes \varepsilon_\Gamma h$$

Morita context

$$\begin{pmatrix} & \alpha_s \\ \beta_t & \end{pmatrix} \begin{pmatrix} V \\ W \end{pmatrix}$$

A gen. by $\alpha_s \beta_t = \alpha_1 s^{-1} t \beta_1 = p(s^{-1}t)$

B gen by $\beta_t \alpha_s = t \beta_1 \alpha_1 s^{-1}$

Rels. $\alpha_s \beta_t$ depends only on $s^{-1}t$

$$\sum \beta_s \alpha_s = e_{11}$$

$$\begin{pmatrix} A & Y \\ X & B \end{pmatrix}$$

$$\boxed{\begin{aligned} X &= \sum \beta_t A = B\beta_1 \\ Y &= \sum A \alpha_s = \alpha_1 B \end{aligned}}$$

March 24, 02

OK back to $\Lambda = \mathbb{C}e_+ \oplus \mathbb{C}e_-$

$$\begin{bmatrix} 1_+ & T' & \beta_+ \\ T & 1_- & \beta_- \\ \alpha_+ & \alpha_- & \rho_\pm \end{bmatrix}\begin{bmatrix} W_+ \\ W_- \\ V \end{bmatrix}$$

What to do?

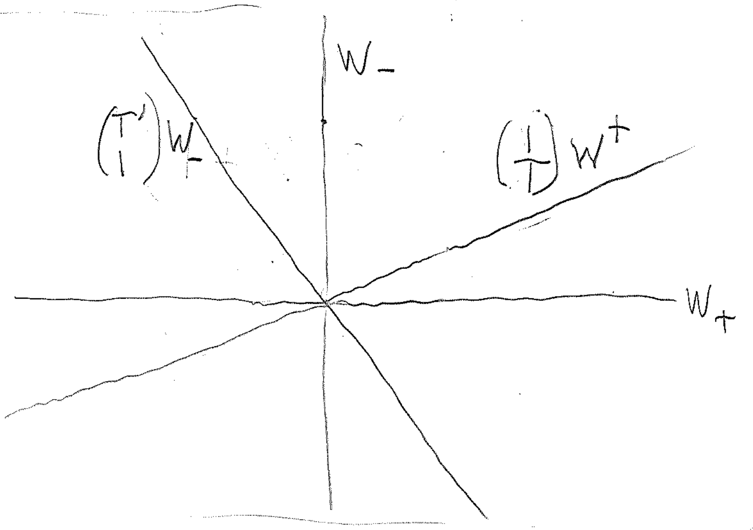
$$\begin{tikzcd} \begin{pmatrix} W_+ \\ W_- \end{pmatrix} & & \begin{pmatrix} W_+ \\ W_- \end{pmatrix} \arrow[ll, "\begin{pmatrix} 1 & T' \\ T & 1 \end{pmatrix}"'] \arrow[dl, "(\alpha_+\ \alpha_-)"] \\ & V \arrow[ul, "\begin{pmatrix} \beta_+ \\ \beta_- \end{pmatrix}"] & \end{tikzcd}$$

Question: Assume that $\begin{pmatrix} \beta_+ \\ \beta_- \end{pmatrix}$, $(\alpha_+\ \alpha_-)$ are bijective. Then you get two splittings on V. Are these related to $\rho_\pm$?

Let
$$\begin{pmatrix} \beta_+ \\ \beta_- \end{pmatrix}^{-1} = (t_+\ t_-) : \begin{pmatrix} W_+ \\ W_- \end{pmatrix} \longrightarrow V$$

so that
$$\begin{pmatrix} \beta_+ \\ \beta_- \end{pmatrix}(t_+\ t_-) = \begin{pmatrix} 1_+ & 0 \\ 0 & 1_- \end{pmatrix} \qquad (t_+\ t_-)\begin{pmatrix} \beta_+ \\ \beta_- \end{pmatrix} = 1_V$$

$$\begin{pmatrix} \beta_+ \\ \beta_- \end{pmatrix}(\alpha_+\ \alpha_-) = \begin{pmatrix} 1_+ & T' \\ T & 1_+ \end{pmatrix}$$



Take $T' = 0$

Two involutions ε and

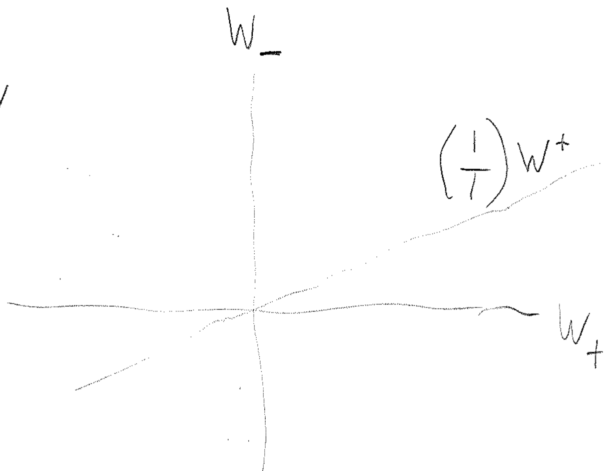
$$F = (1+X)\varepsilon(1+X)^{-1} = \frac{1+X}{1-X}\varepsilon$$

$$\frac{1+X}{1-X} = (1+X)^2 = 1+2X$$

$$\begin{pmatrix} W_+ \\ W_- \end{pmatrix} \xleftarrow{\begin{pmatrix} 1 & 0 \\ T & 1 \end{pmatrix}} \begin{pmatrix} W_+ \\ W_- \end{pmatrix} = V$$

$$\rho_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix}(1\ \ 0)$$

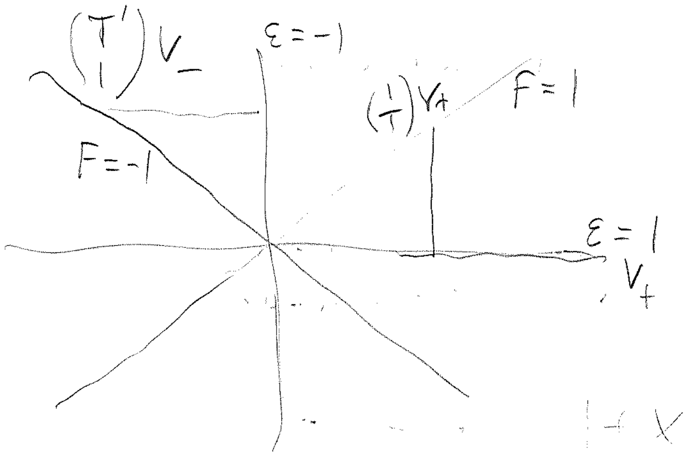
$$\rho_- = \begin{pmatrix} 1 \\ T \end{pmatrix}(1\ \ 0)$$



$$(1+X)\varepsilon(1+X)^{-1} = \begin{pmatrix}1 & 0\\ T & 1\end{pmatrix}\begin{pmatrix}1 & \\ & -1\end{pmatrix}\begin{pmatrix}1 & 0\\ -T & 1\end{pmatrix}$$

$$= \begin{pmatrix}1 & 0\\ T & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ T & -1\end{pmatrix} = \begin{pmatrix}1 & 0\\ 2T & -1\end{pmatrix} = (1+2X)\varepsilon$$

corresp projection is either $\begin{pmatrix}1 & 0\\ T & 0\end{pmatrix}$ or $\begin{pmatrix}0 & 0\\ -T & 1\end{pmatrix}$



$$\begin{pmatrix}1 & T'\\ T & 1\end{pmatrix}\varepsilon\begin{pmatrix}1 & T'\\ T & T\end{pmatrix}^{-1}$$

$1+X$

$$F\begin{pmatrix}1 & T'\\ T & 1\end{pmatrix} = \begin{pmatrix}1 & T'\\ T & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$$

$$F = (1+X)\varepsilon(1+X)^{-1}$$

Hope now that a symmetric position might help.

Situation,

$$\begin{tikzcd} \begin{pmatrix}W_+\\ W_-\end{pmatrix} & & \begin{pmatrix}W_+\\ W_-\end{pmatrix} \arrow[ll, "\begin{pmatrix}1 & -T^*\\ T & 1\end{pmatrix}"'] \arrow[dl, "(\alpha_+\ \alpha_-)"] \\ & V \arrow[ul, "\begin{pmatrix}\beta_+\\ \beta_-\end{pmatrix}"] & \end{tikzcd}$$

What you want:

$$\begin{pmatrix}W_+\\ W_-\end{pmatrix} \xleftarrow{\frac{1-X}{(1-X^2)^{1/2}}} \begin{pmatrix}W_+\\ W_-\end{pmatrix} \xleftarrow{\frac{1+X}{(1-X^2)^{1/2}}} \begin{pmatrix}W_+\\ W_-\end{pmatrix}$$

$$g^{1/2}\varepsilon g^{-1/2} = g\varepsilon = F$$

$\Gamma = \mathbb{Z}/2 = \{1, \varepsilon\}$, $\varepsilon^2 = 1$. $\quad \Lambda = \mathbb{C}\Gamma = \mathbb{C} \oplus \mathbb{C}\varepsilon$

$\Lambda = \mathbb{C}e_+ \oplus \mathbb{C}e_- \qquad e_\pm = \frac{1 \pm \varepsilon}{2}$

So you have two bases $(1, \varepsilon)$ and (e_+, e_-)

$$\begin{pmatrix} 1 \\ \varepsilon \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} e_+ \\ e_- \end{pmatrix} \qquad \begin{pmatrix} e_+ \\ e_- \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ \varepsilon \end{pmatrix}$$

$$\Lambda \otimes V = \begin{pmatrix} 1 \otimes V \\ \varepsilon \otimes V \end{pmatrix} = \begin{pmatrix} e_+ \otimes V \\ e_- \otimes V \end{pmatrix}$$

$$\begin{pmatrix} \iota_1 \\ \iota_\varepsilon \end{pmatrix} = \begin{pmatrix} \iota_+ + \iota_- \\ \iota_+ - \iota_- \end{pmatrix} \qquad \begin{pmatrix} \iota_+ \\ \iota_- \end{pmatrix} = \frac{1}{2}\begin{pmatrix} \iota_1 + \iota_\varepsilon \\ \iota_1 - \iota_\varepsilon \end{pmatrix}$$

$$\cancel{(j_1\ j_\varepsilon)\begin{pmatrix} \iota_+ \\ \iota_- \end{pmatrix} = (j_1, j_\varepsilon)\frac{1}{2}\begin{pmatrix} \iota_1 + \iota_\varepsilon \\ \iota_1 - \iota_\varepsilon \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}}$$

$j_+ \iota_1 = 1$, $j_- \iota_1 = 1$ $\qquad j_1 = \frac{1}{2}(j_+ + j_-)$

$j_+ \iota_\varepsilon = 1$, $j_- \iota_\varepsilon = -1$ $\qquad j_\varepsilon = \frac{1}{2}(j_+ - j_-)$

$$\iota_1 j_1 = \frac{1}{2}(\iota_+ + \iota_-)(j_+ + j_-) = \frac{1}{2}\big((\iota_+ j_+ + \iota_- j_-) + (\iota_- j_+ + \iota_+ j_-)\big)$$

You want to start with W, ε, h sat $h + \varepsilon h \varepsilon = 1_W$, choose a fact. $h = \beta_1\alpha_1 = W \xleftarrow{\beta_1} V \xleftarrow{\alpha_1} W$ and get $p_\pm$ on V.

$p(1) = \alpha_1\beta_1$

$p(\varepsilon) = \alpha_1 \varepsilon \beta_1$

$p_\pm = \alpha_1(1 \pm \varepsilon)\beta_1$

$(1 \pm \varepsilon)h(1 \pm \varepsilon)$

$= h + h\varepsilon h \pm \varepsilon h \pm h\varepsilon$

$= 1 \pm \varepsilon(h + \varepsilon h \varepsilon)$

$= 1 \pm \varepsilon$

$$\big(p(1) \pm p(\varepsilon)\big)^2 = (\alpha_1\beta_1 \pm \alpha_1\varepsilon\beta_1)^2$$
$$= \alpha_1\beta_1\alpha_1\beta_1 \pm \alpha_1\beta_1\alpha_1\varepsilon\beta_1$$
$$\pm \alpha_1\varepsilon\beta_1\alpha_1\beta_1 + \alpha_1\varepsilon\beta_1\alpha_1\varepsilon\beta_1$$
$$= \alpha_1\beta_1 \pm \alpha_1\varepsilon\underbrace{(\varepsilon\beta_1\alpha_1\varepsilon)}\beta_1 \pm \alpha_1\varepsilon\underbrace{(\beta_1\alpha_1)}\beta_1$$
$$= \alpha_1\beta_1 \pm \alpha_1\varepsilon\beta_1 = p(1) \pm p(\varepsilon)$$

Given W, $\varepsilon^2 = 1$, h, $h + \varepsilon h \varepsilon = 1_W$, choose fact

$$h = \beta_1 \alpha_1 : W \xleftarrow{\beta_1} V \xleftarrow{\alpha_1} W$$

$$\begin{pmatrix} W_+ \\ W_- \end{pmatrix} \xleftarrow{\begin{pmatrix} \beta_+ \\ \beta_- \end{pmatrix}} V \xleftarrow{(\alpha_+ \ \alpha_-)} \begin{pmatrix} W_+ \\ W_- \end{pmatrix} \qquad \begin{pmatrix} \beta_+ \\ \beta_- \end{pmatrix} = \begin{pmatrix} e_+ \beta_1 \\ e_- \beta_1 \end{pmatrix}$$

$$(\alpha_+ \ \alpha_-) = (\alpha_1 e_+ \ \ \alpha_1 e_-)$$

then $h = \begin{pmatrix} \beta_+ \alpha_+ & \beta_+ \alpha_- \\ \beta_- \alpha_+ & \beta_- \alpha_- \end{pmatrix}$

and $h + \varepsilon h \varepsilon = 1_W \iff \beta_\pm \alpha_\pm = \frac{1}{2}$.

Better approach $W = \begin{pmatrix} W_+ \\ W_- \end{pmatrix}$, X an odd operator

$X = \begin{pmatrix} 0 & T' \\ T & 0 \end{pmatrix}$ on W. Choose a fact.

$$\begin{pmatrix} W_+ \\ W_- \end{pmatrix} \xleftarrow{\begin{pmatrix} \beta_+ \\ \beta_- \end{pmatrix}} V \xleftarrow{(\alpha_+ \ \alpha_-)} \begin{pmatrix} W_+ \\ W_- \end{pmatrix}$$

of $1 + X = \begin{pmatrix} 1 & T' \\ T & 1 \end{pmatrix} = \begin{pmatrix} \beta_+ \alpha_+ & \beta_+ \alpha_- \\ \beta_- \alpha_+ & \beta_- \alpha_- \end{pmatrix}$

Then you get projections $p_\pm = \alpha_\pm \beta_\pm$ on V.

$$\begin{pmatrix} W_+ \\ W_- \end{pmatrix} \xleftarrow[\sim]{1+X} \begin{pmatrix} W_+ \\ W_- \end{pmatrix}$$

$$(1+X)\varepsilon(1+X)^{-1} \qquad\qquad \varepsilon$$

$$\parallel$$

$$F = \frac{1+X}{1-X}\varepsilon \qquad\qquad F(1+X) = \frac{1+X}{1-X}\varepsilon(1+X)$$

$$= \frac{1+X}{1-X}(1-X)\varepsilon = (1+X)\varepsilon$$

so F is the involution $= +1$ on $\begin{pmatrix} 1 \\ T \end{pmatrix}$
-1 on $\begin{pmatrix} T' \\ 1 \end{pmatrix}$

$$(1+X)^{-1}\varepsilon(1+X)$$

<u>Recap.</u> Start with (W, ε, X), you factor $1+X$

$$\begin{tikzcd}
\begin{pmatrix} W_+ \\ W_- \end{pmatrix} & & \begin{pmatrix} W_+ \\ W_- \end{pmatrix} \arrow[ll, "\begin{pmatrix} 1 & T' \\ T & 1 \end{pmatrix}"'] \arrow[dl, "(\alpha_+ \ \alpha_-)"] \\
& V \arrow[ul, "\begin{pmatrix} \beta_+ \\ \beta_- \end{pmatrix}"] &
\end{tikzcd}
\qquad
\begin{pmatrix} \beta_+\alpha_+ & \beta_+\alpha_- \\ \beta_-\alpha_+ & \beta_-\alpha_- \end{pmatrix} = \begin{pmatrix} 1 & T' \\ T & 1 \end{pmatrix}$$

and you get two projections $p_\pm$ on V $\quad p_\pm = \alpha_\pm\beta_\pm$

Assume $1+X$ invertible (hence also $1-X$ invertible), and suppose V reduced: $(\alpha_+ \ \alpha_-)$ surj, $\begin{pmatrix} \beta_+ \\ \beta_- \end{pmatrix}$ inj. Whence $(\alpha_+ \ \alpha_-)$ $\begin{pmatrix} \beta_+ \\ \beta_- \end{pmatrix}$ invertible.

You have two splittings of W, namely the given ε and the transport of ε by $1+X$.

$$(1+X)\,\varepsilon\,(1+X)^{-1} = \frac{1+X}{1-X}\,\varepsilon$$

$$(1+X)^{-1}\,\varepsilon\,(1+X) = \frac{1-X}{1+X}\,\varepsilon$$

$$\begin{tikzcd}
\begin{pmatrix} W_+ \\ W_- \end{pmatrix} & & \begin{pmatrix} W_+ \\ W_- \end{pmatrix} \arrow[dl] & & \begin{pmatrix} W_+ \\ W_- \end{pmatrix} \arrow[dl] \\
& V \arrow[ul] & & V \arrow[ul] \arrow[ll, shift left] \arrow[ll, shift right] &
\end{tikzcd}$$

$$\begin{pmatrix} V \\ V \end{pmatrix} \xleftarrow{\ \alpha\ } \begin{pmatrix} W_+ \\ W_- \end{pmatrix} \xleftarrow{\begin{pmatrix} \beta_+ & 0 \\ 0 & \beta_- \end{pmatrix}} \begin{pmatrix} V \\ V \end{pmatrix} \xleftarrow{\begin{pmatrix} \alpha_+ & 0 \\ 0 & \alpha_- \end{pmatrix}} \begin{pmatrix} W_+ \\ W_- \end{pmatrix}$$

$$\begin{array}{ccccc}
A \otimes N & \longrightarrow & M & \longrightarrow & \mathrm{Hom}(A, N) \\
a \otimes n & \longmapsto & acn & \longmapsto & (a' \mapsto j a' a c n) \\
& & m & \longmapsto & (a' \mapsto j a' m)
\end{array}$$

Mar 25, 02

Dilation seems to be the point you missed. GNS and all that. Look at GNS for a group Γ. Γ acts on a Hilbert space M, $N \xrightarrow{\iota} M$ is a subspace generating M, $\rho(s^{-1}t) = \iota^* s^{-1} t \iota \in \mathcal{L}(N)$ is a completely positive function from which you can recover M by completion; algebraically, completion means quotient by null vectors for the inner product, image of

$$\begin{array}{ccccc} A \otimes N & \longrightarrow & M & \longrightarrow & \mathrm{Hom}(A, N) \\ a \otimes n & \longmapsto & a\iota n & \longmapsto & (a' \mapsto j a' a \iota n = \rho(a'a)n) \\ & & m & \longmapsto & (a' \mapsto j a' m) \\ t \otimes n & \longmapsto & & & (s \mapsto \rho(st)n) \end{array}$$

Example. $W = \begin{pmatrix} W_+ \\ W_- \end{pmatrix}$ $\quad X = \begin{pmatrix} 0 & T' \\ T & 0 \end{pmatrix}$ then

X can be dilated to an odd involution somehow.

Start with $p \in \mathrm{Hom}_\Lambda(\Lambda \otimes V, \Lambda \otimes V) = \Lambda \otimes \mathrm{Hom}(V, V)$

$$p = e_+ \otimes p_+ + e_- \otimes p_-$$

Replace $\Lambda \otimes V$ by $\begin{pmatrix} V \\ V \end{pmatrix}$ with $e_+ = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $e_- = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

$$W_+ \xleftarrow{\beta_+} V \xleftarrow{\alpha_+} W_+ \qquad \beta_\pm \alpha_\pm = 1_{W_\pm} \qquad p_\pm = \alpha_\pm \beta_\pm$$

$$W_- \xleftarrow{\beta_-} V \xleftarrow{\alpha_-} W_- \qquad X = \beta \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \alpha$$

Conversely given $\begin{pmatrix} W_+ \\ W_- \end{pmatrix}$ + odd op X, you want to dilate which you do by factoring $1+X$ in any way

$$\begin{tikzcd} \begin{pmatrix} W_+ \\ W_- \end{pmatrix} & V \arrow[l, "\begin{pmatrix} \beta_+ \\ \beta_- \end{pmatrix}"'] & \begin{pmatrix} W_+ \\ W_- \end{pmatrix} \arrow[l, "(\alpha_+ \ \alpha_-)"'] \arrow[ll, bend left, "1+X"] \end{tikzcd} \Rightarrow \beta_\pm \alpha_\pm = 1$$

It seems that the Cayley transform is not involved, or relevant. So you assume

$$\begin{pmatrix} W_+ \\ W_- \end{pmatrix} \xleftarrow[\sim]{\begin{pmatrix}\beta_+ \\ \beta_-\end{pmatrix}} V \xleftarrow[\sim]{(\alpha_+ \ \alpha_-)} \begin{pmatrix} W_+ \\ W_- \end{pmatrix} \xleftarrow[\sim]{\begin{pmatrix}\beta_+ \\ \beta_-\end{pmatrix}} V$$

The idea was that this gives two involutions on V, which should be related somehow to $\rho_\pm$

In general

$$\begin{pmatrix} W_+ \\ W_- \end{pmatrix} \xleftarrow{\begin{pmatrix}1 & T' \\ T & 1\end{pmatrix}} V \xleftarrow{(\alpha_+ \ \alpha_-)} \begin{pmatrix} W_+ \\ W_- \end{pmatrix} \xleftarrow{\begin{pmatrix}\beta_+ \\ \beta_-\end{pmatrix}} V$$

$$V \xleftarrow{\alpha_+} W_+ \xleftarrow{\beta_+} V$$

Point
$$\begin{tikzcd} \begin{pmatrix} W_+ \\ W_- \end{pmatrix} & V \arrow[l, "\begin{pmatrix}\beta_+ \\ \beta_-\end{pmatrix}"'] & \begin{pmatrix} W_+ \\ W_- \end{pmatrix} \arrow[l, "(\alpha_+ \ \alpha_-)"'] \arrow[ll, bend left, "1+X"] \end{tikzcd}$$
You need $\begin{pmatrix} V \\ V \end{pmatrix}$ here

So get two involutions F, ε on W where $\varepsilon = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

and
$$F(1+X) = (1+X)\varepsilon$$
$$F = (1+X)\varepsilon(1+X)^{-1} = \frac{1+X}{1-X}\varepsilon$$

Take $V = \begin{pmatrix} W_+ \\ W_- \end{pmatrix}$, $\alpha_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $\alpha_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$\beta_+ = (1 \ \ T')$ $\qquad \alpha_+\beta_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix}(1 \ \ T')$

$\beta_- = (T \ \ 1)$ $\qquad \alpha_-\beta_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix}(T \ \ 1)$

$$\begin{tikzcd} \begin{pmatrix} W_+ \\ W_- \end{pmatrix} & V \arrow[l, "\begin{pmatrix}\beta_+\\ \beta_-\end{pmatrix}"'] & \begin{pmatrix} W_+ \\ W_- \end{pmatrix} \arrow[l, "(\alpha_+ \ \alpha_-)"'] \arrow[ll, bend left] \end{tikzcd}$$

$$1+X = \begin{pmatrix} 1 & T' \\ T & 1 \end{pmatrix}$$

You know that any factorization of $1+X$ leads to two projections on V namely $p_\pm = \alpha_\pm \beta_\pm$

Example: suppose $V = \begin{pmatrix} W_+ \\ W_- \end{pmatrix}$ and

$$\alpha_+ : \begin{pmatrix} W_+ \\ W_- \end{pmatrix} \xleftarrow{\begin{pmatrix}1\\0\end{pmatrix}} W_+ \quad , \quad \alpha_- : \begin{pmatrix} W_+ \\ W_- \end{pmatrix} \xleftarrow{\begin{pmatrix}0\\1\end{pmatrix}} W_-$$

Then $\begin{pmatrix} W_+ \\ W_- \end{pmatrix} \longleftarrow V = \begin{pmatrix} W_+ \\ W_- \end{pmatrix}$ is $\begin{pmatrix}\beta_+\\ \beta_-\end{pmatrix}$ where

$$\beta_+ = (1 \quad 0)\begin{pmatrix} 1 & T' \\ T & 1 \end{pmatrix} = (1 \quad T')$$

$$\beta_- = (0 \quad 1)\begin{pmatrix} 1 & T' \\ T & 1 \end{pmatrix} = (T \quad 1)$$

So
$$\boxed{\begin{aligned} \alpha_+\beta_+ &= \begin{pmatrix}1\\0\end{pmatrix}(1 \quad T') = \begin{pmatrix} 1 & T' \\ 0 & 0 \end{pmatrix} \\ \alpha_-\beta_- &= \begin{pmatrix}0\\1\end{pmatrix}(T \quad 1) = \begin{pmatrix} 0 & 0 \\ T & 1 \end{pmatrix} \end{aligned}}$$
and

$$\alpha_+\beta_- = \begin{pmatrix}1\\0\end{pmatrix}(T \quad 1) = \begin{pmatrix} T & 1 \\ 0 & 0 \end{pmatrix}$$

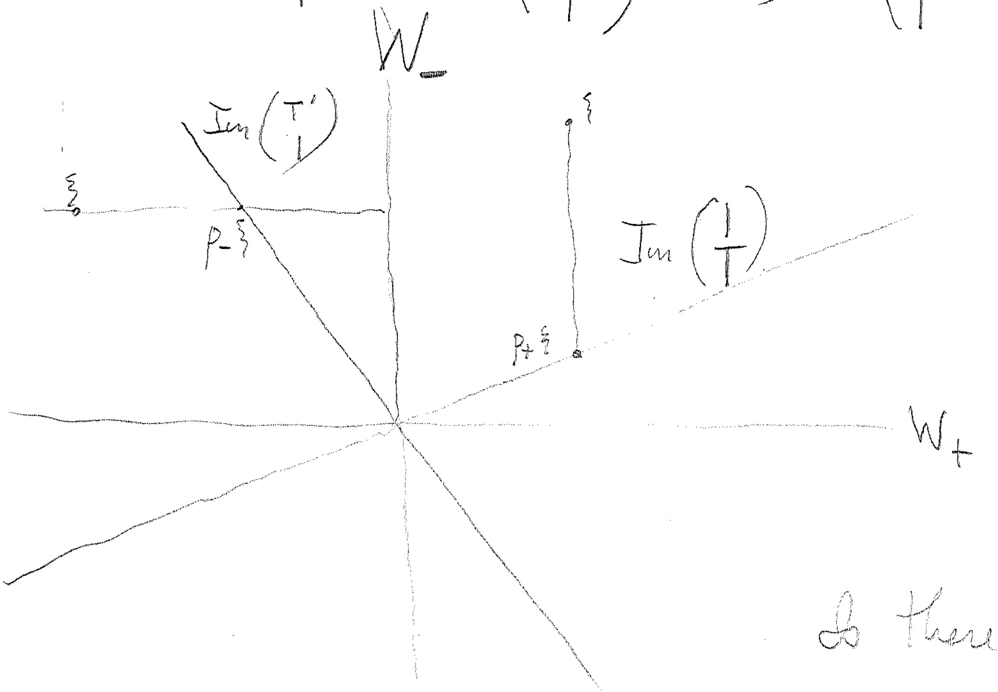
$$\alpha_-\beta_+ = \begin{pmatrix}0\\1\end{pmatrix}(1 \quad T') = \begin{pmatrix} 0 & 0 \\ 1 & T' \end{pmatrix}$$

$$\begin{pmatrix}W_+\\W_-\end{pmatrix} \xleftarrow{\begin{pmatrix}\beta_+\\\beta_-\end{pmatrix} = \begin{pmatrix}1&0\\0&1\end{pmatrix}} \begin{pmatrix}W_+\\W_-\end{pmatrix} \xleftarrow{(\alpha_+\ \alpha_-) = \begin{pmatrix}1&T'\\T&1\end{pmatrix}} \begin{pmatrix}W_+\\W_-\end{pmatrix}$$

$$\boxed{\begin{aligned} p_+ &= \alpha_+\beta_+ = \begin{pmatrix}1\\T\end{pmatrix}(1\ \ 0) = \begin{pmatrix}1&0\\T&0\end{pmatrix} \\ p_- &= \alpha_-\beta_- = \begin{pmatrix}T'\\1\end{pmatrix}(0\ \ 1) = \begin{pmatrix}0&T'\\0&1\end{pmatrix}\end{aligned}}$$

$$\alpha_+\beta_- = \begin{pmatrix}1\\T\end{pmatrix}(0\ \ 1) = \begin{pmatrix}0&1\\0&T\end{pmatrix}$$

$$\alpha_-\beta_+ = \begin{pmatrix}T'\\1\end{pmatrix}(1\ \ 0) = \begin{pmatrix}T'&0\\1&0\end{pmatrix}$$



$$\begin{pmatrix}x\\y\end{pmatrix} \mapsto \begin{matrix}x\\Tx\end{matrix}$$

$$\begin{pmatrix}0&T'\\0&1\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}T'y\\y\end{pmatrix}$$

Is there anything

$$F(1+X) = (1+X)\varepsilon$$

$F = 1$ on $\operatorname{Im}\begin{pmatrix}1\\T\end{pmatrix}$ and $F = -1$ on $\operatorname{Im}\begin{pmatrix}T'\\1\end{pmatrix}$

So you conclude that the Cayley Transform F which $= +1$ on the graph $\begin{pmatrix}1\\T\end{pmatrix}W_+$

$= -1$ on the graph $\begin{pmatrix}T'\\1\end{pmatrix}W_-$

seems unrelated to p_+ and p_-.

$$\Lambda = \mathbb{C}[\mathbb{Z}/2] = \mathbb{C}e_+ \oplus \mathbb{C}e_- \qquad e_\pm = \frac{1\pm\varepsilon}{2}$$

free Λ-module $\quad \Lambda\otimes V = e_+\otimes V \oplus e_-\otimes V$

can identify $\Lambda\otimes V$ with $\begin{pmatrix} V \\ V \end{pmatrix}$ $\quad e_+ = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, e_- = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

endos. of $\Lambda\otimes V$ $\quad \begin{pmatrix} X_+ & 0 \\ 0 & X_- \end{pmatrix} \in \begin{pmatrix} \mathcal{L}(V) & 0 \\ 0 & \mathcal{L}(V) \end{pmatrix}$

$p^2 = p$ same as $p_+, p_- \in \mathcal{L}(V)$.

consider (W, β, α):

$$W \xleftarrow{\beta} \begin{pmatrix} V \\ V \end{pmatrix} \xleftarrow{\alpha} W$$

$$\begin{pmatrix} W_+ \\ W_- \end{pmatrix} \xleftarrow{\begin{pmatrix} \beta_+ & 0 \\ 0 & \beta_- \end{pmatrix}} \begin{pmatrix} V \\ V \end{pmatrix} \xleftarrow{\begin{pmatrix} \alpha_+ & 0 \\ 0 & \alpha_- \end{pmatrix}} \begin{pmatrix} W_+ \\ W_- \end{pmatrix} \qquad \beta_\pm \alpha_\pm = 1_{W_\pm}$$

odd op

$$X = \begin{pmatrix} \beta_+ & 0 \\ 0 & \beta_- \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \alpha_+ & 0 \\ 0 & \alpha_- \end{pmatrix} = \begin{pmatrix} 0 & \beta_+\alpha_- \\ \beta_-\alpha_+ & 0 \end{pmatrix}$$

Moral so far: a retract W of $\Lambda\otimes V$ " following structure; grading $W = \begin{pmatrix} W_+ \\ W_- \end{pmatrix}$ plus odd operator X.

Conversely given such a W, choose a fact. of $1+X$

$$\begin{pmatrix} W_+ \\ W_- \end{pmatrix} \xleftarrow{\begin{pmatrix} \beta_+ \\ \beta_- \end{pmatrix}} V \xleftarrow{(\alpha_+ \ \alpha_-)} \begin{pmatrix} W_+ \\ W_- \end{pmatrix}$$

$$\Longrightarrow \quad \begin{pmatrix} \beta_+\alpha_+ & \beta_+\alpha_- \\ \beta_-\alpha_+ & \beta_-\alpha_- \end{pmatrix} = \begin{pmatrix} 1_{W_+} & T' \\ T & 1_{W_-} \end{pmatrix}$$

so $\begin{matrix} W_+ \\ W_- \end{matrix}$ is a ret of V corresp. to proj. $\quad \begin{matrix} p_+ = \alpha_+\beta_+ \\ p_- = \alpha_-\beta_- \end{matrix}$

At this point ~~you are confused~~. You've found a V with two projections $p_\pm$ yielding a retract $\begin{pmatrix} W_+ \\ W_- \end{pmatrix}$ of $\begin{pmatrix} V \\ V \end{pmatrix}$.

March 27, 02
in the group case
plus Γ action on W

Back to the Morita context
$\Lambda = \mathbb{C}[\Gamma]$. Given $(W \xleftarrow{\beta_1} V \xleftarrow{\alpha_1} W)$

$$\left[\begin{array}{c|c} & \alpha_1 \\ \hline \beta_1 & \end{array}\right]\begin{bmatrix} V \\ W \end{bmatrix}$$

Simplest version might be what? Start with unital ring
$$\begin{bmatrix} \mathbb{C} & 0 \\ 0 & \mathbb{C}[\Gamma] \end{bmatrix}$$
whose (unital) modules are couples $\begin{bmatrix} V \\ W \end{bmatrix}$ consisting of a v.s. and a Γ-module. Adjoin elements α_1, β_1 subject to relations $\sum s\beta_1\alpha_1 s^{-1} = 1_W$. You end up with a Morita context

$$\left[\begin{array}{c|c} A^+ & Y \\ \hline X & B \end{array}\right]$$

A has gen. $\rho(s) = \alpha_1 s \beta_1$

It occurs to me now that the issue here is how to handle the identity $1 \in A^+$ together with the Γ grading.

Suppose $\Gamma = \mathbb{Z}/2$, so that a Γ-graded algebra is a superalgebra $\cancel{A = A_+ \oplus A_-}$ $\quad \cancel{A_+A_+ + A_-A_- \subset A_+}$
$\cancel{A_+A_- + A_-A_+ \subset A_-}$

This is a bad notation. Better is
$$A = A_1 \oplus A_\varepsilon \qquad \begin{array}{l} A_1A_1 + A_\varepsilon A_\varepsilon \subset A_1 \\ A_1A_\varepsilon + A_\varepsilon A_1 \subset A_\varepsilon \end{array}$$

So when you adjoin an identity to A you specify that it has degree 1. This obviously should work for any Γ graded-algebra, with Γ a group.

(Resurrect?) Resurrect the idea of recovering a unital ring from its left regular (left multiplier) ring.

Problem: Form the category of $(V, W, \alpha_1, \beta_1)$
V v.s., W Γ-mod, $W \xleftarrow{\beta_1} V \xleftarrow{\alpha_1} W$ $\mathbb{C}$ linear maps satisfying $\sum s\beta_1\alpha_1 s^{-1} = 1_W$ (Γ finite assumed).
Construct the unital ring R yielding these modules.

To be more specific, you consider the functor

$$(V, W, \alpha_1, \beta_1) \longmapsto \begin{pmatrix} V \\ W \end{pmatrix} \in \mathrm{Mod}(\mathbb{C}) \times \mathrm{Mod}(\mathbb{C}\Gamma)$$

$$(V \;\; W) \begin{bmatrix} \mathbb{C} & \alpha_1 \otimes \mathbb{C}\Gamma \\ \mathbb{C}\Gamma \otimes \beta_1 & \mathbb{C}\Gamma \end{bmatrix} \begin{pmatrix} V \\ W \end{pmatrix} \quad \begin{matrix} A^+ & Y \\ X & B \end{matrix}$$

Is there a kind of tensor algebra here? For a tensor alg of the form $T_A(X) = A \oplus X \oplus X \otimes_A X \oplus \cdots$ you need a bimodule X over A.

$$\begin{pmatrix} A & \\ & B \end{pmatrix} \oplus \begin{pmatrix} & Y \\ X & \end{pmatrix} \oplus \begin{pmatrix} & Y \\ X & \end{pmatrix} \otimes_{\left(\begin{smallmatrix} A & \\ & B \end{smallmatrix}\right)} \begin{pmatrix} & Y \\ X & \end{pmatrix}$$

$$= \begin{pmatrix} Y \otimes_B X & 0 \\ 0 & X \otimes_A Y \end{pmatrix}$$

$$M_2\mathbb{C} \otimes \mathbb{C}\Gamma = \mathbb{C}[M_2 \times \Gamma]$$

Repeat: You have a category of objects $(W, V, \alpha_1, \beta_1)$, where W is a Γ-module, V is a v.s., and $W \xleftarrow{\beta_1} V \xleftarrow{\alpha_1} W$ are linear maps satisfying

$\sum_s s\beta_1\alpha_1 s^{-1} = 1_W$. Problem. Construct a unital ring R whose left modules are exactly these objects. Notice that R is not unique, since $M_A R$ yields the same module category as R. How to make R unique?

Right module version?? You went through this before $(V \otimes \Lambda)p$? W now a right Γ-module with h_1 s.t. $\sum_s w s h_1 s^{-1} = w$ factor $h_1 = W \xrightarrow{\beta_1} V \xrightarrow{\alpha_1} W$

$$w \sum_s s\beta_1\alpha_1 s^{-1} = w \qquad W \xrightarrow{\beta} V \otimes \Lambda \xrightarrow{\alpha} W$$

Mar 27, 02

$$\begin{tikzcd} W \arrow[r, "\beta"] \arrow[dr, "\beta_1"'] & V\otimes\Lambda \arrow[r, "\alpha"] \arrow[d, shift right, "\eta_1"'] & W \arrow[r, "\beta"] & V\otimes\Lambda \\ & V \arrow[u, shift right, "\varepsilon_1"'] \arrow[ur, "\alpha_1"'] & & \end{tikzcd}$$

$$\sum_t f(t)\otimes t \xmapsto{\alpha} \sum_t f(t)\alpha_1 t$$

$$w \xmapsto{\beta} \sum_s ws^{-1}\beta_1 \otimes s$$

$$\sum_t f(t)\otimes t \xmapsto{\alpha} \sum_t f(t)\alpha_1 t \xmapsto{\beta} \sum_s \Big(\sum_t f(t)\alpha_1 \overbrace{ts^{-1}\beta_1}^{p(ts^{-1})}\Big)\otimes s$$

$$\sum_s f(s)\otimes s \xmapsto{\alpha} \sum_s f(s)\alpha_1 s \xmapsto{\beta} \sum_t \sum_s f(s)(\alpha_1 s t^{-1}\beta_1)\otimes t$$

You recall struggling with this stuff, and ultimately settling on $\begin{pmatrix} hBh & hB \\ Bh & B \end{pmatrix}$ pretending $h = h^2$.

Question: Why is all this so hard? Surely the category of $(W, V, \alpha_1, \beta_1)$ should determine a unital ring in a simple way. Maybe via the "fibre functor" approach. Forgetful functor

$$(W, V, \alpha_1, \beta_1) \xmapsto{F} \begin{pmatrix} W \\ V \end{pmatrix}$$

R should be the ring of endos of this fun

$$\begin{pmatrix} \mathbb{C} & \alpha_1 \\ \beta_1 & \Lambda \end{pmatrix} \subset \mathrm{End}\left(F = \begin{pmatrix} F_V \\ F_W \end{pmatrix}\right) = \begin{pmatrix} \mathrm{End}(F_V) & \mathrm{Hom}(F_W, F_V) \\ \mathrm{Hom}(F_V, F_W) & \mathrm{End}(F_W) \end{pmatrix}$$

Let $\begin{pmatrix} A & Y \\ X & B \end{pmatrix}$ be subring of generated by $\mathbb{C}, \alpha_1, \beta_1, \Lambda$

Review the situation: You have cat $\mathcal{U}$ obj $(W, V, \alpha_1, \beta_1)$ and two functors $(W, V, \alpha_1, \beta_1) \longrightarrow \begin{matrix} W \in \mathrm{Mod}(\mathbb{C}\Gamma) \\ V \in \mathrm{Mod}(\mathbb{C}) \end{matrix}$. You now consider morphisms of functors, get $C = \begin{pmatrix} A & Y \\ X & B \end{pmatrix} \cong$ (unital) ring of endos of $\mathcal{U} \longmapsto \begin{pmatrix} V \\ W \end{pmatrix}$.

March 28, 02.

two functors $(W, V, \alpha_1, \beta_1) \xmapsto{F_W} W \in Mod(\mathbb{C}\Gamma)$, $\xmapsto{F_V} V \in Mod(\mathbb{C})$

$$\begin{pmatrix} A & Y \\ X & B \end{pmatrix} = \operatorname{Hom}_{Fun(\mathcal{U}, ab)}\left(\begin{pmatrix} V \\ W \end{pmatrix} \leftarrow \begin{pmatrix} V \\ W \end{pmatrix}\right)$$

notation reversed.

Repeat $F_{\binom{V}{W}} : \mathcal{U} \longrightarrow \begin{pmatrix} V \\ W \end{pmatrix}$ (direct sum)

get unital ring (a Morita context) of endos. of this functor. And you have specific elements

$$\begin{pmatrix} 1_V & \alpha_1 \\ \beta_1 & \{s_W\} \end{pmatrix} \in \begin{pmatrix} A & Y \\ X & B \end{pmatrix} \qquad \sum s\beta_1\alpha_1 s^{-1} = 1_W$$

You want to show that these elements generate $\begin{pmatrix} A & Y \\ X & B \end{pmatrix}$

Let $\begin{pmatrix} A' & Y' \\ X' & B' \end{pmatrix}$ be the ring defined by gens rels. Then there is a canonical morphism

$$\begin{pmatrix} A' & Y' \\ X' & B' \end{pmatrix} \longrightarrow \begin{pmatrix} A & Y \\ X & B \end{pmatrix}$$

which induces an equivalence of left modules. So suppose $R' \xrightarrow{u} R$ is a unital alg map inducing an equivalence $Mod(R') \longleftarrow Mod(R)$.

$$Mod(S) \underset{u_*}{\overset{u_!}{\rightleftarrows}} Mod(R) \quad (u^* \text{ leftward})$$

$$\operatorname{Hom}(FX, Y) = \operatorname{Hom}(X, GY)$$

$$\begin{array}{c} \operatorname{Hom}(FX, FX') = \operatorname{Hom}(X, GFX') \\ S\uparrow \\ \operatorname{Hom}(X, X') \end{array}$$

March 29, 02

$R \xrightarrow{u} S$ unital ring map

$\mathrm{Mod}(R) \underset{u^*}{\overset{u_!}{\rightleftarrows}} \mathrm{Mod}(S)$ u^* rest. of scalars $u^*(N) = N$ with mult $r\cdot n = u(r)n$

$u_!(M) = S\otimes_R M$

adjunction maps $u_!u^*N = S\otimes_R N \longrightarrow N$, $s\otimes n \mapsto sn$ $S\otimes_R S \xrightarrow{\sim} S$, $s\otimes s' \mapsto ss'$

$M \longrightarrow u^*u_!M = S\otimes_R M$

$m \longmapsto 1\otimes m$

$R \xrightarrow{\sim} S\otimes_R R \to S$

$r \longmapsto 1_S\otimes r$, $s\otimes r \mapsto su(r)$

Next look at $\mathrm{Mod}(R) \underset{u_*}{\overset{u^*}{\leftrightarrows}} \mathrm{Mod}(S)$ $u_*(M) = \mathrm{Hom}_R(S,M)$

$u^*u_*(M) = \mathrm{Hom}_R(S,M) \longrightarrow M$ $\mathrm{Hom}_R(S,M) \to \mathrm{Hom}_R(R,M)$

$f \longmapsto f(1_S)$

$N \longrightarrow u_*u^*N = \mathrm{Hom}_R(S,N)$ $s' \mapsto (s \mapsto ss')$

$n \mapsto (s\mapsto sn)$ $S \xrightarrow{\sim} \mathrm{Hom}_R(S,S)$

~~$R \to S$ unital ring map~~ $\beta: 1_M \longrightarrow GF M$ ($u^*u_!$)

~~$\mathrm{Mod}(R) \underset{u^*}{\overset{u_!}{\rightleftarrows}} \mathrm{Mod}(S)$ $u_!(M) = S\otimes_R M$, $M = R\otimes_R M \xrightarrow{u\otimes 1} S\otimes_R M$~~

$R \xrightarrow{u} S$ $\mathrm{Mod}(R) \underset{u^*}{\overset{u_!}{\rightleftarrows}} \mathrm{Mod}(S)$ $u_!M = S\otimes_R M$

$\beta: M \longrightarrow u^*u_!M$ should be $M = R\otimes_R M \xrightarrow{u\otimes 1} S\otimes_R M$

$m \mapsto 1_R\otimes m \longmapsto 1_S\otimes m$

u^* equiv. $\Longrightarrow$ β isom. $\Longrightarrow$ $R \xrightarrow{\sim} S\otimes_R R \xrightarrow{\sim} S$

$r \longmapsto 1_S\otimes r \longmapsto u(r)$

$\mathrm{Hom}(\overset{u_!}{F}X, Y) = \mathrm{Hom}(X, \overset{u^*}{G}Y)$ $\mathrm{Hom}(FGY', Y) = \mathrm{Hom}(GY', GY)$

$\wr\uparrow$ G f.f.

$\mathrm{Hom}(Y', Y)$

$\mathrm{Hom}(FX, FX') = \mathrm{Hom}(X, GFX')$

F fully faithful $\wr\uparrow$ $\sim$

$\mathrm{Hom}(X, X')$ $\Longrightarrow X' \xrightarrow{\sim} GFX'$

F fully faithful $\Longleftrightarrow$ $\beta: 1 \xrightarrow{\sim} GF$

G ——— $\Longleftrightarrow$ $\alpha: FG \xrightarrow{\sim} 1$

$R \xrightarrow{u} S$ $\qquad$ $\mathrm{Mod}(R) \underset{u^*}{\overset{u_!}{\rightleftarrows}} \mathrm{Mod}(S)$

u^* = restriction of scalars: $u^*N = N$ with $r.n = u(r)n$

$u_!$ = extension of —— : $u_!M = S \otimes_R M$.

adj arrows $\quad u_! u^* N = S \otimes_R N \xrightarrow{\alpha} N$

$s \otimes n \longmapsto sn$

$\beta : M \longrightarrow u^* u_! M : \quad M = R \otimes_R M \xrightarrow{u \otimes 1} S \otimes_R M$

Category theory tells us that u^* is an equiv. $\iff$ α, β are isos, and if so then $u_!$ is a quasi-inverse to u^*.

Take $M = R$: $\quad R \cong R \otimes R \xrightarrow[\sim]{u \otimes 1} S \otimes_R R \xrightarrow{\sim} S$

$r \quad 1_R \otimes r \quad 1_S \otimes r \mapsto u(r)$

u^* rest. of scalars (equivalence)

u_* coextension of scalars $\quad u_*M = \mathrm{Hom}_R(S, M)$

$\alpha : u^* u_* M \xrightarrow{\sim} M : \quad \mathrm{Hom}_R(S, M) \xrightarrow{\sim} \mathrm{Hom}_R(R, M) = M$

$\beta : N \longrightarrow u_* u^* N = \mathrm{Hom}_R(S, N) \qquad f \longmapsto fu$

How to make sense of this?

$\mathrm{Hom}(FX, Y) \to \mathrm{Hom}(GFX, GY) \longrightarrow \mathrm{Hom}(X, GY)$

You are guessing that $\alpha : u^* u_* M \longrightarrow M$ is ...

$(u^*) \mathrm{Hom}_R(S, M) \longrightarrow M \quad , \quad f \longmapsto f(1_S)$

$(u^*) \mathrm{Hom}_R(S, M) \longrightarrow \mathrm{Hom}_R(R, M) \longrightarrow M$

$(s \mapsto f(s)) \longmapsto (r \mapsto fur) \longmapsto f(u 1_R) = f(1_S)$

How do I see the adjunction map?

$$\mathrm{Hom}_S(N, \underbrace{\mathrm{Hom}_R(S, M)}_{u_* M}) \simeq \mathrm{Hom}_R(\overbrace{S \otimes_S N}^{u^* N \,=}, M)$$

So now take $N = u_* M$. and look for the image of the identity.

$$\mathrm{Hom}_S(N, \mathrm{Hom}_R(S, M)) \longrightarrow \mathrm{Hom}_R(S \otimes_S N, M) = \mathrm{Hom}_R(N, M)$$

$$(s, n) \mapsto f(s, n) \longmapsto (s \otimes n \mapsto f(s, n)) \mapsto f(1, n)$$

So the adj. map α will ~~a~~ occur with $N = \mathrm{Hom}_R(S, M)$

$$\mathrm{Hom}_S(N, \mathrm{Hom}_R(S, M)) = \mathrm{Hom}_R(N, M)$$

$$f(s, n) \longmapsto f(1, n)$$

$$\begin{array}{ccc} \mathrm{Hom}_S(\mathrm{Hom}_R(S, M), \mathrm{Hom}_R(S, M)) & \xrightarrow{\sim} & \mathrm{Hom}_R(\mathrm{Hom}_R(S, M), M) \\ (s, \nu) \mapsto f(s, \nu) \in M \quad \Big\downarrow & & \\ \mathrm{Hom}_S(S \otimes_S \mathrm{Hom}_R(S, M), M) & \longrightarrow & \mathrm{Hom}_S(\mathrm{Hom}_R(S, M), M) \end{array}$$

$$1 \otimes \nu \longleftarrow \nu$$

$$s \otimes \nu \longmapsto f(s, \nu) \longmapsto f(1_S, \nu)$$

adjoints.

$$\begin{array}{ccc} \mathrm{Hom}(GY, GY') & = & \mathrm{Hom}(FGY, Y') \\ \uparrow S & \nearrow \sim & \\ \mathrm{Hom}(Y, Y') & & \end{array} \quad \Rightarrow \alpha : FGY \xrightarrow{\sim} Y$$

$$\beta : N \longrightarrow u_* u^* N = \mathrm{Hom}_R(S, N) \qquad n \longmapsto (s \longmapsto sn)$$

Now you want to return to the Morita equivalence

$$\begin{pmatrix} 1_V & \alpha_1 \\ \beta_1 & \{s_W\} \end{pmatrix} \begin{pmatrix} V \\ W \end{pmatrix} \qquad \sum s \beta_1 \alpha_1 s^{-1} = 1_W$$

Back to $\Lambda = \mathbb{C}e_+ \oplus \mathbb{C}e_- = \mathbb{C}1 \oplus \mathbb{C}s$

You want to set up the $\quad e_\pm = \frac{1 \pm s}{2} \quad s^2 = 1$

Morita context. $\begin{bmatrix} 1_V & \alpha_+ & \alpha_- \\ \beta_+ & 1_{W_+} & \\ \beta_- & & 1_{W_-} \end{bmatrix} \begin{bmatrix} V \\ W_+ \\ W_- \end{bmatrix}$ cat $V, W_+, W_-, \alpha_\pm, \beta_\pm$

$\beta_\pm \alpha_\pm = 1_{W_\pm}$

$$\begin{pmatrix} W_+ \\ W_- \end{pmatrix} \xleftarrow{\begin{pmatrix}\beta_+ \\ \beta_-\end{pmatrix}} \begin{pmatrix} V \\ V \end{pmatrix} \xleftarrow{(\alpha_+ \ \alpha_-)} \begin{pmatrix} W_+ \\ W_- \end{pmatrix}$$

It looks like you might find the sort of proof you want. R defined by generators and relation, S = ring of endos of the forgetful functor.

Go back to $\begin{bmatrix} 1_V & \alpha_1 \\ \beta_1 & \Gamma \end{bmatrix} \begin{bmatrix} V \\ W \end{bmatrix}$ $\qquad \sum s\beta_1\alpha_1 s^{-1} = 1_W$

$\mathcal{U}$ = cat of objects $(W, V, \ W \xleftarrow{\beta_1} V \xleftarrow{\alpha_1} W)$

Let's go over the grading on the ring with these generators and relations $\quad M_2\mathbb{C} \times \mathbb{C}\Gamma = \mathbb{C}[\{e_{ij}\} \times \Gamma]$

$$1_V \overset{\Delta}{\longmapsto} e_{11} \otimes 1_\Gamma \otimes 1_V$$

$$\alpha_1 \longmapsto e_{12} \otimes 1_\Gamma \otimes \alpha_1$$

$$\beta_1 \longmapsto e_{21} \otimes 1_\Gamma \otimes \beta_1$$

$$s \longmapsto e_{22} \otimes s \otimes s$$

$$\begin{pmatrix} A & Y \\ X & B \end{pmatrix}$$

$$\underbrace{\sum_s s\beta_1\alpha_1 s^{-1}}_{1_W} \longmapsto e_{22} \otimes 1_\Gamma \otimes \underbrace{\sum_s s\beta_1\alpha_1 s^{-1}}_{1_W}$$

March 30, 02.

Review the Morita equivalence

Γ group (finite)

$\mathcal{W}$: object consists of Γ-module W plus $\mathbb{C}$-linear operator h satisfying $\sum_s shs^{-1}w = w \quad \forall w \in W$.

$\mathcal{V}$: object consists of a v.s. V + ops $p(s)$ $s\in\Gamma$ satisf.

(i) $\displaystyle\sum_{u=st} p(s)p(t) = p(u) \qquad \left(\sum_t p(st^{-1})p(t) = p(s)\right)$

(ii) $V = \sum_s p(s)V \quad , \quad \bigcap_s \operatorname{Ker} p(s) \text{ on } V = 0$

$\mathcal{U}$: object $(W, V, \beta_1, \alpha_1)$, W Γ-module, V v.s,

$W \xleftarrow{\beta_1} V \xleftarrow{\alpha_1} W \qquad \mathbb{C}$-linear maps.

(i) $\displaystyle\sum_s s\beta_1\alpha_1 s^{-1} = 1_W$

(ii) β_1 injective, α_1 surjective

---

2 functors

$\mathcal{U} \longrightarrow \mathcal{W}$ (forget V functor)

$(W, V, \beta_1, \alpha_1) \longmapsto (W, h)$ where $h = \beta_1\alpha_1$

equivalence because $V = \operatorname{Im} \beta_1\alpha_1$.

$\mathcal{U} \longrightarrow \mathcal{V}$ (forget W)

$(W, V, \beta_1, \alpha_1) \longmapsto \left(V, p(s) = \alpha_1 s \beta_1 : V \xleftarrow{\alpha_1} W \xleftarrow{\beta_1} V\right)$

$\displaystyle\sum_t p(st^{-1})p(t) = \sum_t \alpha_1 s(t^{-1}\beta_1\alpha_1 t)\beta_1 = \alpha_1 s\beta_1 = p(s).$

$(\forall s)$, $0 = p(s)v = \alpha_1 s\beta_1 v \Rightarrow \cancel{\forall s} \sum_s (s^{-1}\beta_1\alpha_1 s)\beta_1 v = 0$

$\Rightarrow \beta_1 v = 0 \Rightarrow v.$

$v = \alpha_1 w = \displaystyle\sum_s \underbrace{\alpha_1 s\beta_1}_{p(s)}\underbrace{\alpha_1 s^{-1} w}_{V}$

Use $\{p(s)\}$ to get a projection p on the free Γ-module

$$\Lambda\otimes V = \bigoplus_t t\otimes V = \Big\{\sum_t t\otimes f(t) \,\Big|\, f:\Gamma\to V\Big\}$$

$$\boxed{p\Big(\sum_t t\otimes f(t)\Big) = \sum_s s\otimes\sum_t p(s^{-1}t)f(t)}$$

$$p(t\otimes v) = \sum tu^{-1}\otimes p(u)v \qquad s=tu^{-1},\ u=s^{-1}t$$
$$= \sum_s s\otimes p(s^{-1}t)v$$

$(\forall u\in\Gamma)\ pu=up, \quad p^2=p \qquad \sum_t p(s^{-1}t)p(t^{-1}u) = p(s^{-1}u)$

Corresp. to proj p is a Γ-module Retract W of

$$\begin{tikzcd}
W & \Lambda\otimes V \arrow[l, "\beta"'] \arrow[d, equal] & W \arrow[l, "\alpha"'] \arrow[ddl, "\alpha_1"] \\
 & \bigoplus s\otimes V \arrow[d, shift left, "j_1"] & \\
 & V \arrow[u, shift left, "\iota_1"] \arrow[uul, "\beta_1"] &
\end{tikzcd}$$

β,α Γ-maps.
$\beta\alpha = 1_W$
$p=\alpha\beta$.

$\iota_1 v = 1\otimes v \qquad \beta=\beta_1\iota \qquad \alpha_1 = j_1\alpha$

$j_1\sum t\otimes f(t) = f(t)$

$$1_{\Lambda\otimes V} = \sum_s \iota_s j_s = \sum_s s\,\iota_1 j_1 s^{-1}$$

$$1_W = \beta\, 1_{\Lambda\otimes V}\,\alpha = \sum_s s\,\beta\iota_1 j_1\alpha\, s^{-1} = \sum_s s\,\beta_1\alpha_1 s^{-1}$$

$$\alpha_1 W = \sum_s p(s)V \qquad \operatorname{Ker}\beta_1 = \bigcap_s \operatorname{Ker} p(s)$$

Assembly $\quad X=\bigcup U_i \qquad E|U_i \xleftarrow{\beta_i} \Lambda\otimes V_i \xleftarrow{\alpha_i} E|U_i$

$$E \xleftarrow{(\cdots\chi_i\beta_i\cdots)} \underbrace{\begin{pmatrix}\vdots\\ \Lambda\otimes V_i\\ \vdots\end{pmatrix}}_{\Lambda\otimes\bigoplus V_i} \xleftarrow{\begin{pmatrix}\vdots\\ \chi_i\alpha_i\\ \vdots\end{pmatrix}} E$$

April 1, 02 IDEA in Serre thm. proof that you construct a factorization of the identity thru a free module.

$$E \xleftarrow{(\cdots X_i \beta_i \cdots)} \Lambda \otimes \bigoplus_i V_i \xleftarrow{\begin{pmatrix} \vdots \\ \alpha_i X_i \\ \vdots \end{pmatrix}} E$$

Line Bdl/S^2 + Poisson

$$a \quad \forall n, N \quad \left|\left(\frac{d}{dx}\right)^n f(x)\right| \leq \frac{C_{n,N}}{(1+|x|)^N} \qquad f \in \mathcal{S}(\mathbb{R})$$

$$f(x) \longmapsto f_4(x) = \sum_{n \in \mathbb{Z}} f(x+n) \quad - \quad C^\infty, \text{periodic.}$$

$$\hat{f}(y) = \int_{-\infty}^{\infty} e^{-2\pi i x y} f(x)\, dx \qquad \int_0^1 e^{-2\pi i x m} f_4(x)\, dx = \hat{f}_4$$

$$f_4(x) = \sum_m e^{2\pi i m x} \hat{f}_4(m)$$

$$= \sum_m \int_0^1 e^{2\pi i m (x-x')} \sum_n f(x'+n)\, dx'$$

$$= \sum_n \sum_m \underbrace{\int_0^1 e^{2\pi i m(x-x')} f(x'+n)\, dx'}$$

$$y = x'+n \qquad x' = y-n$$

$$\sum_m \sum_n \int_n^{n+1} e^{2\pi i m(x-y+\cancel{n})} f(y)\, dy$$

$$= \sum_m e^{2\pi i m x} \underbrace{\sum_n \int_n^{n+1} e^{-2\pi i m y} f(y)\, dy}$$

$$\int_{-\infty}^{\infty} e^{-2\pi i m y} f(y)\, dy = \hat{f}(m)$$

$$\tilde{f}(x,y) = \sum_m e^{2\pi i m y} f(x+m) \qquad \text{period 1 in } y$$

$$e^{2\pi i x y} \tilde{f}(x,y) = \sum_m e^{2\pi i (x+m)} f(x+m) \qquad \text{period 1 in } x$$

$$\tilde{f}(x,y) = \tilde{f}(x, y+m) \qquad e^{2\pi i (x+m) y} \tilde{f}(x+m, y) = \cancel{e^{2\pi i x y}} \tilde{f}(x,y)$$

$$\boxed{\tilde{f}(x+m, y+n) = e^{-2\pi i m y} \tilde{f}(x,y)}$$

Claim $f \longmapsto \tilde{f}$ $\quad \mathcal{S}(\mathbb{R}) \xrightarrow{\sim} \Gamma(T^2, L)$

$$F(x,y) \in C^\infty(\mathbb{R}^2), \quad F(x+m, y+n) = e^{-2\pi i m y} F(x,y).$$

$$F(x,y) = \sum_m e^{2\pi i m y} \underbrace{\int_0^1 e^{-2\pi i m y'} F(x,y')\,dy'}_{f_m(x)}$$

$$f_m(x+1) = \int_0^1 e^{-2\pi i m y'} \underbrace{F(x+1,y')}_{e^{-2\pi i y'} F(x,y)}\,dy' = f_{m+1}(x)$$

$$f_m(x) = f_{m-1}(x+1) = \cdots = f_0(x+m).$$

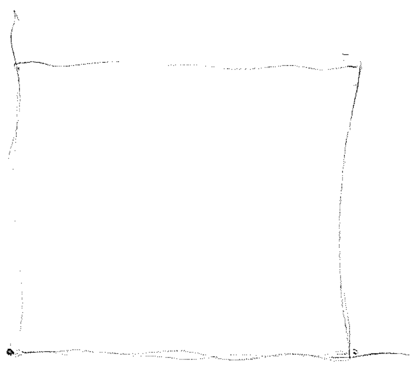
$$\therefore \quad F(x,y) = \widetilde{f_0}(x)$$

---

degree argument. $\quad \psi \in \Gamma(L) \quad \psi \in C^\infty(\mathbb{R}^2),$

$$\psi(x+m, y+n) = e^{-2\pi i m y}\psi(x,y)$$

$\psi \neq 0.$

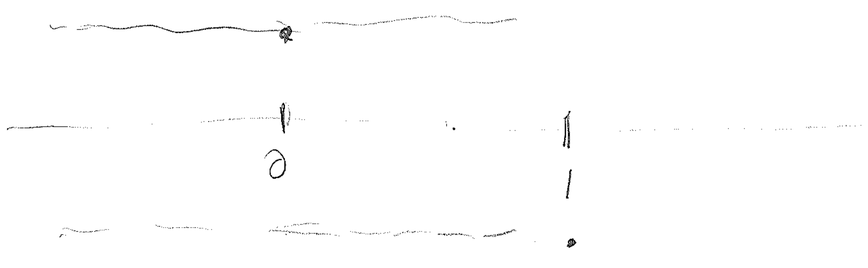


$$(\psi^{-1} d\psi) = (\psi^{-1}\partial_x \psi)\,dx + (\psi^{-1}\partial_y \psi)\,dy$$

$$\psi(x+1,y) = \psi(x,y)\, e^{-2\pi i m y}$$

$$\frac{d\psi(x+1,y)}{\psi(x+1,y)} - \frac{d\psi(x,y)}{\psi(x,y)} = 2\pi i m\,dy$$

Mobius: $\quad \Gamma(M) = \left\{ \begin{array}{c} \psi \in C^\infty(\mathbb{R}) \\ \psi = \overline{\psi} \end{array} \middle| \ \psi(x+1) = -\psi(x) \right\}.$



C Go over the action of H on the principal

$\mathbb{T}$-bundles. What is H? A central extension with cross section whose 2 cocycle is bilinear.

$$\mathbb{T} \longrightarrow H \underset{\curvearrowleft s}{\longrightarrow} \mathbb{R}^2 \qquad s(x_1,y_1)\, s(x_2,y_2) = e^{2\pi i y_1 x_2} s(x_1+x_2, y_1+y_2)$$

$\mathrm{Lie}(H) =$ basis $(\partial_\theta, \partial_x, \partial_y)$. Start from D_x, D_y

$$\partial_x \tilde{f}(x,y) = \sum_m e^{2\pi i m y} \partial_x f(x+m)$$

$$e^{2\pi i x y}\tilde{f}(x,y) = \sum_m e^{2\pi i (x+m) y} f(x+m)$$

$$\boxed{\begin{aligned} D_x &= \partial_x \\ D_y &= \partial_y + 2\pi i x \end{aligned}}$$

$$e^{2\pi i x y}\left(\partial_y + 2\pi i x\right)\tilde{f} = \sum_m e^{2\pi i (x+m) y}\, 2\pi i (x+m) f(x+m)$$

$$\left(\partial_y + 2\pi i x\right)\tilde{f} = \left(2\pi i\, x f(x)\right)^{\sim}$$

Operators $D_x = \partial_x$, $D_y = \partial_y + 2\pi i x$, $[D_x, D_y] = 2\pi i$

on

$$e^{a\partial_x} e^{b(\partial_y + 2\pi i x)} = e^{a\partial_x} e^{2\pi i b x} e^{b\partial_y}$$

$$= e^{2\pi i a b} e^{a\partial_x} e^{b\partial_y}$$

Look at autom. condition

$$\psi(x+m, y+n) = e^{-2\pi i m y}\psi(x,y)$$

or

$$\psi(x,y) = e^{2\pi i m y} e^{m\partial_x} e^{n\partial_y}\psi(x,y)$$

$$= e^{m(\partial_x + 2\pi i y)} e^{n\partial_y}$$

$$\left[\begin{aligned} D_x &= \partial_x \\ D_y &= \partial_y + 2\pi i x \end{aligned} \;,\; \begin{aligned} \nabla_x &= \partial_x + 2\pi i y \\ \nabla_y &= \partial_y \end{aligned}\right] = 0$$

$$[D_x, D_y] = 2\pi i \qquad\qquad [\nabla_x, \nabla_y] = -2\pi i$$

d At some point you should work on simple cases for the $SL(2,\mathbb{R})$ (projective) action.

First point is that quadratic expressions in D_x, D_y extend the projective translation action of $\mathbb{R}^2$ to $SL(2,\mathbb{R}) \ltimes \mathbb{R}^2$.

$$\left[\frac{1}{4\pi i} D_x^2, \begin{pmatrix} D_x \\ D_y \end{pmatrix}\right] = \begin{pmatrix} 0 \\ D_x \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} D_x \\ D_y \end{pmatrix}$$

$$\left[\frac{1}{2\pi i}(D_x D_y + D_y D_x), \begin{pmatrix} D_x \\ D_y \end{pmatrix}\right] = \begin{pmatrix} -2 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} D_x \\ D_y \end{pmatrix}$$

$$\left[\frac{1}{4\pi i} D_y^2, \begin{pmatrix} D_x \\ D_y \end{pmatrix}\right] = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} D_x \\ D_y \end{pmatrix}$$

$$[D_x^2, D_y^2] = 4\pi i (D_x D_y + D_y D_x)$$

$$[D_x^2, D_y] = D_x 2\pi i + 2\pi i D_x = 4\pi i D_x$$

$$\left[\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}\right] = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\left[\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right] \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}$$

$$\left[\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}\right] = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

On $\mathcal{S}(\mathbb{R})$ $D_x = \frac{d}{dx}$ $D_y = 2\pi i x$ so it should be easy to exponentiate $\frac{1}{4\pi i} D_y^2 = \frac{1}{\cancel{4\pi i}} \cancel{2\pi i}\, 2\pi i\, x^2 = \pi i x^2$ to $e^{\frac{t}{4\pi i} D_y^2} = e^{i\pi t x^2}$. Check this $e^{-i\pi t x^2} \frac{d}{dx} e^{i\pi t x^2} = \frac{d}{dx} + 2\pi i t x$ $= D_x + t D_y$ and $e^{-\pi i t x^2}\, 2\pi i x\, e^{\pi i t x^2} = 2\pi i x = D_y$. So you have

$$e^{-i\pi t x^2}\begin{pmatrix} D_x \\ D_y \end{pmatrix} e^{i\pi t x^2} = \begin{pmatrix} D_x + t D_y \\ D_y \end{pmatrix} = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}\begin{pmatrix} D_x \\ D_y \end{pmatrix}$$

So what can you say to add understanding? Let's go over $SL(2,\mathbb{Z})$ "action". Begin with $\psi_0(x+m, y+n) = e^{-2\pi i m y}\psi_0(x,y)$ set $\psi_1(x,y) = e^{2\pi i x y}\psi_0(x,y)$. Then $\psi_1(x+m, y+n) = e^{2\pi i (x+m)(y+n)}\psi_0(x+m, y+n)$
$= e^{2\pi i (x+m)(y+n)} e^{-2\pi i m y}\psi_0(x,y) = e^{2\pi i x y + 2\pi i n x}\psi_0(x,y) = e^{2\pi i n x}\psi_1(x,y)$

e Let's work on realizations of the basic representation of the CCR (dim 1).

position $L^2(\mathbb{R}, dx)$ = with $p =$

momentum $L^2(\mathbb{R}, dy)$.

holomorphic representation $\|f\|^2 = \iint e^{-|z|^2} |f(z)|^2 \frac{dx\,dy}{\pi}$

annih. $a = \partial_z$ $[a, a^*] = 1$

creation $a^* = z$

Let's go over the modular property

$$\mathcal{L} = \{ \psi \in C^\infty(\mathbb{R}^2) \mid \psi(x+m, y+n) = e^{-2\pi i m y} \psi(x,y) \}$$

You want to apply elts of $SL(2,\mathbb{Z})$ to ψ and then adjust the multiplier so as to get back in $\mathcal{L}$.

$$g = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} : \mathbb{R}^2 \longrightarrow \mathbb{R}^2, \quad g^* x = x + y, \quad g^* y = y$$

$$\boxed{(g^*\psi_0)(x,y) = \psi_0(x+y, y).}$$

Let $\psi_1(x,y) = \psi_0(x+y, y)$

$$\begin{aligned}
\psi_1(x+m, y+n) &= \psi_0(x+y+m+n, y+n) \\
&= e^{-2\pi i (m+n) y} \psi_0(x+y, y) \\
&= e^{-2\pi i (m+n) y} \psi_1(x,y)
\end{aligned}$$

$$\psi_2(x,y) = e^{h(y)} \psi_1(x,y)$$

$$\psi_2(x+m, y+n) = e^{h(y+n) - 2\pi i (m+n) y - h(y)} \psi_2(x,y)$$

$\therefore$ you want $h(y+n) - h(y) - 2\pi i (m+n) y = -2\pi i m y$

$$h(y+n) - h(y) = 2\pi i n y$$

$$\frac{(y+n)(y+n-1)}{2} - \frac{y(y-1)}{2} = \frac{1}{2}\left\{ \cancel{y^2} + 2yn + n^2 - \cancel{y} - n - \cancel{y^2} + \cancel{y} \right\} = yn + \frac{n^2 - n}{2}$$

f) $\boxed{\psi_0 \in \mathcal{L} \Rightarrow \psi_2(x,y) = e^{2\pi i\left(\frac{y(y-1)}{2}\right)} \psi_0(x+y,y) \quad \in \mathcal{L}}$

Check $\psi_2(x+m,y+n) = e^{2\pi i \frac{(y+n)(y+n-1)}{2}} \psi_0(x+y+m+n, y+n)$

$$= e^{2\pi i \frac{(y+n)(y+n-1)}{2} - 2\pi i (m+n) y} \psi_0(x+y,y)$$

$$= e^{2\pi i \left[\frac{(y+n)(y+n-1)}{2} - \frac{y(y-1)}{2} - ny\right]} e^{2\pi i \frac{y(y-1)}{2}} e^{-2\pi i m y} \psi_0(x+y,y)$$

$\frac{n^2-n}{2}$

$$= e^{-2\pi i m y} \psi_2(x,y)$$

Next $g = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ $\quad g^*x = x$, $\quad g^*y = x+y$ $\quad \boxed{\psi_1(x,y) = \psi_0(x,x+y)}$

$\psi_1(x+m,y+n) = \psi_0(x+m, x+y+m+n) = e^{-2\pi i m(x+y)} \psi_1(x,y)$

$\psi_2(x,y) = e^{+2\pi i xy} \psi_1(x,y)$

$$\psi_2(x+m,y+n) = e^{+2\pi i (x+m)(y+n)} \psi_1(x+m,y+n)$$

$$= e^{+2\pi i [xy + xn + my] - 2\pi i m(x+y)} \psi_1(x,y)$$

$$= e^{-2\pi i [xn + my + mx + my]} e^{-2\pi i xy} \psi_1(x,y) \quad ?$$

$$\frac{\psi_1(x+m,y+n)}{\psi_1(x,y)} = \frac{\psi_0(x+m, x+m+y+n)}{\psi_0(x,x+y)} = \frac{\psi_0(x+m,x+y)}{\psi_0(x,x+y)}$$

$$= e^{-2\pi i m(x+y)}$$

Note this is independent of n. What do you want? A multiplier that will convert from $e^{-2\pi i m(x+y)}$ to $e^{-2\pi i m y}$, yet keep the y periodicity $e^{-2\pi i \frac{x(x-1)}{2}}$

g) $\psi_2(x,y) = e^{h(x)}\psi_1(x,y)$

$$\psi_2(x+m,y+n) = e^{h(x+m)}\psi_1(x+m,y+n)$$

$$= e^{h(x+m)} e^{-2\pi i m(x+y)}\psi_1(x,y) = e^{h(x+m)-2\pi i m(x+y)-h(x)}\psi_2(x,y)$$

You want $h(x+m)-h(x) = 2\pi i m x$ $\therefore$ $h(x) = 2\pi i \frac{x(x-1)}{2}$

Claim:

$$\psi_2(x,y) = e^{2\pi i \frac{x(x-1)}{2}}\psi_0(x,x+y) \in \mathcal{L}$$

$$\frac{\psi_2(x+m,y+n)}{\psi_2(x,y)} = e^{2\pi i\left[\frac{(x+m)(x+m-1)}{2} - \frac{x(x-1)}{2}\right]} \overbrace{\frac{\psi_0(x+m, x+y+\cancel{m+n})}{\psi_0(x,x+y)}}^{e^{-2\pi i m(x+y)}}$$

$$e^{2\pi i\left[\frac{\cancel{x^2}+2\cancel{x}m+m^2-\cancel{x}-m}{2}\right]} e^{-2\pi i m(\cancel{x}+y)} = e^{-2\pi i m y}$$

So what you do now is to take a general $\begin{pmatrix} a & b \\ c & d\end{pmatrix} \in SL(2,\mathbb{Z})$ and transform $\psi_1(x,y) = \psi_0(ax+by, cx+dy)$

$$\psi_1(x+m,y+n) = \psi_0\left(\begin{pmatrix} a & b \\ c & d\end{pmatrix}\begin{pmatrix} x \\ y\end{pmatrix} + \begin{pmatrix} a & b \\ c & d\end{pmatrix}\begin{pmatrix} m \\ n\end{pmatrix}\right)$$

$$= \psi_0(ax+by+am+bn, cx+dy+\cancel{cm+dn})$$

$$= e^{-2\pi i\boxed{(cx+dy)}\boxed{(am+bn)}}\psi_0(ax+by, cx+dy)$$

$$= e^{-2\pi i\boxed{(cx+dy)}\boxed{(am+bn)}}\psi_1(x,y)$$

$$\psi_2(x,y) = e^{h(x,y)}\psi_1(x,y)$$

$$\psi_2(x+m,y+n) = e^{h(x+m,y+n)}\psi_1(x+m,y+n)$$

$$= e^{h(x+m,y+n)} e^{-2\pi i\boxed{(cx+dy)}\boxed{(am+bn)}} e^{-h(x,y)}\psi_2(x,y)$$

$\therefore$ want $h(x+m,y+n) - h(x,y) = 2\pi i\boxed{(cx+dy)}\boxed{(am+bn)} - 2\pi i m y$

$$\begin{pmatrix} m & n\end{pmatrix}\begin{pmatrix} a \\ b\end{pmatrix}\begin{pmatrix} c & d\end{pmatrix}\begin{pmatrix} x \\ y\end{pmatrix} = \begin{pmatrix} m & n\end{pmatrix}\begin{pmatrix} ac & ad \\ bc & bd\end{pmatrix}\begin{pmatrix} x \\ y\end{pmatrix}$$

$$\begin{pmatrix} ac & ad \\ bc & bd\end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 0 & 0\end{pmatrix} = \begin{pmatrix} ac & bc \\ bc & bd\end{pmatrix} \text{ symmetric!}$$

h)

$$\psi_1(x,y) = \psi_0(-x,-y)$$

$$\psi_1(x+m,y+n) = \psi_0(-x-m,-y-n) = e^{-2\pi i(-m)(-y)}\psi_0(-x,-y)$$

$$= e^{-2\pi i m y}\psi_1(-x,-y)$$

---

you need to understand solutions of

$$h(x+m,y+n) - h(x,y) = (x\ \ y)\begin{pmatrix}\alpha & \beta \\ \beta & \gamma\end{pmatrix}\begin{pmatrix}m \\ n\end{pmatrix}$$

where $\alpha, \beta, \gamma \in \mathbb{Z}$, the point is to generalize the case $h(x,y) = \frac{x(x-1)}{2}$. Constraints on $\begin{pmatrix}\alpha & \beta \\ \beta & \gamma\end{pmatrix} = \begin{pmatrix}ac & bc \\ bc & bd\end{pmatrix}$

$$\underbrace{\begin{pmatrix}a \\ b\end{pmatrix}(c\ \ d)}_{\text{rank 1 op.}} = \begin{pmatrix}0 & 1 \\ 0 & 0\end{pmatrix} + \begin{pmatrix}\alpha & \beta \\ \beta & \gamma\end{pmatrix}$$

$$\frac{1}{2}\begin{pmatrix}x \\ y\end{pmatrix}^t\begin{pmatrix}\alpha & \beta \\ \beta & \gamma\end{pmatrix}\begin{pmatrix}x \\ y\end{pmatrix} - \frac{1}{2}\begin{pmatrix}\alpha \\ \gamma\end{pmatrix}^t\begin{pmatrix}x \\ y\end{pmatrix}$$

separate into $\begin{pmatrix}\alpha & 0 \\ 0 & 0\end{pmatrix} + \begin{pmatrix}0 & \beta \\ \beta & 0\end{pmatrix} + \begin{pmatrix}0 & 0 \\ 0 & \gamma\end{pmatrix}$

$$\begin{pmatrix}a & b \\ c & d\end{pmatrix} = \begin{pmatrix}0 & 1 \\ -1 & 0\end{pmatrix} \qquad \begin{pmatrix}\alpha & \beta \\ \beta & \gamma\end{pmatrix} = \begin{pmatrix}0 \\ -1\end{pmatrix}(-1\ \ 0) - \begin{pmatrix}0 & 1 \\ 0 & 0\end{pmatrix}$$

$$= \begin{pmatrix}0 & -1 \\ -1 & 0\end{pmatrix}$$

$$\frac{1}{2\pi i}h(x,y) = \frac{1}{2}(x\ \ y)\begin{pmatrix}0 & -1 \\ -1 & 0\end{pmatrix}\begin{pmatrix}x \\ y\end{pmatrix} = -xy$$

$$\psi_1(x,y) = \psi_0(ax+by,\ cx+dy) = \psi_0(y,-x)$$

$$\boxed{\psi_2(x,y) = e^{-2\pi i x y}\psi_0(y,-x)}$$

ψ

i) Given $\psi_0 \in \mathcal{L}$ let

$$\psi_1(x,y) = e^{-2\pi i xy}\,\psi_0(y,-x)$$

let
$$\psi_2(x,y) = e^{-2\pi i xy}\,\psi_1(y,-x)$$
$$= e^{-2\pi i xy}\, e^{-2\pi i y(-x)}\,\psi_0(-x,-y) = \psi_0(-x,-y).$$
$$\psi_3(x,y) = e^{-2\pi i xy}\,\psi_2(y,-x)$$
$$= e^{-2\pi i xy}\,\psi_0(-y,x)$$
$$\psi_4(x,y) = e^{-2\pi i xy}\,\psi_3(-y,-x)$$
$$= e^{-2\pi i xy}\, e^{-2\pi i(-y)(-x)}\,\psi_0(x,y)$$

---

discuss abstractly the situation

$\psi \in C^\infty(\mathbb{R}^2)$ $\quad g \in SL(2,\mathbb{R})$ $\quad$ define $T_g\psi$

$T_g(\psi) = e^{Q_g}(\psi \circ g)$ or $= e^{Q_g}\cdot g^*(\psi)$, here Q_g is a quadratic function on $\mathbb{R}^2$.

Do first for the connection form $A = 2\pi i\, x\,dy$.

$$(d+A)(e^Q g^*\psi) \overset{?}{=} e^Q g^*(d\psi + A\psi)$$

$$\| \qquad\qquad\qquad\qquad \|$$

$$e^Q(d + dQ + A)\,g^*\psi \qquad e^Q(d\,g^*\psi + g^*A\cdot g^*\psi)$$

$$\|$$

$$e^Q(d + g^*A)\,g^*\psi$$

$$dQ + A = g^*A \qquad dQ = g^*A - A.$$

$$T_g = e^{Q_g}g^* \qquad T_{g_1g_2} = e^{Q_{g_1g_2}}\,g_2^*g_1^*.$$

$$T_{g_1}T_{g_2} = e^{Q_{g_1}}g_1^*\,e^{Q_{g_2}}g_2^* = e^{Q_{g_1} + g_1^*Q_{g_2}}\,g_1^*g_2^*$$

j) $$g^* x = ax + by$$
$$g^* y = cx + dy$$

$$g^* \psi(x,y) = \psi(ax+by,\ cx+dy)$$

$$e^{Q_g} = \exp 2\pi i \frac{1}{2}\left\{ \begin{pmatrix} x \\ y \end{pmatrix}^t \begin{pmatrix} ac & bc \\ bc & bd \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} - (acx + bdy) \right\}$$

$$\begin{pmatrix} x \\ y \end{pmatrix}^t \begin{pmatrix} ac & bc \\ bc & bd \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = acx^2 + 2bcxy + bdy^2$$

$$\frac{1}{2\pi i} Q_{g_1} = a_1c_1 \frac{x^2 - x}{2} + b_1c_1 xy + b_1d_1 \frac{y^2 - y}{2}$$

$$\frac{1}{2\pi i} Q_{g_2} = a_2c_2 \frac{x^2 - x}{2} + b_2c_2 xy + b_2d_2 \frac{y^2 - y}{2}$$

$$\frac{1}{2\pi i} g_1^* Q_{g_2} = a_2 c_2$$

$$g_1^* \left(\frac{1}{2} \begin{pmatrix} x \\ y \end{pmatrix}^{t} \begin{pmatrix} a_2c_2 & b_2c_2 \\ b_2c_2 & b_2d_2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \right)$$

$$= \frac{1}{2} \begin{pmatrix} x \\ y \end{pmatrix}^{t} \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}^t \begin{pmatrix} a_2c_2 & b_2c_2 \\ b_2c_2 & b_2d_2 \end{pmatrix} \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$ very messy

---

$$A = 2\pi i\, x\,dy \qquad dQ_g = g^*A - A$$

$$dQ_1 = g_1^* A - A \qquad dQ_2 = g_2^* A - A$$

$$g_1^* dQ_2 = g_1^* g_2^* A - g_1^* A$$

$$g_1^* \psi(x,y) = \psi(a_1 x + b_1 y,\ c_1 x + d_1 y)$$

$$g_2^* g_1^* \psi(x,y) = \psi(a_1 \overbrace{g_2^* x}^{a_2 x + b_2 y} + b_1 \overbrace{g_2^* y}^{a_2 x + b_2 y},\ c_1 g^* x + d_1 g^* y)$$

$$= \psi((a_1 a_2 + b_1 d_2)x + (a_1 b_2 + b_1 b_2) y,$$

$$g_2^* g_1^* = (g_1 g_2)^*$$

k) *Check carefully*

$$g_1^* \psi(x,y) = \psi_1(a_1 x + b_1 y,\ c_1 x + d_1 y)$$

$$g_2^* g_1^* \psi(x,y) = \psi_2\big(a_1(a_2 x + b_2 y) + b_1(c_2 x + d_2 y),\ c_1(a_2 x + b_2 y) + d_1(c_2 x + d_2 y)\big)$$

$$= \psi_2\big((a_1 a_2 + b_1 c_2)x + (a_1 b_2 + b_1 d_2)y,\ (c_1 a_2 + d_1 c_2)x + (c_1 b_2 + d_1 d_2)y\big)$$

$$g_1 g_2 = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}\begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} = \begin{pmatrix} a_1 a_2 + b_1 c_2 & a_1 b_2 + b_1 d_2 \\ c_1 a_2 + d_1 c_2 & c_1 b_2 + d_1 d_2 \end{pmatrix}$$

$$dQ_1 = g_1^* A - A \qquad dQ_2 = g_2^* A - A$$

$$g_2^* dQ_1 = g_2^* g_1^* A - g_2^* A \implies g_2^* dQ_1 + dQ_2 = \underbrace{g_2^* g_1^*}_{(g_1 g_2)^*} A - A$$

$$g_2^* Q_1 + Q_2$$

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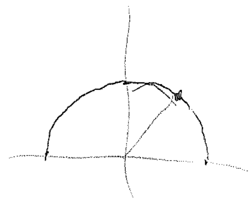
generators. $\omega = e^{2\pi i/3}$



$$\omega^2 + \omega + 1 = 0$$

$$\frac{a\omega + b}{c\omega + d} = \omega$$

$$a\omega + b = c\omega^2 + d\omega$$

$$(a-d)\omega + b = c(-\omega - 1)$$

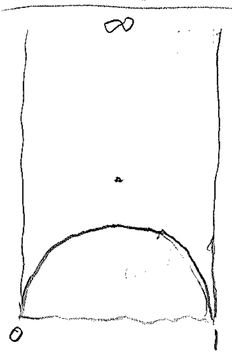
$$(a-d+c)\omega + b + c = 0$$

$$d = a + c \qquad 0 = b + c \qquad \begin{pmatrix} 1 & +1 \\ -1 & 0 \end{pmatrix}$$

$$\frac{\omega+1}{-\omega} = \omega \ ? \qquad \text{Yes.}$$

$$\begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \therefore \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix} \text{ order } 6$$



$$\begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix} \text{ order } 6$$

$$\begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

mult by $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ to get

$$\begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix} \qquad \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix} \text{ order } 3$$

$\ell)$ $\begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$ order 3

$$\begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix} \times \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}, \quad \begin{pmatrix} ac & bc \\ bc & bd \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ -1 & 1 \end{pmatrix}$$

$$\frac{1}{2\pi i} Q = -xy + \frac{y(y-1)}{2}.$$

$g^* x = -y$

$g^* y = x - y$

You have $T_g \psi = e^{Q} g^* \psi$

$T_g^2 \psi = e^{Q + g^* Q} g^* g^* \psi$ etc.

$$Q' = -xy + \frac{1}{2}y^2 - \frac{1}{2}y$$

$$g^* Q' = +(+y)(x-y) + \frac{1}{2}(x-y)^2 - \frac{1}{2}(x-y)$$

$$= \cancel{xy} - y^2 + \frac{1}{2}x^2 - \cancel{xy} + \frac{1}{2}y^2 - \frac{1}{2}x + \frac{1}{2}y$$

$$= \frac{1}{2}x^2 - \frac{1}{2}y^2 - \frac{1}{2}x + \frac{1}{2}y$$

$$g^* g^* Q' = \frac{1}{2}(-y)^2 - \frac{1}{2}(x-y)^2 - \frac{1}{2}(-y) + \frac{1}{2}(x-y)$$

$$= \cancel{\frac{1}{2}y^2} - \frac{1}{2}x^2 + xy - \cancel{\frac{1}{2}y^2} + \cancel{\frac{1}{2}y} + \frac{1}{2}x - \cancel{\frac{1}{2}y}$$

$$= xy - \frac{1}{2}x^2 + \frac{1}{2}x$$

$$g_1 = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}, \quad g_2 = \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} \qquad g_1 g_2 = \begin{pmatrix} a_1a_2 + b_1c_2 & a_1b_2 + b_1d_2 \\ c_1a_2 + d_1c_2 & c_1b_2 + d_1d_2 \end{pmatrix}$$

$$g_1^* \psi(x,y) = \psi(a_1x + b_1y, c_1x + d_1y)$$

$$g_2^* g_1^* \psi(x,y) = g_2^* \psi(a_1x + b_1y, c_1x + d_1y)$$

$$= \psi(a_1(a_2x + b_2y) + b_1(c_2x + d_2y), c_1(a_2x + b_2y) + d_1(c_2x + d_2y))$$

$$= \psi((a_1a_2 + b_1c_2)x + (a_1b_2 + b_1d_2)y, (c_1a_2 + d_1c_2)x + (c_1b_2 + d_1d_2)y)$$

m) $T_{g_1} = e^{Q_1} g_1^*$, $T_{g_2} T_{g_1} = e^{Q_2} g_2^* e^{Q_1} g_1^*$

$Q_i' = \frac{1}{2\pi i} Q_i$

$$T_{g_2} T_{g_1} = e^{Q_2 + g_2^* Q_1} g_2^* g_1^* = e^{Q_2 + g_2^* Q_1} (g_1 g_2)^*$$

$$Q_1' = \frac{1}{2}\begin{pmatrix} x \\ y \end{pmatrix}^t \begin{pmatrix} a_1 c_1 & b_1 c_1 \\ b_1 c_1 & b_1 d_1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} - \frac{1}{2}(a_1 c_1 x + b_1 d_1 y)$$

$$Q_1' = \frac{1}{2} a_1 c_1 x^2 + b_1 c_1 xy + \frac{1}{2} b_1 d_1 y^2 - \frac{1}{2} a_1 c_1 x - \frac{1}{2} b_1 d_1 y$$

$$g_2^* Q_1' = \frac{1}{2} a_1 c_1 (a_2 x + b_2 y)^2 = \frac{1}{2} a_1 c_1 (a_2^2 x^2 + 2 a_2 b_2 xy + b_2^2 y^2)$$

$$+ b_1 c_1 (a_2 x + b_2 y)(c_2 x + d_2 y) = b_1 c_1 (a_2 c_2 x^2 + a_2 d_2 xy + b_2 c_2 xy + b_2 d_2 y^2)$$

$$+ \frac{1}{2} b_1 d_1 (c_2 x + d_2 y)^2 = \frac{1}{2} b_1 d_1 (c_2^2 x^2 + 2 c_2 d_2 xy + d_2^2 y^2)$$

$$- \frac{1}{2} a_1 c_1 (a_2 x + b_2 y) = -\frac{1}{2} a_1 c_1 a_2 x - \frac{1}{2} a_1 c_1 b_2 y$$

$$- \frac{1}{2} b_1 d_1 (c_2 x + d_2 y) = -\frac{1}{2} b_1 d_1 c_2 x - \frac{1}{2} b_1 d_1 d_2 y$$

$$Q_2' = \frac{1}{2} a_2 c_2 x^2 + b_2 c_2 xy + \frac{1}{2} b_2 d_2 y^2 - \frac{1}{2} a_2 c_2 x - \frac{1}{2} b_2 d_2 y$$

<u>coeff of x^2</u> $= \overset{1}{\frac{1}{2} a_1 c_1 a_2^2} + \overset{3}{b_1 c_1 a_2 c_2} + \overset{2}{\frac{1}{2} b_1 d_1 c_2^2} + \overset{3'}{\frac{1}{2} a_2 c_2}$

Compare with $\frac{1}{2}(a_1 a_2 + b_1 c_2)(c_1 a_2 + d_1 c_2) = \frac{1}{2}\left[\overset{1}{a_1 a_2 c_1 a_2} + \overset{4'}{a_1 a_2 d_1 c_2} \right.$
$\left. \underset{4}{b_1 c_2 c_1 a_2} + \underset{2}{b_1 c_2 d_1 c_2}\right]$

$(3) + (3') = a_2 c_2 \left(b_1 c_1 + \frac{1}{2}\right)$

$(4) + (4') = a_2 c_2 (b_1 c_1 + a_1 d_1)\frac{1}{2}$

$\frac{b_1 c_1 + a_1 d_1}{2} = \left(2 b_1 c_1 + \underbrace{a_1 d_1 - b_1 c_1}_{1}\right)\frac{1}{2}$ YES

<u>coeff of y^2</u> $= \overset{1}{\frac{1}{2} a_1 c_1 b_2^2} + b_1 c_1 b_2 d_2 + \overset{2}{\frac{1}{2} b_1 d_1 d_2^2} + \frac{1}{2} b_2 d_2$

compare to $\frac{1}{2}(a_1 b_2 + b_1 d_2)(c_1 b_2 + d_1 d_2) = \overset{1}{\frac{1}{2} a_1 c_1 b_2^2} + \frac{1}{2} a_1 b_2 d_1 d_2 + \frac{1}{2} b_1 c_1 b_2 d_2$ $\overset{2}{+ \frac{1}{2} b_1 d_1 d_2^2}$

$$b_2 d_2 \left(b_1 c_1 + \frac{1}{2}\right) \overset{?}{=} b_2 d_2 \left(\frac{1}{2} a_1 d_1 + \frac{1}{2} b_1 c_1\right) \quad \text{Yes.}$$

<u>coeff of xy</u> $= \overset{1}{a_1 c_1 a_2 b_2} + \overset{2}{b_1 c_1 a_2 d_2} + b_1 c_1 b_2 c_2 + \overset{3}{b_1 d_1 c_2 d_2} + b_2 c_2$

compare to $(a_1 b_2 + b_1 d_2)(c_1 a_2 + d_1 c_2) = \overset{1}{a_1 c_1 a_2 b_2} + \underbrace{a_1 d_1}_{b_1 c_1 + 1} b_2 c_2 + \overset{2}{b_1 c_1 a_2 d_2} + \overset{3}{b_1 d_1 c_2 d_2}$

coeff of x $= -\frac{1}{2} a_2 c_2 - \frac{1}{2} a_1 c_1 a_2 - \frac{1}{2} b_1 d_1 c_2$

compare to $-\frac{1}{2}(a_1 a_2 + b_1 c_2)$

doesn't work.

n) Return to order 3 elt.

$$\begin{pmatrix} 0 & -1 \\ +1 & -1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \end{pmatrix} \qquad g = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix} \qquad \begin{matrix} g^* x = -y \\ g^* y = x - y \end{matrix}$$

$$\begin{pmatrix} 0 & -1 \\ +1 & -1 \end{pmatrix}\begin{pmatrix} -1 \\ -1 \end{pmatrix} = \begin{pmatrix} +1 \\ 0 \end{pmatrix} \qquad T_g = e^Q g^*$$

$$\begin{pmatrix} 0 & -1 \\ +1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ +1 \end{pmatrix} \qquad \begin{pmatrix} ac & bc \\ bc & bd \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ -1 & 1 \end{pmatrix}$$

$$A' = x\,dy \qquad g^*A' = (-y)\,d(x-y) = -y\,dx + y\,dy$$

$$g^*A' - A' = -x\,dy - y\,dx + y\,dy$$

$$= d\left(-xy + \frac{y^2}{2}\right)$$

$$\frac{1}{2}\begin{pmatrix} x \\ y \end{pmatrix}^t \begin{pmatrix} 0 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = -xy + \frac{y^2}{2}$$, you actually choose $-xy + \frac{y(y-1)}{2}$ to respect "$\mathbb{Z}$"

Claim that $\psi(x,y) \longmapsto e^Q g^* \psi(x,y)$

$$= e^{2\pi i\left(-xy + \frac{y(y-1)}{2}\right)} \psi(-y, x-y)$$ carries $\mathcal{L}$ into $\mathcal{L}$

You want to check carefully that this transformation has order 3, and that it commutes with $\psi(x,y) \longrightarrow \psi(-x,-y)$.

This seems unlikely because of the $\frac{y(y-1)}{2}$ term.

Let's try the order 6 elt.

$$\begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix},$$

$$g = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix} \qquad \begin{matrix} g^* x = y \\ g^* y = y - x \end{matrix} \qquad \begin{pmatrix} ac & bc \\ bc & bd \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ -1 & 1 \end{pmatrix}$$

so $\frac{1}{2\pi i} Q = Q' = -xy + \frac{y(y-1)}{2}$ and it should be true that

$$\psi(x,y) \xmapsto{T_g} e^{2\pi i\left(-xy + \frac{y(y-1)}{2}\right)} \psi(y, y-x)$$ maps $\mathcal{L}$ to $\mathcal{L}$.

c) To compute $T_g^3 = e^{Q + g^*Q + (g^*)^2 Q}(g^*)^3$

$$Q' = -xy + \frac{y^2}{2} - \frac{y}{2}$$

$$g^*Q' = -y(y-x) + \frac{(y-x)^2}{2} - \frac{y-x}{2}$$

$$= \cancel{xy} - y^2 + \frac{y^2}{2} - \cancel{xy} + \frac{x^2}{2} + \frac{x}{2} - \frac{y}{2}$$

$$= \frac{x^2}{2} - \frac{y^2}{2} + \frac{x}{2} - \frac{y}{2}$$

$$g^*g^*Q' = \frac{y^2}{2} - \frac{(y-x)^2}{2} + \frac{y}{2} - \frac{y-x}{2}$$

$$= \cancel{\frac{y^2}{2}} - \cancel{\frac{y^2}{2}} + xy - \frac{x^2}{2} + \frac{x}{2}$$

$$= xy - \frac{x^2}{2} + \frac{x}{2}$$

$$Q' + g^*Q' + (g^*)^2 Q' = \left.\begin{array}{l} -\cancel{xy} + \cancel{\frac{y^2}{2}} - \frac{y}{2} \\ \cancel{\frac{x^2}{2}} - \cancel{\frac{y^2}{2}} + \frac{x}{2} - \frac{y}{2} \\ \cancel{xy} - \cancel{\frac{x^2}{2}} + \frac{x}{2} \end{array}\right\} = x - y$$

Check again for $g = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$ $\quad g^*x = -y$, $g^*y = x - y$ $\quad \begin{pmatrix} ac & bc \\ bc & bd \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ -1 & 1 \end{pmatrix}$

$$Q' = -xy + \frac{y^2}{2} - \frac{y}{2}$$

$$g^*Q' = -(-y)(x-y) + \frac{(x-y)^2}{2} - \frac{x-y}{2}$$

$$= \cancel{xy} - y^2 + \frac{x^2}{2} - \cancel{xy} + \frac{y^2}{2} - \frac{x}{2} + \frac{y}{2}$$

$$= \frac{x^2}{2} - \frac{y^2}{2} - \frac{x}{2} + \frac{y}{2}$$

$$g^*(g^*Q') = \frac{(-y)^2}{2} - \frac{(x-y)^2}{2} - \frac{(-y)}{2} + \frac{x-y}{2}$$

$$= \cancel{\frac{y^2}{2}} - \frac{x^2}{2} + xy - \cancel{\frac{y^2}{2}} + \cancel{\frac{y}{2}} + \frac{x}{2} - \cancel{\frac{y}{2}}$$

$$= xy - \frac{x^2}{2} + \frac{x}{2}$$

$-\cancel{xy} + \cancel{\frac{y^2}{2}}$	$-\frac{y}{2}$
$\cancel{\frac{x^2}{2}} - \cancel{\frac{y^2}{2}}$	$-\frac{x}{2} + \frac{y}{2}$
$\cancel{xy} - \cancel{\frac{x^2}{2}}$	$\frac{x}{2}$

p) $g = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ $\quad g^*x = y$, $g^*y = -x$ $\quad \begin{pmatrix} ac & bc \\ bc & bd \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$

$$\psi(x,y) \longmapsto e^{-2\pi i xy}\psi(y,-x), \qquad g^*\psi(y,-x) = \psi(-x,-y)$$

$$Q' = -xy$$

$$g^*Q' = -(y)(-x) = xy$$

$$Q' + g^*Q' = 0$$

$\therefore T_g^2\psi(x,y) = \psi(-x,-y)$

$\therefore$ this T_g has order 4.

What structure do you have? You have two generators for $SL(2,\mathbb{Z})$ namely $s = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ of order 4 and

$t = \begin{pmatrix} 0 & +1 \\ -1 & +1 \end{pmatrix}$ of order 6 $\qquad s^2 = t^3 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$

$f \in \mathcal{S}(\mathbb{R}) \qquad \tilde{f}(x,y) = \sum_m e^{2\pi i my} f(x+m) \in C^\infty(\mathbb{R}^2)$

$$\boxed{\begin{aligned} \tilde{f}(x, y+n) &= \tilde{f}(x,y) \\ \tilde{f}(x+m, y) &= e^{-2\pi i my}\tilde{f}(x,y) \end{aligned}}$$

$\tilde{f}(x+m, y+n) = e^{-2\pi i my}\tilde{f}(x,y)$ automorphic type condition

(because $\quad e^{2\pi i xy}\tilde{f}(x,y) = \sum_m e^{2\pi i(x+m)y} f(x+m)$

$$e^{2\pi i(x+m)y}\tilde{f}(x+m,y) = e^{2\pi i xy}\tilde{f}(x,y)$$

Claim. $f \longmapsto \tilde{f}$, $\quad \mathcal{S}(\mathbb{R}) \xrightarrow{\sim} \mathcal{L} = \{\psi(x,y) \in C^\infty(\mathbb{R}^2) \mid \text{autom. condition}\}$

$$f(x) = \int_{\mathbb{R}/\mathbb{Z}} \tilde{f}(x,y)\,dy \qquad \psi(x+m,y) = e^{-2\pi i my}\psi(x,y)$$

$\psi \in \mathcal{L}$

$$f_m(x) = \int_{\mathbb{R}/\mathbb{Z}} e^{-2\pi i my}\psi(x,y)\,dy$$

$$f_m(x-m) = \int_{\mathbb{R}/\mathbb{Z}} e^{-2\pi i my}\underbrace{\psi(x-m,y)}_{e^{2\pi i my}\psi(x,y)}\,dy = f(x)$$

$$f_m(x) = f(x+m).$$

8

Next explain $\mathcal{L} = \Gamma(\mathbb{T}^2, L)$

$\mathcal{L}$ is a module over $C^\infty(\mathbb{T}^2) = \{\varphi \in C^\infty(\mathbb{R}^2) \mid \varphi(x+m, y+n) = \varphi(x,y)\}$

Locally over $\mathbb{T}^2$, $\mathcal{L}$ is a free module of rank 1 over $C^\infty(\mathbb{T}^2)$. Maybe you're saying that

$$\mathcal{L}(U) = \{\psi(x,y) \in C^\infty(\pi^{-1}U) \mid \text{autom. condition}\}$$

if U small enough to lift

$$\begin{tikzcd} \tilde{U} \arrow[d, "\wr"'] & \subset & \mathbb{R}^2 \arrow[d] \\ U & \subset & \mathbb{T}^2 \end{tikzcd}$$

$$\pi^{-1}U = \tilde{U} \times \mathbb{Z}^2$$

$$\mathcal{L}(U) = \{\psi(x,y) \in C^\infty(\tilde{U} \times \mathbb{Z}^2) \mid \text{autom.}\} \simeq C^\infty(\tilde{U})$$

$$L(\pi(x_0,y_0)) = \{\overset{\psi(x_0+m,\, y_0+n)}{\text{functions}} \text{ on } \pi^{-1}\pi(x_0,y_0) = (x_0,y_0) + \mathbb{Z}^2 \text{ sat.}\}$$

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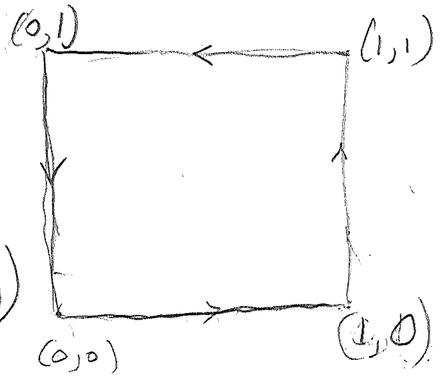
nontrivial means $\exists\ \psi \in \mathcal{L}$ nowhere vanishing



$$\psi(x+1,y) = e^{-2\pi i y}\psi(x,y)$$

$$\frac{d\psi}{\psi}(x+1,y) = -2\pi i\, dy + \frac{d\psi}{\psi}(x,y)$$

$$\int_{y=0}^{y=1}$$

$$\tau = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}(\tau) = \frac{-1}{\tau - 1} \qquad \tau^2 - \tau + 1 = 0$$

$$\tau = \frac{-1 \pm \sqrt{1-4}}{2} = \frac{-1 \pm \sqrt{3}\, i}{2}$$

$$\tau = \frac{-1 + \sqrt{3}\, i}{2} \quad \text{if } \tau \in UHP.$$

(1)

(A) Schwartz class $\mathcal{S}(\mathbb{R}) = \left\{ f(x) \in C^\infty(\mathbb{R}) \;\middle|\; \begin{matrix} \forall m,n \in \mathbb{N} \\ \left| x^m \frac{d^n}{dx^n} f(x) \right| \leq C_{m,n,f} \end{matrix} \right\}$

$$f(x) \in \mathcal{S}(\mathbb{R}) \quad \xmapsto{F.T} \quad \hat{f}(y) = \int_{-\infty}^{\infty} e^{-2\pi i xy} f(x)\, dx$$

Poisson summation $\downarrow$ $\qquad\qquad$ $\downarrow$ restrict from $\mathbb{R}$ to $\mathbb{Z}$

$$\boxed{\sum_{m \in \mathbb{Z}} f(x+m) = \sum_{n \in \mathbb{Z}} e^{2\pi i n x} \hat{f}(n)} \qquad \hat{f}(n)$$

(B) Ex. $\quad f(x) = e^{-\frac{\pi x^2}{t}} \qquad \hat{f}(y) = \sqrt{t}\, e^{-\pi y^2 t} \qquad \mathrm{Re}(t) > 0$

$$\sum_m e^{-\frac{\pi (x+m)^2}{t}} = \sqrt{t} \sum_n e^{2\pi i n x} e^{-\pi n^2 t}$$

$x = 0$ $\qquad \sum_m e^{-\frac{\pi m^2}{t}} = \sqrt{t} \sum_n e^{-\pi n^2 t}$ $\quad \rightsquigarrow$ Func. Eqn ζ, Quad. Reciprocity

(C) Proof: $\quad \sum_{m \in \mathbb{Z}} f(x+m) = \sum_n e^{2\pi i n x} a_n$

$$a_n = \int_0^1 e^{-2\pi i n x} \sum_m f(x+m)\, dx = \sum_m \int_m^{m+1} e^{-2\pi i n (x \cancel{-m})} f(x \cancel{+m})\, dx$$

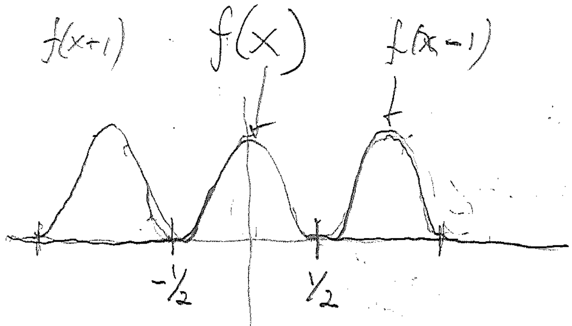
$$= \int_{-\infty}^{\infty} e^{-2\pi i n x} f(x)\, dx = \hat{f}(n).$$

② Electrical Engineering

x = Frequency

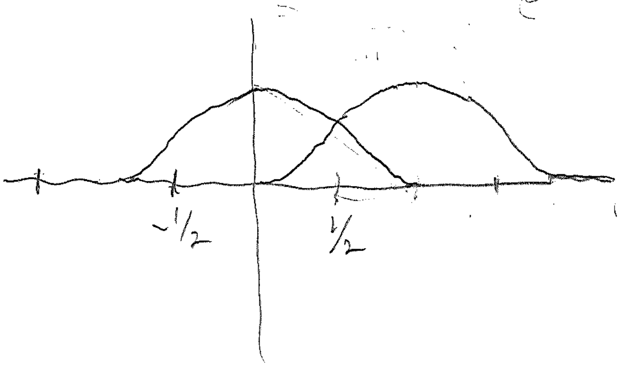
y = time

$\hat{f}(y)$



Support of F.T.

principles: if bandwidth of signal is $\subset (-\frac{1}{2}, \frac{1}{2})$ then there is no loss of information on applying | P.S. (not | restricting from $\mathbb{R}$ to $\mathbb{Z}$



$\hat{f}(y) \longmapsto \hat{f}(n)$ sampling (Analog $\rightarrow$ Digital)

opposite process: interpolation.

$$\frac{\sin \pi y}{\pi y} = \begin{cases} 0 & 0 \neq y \in \mathbb{Z} \\ 1 & 0 = y. \end{cases}$$

$$\hat{f}(y) \overset{?}{=} \sum_n \frac{\sin \pi(y-n)}{\pi(y-n)} \hat{f}(n)$$

true if Supp $f(x) \subset (-\frac{1}{2}, \frac{1}{2})$

3

$f(x) \in \mathcal{S}(\mathbb{R})$, <u>twist</u> P.S. by a character $m \mapsto e^{2\pi i m y}$

$$\tilde{f}(x,y) = \sum_m e^{2\pi i m y} f(x+m) \in C^\infty(\mathbb{R}^2); \quad \tilde{f}(x,y+n) = \tilde{f}(x,y)$$

$$e^{2\pi i x y}\tilde{f}(x,y) = \sum_m e^{2\pi i (x+m) y} f(x+m), \quad \tilde{f}(x+m,y) = e^{-2\pi i m y}\tilde{f}(x,y)$$

<u>Def.</u> $$\mathcal{L} = \left\{ \psi(x,y) \in C^\infty(\mathbb{R}^2) \;\middle|\; \underbrace{\psi(x+m,y+n) = e^{-2\pi i m y}\psi(x,y)}_{\text{automorphic condition}} \right\}$$

transform: $\mathcal{S}(\mathbb{R}) \longrightarrow \mathcal{L}$, $f(x) \longmapsto \tilde{f}(x,y)$

Claim this $\cong$. The inverse map is $\psi(x,y) \longmapsto \int_{\mathbb{R}/\mathbb{Z}} \psi(x,y)\,dy$

$$\psi(x,y) = \sum_m e^{2\pi i m y} a_m(x)$$

$$a_m(x) = \int_{\mathbb{R}/\mathbb{Z}} e^{-2\pi i m y}\psi(x,y)\,dy = \int_{\mathbb{R}/\mathbb{Z}} \psi(x+m,y)\,dy = a_0(x+m).$$

$$\psi(x,y) = \sum_m e^{2\pi i m y} a_0(x+m). = \tilde{a}_0(x,y)$$

$$\int_{\mathbb{R}/\mathbb{Z}} e^{2\pi i x y}\tilde{f}(x,y)\,dx = \int_{\mathbb{R}/\mathbb{Z}} e^{2\pi i x y}\sum_m e^{2\pi i m y} f(x+m)\,dx$$

$$= \sum_m \int_m^{m+1} e^{2\pi i (x+m) y} f(x+m)\,dx = \int_{-\infty}^{\infty} e^{2\pi i x y} f(x)\,dx = \hat{f}(-y)$$

(4) Next Explain why $\mathcal{L}$ is the space of C^∞ sections of a complex C^∞ line bundle L over $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$.

$$\pi : \mathbb{R}^2 \longrightarrow \mathbb{R}^2/\mathbb{Z}^2 \qquad \pi(x,y) = (x,y) + \mathbb{Z}^2$$

For any open set $\mathcal{U} \subset \mathbb{T}^2$, let

$$\mathcal{L}(\mathcal{U}) = \{ \psi(x,y) \in C^\infty(\pi^{-1}\mathcal{U}) \mid \text{autom. cond.} \}$$

$\mathcal{L}(\mathcal{U})$ is naturally a module over

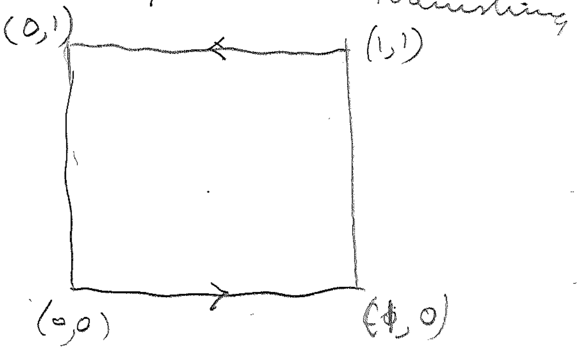
$$C^\infty(\mathcal{U}) = \{ \varphi(x,y) \in C^\infty(\pi^{-1}\mathcal{U}) \mid \varphi(x+m, y+n) = \varphi(x,y) \}$$

If $\mathcal{U}$ admits a lifting
$$\begin{array}{ccc} \tilde{\mathcal{U}} & \subset & \mathbb{R}^2 \\ s\downarrow & & \downarrow\pi \\ \mathcal{U} & \subset & \mathbb{T}^2 \end{array}$$

$$\pi^{-1}\mathcal{U} = \tilde{\mathcal{U}} \times \mathbb{Z}^2$$

$$\mathcal{L}(\mathcal{U}) \simeq C^\infty(\tilde{\mathcal{U}})$$

L non-trivial: trivial means $\exists\, \psi \in \mathcal{L}$ nowhere vanishing



$$\psi(x+1,y) = e^{-2\pi i y}\psi(x,y)$$

$$\frac{d\psi}{\psi}(x+1,y) = -2\pi i\, dy + \frac{d\psi}{\psi}(x,y)$$

$$\int_{\square} \frac{d\psi}{\psi} = -2\pi i$$

$$\{\text{line bundles on } X\} = H^2(X, \mathbb{Z})$$

(5) Operators

$$\mathcal{S}(\mathbb{R}),\quad f \mapsto \frac{d}{dx} f \qquad \text{and} \qquad f \mapsto 2\pi i x f$$

$$\left[\frac{d}{dx}, 2\pi i x\right] = 2\pi i$$

$$\partial_x \tilde{f}(x,y) = \sum_m e^{2\pi i m y} \frac{df}{dx}(x+m) \qquad \widetilde{\frac{df}{dx}} = \partial_x \tilde{f}$$

$$\partial_y\left(e^{2\pi i x y}\tilde{f}(x,y)\right) = \partial_y \sum_m e^{2\pi i (\cancel{x}+m)y} f(x+m)$$

$$= \sum_m e^{2\pi i (\cancel{x}+m)y}\, 2\pi i (x+m) f(x+m)$$

$$e^{-2\pi i x y}\partial_y e^{2\pi i x y}\tilde{f}(x,y) \doteq (2\pi i x f)^{\sim} \qquad (2\pi i f)^{\sim} = (\partial_y + 2\pi i x)\tilde{f}$$

The ops. $D_x = \partial_x$, $D_y = \partial_y + 2\pi i x$ on $\mathcal{L}$ define a <u>connection</u> on L. Lifting of the vector fields ∂_x, ∂_y to $\mathcal{L}$.

action of Heisenberg group on $\mathcal{L}$

$$e^{aD_x} e^{bD_y}\psi(x,y) = e^{a\partial_x} e^{b 2\pi i x} e^{b\partial_y}\psi(x,y)$$

$$= e^{2\pi i ab} e^{2\pi i b x}\psi(x+a, y+b).$$

$$\left[\begin{array}{lcl} D_x = \partial_x & , & \nabla_x = \partial_x + 2\pi i y \\ D_y = \partial_y + 2\pi i x & & \nabla_y = \partial_y \end{array}\right] = 0$$

$$e^{m\nabla_x} e^{n\nabla_y}\psi(x,y) = e^{m\partial_x + m 2\pi i y} e^{n\partial_y}\psi(x,y)$$

$$= e^{2\pi i m y}\psi(x+m, y+n)$$

(6) familiar representations of the CCR in QM

p momentum $\qquad [p, q] = \frac{1}{i}$
q position

1) $\mathcal{H} = L^2(\mathbb{R}, dx) \qquad q = x \qquad p = \frac{1}{i}\frac{d}{dx}$

2) $a = ip + \frac{q}{2}, \quad a^* = -ip + \frac{q}{2} \qquad [a, a^*] = 1.$

$$\mathcal{H} = \left\{ f(z) \text{ entire} \;\middle|\; \|f\|^2 = \int e^{-|z|^2} |f(z)|^2 \frac{d^2 z}{\pi} < \infty \right\}$$

$$a = \frac{d}{dz} \qquad a^* = z$$

3) $\mathcal{H} = L^2(\mathbb{T}^2, L), \qquad [D_x, D_y] = 2\pi i$

7 modular property of $\mathcal{L}$ over $\mathbb{T}^2$

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})$$

$$g^*\psi(x,y) = \psi(ax+by,\ cx+dy)$$

$$Q = \frac{1}{2}\begin{pmatrix} x \\ y \end{pmatrix}^t \begin{pmatrix} ac & bc \\ bc & bd \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} - \frac{1}{2}(acx + bdy)$$

$$\psi \in \mathcal{L} \Longrightarrow e^{2\pi i Q} g^*\psi \in \mathcal{L}.$$

$$g = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad g^*\psi(x,y) = \psi(y,-x)$$

$$\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \qquad Q = -xy$$

$$\boxed{\psi \in \mathcal{L} \Rightarrow e^{-2\pi i xy}\psi(y,-x) \in \mathcal{L}}$$

$$g = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \qquad g^*\psi(x,y) = \psi(x, x+y)$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \qquad Q = \frac{x^2 - x}{2}$$

$$\boxed{\psi \in \mathcal{L} \Longrightarrow e^{2\pi i \frac{x^2-x}{2}}\psi(x,x+y) \in \mathcal{L}}$$