Serre thm. X compact (Hausdorff), there is an equivalence between f.g. proj modules over $\mathbb{C}(X)$ and v.b. over X. given by $E/X \longmapsto \Gamma(X, E)$

step 1: Any E is a retract (direct summand) of a trivial v.b. $E \xleftarrow{j} X \times V \xleftarrow{i} E$ $\exists$ v.b. maps i, j sat $ji = 1_E$. (Alt form: $\exists$ a v.b. E' together with an isom. $E \oplus E' = X \times V$.)

Proof. Locally E is trivial $\Rightarrow$ E is a retract of a triv. bdl $\exists\, U_\mu$ finite open and maps $E_{U_\mu} \xleftarrow{j_\mu} U_\mu \times V_\mu \xleftarrow{i_\mu} E_{U_\mu}$.

Let $1 = \sum_\mu \chi_\mu^2$ be a part. of 1 $\operatorname{Supp} \chi_\mu \subset U_\mu$.

$$\begin{tikzcd} E & X \times \begin{pmatrix} \vdots \\ \oplus \\ V_\mu \\ \oplus \\ \vdots \end{pmatrix} \arrow[l, "(\cdots \chi_\mu j_\mu \cdots)"'] & E \arrow[l, "\begin{pmatrix} \vdots \\ \chi_\mu i_\mu \\ \vdots \end{pmatrix}"'] \end{tikzcd}$$

$$\sum \chi_\mu j_\mu i_\mu \chi_\mu = \sum \chi_\mu^2 = 1$$

Step 2: Given a cont. family of projections $\overset{= e_x^2}{e_x} \in \operatorname{End}(V)$ locally $\exists$ a cont family of invertible op's u_x such that $u_x e_x u_x^{-1}$ is constant.

Replace $e^2 = e$ by $F^2 = 1$ $F = 2e - 1$

cont family of involutions fix basept. $\varepsilon = F$ at basept. Want to const. $u \varepsilon u^{-1} = F$.

ε, F gen. dihedral gp.

$g = F\varepsilon$ $\varepsilon g \varepsilon^{-1} = \varepsilon F = g^{-1}$

Assume $g^{1/2} \ni$ $(g^{1/2})^2 = g$, $\varepsilon g^{1/2} \varepsilon^{-1} = g^{-1/2}$ $\Big\}$ $g^{1/2} \varepsilon g^{-1/2} = g^{1/2} g^{1/2} \varepsilon = g\varepsilon = F$

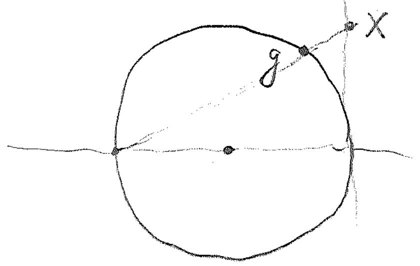
$g = 1+x$ $\qquad g^{s} = \sum_{n\geq 0} \frac{s(s-1)\cdots(s-n+1)}{n!} x^n \qquad \|x\|<1.$

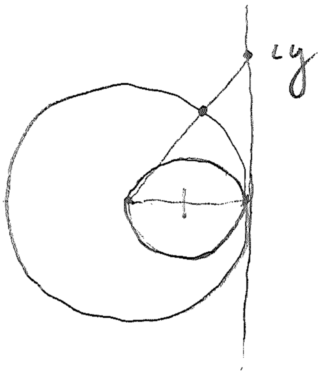
Cayley transform $\quad g = \frac{1+X}{1-X} = \begin{pmatrix}1 & 1\\ -1 & 1\end{pmatrix} X$ ~~[illegible]~~ $=$

$$X = \begin{pmatrix}1 & -1\\ 1 & 1\end{pmatrix} g = \frac{g-1}{g+1}$$

Claim $\|g-1\|$ small $\Longrightarrow$ X defined and $g = \frac{1+X}{1-X}$

$g^{1/2} = \frac{1+X}{\sqrt{1-X^2}} \qquad \varepsilon g^{1/2} \varepsilon^{-1} = \frac{1-X}{\sqrt{1-X^2}} = g^{-1/2}$





$$\frac{1+iy}{1-iy} = \frac{1-y^2+2iy}{1+y^2}$$

$g = F\varepsilon \qquad \|g-1\|$ small. $\qquad g$

$X =$ ~~[illegible]~~ $\frac{g-1}{g+1} = 1 - \frac{2}{g+1} \qquad 1-X = \frac{2}{g+1} \qquad \frac{2}{1-X} = \frac{g+1}{\cancel{2}}$

$g = \frac{2}{1-X} - 1 = \frac{1+X}{1-X}$

problem – if you want to use $g^{1/2} = \frac{1+X}{\sqrt{1-X^2}}$, you must handle the denominator.

The best method is to use the exponential map

$$\exp : \mathrm{End}(V) \longrightarrow \mathrm{Aut}(V)$$

which on a small enough star like nbd of 0 respects the group laws in the radial direction & is a diffeo near 0. This should yield $g^{1/2}$ with $\varepsilon g^{1/2} \varepsilon^{-1} = g^{-1/2}$, since square roots will be unique.

Heisenberg Lie alg. basis X, Y, H

relations $[X,Y] = H \quad [X,H] = [Y,H] = 0$

$$\exp(aX + bY + cH) = e^{aX} e^{bY} e^{\left(-\frac{ab}{2} + c\right)H}$$

$$e^{X+Y} \qquad \frac{X^2 + XY + YX + Y^2}{2}$$

$$\qquad\qquad \parallel$$

$$e^{X} e^{Y} e^{-\frac{1}{2}H} \qquad \frac{X^2}{2} + XY + \frac{Y^2}{2} - \frac{XY - YX}{2}$$

$$\boxed{e^{aX} e^{bY} = e^{aX + bY + \frac{ab}{2}H}}$$

$$\boxed{e^{bY} e^{aX} = e^{aX + bY - \frac{ab}{2}H}}$$

$$e^{aX} e^{bY} e^{-aX} e^{-bY} = e^{ab[X,Y]}$$

3 dim'l Lie alg. Except you are missing the $2\pi i$. What you forgot is the fact that bilinear forms give 2-cocycles. Review group cohomology.

$$\begin{tikzcd} H \arrow[r, tail, "i"] & E \arrow[r, two heads] & G \arrow[l, bend right, "s"'] \end{tikzcd} \qquad H \text{ abelian}$$

$$s(g_1) s(g_2) \; s(g_1 g_2)^{-1} = i f(g_1, g_2)$$

$$s(g_1) s(g_2) = f(g_1, g_2) \, s(g_1 g_2)$$

$$\begin{aligned}
\big(s(g_1) s(g_2)\big) s(g_3) &= f(g_1, g_2) \, s(g_1 g_2) \, s(g_3) \\
&= \boxed{f(g_1, g_2) \, f(g_1 g_2, g_3)} \, s(g_1 g_2 g_3) \\
&= s(g_1) \, f(g_2, g_3) \, s(g_2 g_3) \\
&= \boxed{{}^{g_1} f(g_2, g_3) \, f(g_1, g_2 g_3)} \, s(g_1 g_2 g_3)
\end{aligned}$$

$$\boxed{g_1 f(g_2, g_3) - f(g_1 g_2, g_3) + f(g_1, g_2 g_3) - f(g_1, g_2) = 0}$$

G abelian , H central

$$f(y,z) - f(x+y,z) + f(x,y+z) - f(x,y) = 0$$

$$\check{y}z - (x+\check{y})z + x(\check{y}+z) - x\check{y} = 0$$

This is true for any bilinear $f(x,y)$.

$$f(y,z) - f(x+y,z) = -f(\overset{x}{\cancel{y}},z)$$

$$f(x,y+z) - f(x,y) = f(x,z)$$

Your are dealing with $\mathbb{R}^2$ the group, so you need to understand bilinear form on $\mathbb{R}^2$ 4 dim'l space

$$\begin{tikzcd}
\mathbb{R}H \arrow[r, tail] & \mathrm{Heis} \arrow[r, two heads] & \mathbb{R}X+\mathbb{R}Y \\
\mathfrak{h} & \ell & \mathfrak{g} \arrow[l, bend right, "s"']
\end{tikzcd}$$

$$[s(x), s(y)] = f(x,y) + s([x,y])$$

~~$[s(x),[s(y),s(z)]] = [s(x), f(y,z) + s([y,z])]$~~

~~$= \cancel{x} f(y,z) + [s(x), s([y,z])]$~~

$$[s(x),[s(y),s(z)]] = [s(x), f(y,z) + s([y,z])]$$

$$= x f(y,z) + f(x,[y,z]) + s([x,[y,z]])$$

$$[[s(x),s(y)],s(z)] = [f(x,y) + s([x,y]), s(z)]$$

$$= -z f(x,y) + f([x,y],z) + s([[x,y],z])$$

$$[s(y),[s(x),s(z)]] = [s(y), f(x,z) + s([x,z])]$$

$$= y f(x,z) + f(y,[x,z]) + s([y,[x,z]])$$

If $\mathfrak{g}$ abelian

$$x f(y,z) = y f(x,z) - z f(x,y)$$

If $\mathfrak{h}$ central only condition is $f(x,y)$ skew-symm. bilinear.

You reviewed ~~the~~ 2 cocycles for group + Lie alg extensions, especially central extensions of an abelian group ~~+~~ Lie alg.

$$\begin{tikzcd} M \arrow[r, tail] & E \arrow[r] & G \arrow[l, bend right, "s"'] \end{tikzcd} \qquad s(x)\,s(y) = f_2(x,y)\,s(xy)$$

$$x f_2(y,z) - f_2(xy,z) + f_2(x,yz) - f_2(x,y) = 0$$

$\Rightarrow$ bilinear f_2 is a 2-cocycle.

$$x f_1(y) - f_1(xy) + f_1(x) = 0 \qquad \text{linear } f_1 \text{ is a 1-cocycle.}$$

$$f_2(x,y) = f_1(y) - f_1(xy) + f_1(x)$$ is deviation of f_1 from being a group homom.

Lie ~~g~~ case
$$\begin{tikzcd} h \arrow[r, tail] & l \arrow[r] & g \arrow[l, bend right, "s"'] \end{tikzcd}$$

$$[s(x), s(y)] = f_2(x,y) + s([x,y])$$

$$s'(x) = f_1(x) + s(x)$$

$$[s'(x), s'(y)] = f_2'(x,y) + s'([x,y])$$

$$[f_1(x)+s(x), f_1(y)+s(y)] = f_2'(x,y) + f_1([x,y]) + s([x,y])$$

$$\underbrace{x f_1(y) - y f_1(x) - f_1([x,y])}_{\text{coboundary of } f_1} \doteq f_2'(x,y) - f_2(x,y)$$

If h central + g abelian then all $\overset{\text{linear}}{f_1(x)}$ are cocycles. and all skew-symm $\overset{\text{bilinear}}{f_2(x,y)}$ are cocycles.

You now would like to construct the complex line bundle of degree 1 over $\mathbb{T}^2$ as a homogeneous space of the Heisenberg group K by a $\mathbb{Z}^3$. What's the best way to proceed?

use model

$$f(x) \longmapsto \tilde{f}(x,y) = \sum_m e^{2\pi i m y} f(x+m)$$

$$\tilde{f}(x,y) = \tilde{f}(x,y+1) \qquad \tilde{f}(x,y) = e^{2\pi i y} \tilde{f}(x+1,y)$$

note that $(\partial_x f)(x) \longmapsto (\partial_x \tilde{f})(x,y)$

$$(\partial_y \tilde{f})(x,y) = \sum_m e^{2\pi i m y} \begin{pmatrix} 2\pi i m \\ +2\pi i x \end{pmatrix} f(x+m)$$

$$+ (2\pi i x \tilde{f})(x,y)$$

$${}^{2\pi i}\widetilde{(x f)}(x,y) = (\partial_y + 2\pi i x)\tilde{f}$$

$$\widetilde{\partial_x f} = \partial_x \tilde{f}$$

$$\nabla_x = \partial_x$$

$$\nabla_y = \partial_y + 2\pi i x$$

$$[\nabla_x, \nabla_y] = 2\pi i$$

So on $\mathcal{S}(\mathbb{R})$ define

$$\nabla_x f = \partial_x f \qquad ?$$

$$\nabla_y f = (2\pi i x) f$$

$$\widetilde{2\pi i x f}(x,y) = \sum_m{}' e^{2\pi i m y} 2\pi i (m+x) f(m+x)$$

Let's use the $\mathcal{S}(\mathbb{R})$ model for space of sections

$$\widetilde{2\pi i x f(x)} = \sum_m e^{2\pi i m y} 2\pi i (x+m) f(x+m)$$

$$\partial_y \widetilde{f(x)} = \sum_m e^{2\pi i m y} 2\pi i m\, f(x+m)$$

$$(\partial_y + 2\pi i x)\tilde{f} = (2\pi i x f)^{\sim}$$

so what can you do?

Question: You want the Heisenberg group to act transitively on the ~~plane~~ principal $U(1)$ bundle of degree 1 over $\mathbb{T}^2$.

You want a direct approach to this line bundle by first describing it when pulled back to $\mathbb{R}^2$, then you get the line bundle + connection by taking a suitable quotient by $\mathbb{Z}^2$.

Consider $\nabla = d + 2\pi i x\, dy$ on the trivial line bundle over $\mathbb{R}^2$. curvature $\nabla^2 = 2\pi i\, dx\, dy$, $\frac{i}{2\pi} 2\pi i\, dx\, dy = -dx\, dy$ ~~these~~ represents generator of $H^2(\mathbb{R}^2/\mathbb{Z}^2, \mathbb{Z})$. You are looking at $\mathbb{R}^2$ with connection $\nabla = \partial_x dx + (\partial_y + 2\pi i x) dy$. What's important? holonomy?

There is something called the holonomy group - you fix a basepoint and then do parallel transport along loops, getting a subgroup of autos of the fibre over the basepoint. This is a quotient of π_1 when curvature $= 0$.

It should be true that the holonomy group in the case of $\mathbb{R}^2$ with $\nabla = d + 2\pi i x dy$ is the circle, not interesting. You would like to recover the Heisenberg group.

Look at $\mathcal{H}$ acting on itself by left translation. $\mathcal{H}$ here should be the canonical extension

$$0 \longrightarrow \mathbb{T} \longrightarrow \mathcal{H} \longrightarrow \mathbb{R} \times \hat{\mathbb{R}} \longrightarrow 0$$

Recall that there is an extension when you have a bilinear map [redacted]

$$(\mathbb{R} \times \hat{\mathbb{R}}) \times (\mathbb{R} \times \hat{\mathbb{R}}) \longrightarrow \mathbb{T}$$

$$e^{aX} e^{bY} \quad (e^{cX} e^{dY}) \mapsto e^{(a+c)X} e^{(b+d)Y} e^{-bc[X,Y]}$$

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So it seems that you first need to understand completely the link between $\mathcal{H}$ and the line bundle with connection on $\mathbb{R}^2$. What you need to understand better is the $SL(2,\mathbb{R})$ action. And maybe it is related to the different connection forms (quadratic)

Guess $\mathcal{H} = \mathbb{T} \times \mathbb{R} \times \hat{\mathbb{R}}$ = unit circle bundle

$\mathbb{Z}^2$ free abelian, $\mathcal{H}/\mathbb{Z}^2$ unit circle bundle over $\mathbb{T}^2$.

Start with $\mathbb{R}^2$, trivial line bundle $\mathbb{R}^2 \times \mathbb{C}$ equipped with connection $d + 2\pi i x dy$, curvature $2\pi i dx dy$, vector fields $X = \partial_x$, $Y = \partial_y$ on $\mathbb{R}^2$

$f(x,y)$ section of trivial line bundle is flat along a curve $x(t), y(t)$ when

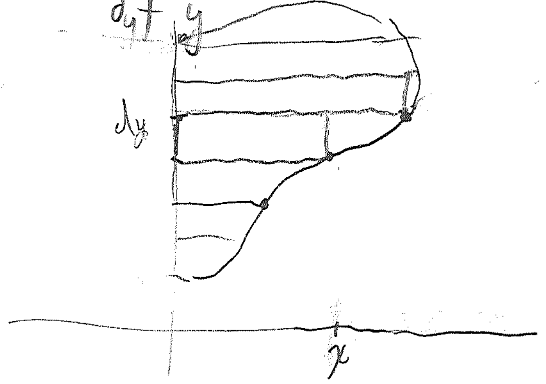
$$\frac{df}{dt} + 2\pi i x \frac{dy}{dt} f = 0$$

linear DE for f over the curve

means $\partial_x f \, \dot{x} + 2\pi i \, x \dot{y} f = 0$

$$f(t) = e^{-\int_0^{(x,y)} 2\pi i x \dot{y} \, dt} f(0)$$

$\partial_y f \, \dot{y}$



$$\oint x \, dy = \iint dx dy$$

Next look at $SL(2,\mathbb{R})$ action on $\mathbb{R}^2$. Better first look at different gauge $\omega = dx dy$

$$x dy - d(xy) = -y dx$$

so conjugating $e^{+2\pi i xy} (d + 2\pi i x dy) e^{-2\pi i xy} = d + 2\pi i(-d(xy) + x dy)$

$$= d - 2\pi i y dx$$

Now comes obvious question: how quadratic functions in x, y yield gauge equiv connections.

You consider $\mathbb{R}^2$, the trivial line bundle ~~over~~ $L = \mathbb{R}^2 \times \mathbb{C}$ over $\mathbb{R}^2$, and YM connections on L, which means ~~this I think that~~ roughly that the curvature is harmonic (translation invariant) and the connection form is $\perp$ to the gauge transf ~~orbit~~.

$$\text{connection} \quad \nabla = d + A = dx\,\partial_x + dy\left(\partial_y + 2\pi i x\right)$$

what is your idea? $dA = 2\pi i \, dx\, dy$, so your connection form can ?

you've forgotten the idea that a connection is given by a connection form on the principal bundle. Your principal bundle should be $\mathbb{R}^2 \times \mathbb{T}$ coords $x, y, ~~\theta~~\,\varphi$

$$d\theta - \check{A} = d\varphi - 2\pi i x\,dy \quad ?$$

It shouldn't be difficult. You have the 3 man. $\mathbb{R}^2 \times \mathbb{T}$, circle bundle over the plane. coordinates $x, y, ~~\theta~~\,\varphi$ where $\varphi \in \mathbb{R}/\mathbb{Z}$, so actually only $d\theta$ is well defined: the conn. form is a 1-form on $\mathbb{R}^2 \times \mathbb{T}$ restricting to fund. class of the fibre at each (x,y).

$$\theta = d\varphi + P\,dx + Q\,dy$$

whose diffl comes from the base

$$d\theta = dP\,dx + dQ\,dy = \left(-\frac{\partial P}{\partial y} + \frac{\partial Q}{\partial x}\right) dx\,dy$$

When you do $\parallel$ transp along a curve you lift the tangent vector to the curve up to a horizontal vector, i.e. in the kernel of θ.

Consider $\mathbb{R}^2$, the trivial line bundle over it, a volume elt. a connection for A whose curvature dA is the volume ~~form~~ ω A is unique up to the differential of a function df. Point is that ω is translation invariant but A is not

You need to get the formalism straight, because you are dealing with a connection - this should be written $d+A$, the arbitrariness in the connection is due to the action of gauge transfs. $g^{-1}(d+A)g$, equal in the trivial line bundle case to $A+g^{-1}dg$ and $g^{-1}dg = d(\log g)$. Any f on $\mathbb{R}^2$ is $\log g$ with $g=e^f$. So you are looking at all 1-form A modulo all df. Poincaré lemma says the curvature dA determines the possible classes of $d+A$ mod gauge transf.

Now look at translation invariance. The invariant diffl forms are spanned by 1, dx, dy, $dxdy$. Look at A such that $dA = dxdy$. Examples: $A = x\,dy$ or $-y\,dx$ not translation invariant. $xdy - (-ydx) = d(xy)$ So there is a unique $A = Pdx + Qdy$ modulo $\{df\}$ with $dA = dxdy$.

How can you organize the choices? Since P, Q are not translation invariant the simplest thing is to require translation invariance modulo constants. Get basis xdx, ydx, xdy, ydy

So it seems that our possible A is an affine hyperplane in a tensor product $V \otimes dV$ where $V = \mathbb{R}x + \mathbb{R}y$. You maybe have $S^2V \longrightarrow V\otimes V \longrightarrow \underset{\omega}{\Lambda^2 V}$

Yes. This looks very reasonable. S^2V is the space of quadratic functions, span of $(ax+by)^2$.

Review. $\mathbb{R}^2$, trivial bundle over $\mathbb{R}^2$, translation inv. volume form $\omega \in \Lambda^2 V$, $V = \mathbb{R}x \oplus \mathbb{R}y$. Possible connection forms are elements of $V\otimes V$, basis $\begin{pmatrix} x \\ y \end{pmatrix} \times \begin{vmatrix} dx \\ dy \end{vmatrix}$ and curvature is the $\overset{\text{canon.}}{\text{map}}$ $V\otimes V \longrightarrow \Lambda^2 V$, whose kernel is S^2V

then you fix ω and look at its inverse image

So what to do? You have this connection form $A_{ij} x^i dx^j$. You can do ∥ transport along curve. Let's try to get the ∥ transport along lines through 0.

$$x = at \qquad y = bt$$

$$\partial_t \, \sigma(tx, ty) + \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$ use vector notation

$$\partial_t \, \sigma(t\underline{x}) + t\, \underline{x}^t A \underline{x} \, \sigma(t\underline{x}) = 0$$

$$\sigma(t\underline{x}) = \sigma(0)\, e^{-\frac{1}{2} \underline{x}^t A \underline{x}\, t^2}$$

so the appropriate "exp" map seems to be $\sigma(\underline{x}) = \sigma(0) e^{-\frac{1}{2}\underline{x}^t A \underline{x}}$

A should be purely imaginary. Actually ~~it looks like~~ you should calculate ∥ transport along the linear path

$$\underline{x}(t) = \underline{x}_0 + t \text{[illegible]}$$

$$\partial_t \sigma(\underline{x}_0 + t\underline{v}) + (\underline{x}_0 + t\underline{v})^* A (\underline{v})\, \sigma(\underline{x}_0 + t\underline{v}) = 0$$

$$\sigma(\underline{x}_0 + t\underline{v}) = e^{-\left\{ t\, \underline{x}_0^* A \underline{v} + \frac{t^2}{2} \underline{v}^* A \underline{v} \right\}} \sigma(\underline{x}_0) \qquad \begin{pmatrix} x \\ y \end{pmatrix}^* \begin{pmatrix} 0 & 1 \\ & 0 \end{pmatrix} \begin{pmatrix} dx \\ dy \end{pmatrix}$$

Go over what you've learned. Consider trivial line bundle over $V = \mathbb{R}^2$ with connection $d + A$, $A \in V^* \otimes dV^*$

$$0 \longrightarrow S^2 V^* \xhookrightarrow{d} V^* \otimes dV^* \xrightarrow{d} \Lambda^2 dV^* \longrightarrow 0.$$ Specify curvature, the connections of interest are a coset mod S^2V^*. parallel transport along a curve γ is just $\exp\left(-\int_\gamma A\right)$, you want this for line segments $v_0 + tv$ $0 \le t \le 1$.

Jan 18, 02

$$\int_0^1 A(v_0+tv, v)\,dt = A(v_0,v) + \tfrac{1}{2}A(v,v)$$

Try to straighten this out. You have some kind of action of elements $v \in V$ on $V \times \mathbb{C}$, which should turn out to be a central extension of the abelian Lie gp $\wedge V$ by $\mathbb{T}$. Given $v \in V$ it acts on V by translation $v_0 \longmapsto v_0 + v$. Use the linear path $v_0 + tv$, and ~~[illegible]~~ $\|$ transport along this path which is $\boxed{\exp\{A(v_0,v) + \tfrac{1}{2}A(v,v)\}}$ The action of v on $V \times \mathbb{C}$ sends (v_0, ξ) to $(v_0 + v, \wedge\, \xi)$

note $\tfrac{1}{2}A(v_0,v_0) + A(v_0,v) + \tfrac{1}{2}A(v,v) = \tfrac{1}{2}A(v_0+v, v_0+v)$

$\bigcirc$if A were symmetric

V 2 dim'l real v.s. equipped with $0 \neq \omega \in \Lambda^2 V^*$

$$\begin{tikzcd}
0 \arrow[r] & S^2V^* \arrow[r] & V^* \otimes V^* \arrow[r] \arrow[d, phantom, "\wr"] & \Lambda^2 V^* \arrow[r] & 0 \\
0 \arrow[r] & S^2V^* \arrow[r, "d"] & V^* \otimes dV^* \arrow[r, "d"] & \Lambda^2 dV^* \arrow[r] & 0 \\
& & A \arrow[u, phantom, sloped, "\in"] & \omega \arrow[u, phantom, sloped, "\in"] &
\end{tikzcd}$$

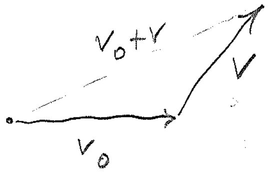
to associate to $v \in V$ a $\|$ transp. diffeo on $V \times \mathbb{T}$

given $v_0 \in V$ take ~~[illegible]~~ $\|$ transp. on linear segment

$v_0 + tv \qquad 0 \leq t \leq 1$. This means ~~[illegible]~~

$$2\pi i \int_0^1 A(v_0+tv, v)\,dt = 2\pi i\left(A(v_0,v) + \tfrac{1}{2}A(v,v)\right)$$

Forget $\mathbb{O}$ which should be irrelevant for $\|$ transport.



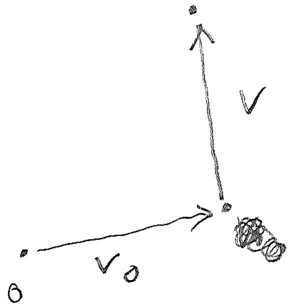
area of Δ.

What seems to happen: Use ∥ transport along lines thru 0 to trivialize the line bundle. This means you join v to 0 via tv. ∥ transport gives the phase $\int_0^1 A(tv, v)\,dt = \frac{1}{2} A(v,v)$ (times $\pm 2\pi i$)

What are you trying to do? You have the line bundle $V \times \mathbb{C}$ with connection $d + A$, which allows ∥ transp along curves. You would like to lift translations on V, the translation action of V on itself to the line bundle, but there's an obstruction.

Consider $v_0 + tv$ $0 \le t \le 1$. The parallel transp, for the line bundle along this curve is $\pm 2\pi i \underbrace{\int_0^1 A(v_0+tv, v)\,dt}_{}$

$$A(v_0,v) + \frac{1}{2} A(v,v)$$



the v_0 segment shifts phase by $\frac{1}{2}A(v_0,v_0)$, then v segment adds

$$A(v_0, v) + \frac{1}{2} A(v,v)$$

If you split A into Symm. + skew symm. parts then

symm part $\frac{1}{2} A(v_0,v_0) + A(v_0,v) + \frac{1}{2} A(v,v)$ yields $\frac{1}{2}A(v_0+v, v_0+v)$

skew part $\frac{1}{2}\left(A(v_0,v) - A(v,v_0)\right)$, should be area of 

What can you hope for? to identify $V \times \mathbb{T}$ with the Heisenberg group.

Let's review V 2-dim real v.s. $V \times \mathbb{C}$ trivial line bundle over V, connection $d \overset{2\pi i}{+} A$ where $A \in V^* \otimes dV^*$

$A_{ij}\, x_i\, dx_j$ Get ∥ transport $2\pi i \int_\gamma A(v(t), v'(t))\,dt$

$v(t) = v_0 + tv$ $\int_0^1 A(v_0 + tv, v)\,dt = A(v_0,v) + \frac{1}{2}A(v,v)$

define an action somewhere. Take $V \oplus \mathbb{R}$

$$T_v \begin{pmatrix} v_0 \\ c \end{pmatrix} = \begin{pmatrix} v + v_0 \\ c + A(v_0, v) + \frac{1}{2} A(v,v) \end{pmatrix}$$

$$T_{v'} T_{v''} \begin{pmatrix} v_0 \\ c \end{pmatrix} = T_{v'} \begin{pmatrix} v_0 + v'' \\ c + A(v_0, v') + \frac{1}{2} A(v', v') \end{pmatrix}$$

$$= \begin{pmatrix} v_0 + v'_3 + v'' \\ c + A(v_0, v') + \frac{1}{2} A(v', v') \\ + A(v_0 + v', v'') + \frac{1}{2} A(v'', v'') \end{pmatrix}$$

$$\overset{\checkmark}{c} + A(\overset{\checkmark}{v_0}, v') + A(\overset{\checkmark}{v_0}, v'') + A(v', v'')$$
$$+ \frac{1}{2} A(\overset{\checkmark}{v'}, v') + \frac{1}{2} A(\overset{\checkmark}{v''}, v'')$$

$$T_{v'+v''} \begin{pmatrix} v_0 \\ c \end{pmatrix} = \begin{pmatrix} v_0 + v' + v'' \\ \overset{\checkmark}{c} + A(\overset{\checkmark}{v_0}, v' \overset{\checkmark}{+} v'') + \frac{1}{2} A(v' + v'', v' + v'') \end{pmatrix}$$

difference is $A(v', v'') - \frac{1}{2} A(v', v'') - \frac{1}{2} A(v'', v')$

$$= \frac{1}{2} \left(A(v', v'') - A(v'', v') \right)$$

You believe that the line bundle is Heisenberg group which you define as $V \times \mathbb{T}$ with a certain product $(v, \zeta) \cdot (v', \zeta') = (v + v', e^{-\pi \omega(v, v')} \zeta \zeta')$. It seems that $SL(V)$ acts on the Heisenberg group. But then you have

Recall 2-cocycle $f : V^2 \longrightarrow$ abel. W triv. act

$$\underbrace{f(v_2, v_3) - f(v_1 + v_2, v_3)}_{- f(v_1, v_3)} + \underbrace{f(v_1, v_2 + v_3) - f(v_1, v_2)}_{f(v_1, v_3)} = 0$$

1- coboundary $(\delta h)(v_1, v_2) = h(v_2) - h(v_1+v_2) + h(v_1)$

~~Thus~~ If h a function on V, ~~[illegible]~~ then h called <u>quadratic</u> when $h(v_1+v_2) - h(v_1) - h(v_2)$ is add bilinear

So apply this to $\mathbb{R} \longrightarrow ? \longrightarrow V$ and you get a Lie group given by $V \times \mathbb{R}$ with mult.

$$(v, c)\cdot(v', c') = (v+v', f(v,v') + c + c')$$

What about h? You get an isom. gp ~~where~~ ifrom $f' = f + \delta h$ where $h : V \to \mathbb{R}$ quadratic.

$$(v,c) \quad (v',c') \longmapsto (v+v', f'(v,v') + c + c')$$

$$(v+v', h(v+v') + f'(v,v') + c + c')$$

$$(v, h(v)+c)\cdot(v', h(v')+c') \longmapsto (v+v', f(v,v') + h(v) + h(v') + c + c')$$

[redacted]

$$f' - f = \delta h$$

The other point is the exact sequence

$$\begin{tikzcd} S^2V^* \arrow[r, tail] & V^*\otimes V^* \arrow[r, two heads] & \Lambda^2 V^* \\ & f & \omega \end{tikzcd}$$

So there is a canonical choice for f, namely its skew symm.

$$f(v,v') - \frac{1}{2}\big(f(v,v') + f(v',v)\big) = \frac{1}{2}\big(f(v,v') - f(v',v)\big)$$

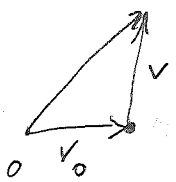
It should be clear that the group of autos of V preserving f acts on the extension groups, so using <u>ω = skew symm of f</u> yields action of $\mathrm{Aut}(V, \omega)$

Now that you have $E = V \times \mathbb{R}$, can you relate it to your line bdle over V with connection? Eventually you want, $\mathbb{R}$ to become $\mathbb{R}/\mathbb{Z} \simeq \mathbb{T}$.

Go back to conn. $D = d + A$ $\quad A \in V^* \otimes dV^* = V^* \otimes V^*$

do || transp along curves. You want to control

$$\int_{\text{path } \gamma} A$$



$v_0 + tv$ $\quad 0 \le t \le 1$
path.

$$\int_0^1 A(v_0+tv, v)\, dt = A(v_0, v) + \tfrac{1}{2} A(v,v)$$

Let's try to make an action of your $(V \times \mathbb{R})_\omega$ Heisenberg group on this line bundle

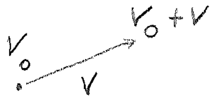
$$T_{(v,c)}(v', c') = (v+v', A(v',v) + \tfrac{1}{2} A(v,v) + c + c')$$

Heisenberg $\quad (v,c)(v',c') = (v+v', \underset{\text{bilinear}}{\underline{\overset{*}{f(v,v')}}} + c + c')$

[redacted]

What is the best you can do with || transport.

$$T_v(v_0, c_0) = (v_0+v, A(v_0,v) + \tfrac{1}{2} A(v,v) + c_0)$$



Your problem seems to be with basepoint. The connection form $A \in V^* \otimes dV^*$ appears to require ~~specify~~ an origin for your 2-plane, i.e. where the elements of V^* vanish

---

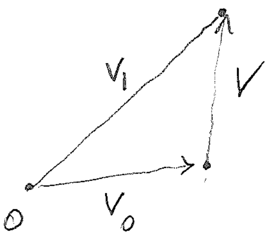
Go back to the original geometry, you have a 2-plane V equipped with a volume 2-form, translation invariant

Origin is not specified when $A = y\,dx$

There's so much to understand.

Go back to the Weil alg., recall this. principal G bundle $G \longrightarrow P \longrightarrow B$, not really relevant since you consider line bundles.

Go back to V 2-plane with volume form $\omega \in \Lambda^2 dV^*$ and a connection $A \in V^* \otimes dV^*$, $dA = \omega$. Consider $\parallel$ transport $\exp 2\pi i \int_\gamma A$.



$$\int_0^1 A(tv_0, v_0)dt = \tfrac{1}{2} A(v_0, v_0) \quad \text{sim for } v_1$$

$$\int_0^1 A(v_0 + tv, v)dt = A(v_0, v) + \tfrac{1}{2} A(v, v)$$

you want to compare $\rightsquigarrow \frac{1}{2} A(v_0, v_0) + A(v_0, v) + \frac{1}{2} A(v, v)$

with $\frac{1}{2} A(v_0 + v, v_0 + v) = \frac{1}{2} A(v_0, v_0) + \frac{1}{2}(A(v_0, v) + A(v, v_0)) + \frac{1}{2} A(v_1, v_1)$

difference is $\frac{1}{2}(A(v_0, v) - A(v, v_0))$. ~~Still ca~~ ~~Still don't~~ get the gen

---

Try something else, namely, operators on sections. V, functions on V, connection $d + A$, look at flow with velocity v, takes $v_0 \longmapsto v_0 + tv$, corresp. v.f. is ∂_v, but now lift this vector field into the principal bundle, which means you get $\partial_v + A(-, v)$

a little review

$f \in \mathcal{S}(\mathbb{R})$ $\qquad \tilde{f}(x,y) = \sum_{m\in\mathbb{Z}} e^{2\pi i m y} f(x+m)$ $\quad$ per in y

$$e^{2\pi i x y}\hat{f}(x,y) = \sum_{m\in\mathbb{Z}} e^{2\pi i (x+m) y} f(x+m) \quad \text{per in } x$$

$$\tilde{f}(x,y+1) = \tilde{f}(x,y) = e^{2\pi i y}\tilde{f}(x+1,y)$$

$\mathcal{A} =$ periodic in x,y fns.

$\mathcal{M} = \{ s(x,y) \text{ as above} \}$

connection $\quad \partial_x \tilde{f}(x,y) = \widetilde{\frac{d}{dx}f}(x,y)$

$$\left(\partial_y \tilde{f}\right)(x,y) = \sum_m e^{2\pi i m y}\, 2\pi i m\, f(x+m)$$

$$\left(2\pi i \hat{x} \tilde{f}\right)(x,y) = \sum_m e^{2\pi i m y}\, 2\pi i x\, f(x+m)$$

$$\left(\partial_y + 2\pi i x\right)\tilde{f} = \left(2\pi i x f\right)^{\sim}$$

suggests $\mathcal{M}$ closed under $\begin{cases} \nabla_x = \partial_x \\ \nabla_y = \partial_y + 2\pi i \hat{x} \end{cases}$

$$\partial_y\left(e^{2\pi i \hat{x}\hat{y}} s(x,y)\right) = \partial_y\left(e^{2\pi i x y}\, e^{2\pi i y}\, s(x+1,y)\right)$$

$$\nabla_y s(x,y) = \nabla_y\left(e^{2\pi i y}\, s(x+1,y)\right)$$

$$= e^{2\pi i y}\left(\underbrace{\left(\overset{+2\pi i}{\partial_y} + 2\pi i x\right) s(x+1,y)}_{\left(\nabla_y s\right)(x+1,y)}\right)$$

$\left(\nabla_y s\right)(x,y)$

$= \left(\partial_y s\right)(x,y) + 2\pi i\, x\, s(x,y)$

$$\nabla_y\left(e^{2\pi i y} s(x+1,y)\right) = e^{2\pi i y}\left(\partial_y \overbrace{+ 2\pi i + 2\pi i x}^{2\pi(x+1)}\right) s(x+1,y)$$

$$= e^{2\pi i y}\left(\nabla_y s\right)(x+1,y)$$

$$e^{v\cdot\nabla} = e^{a\nabla_x + b\nabla_y} = e^{a\nabla_x} e^{b\nabla_y} e^{-\frac{1}{2}ab\,2\pi i}$$

$$\left(e^{(a,b)\cdot\nabla} s\right)(x,y) = e^{a\partial_x}\left(e^{b\partial_y} e^{b2\pi i x - \pi a b i} s\right)(x,y)$$

$$= e^{-i\pi ab}$$

?

$$e^{a\nabla_x + b\nabla_y} = e^{b\nabla_y} e^{a\nabla_x} e^{-\pi i ab}$$

$$\left(e^{a\nabla_x} s\right)(x,y) = s(x+a,y)$$

$$e^{b\nabla_y} s = e^{b(\partial_y + 2\pi i x)} s = e^{b\partial_y} + e^{2\pi i b x} s$$

$$= e^{2\pi i b x} s(x, y+b)$$

answer $e^{-\pi i a b} e^{2\pi i b x} s(x+a, y+b)$

$$\tilde{f}(x,y) = \sum_m e^{2\pi i m y} f(x+m)$$

$$\nabla_x \tilde{f} = \widetilde{\partial_x f} \qquad \nabla_y \tilde{f} = (2\pi i x f)^\sim$$

$$e^{a\partial_x + b2\pi i x} f = e^{b2\pi i x} e^{a\partial_x} e^{\overbrace{+\frac{1}{2}[a\partial_x,\, 2\pi i b x]}^{\pi i a b}} f$$

$$\boxed{e^{a\partial_x + 2\pi i b x} f = e^{\pi i a b} e^{2\pi i b x} f(x+a)}$$

Aside from the messy calculations it seems that you do get an action of the Heisenberg Lie grp on $\Gamma(T_2, L)$

880

$$f \in \mathcal{S}(\mathbb{R}) \qquad f \mapsto \tilde{f}(x,y) = \sum_{m \in \mathbb{Z}} e^{2\pi i m y} f(x+m)$$

$$\tilde{f}(x,y+1) = \tilde{f}(x,y) = e^{2\pi i y}\, \tilde{f}(x+1,y)$$

$$e^{2\pi i x y}\, \tilde{f}(x,y) = e^{2\pi i (x+1) y}\, \tilde{f}(x+1,y)$$

$\mathcal{L}$ = sp of smooth $s(x,y)$ on $\mathbb{R}^2$ having these period. prop.

$\mathcal{L}$ is a module over $C^\infty(\mathbb{T}^2)$ $\qquad \mathbb{T} = \mathbb{R}/\mathbb{Z}$

---

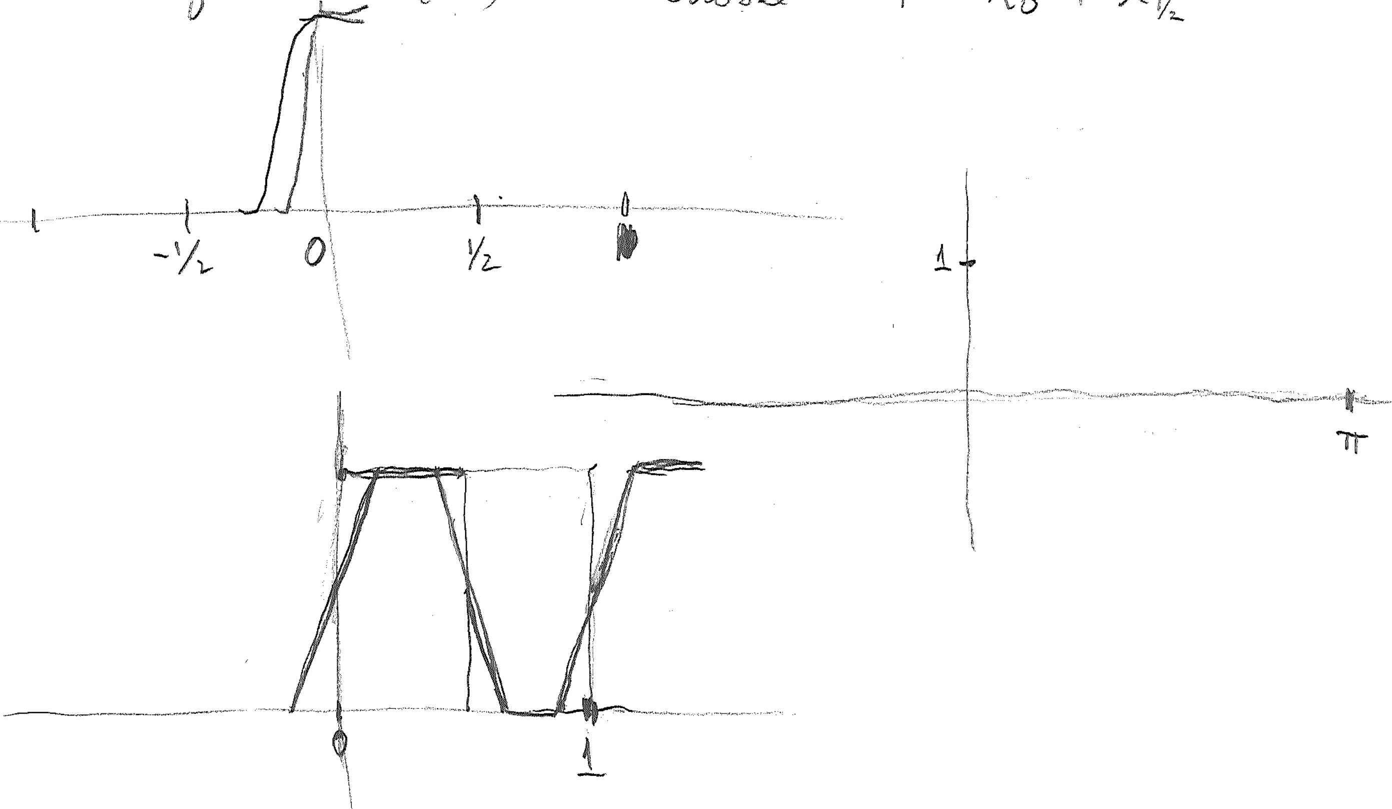
idea: ~~could there exist a direct embedding~~

can you obtain your connection on the line bundle $\mathcal{L}$ over $\mathbb{T}^2$ as a Grassmannian connection for a direct embedding in a trivial rank 2 bdl.

---

what's important now is the action of the Heisenberg group on the line bundle

but first you need for your lecture to embed $\mathcal{L}$ as a summand of a $C^\infty(\mathbb{T}^2)^{\oplus 2}$. Choose $1 = \chi_0 + \chi_{1/2}$



Idea: A partition $\sum \chi_\mu^2 = 1$ is appropriate for Hilbert space ^ situations. Given local isometric embeddings $E_{U_\mu} \xrightarrow{s_\mu} U_\mu \times H_\mu$, then $\sum \chi_\mu s_\mu : E \to X \oplus H_\mu$ should be isometric. inner product. But a partition

$\sum \chi_\mu \chi'_\mu = 1$ might fit better the shrinking idea for coverings. ?

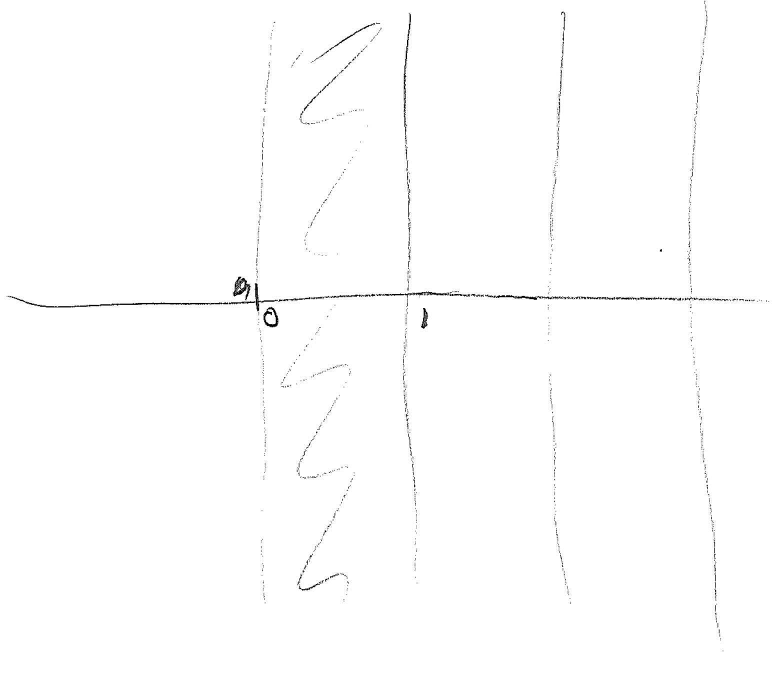
So what you want now is local trivialization

Remove $x = 0 \in \mathbb{R}/\mathbb{Z}$. $T^2 \xrightarrow{x} \mathbb{R}/\mathbb{Z}$

$$\begin{tikzcd}
F(x,y) \text{ smooth on } & (\mathbb{R}-\mathbb{Z}) \times \mathbb{R} \arrow[r, phantom, "\subset"] \arrow[d] & \mathbb{R}^2 \arrow[d] \\
& \mathbb{R}/\mathbb{Z} - \{0\} \times \mathbb{R}/\mathbb{Z} \arrow[r, phantom, "\subset"] & T^2
\end{tikzcd}$$

$F(x,y)$ smooth function on $\{(x,y) \mid x \notin \mathbb{Z}$

$$F(x,y+1) = F(x,y) = e^{2\pi i y} F(x+1,y) \qquad \begin{matrix} x \in \mathbb{R}-\mathbb{Z} \\ y \in \mathbb{R} \end{matrix}$$



F equivalent to smooth fn. on $(0,1) \times \mathbb{R}$

You have $L_U \simeq C(U, C(\mathbb{R}/\mathbb{Z}))$

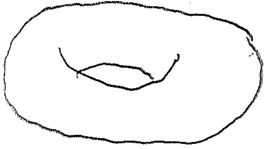
Let's decide whether the Heisenberg group acts on L.

First show $\mathcal{L}$ is locally an $\mathcal{A} = C(\mathbb{T}^2)$ module of rank 1. You need $\overset{\text{module}}{\wedge}$ maps $\mathcal{A} \to \mathcal{L}$ and $\mathcal{L} \to \mathcal{A}$

Can there exist a mod. map $\mathcal{A} \to \mathcal{L}$ nonvanishing

W i.e. an $F(x,y) \in \mathcal{L}$ nonvanishing. If so you can take the logarithm and get $G(x,y)$ smooth on $\mathbb{R}^2$ (cont. at least) satisfying $G(x,y+1) = G(x,y) = 2\pi i y + G(x+1,y)$

Have cy



.

Have $F(x,y+1) = F(x,y) = e^{2\pi i y} F(x+1,y)$. Assume $F(x,y) \neq 0 \quad \forall x,y \in \mathbb{R}^2 \Longrightarrow \exists\, G(x,y) \ni e^G = F$, G continuous smooth etc. Then $e^{G(x,y+1) - G(x,y)} = 1$

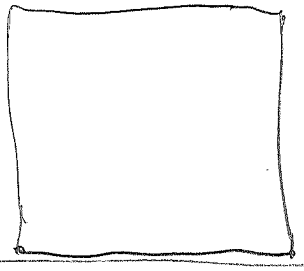
$$\Longrightarrow \left| \begin{array}{l} G(x,y+1) - G(x,y) = 2\pi i k \qquad \text{sin} \\ G(x,y) - G(x+1,y) = 2\pi i y + 2\pi i \ell \end{array} \right.$$

apply ∂_x.

$(\partial_x G)(x,y)$ doubly periodic

$(\partial_y G)(x,y)$ periodic in y

$(\partial_y G)(x,y) - (\partial_y G)(x+1,y) = 2\pi i\, dy$



$\int dG$

$$\int_{(0,1)}^{(1,1)} dG = G(1,1) - G(0,1)$$

F non-vanishing

$$\frac{dF}{F}(x,y+1) = \frac{dF}{F}(x,y) = 2\pi i\, dy + \frac{dF}{F}(x+1,y)$$

Assume L has a non-vanishing section $F(x,y) \neq 0$ on $\mathbb{R}^2$. View $x \mapsto F(x,y)$ as a path in the loop space of $\mathbb{C}^\times$. The degree of this path is independent of x

$$\boxed{\int_0^1 d\log F(x,y) = \int_0^1 F(x,y)^{-1}\partial_y F(x,y)\,dy \quad \text{divided by } 2\pi i}$$

But

$$d\log F(x,y) = d\log\left(e^{2\pi i y}F(x+1,y)\right)$$

$$= 2\pi i\,dy + d\log F(x+1,y)$$

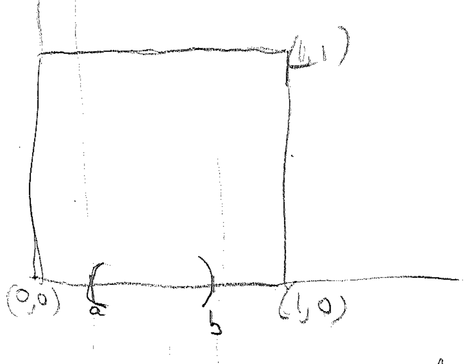
$$\int_0^1 \quad " \quad = 2\pi i + \int_0^1 \text{———}$$

Question: Since L is supposed to have a connection with some sort of translational invariance properties, might there be a way to exploit translation invariance to construct a canonical direct embedding of L into a trivial bundle, possibly infinite diml?

Related ideas. Mumford's theory of Θ functions involving Heisenberg groups?

Earlier question of a direct embedding of L into a trivial bundle such that the YM connection is the Grassmannian conn.

$\mathcal{a} = C^\infty(T^2)$ = smooth periodic on $\mathbb{R}^2$



$a \leq x \leq b$, $y \in \mathbb{R}$

I open interval in $\mathbb{R}$ of line <1

$(I \times \mathbb{Z}) \times \mathbb{R}$

open in

~~Do you need~~ Can you set this up without referring to open sets?

Go back to the connection etc.

$$F(x,y+1) = F(x,y) = e^{2\pi i y} F(x+1,y)$$

$$F(x,y) = \sum_{m\in\mathbb{Z}} e^{2\pi i m y} f(x+m)$$

$$f_m(x) = \oint e^{-2\pi i m y} F(x,y)\,dy = \oint e^{-2\pi i m y + 2\pi i y} F(x+1,y)\,dy$$

$$= f_{m-1}(x+1) = \cdots = f_0(x+m).$$

$F \in \mathcal{L} \implies \partial_x F \in \mathcal{L}$ in fact $F = \tilde{f} \implies \partial_x F = \widetilde{\partial_x f}$

$$F \in \mathcal{L} \implies e^{2\pi i xy} F(x,y) = e^{2\pi i (x+1) y} F(\cancel{x+1}, y)$$

$$\underbrace{e^{-2\pi i xy} \partial_y e^{2\pi i xy}} F(x,y) = e^{-2\pi i xy} \partial_y e^{+2\pi i xy} e^{2\pi i y} F(\overset{+1}{x}, y)$$

$$\left(\partial_y + 2\pi i x\right) F(x,y) = \left(\partial_y + 2\pi i x\right) e^{2\pi i y} F(x+1,y)$$

$$= e^{2\pi i y}\left(\partial_y + 2\pi i(x+1)\right) F(x+1,y)$$

$$\therefore \quad \nabla_y = \partial_y + 2\pi i x \quad \text{preserves} \quad \mathcal{L}$$

$$\nabla_y \tilde{f}(x,y) = \sum_{m\in\mathbb{Z}} \left(\partial_y + 2\pi i x\right)\left(e^{2\pi i m y} f(x+m)\right)$$

$$= \sum_m e^{2\pi i m y} \underbrace{\left(\cancel{\partial_y} + 2\pi i(x+m)\right) f(x+m)}_{(2\pi i x f)(x+m)}.$$

$$\boxed{\nabla_y \tilde{f} = \widetilde{2\pi i x f}}$$

Is it clear that $D_x = \partial_x$, $D_y = \partial_y + 2\pi i x$ define a connection on the module $\mathcal{L}$ over the ring $\mathcal{A} = C^\infty(T^2)$.

$f \in \mathcal{A}$ $\quad s \in L$

$$D(fs) = df\, s + f\, Ds$$

$$D_{\substack{x\\y}}(fs) = (\partial_x f)s + f D_x s$$

$$Ds = dx(D_x s) + dy(D_y s)$$

---

curvature

$$\boxed{D = d + 2\pi i x\, dy}$$

$$D^2 = 2\pi i\, dx dy$$

A connection on the base allows us to lift vector fields on the base to transverse vector fields in the total space.

∂_x , ∂_y on $\mathbb{T}^2$

$$\exp(v.D) \ ?$$

---

digress Mumford Θ fns. I think he considered finite Heisenberg groups, that is, the canonical central extension of $A \times A^\vee$ by appropriate μ (roots of 1), where A is a finite subgroup of an abelian variety.

It seems that Manin has ~~a~~ good noncomm analogue of Mumford's theory.

∂_x lifts to $D_x = \partial_x$ on elts of $\mathcal{L}$

$\partial_y \qquad\qquad D_y = \partial_y + 2\pi i x = e^{-2\pi i xy}\,\partial_y\, e^{2\pi i xy}$

$$e^{-2\pi i xy}\,\partial_y\, \underbrace{e^{2\pi i xy}\tilde{f}(x,y)}_{\sum\limits_m e^{2\pi i (x+m)y} f(x+m)} = e^{-2\pi i xy} \sum_m e^{2\pi i (x+m)y}\, 2\pi i (x+m)\, f(x+m)$$

$$= (2\pi i x f)^\sim$$

$$D_y \tilde{f} = (2\pi i x f)^\sim$$

~~[illegible]~~ Next you want a vector $v = \binom{a}{b} \in \mathbb{R}^2$

translation op. $f(x,y) \longmapsto f(x+a, y+b)$

$$\left(e^{a\partial_x + b\partial_y} f\right)(x,y) \;/\!/$$

$$\phi(t,x,y) = e^{t(a\partial_x + b\partial_y)} f \qquad \partial_t \phi = (a\partial_x + b\partial_y)\phi$$

$$\phi\big|_{t=0} = f$$

So you have a lot of work left.

$$[D_x, D_y] = [\partial_x, \partial_y + 2\pi i x] = 2\pi i$$

$$e^{(aD_x + bD_y)} F(x,y) = e^{bD_y} e^{aD_x} e^{\frac{1}{2}[aD_x, bD_y]} F$$

$$= e^{\pi i ab} e^{b 2\pi i x} F(x+a, y+b)$$

$$\underline{e^{(aD_x + bD_y)}} F(x,y) = e^{\pi i ab} e^{2\pi i x b} F(x+a, y+b)$$?!

Basically you should work with $e^{aD_x} \cdot e^{bD_y}$

~~As enoides~~ A typical element of the H-group is

$e^{aX} e^{bY} e^{cH}$ and the product is

$$\left(e^{aX} e^{bY} e^{cH}\right)\left(e^{a'X} e^{b'Y} e^{c'H}\right) = e^{(a+a')X} e^{(b+b')Y} e^{(-ba'+c+c')H}$$

Consider now $e^{aD_x} e^{bD_y} f(x) = e^{aD_x} e^{b 2\pi i x} f(x)$

$$= e^{2\pi i (x+a) b} f(x+a)$$

$$e^{bD_y} e^{aD_x} f(x) = e^{bD_y} f(x+a) = e^{2\pi i b x} f(x+a)$$

Is $e^{aD_x} e^{bD_y} = e^{\underbrace{[aD_x, bD_y]}_{2\pi i ab}} e^{bD_y} e^{aD_x}$?

Program: Work out the action of the Heisenberg group on $\mathcal{L}$

Question: How can you find the stabilizer of a point, say $(0,0) \in \mathbb{R}^2$.

Review If $f(x) \in \mathcal{S}(\mathbb{R})$, then

$$\tilde{f}(x,y) = \sum_m e^{2\pi i m y} f(x+m) \quad \in \mathcal{L} \quad \text{where}$$

$$\mathcal{L} = \left\{ F(x,y) \in C^\infty(\mathbb{R}^2) \;\middle|\; \begin{array}{l} F(x,y+1) = F(x,y) \\ F(x,y) = e^{2\pi i y} F(x+1,y) \end{array} \right\}$$

conversely if F

$\mathcal{S}(\mathbb{R})$ Schwartz space of rapidly dec. smooth fns on $\mathbb{R}$

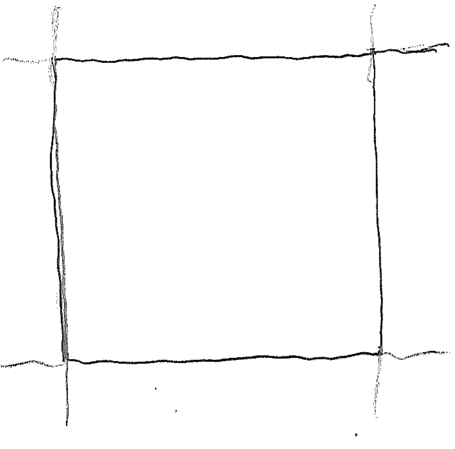
$$\mathcal{L} = \left\{ F(x,y) \in C^\infty(\mathbb{R}^2) \;\middle|\; F(x,y+1) = F(x,y) = e^{2\pi i y} F(x+1,y) \right\}$$

then $f \in \mathcal{S}(\mathbb{R}) \Longrightarrow \tilde{f}(x,y) = \sum_m e^{2\pi i m y} f(x+m) \in \mathcal{L}$

$$F \in \mathcal{L} \Longrightarrow \tilde{F}(x) = \oint F(x,y)\,dy \in \mathcal{S}(\mathbb{R})$$

and these two maps are inverses of each other.

$\mathcal{L}$ is a module over $\mathcal{a} = C^\infty(T^2)$. Claim it's a fg proj. $\mathcal{a}$ module, i.e. a direct summand (retract) of a free f.g. $\mathcal{a}$-module



Consider open $\mathcal{U} \subset T^2$

$$\mathcal{L}(\mathcal{U}) = \left\{ F \in C^\infty(\pi^{-1}\mathcal{U}) \;\middle|\; \begin{array}{l}\text{same}\\ \text{per cond.}\end{array} \right\}.$$

e.g. $\mathcal{U} = \{ (x+\mathbb{Z}, y+\mathbb{Z}) \mid x \notin \mathbb{Z} \}$.

$$\simeq (\mathbb{R}/\mathbb{Z} - 0) \times \mathbb{R}/\mathbb{Z}$$

$$\pi^{-1}\mathcal{U} = \coprod_{n \in \mathbb{Z}} (n, n+1) \times \mathbb{R}$$

$F \in C^\infty(\pi^{-1}\mathcal{U})$ $\quad F(x,y) \quad x \in (n,n+1). \quad y \in \mathbb{R}/\mathbb{Z}$

$$\mathcal{L}(\pi^{-1}\{x+\mathbb{Z} \neq \mathbb{Z}\}) \xrightarrow{\sim} \{ F(x,y) \in C^\infty((0,1) \times \mathbb{R}/\mathbb{Z})$$

Idea about the transform $f(x) \mapsto \tilde{f}(x,y)$, what might it mean to localize at a point $(x,y) \in T^2$, wavelet with position x and momentum y. you are reminded of Hormander's partition of unity, rather his localization theory behind Fourier integral operators.

Another idea - coherent states, coherent state representation is complete but not orthogonal,

---

connection on $\mathcal{L}$, you seek operators D_x, D_y on $\mathcal{L}$ compatible w. ∂_x, ∂_y on $\mathcal{A}$ in the sense that Liebniz holds:

$$D_x(f\psi) = \partial_x f\,\psi + f D_x\psi \quad \text{also for } D_y$$

because of isom. $\mathcal{S}(\mathbb{R}) \xrightarrow{\sim} \mathcal{L}$ you can look for D_x, D_y on $\mathcal{S}$:

$$\tilde{f}(x,y) = \sum_m e^{2\pi i m y} f(x+m)$$

clearly if $\partial_x \tilde{f} = \widetilde{\frac{d}{dx} f} \quad \therefore \boxed{D_x\psi = \partial_x\psi}$

$$\partial_y \tilde{f}(x,y) = \sum_m e^{2\pi i m y}\, 2\pi i m\, f(x+m)$$

$$2\pi i x\,\tilde{f}(x,y) = \sum_m e^{2\pi i m y}\, 2\pi i x\, f(x+m)$$

$$(\partial_y + 2\pi i x)\tilde{f}(x,y) = \sum_m e^{2\pi i m y}\, 2\pi i (x+m) f(x+m) = (2\pi i x f)^{\sim}$$

$$\therefore \boxed{D_y\psi = (\partial_y + 2\pi i x)\psi} \qquad [D_x, D_y] = 2\pi i$$

$$e^{X+Y}e^{\frac{1}{2}[X,Y]} = e^X e^Y$$

$$\frac{X^2+XY+YX+Y^2}{2} \qquad \frac{X^2}{2} + XY + \frac{Y^2}{2}$$

So now you have good control over the connection

$$e^{aD_x} e^{bD_y}\psi(x,y) =$$

IDEA. You get a representation of the Heisenberg group on $\mathcal{L} \simeq \mathcal{S}$. This should be irreducible, at least the L^2 version is.

What about L^2 completions?

There should be an L^2 norm on $\mathcal{L}$, such that the representation of the Heisenberg group is unitary, just because $\parallel$ transport should preserve norm square of a section, and volume is translation invariant. You are hoping that $\mathcal{L}$ is a kind of tensor product of $C^\infty(\mathbb{R}/\mathbb{Z})$ with $C^\infty(\mathbb{Z})$, rapidly decreasing functions on $\mathbb{Z}$. If so, then $\mathcal{L}$ should have an interpretation as kernels.

$\mathcal{L}$ is some twisted version of $C^\infty(T) \otimes C^\infty(T)$, better is the idea that $\mathcal{L}$ is line bundle over $C^\infty(T^2)$, where the first circle T is $\mathbb{R}/\mathbb{Z}$ and the second circle is the character group of $\mathbb{Z}$. $\mathcal{L}$ is a K-theory correspondence linking the classifying space $B\mathbb{Z} = \mathbb{R}/\mathbb{Z}$ with the group ring $C_r(\mathbb{Z})$ which is $C(\check{\mathbb{Z}}) = \mathbb{C}(\mathbb{T})$

Also known as Poincare divisor

What's important: $\mathcal{L}$ is an irred rep of Heisenberg gp.

---

to construct L a complex line bundle over T^2 equipped with a connection, such that $\mathcal{L}$ can be identified with the space of sections of L over T^2.

first attempt is to define the fibre of L over a point $(x,y) \in T^2$ and the space $\mathcal{L}$ of all sections modulo sections vanishing at (x,y). This is an intrinsic definition, and it should be clear that you get a locally trivial fibre bundle of complex lines. Local trivs come from nonvanishing $\phi \in \mathcal{L}$ at $(x,y) \in T^2$.

the operators D_x, D_y on $\mathcal{L}$ which cover ∂_x, ∂_y on $\mathcal{A}$ should define a Lie rep. of Heisenberg on $\mathcal{L}$ covering the translation action.

Let's now do this construction carefully. You have a specific $\mathcal{A}$ module $\mathcal{L}$. consisting of all smooth $\psi(x,y)$ on $\mathbb{R}^2$ sat. $\psi(x,y+1) = \psi(x,y) = e^{2\pi i y}\psi(x+1,y)$.

Introduce an equiv rel. $\psi \sim \psi'$ iff equal at (x,y)

But it should be possible to say what $L_{(x,y)}$ is.

$\mathcal{L}(U) = \{\psi(x,y) \in C^\infty(\pi^{-1}U) \mid$ automorphy conditions hold$\}$

$L_{(\bar{x}_0,\bar{y}_0)} = \{\psi(x,y)$ functions on $\pi^{-1}(\bar{x}_0,\bar{y}_0)$ sat etc.

$$\begin{tikzcd} L = \mathbb{R}^2 \times^{\mathbb{Z}^2} \mathbb{C} \arrow[d] \\ T^2 \end{tikzcd}$$

NO this looks like a flat v.b.

$L_{(x_0+\mathbb{Z},\, y_0+\mathbb{Z})} = \{\psi(x,y)$ fn on $(x_0+\mathbb{Z})\times(y_0+\mathbb{Z})$ sat. period. cond.
$\psi(x,y+1) = \psi(x,y) = e^{2\pi i y}\psi(x+1,y)\}$

can phrase it as $\psi(x_0+m, y_0+n) = e^{-2\pi i m y_0}\psi(x_0,y_0)$

So progress was made on going from $\mathcal{L}$ to L.

$\mathcal{O}(U)$ = smooth functions $f(x,y)$ on $\pi^{-1}U \subset \mathbb{R}^2$ sat
$f(x+m, y+n) = f(x,y)$

$\mathcal{L}(U)$ ψ

$\mathcal{L}(\pi x_0, \pi y_0) = \{\psi(x,y)$ on $(x_0+\mathbb{Z}, y_0+\mathbb{Z}) \mid \psi(x_0+m, y_0+n) = \hat{e}^{2\pi i x_0}\psi(x,y)\}$

1-jets for L J_1L $0 \to L\otimes T^* \to J_1L \to L \to 0$

$J_1L_{(\pi x_0, \pi y_0)} = \mathcal{L}/\{\psi \in \mathcal{L} \mid \psi$ vanishes to 2nd order at ~~$(\pi x_0, \pi y_0)$~~ $(x_0, y_0)\}$

$J_1\mathcal{L} \overset{\text{def}}{=}$ sections of J_1L over T^2

Let's forget the periodicity conditions for the moment and focus on what happens over $\mathbb{R}^2$. Then

$\mathcal{O}, \mathcal{L}$

---

perhaps it would better to review holom. function picture for the harmonic oscillator (simple).

$$\|f\|^2 = \int |f(z)|^2 e^{-|z|^2} \boxed{\frac{dx\,dy}{\cancel{2}\pi}}$$

$$\iint e^{-x^2-y^2}\,dx\,dy$$

$$= 2\pi\int_0^\infty e^{-r^2}\,r\,dr$$

$$= 2\pi\left[-\frac{e^{-r^2}}{2}\right]_0^\infty = \pi$$

$dz = dx + i\,dy$

$d\bar z = dx - i\,dy$

$\underline{d\bar z\, dz}$

$$\frac{d\bar z\, d\bar z}{2i\pi} = +2i\frac{dx\,dy}{\pi}$$

$[a, a^*] = 1.$ $\qquad a|0\rangle = 0 \qquad \langle 0|0\rangle = 1$

$$\langle 0|a^n a^{*n}|0\rangle = n!$$

$$\langle f|f\rangle = \int_{\mathbb{C}} |f|^2 e^{-|z|^2}\frac{dx\,dy}{\pi}$$

reproducing kernel. ~~$f(g)$~~ $f(w) = \langle K(\bar w, z)|f\rangle$

$$f(w) = \int_{\mathbb{C}} e^{w\bar z} f(z)\,\frac{dx\,dy}{\pi}$$

Idea action of $\mathbb{Z}^2$ on $\mathcal{S}(\mathbb{R})$, one generator $f(x) \mapsto f(x+1)$, the other $f(x) \mapsto e^{2\pi i x} f(x)$, the quotient by the first sends $f(x)$ to the periodic function $\sum_{m\in\mathbb{Z}} f(x+m)$, the quotient by the second sends $f(x)$ to the measure $\sum_{n\in\mathbb{Z}} \delta(x-n)f(x)$ supported on $\mathbb{Z}$. Both lie outside $\mathcal{S}(\mathbb{R})$.

Can you take the ~~quotient~~ of $\mathcal{S}(\mathbb{R})$ by $\mathbb{Z}^2$?

Program for today Wed Jan 23, 02. Get a good understanding of the line bundle L over T^2, its connection and the action of the Heisenberg group.

You have a representation the group $\mathbb{Z}^2$ on $\mathcal{L}$, ideally this is equivalent to a module structure for $\mathcal{L}$ over the group ring of $\mathbb{Z}^2$ which is the ring $\mathcal{A}$ of functions on T^2. The quotient of $\mathcal{L}$ the group $\mathbb{Z}^2$ should be $\mathcal{L}/\mathcal{I}\mathcal{L}$ where $\mathcal{I}$ is the augmentation ideal in the group ring $\mathcal{A}$, in other words the fibre $L_{(0,0)}$.

OK now you have the problem of making sense out of the statement that ~~$\sum_m f(x+m)$ is the quotient~~

$$f(x) \longmapsto \sum_m f(x+m)$$

$$\mathcal{S}(\mathbb{R}) \longrightarrow C^\infty(\mathbb{R}/\mathbb{Z})$$

is the quotient of the $\mathbb{Z}$-translation action on $\mathcal{S}(\mathbb{R})$.

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p 859 for $e^{\pi i y(y-1)}\psi(x,y) \overset{\text{def}}{=} \tilde{\psi}(x,y)$

satisfies $\tilde{\psi}(x+1, y+1) = \tilde{\psi}(x,y)$

p 883 question: direct embedding of L into an (infinite dimensional?) trivial bundle over T^2 such that the Grassmannian connection yields the desired connection.

To understand L well enough to describe its connection. Start with $\mathcal{L} = \{\psi \text{ smooth on } \mathbb{R}^2 \mid \psi(x+m, y+n) = e^{-2\pi i m y}\psi(x,y)\}$

$$D_x = \partial_x \qquad D_y = \partial_y + 2\pi i x$$

Go back to the case without $\mathbb{Z}^2$ action. $C^\infty(\mathbb{R}^2)$ with the operators D_x and D

How should you proceed? It's clear that when you say $\mathcal{L}$ is $C^\infty(\mathbb{R}^2)$ without $\mathbb{Z}^2$ conditions that you have the sections of the trivial line bundle. So it should be then a matter of ~~differ~~ identifying L with the Heisenberg group.

$$e^{2\pi i c} e^{bD_y} e^{aD_x}\psi(x,y) = e^{2\pi i c}\underbrace{e^{b(\partial_y + 2\pi i x)}\underbrace{e^{a\partial_x}\psi(x,y)}_{\psi(x+a,y)}}$$

$$= e^{2\pi i c} e^{2\pi i b x}\psi(x+a, y+b)$$

check that

$$D_{\substack{x\\y}}\left(f(x,y)\psi(x,y)\right) = \begin{matrix}\partial_x\\\partial_y\end{matrix}\Big| f(x,y)\cdot\psi(x,y) + f(x,y) D_{\substack{x\\y}}\psi$$

$$e^{aD_x}(f\psi) = \left(e^{a\partial_x}f\right)\left(e^{aD_x}\psi\right)$$

$$D_x(f\psi) = (\partial_x f)\psi + fD_x\psi$$

$$\frac{a^n}{n!}D_x^n(f\psi) = \frac{a^n}{n!}\sum_{k=0}^{n}\binom{n}{k}\partial_x^k f\, D_x^{n-k}\psi$$

$$= \sum_{k+l=n}\frac{a^k}{k!}\partial_x^k f\,\frac{a^l}{l!}D_x^l\psi$$

---

what's next?

$$L_{(x_0+\mathbb{Z},\, y_0+\mathbb{Z})} = \Big\{\psi(x,y) \text{ on } (x_0+\mathbb{Z})\times(y_0+\mathbb{Z})$$
$$\psi(x+m, y+n) = e^{-2\pi i m y_0}\psi(x,y)\Big\}$$

what is the isotropy group for a non zero elt of $L_{(0+\mathbb{Z},\, 0+\mathbb{Z})}$

$\Rightarrow a, b \in \mathbb{Z}$

$$e^{2\pi i c}\cancel{e^{2\pi i b}}\underbrace{\psi(x+a, y+b)}_{\cancel{e^{-2\pi i a y}}\psi(x,y)}.$$

Look at $\mathcal{A} = C^\infty(\mathbb{R}^2)$ $\quad \partial_x, \partial_y$

$\mathcal{L} = C^\infty(\mathbb{R}^2)$ $\quad D_x = \partial_x, \; D_y = \partial_y + 2\pi i x$

$$e^{a\partial_x}e^{b\partial_y}f(x,y) = f(x+a, y+b).$$

$$e^{bD_y}e^{aD_x}\psi(x,y) = e^{2\pi i b x}\psi(x+a, y+b)$$

Idea: You want to view the isomorphism $\mathcal{S}(\mathbb{R}) \simeq \mathcal{L}$ as providing a way to localize an $\overbrace{f(x)}^{\text{element}} \in \mathcal{S}(\mathbb{R})$ into pieces with approximately defined time + frequency. ~~time ≈ x~~. This may not work because position would would be a point on the circle, momentum ?

Usual Heisenberg $2\pi i x$, ∂_x

Question: Suppose you take partitions of unity on the x and y circles and form the product to get a partition of unity on T^2. Is there an interesting description of what it means for an element of $\mathcal{S}(\mathbb{R})$ to be supported in a nbd of a point of T^2 ?

think of time and frequency for a discrete signal, time is measured in seconds, frequency in either | cycles / radians per second so there's an obvious duality (pairing) which is dimensionless. The x coord circle gives the phase shift of the discrete signal, the y coord is really the frequency, this is is clear from $\sum_m e^{2\pi i m y} f(x+m)$ Fourier transform of the discrete signal $m \mapsto f(x+m)$ x is phase shift.

Question: Do the elements $\psi(x,y) \in \mathcal{L}$ admit a natural interpretation as kernels?

Next look at the case where $\mathbb{Z}^2$ is ignored:

$$\mathcal{A} = \{ f(x,y) \in C^\infty(\mathbb{R}^2) \}$$

$$\mathcal{L} = \{ \psi(x,y) \in C^\infty(\mathbb{R}^2) \}$$

$\mathcal{L}$ = space of sections of trivial line bdle. over $\mathbb{R}^2$.

connection $D_x = \partial_x$, $D_y = \partial_y + 2\pi i x$ on $\mathcal{L}$

means Leibniz

$$D_{\substack{x\\y}}(f\psi) = \partial_{\substack{x\\y}} f\, \psi + f D_{\substack{x\\y}} \psi$$

recall

$$e^{a\partial_x} e^{b\partial_y} f(x,y) = f(x+a, y+b)$$

$$e^{aD_x} e^{bD_y} \psi(x,y) = e^{a\partial_x} e^{b\partial_y} e^{2\pi i b x} \psi(x,y)$$

$$= e^{a\partial_x} e^{2\pi i b x} \psi(x, y+b)$$

$$= e^{2\pi i b(x+a)} \psi(x+a, y+b)$$

$$\boxed{e^{aD_x} e^{bD_y} \psi(x,y) = e^{2\pi i ab}\ e^{2\pi i b x} \psi(x+a, y+b)}$$

Check

$$e^{aD_x} e^{bD_y} \psi = e^{[aD_x, bD_y]} e^{bD_y} e^{aD_x} \psi(x,y)$$

$$= e^{2\pi i ab}\ e^{b 2\pi i x}\ e^{b\partial_y} e^{a\partial_x} \psi(x,y)$$

$$= e^{2\pi i ab}\ \underbrace{e^{2\pi i b x} \psi(x+a, y+b)}_{}.$$

How to handle Λ ?

$$\cancel{e^{2\pi i(y+b)}\ e^{2\pi i b(x+1)}\ \underbrace{\psi(x+1+a, y+b)}_{e^{-2\pi i(y+b)}\psi(x+a, y+b)} \overset{?}{=} e^{2\pi i y}\ e^{2\pi i b x} \psi(x+a, y+b)}$$

$$e^{2\pi i b(x+1)}\ \underbrace{\psi(x+1+a, y+b)}_{e^{-2\pi i(y+b)}\psi(x+a, y+b)} = e^{2\pi i b x} e^{-2\pi i y} \psi \qquad ?$$

$$\psi'(x,y) = e^{2\pi i b x}\,\psi(x+a, y+b)$$

$$\psi'(x+1,y) = e^{2\pi i b x + 2\pi i b}\,\psi(x+a+1, y+b)$$

$$e^{2\pi i y}\psi'(x+1,y) = e^{\substack{2\pi i b x + 2\pi i b \\ +2\pi i y}}\, e^{-2\pi i (y+b)}\,\psi(x+a, y+b)$$

$$\psi(x,y+1) = \Big|\,\psi(x,y) = e^{2\pi i y}\,\psi(x+1,y)\Big)\Big|$$

$$\partial_y\psi(x,y) = \underbrace{e^{2\pi i y}\big(2\pi i\,\psi(x+1,y) + \partial_y\psi(x+1,y)\big)}$$

$$(\partial_y + 2\pi i x)\psi(x,y) = \wedge + e^{2\pi i y}\,2\pi i x\,\psi(x+1,y)$$

$$= e^{2\pi i y}\big(2\pi i(x+1)\psi(x+1,y) + \partial_y\psi(x+1,y)\big)$$

Assume $\psi(x,y) = \psi(x,y+1)$

$$e^{2\pi i x y}\psi(x,y) = e^{2\pi i (x+1) y}\,\psi(x+1,y)$$

$$e^{bD_y}\psi(x,y) = e^{2\pi i b x}\,e^{b\partial_y}\,\psi(x,y)$$

$$\psi_1(x,y) = e^{2\pi i b x}\,\psi(x, y+b)$$

$$e^{2\pi i y}\psi_1(x+1,y) = \cancel{e^{2\pi i y}}\quad e^{2\pi i b(x+1)}\underbrace{\psi(x+1, y+b)}_{e^{-2\pi i (y+b)}\psi(x,y+b)}$$

$$= e^{2\pi i b x}\,\psi(x,y+b) = \psi_1(x,y).$$

$$\psi(x,y) = \sum_{m\in\mathbb{Z}} e^{2\pi i m y} f(x+m)$$

$$\oint |\psi(x,y)|^2 dy = \sum_m |f(x+m)|^2 \qquad \therefore \|\psi\|_{L^2}^2 = \|f\|_{L^2}^2$$

IDEA look at points of order $|N$ on T^2, should yield DFT, Mumford type Θ functions, could these yield a direct embedding of L into a trivial bundle?

You now have Heis gp acting on $\mathcal{L}$, $\mathcal{Q}$ and you it would like to get control of L. Given a point of T^2 ie cosets $(x_0+\mathbb{Z}, y_0+\mathbb{Z})$ then the fibre of L over the point is the space of functions on $(x_0+\mathbb{Z})\times(y_0+\mathbb{Z})$ satisfying the autom. condition.

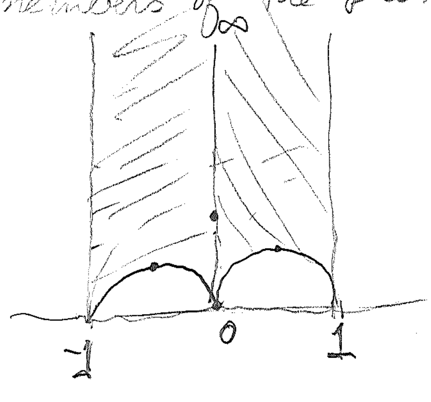
Then look at 1-jets.

You ought to explore what to do. Your aim is the $SL(2,\mathbb{Z})$ action on L over T^2. Digress a bit on approaches to this problem.

First discuss $SL(2,\mathbb{Z})$ action on T^2. This is obvious from $\mathbb{Z}^2 \rightarrowtail \mathbb{R}^2 \twoheadrightarrow T^2$ $\qquad T^2 \xleftarrow{\sim} \mathbb{Z}^2 \otimes_{\mathbb{Z}} \mathbb{R}/\mathbb{Z}$

It should be possible to use the triangulation of the UHP to reduce the general $SL(2,\mathbb{Z})$ auto to the simple cases $\begin{pmatrix}1&1\\0&1\end{pmatrix}$ and $\begin{pmatrix}1&0\\1&1\end{pmatrix}$.

The triangulation has for its vertices the set $P_1\mathbb{Q} = \mathbb{Q}\cup\{\infty\}$ of rank 1 summands, call these lines in $\mathbb{Z}^2$, two lines form a 1 simplex iff their sum is $\mathbb{Z}^2$. Given ~~two~~ a pair complementary lines there exactly two lines complementary to the members of the pair.



Now you want to extend the transformation

$$\psi(x,y) \longmapsto e^{2\pi i xy}\psi(x,y) = \phi(x,y)$$

which links different automorphy conditions

For ψ you have $\boxed{\psi(x+m,y+n) = e^{-2\pi i m y}\psi(x,y)}$

$$\phi(x,y) = e^{2\pi i x y}\psi(x,y) = e^{2\pi i x y}e^{2\pi i y}\psi(x+1,y) = \phi(x+1,y)$$

$$\phi(x,y+1) = e^{2\pi i x(y+1)}\psi(x,y+1) = e^{2\pi i x}\underbrace{\left(e^{2\pi i x y}\psi(x,y)\right)}_{\phi(x,y)}$$

$$\phi(x+1,y) = \phi(x,y)$$
$$\phi(x,y+1) = e^{2\pi i x}\phi(x,y)$$

$$\boxed{\phi(x+m,y+n) = e^{2\pi i n x}\phi(x,y)}$$

$$\psi_1(x,y) = e^{2\pi i b x}\psi(x+a,y+b)$$

$$\psi_1(x+1,y) = e^{2\pi i b x + \cancel{2\pi i b}}\underbrace{\psi(x+a+1,y+b)}_{e^{-2\pi i(y+\cancel{b})}\psi(x+a,y+b)}$$

$$= e^{-2\pi i y}\left(e^{2\pi i b x}\psi(x+a,y+b)\right) = e^{-2\pi i y}\psi_1(x,y)$$

$$e^{2\pi i x y + 2\pi i b x}\psi(x+a,y+b)$$

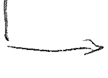
$e^{2\pi i(x+a)(y+b)}\psi(x+a,y+b)$ per. in x

$$e^{-2\pi i a(y+b)}$$

---

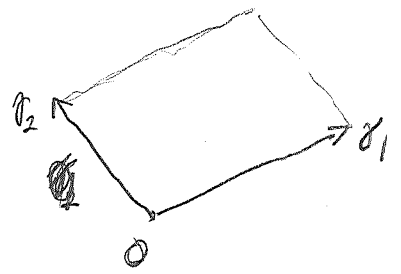
propose to transform by a simple element of $SL(2,\mathbb{Z})$. what ideas? $\mathbb{Z}^2\otimes\mathbb{R}/\mathbb{Z} = T^2$, you want a coordinate free approach, start with a real 2 plane V and a lattice Γ. Suppose V oriented

Start with a real 2 plane V, volume element, and a lattice Γ such that V/Γ has volume 1. ~~Consider~~ Choose as basis for Γ: γ_1, γ_2 such that $\gamma_1 \wedge \gamma_2 = \text{vol}$. Then you get an standard picture

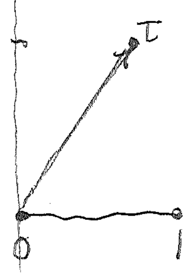


$V = \mathbb{R}^2$ $\gamma_1 = (1,0)$
$\Gamma = \mathbb{Z}^2$ $\gamma_2 = (0,1)$

In general given V, vol, $\gamma_1 \wedge \gamma_2 =$ vol then your lattice is $\{m\gamma_1 + n\gamma_2\}$. ~~you can think of~~ you pick γ_1 first, then γ_2 is on the "positive" side

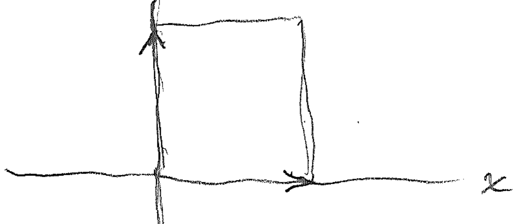


The usual conventions require periodicity wrt γ_1, especially if you take a complex plane you take $\gamma_1 = 1$, $\gamma_2 \in$ UHP.



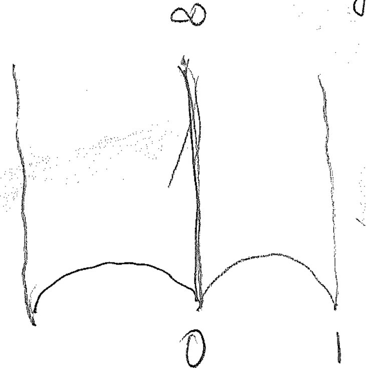
Something is wrong here. Volume = Im τ. You are introducing a conformal structure on V, it shouldn't be part of the structure at this point. So

So begin with



This is your basic geometry. What's important?

Recall that given complementary lines in the lattice Γ there are two directions ~~you can~~ to go in the the triangulations



$\infty \quad \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$0 \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$1 \quad \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

What you would like to do is to replace the condition defining $\mathcal{L}$:

$$\psi(x+m, y+n) = e^{-2\pi i m y}\psi(x,y)$$

Let $\phi(x,y) = \psi(x+y, y)$

$$\phi\begin{pmatrix} x \\ y \end{pmatrix} = \psi\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

$$\phi(x+1, y) = \psi(x+1+y, y) = e^{-2\pi i y}\psi(x+y, y) = e^{-2\pi i y}\phi(x,y)$$

$$\phi(x, y+1) = \psi(x+y+1, y+1) ~~= e^{-2\pi i(y+1)}\psi(x+y~~$$

$$= \psi(x+y+1, y) = e^{-2\pi i y}\psi(x+y, y) = e^{-2\pi i y}\phi(x,y)$$

so you find that $\phi(x+1, y) = \phi(x, y+1)$ so it seems that $\phi(x,y)$ is fixed under $(x,y) \longmapsto (x+m, y-m)$

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

corresp to fact that $\psi(x,y)$ is fixed under $(x,y) \longmapsto (x, y+1)$.

$$\phi(x,y) = \psi(x+y, y) = e^{2\pi i y}\psi(x+y+1, y)$$

$$= e^{2\pi i y}\psi(x+y+1, y+1) = e^{2\pi i y}\phi(x, y+1)$$

$$= e^{2\pi i y}\left(e^{2\pi i y}\phi(x, y+2)\right)$$

$$\phi(x,y) = \psi(x+y, y) = e^{2\pi i y}\psi(x+y+1, y)$$

$$= e^{2\pi i y}\phi(x+1, y)$$

$$\phi(x+1, y) = \psi(x+1+y, y) = e^{-2\pi i y}\psi(x+y, y) = e^{-2\pi i y}\phi(x,y)$$

$$\phi(x,y) = e^{2\pi i y}\phi(x+1, y) = \left(e^{2\pi i y}\right)^2\phi(x+2, y)$$

$$\phi(x,y) \qquad = \left(e^{2\pi i y}\right)^2\phi(x, y+2)$$

$$\psi(x, y+1) = \psi(x,y) \qquad \psi(x+1,y) = e^{-2\pi i y}\psi(x,y)$$

$$\boxed{\psi(x+m, y+n) = e^{-2\pi i y m}\psi(x,y)}$$

Put $\phi(x,y) = \psi(x+y, y)$

$$\phi(x+1,y) = \psi(x+1+y,y) = e^{-2\pi i y}\psi(x+y,y) = e^{-2\pi i y}\phi(x,y)$$

$$\phi(x+m,y) = e^{-2\pi i m y}\phi(x,y)$$

$$\phi(x,y+1) = \psi(x+y+1,y+1) = \cancel{\psi(x+y+1}$$

$$= \psi(x+y+1,y) = e^{-2\pi i y}\psi(x+y,y) = e^{-2\pi i y}\phi(x,y)$$

$$\cancel{\phi(x,y+n) = e^{-2\pi i(n-1)y}\phi(x,y+n-1)}$$

$$\phi(x,y+n) = \phi(x,(y+n-1)+1) = e^{-2\pi i(y+n-1)}\phi(x,y+n-1)$$

$$= e^{-2\pi i y}\phi(x,y+n-1)$$

clearer
~~better~~ would be

$$\phi(x,y+1) = e^{-2\pi i y}\phi(x,y)$$

$$\phi(x,y+2) = e^{-2\pi i(y+1)}\phi(x,y+1)$$

$$\phi(x,y+3) = e^{-2\pi i(y+2)}\phi(x,y+2)$$

so it seems that you get

$$\phi(x+m,y) = e^{-2\pi i \overset{m}{y}}\phi(x,y)$$

$$\phi(x,y+n) = e^{-2\pi i \overset{n}{y}}\phi(x,y)$$

$$\therefore \quad \phi(x+m,y+n) = e^{-2\pi i n y}\phi(x+m,y) = e^{-2\pi i n y}e^{-2\pi i m y}\phi(x,y)$$

$$\boxed{\phi(x+m,y+n) = e^{-2\pi i(m+n)y}\phi(x,y)}$$

$$\psi(x+1,y+1) = \psi(x+1,y) = e^{-2\pi i y}\psi(x,y)$$

Discuss, examine, what happens with, the action of SL(2, Z). At the moment you have a picture of the line bundle L given by its space of sections $\mathcal{L}$.

Maybe it might be a good idea to get a clean description of L. Go back to the idea of ~~doing things~~ working ~~of $\mathcal{L}$~~ over the plane ignoring the automorphy conditions.

Begin with $\mathcal{L} = C^\infty(\mathbb{R}^2)$ $\quad D_x = \partial_x \quad D_y = \partial_y + 2\pi i x$

then $e^{aD_x} e^{bD_y} \psi(x,y) = e^{a\partial_x} e^{b(2\pi i x)} e^{b\partial_y} \psi(x,y)$

$$= e^{a\partial_x} e^{2\pi i b x} \psi(x, y+b) = e^{2\pi i b(x+a)} \psi(x+a, y+b)$$

$$e^{bD_y} e^{aD_x} \psi(x,y) = e^{2\pi i bx} \psi(x+a, y+b)$$

$$e^{aD_x} e^{bD_y} \psi = e^{2\pi i ab} e^{bD_y} e^{aD_x} \psi$$

Make this intrinsic V real 2-plane, connection D on the trivial line bundle over V, $D = d + A$

in ~~our~~ your case $V = \mathbb{R}^2$, $A = 2\pi i x\,dy$, but you have already seen that it's nice to use the gauge transf.

$\psi(x,y) \longmapsto e^{2\pi i xy} \psi(x,y)$

$$e^{2\pi i xy} (d + 2\pi i x\,dy) e^{-2\pi i xy}$$

$$= d - 2\pi i\, d(xy) + 2\pi i x\,dy = d - 2\pi i y\,dx$$

The arbitrariness in A is the differential of a quadratic function + linear function (?)

$$e^{-\alpha\binom{x}{y}} (d + A) e^{\alpha\binom{x}{y}} = d + \left(\alpha \binom{dx}{dy} + A\right)$$

How to study? ~~Begin with~~ the 2 plane $V = \mathbb{R}^2$ and a nonzero volume forms ω. Choose a connection on the trivial line bundle $d + P\,dx + Q\,dy$ ~~whose~~ with curvature $\omega = 2\pi i\, dx\, dy$ e.g. $d + 2\pi i x\,dy$. Subtract

$$A = P\,dx + (Q - 2\pi i x)\,dy \qquad \partial_x Q = \partial_y P$$

$$dA = (\partial_x Q - \partial_y P - 2\pi i)\,dx\,dy \quad ?$$

$$d(P\,dx + Q\,dy) = (-\partial_y P + \partial_x Q)\,dx\,dy \quad ?$$

Repeat. $V = \mathbb{R}^2$ with volume $2\pi i\,dx\,dy$

Conn. $D = d + A = d + P\,dx + Q\,dy \ni dA = \underbrace{(-\partial_y P + \partial_x Q)}_{2\pi i}\,dx\,dy$

By Poincaré lemma

$$A = 2\pi i x\,dy + df$$ where $f(x,y)$ is unique up to an additive constant.

Repeat $V = \mathbb{R}^2$ with volume $dx\,dy$, consider a connection $D = d + A$ on the trivial line bundle $V \times \mathbb{C}$ with curvature $D^2 = dA = 2\pi i\,dx\,dy$, e.g. $A = 2\pi i x\,dy$. Let $A_1 = P\,dx + Q\,dy$ be another connection form with same curvature. Poincaré lemma $\Longrightarrow A_1 = 2\pi i x\,dy + df$ where f is unique up to an add. const.

Philosophy here: ω is translation invariant, so you would like A to be translation invariant, impossible, so do the best you can. From the 2 examples $A = 2\pi i x\,dy$ and $-2\pi i y\,dx$ you have translation invariance in the y and x directions resp. So it becomes clear that you want to choose a line (one diml subspace of V) and require A to be invariant under translations parallel to this line

$$A = P(x,y)\,dx + Q(x,y)\,dy \qquad l = \mathbb{R}(r,s)$$

$$\overset{\parallel}{} \; P(x+tr, y+ts)\,dx + Q(x+tr, y+ts)\,dy$$

so P, Q must be constant along cosets of l

i.e. they are functions on V/l

So it seems you have a natural class of connections to pin down generalizing your two examples $-2\pi i y\,dx$, $2\pi i x\,dy$

Recall exact sequence

$$\begin{tikzcd} S^2V^* \arrow[r, tail] & V^*\otimes V^* \arrow[r] \arrow[d, phantom, "\cong" rotate=90] & \Lambda^2V^* \arrow[d, phantom, "\parallel" rotate=90] \\ S^2V^* \arrow[r, "d"] & V^*\otimes dV^* \arrow[r, "d"] & \Lambda^2 dV^* \arrow[d, phantom, "\ni" rotate=-90] \\ & & \omega \end{tikzcd}$$

looking at $d^{-1}\omega$ which is acted by $SL(V,\omega)$

$-y\,dx$ Question: Does $-y\,dx$ have an intrinsic meaning? This is a 1-form on V

$V = \mathbb{R}^2$ coords x, y $\omega = dx\,dy$

you want to understand 1-forms A on V such that $dA = \omega$. Given such a 1-form A_0, Poincaré's lemma says any other has the form $A = A_0 + df$, f unique modulo constants. Consider translations on V: $(x,y) \mapsto (x+a, y+b)$ These preserve ω, hence translations act on $\{A \mid dA = \omega\}$.

It seems you want to choose A_0 so that it's orbit under the translation group is as small as possible. For example if $A_0 = x\,dy$, then A_0 is ~~presso~~ fixed under $y \mapsto y+b$, so the orbit is $\{(x+a)\,dy \mid a \in \mathbb{R}\}$., 1-diml affine line

Next look at the action of $SL(2,\mathbb{R})$, and maybe $SL(2,\mathbb{R}) \ltimes \mathbb{R}^2$

You should have an action of $SL(2,\mathbb{R}) \ltimes \mathbb{R}^2$ on the space of connections $d+A$ on the trivial line bundle $\mathbb{R}^2 \times \mathbb{C}$ having curvature ω

$$\begin{tikzcd} \mathbb{R}^2 \times \mathbb{C} \arrow[d] \\ \mathbb{R}^2 \end{tikzcd}$$

$\mathbb{R}^2 \times \mathbb{T}$ coords x, y, θ

$\mathbb{R}^2$ " x, y

you look for a connection form $d\theta + (\quad)dx + (2\pi i x)dy$

you probably want to replace $SL(2,\mathbb{R})$ with its Lie alg.

IDEA: $SL(2,\mathbb{R})$ operates on the circle $\mathbb{P}^1(\mathbb{R})$

Go back to the trivial line bundle over $\mathbb{R}^2$ whose sections are just smooth functions $\psi(x,y)$.

This time you want both translations and symplectic linear transformations on $\mathbb{R}^2$. Lie alg of $SL(2,\mathbb{R})$ should give rise to three vector fields on $\mathbb{R}^2$. $x\partial_x - y\partial_y$, $x\partial_y$, $y\partial_x$

$$\begin{pmatrix} & \\ & \end{pmatrix}\begin{pmatrix} & \\ & \end{pmatrix}$$ basis of $\overset{sl}{\cancel{End}}(V)$ = endos of trace 0

Lie of translations has basis ∂_x, ∂_y

$$[x\partial_y, y\partial_x] = [x\partial_y, y]\partial_x + y[x\partial_y, \partial_x]$$
$$= x\partial_x - y\partial_y$$

Now take the connection $\boxed{d + 2\pi i x\,dy}$ on
$$\begin{tikzcd} L = \mathbb{R}^2 \times \mathbb{C} \arrow[d] \\ \mathbb{R}^2 \end{tikzcd}$$

~~A vector field?~~ Start with $\mathbb{R}^2$ equipped with the infinitesimal translation action = the Lie alg (abelian) with basis ∂_x, ∂_y (vector fields). Then enlarge the action to include the infinitesimal action of $SL(2,\mathbb{R})$. So then you have 5 vector fields $\partial_x, \partial_y, x\partial_y, y\partial_x$, $x\partial_x - y\partial_y$ on $\mathbb{R}^2$. It might be true the stabilizer of the connection is the largest Lie subalgebra ~~such that the lift~~ whose lift to L preserves bracket ⓘ?

Review. You have brought in $SL(2,\mathbb{R})$. Begin with $\mathbb{R}^2$ and volume $\omega = dx\,dy$, symmetry group $SL(2,\mathbb{R}) \ltimes \mathbb{R}^2$. Describe the action by the Lie alg. basis ∂_x, ∂_y, $x\partial_y$, $y\partial_x$, $x\partial_y - y\partial_y$

$\mathcal{L}_X = [d, \iota_X]$. $\quad d\iota(x\partial_x - y\partial_y)\omega = d(x\,dx + y\,dy) = 0$

Now look at connections on the trivial line bundle over $\mathbb{R}^2$.

What is the meaning of $[D, \iota_X]$? Probably this is the lift of $\mathcal{L}_X$.

$$[D, \iota_X] = [d + A, \iota_X] = \mathcal{L}_X - \iota_X A$$

Look at $D = d + 2\pi i x\,dy$

$$\iota_X D - D\iota_X = X + (2\pi i)\,\iota_X(x\,dy) \qquad -\iota_X A$$

$X = \partial_x$	get	$\partial_x + (2\pi i)\,0$	D_x	0
$X = \partial_y$	"	$\partial_y + 2\pi i x$	D_y	$-2\pi i x$
$x\partial_y$		$x\partial_y + (2\pi i)x^2$	xD_y	$-2\pi i x^2$
$y\partial_x$		$y\partial_x$	yD_x	0
$x\partial_x - y\partial_y$		$x\partial_x - y\partial_y + (2\pi i)(-xy)$	$xD_x - yD_y$	$2\pi i x y$

So $-\iota_X A$ is the obstruction to X preserving the connection A.

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need a discussion about what to expect, the problem is that the symmetries of the original situation $\mathbb{R}^2, \omega$ do not preserve A. let's start with $\mathbb{R}^2, \omega$ and a choice for A, say $2\pi i x\,dy$. Take a symmetry of $(\mathbb{R}^2, \omega)$ say a translation $(x,y) \longmapsto (x+a, y+b)$. A is not preserved $d + 2\pi i x\,dy \longmapsto d + 2\pi i(x+a)dy$, but you can combine this symmetry with a gauge transformation

$$d + 2\pi i x\,dy + 2\pi i a\,dy \longmapsto \underbrace{e^{2\pi i a y}\left(d + 2\pi i \overset{x+a}{\cancel{x}}\,dy\right) e^{-2\pi i a y}}_{\substack{\| \\ d + 2\pi i x\,dy}}$$

to preserve the connection. Thus you can lift translations on $(\mathbb{R}^2, \omega)$ to autos of the line bundle + connection $(\mathbb{R}^2 \times \mathbb{C},\ d + 2\pi i x\,dy)$. The gauge transf is unique up to a phase factor, so there is a central extension of $\mathbb{R}^2$ acting on L.

Find formula

$$\begin{tikzcd}
\psi(x,y) \arrow[r, mapsto] \arrow[dd, mapsto] & e^{2\pi i a y}\psi(x+a, y+b) \arrow[d, mapsto] \\
 & e^{2\pi i a y}\left(\partial_y + 2\pi i(x+a)\right)\psi(x+a, y+b) \\
(\partial_y + 2\pi i x)\psi(x,y) \arrow[r, mapsto] & e^{2\pi i a y}\left(\partial_y + 2\pi i(x+a)\right)\psi(x+a, y+b)
\end{tikzcd}$$

Check: $e^{a\partial_x} e^{2\pi i b x} e^{b\partial_y} \psi(x,y)$

$$= e^{a\partial_x} e^{2\pi i b x}\psi(x, y+b) = e^{2\pi i b(x+a)}\psi(x+a, y+b) \quad ?$$

$\psi \in C^\infty(\mathbb{R}^2)$

$$D\psi = dx\, \partial_x\psi + \partial_y\psi dy + 2\pi i x\psi dy$$
$$= dx\,(\partial_x\psi) + dy\,(\partial_y\psi + 2\pi i x\psi)$$

$$T_{a,b}\psi(x,y) = \psi(x+a, y+b) = e^{a\partial_x} e^{b\partial_y}\psi(x,y).$$

$(x,y) \mapsto (x+a, y+b)$ $\qquad D\psi($

$\psi(x,y) \mapsto \psi(x+a, y+b)$

$$D\psi(x,y) = dx\, \partial_x\psi(x,y) + dy\left(\partial_y\psi + 2\pi i x\right)\psi(x,y)$$
$$\Big\downarrow$$
$$D\psi(x+a, y+b) = dx\, \partial_x\psi(x+a, y+b) + dy\left(\partial_y + 2\pi i x + 2\pi i a\right)\psi(x+a, y+b)$$

So your notation is confused. You have to keep straight $D_y\psi(x+a, y+b) = (\partial_y\psi)(x+a, y+b) + 2\pi i x\psi(x+a, y+b)$

and $(D_y\psi)(x+a, y+b) = (\partial_2\psi)(x+a, y+b) + 2\pi i(x+a)\psi(x+a, y+b)$

Maybe you should be working with operators on functions and evaluation at points $\langle x, y|$. Operators should act on [illegible] $C^\infty(\mathbb{R})$, possibly $\Omega(\mathbb{R})$

First you need the connection

Start again with the manifold $\mathbb{R}^2$ and "vol" $2\pi i\, dx dy$

Have symmetries from $SL(2,\mathbb{R}) \ltimes \mathbb{R}^2$, but we will look first at translations. Want connection $d+A$ with curvature ω, but $d+A$ will not be translation invariant. Possible to correct by using a gauge transf. Examples. $A = 2\pi i x dy$

$$e^{a\partial_x} e^{b\partial_y}\, x dy = e^{a\partial_x} x dy\, e^{b\partial_y}$$
$$= (x+a)\, dy\, e^{a\partial_x} e^{b\partial_y}$$

$$\cancel{e^{a\partial_x} e^{b\partial_y}(d + 2\pi i x dy) = \underbrace{(d + 2\pi i x + 2\pi i a)}_{e^{-2\pi i a x}} e^{a\partial_x} e^{b\partial_y}}$$

$$e^{a\partial_x} e^{b\partial_y}(d + 2\pi i x\,dy) = \underbrace{(d + 2\pi i x\,dy + 2\pi i a\,dy)}_{e^{-2\pi i a y}(d+2\pi i x\,dy)e^{2\pi i a y}} e^{a\partial_x} e^{b\partial_y}$$

so $e^{+2\pi i a y} e^{a\partial_x} e^{b\partial_y}$ leaves $d + 2\pi i x\,dy$ fixed

This mean that $\partial_x + 2\pi i y, \partial_y$ commute with $d + 2\pi i x dy$

$\partial_y + 2\pi i x, \partial_x$

$$[\partial_x + 2\pi i y, \partial_y + 2\pi i x] = 2\pi i - 2\pi i = 0$$

You have reached a difficulty, you did not expect to worry about left + right when dealing with symmetries of L. Go back and review.

$$\mathcal{A} = C^\infty(T^2) = \{ f \in C^\infty(\mathbb{R}^2) \mid e^{a\partial_x} e^{b\partial_x} f = f \}.$$
for $a, b \in \mathbb{Z}$

$$\mathcal{L} = \{ \psi \in C^\infty(\mathbb{R}^2) \mid \underbrace{\psi(x, y+1) = \psi(x,y)} = e^{2\pi i y}\psi(x+1,y) \}.$$

$$e^{b\partial_y}\psi = \psi \qquad b \in \mathbb{Z}$$

$$e^{a(\partial_x + 2\pi i y)}\psi = \psi \qquad a \in \mathbb{Z}$$

---

begin with $\mathbb{R}^2$ with infinitesimal translations $a\partial_x + b\partial_y$ and 2-form $\omega = 2\pi i\,dx\,dy$ which is translation invariant. $\mathcal{L}(X) = \iota_X d + d\iota_X$. Choose a connection on the trivial line bundle with curvature ω, e.g. $d + 2\pi i x dy$. What is a connection:

$$D(f\psi) = df\,\psi + f D\psi$$

$$D_X(f\psi) = Xf\,\psi + f D_X\psi$$

Idea: the connection is not translation invariant, but this can be corrected by a gauge transf.

$$\mathcal{L}(\partial_x)(d + 2\pi i x\,dy) = 2\pi i\,dy$$

$$\mathcal{L}(a\partial_x + b\partial_y)(\qquad) = 2\pi i a\,dy$$

$$[d + 2\pi i x\,dy, 2\pi i a y] = 2\pi i a\,dy$$

$$\left(\mathcal{L}(a\partial_x + b\partial_y) + ad(2\pi i a y)\right)(d + 2\pi i x\,dy) = 0$$

You seem to need practice. Start with the Lie group $\mathbb{R}^2$, the invariant vector field ∂_x, ∂_y giving translation $\langle x,y| e^{a\partial_x} e^{b\partial_y} \psi = \langle x+a, y+b | \psi \rangle$.

Fix $A = 2\pi i x\,dy$ a 1-form such that $D = d + A$ is a connection on the trivial ~~[illegible]~~ line bundle, whose curvature is $dA = 2\pi i\,dx\,dy = \omega$, Now ~~[illegible]~~ ω is translation invariant, but A isn't

$$e^{a\partial_x} e^{b\partial_y} x\,dy = e^{a\partial_x} x\, e^{b\partial_y} dy = (x+a)\, e^{a\partial_x}\, d(y+b)\, e^{b\partial_y}$$

$$(x+a)\,dy\, e^{a\partial_x} e^{b\partial_y}$$

$$e^{a\partial_x} e^{b\partial_y}\, x\,dy\, e^{-b\partial_y} e^{-a\partial_x} = (x+a)\,dy$$

$$e^{a\partial_x} e^{b\partial_y} (d + 2\pi i x\,dy)\, e^{-b\partial_y} e^{-a\partial_x} = d + 2\pi i (x+a)\,dy$$

$$e^{-2\pi i a y} (d + 2\pi i x\,dy)\, e^{+2\pi i a y} = d + 2\pi i (a+x)\,dy$$

Therefore you find-

$$\boxed{e^{2\pi i a y} e^{a\partial_x} e^{b\partial_y}}\,(d + 2\pi i x\,dy) = (d + 2\pi i x\,dy)\, e^{2\pi i a y} e^{a\partial_x} e^{b\partial_y}$$

Although the connection $d + 2\pi$

$$e^{a(\partial_x + 2\pi i y)} e^{b\partial_y} \quad \text{commutes with} \quad d + 2\pi i x\,dy$$

$$[\partial_x + 2\pi i y, \partial_y] = -2\pi i$$

What's strange at this point is that

$$\left[\ \boxed{\begin{matrix}\partial_x + 2\pi i y \\ \partial_y\end{matrix}}\ ,\ \boxed{\begin{matrix}\partial_x \\ \partial_y + 2\pi i x\end{matrix}}\ \right] = 0$$

these also generate one

these generate a Heisenberg alg. So one should be a left action and the other should be a right action.

Next comes $SL(2,\mathbb{R})$ action. 3 v.f. [redacted] $x\partial_y,\ y\partial_x,\ [x\partial_y, y\partial_x] = x\partial_x - y\partial_y$

$x\partial_y,\ y\partial_x,\ x\partial_x - y\partial_y$ are vector fields on $\mathbb{R}^2$ fixing 0.

$d + 2\pi i x\, dy$ $\qquad\qquad$ $e^{tx\partial_y}$

$[x\partial_y,\ d + 2\pi i x\, dy]$?

everything should be an operator on $\Omega(\mathbb{R}^2)$. A vector field is first order diff'l operator (derivation) on $\Omega^0(\mathbb{R}^2)$, but it has a natural extension to all forms as $\mathcal{L}(X) = [d, i(X)]$

So it seems that $[x\partial_y,\ \ ]$ means $\mathcal{L}(x\partial_y)$

$$\mathcal{L}(x\partial_y)(x\,dy) = \cancel{(\mathcal{L}(x\partial_y)x)}\,dy + x(\mathcal{L}(x\partial_y)\,dy)$$

$$= x\,d\,\mathcal{L}(x\partial_y)y = x\,dx$$

$$\mathcal{L}(x\partial_y)\ x\,dy = d(i(x\partial_y)x\,dy) + i(x\partial_y)\,dx\,dy$$

$$= d(x^2) - dx\,x = x\,dx$$

$$\mathcal{L}(y\partial_x)(x\,dy) = y\,dy + x\,0 = y\,dy$$

$$(d\,\cancel{\iota(y\partial_x)} + \iota(y\partial_x)\,d)(x\,dy) = \iota(y\partial_x)\,dx\,dy = y\,dy$$

$$\mathcal{L}(x\partial_x)(x\,dy) = x\,dy$$

$$\mathcal{L}(y\partial_y)(x\,dy) = 0\,dy + x\,\mathcal{L}(y\partial_y)\,dy = x\,dy.$$

$$\mathcal{L}(x\partial_x - y\partial_y)(x\,dy) = 0$$

Check. $\mathcal{L}(x\partial_y)\,\mathcal{L}(y\partial_x)(x\,dy) = \mathcal{L}(x\partial_y)\,y\,dy = x\,dy + y\,dx$

$\mathcal{L}(y\partial_x)\,\mathcal{L}(x\partial_y)(x\,dy) = \mathcal{L}(y\partial_x)\,x\,dx = y\,dx + x\,dy$

Summarize

$$\mathcal{L}(x\partial_y)(x\,dy) = x\,dx$$
$$\mathcal{L}(y\partial_x)(x\,dy) = y\,dy$$
$$\mathcal{L}(x\partial_x - y\partial_y)(x\,dy) = 0$$

Now you

apparently the vector field $x\partial_x - y\partial_y$ respects $D = d + 2\pi i x\,dy$

Good question: What is $e^{t(x\partial_x - y\partial_y)}$ applied to $f \in \Omega^0(\mathbb{R}^2)$ 9/11

$$e^{tx\partial_x} x = \sum \frac{t^n}{n!}(x\partial_x)^n x = e^t x$$

$$e^{t(x\partial_x - y\partial_y)}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} e^t x \\ e^{-t} y \end{pmatrix}$$

$sl(2,\mathbb{R})$ acts on $\Omega(\mathbb{R}^2)$ $\quad X \mapsto \mathcal{L}(X)$

has basis $x\partial_y, y\partial_x, x\partial_x - y\partial_y$

$$\mathcal{L}(x\partial_y)(d + 2\pi i x dy) = 2\pi i \underbrace{\mathcal{L}(x\partial_y)(xdy)}_{\cancel{\mathcal{L}(x\partial_y)x}\, dy + x\underbrace{\mathcal{L}(x\partial_y)dy}_{xdx}}$$

$$\mathcal{L}(y\partial_x)(d + 2\pi i x dy) = 2\pi i\Big[\underbrace{\mathcal{L}(y\partial_x)x}_{ydy}\, dy + x\,\cancel{\mathcal{L}(y\partial_x)dy}\Big]$$

$$\boxed{\begin{aligned}\mathcal{L}(x\partial_y)(d + 2\pi i x dy) &= 2\pi i(xdx)\\ \mathcal{L}(y\partial_x)(\text{———}) &= 2\pi i(ydy)\end{aligned}}$$

$$d\,\iota(x\partial_x)\,xdy + \iota(x\partial_x)\,dxdy = xdy$$

$$d\,\iota(y\partial_y)\,xdy + \iota(y\partial_y)\,dxdy = \boxed{d(yx) - dx\,y}$$

$$d\,i(x\partial_y)\,xdy + \iota(x\partial_y)\,dxdy$$

$$d(x^2) - xdx = xdx.$$

So what you want to adjust by a gauge transf.

~~$e^{2\pi i xy}(d + 2\pi i x dy)e^{-2\pi i xy} = d + 2\pi i x dy - 2\pi i d(xy) = d - 2\pi i y dx$~~

$$e^{-2\pi i \frac{x^2}{2}}(d + 2\pi i x dy)e^{2\pi i \frac{x^2}{2}} = d + 2\pi i x dy + 2\pi i x dx$$

$$\boxed{\begin{aligned}&\underbrace{\mathcal{L}(x\partial_y)(d + 2\pi i x dy)}_{2\pi i(xdx)} + \underbrace{[\pi i x^2, d + 2\pi i x dy]}_{-\pi i[d, x^2] = -2\pi i x dx.} = 0\\ &\mathcal{L}(y\partial_x)(d + 2\pi i x dy) + [\pi i y^2, d + 2\pi i x dy] = 0\\ &\mathcal{L}(x\partial_x - y\partial_y)(d + 2\pi i x dy) = 0\end{aligned}}$$

Review. plane $\mathbb{R}^2$, action of $SL(2,\mathbb{R}) \ltimes \mathbb{R}^2$

Lie algebra $\partial_x, \partial_y, x\partial_y, y\partial_x, x\partial_x - y\partial_y$

$\omega = 2\pi i\, dx dy$ invariant

connection $d + x\,dy$ is not invariant but can be corrected by gauge transf.

$$[x\partial_y, \partial_x] = -\partial_y \qquad [y\partial_x, \partial_x] = 0$$

$$[x\partial_y, \partial_y] = 0 \qquad [y\partial_x, \partial_y] = -\partial_x$$

$$[x\partial_y, x] = 0 \qquad [y\partial_x, x] = y$$

$$[x\partial_y, y] = x \qquad [y\partial_x, y] = 0$$

$$ad(x\partial_y)\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ x \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

$$ad(y\partial_x)\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

$$\left[\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right] = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\left[x\partial_x - y\partial_y, \begin{pmatrix} x \\ y \end{pmatrix}\right] = \begin{pmatrix} x \\ -y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

It's time now to calculate. Manifold $\mathbb{R}^2$, to be playing with operators on $C^\infty(\mathbb{R}^2)$, more generally, $\Omega(\mathbb{R}^2)$. x, y coord fns

If $X = f\partial_x + g\partial_y$ is a vector field on $\mathbb{R}^2$, then it acts on $\Omega(\mathbb{R}^2)$ by $\mathcal{L}(X)\xi = d\,\iota(X)\xi + \iota(X)\,d\xi$

trivial (complex) line bundle over $\mathbb{R}^2$

connection $d + A$ $\qquad A \in \Omega^1(\mathbb{R}^2)$ $\qquad dA = \omega = (2\pi i)dx\,dy$

Describe your aim. Take $A = x\,dy$ $\qquad d$

$$\mathcal{L}(a\partial_x + b\partial_y)\, dx\,dy = d\,\iota(a\partial_x + b\partial_y)\,dx\,dy = d(a\,dy - b\,dx) = 0.$$

$$\mathcal{L}(a\partial_x+b\partial_y)\,(d+x\,dy) = \begin{matrix} d\,\iota(a\partial_x+b\partial_y)\,x\,dy\ + \\ \iota(a\partial_x+b\partial_y)\,dx\,dy\ \cancel{+} \end{matrix}$$

$$= d(\cancel{bx}) + a\,dy - \cancel{b\,dx} \quad = a\,dy$$

$$e^{ay}(d+x\,dy)e^{-ay} = d - a\,dy + x\,dy$$

$$\left[\mathcal{L}(a\partial_x+b\partial_y) + \mathrm{Ad}(e^{ay})\right](d+x\,dy) = d+x\,dy$$

Aim: to link with what you did before.

sections $\mathcal{L} = \{\psi \in C^\infty(\mathbb{R}^2) \mid \text{autom. cond.}\}$? Ex.

$f \in \mathcal{S}(\mathbb{R})$ $$\tilde{f} = \sum_{m\in\mathbb{Z}} \cancel{e^{\partial_x}}\, e^{m(\partial_x + 2\pi i y)} f$$

$$\tilde{f}(x,y) = \sum_m e^{2\pi i m y} f(x+m)$$

$$= \sum_m e^{2\pi i m y} e^{m\partial_x} f(x) = \sum_m e^{m(\partial_x + 2\pi i y)} f(x).$$

Don't worry about notation for $\mathcal{S}(\mathbb{R})$, rather get the automorphy condition straight.

$$\boxed{e^{\partial_y}\tilde{f} = \tilde{f}} \qquad e^{\partial_x} e^{2\pi i x y}\tilde{f} = e^{2\pi i x y}\tilde{f}$$

$$e^{2\pi i (\cancel{x+1}) y} e^{\partial_x}\tilde{f} = \cancel{e^{2\pi i x y}}\tilde{f}$$

$$\boxed{e^{2\pi i y} e^{\partial_x}\tilde{f} = \tilde{f}}$$

$$\cancel{\psi(x+1,y) = e^{2\pi i y}\psi(x,y)}$$ Almost there.

$$\mathcal{L} = \{\psi \in C^\infty(\mathbb{R}^2) \mid e^{\partial_y}\psi = \psi,\quad e^{2\pi i y}e^{\partial_x}\psi = \psi\}$$

$$\tilde{f} = \sum_m e^{2\pi i y m} e^{m\partial_x} f = \sum_m e^{m(\partial_x + 2\pi i y)} f$$

$$\boxed{\partial_x\tilde{f} = \widetilde{\partial_x f}} \qquad \underset{2\pi i x\tilde{f}}{\partial_y\tilde{f}} = \sum_m e^{m\partial_x} e^{m 2\pi i y} \underset{+\,2\pi i x}{(2\pi i m)} f = \widetilde{(2\pi i x) f}$$