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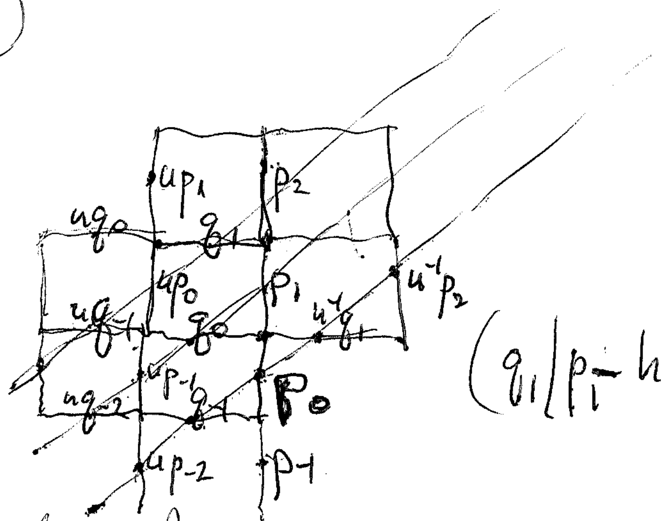

$$\begin{pmatrix} p_1 \\ q_0 \end{pmatrix} = \begin{pmatrix} k_1 & h_1 \\ -\bar h_1 & k_1 \end{pmatrix}\begin{pmatrix} up_0 \\ q_1 \end{pmatrix}$$

$$(q_1 | p_1 - h_1 q_1) = 0 \qquad p_1 - h_1 q_1 = k_1 u p_0$$

So I am looking a direct sum of 2x2 unitaries matrices of the form

$$\begin{pmatrix} k_n & h_n \\ -\bar h_n & k_n \end{pmatrix} = \begin{pmatrix} \sqrt{1-|h_n|^2} & h_n \\ -\bar h_n & \sqrt{1-|h_n|^2} \end{pmatrix}$$

ignore phase of h.

$$\begin{pmatrix} \sqrt{1-t^2} & t \\ -t & \sqrt{1-t^2} \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

$$= \exp \theta \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

I that $(h_n) \in \ell^2$ means that $\underline{u \text{ is } \equiv 1}$ modulo Hilbert Schmidt. No probably it means that $u \equiv$ standard u_0 mod Hilbert Schmidt where u_0 has all $h_n = 0$. What is u_0?

$up_n = p_{n+1} \quad \forall n \qquad p_n = u p_{n-1}$

$q_n = q_{n-1} \qquad q_n = q_{n-1}$

(31) It seems like you have

$$p_0 \xmapsto{u} u p_0 = p_1 = u^{-1} p_2$$

$$u^{-1} q_1 \xmapsto{u} q_1 = q_{-1}$$

$$u p_2 \quad q_{-1} \quad p_0 \quad u^{-1} q_1 \quad u^{-1} p_2$$

orthonormal basis

So it looks like we have a 2 step

So we seem to have a perturbation situation

u, u_0 are congruent modulo the ideal of Hilbert-Schmidt operators. A natural question is whether a good scattering situation arises. The existence of wave operators $\lim_{n\to\infty} u_0^{-n} u^n$. Look at the resolvents.

$$\lambda - u = (\lambda - u_0) - (u - u_0)$$

$$\frac{1}{\lambda - u} = \frac{1}{\lambda - u_0} + \frac{1}{\lambda - u_0}(u - u_0)\frac{1}{\lambda - u_0} + \cdots$$

Look at the continuous case. $\lambda \begin{pmatrix} p \\ q \end{pmatrix} = \underbrace{\begin{pmatrix} \partial_x & h \\ -h & -\partial_x \end{pmatrix}}_{D} \begin{pmatrix} p \\ q \end{pmatrix}$

here $h(x)$ is a $(\ell^2?)$ function of x.

$$\frac{1}{\lambda - D} = \frac{1}{\lambda - D_0} + \frac{1}{\lambda - D_0}\begin{pmatrix} 0 & h \\ h & 0 \end{pmatrix}\frac{1}{\lambda - D_0} + \cdot$$

We would like to understand the operator

$$\frac{1}{\lambda - D_0}\begin{pmatrix} 0 & h \\ h & 0 \end{pmatrix} \qquad \left(x \left| \frac{1}{\lambda - \partial_x} \right| x' \right) = \begin{cases} e^{\lambda(x-x')} & x > x' \\ 0 & x < x' \end{cases}$$

Put $\lambda = i\omega$. To solve

$$(-i\omega + \partial_x)\,\phi(x) = \delta(x) \qquad s = -i\omega$$

$$\phi(x) = \text{const}\; e^{i\omega x} \quad \text{for } x \neq 0.$$

$$\operatorname{Im}(\omega) > 0 \implies \phi(x) = \begin{cases} e^{i\omega x} & \text{for } x > 0 \\ 0 & \text{for } x < 0 \end{cases}$$

The Green's function is $\phi(x-x')$. We are interested in the kernel

$$K(x,x') = \phi(x-x')\,h(x')$$

$$\cancel{\iint dx\,dx' \; K(x,x')\,\overline{K(x',x)}}$$

$$\cancel{= \iint dx\,dx' \; \phi(x-x')\,h(x')\,\overline{\phi(x'-x)}\,\overline{h(x)}}$$

$$\cancel{= \iint dx\,dy \; \phi(y)\,\overline{\phi(-y)}\,h(x-y)\,\overline{h(x)}}$$

$y = x - x'$
$x' = x - y$

$$\operatorname{tr}(K^*K) = \int dx \int \underbrace{K^*(x,x')}_{\overline{K(x',x)}}\, K(x',x)\, dx'$$

$$= \iint dx\,dx' \; |K(x',x)|^2$$

$y = x - x'$
$x' = x - y$

$$\iint dx\,dx' \; |\phi(x-x')\,h(x')|^2 = \iint dx\,dy \; |\phi(y)|^2\,|h(x-y)|^2$$

(33) $\mathrm{tr}(K^*K) = \iint dx\,dx' \left|\phi(x-x')\,h(x')\right|^2$

$$\cancel{= \int_{-\infty}^{\infty} |h(x)|^2 dx \int}$$

$y = x - x'$

$dy = dx$

$$= \int_{-\infty}^{\infty} dx' \,|h(x')|^2 \int_{-\infty}^{\infty} dx \,|\underbrace{\phi(x-x')}|^2$$

$e^{-s(x-x')}$ $\quad x > x'$

0 $\quad x < x'$

$$= \int dy\, |\phi(y)|^2 = \int_0^{\infty} dy\; e^{\overset{-\mathrm{Im}(\omega)}{\overset{\shortparallel}{2\mathrm{Re}(i\omega y)}}}$$

$$= \frac{1}{2\,\mathrm{Im}(\omega)}$$

So K is Hilbert-Schmidt $\Longleftrightarrow$ $h(x) \in L^2$

Review: Consider 1 dim Dirac Equation

$$\begin{pmatrix} p_n \\ q_n \end{pmatrix} = \frac{1}{k_n}\begin{pmatrix} 1 & h_n \\ \bar{h}_n & 1 \end{pmatrix}\begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix}\begin{pmatrix} p_{n-1} \\ q_{n-1} \end{pmatrix}$$

$$\partial_x \begin{pmatrix} p_x \\ q_x \end{pmatrix} = \begin{pmatrix} i\lambda & h_x \\ \bar{h}_x & -i\lambda \end{pmatrix}\begin{pmatrix} p_x \\ q_x \end{pmatrix} \qquad \Big(\partial_x \quad -h_x$$

$$\begin{pmatrix} i\lambda & 0 \\ 0 & -i\lambda \end{pmatrix}\begin{pmatrix} p \\ q \end{pmatrix} = \Big(\partial_x p \qquad ?$$

$$\begin{pmatrix}\partial_x & -h_x\\ +\bar{h}_x & -\partial_x\end{pmatrix}\begin{pmatrix}p\\ q\end{pmatrix} = i\lambda\begin{pmatrix}p\\ +q\end{pmatrix}$$

$$\underset{\substack{\uparrow\\ \text{should be}\\ \omega,\,k\,?}}{i\lambda}\,\phi = \underbrace{\begin{pmatrix}\partial_x & -h\\ \bar{h} & -\partial_x\end{pmatrix}}_{\text{skew adjoint}}\phi$$

$$\begin{pmatrix}\partial_x - i\omega & h\\ \bar{h} & \partial_x + i\omega\end{pmatrix}\phi = 0.$$

~~[illegible]~~ You are trying to understanding this Dirac equation, the scattering.

~~You need~~ ~~[illegible]~~ ~~to pass between~~

You have to recall the eigenvector picture for scattering. You have a wave eqn.

~~[illegible]~~ $\partial_t \psi = X\psi$ with $X^* = -X$

$\psi = e^{-i\omega t}\phi$ $\qquad \overset{\lambda}{-i\omega}\,\phi = X\phi$

$$\frac{1}{\lambda - X} = \frac{1}{\lambda - X_0} + \frac{1}{\lambda - X_0}V\frac{1}{\lambda - X_0} + \cdots$$

So you have a wave equation ~~with particles~~ on the line where the field has two components, one right moving, the other left moving. And there's a perturbation h, reflection between the two. You want $G = \frac{1}{\lambda - D}$

where $D = D_0 + V$. Then formally.

$$\frac{1}{\lambda - D} = \frac{1}{\lambda - D_0 - V} = \frac{1}{\lambda - D_0} + \frac{1}{\lambda - D_0} V \frac{1}{\lambda - D_0} + \cdots$$

$$G = G_0 + G_0 V G_0 + (G_0 V)^2 G_0 + \cdots$$

$$= \frac{1}{1 - G_0 V} G_0$$

$$D_0 = \begin{pmatrix} \partial_x & 0 \\ 0 & -\partial_x \end{pmatrix} \qquad V = \begin{pmatrix} 0 & h \\ -\bar{h} & 0 \end{pmatrix}$$

$$\lambda - D_0 = \begin{pmatrix} \lambda - \partial_x & 0 \\ 0 & \lambda + \partial_x \end{pmatrix}$$

$\lambda \notin i\mathbb{R}$

$$G_0 = \begin{pmatrix} \frac{1}{\lambda - \partial_x} & 0 \\ 0 & \frac{1}{\lambda + \partial_x} \end{pmatrix}$$

$$\left(x \left| \frac{1}{\partial_x + \lambda} \right| 0 \right) = \begin{cases} e^{-\lambda x} & x > 0 \\ 0 & x < 0 \end{cases}$$

Assume $Re(\lambda) > 0$

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$$\left(x \middle| \frac{1}{-\partial_x+\lambda} \middle| 0\right) = \begin{cases} 0 & x>0 \\ e^{\lambda x} & x<0 \end{cases}$$

$$G_0 V = \begin{pmatrix} \frac{1}{\lambda-\partial_x} & \\ & \frac{1}{\lambda+\partial_x} \end{pmatrix} \begin{pmatrix} & h \\ -\bar{h} & \end{pmatrix}$$

$$= \begin{pmatrix} 0 & \frac{1}{\lambda-\partial_x} h \\ -\frac{1}{\lambda+\partial_x}\bar{h} & 0 \end{pmatrix}$$

The Hilbert Schmidt norm of an operator is the l^2 norm of ~~the~~ its matrix components.

$$K(x,x') = \begin{cases} e^{\lambda(x-x')} h(x') & x<x' \\ 0 & x>x' \end{cases}$$

$$\iint_{x<x'} |e^{\lambda(x-x')}|^2 |h(x')|^2 \, dx\, dx'$$

$$\parallel$$

$$\underbrace{\int_{-\infty}^{\infty} dx' \, |h(x')|^2}_{\|h\|^2} \underbrace{\int_{-\infty}^{x'} e^{2\mathrm{Re}(\lambda)(x-x')} dx}_{\int_0^\infty e^{-2\mathrm{Re}(\lambda) y} dy \,=\, \frac{1}{2\mathrm{Re}(\lambda)}}$$

(37) The wave operators intertwine u and u_0. If $W_+ = \lim_{n\to+\infty} u_0^{-n} u^n$ $\exists$

then $u_0^{-1} W_+ u = \lim_{n\to+\infty} u_0^{-n-1} u^{n+1} = W_+$. But the problem is that $W_\pm$ might fail to be unitary. If you have a unitary intertwining u and u_0, then it intertwines the resolvents.

Return to $X = X_0 + V$ $\qquad u^t = e^{tX}$,

$u^t = e^{tX} = e^{t(X_0+V)}$, $\qquad u_0^{-t} u^t = e^{-tX_0} e^{t(X_0+V)}$,

what to do? What is the scattering operator?

$$W_+ = \lim_{t\to+\infty} e^{-tX_0} e^{tX}$$

$$W_- = \lim_{t\to-\infty} e^{-tX_0} e^{tX}$$

$$W_+ W_-^{-1} = \lim_{\substack{\longrightarrow \\ t_1\to+\infty}} \lim_{\substack{\longrightarrow \\ t_2\to-\infty}} e^{-t_1X_0} e^{t_1X} e^{-t_2X} e^{+t_2X_0}$$

$$= \lim_{t\to+\infty} e^{-tX_0} e^{+2tX} e^{-tX_0}$$

Formula for W_+?

$$e^{tX} = e^{tX_0} + \int_0^t dt_1\, e^{(t-t_1)X_0} V e^{t_1X_0} + \cdots$$

$$e^{-tX_0} e^{tX} = 1 + \int_0^t dt_1\, e^{-t_1X_0} V e^{t_1X_0} +$$

$$+ \iint_{0<t_1<t_2<t} e^{(-t_2)X_0} V e^{(t_2-t_1)X_0} V e^{(t_1)X_0}\, dt_1\, dt_2$$

(45) Check
$$(z^k, g_n) = c^2\left(-\frac{s^2}{c^2+s^2 n}\right) + s^2\left(1 - \frac{s^2}{c^2+s^2 n} n\right)$$
$$= \frac{1}{c^2+s^2 n}\left(-c^2 s^2 + s^2\left(c^2 + \cancel{s^2 n} - \cancel{s^2 n}\right)\right) = 0.$$

$$(g_n | g_n) = (1 | g_n) = c^2 1 + s^2 1\left(1 - \frac{s^2}{c^2+s^2 n} n\right)$$
$$= \frac{1}{c^2+s^2 n}\left(c^2 + s^2\left(c^2 + s^2 n - s^2 n\right)\right)$$
$$= \frac{c^2(1+s^2)}{c^2+s^2 n} = \frac{(1-s^2)(1+s^2)}{1-s^2+s^2 n}$$

I think that
$$\tilde{g}_n = 1 - \frac{s^2}{c^2+s^2 n}(z + \cdots + z^n)$$
then
$$\tilde{p}_n = z^n - \frac{s^2}{c^2+s^2 n}(z^{n-1} + \cdots + 1)$$

$$\begin{pmatrix} \tilde{p}_n \\ \tilde{g}_n \end{pmatrix} = \begin{pmatrix} 1 & h_n \\ \bar{h}_n & 1 \end{pmatrix} \begin{pmatrix} z\tilde{p}_{n-1} \\ \tilde{g}_{n-1} \end{pmatrix}$$

$$\tilde{g}_n - \tilde{g}_{n-1} = \bar{h}_n z \tilde{p}_{n-1}$$
$$\|$$
$$-\frac{s^2}{c^2+s^2 n}(z + \cdots + z^n) + \frac{s^2}{c^2+s^2(n-1)}(z + \cdots + z^{n-1})$$

so it appears that $\bar{h}_n = \dfrac{-s^2}{c^2+s^2 n}$

(46) ~~Let's look a $d\mu$ of the form~~

Begin with

$L^2(S^1, d\mu)$ assume $(h_n)_{n\geq 1}$ ℓ^2

so get $g_\infty \in H^2(S^1, d\mu) \ominus z H^2(S^1, d\mu)$.

You know $|g_\infty|^2 \, d\mu = \frac{d\theta}{2\pi}$.

Assume $d\mu$ abs. cont ~~wrt the~~ wrt Leb.

$$\therefore \; d\mu = \frac{1}{|g_\infty|^2} \frac{d\theta}{2\pi}$$

Better might be to begin with $\rho \frac{d\theta}{2\pi}$ and ask when $\rho = \frac{1}{|g_\infty|^2}$ on S^1, g_∞ analytic on D non vanishing. Look at $-\log \rho$ harmonic on disk. Look for simplest cases.

Suppose $d\mu = \rho \frac{d\theta}{2\pi}$ $\rho(\theta) > 0$ on S^1.

$-\log \rho$ real function on S^1. When is it real part of an analytic fn. on D.

Begin with simplest ~~situation~~ of the moment problem on S^1. Take $d\mu = \rho \frac{d\theta}{2\pi}$ ρ smooth >0.

$-\ln \rho$ smooth $\mathbb{R}$-valued on S^1, so get

$-\ln \rho = g(z) + \overline{g(z)}$ $g(z)$ smooth on S^1 extending anal ~~to~~ to D.

(47) g unique up to $i\mathbb{R}$. Then

$$\rho = \frac{1}{|q|^2} \qquad q = e^{g} \quad \text{smooth on } \bar{D}$$
anal. inside.

Then $d\mu = \dfrac{1}{|q|^2}\dfrac{d\theta}{2\pi}$

$$L^2(S^1, d\mu) \qquad\qquad L^2\left(S^1, \frac{d\theta}{2\pi}\right)$$

You need to know that $q\, H^+ = H^+$.
You know that $q = e^{g}$ smooth on $\bar{D}$
analytic in D. So q, q^{-1} both
map H^+ to H^+.

$$L^2\left(S^1, \frac{d\theta}{2\pi}\right) \xrightarrow{\bar{q}\,\cdot} L^2(S^1, d\mu) \xleftarrow[\sim]{q\,\cdot} L^2\left(S^1, \frac{d\theta}{2\pi}\right)$$
$$S = \frac{\bar{q}}{q}$$

inside $L^2(S^1, d\mu)$ $\qquad q = \xi_-, \ \bar{q} = \xi_+$

$$F_{mn} = \left(u^{<-m}\xi_-\right)^{\perp} \cap \left(u^{>n}\xi_+\right)^{\perp}$$
$$\longrightarrow \left(z^{-m}H_-\right)^{\perp} \cap \left(z^{n+1}\frac{\bar{q}}{q}H_+\right)^{\perp}$$
$$= z^{-m}H_+ \cap z^{n+1}\frac{\bar{q}}{q}H_-$$
$$\xrightarrow{\sim} z^{-m}qH_+ \cap z^{n+1}\bar{q}H_- \qquad \{1, z, \dots, z^{m+n}\}.$$
$$= z^{-m}H_+ \cap z^{n+1}H_- \xrightarrow{\sim} \overbrace{H_+ \cap z^{m+n+1}H_-}$$

(48) Inverse scattering problem.

In the case of $d\mu = \rho \frac{d\theta}{2\pi}$ ρ smooth, >0,

write $-\log \rho = g + \bar{g}$ g extends smoothly to analytic fn. in $\mathbb{D}$

$q = e^{g}$ q smooth on $\bar{D}$ analytic + nonvanishing on D

Then in $L^2(S^1, d\mu)$ you have $\xi_+ = q$, $\xi_- = \bar{q}$

sat $\xi_+^* z^n \xi_+ = \int \bar{q} z^n q \frac{1}{|q|^2} \frac{d\theta}{2\pi} = \delta_n$, also for q_-

and

$$F_{mn} = \left(z^{<-m}\xi_+\right)^\perp \cap \left(z^{>n}\xi_+\right)^\perp \quad \text{in } L^2(S^1, d\mu)$$

$$\xrightarrow{\sim} \left(z^{-m} H_-\right)^\perp \cap \left(z^{n+1}\frac{\bar{q}}{q} H_+\right)^\perp \quad L^2(S^1, d\mu) \text{ where } \xi_- = 1$$

$$= z^{-m} H_+ \cap z^{n+1}\frac{\bar{q}}{q} H_-$$

$$= \frac{1}{q}\Big(\underbrace{z^{-m} H_+ \cap z^{n+1} H_-}_{\mathbb{C} z^{-m} + \cdots + \mathbb{C} z^n}\Big)$$

Next I want the 2 sided version.

So suppose you ~~take~~ have $(h_n)_{n\in\mathbb{Z}}$

and form the corresponding $(H, 4, p_{mn}, q_{mn})$.

Assume (h_n) in ℓ^2 whence you get

4 ξ's in and out, left and right

(49) How to organize? ~~...~~ Consider case finitely ~~...~~ many $h_n \neq 0$. Scattering ~~...~~ matrix 2×2 unitary, and it's equivalent to a tranfer matrix from the left to the right. Guess what happens is that the two in ξ's gives $H \simeq L^2(S^1)^{\oplus 2}$ and there's a corresp. out rep. So in fact we have ~~...~~ a vector version.

Need an example. Take a single non zero h, say h_1. Better suppose $h_n = 0$ for $|n| \geq N$. The recursion relations

$$\begin{pmatrix} p_n \\ q_n \end{pmatrix} = \frac{1}{k_n}\begin{pmatrix} 1 & h_n \\ \bar{h}_n & 1 \end{pmatrix}\begin{pmatrix} u p_{n-1} \\ q_{n-1} \end{pmatrix}$$

become

$$p_n = u p_{n-1} \qquad \text{when } h_n = 0.$$
$$q_n = q_{n-1}$$

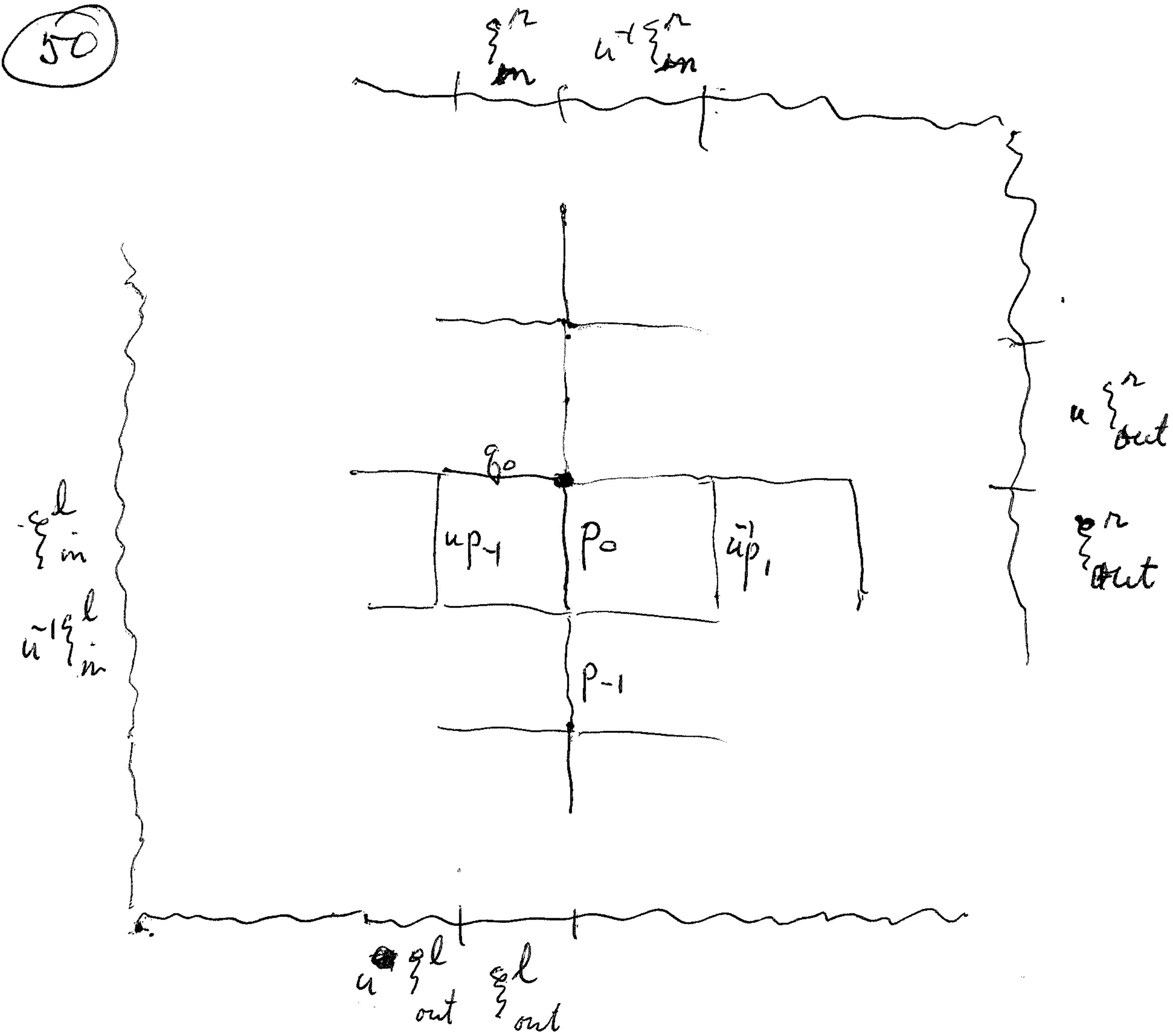
So if $h_n = 0$ for $|n| \geq N$, then we have

q_n indep. of n for $n \gg 0$ and for $n \ll 0$

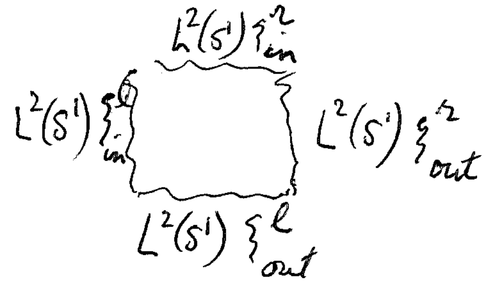
$u^{-n} p_n$ ———— ————.

so we have 4 unit vectors defined.

Draw picture

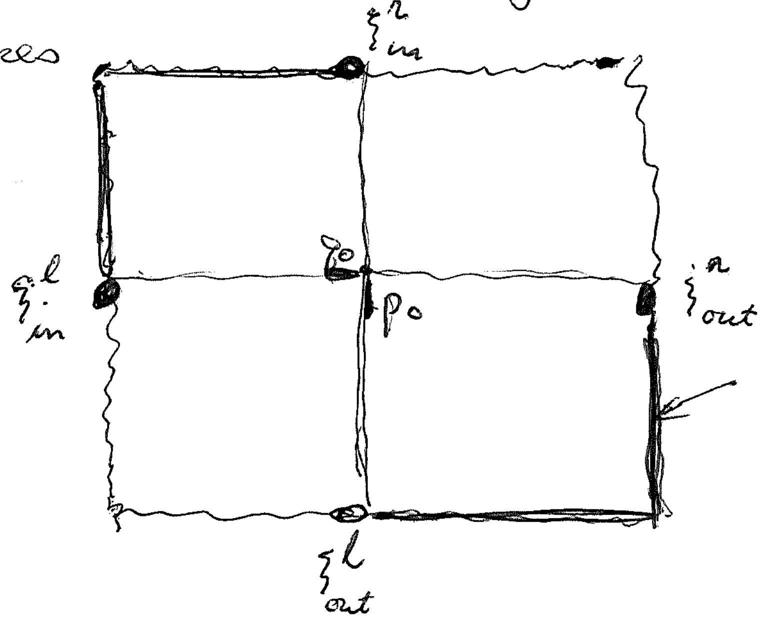


You can see the scattering matrix – it's the square

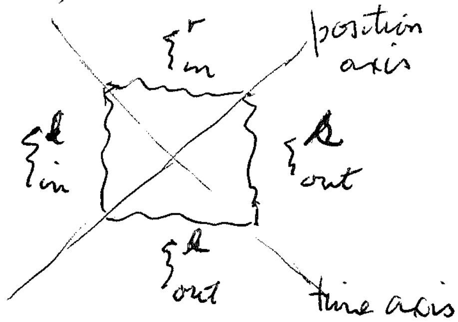


Idea causality and light cone idea, this is the ~~statement that~~ orthogonality ~~statement~~ for the p_{mn}, q_{mn}. If you are at a point mn then ~~the~~ elts in the forward cone ~~and elts~~ are orthogonal to elements in the backwards cone

(51) Pick ~~the~~ incoming and outgoing subspaces



Anyway you ~~seem to~~ have this S matrix, 2×2 unitary values in $L^\infty(S^1)$



Example.

$$\begin{pmatrix} p_n \\ q_n \end{pmatrix} = \frac{1}{k_n}\begin{pmatrix} 1 & h_n \\ \bar h_n & 1 \end{pmatrix}\begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} p_{n-1} \\ q_{n-1} \end{pmatrix}$$

$$\begin{pmatrix} p_x \\ q_x \end{pmatrix} = \frac{1}{k_{x\Delta x}}\begin{pmatrix} 1 & h_x\Delta x \\ \bar h_x\Delta x & 1 \end{pmatrix}\begin{pmatrix} z^{\Delta x} & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} p_{x-\Delta x} \\ q_{x-\Delta x} \end{pmatrix}$$

$$\frac{z^{\Delta x}-1}{\Delta x} \longrightarrow \log z = ik.$$

$$\partial_x \begin{pmatrix} p_x \\ q_x \end{pmatrix} = \begin{pmatrix} ik & h_x \\ \bar h_x & 0 \end{pmatrix}\begin{pmatrix} p_x \\ q_x \end{pmatrix}$$

If $h=0$, get

$$\begin{pmatrix} \partial_x - ik & 0 \\ 0 & \partial_x \end{pmatrix}\begin{pmatrix} p \\ q \end{pmatrix} = 0$$

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so $\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} Ae^{ikx} \\ B \end{pmatrix}$

$$\begin{pmatrix} \partial_x - ik & 0 \\ 0 & \partial_x \end{pmatrix}\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} 0 & h \\ \bar{h} & \partial \end{pmatrix}\begin{pmatrix} p \\ q \end{pmatrix}$$

Set up an integral equation ~~from~~ which will give the solution with the the desired boundary conditions.

$$(\partial_x - ik)^{-1}(x,0) = \begin{cases} e^{ikx} & x>0 \\ 0 & x<0 \end{cases} \quad \text{for } \operatorname{Im}(k)>0$$

$$= \begin{cases} 0 & x>0 \\ -e^{ikx} & x<0 \end{cases} \quad \text{for } \operatorname{Im}(k)<0$$

So we have a puzzle with ∂_x^{-1}

Actually you probably choose what ~~the~~ the function according to the four boundary conditions.

$$\begin{pmatrix} \partial_x - ik & 0 \\ 0 & \partial_x \end{pmatrix}\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} 0 & h \\ \bar{h} & 0 \end{pmatrix}\begin{pmatrix} p \\ q \end{pmatrix}$$

$$D_0 \psi = V \psi$$

?

$G_0(D_0 - V)\psi$

(53)

~~To st~~ To solve $D_0\psi = \gamma$. Suppose $D_0 G_0 = 1$, then

$$D_0(\psi - G_0\gamma) = \gamma - \gamma = 0.$$

Thus $\psi = G_0\gamma$ is a solution and we can make ψ satisfy given b.c.'s by requiring G_0 to.

Now consider solving $(D_0 - V)\psi = 0$ with given b.c.'s. Let $G_0 \ni D_0 G_0 = 1$ & G_0 sats given b.c.'s ~~Then~~ Suppose can solve $(1 - G_0 V)\psi = 0$. Then $\psi = G_0 V\psi$ so ψ sats given b.c. Also

$$D_0\psi = D_0 G_0 V\psi = V\psi.$$

So to solve $(D_0 - V)\psi = 0$ with a given b.c. you first solve $D_0 G_0 = 1$ with this b.c. and then solve $(1 - G_0 V)\psi = 0$.

Next you want ~~[illegible]~~ wave operators - intertwining ~~[illegible]~~ operators. Can you construct intertwining operators for D_0 and $D_0 - V$?

Suppose $T D_0 = (D_0 - V)T$, i.e.

$$[D_0, T] = VT$$

(54) Presumably this leads to Dyson's expansion. Thus if

$$W_+ = \lim_{t\to\infty} e^{-tD_0} e^{t(D)} \qquad \exists$$

Then $e^{-sD_0} W_+ = \lim_{t\to\infty} e^{-(s+t)D_0} e^{tD}$ $\qquad t' = s+t$, $t = t'-s$

$$= \lim_{t'\to\infty} e^{-t'D_0} e^{(t'-s)D}$$

$$= \underline{\lim_{t'\to\infty} e^{-t'D_0} e^{t'D}}\; e^{-sD} = W_+ e^{-sD}$$

$$TD_0 = DT \quad ?$$

$$GT = TG_0$$

$z^n z^{-n+1}$

$$\begin{pmatrix} z^{-n} p_n \\ q_n \end{pmatrix} = \frac{1}{k_n} \begin{pmatrix} z^{-n} & z^{-n} h_n \\ \bar{h}_n & 1 \end{pmatrix} \begin{pmatrix} z\, p_{n-1} \\ q_{n-1} \end{pmatrix}$$

$$= \frac{1}{k_n} \begin{pmatrix} 1 & z^{-n} h_n \\ z^n \bar{h}_n & 1 \end{pmatrix} \begin{pmatrix} z^{-n+1} p_{n-1} \\ q_{n-1} \end{pmatrix}$$

Thus we have a composition. You in a group sort of $SU(1,1)$, but over Laurent series, i.e. matrix functions of z. So what convergence results can you expect?

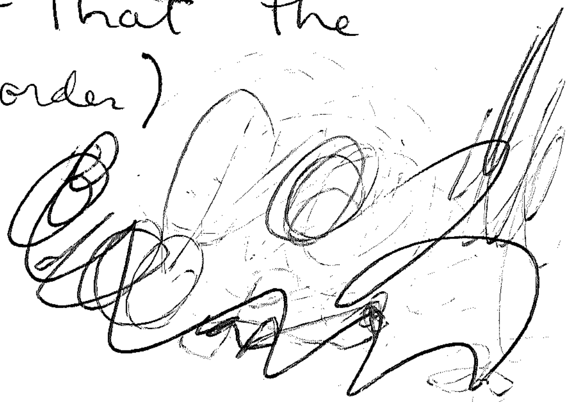
Thus you work with 2×2 matrices of functions on S^1

$$\begin{pmatrix} p_0 \\ q_0 \end{pmatrix} = \frac{1}{k_1}\begin{pmatrix} 1 & -h_1 z^{-1} \\ -\bar{h}_1 z & 1 \end{pmatrix}\begin{pmatrix} z^{-1} p_1 \\ q_1 \end{pmatrix}$$

$$= \frac{1}{k_1}\begin{pmatrix} 1 & -h_1 z^{-1} \\ -\bar{h}_1 z & 1 \end{pmatrix}\frac{1}{k_2}\begin{pmatrix} 1 & -h_2 z^{-2} \\ -\bar{h}_2 z^2 & 1 \end{pmatrix}\begin{pmatrix} z^{-2} p_2 \\ q_n \end{pmatrix}$$

Basically you want to know that the infinite matrix product (in order)

$$\prod_{n=1}^{\infty} \frac{1}{k_n}\begin{pmatrix} 1 & h_n z^{-n} \\ -\bar{h}_n z^n & 1 \end{pmatrix}$$



converges in this $SU(1,1)(z)$ group.

Question: Fix z on S^1, ask about convergent inf. product expansions in $SU(1,1) \simeq SL_2(\mathbb{R})$. It should be true that ~~[illegible]~~ absolute convergence ~~[illegible]~~ in the Lie algebra should be sufficient. i.e. Given $g_n \in G$ such that $\sum_n \|g_n - 1\| < \infty$, then

$$\lim_{n\to\infty} \prod_{j=1}^{n} g_j \quad \exists.$$

$$\prod_{n=1}^{\infty} (1 + x_n)$$

$$\prod_{n=1}^{\infty} k_n^{-1}\begin{pmatrix}1 & h_n z^{-n}\\ -\bar{h}_n z^n & 1\end{pmatrix}$$

$$T_n = k_n^{-1}\begin{pmatrix}1 & h_n z^{-n}\\ \bar{h}_n z^n & 1\end{pmatrix} T_{n-1} \qquad n \geq 1.$$

$$T_0 = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} \qquad \in \begin{pmatrix}F_0 & 0\\ 0 & F_0\end{pmatrix}$$

$$T_1 = k_1^{-1}\begin{pmatrix}1 & h_1 z^{-1}\\ \bar{h}_1 z & 1\end{pmatrix} \in \begin{pmatrix}F_0 & \bar{F}_1\\ F_1 & F_0\end{pmatrix}$$

$$T_2 = k_2^{-1}k_1^{-1}\begin{pmatrix}1 & h_2 z^{-2}\\ \bar{h}_2 z^2 & 1\end{pmatrix}\begin{pmatrix}1 & h_1 z^{-1}\\ \bar{h}_1 z & 1\end{pmatrix}$$

$$\cancel{\in \begin{pmatrix}1 & h_2 z^{-2}\\ \bar{h}_2 z^2 & 1\end{pmatrix}\begin{pmatrix}F_0 & \bar{F}_1\\ F_1 & F_0\end{pmatrix}}$$

$$= \begin{pmatrix}1 + h_2 \bar{h}_1 z^{-1} & h_1 z^{-1} + h_2 z^{-2}\\ \bar{h}_2 z^2 + \bar{h}_1 z & 1 + \bar{h}_2 h_1 z\end{pmatrix}$$

$$T_3 \quad \begin{pmatrix}1 & h_3 z^{-3}\\ \bar{h}_3 z^3 & 1\end{pmatrix}\begin{pmatrix} & \\ & \end{pmatrix}$$

$$\begin{pmatrix}\langle 1, z^{-1}\rangle & \langle z^2, z, z, 1\rangle\\ \langle z^3, z^2, z\rangle & \end{pmatrix}$$

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$$T_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$T_1 = k_1^{-1} \begin{pmatrix} 1 & h_1 z^{-1} \\ \bar{h}_1 z & 1 \end{pmatrix} \in \begin{pmatrix} \langle 1 \rangle & \langle z^{-1} \rangle \\ \langle z \rangle & \langle 1 \rangle \end{pmatrix}$$

$$T_2 \in \begin{pmatrix} \langle 1 \rangle & \langle z^{-2} \rangle \\ \langle z^2 \rangle & \langle 1 \rangle \end{pmatrix} \begin{pmatrix} \langle 1 \rangle & \langle z^{-1} \rangle \\ \langle z \rangle & \langle 1 \rangle \end{pmatrix}$$

$$\begin{pmatrix} 1 & z^{-3} \\ z^3 & 1 \end{pmatrix} \begin{pmatrix} \langle 1, z^{-1} \rangle & \langle z^{-1}, z^{-2} \rangle \\ \langle z^2, z \rangle & \langle z, 1 \rangle \end{pmatrix}$$

$$\begin{pmatrix} 1 & \langle z^4 \rangle \\ \langle z^4 \rangle & 1 \end{pmatrix} \begin{pmatrix} \langle 1, z^{-1}, z^{-2} \rangle & \langle z^{-1}, z^{-2}, z^{-3} \rangle \\ \langle z^3, z^2, z \rangle & \langle z^2, z, 1 \rangle \end{pmatrix}$$

$$\begin{pmatrix} \langle 1, z^{-1}, z^{-2}, z^{3} \rangle & \langle z^{-1}, z^{-2}, z^{-3}, z^{-4} \rangle \\ \langle z^4, z^3, z^2, z \rangle & \langle z^3, z^2, z, 1 \rangle \end{pmatrix}$$

$$T_n \in \begin{pmatrix} \overline{F_{n-1}} & \overline{z F_{n-1}} \\ z F_{n-1} & F_{n-1} \end{pmatrix}$$

$z^{n+1} z^{-1} \langle 1, z^{-1}, \dots, z^{-n+1} \rangle$

$\langle z^n, z^{n-1}, \dots, z \rangle$

$\parallel$

$z F_{n-1}$

$$T_{n+1} \in \begin{pmatrix} 1 & \langle z^{-n-1} \rangle \\ \langle z^{n+1} \rangle & 1 \end{pmatrix} \begin{pmatrix} \overline{F_{n-1}} & \overline{z F_{n-1}} \\ z F_{n-1} & F_{n-1} \end{pmatrix}$$

$$\subset \begin{pmatrix} & \\ & F_{n-1} + z F_{n-1} \end{pmatrix}$$

$$T_{n+1} \in \begin{pmatrix} 1 & \langle z^{-n-1} \rangle \\ \langle z^{n+1} \rangle & 1 \end{pmatrix} \begin{pmatrix} \langle z^{-n+1}, \dots, z^{-1}, 1 \rangle & \langle z^{-1}, \dots, z^{-n} \rangle \\ \langle z, \dots, z^n \rangle & \langle 1, \dots, z^{n-1} \rangle \end{pmatrix}$$

$$\in \begin{pmatrix} \langle 1, z^{-1}, \dots, z^{-n+1}, z^{-n} \rangle & \langle z^{-1}, \dots, z^{-n-1} \rangle \\ \langle z, \dots, z^n, z^{n+1} \rangle & \langle 1, z, \dots, z^n \rangle \end{pmatrix}$$

So if T_∞ exists it has the form

$$\begin{pmatrix} \bar{d} & \bar{c} \\ c & d \end{pmatrix}$$

$c(z)$ analytic in D

$c(0) = 0$.

$d(z)$ analytic in D

$d(0) = (\prod k_n)^{-1}$

$$|d|^2 - |c|^2 = 1.$$

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$m < n$

$$\begin{pmatrix} 1 & h_n z^{-n} \\ \bar{h}_n z^n & 1 \end{pmatrix}\begin{pmatrix} 1 & h_{n-1} z^{-n+1} \\ \bar{h}_{n-1} z^{n-1} & 1 \end{pmatrix} \cdots \begin{pmatrix} 1 & h_m z^{-m} \\ \bar{h}_m z^m & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & z^{-m+1} \\ z^{m-1} & 1 \end{pmatrix}\begin{pmatrix} 1 & z^{-m} \\ z^{m} & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & z^{-m+2} \\ z^{m-2} & 1 \end{pmatrix}\begin{pmatrix} 1, z & z^{-m}, z^{-m+1} \\ z^{m-1}, z^{m} & 1, z^{-1} \end{pmatrix}$$

not correct

$$\begin{pmatrix} 1, z, z^2 & z^{-m}, z^{-m+1}, z^{-m+2} \\ z^{m-2}, z^{m-1}, z^{m} & 1, z^{-1}, z^{-2} \end{pmatrix}$$

$$\begin{pmatrix} 1 & z^{-m-1} \\ z^{m+1} & 1 \end{pmatrix}\begin{pmatrix} 1 & z^{-m} \\ z^{m} & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & z^{-m-2} \\ z^{m+2} & 1 \end{pmatrix}\begin{pmatrix} 1, z^{-1} & z^{-m}, z^{-m-1} \\ z^{m}, z^{m+1} & 1, z \end{pmatrix}$$

$$\begin{pmatrix} 1, z^{-1}, z^{-2} & \\ z^{m}, z^{m+1}, z^{m+2} & 1, z, z^2 \end{pmatrix}$$

Conclude

$$\begin{pmatrix}1 & h_n z^{-n}\\ \bar h_n z^n & 1\end{pmatrix}\cdots\cdots\begin{pmatrix}1 & h_m z^{-m}\\ \bar h_m z^m & 1\end{pmatrix}$$

$$\in \begin{pmatrix} 1, z^{-1},\dots, z^{-(n-m)} & z^{-n},\dots,z^{-m}\\ z^n,\dots,z^m & 1,z,\dots,z^{n-m}\end{pmatrix}$$

$$\cancel{z^{n+1}\begin{pmatrix}(1, z^{-1},\dots,z^{-n+m})\\ (z^n,\dots,z^m)\end{pmatrix} = \begin{pmatrix}(z^{n+1},\dots,z^{m+1})\\ (z^n,\dots,z^m)\end{pmatrix}}$$

Go back to continuous case

$$\partial_x\begin{pmatrix}p\\q\end{pmatrix} = \begin{pmatrix} ik & h\\ \bar h & 0\end{pmatrix}\begin{pmatrix}p\\q\end{pmatrix}$$

propagation given by time ordered exponential

$$e^{T\int \begin{pmatrix} ik & h_x\\ \bar h_x & 0\end{pmatrix}dx}$$

except you want to replace p_x by $e^{-ikx}p_x$

$$\partial_x\begin{pmatrix}e^{-ikx}p_x\\ q_x\end{pmatrix} = \begin{pmatrix} -ik\cancel{e^{-ikx}}p_x + e^{-ikx}(i\cancel{k}p + hq)\\ \bar h e^{ikx}e^{-ikx}p\end{pmatrix}$$

$$= \begin{pmatrix}0 & h_x e^{-ikx}\\ \bar h_x e^{ikx} & 0\end{pmatrix}\begin{pmatrix}e^{-ikx}p_x\\ q_x\end{pmatrix}$$

$$\begin{pmatrix}e^{-ika}p_a\\ q_a\end{pmatrix} = e^{T\int_a^b \begin{pmatrix}0 & h_xe^{-ikx}\\ \bar h_x e^{ikx} & 0\end{pmatrix}dx}\begin{pmatrix}e^{-ikb}p_b\\ q_b\end{pmatrix}$$

(61) Guess

$$e^{T\int_a^b \begin{pmatrix} 0 & h_x e^{-ikx} \\ \bar{h}_x e^{ikx} & 0 \end{pmatrix} dx} = \begin{pmatrix} \bar{d} & \bar{c} \\ c & d \end{pmatrix}$$

where $c = \int_a^b \bar{h}_x e^{ikx} dx + \text{3rd order}$

$$d = 1 + \int_a^b dx_1 \int_{x_1}^b dx_2 \; \bar{h}_{x_1} e^{ikx_1} h_{x_2} e^{-ikx_2}$$

3rd order in c $= \int_a^b dx_1 \int_{x_1}^b dx_2 \int_{x_2}^b dx_3 \; \bar{h}_{x_1} e^{ikx_1} h_{x_2} e^{-ikx_2} \bar{h}_{x_3} e^{ikx_3}$

$a \le x_1 \le x_2 \le x_3 \le b$

This seems correct but it looks strange

$$e^{ik(x_1 - x_2 + x_3)}$$

First order in c. $\int_a^b \bar{h}_x e^{ikx} dx \qquad e^{ikx} \quad x \in [a,b]$

2nd $\iint \cdots e^{ik(x_1 - x_2)}$

$a \le x_1 \le x_2 \le b \qquad a - b \le x_1 - x_2 \le 0$

You've reversed the order of matrix mult.

$a \le x_1 \le x_2 \le x_3 \le b$

$a \le x_1 - x_2 + x_3 \le b$

So what should happen is that the propagator from $x=a$ to $x=b$

62 $\Phi_{b,a}$: $\begin{pmatrix} e^{-ikb} p_b \\ q_b \end{pmatrix} = \Phi_{b,a} \begin{pmatrix} e^{-ika} p_a \\ q_a \end{pmatrix}$

should have the form $\begin{pmatrix} \bar d & \bar c \\ c & d \end{pmatrix}$

where $c = \int_a^b e^{ikx} (\sim) \, dx$ F.T. has supp. $[a,b]$

$$d = 1 + \int_0^{b-a} e^{ikx} (\cdot\cdot) \, dx$$

Next question: Given $\Phi_{\infty,-\infty}$ can you construct the factorization $\Phi_{\infty,-\infty} = \Phi_{\infty,0} \Phi_{0,-\infty}$

$$\begin{pmatrix} \bar d_+ & \bar c_+ \\ c_+ & d_+ \end{pmatrix} \begin{pmatrix} \bar d_- & \bar c_- \\ c_- & d_- \end{pmatrix} = \begin{pmatrix} \bar d & \bar c \\ c & d \end{pmatrix}$$

$c = c_+ \bar d_- + d_+ c_-$ $\qquad c_+ \in H_+ , d_+ \in 1 + H_+$

$d = c_+ \bar c_- + d_+ d_-$ $\qquad c_- \in H_- , d_- \in 1 + H_-$

So things are very interesting. Notice that you want a factorization within the $SU(1,1)$ group, and not the $SU(2)$

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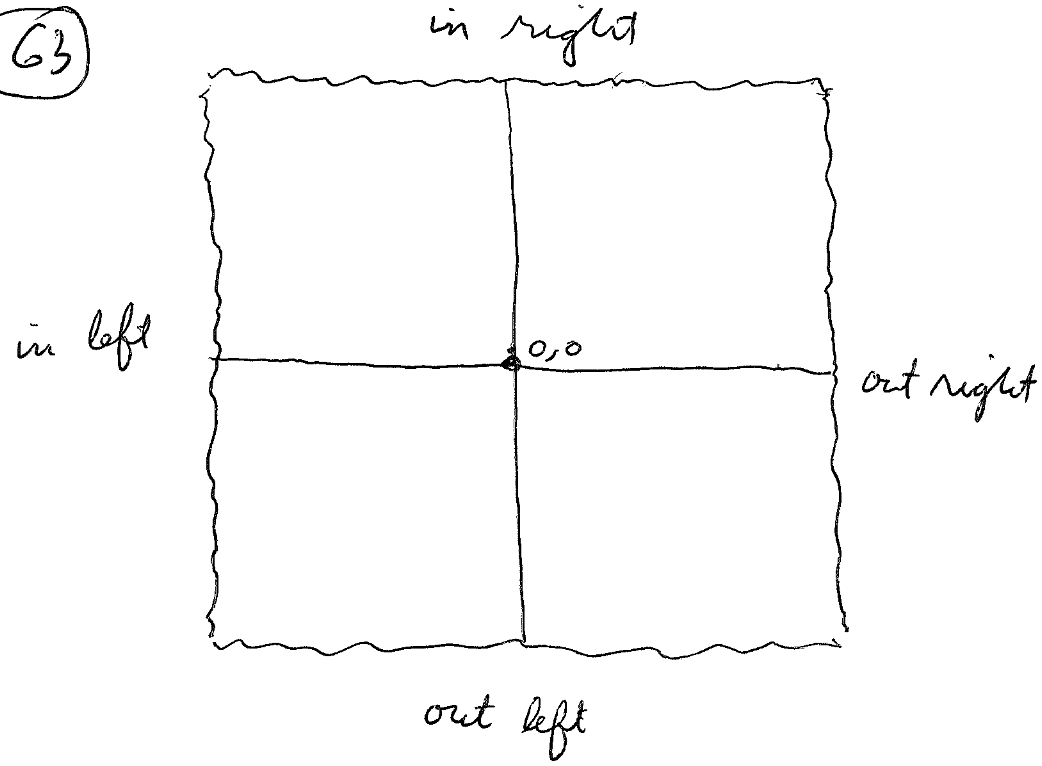
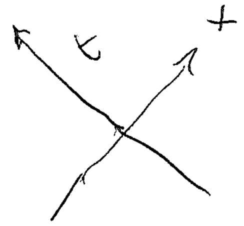




(63) Recursion relation

$z^n \cdot z^{-n+1}$

$$\begin{pmatrix} z^{-n} p_n \\ q_n \end{pmatrix} = \frac{1}{k_n} \begin{pmatrix} z^{-n} & z^{-n} h_n \\ \bar{h}_n & 1 \end{pmatrix} \begin{pmatrix} z\, p_{n-1} \\ q_{n-1} \end{pmatrix}$$

m

$$= \frac{1}{k_n} \begin{pmatrix} 1 & h_n z^{-n} \\ \bar{h}_n z^n & 1 \end{pmatrix} \begin{pmatrix} z^{-(n-1)} p_{n-1} \\ q_n \end{pmatrix}$$

$$T_{nm} = \frac{1}{k_n} \begin{pmatrix} 1 & h_n z^{-n} \\ \bar{h}_n z^n & 1 \end{pmatrix} \frac{1}{k_{n-1}} \begin{pmatrix} 1 & h_{n-1} z^{-n+1} \\ \bar{h}_{n-1} z^{n-1} & 1 \end{pmatrix} \cdots \frac{1}{k_{m+1}} \begin{pmatrix} 1 & h_{m+1} z^{-m} \\ \bar{h}_{m+1} z^{m+1} & 1 \end{pmatrix}$$

$$\begin{pmatrix} z^{-n} p_n \\ q_n \end{pmatrix} = T_{nm} \begin{pmatrix} z^{-m} p_m \\ q_m \end{pmatrix}$$

Moreover

$$T_{nm} = \begin{pmatrix} \bar{d} & \bar{c} \\ c & d \end{pmatrix} \qquad |d|^2 - |c|^2 = 1$$

$c \in \langle z^n, z^{n-1}, \dots, z^{m+1} \rangle$ | $d \in \langle 1, z, \dots, z^{n-m} \rangle$.

Maybe what's important is the projector, reproducing kernel for the subspace. ~~Then~~ You recall that for $H_+ / SH_+ = H_+ \cap SH_-$ the reproducing kernel is something like

$$\frac{1 - S(z)\overline{S(w)}}{1 - z\bar{w}}$$

Other ideas, these from yesterday's walk. Recall the correspondence between unitaries and transfer

(64) ~~[illegible]~~ Ready to finish. Ideas to use: scattering-transfer matrix picture as Lagrangian subspace of ~~Hermitian~~ Krein space, projector = reproducing kernel ~~[illegible]~~ ~~[illegible] This is the key to use a function~~ ~~[illegible]~~ You need to refine an argument like

$$F_{mn} = \{u^{<-m}\xi_-, u^{>n}\xi_+\}^\perp$$

$$\xrightarrow{\sim} (z^{-m}H_- + z^n \nu H_+)^\perp$$

$$= z^{-m}H_+ \cap z^n \nu H_- \qquad \nu = \frac{\bar{\delta}}{\delta}$$

$$\xrightarrow{\sim} z^{-m} q H_+ \cap z^n \bar{q} H_-$$

$$= z^{-m} H_+ \cap z^n H_-$$

$$= \mathrm{span}\{z^{-m}, z^{-m+1}, \ldots, z^n\}.$$

to an actual formula for the orthogonal projector onto F_{mn}. ~~So you need a~~ ~~comp~~ ~~You need to mix the scattering picture~~

~~[illegible]~~ Scattering picture: ~~[illegible]~~

Two copies ~~[illegible]~~ $L^2(S^1)\xi_-$ $\quad L^2(S^1)\xi_+$ glued together by ~~an isomorphism~~ a unitary

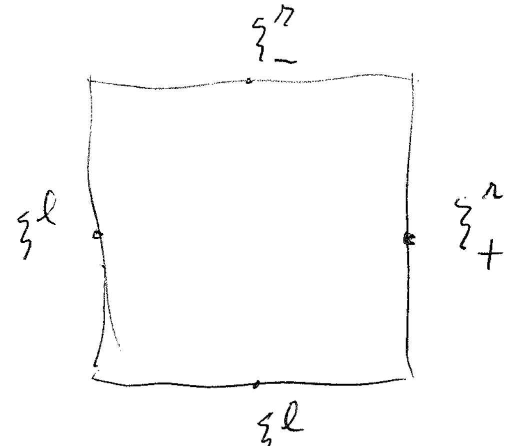
Scattering picture in the moment problem. building up from $n=0$ end and down from the $n=\infty$ end. ~~[illegible]~~ Look at $n=+\infty$ end. You get $L^2(S^1)\xi_- \xrightarrow[\sim]{\nu} L^2(S^1)\xi_+$ which can be replaced by ~~[illegible]~~ $L^2(S^1)\xi_- \oplus L^2(S^1)\xi_+$ and Γ_ν. Krein space + Lag. subspace.

(65) Other ingrediant is the ~~filtration~~; bifiltration $(z^{-m}H_-\xi_-)^\perp \oplus (z^{n}H_+\xi_+)^\perp$

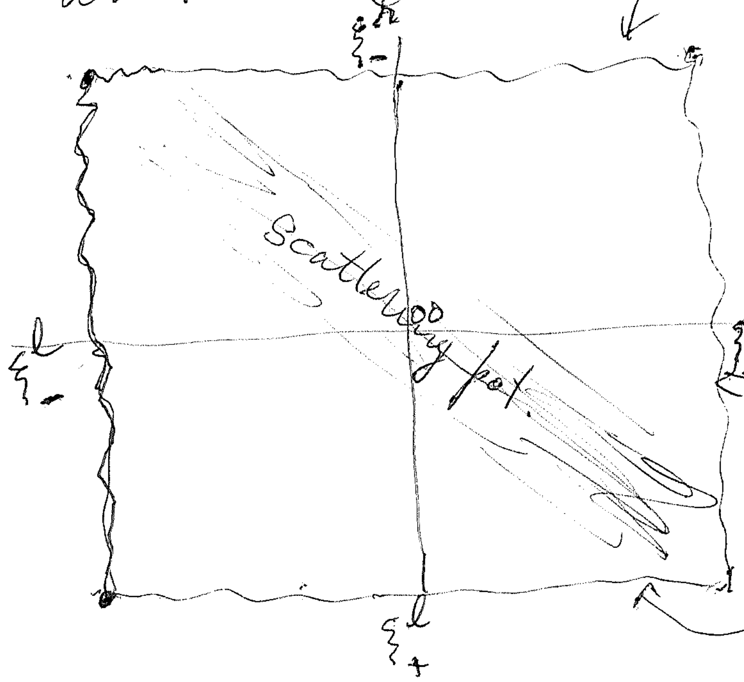
$= z^{-m}H_+\xi_- \oplus z^{n}H_-\xi_+$.

Recap. Structure is $L^2(S^1)\xi_-$ with $z^{-m}H_+\xi_-$ direct sum $L^2(S^1)\xi_+$ with $z^{n}H_-\xi_+$ and Γ_ν. Important point is that ν has Birkhoff factorization $\nu = \frac{\bar{g}}{g}$.

Next what. This time have



Here's what I learned



note u moves this orth seg to the left, and it's on the right side of the scattering so this side is incoming on the right

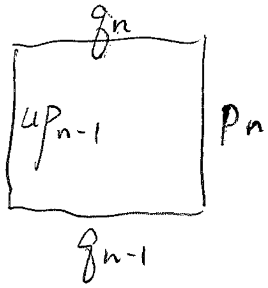
u moves this up which is outgoing on the right

moves to left which is outgoing

For each side you should have a half space which is in the opposite direction of scattering obs.

So you want $H_-\xi^l_-$ $H_+\xi^l_+$ $H_-\xi^r_-$ $H_+\xi^r_+$

(66) Let's try to fit things together. ~~The~~ You have the basic square describing the unitary S-matrix (from in to out) and the transfer matrix from left ~~to~~ right, but the entries are functions on the circle



$$(q_n, p_n) = h_n$$

$$\cancel{\begin{pmatrix} p_n \\ u p_{n-1} \end{pmatrix} = \begin{pmatrix} & -h_n \\ & \end{pmatrix} \begin{pmatrix} q_n \\ q_{n-1} \end{pmatrix}}$$

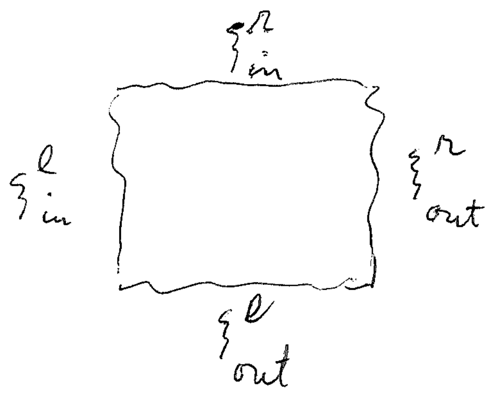
$$(q_n | p_n - h_n q_n) = 0$$

$$p_n - h_n q_n = k_n u p_{n-1}$$

$$\begin{pmatrix} p_n \\ q_{n-1} \end{pmatrix} = \begin{pmatrix} k_n & h_n \\ -\bar{h}_n & k_n \end{pmatrix} \begin{pmatrix} u p_{n-1} \\ q_n \end{pmatrix}$$

$$q_{n-1} = -\bar{h}_n u p_{n-1} + k_n q_n$$

$(q_{n-1} | u p_{n-1}) =$ [redacted] $(-\bar{h}_n u p_{n-1}, u p_{n-1}) = -h_n$



$$\begin{pmatrix} \xi^r_{out} \\ \xi^l_{out} \end{pmatrix} = \underbrace{\begin{pmatrix} \bar{d} & \bar{c} \\ c & d \end{pmatrix}}_{\text{NO, need unitary}} \begin{pmatrix} \xi^l_{in} \\ \xi^r_{in} \end{pmatrix}$$

(67) ~~[illegible]~~

$$S = \Theta_{h_n} \Theta_{h_{n-1}} \cdots \Theta_{h_m}$$

$$\Theta_h = \frac{1}{k}\Big(\quad hz$$

Consider.

$$\begin{pmatrix} z^{-n} p_n \\ q_n \end{pmatrix} = \frac{1}{k_n} \begin{pmatrix} z^{-n} & h_n z^{-n} \\ \bar{h}_n & 1 \end{pmatrix} \begin{pmatrix} \cancel{z} p_{n-1} \\ q_{n-1} \end{pmatrix}$$

disc 1 diml Dirac Eq eigenvalue z.

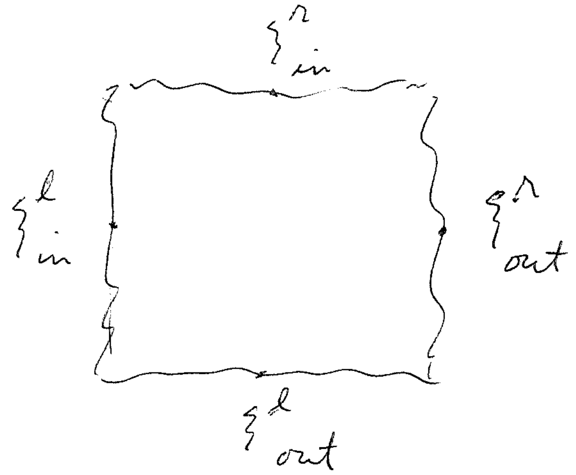
$$= \frac{1}{k_n} \begin{pmatrix} 1 & h_n z^{-n} \\ \bar{h}_n z^n & 1 \end{pmatrix} \begin{pmatrix} z^{-(n-1)} p_{n-1} \\ q_{n-1} \end{pmatrix}$$

so for $|n| \gg 0$ $\quad z^{-n} p_n$ and q_n are constant in n. ~~Basic scatter~~

left $\qquad$ ~~scattering~~ interaction $\qquad$ right

$$\begin{pmatrix} z^n \\ 0 \end{pmatrix} \longleftrightarrow \begin{pmatrix} A z^n \\ B \end{pmatrix} = \begin{pmatrix} z^n \\ 0 \end{pmatrix} A + \begin{pmatrix} 0 \\ 1 \end{pmatrix} B$$

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} \longleftrightarrow \begin{pmatrix} C z^n \\ D \end{pmatrix} = \begin{pmatrix} z^n \\ 0 \end{pmatrix} C + \begin{pmatrix} 0 \\ 1 \end{pmatrix} D$$



what is ξ^r_{in} ?

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} T \longleftrightarrow \begin{pmatrix} z^n \\ 0 \end{pmatrix} R + \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

ξ^l_{in}

$$\begin{pmatrix} z^n \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} R' \longleftrightarrow \begin{pmatrix} z^n \\ 0 \end{pmatrix} T'$$

$$\xi^r_{in} \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix} \frac{1}{D} \longleftrightarrow \begin{pmatrix} z^n \\ 0 \end{pmatrix} \frac{C}{D} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} z^n \\ 0 \end{pmatrix} \longleftrightarrow \begin{pmatrix} z^n \\ 0 \end{pmatrix} A + \begin{pmatrix} 0 \\ 1 \end{pmatrix} B$$

$$- \quad + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \frac{B}{D} \longleftrightarrow \begin{pmatrix} z^n \\ 0 \end{pmatrix} \frac{+BC}{D} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} B$$

$$\begin{pmatrix} z^n \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \end{pmatrix} \frac{B}{D} \longleftrightarrow \begin{pmatrix} z^n \\ 0 \end{pmatrix} \frac{1}{D} \quad \text{is } \xi^l_{in}$$

So the scattering matrix should be roughly

$$\begin{pmatrix} \frac{1}{D} & \frac{C}{D} \\ -\frac{B}{D} & \frac{1}{D} \end{pmatrix}$$

~~which doesn't agree with~~

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \cdots \frac{1}{k_n} \begin{pmatrix} 1 & h_n z^{-n} \\ \bar{h}_n z^n & 1 \end{pmatrix} \frac{1}{k_{n-1}} \begin{pmatrix} 1 & h_{n-1} z^{-n+1} \\ \bar{h}_{n-1} z^{n-1} & 1 \end{pmatrix} \cdots$$

should be the transfer from left to right.

If (h_n) supp in $[m, n]$, then $\begin{pmatrix} \bar{D} & \bar{C} \\ C & D \end{pmatrix}$ with $C \in \langle z^n, z^m \rangle$ and $D \in \langle 1, \ldots, z^{n-m} \rangle$

$$\begin{pmatrix}\bar d & \bar c \\ c & d\end{pmatrix} \in \begin{pmatrix} z^{-n+m\cancel{-1}}\cancel{z}\langle z^{n-m},\dots,1\rangle & z^{-n}\langle z^{n-m},\dots,1\rangle \\ z^{m\cancel{-1}}\cancel{z}\langle z^{n-m},\dots,1\rangle & \langle z^{n-m},\dots,1\rangle \end{pmatrix}$$

$$= \begin{pmatrix} \overline{\langle z^{n-m},\dots,1\rangle} & \overline{\langle z^{+m},\dots,z^{+n}\rangle} \\ \langle z^{n},\dots,z^{m}\rangle & \langle z^{n-m},\dots,1\rangle \end{pmatrix}$$

Conclude

$$\frac{1}{k_n}\begin{pmatrix} z & h_n \\ \bar h_n z & 1\end{pmatrix}\cdots\frac{1}{k_m}\begin{pmatrix} z & h_m \\ \bar h_m z & 1\end{pmatrix} \in \begin{pmatrix} z\langle z^{n-m},\dots,1\rangle & \langle z^{n-m},\dots,1\rangle \\ z\langle z^{n-m},\dots,1\rangle & \langle z^{n-m},\dots,1\rangle\end{pmatrix}$$

$$\frac{1}{k_n}\begin{pmatrix} 1 & h_n z^{-n} \\ \bar h_n z^{n} & 1\end{pmatrix}\cdots\frac{1}{k_m}\begin{pmatrix} 1 & h_m z^{-m} \\ \bar h_m z^{m} & 1\end{pmatrix} \in \begin{pmatrix} \overline{\langle z^{n-m},\dots,1\rangle} & \overline{\langle z^{n},\dots,z^{m}\rangle} \\ \langle z^{n},\dots,z^{m}\rangle & \langle z^{n-m},\dots,1\rangle\end{pmatrix}$$

because

$$\begin{pmatrix} z^{-n} & 0 \\ 0 & 1\end{pmatrix}\frac{1}{k_n}\begin{pmatrix} 1 & h_n \\ \bar h_n & 1\end{pmatrix}\begin{pmatrix} z & 0 \\ 0 & 1\end{pmatrix}\begin{pmatrix} z^{n-1} & 0 \\ 0 & 1\end{pmatrix} = \frac{1}{k}\begin{pmatrix} 1 & h_n z^{-n} \\ \bar h_n z^{n} & 1\end{pmatrix}$$

Focus on the problem at hand, i.e.

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Review.

$$\begin{pmatrix} z^{-n} & 0\\ 0 & 1\end{pmatrix}\frac{1}{k_n}\begin{pmatrix}1 & h_n\\ \bar h_n & 1\end{pmatrix}\begin{pmatrix} z & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix} z^{n-1} & 0\\ 0 & 1\end{pmatrix} = \frac{1}{k_n}\begin{pmatrix}1 & h_n z^{-n}\\ \bar h_n z^{n} & 1\end{pmatrix}$$

$$\begin{pmatrix} z^{-n} & 0\\ 0 & 1\end{pmatrix}\frac{1}{k_n}\begin{pmatrix} z & h_n\\ \bar h_n z & 1\end{pmatrix}\cdots\frac{1}{k_m}\begin{pmatrix} z & h_m\\ \bar h_m z & 1\end{pmatrix}\begin{pmatrix} z^{m-1} & 0\\ 0 & 1\end{pmatrix}$$

$$\in \begin{pmatrix} z^{-n} & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix} \cancel{z}\langle z^{n-m},\dots,1\rangle & \langle z^{n-m},\dots,1\rangle\\ \cancel{z}\langle z^{n-m},\dots,1\rangle & \langle z^{n-m},\dots,1\rangle\end{pmatrix}\begin{pmatrix} z^{m\cancel{-1}} & 0\\ 0 & 1\end{pmatrix}$$

$$= \begin{pmatrix} \overline{\langle z^{n-m},\dots,1\rangle} & \overline{\langle z^{n},\dots,z^{m}\rangle}\\ \langle z^{n},\dots,z^{m}\rangle & \langle z^{n-m},\dots,1\rangle\end{pmatrix}$$

---

begin with a finite h-sequence, construct [redacted] the Hilbert space H, U, p_{mn}, q_{mn}. This you do by orthonormal bases. But spectral theory in principle gives a Hilbert space whose elements are functions [redacted] on S^1, in general L^2 fns wrt measure. [redacted] Use scattering to realize [redacted] $\begin{pmatrix} p_n\\ q_n\end{pmatrix}$ by nice functions. ~~Then you get the idea that~~

Idea: the scattering matrix depends only on the reflection coefficient. The whole scattering picture.

Picture: [redacted] a summable [redacted] sequence (h_n) should yield a transfer matrix $\begin{pmatrix} \bar d & \bar c\\ c & d\end{pmatrix}$ where c is an l^1 Fourier series, hence $c(z)$ is continuous on S^1, and $|c(z)| < 1$.

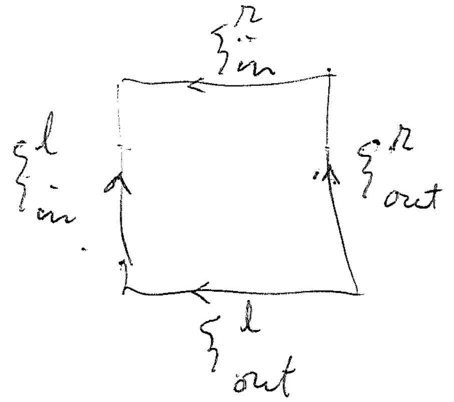
(74) idea. begin with reflection coefficient $c(z)$, ℓ^1 Fourier series, $|c(z)| < 1$.

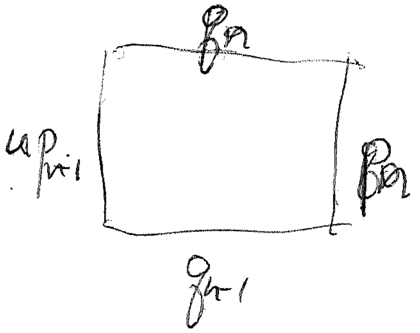
$$L^2(S^1) \xrightarrow{a} H \xleftarrow{b} L^2(S^1)$$

$$a1 = \xi^r_{in} = \lim q_n$$

$$b1 = \xi^r_{out} = \lim u^{-n} p_n$$

$$c = b^* a$$





$$(q_n, p_n) = h_n$$

$$p_n - q_n \underbrace{(q_n, p_n)}_{h_n} = k_n u p_{n-1}$$

$$q_n - p_n (p_n, q_n) = k_n q_{n-1}$$

i.e. $$\frac{1}{k_n}\begin{pmatrix} 1 & -h_n \\ -\bar{h}_n & 1 \end{pmatrix}\begin{pmatrix} p_n \\ q_n \end{pmatrix} = \begin{pmatrix} u p_{n-1} \\ q_{n-1} \end{pmatrix}$$

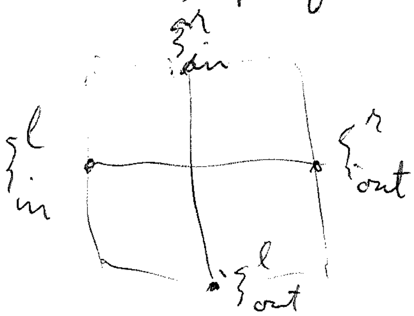
$$p_n = k_n u p_{n-1} + h_n q_n$$

$$q_{n-1} = -\bar{h}_n u p_{n-1} + k_n q_n \quad ?$$

$$q_{n-1} = \left(q_n - [k_n u p_{n-1} + h_n q_n]\,\bar{h}_n\right)/k_n$$

$$= -\bar{h}_n u p_{n-1} + \frac{1 - |h_n|^2}{k_n} q_n \qquad k_n$$

(75) Start with (h_n) summable, construct H, u, p_n, q_n. Then ~~there~~ $\exists$ limits



$$\xi^r_{in} = \lim_{n\to\infty} q_n \qquad \xi^r_{out} = \lim_{n\to\infty} u^{-n} p_n$$

$$\xi^l_{out} = \lim_{n\to-\infty} q_n \qquad \xi^l_{in} = \lim_{n\to-\infty} u^{-n} p_n$$

$$\begin{pmatrix} \xi^r_{out} \\ \xi^r_{in} \end{pmatrix} = \begin{pmatrix} \bar d & \bar c \\ c & d \end{pmatrix} \begin{pmatrix} \xi^l_{in} \\ \xi^l_{out} \end{pmatrix}$$

$$\frac{1}{d}\xi^r_{in} - \frac{c}{d}\xi^l_{in} = \xi^l_{out}$$

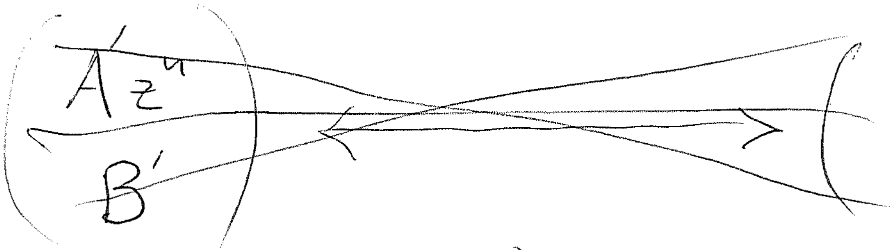
$$\xi^r_{out} = \bar d \xi^l_{in}$$

$$\bar d\left(\frac{1}{d}\xi^r_{in} - \frac{c}{d}\xi^l_{in}\right)$$

$$\bar d - \frac{|c|^2}{d} = \frac{1}{d}$$

$$\begin{pmatrix} \xi^r_{out} \\ \xi^l_{out} \end{pmatrix} = \begin{pmatrix} \frac{1}{d} & \frac{\bar c}{d} \\ -\frac{c}{d} & \frac{1}{d} \end{pmatrix} \begin{pmatrix} \xi^l_{in} \\ \xi^r_{in} \end{pmatrix}$$

Can you see that d is nonvanishing, either in or outside of $\mathbb{D}$! Eigenvalue equation.



Let
$$\begin{pmatrix} p_n \\ q_n \end{pmatrix} = \begin{pmatrix} A' z^n \\ B' \end{pmatrix} \qquad n \ll 0$$
$$= \begin{pmatrix} A z^n \\ B \end{pmatrix} \qquad n \gg 0$$

Then
$$\begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} \bar d & \bar c \\ c & d \end{pmatrix} \begin{pmatrix} A' \\ B' \end{pmatrix}$$

Reason: If the transfer is zero for a particular value of z, then

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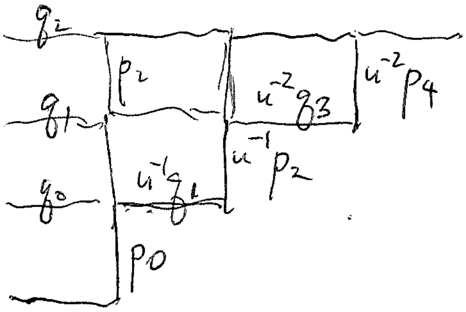
Consider the Dirac equation, i.e. the eigenvector eqn. for a specific z. Suppose $h_n = 0$ for $|n| >> 0$. You have a 2 dim'l space of solutions. Recall that a staircase gives rise to an orthonormal basis. You should know when a solution is square summable.



so need $z^{-n} q_{2n+1}$, $z^{-n} p_{2n}$ to be an ℓ^2 sequence, but for large n $q_n = \alpha$ is constant and $z^{-n} p_n = \beta$ is constant. Thus you want $z^{-n}\alpha$ and $z^{n}\beta$ to be ℓ^2 as $n \to \infty$. So only possible when

$|z| < 1$ and $\alpha = 0$

$|z| > 1$ and $\beta = 0$.

In general

$$\begin{pmatrix} A' z^n \\ B' \end{pmatrix} \xleftarrow{-\infty \leftarrow n} \begin{pmatrix} p_n \\ q_n \end{pmatrix} \overset{n \to +\infty}{\sim} \begin{pmatrix} A z^n \\ B \end{pmatrix}$$

So the ℓ^2 condition is clearer if you use $\begin{pmatrix} z^{1/2} & 0 \\ 0 & z^{-1/2} \end{pmatrix}$.

So what to understand?

Situation so far. You know essentially that the situation depends on the reflection coefficient. You suppose given $H = L^2(S^1)\xi_{in}^{n} + L^2(S^1)\xi_{out}^{n}$ glued together via a contraction γ, which is a summable Fourier with $|\gamma(z)| < 1$ on S^1. Dilate contraction

$$X \overset{a}{\underset{b}{\rightrightarrows}} H$$

$$b^* a = \gamma$$

$$\begin{tikzcd}
L^2(S^1)\xi_- \arrow[dr, "a"] & \\
& H \\
L^2(S^1)\xi_+ \arrow[ur, "b"'] &
\end{tikzcd}$$

$$\Big\| \overset{a}{\overset{\shortparallel}{\xi_-}} f + \overset{b}{\overset{\shortparallel}{\xi_+}} g \Big\|^2$$

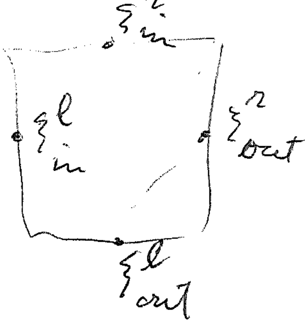
$$= \|f\|^2 + \big(\overbrace{\xi_+^* \xi_-}^{b^*a} f, g\big) + \big(g, \overbrace{\xi_+^* \xi_-}^{b^*a}\big) + \|g\|^2$$

$$= \|\gamma f + g\|^2 + \|(1-\gamma^*\gamma)^{1/2} f\|^2$$

What seems to happen. There is maybe a new phenomenon here in that $(1-\gamma^*\gamma)^{1/2}$ is replaced by something holom. d.

So maybe this is the point. Anyway suppose given $c(z) = \sum_{n\in\mathbb{Z}} c_n z^n$ $\quad \sum |c_n| < \infty$ such that $|c(z)| < 1$ for $z \in S^1$. Then $c(z)$ is continuous so $|c(z)| < 1-\varepsilon$ and there is a unique $d(z)$ analytic nonvanishing + ?

$$\begin{pmatrix}\xi^r_{out}\\ \xi^r_{in}\end{pmatrix} = \begin{pmatrix}\bar d & \bar c\\ c & d\end{pmatrix}\begin{pmatrix}\xi^l_{in}\\ \xi^l_{out}\end{pmatrix}$$



$$\frac{1}{d}\xi^r_{in} - \frac{c}{d}\xi^l_{in} = \xi^l_{out}$$

$$\xi^r_{out} = \bar d\,\xi^l_{in} + \bar c\left(\frac{1}{d}\xi^r_{in} - \frac{c}{d}\xi^l_{in}\right)$$

$$= \frac{\bar c}{d}\xi^r_{in} + \frac{1}{d}\xi^l_{in}$$

$$\begin{pmatrix}\xi^r_{out}\\ \xi^l_{out}\end{pmatrix} = \begin{pmatrix}\frac{1}{d} & \frac{\bar c}{d}\\ -\frac{c}{d} & \frac{1}{d}\end{pmatrix}\begin{pmatrix}\xi^l_{in}\\ \xi^r_{\cancel{out}\ in}\end{pmatrix}$$

reflection coefficient on right is $\frac{\bar c}{d}$

$$\begin{pmatrix}\xi^l_{in}\\ \xi^r_{in}\end{pmatrix} = \begin{pmatrix}\frac{1}{d} & -\frac{\bar c}{d}\\ \frac{c}{d} & \frac{1}{d}\end{pmatrix}\begin{pmatrix}\xi^r_{out}\\ \xi^l_{out}\end{pmatrix}$$

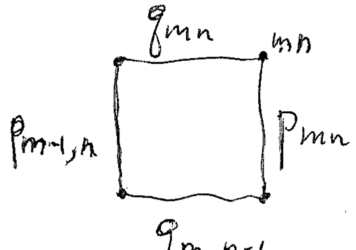
79 ~~Go back to the trans~~

Review. Given (h_n) ~~summable~~ arbitrary you can construct (H, u, p_{mn}, q_{mn}). When (h_n) summable you get transfer matrix form left to rt

$$\begin{pmatrix}\bar d & \bar c\\ c & d\end{pmatrix} = \cdots k_n\begin{pmatrix}1 & h_n z^{-n}\\ \bar h_n z^n & 1\end{pmatrix}\cdot\frac{1}{k_{n-1}}\begin{pmatrix}1 & h_{n-1}z^{-n+1}\\ \bar h_{n-1} z^{n-1} & 1\end{pmatrix}\cdots$$

which should be a continuous function on S^1 values in $SU(1,1)$. basic properties c abs. conv. FS. $|c(z)| < 1$. d analytic in D non-vanishing.

You've shifted away from H to the eigenvalue equation. ~~Let us consider which~~ Go back to Hilbert space. Say we have (H, u, p_{mn}, q_{mn})

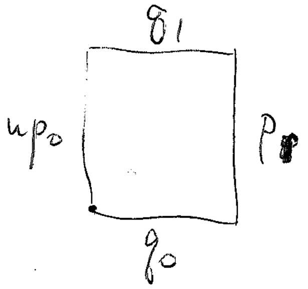


$$\begin{pmatrix}p_{mn}\\ q_{m,n-1}\end{pmatrix} = \begin{pmatrix}k & h\\ -\bar h & k\end{pmatrix}\begin{pmatrix}p_{m-1,n}\\ q_{mn}\end{pmatrix}$$

$$(q_{mn}|p_{mn} - h q_{mn}) = (q_{mn}|k p_{m-1,n}) = 0$$

$$\| $$

$$(q_{mn}|p_{mn}) - h$$



$$\begin{pmatrix}p_1\\ q_0\end{pmatrix} = \begin{pmatrix}k & h\\ -\bar h & k\end{pmatrix}\begin{pmatrix}up_0\\ q_1\end{pmatrix}$$

$$k q_1 = q_0 + \bar h u p_0$$

$$p_1 = k u p_0 + \frac{h}{k}(q_0 + \bar h u p_0) = \overbrace{\left(k + \frac{|h|^2}{k}\right)}^{\frac{1}{k}} u p_0 + \frac{h}{k} q_0$$

$$\begin{pmatrix}p_1\\ q_1\end{pmatrix} = \frac{1}{k}\begin{pmatrix}1 & h\\ \bar h & 1\end{pmatrix}\begin{pmatrix}up_0\\ q_0\end{pmatrix}$$

(80) Formulate: Given (h_n) summ. get ~~left~~ to right scattering transfer matrix

$$\begin{pmatrix}\bar d & \bar c\\ c & d\end{pmatrix} = \cdots\cdots \underbrace{\frac{1}{k_n}\begin{pmatrix}1 & h_n z^{-n}\\ \bar h_n z^{n} & 1\end{pmatrix}}\frac{1}{k_{n-1}}\begin{pmatrix}1 & h_{n-1} z^{-n+1}\\ \bar h_{n-1} z^{n-1} & 1\end{pmatrix}\cdots\cdots$$

$$\cdots\cdots\Theta_n(h,z)\cdot\Theta_{n-1}(h,z)\cdots\cdots$$

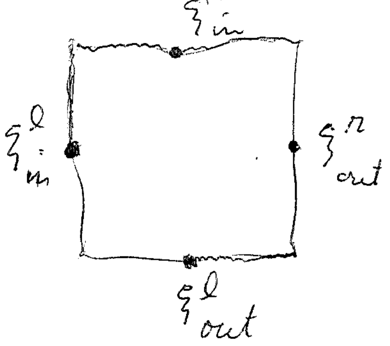
Suppose you factor this

$$\begin{pmatrix}\bar d & \bar c\\ c & d\end{pmatrix} = \underbrace{\begin{pmatrix}\bar d_+ & \bar c_+\\ c_+ & d_+\end{pmatrix}}\begin{pmatrix}\bar d_- & \bar c_-\\ c_- & d_-\end{pmatrix}$$

$$\prod_{n>0}\Theta_n(h,z)\ \prod_{0\geq n}\Theta_n(h,z)$$

Know that $c_+ \in \langle z, z^2, \dots\rangle$, to first order in h it should be $\sum_{n>0}\bar h_n z^n$. The question is how to construct this factorization. The hope would be to do things in the unitary picture, i.e. direct reduction to the usual Birkhoff factorization.

Let's pursue the unitary picture. You have in H the vector $\xi^{r,l}_{in,out}$



$$\begin{pmatrix}\xi^r_{out}\\ \xi^r_{in}\end{pmatrix} = \begin{pmatrix}\bar d & \bar c\\ c & d\end{pmatrix}\begin{pmatrix}\xi^l_{in}\\ \xi^l_{out}\end{pmatrix}$$

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$$\xi^r_{in} = c\,\xi^l_{in} + d\,\xi^l_{out}$$

$$\xi^l_{out} = \frac{1}{d}\xi^r_{in} - \frac{c}{d}\xi^l_{in} \qquad \frac{|d|^2-|c|^2}{d}$$

$$\xi^r_{out} = \bar{d}\,\xi^l_{in} + \bar{c}\left(\frac{1}{d}\xi^r_{in} - \frac{c}{d}\xi^l_{in}\right)$$

$$= \left(\tfrac{1}{d}\right)\xi^l_{in} + \frac{\bar{c}}{d}\xi^r_{in}$$

$$\begin{pmatrix} \xi^r_{out} \\ \xi^l_{out} \end{pmatrix} = \begin{pmatrix} \frac{1}{d} & \frac{\bar{c}}{d} \\ -\frac{c}{d} & \frac{1}{d} \end{pmatrix} \begin{pmatrix} \xi^l_{in} \\ \xi^r_{in} \end{pmatrix}$$

Is there a simple way to do this calc?

$$\boxed{\begin{aligned} \begin{pmatrix} \xi^r_+ \\ \xi^r_- \end{pmatrix} &= \begin{pmatrix} \bar{d} & \bar{c} \\ c & d \end{pmatrix} \begin{pmatrix} \xi^l_- \\ \xi^l_+ \end{pmatrix} \\ \begin{pmatrix} \xi^r_+ \\ \xi^l_+ \end{pmatrix} &= \begin{pmatrix} \frac{1}{d} & \frac{\bar{c}}{d} \\ -\frac{c}{d} & \frac{1}{d} \end{pmatrix} \begin{pmatrix} \xi^l_- \\ \xi^r_- \end{pmatrix} \end{aligned}}$$

$$\begin{vmatrix} \frac{1}{d} & \frac{\bar{c}}{d} \\ -\frac{c}{d} & \frac{1}{d} \end{vmatrix} = \frac{1+|c|^2}{d^2} = \frac{d\bar{d}}{d^2} = \frac{\bar{d}}{d}$$

this loop has degree zero since d is nonvanishing on $\bar{D}$.

So what to do? ~~You want to find a way~~ ~~[illegible]~~ You have to decide if there is a Birkhoff factorization for the transfer matrix. ~~[illegible]~~ The unitary scattering matrix has a usual Birkhoff Fact. Is this useful ~~in~~ ~~[illegible]~~ rank 2 ~~[illegible]~~ situation ~~[illegible]~~

$$\begin{matrix} L^2(S^1)\,\xi^r_+ \\ \oplus \\ L^2(S^1)\,\xi^l_+ \end{matrix} \quad \sim \quad \begin{matrix} L^2(S^1)\,\xi^l_- \\ \oplus \\ L^2(S^1)\,\xi^r_- \end{matrix}$$

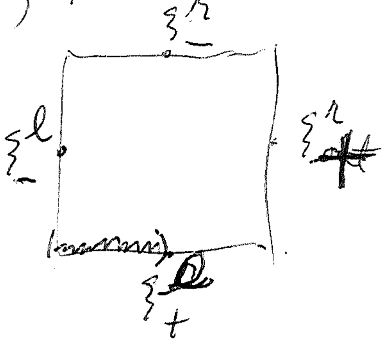
(82) Perhaps there's a better viewpoint using subspaces. Replace unitary or pseudo-unitary by Lagrangian subspace.

Consider the free ~~situation~~ - all $h_n = 0$.

i.e. $\begin{pmatrix} u^{-n} p_n \\ q_n \end{pmatrix}$ independent of n. Hilbert space H has orth basis. $\begin{cases} u^n \xi_- & \xi_- = q_n \ \forall n \\ u^n \xi_+ & \xi_+ = u^{-n} p_n \ \forall n \end{cases}$

S-matrix has zero reflection, transmission = 1

S-matrix is the identity?



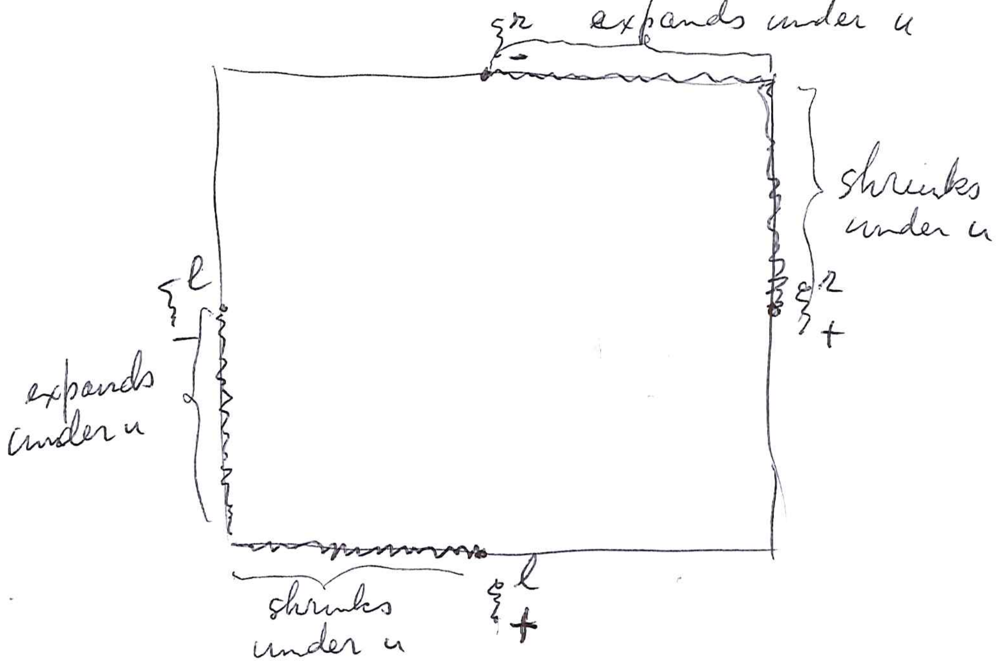
$$\begin{pmatrix} \xi^r_+ \\ \xi^l_+ \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \xi^l_- \\ \xi^r_- \end{pmatrix}$$

general case $\begin{pmatrix} \frac{1}{d} & \frac{\bar{c}}{d} \\ -\frac{c}{d} & \frac{1}{d} \end{pmatrix}$ The way to think: You have this rank 2 module over functions on S^1, actually it has rank 4 with a with the graph of S as a subspace. Basic idea should be something Grassmannian like, more precisely a pair of complementary subspaces.

Think of $L^2(S^1)\xi_- \oplus L^2(S^1)\xi_+$. No. Think of

$$L^2(S^1)\xi^l_- \oplus L^2(S^1)\xi^r_- \sim L^2(S^1)\xi^r_+ \oplus L^2(S^1)\xi^l_+$$

there being natural half spaces associated with these subspaces.



So the idea is you have this rank 2 "u-space"

$$S : L^2(S^1)\xi^r_+ \oplus L^2(S^1)\xi^l_+ \simeq L^2(S^1)\xi^l_- \oplus L^2(S^1)\xi^r_-$$

and certain obvious half spaces. First take $h_n = 0$ $\forall n$. Then $\xi^r_+ = \xi^l_-$, $\xi^l_+ = \xi^r_-$ $S \simeq 1$.

What might make this clearer?

~~scattering pre Tran~~

Idea: T = transfer matrix $\begin{pmatrix} \bar d & \bar c \\ c & d \end{pmatrix}$, a loop in $SU(1,1)$, an isom from the left Krein space $L^2(S^1)\xi^l_+ \oplus L^2(S^1)\xi^l_-$ to the right Krein space $L^2(S^1)\xi^r_+ \oplus L^2(S^1)\xi^r_-$. You seek a factorization $T = T_{\geq n} T_{<n}$ for each n, a Birkhoff factorization in the Krein setting. What's the model? The case $h = 0$.

Examine what happens to first order in h

$$T = \begin{pmatrix} 1 & \sum h_n z^{-n} \\ \sum \bar h_n z^n & 1 \end{pmatrix} = \begin{pmatrix} 1 & h(z^{-1}) \\ \bar h(z) & 1 \end{pmatrix} = \begin{pmatrix} 1 & h_+(z^{-1}) \\ \bar h(z)_+ & 1 \end{pmatrix} \Big($$

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transfer matrix $T=\begin{pmatrix}\bar d & \bar c\\ c & d\end{pmatrix}$ from left to right

$$T: L^2(S^1)\xi_+^\ell \oplus L^2(S^1)\xi_-^\ell \xrightarrow{\sim} L^2(S^1)\xi_+^r \oplus L^2(S^1)^r_-$$

$$T = \cdots \frac{1}{k_n}\begin{pmatrix}1 & h_n z^{-n}\\ \bar h_n z^n & 1\end{pmatrix}\frac{1}{k_{n-1}}\begin{pmatrix}1 & h_{n-1} z^{-n+1}\\ \bar h_{n-1} z^{n-1} & 1\end{pmatrix}\cdots$$

You wish to ~~factor~~ construct the factorization: $T = T_{\geq 0}\, T_{<0}$ where

$T_{\geq 0}$ assoc. to sequence $(h_n)_{n\geq 0}$ extended by 0

$T_{<0}$ ——— $(h_n)_{n<0}$ ———.

but starting with T. To first order in h we have

$$T = \begin{pmatrix}1 & h(z^{-1})\\ \bar h(z) & 1\end{pmatrix} \qquad \bar h(z) = \sum_{n\in\mathbb{Z}} \bar h_n z^n$$

$$= \begin{pmatrix}1 & h_+(z^{-1})\\ \bar h_+(z) & 1\end{pmatrix}\begin{pmatrix}1 & h_-(z^{-1})\\ \bar h_-(z) & 1\end{pmatrix} \qquad \bar h_+(z) = \sum_{n>0} \bar h_n z^n$$

Is there a nice interpretation? You have a splitting on the Lie algebra level.

First you have to understand T. T is not ~~the most~~ a general loop in $SU(1,1)$ because of the analyticity + ~~more~~ non-vanishing conditions in d.

$$\frac{1}{\sqrt{1-|\bar h(z)|^2}}\begin{pmatrix}1 & h(z^{-1})\\ \bar h(z) & 1\end{pmatrix}$$

(85) Put into words what you want.

You have $T = \begin{pmatrix} \bar{d} & \bar{c} \\ c & d \end{pmatrix}$ transfer matrix $l \leftrightarrow r$

you want to construct the fact. $T = T_{\geq 0} T_{\leq 0}$

$$T_{\geq 0} = \begin{pmatrix} \bar{d}_+ & \bar{c}_+ \\ c_+ & d_+ \end{pmatrix} \qquad T_{<0} = \begin{pmatrix} \bar{d}_- & \bar{c}_- \\ c_- & d_- \end{pmatrix}$$

Check

$$\begin{pmatrix} z^{-n} & 0 \\ 0 & 1 \end{pmatrix} \frac{1}{k_n} \begin{pmatrix} z & h_n \\ \bar{h}_n z & 1 \end{pmatrix} \begin{pmatrix} z^{+n-1} & 0 \\ 0 & 1 \end{pmatrix} = \frac{1}{k_n} \begin{pmatrix} 1 & h_n z^{-n} \\ \bar{h}_n z^n & 1 \end{pmatrix}$$

$$\begin{pmatrix} z^{-n} & 0 \\ 0 & 1 \end{pmatrix} \frac{1}{k_n} \begin{pmatrix} z & h_n \\ \bar{h}_n z & 1 \end{pmatrix} \cdots \frac{1}{k_m} \begin{pmatrix} z & h_m z^{-1} \\ \bar{h}_m z & 1 \end{pmatrix} \begin{pmatrix} z^{m-1} & 0 \\ 0 & 1 \end{pmatrix} = T_{nm}$$

$n > m$

$$\in \begin{pmatrix} z^{-n} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} z\langle z^{n-m}, \dots, 1\rangle & \langle z^{n-m}, \dots, 1\rangle \\ z\langle z^{n-m}, \dots, 1\rangle & \langle z^{n-m}, \dots, 1\rangle \end{pmatrix} \begin{pmatrix} z^{m-1} & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \langle 1, \dots, z^{-n+m}\rangle & \langle z^{-m}, \dots, z^{-n}\rangle \\ \langle z^n, \dots, z^m\rangle & \langle z^{n-m}, \dots, 1\rangle \end{pmatrix}$$

So d should always be analytic in D and invertible. This is a feature I don't understand,

$$\begin{pmatrix} \bar{d}_+ & \bar{c}_+ \\ c_+ & d_+ \end{pmatrix} \begin{pmatrix} \bar{d}_- & \bar{c}_- \\ c_- & d_- \end{pmatrix} = \begin{pmatrix} & \\ c_+\bar{d}_- + d_+c_- & c_+\bar{c}_- + d_+d_- \end{pmatrix}$$

?

86 Is there a relation between factoring T and splitting, i.e. a direct sum decomposition. Review in the case of the loop group. $H = H_+ \oplus H_-$ basic splitting, $g \in$ loop gp, a clutching function.

Suppose $g = g_+ g_-$ where $g_\pm : H_\pm \xrightarrow{\sim} H_\pm$

$$H_+ \cap g H_- \xrightarrow{\sim} g_+^{-1} H_+ \cap g_- H_- = H_+ \cap H_- = \overset{\circ}{H}$$

typical situation $H = H_+ \oplus H_-$

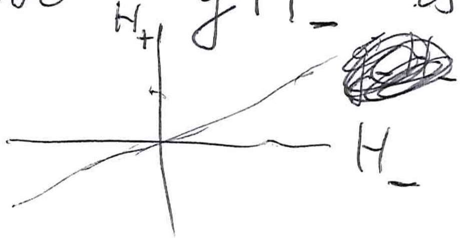
$g \in LG$ clutching fn. $\mathcal{E}_g$

$$\Gamma(\mathcal{E}_g) = \left\{ \begin{pmatrix} f_+ \\ f_- \end{pmatrix} \middle| \; f_+ = g f_- \right\}$$

$$0 \hookrightarrow H_+ \cap g H_- \to \begin{matrix} H_+ \\ \oplus \\ H_- \end{matrix} \xrightarrow[(in_+ \; -g\, in_-)]{} H \longrightarrow H \big/ {H_+ + gH_-} \to 0$$

If $g = g_+ g_-$ where $g_\pm \; H_\pm \xrightarrow{\sim} H_\pm$

then gH_- and ~~is complementary to~~ H_+ are complementary, so gH_- is another complement to H_+



(87) Back to rank 1, to the simplest loop gp.

Begin with $g : S^1 \longrightarrow \mathbb{C}^\times$, better

$g \in L^\infty(S^1)^\times$, have polar decomp.

$g = \frac{g}{|g|} \cdot |g|$. Description of unitarys

$g \in L^\infty(S^1)$ in terms of "outgoing" subspaces modulo constants in S^1.

Example: $L^2(S^1, d\mu)$ $\quad d\mu = \rho \frac{d\theta}{2\pi}$ $\quad \int_{S^1} d\mu = 1$

where $\rho > 0$ suff. cont. In this case

$-\log \rho = f + \bar{f}$ f anal. in D, $d\mu = \frac{1}{|g|^2} \frac{d\theta}{2\pi}$, $g = e^f$

etc.

$$L^2(S^1)\xi_+ \xrightarrow[\sim]{\bar{g}\cdot} L^2(S^1, d\mu) \xleftarrow[\sim]{g\cdot} L^2(S^1)\xi_-$$

$$\xi_+ \longmapsto \bar{g}, \quad g \longleftarrow\!\!\shortmid \xi_-$$

$S = \frac{\bar{g}}{g}$ this is my g, it involves $\mathrm{Im}(f)$, the conjugate fn. to $-\log\rho$. ~~As what a~~ What are you missing? Something in the geometry concerning $H = H_+ \oplus H_-$ and SH_+.

~~Whether~~ Have map

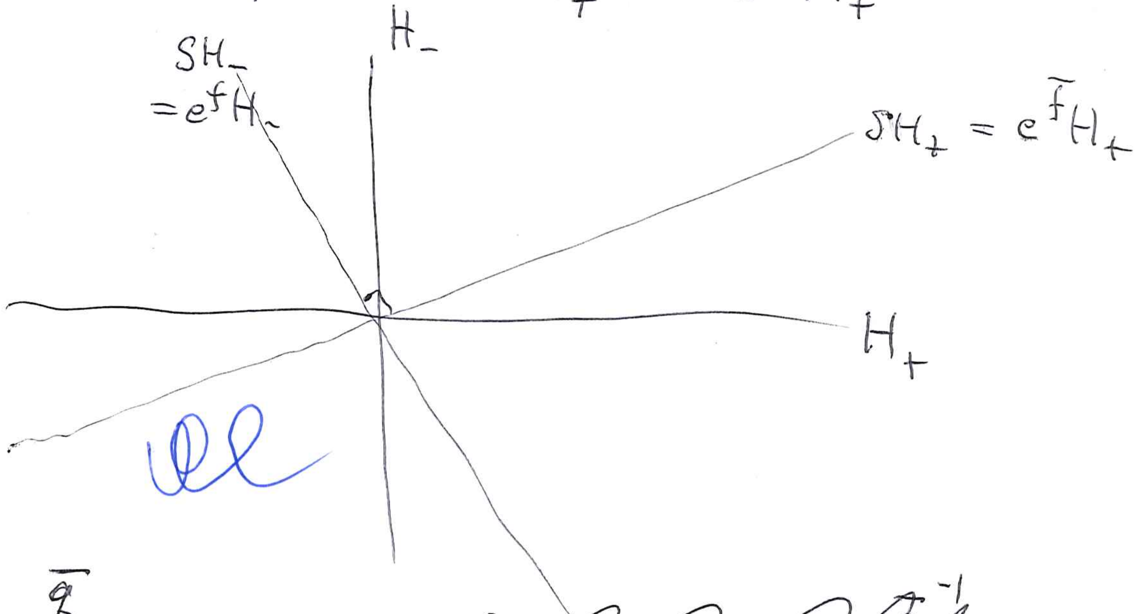
$$S \longmapsto SH_+ = \overset{(2)}{\bigoplus_{n \geq 0}} z^n S$$

from unitary loops to outgoing subspaces. This is essentially an isom. ~~If we mod~~ ~~as better spaces~~ What happens for $S = \frac{\bar{g}}{g}$

(88) You want to understand what happens in Fock space.

Return to $L^2(S^1) = H_+ \oplus H_-$ acted on by loop groups. Consider $S = \frac{e^{+\bar f}}{e^f} = e^{-2i\,\mathrm{Im}(f)}$ where f is holomorphic in D, $f(z) = \sum_{n \geq 0} f_n z^n$

What is SH_+? $SH_+ = e^{\bar f} H_+$



$S = \frac{\bar g}{g}$ $\cancel{SH_+ \cap H_-}$

$$\frac{\bar g}{g} H_+ \underset{+}{\cap} H_- = \bar g H_+ \cap H_- \xleftarrow{\bar g \cdot} \underbrace{H_+ \cap \underbrace{\tfrac{1}{\bar g} H_-}_{H_-}}_{=0}$$

So what you want to see is the operator whose graph is SH_+. The Toeplitz operator belonging to S.

(89) So what's going on. I recall ~~that~~ something about rank 1 loop group. ~~That~~ Inverse scattering in the moment ~~situation~~ problem.

Begin with $d\mu = \rho \frac{d\theta}{2\pi}$ $\qquad \rho$ smooth, >0, $\int \rho \frac{d\theta}{2\pi} = 1$.

Then $\quad -\log\rho = \sum_{n\in\mathbb{Z}} c_n z^n \qquad \overline{c_n} = c_{-n}$

$$= f(z) + \overline{f(z)} \qquad f(z) = \frac{c_0}{2} + \sum_{n\geq 1} c_n z^n + \text{imag. const.}$$

$$\rho = \frac{1}{|g|^2} \qquad g = e^f$$

$$\begin{array}{ccccc} L^2(S^1) & \xrightarrow{\sim} & L^2(S^1, d\mu) & \xleftarrow{\sim} & L^2(S^1) \\ 1 & \longmapsto & g = \xi_-, \ \xi_+ = \bar{g} & \longleftarrow & 1 \end{array}$$

$$S = \frac{\bar g}{g} = e^{\bar f - f} = e^{-2i(\mathrm{Im} f)}$$

Conversely, suppose given $S(z)$ smooth ~~loop~~ map $S: S^1 \to S^1$ of degree zero. $\frac{1}{2\pi i}\log S$ real defined up to $\mathbb{Z}$. $\quad \frac{1}{2\pi i}\log S = \sum c_n z^n \qquad \overline{c_n} = c_{-n}$

$$\log S = \sum c_n z^n \qquad -\overline{c_n} = +c_{-n}$$

c_0 defined mod $2\pi i\mathbb{Z}$,

$$-f = +\frac{c_0}{2} + \sum_{n\geq 1} c_n z^n$$

$$+\bar f = \frac{c_0}{2} + \sum_{n\leq -1} c_{+n} z^{+n} \qquad \therefore f - \bar f = \log S.$$

(90) In the end you have a function $g = e^f$ analytic and invertible on D with some niceness about the boundary behavior. Measure $d\mu = \frac{1}{|g|^2}\frac{d\theta}{2\pi}$ (adjust real constant), scattering $S = \frac{\bar{g}}{g}$.

What next? Construct sequence (h_n)

Summarize. You are concerned with the moment problem for a measure on S^1.

<u>You have</u> $d\mu \longleftrightarrow (h_n)_{n\geq 1}$ bijection.

Start with $L^2(S^1, d\mu) = H$, $u = z$, $\xi = 1$.

$X = \operatorname{Ker} \xi^* = \xi^\perp$. $uX = (u\xi)^\perp$. You have a partial unitary $X \overset{a}{\underset{b}{\rightrightarrows}} H$ a inclusion, $b = ua$

$$H = aX \oplus \overbrace{\mathbb{C}\xi}^{V_+} = bX \oplus \overbrace{\mathbb{C}u\xi}^{V_-}$$

eigenvalue equation. be concentrate on the Green's function. There should be a Green's function.

Actually theory says that by supplying a b. cond. for the partial unitary you get an operator which then has resolvent.

$$H = aX \oplus \mathbb{C}\xi_+ = bX \oplus \mathbb{C}\xi_-$$

$$c_h = ba^* \oplus \xi_- h \xi_+^* \qquad \text{for } |h| \leq 1$$

is a contraction operator

$$\xi_+^* \frac{1}{z - c_h} = \xi_+^* \frac{1}{z - c_0 - \xi_- h \xi_+^*} = \xi_+^* \frac{1}{z-c_0} + \xi_+^* \frac{1}{z-c_0}\xi_- h \xi_+^* \frac{1}{z-c_0}$$

$$= \frac{1}{1 - \xi_+^*(z-c_0)^{-1}\xi_- h}\, \xi_+^* \frac{1}{z-c_0}$$

(91) $S_h(z^{-1}) = \dfrac{1}{1-S_0(z^{-1})h}\, S_0(z^{-1})$?

$$\xi_+^* \frac{1}{z-c_h}\xi_-$$

Actually you should get straight the spectrum. If you are dealing with contraction operators, then the spectrum is inside $D : |z|<1$. The problem concerns c, c^*.

$$c_h = ba^* + \xi_- h \xi_+^*$$

$$Y = aX \oplus \mathbb{C}\xi_+ = bX \oplus \mathbb{C}\xi_-$$

$$\xi_+^*\frac{1}{z-c_h} = \xi_+^*\frac{1}{z-c_0} + \left(\xi_+^*\frac{1}{z-c_0}\xi_- h\right)\xi_+^*\frac{1}{z-c_0} + \cdot$$

$$\xi_+^* G_h = \frac{1}{1-\xi_+^* G_0 \xi_- h}\,\xi_+^* G_0$$

$\dfrac{1}{z-c_0}$

$\xi_+^* G_0 \xi_-$

$$\xi_+^* G_h \xi_- = \frac{1}{1-\xi_+^* G_0 \xi h}\,\xi_+^* G_0 \xi_- =$$

Put $h=1$.

$$\xi^*\frac{1}{z-u}u\xi = \frac{S_0}{1-S_0}$$

$$\frac{2S_0}{1-S_0}+1 = \frac{1+S_0}{1-S_0}$$

$$\xi^*\frac{z+u}{z-u}\xi = \frac{1+S_0}{1-S_0} \qquad \frac{2u}{z-u}+1 = \frac{2u+z-\cancel{u}}{z-u}$$

---

concentrate - start again. What is the aim? Given smooth $S : S^1 \to S^1$ of degree zero. Then $\log S$ is an $i\mathbb{R}$-valued smooth function on S^1 determined modulo $2\pi i\mathbb{Z}$. $\therefore\ S = \dfrac{\bar{g}}{g}$ where $g = e^f$ $\quad f - \bar{f} = \log S$. Then can form

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$$d\mu = \frac{1}{|g|^2}\frac{d\theta}{2\pi}$$ normalized so that $\int d\mu = 1$.

$$\begin{array}{ccccc} L^2(S^1) & \xrightarrow{\wedge} & L^2(S^1, d\mu) & \xleftarrow{\sim} & L^2(S^1) \\ 1 & & \bar{g}\ ,\ g & \longleftarrow & 1 \end{array}$$

If S smooth unitary loop of degree zero, then
$S = \frac{\bar{g}}{g}$ where $\left(\begin{array}{l} g\cdot : H_+ \xrightarrow{\sim} H_+ \\ \bar{g}\cdot : H_- \xrightarrow{\sim} H_- \end{array}\right.$

so

$$SH_+ \oplus H_- = H.$$

Also get $SH_+ \cap z^n H_-$.

Begin again. $S : S^1 \to S^1$ smooth. Can write $S = \frac{\bar{g}}{g}$ $\quad g = e^f$ $\quad$ f analytic in D smooth on $\bar{D}$

g normalized so that $\int \frac{1}{|g|^2}\frac{d\theta}{2\pi} = 1$.

$H = L^2(S^1)$ $\qquad \xi_- = 1$ $\qquad \xi_+ = \frac{\bar{g}}{g} = S$.

$H = L^2(S^1, \overbrace{\frac{1}{|g|^2}\frac{d\theta}{2\pi}}^{d\mu})$ $\qquad \xi_- = g$ $\qquad \xi_+ = \bar{g}$

Green's functions

Suppose you start with the prob. measure $d\mu = \frac{1}{|g|^2}\frac{d\theta}{2\pi}$, $H = L^2(S^1, \frac{1}{|g|^2}\frac{d\theta}{2\pi})$

$\xi = 1$, $u\xi = z$ $\qquad aX = (\mathbb{C}\xi)^\perp$ $\qquad bX = (\mathbb{C}u\xi)^\perp$

$ua = b$

93 You want to consider the port, partial unitary. Review scattering

$$(a\lambda - b)x = -\sigma_+ + \sigma_-$$

$$H = Y^\perp \oplus X \oplus V^+$$
$$= Y^\perp \oplus uX \oplus V^-$$

$\lambda\xi + \lambda x + \lambda\sigma_+$

$u(\xi) + u(x) + u(\sigma_+)$

$\lambda(\xi) + \lambda u(x_2) + \lambda\sigma_-$

$$\xi + x + \sigma_+ = \xi + ux' + \sigma_-$$

$$u\xi + ux + u\sigma_+ = \lambda\xi + \lambda ux' + \lambda\sigma_-$$

project both sides onto uX.

$$ux \doteq \lambda ux' \qquad \therefore\ x = \lambda x'$$

$$\lambda x' + \sigma_+ = ux' + \sigma_-$$

$$(\lambda - u)x = -\sigma_+ + \sigma_-$$

$$(\lambda a - b)x = -\sigma_+ + \sigma_-$$

$$(\lambda - a^*b)x = a^*\sigma_-$$

$$x = \frac{1}{\lambda - a^*b}a^*\sigma_- = a^*\left(\frac{1}{\lambda - ba^*}\right)\sigma_-$$

$$\sigma_+ = \sigma_- - (\lambda a - b)a^*\left(\frac{1}{\lambda - ba^*}\right)\sigma_-$$

$$= \left\{\lambda - \cancel{ba^*} - \lambda aa^* + \cancel{ba^*}\right\}\frac{1}{\lambda - ba^*}\sigma_-$$

$$= \lambda(1 - aa^*)\frac{1}{\lambda - ba^*}\sigma_-$$

$$v_- = \xi_- \xi_-^* v_-$$

$$\xi_+^* v_+ = \xi_+^* \frac{1}{1-\lambda^{-1} b a^*} \xi_- \, \xi_-^* v_-$$

scattering for $|\lambda| > 1$

$$(\lambda a - b)x = -v_+ + v_-$$

$$b^*(\lambda a - b)x = -b^* v_+$$

$$(1-\lambda b^* a)x = b^* v_+$$

$$x = b^*(1-\lambda ab^*)^{-1} v_+$$

$$v_- = v_+ + (\lambda a - b)\, b^*(1-\lambda ab^*)^{-1} v_+$$

$$= \{(1-\cancel{\lambda ab^*}) + \cancel{\lambda ab^*} - bb^*\}(1-\lambda ab^*)^{-1} v_+$$

$$= (1-bb^*)(1-\lambda ab^*)^{-1} v_+$$

$$\xi_-^* v_- = \xi_-^*(1-\lambda ab^*)^{-1}\xi_+ \, \xi_+^* v_+$$

scat for $|\lambda| < 1$.

Question: Take partial unitary $X \overset{a}{\underset{b}{\rightrightarrows}} H = L^2(S^1, d\mu)$ and you get a response fn. Somehow you have replaced $L^2(S^1, d\mu)$ where $p_0 = q_0 = 1$ by a ~~ranke~~ rank 2 scattering situation

What is the Pick fn. belonging to $d\mu$?

$$\int \frac{\qquad}{}$$

(95) Anyway ~~const~~ take $H = L^2(S^1, d\mu)$ $\quad d\mu = \frac{1}{|\eta|^2}\frac{d\theta}{2\pi}$

$$H = aX \oplus \mathbb{C}\eta = bX \oplus \mathbb{C}u\eta.$$

$$c_h = \underbrace{ba^*}_{u - u\eta\eta^*} + u\eta h\eta^*$$

$$c_s = u + s\, u\eta\eta^*$$

Begin with

$$L^2(S^1) \xrightarrow{\sim} L^2(S^1, d\mu) \xleftarrow{\sim} L^2(S^1)$$

$$1 \longmapsto \eta, \ \bar{\eta} \longleftarrow 1$$

~~Pick function assoc. to $d\mu$ is essentially~~

~~$\sum_{n\in\mathbb{Z}} \mu_n z^n$ $\quad \mu_n = \int \zeta^{-n} d\mu$~~

~~$\sum_{n\in\mathbb{Z}} z^n \int \zeta^{-n} d\mu = \int \Big[\sum_{n\geq 0} (z\zeta^{-1})^n + \sum_{n>0} (\bar{z}^{-1}\zeta)^n \Big] d\mu$~~

~~$= \int \left(\frac{1}{1 - z\zeta^{-1}} + \frac{\bar{z}\zeta}{1 - \bar{z}\zeta} \right) d\mu = \int \frac{1 - |z|^2}{|1 - z\zeta^{-1}|^2} d\mu$~~

~~$= \int \left(\frac{1}{1 - z\zeta^{-1}} + \frac{z^{-1}\zeta}{1 - z^{-1}\zeta} \right) d\mu$~~

~~$= \frac{1 - z^{-1}\zeta + (1 - z\zeta^{-1})z^{-1}\zeta}{} = 0.$~~

(96) You want to start with a smooth scattering situation. Just what does this mean? You have a real problem here, that you ~~..~~ keep finding.

I want to make a break-through. How. You should have all the ingredients.

Here's the problem: ~~...~~ Consider $d\mu = \rho \frac{d\theta}{2\pi}$ ρ smooth > 0, $\int \rho \frac{d\theta}{2\pi} = 1$. Construct the sequence of orthogonal polys p_n, q_n, h_n. The problem is to show $\sum |h_n| < \infty$.

The idea should involve partial unitaries.

Idea to try. Remove the boundary condition $p_0 = q_0 = 1$, get partial unitary which can be dilated to a unitary. ~~This extends~~

Do some exercises. ~~.....~~

$$a^*a = b^*b = 1.$$

$$X \underset{b}{\overset{a}{\rightrightarrows}} X \qquad Y = aX \oplus \mathbb{C}\xi_+ = bX \oplus \mathbb{C}\xi_-$$

$$c_h = \overbrace{ba^*}^{c_0} + \xi_- h \xi_+^* \qquad \text{contraction for } |h| \le 1.$$

$$\xi_+^* \frac{1}{\lambda - c_h} = \xi_+^* \frac{1}{\lambda - c_0} + \xi_+^* \frac{1}{\lambda - c_0} \xi_- h \xi_+^* \frac{1}{\lambda - c_0} + \cdots$$

$$= \frac{1}{1 - \xi_+^* \frac{1}{\lambda - c_0} \xi_- h} \xi_+^* \frac{1}{\lambda - c_0}$$

$$S_h = \frac{1}{1 - S_0 h} S_0 \qquad S_1 = \frac{1}{1 - S_0} S_0$$

$$1 + S_1 = \frac{1}{1 - S_0}$$

(97) Get the geometry straight.

First do the measure case. $\qquad g = e^f$

$$d\mu = \rho \frac{d\theta}{2\pi}, \quad \rho = \frac{1}{|g|^2} \qquad -\log \rho = f + \bar{f}$$

$$\| \qquad \sum_{n\in\mathbb{Z}} c_n e^{in\theta}, \quad \bar{c}_n = c_{-n}$$

$$f = \overbrace{\underset{\text{const}}{\text{real}}}^{r} + \sum_{n\geq 1} c_n e^{in\theta}$$

and r is to be adjusted so that $\int \rho \frac{d\theta}{2\pi} = 1$.

$$S = \frac{\bar{g}}{g} = e^{\bar{f} - f} \qquad g = e^f$$

important is
$$\cdot g : H_+ \xrightarrow{\sim} H_+$$
$$\cdot \bar{g} : H_- \xrightarrow{\sim} H_-$$

$\boxed{g \text{ is a kind of de Branges function}}$

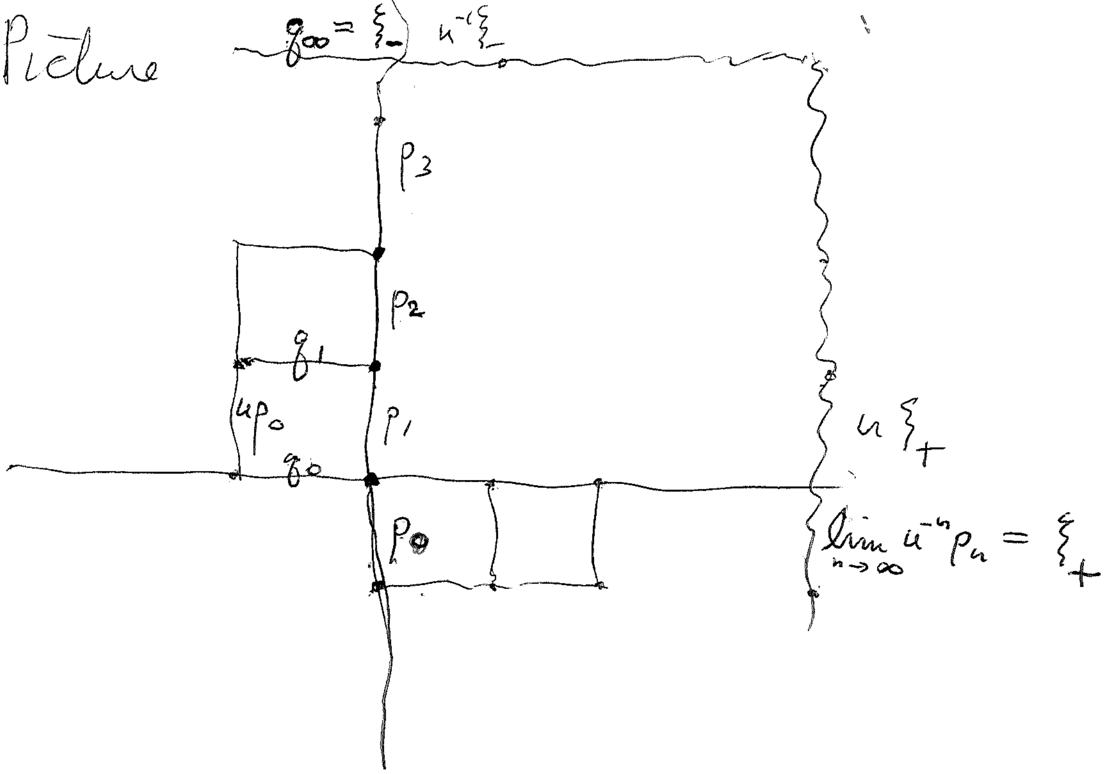
So what next.

What do we have? $H = L^2(S^1, d\mu)$, u = mult. by z, $\xi_- = g \in H$, $\xi_+ = \bar{g} \in H$.

$$L^2(S^1)g \xrightarrow{\sim} H \xleftarrow{\sim} L^2(S^1)\bar{g}$$

somehow you want to pass from this situation to a partial unitary. You want a decreasing filtration on H that corresponds to the Schur expansion of the response function of this partial unitary. This decreasing filtration should be orthogonal to the orth poly increasing filtration.

(98) Picture



$$F_{00} = \{u^{<0}\xi_-, u^{>0}\xi_+\}^\perp$$ should be 1 dimensional.

$$F_{mn} = \{u^{<-m}\xi_-, u^{>n}\xi_+\}^\perp$$ should be $m+n+1$ diml

$$\Big\downarrow S$$

$$\{z^{<-m}, z^{>n}\tfrac{\bar q}{q}\}^\perp$$

$$= \{z^{-m}H_-, z^{n+1}\tfrac{\bar q}{q}H_+\}^\perp$$

$$= z^{-m}H_+ \cap z^{n+1}\tfrac{\bar q}{q}H_-$$

$$\xrightarrow{\sim} z^{-m}qH_+ \cap z^{n+1}\bar q H_- = z^{-m}H_+ \cap z^{n+1}H_-$$

$$= \langle z^m, \dots, z^n\rangle$$

This is all OK, but you want to work from the scattering end.

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Thus you need

$$\overline{F^{\perp}_{mn} = u^{<-m}\xi_- + u^{>n}\xi_+}$$

$$\Big\downarrow S$$

$$\overline{z^{-m}H_- + z^{n+1}SH_+}$$

the closure should be unnecessary.

Main case might be $F^{\perp}_{00} = \overline{H_- + zSH_+}$

So it's clear that you ~~have~~ should complete control. The issue is how to handle, control $z^k H_- + SH_+$

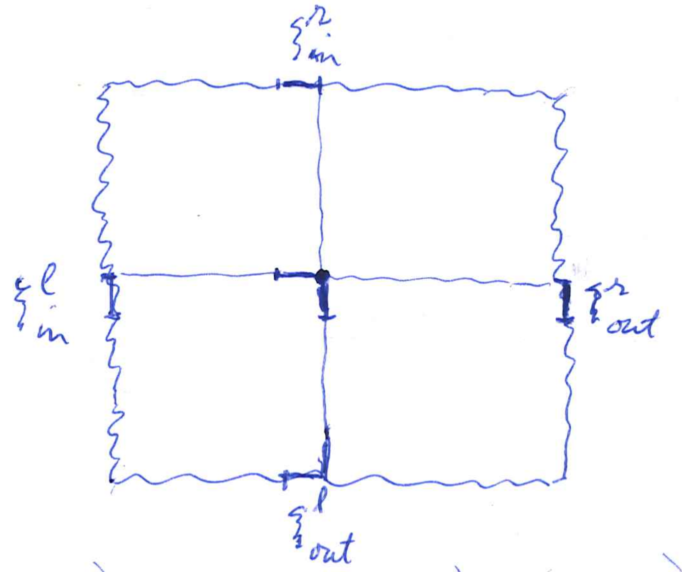
$k \geq 0$. The first thing then to understand is $H_- + SH_+$. This is H, but there ought to be "angles" somewhere.

I expect determinants to appear too.

Δular matrices. If one modifies the inner product by g, then you get the orth basis z^n. Let's figure out the analogue of orth. polys in our case. We have decreasing filt.

$\boxed{\text{OKAY!}}$ Basic idea. Relate orthonormal bases. $\{p_0, p_1, \cdots ; u^{-1}\xi_-, u^{-2}\xi_-, \cdots\}$ to the basis (not orthonormal) $\{u^{-1}\xi_-, u^{-2}\xi_-, \cdots ; \xi_+, u\xi_+, \cdots\}$

(100) Digress on the scattering. $\bar{d} - \frac{|c|^2}{d} = \frac{1}{d}$



$$\xi^r_{in} = c\,\xi^l_{in} + d\,\xi^l_{out}$$

$$\xi^l_{out} = \frac{1}{d}\xi^r_{in} - \frac{c}{d}\xi^l_{in}$$

$$\xi^r_{out} = \bar{d}\,\xi^l_{in} + \bar{c}\left(\frac{1}{d}\xi^r_{in} - \frac{c}{d}\xi^l_{in}\right)$$

$$= \frac{\bar{c}}{d}\xi^r_{in} + \frac{1}{d}\xi^l_{in}$$

$$\begin{pmatrix} \xi^r_{out} \\ \xi^r_{in} \end{pmatrix} = \begin{pmatrix} \bar{d} & \bar{c} \\ c & d \end{pmatrix} \begin{pmatrix} \xi^l_{in} \\ \xi^l_{out} \end{pmatrix}$$

$$\begin{pmatrix} \xi^r_{out} \\ \xi^l_{out} \end{pmatrix} = \underbrace{\begin{pmatrix} \frac{1}{d} & \frac{\bar{c}}{d} \\ -\frac{c}{d} & \frac{1}{d} \end{pmatrix}}_{S} \begin{pmatrix} \xi^l_{in} \\ \xi^r_{in} \end{pmatrix}$$

S smooth loop in $U(2)$.

You want to think of S as a perturbation of $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. There should be half spaces. Guess

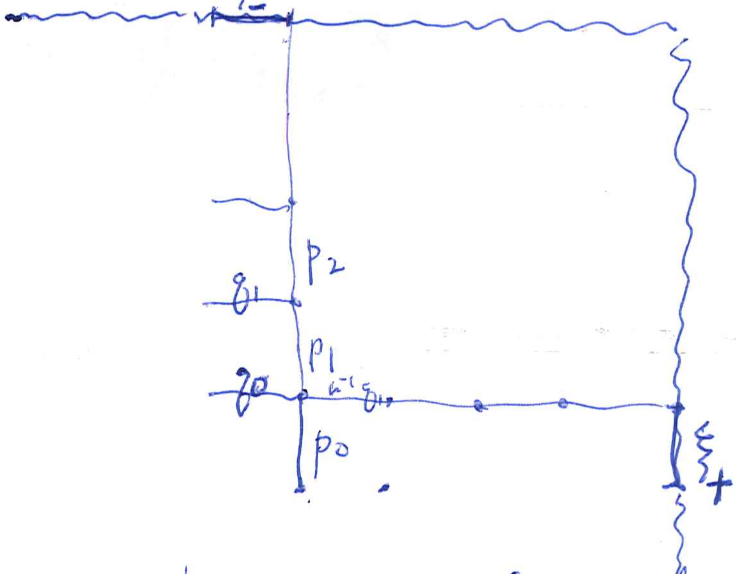
$H_+\xi^r_{in} \oplus H_-\xi^l_{in}$ versus $H_+\xi^r_{out} \oplus H_-\xi^l_{out}$

incoming outgoing

Conjecture that

$S\begin{pmatrix} H_-\xi^l_{in} \\ H_+\xi^r_{in} \end{pmatrix}$ is complementary to $\begin{pmatrix} H_+\xi^r_{out} \\ H_-\xi^l_{out} \end{pmatrix}$

(101) Back to rank 1.



We want to relate the orthonormal basis $\{u^{-1}\xi_-, u^{-2}\xi_-, \cdots; p_1, p_2, \cdots\}$ to the basis

$u^{-1}\xi_-, u^{-2}\xi_-, \cdots; u\xi_+, u^2\xi_+, \cdots\}$ and this in turn to $\{u^{-1}q_1, u^{-2}q_2, \cdots, u\xi_+, u^2\xi_+, \cdots\}$.

Notice that the orth basis [redacted]

$$u^{-2}\xi_-, u^{-1}\xi_-, p_0, p_1, p_2, \cdots$$

is what you get by using the ~~increasing~~ filtration

$$z^{-2}H^2_+ \supset \underset{\uparrow \atop u^{-1}q_\infty}{z^{-1}H^2_+ \supset} H^2_+ \underset{\uparrow \atop q_\infty}{\supset} zH^2_+ \underset{\uparrow \atop uq_\infty}{\supset} z^2H^2_+$$

Question: Recall condition $c^n\xi \to 0$ all ξ, which implies isometric embedding

$$\frac{1}{\lambda - c} \qquad Y = aX \oplus \mathbb{C}\xi_+ = bX \oplus \mathbb{C}\xi_-$$

$$c = ba^* \qquad 1 - aa^* = 1 - ab^*ba^*$$

$$\left\| \xi_+^* \frac{1}{\lambda - ba^*} y \right\|^2 = y^* \frac{1}{\bar{\lambda} - ab^*} \overbrace{\xi_+ \xi_+^*} \frac{1}{\lambda - ba^*} y$$

(102)

$$\frac{1}{\lambda - c^*}(1-c^*c)\frac{1}{\lambda - c} \qquad \underbrace{\frac{1}{\lambda - c} + \frac{c^*}{1-\lambda c^*}}$$

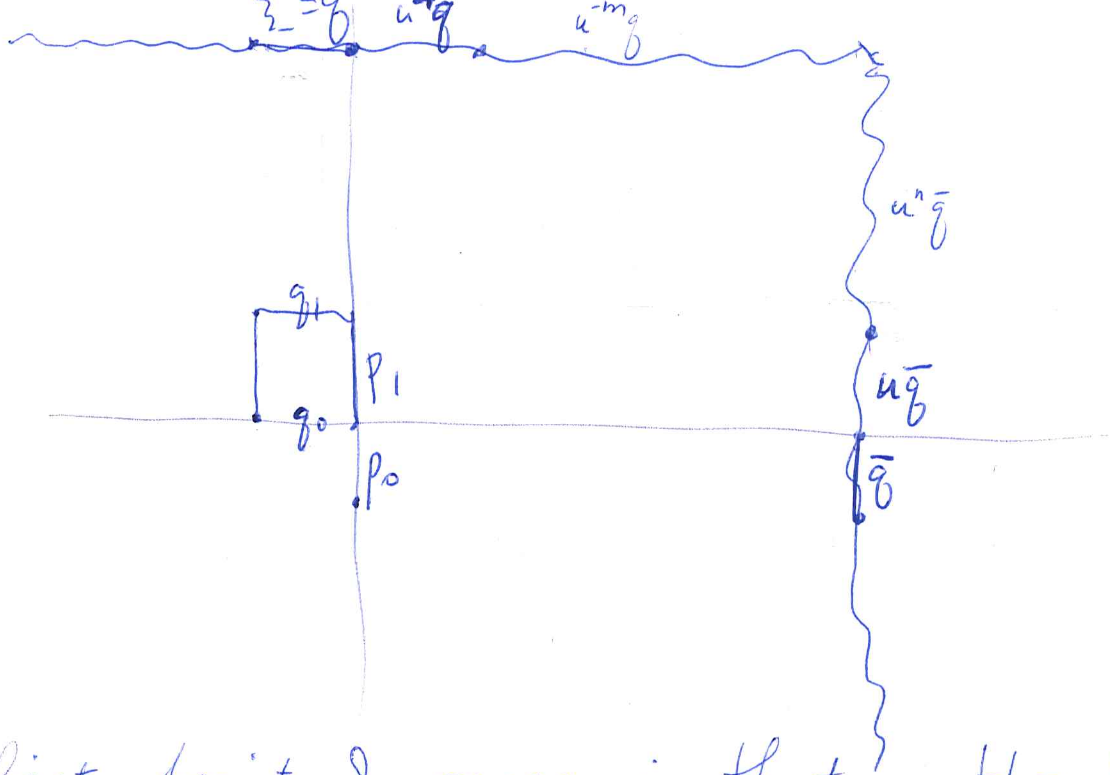
$$= \frac{1}{1-\lambda c^*}\Big(\underbrace{c^*(\lambda - c) + 1-\lambda c^*}_{1-c^*c}\Big)\frac{1}{\lambda - c}$$

How does this work in the smooth case? What can you hope for?

Situation: $S = \frac{\bar{q}}{q}$ Geometry

$$\overset{\xi_-}{\downarrow} \qquad\qquad\qquad\qquad\qquad \overset{\xi_+}{\downarrow}$$

$$L^2(S^1)q \;\subset\!\xrightarrow{\sim}\; L^2(S^1, d\mu) \;\xleftarrow{\sim}\!\supset\; L^2(S^1)\bar{q}$$



first point I guess is that $H = L^2(S^1; d\mu)$ has the orthonormal basis $u^n q$, $n \in \mathbb{Z}$. and also the orth basis $u^n\bar{q}$, $n \in \mathbb{Z}$.

S = mult. by $\frac{\bar{q}}{q}$ commutes with u and transforms $u^n q \longmapsto u^n \bar{q}$. Matrix

$$(u^m\bar{q})^*(u^n q) = \int \quad ?$$

(103) orth basis $u^n g \quad n \in \mathbb{Z}$

orth basis $u^n \bar{g} \quad n \in \mathbb{Z}$.

$$(u^m \bar{g})^* (u^n g) = \int g \, \cancel{z}^{-m+n} g \, \frac{d\theta}{|g|^2 2\pi}$$

$$= \int z^{-m+n} \frac{g}{\bar{g}} \frac{d\theta}{2\pi}$$

---

matrix of inner products.

$$(u^{-m} \xi_-)^* (u^n \xi_+) = \int \overline{z^{-m} g} \, z^n \bar{g} \, \frac{d\theta}{|g|^2 2\pi}$$

$$= \int z^{m+n} \frac{\bar{g}}{g} \frac{d\theta}{2\pi} \quad \text{seems good.}$$

So we have this S operator and we ?

~~apply~~

$$\begin{array}{ccccc} L^2(S^1)\xi_- & \xrightarrow{\sim} & L^2(S^1, d\mu) & \xleftarrow{\sim} & L^2(S^1)\xi_+ \\ \cup & & & & \cup \\ H_- \xi_- & & & & H_+ \xi_+ \end{array}$$

these two subspaces should be complementary inside $L^2(S^1, d\mu)$. Check. Apply ~~outgoing~~ ~~rep.~~ incoming rep. $\xi_- \longmapsto 1$, $\xi_+ \longmapsto \frac{\bar{g}}{g}$

$$H_- \xi_- \longmapsto H_- , \quad H_+ \xi_+ \longmapsto H_+ \frac{\bar{g}}{g}$$

actually its seems simpler to have ?

$H_- \xi_- \longmapsto H_- g$

(104) Anyway the point you might concentrate upon is what is true about ?

So you have learned something about the operator which takes

Yesterday summary: Consider smooth rank 1 scattering, by this I means ~~you have a~~ one has a smooth $S : S^1 \to S^1$ of degree 0 (to begin with), then you use S to glue together two copies of $L^2(S^1)$, ~~maybe~~ better to say you consider $L^2(S^1)$ with the half spaces $\underset{\text{incoming}}{H_-}$ and $\underset{\text{outgoing}}{SH_+}$.

0 case considered before is when $H_- \perp SH_+$ i.e. $H_+ \supset SH_+$. This means S extends analytically to D. ~~This is a special case~~ Then S has degree n.

So one new point is that H

~~Difficulty of L-Scattering~~ You are trying to understand [illegible] Loop group - abelian Lie group, self dual,

Try to understand the situation. You have outgoing subspaces

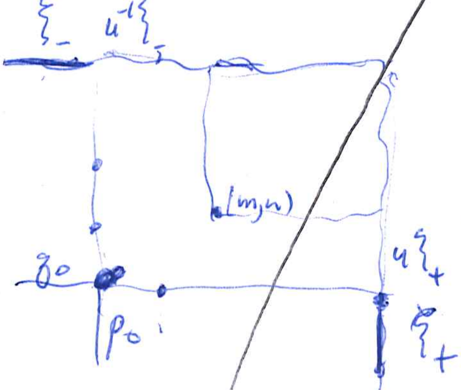
(105) Review the loop group representation

What is your aim? Situation: Smooth loop S of degree zero, ~~what the to get~~ gives rise to scattering situation $H = L^2(S^1)$ incoming subspace H_- outgoing subspace SH_+. Also you have factorization $S = \frac{\bar{g}}{g}$ with g smooth invertible on $\bar{D}$ and analytic in D. This gives isom. $H \simeq L^2(S^1, \frac{1}{|g|^2}\frac{d\theta}{2\pi})$ ~~with~~ such that $\xi_- \longleftrightarrow g$, $\xi_+ \longleftrightarrow \bar{g}$. Your problem?

You have a bifiltration of H

Start again. Given a smooth loop of degree 0 then you get a scattering situation:
$H = L^2(S^1)$ with $\boxed{H_{in} = H_-, \quad H_{out} = SH_+}$.
~~and~~ Also you get bifiltration complementary

$$F_{mn} = \{u^{<-m}\xi_-, u^{>n}\xi_+\}^\perp$$



$$= \left(u^{-m}H_-\xi_- + u^{n+1}H_+\xi_+\right)^\perp$$

$$= u^{-m}H_+\xi_- \cap u^{n+1}H_-\xi_-$$

$$\xrightarrow{\sim} u^{-m}H_+ g \cap u^{n+1}H_-\bar{g}$$

$$\cong z^{-m}H_+ \cap z^{n+1}H_-$$

$$= \langle z^{-m}, \ldots\ldots, z^n \rangle$$

(106) The thing to understand is the orthogonal projection onto F_{mn}.

Repeat. Given smooth loop $S: S^1 \to S^1$ of degree 0 you get a scattering situation $H = L^2(S^1)$, H_- incoming, SH_+ outgoing (complementary) and bifiltration

$$F_{mn} = z^{-m} F_{m+n}$$

$$F_{mn} = z^{-m}H_+ \cap z^{n+1}H_-$$

In general it should be

$$F_{mn} = z^{-m}SH_+ \cap z^{n+1}H_- \qquad z^{-m}\frac{1}{q}H_+ \cap z^{n+1}\frac{1}{\bar{q}}H_-$$

$$F_{mn} = z^{-m}\frac{\bar{q}}{q}H_+ \cap z^{n+1}H_- \subset L^2(S^1)$$

$$\uparrow \bar{q}$$

$$z^{-m}\frac{1}{q}H_+ \cap z^{n+1}\frac{1}{\bar{q}}H_- \subset L^2\left(S^1, \frac{1}{|q|^2}\frac{d\theta}{2\pi}\right)$$

Notation is terrible.

(107) Start again. You are given $S : S^1 \to S^1$ a smooth loop of degree zero, it gives rise to a scattering situation:

$$H = L^2(S^1), \quad H_{out} = H_+, \quad H_{in} = SH_-$$

i.e. $\xi_+ = 1$, $\xi_- = S$. Now degree zero says that $H = SH_- \oplus H_+$ not orthog.

$F_{mn} =$

Let's start with $H = L^2(S^1, d\mu)$, where

$$d\mu = \frac{1}{|q|^2}\frac{d\theta}{2\pi} \qquad S = \frac{\bar{q}}{q} \qquad q \text{ normalized so that } \int d\mu = 1$$

Then $\xi_- = q \qquad \xi_+ = \bar{q}$

$$L^2(S^1)\xi_- \xrightarrow{\sim} H \xleftarrow{\sim} L^2(S^1)\xi_+ \qquad ?$$

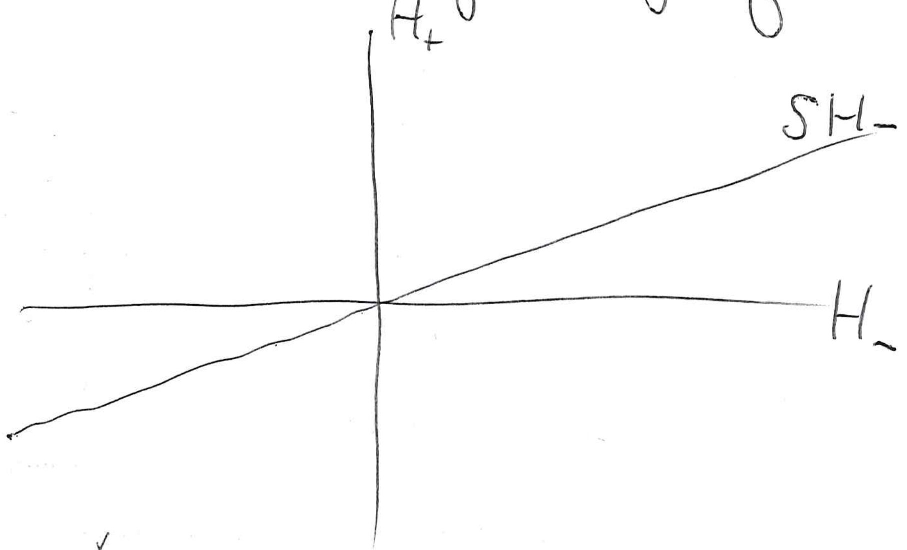
$$d\mu = \rho\frac{d\theta}{2\pi} \qquad F_{mn} = \langle z^{-m}, \dots, z^n \rangle \subset L^2(S^1, d\mu)$$

$$q_n \in F_{0n} \ominus z F_{0,n-1}$$

$$p_n = z^{+n}\bar{q}_n \in \bar{z}^n\left(F_{n0} \ominus \bar{z} F_{n-1,0}\right) = F_{0n} \ominus F_{0,n-1}$$

Suppose you form.

(108) Describe geometry of $H = SH_- \oplus H_+$



Actually you have good control over S.

V vector space, ΛV corresp. "Fock" space, have $GL(V)$, $gl(V)$ acting on ΛV, ΛV contains lines $\Lambda^{max} W$ for each $W \subset V$. Graeme's idea that over $Grass(V)$ you have a holom. line bundle such that ΛV is the space of holom. sections.

Geometric situation: Think carefully. You are dealing with rank 1 S. The ingredients are $L^2(S^1)$, $H_+^{\#}$, S. On the subspace level

Review: Your persistent problem ~~S~~ is to show the h-sequence for a smooth loop is at least summable. ~~However (h_n) is some sort~~ h_n is some function of S, specifically some angle related to the subspaces SH_-, H_+

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2 sided case. Start with ~~g_n~~

$$\begin{pmatrix} u^{-n}p_n \\ q_n \end{pmatrix} = \underbrace{\begin{pmatrix} u^{-n} & 0 \\ 0 & 1 \end{pmatrix} \frac{1}{k_n}\begin{pmatrix} 1 & h_n \\ \bar h_n & 1 \end{pmatrix}\begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} u^{n-1} & 0 \\ 0 & 1 \end{pmatrix}}_{\frac{1}{k_n}\begin{pmatrix} 1 & h_n u^{-n} \\ \bar h_n u^n & 1 \end{pmatrix}} \begin{pmatrix} u^{-n+1}p_{n-1} \\ q_{n-1} \end{pmatrix}$$

~~If $h_n \neq 0$, ... do~~

$$\begin{pmatrix} u^{-n}p_n \\ q_n \end{pmatrix} = T_{nm}\begin{pmatrix} u^{-m+1}p_{m-1} \\ q_{m-1} \end{pmatrix}$$

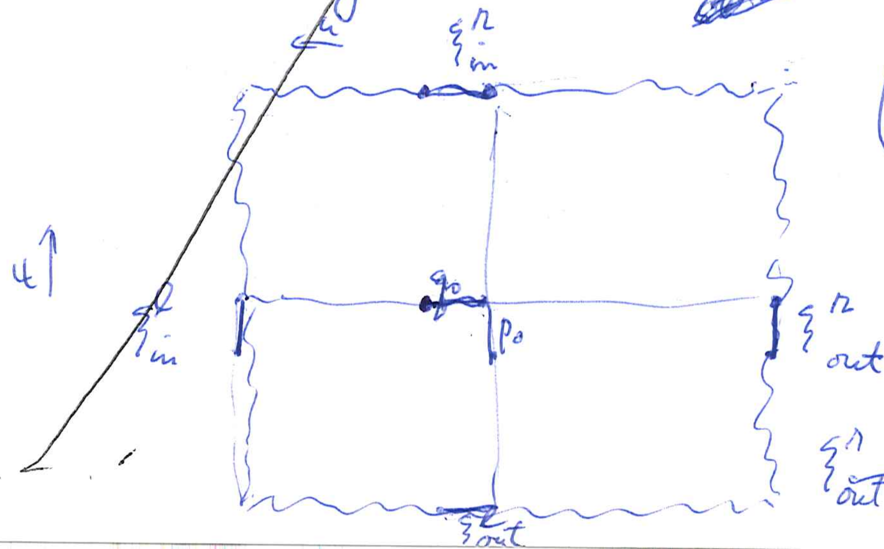
$$T_{nm} \in \begin{pmatrix} \overline{\langle z^0, \ldots, z^{n-m}\rangle} & \overline{\langle z^n, \ldots, z^m\rangle} \\ \langle z^n, \ldots, z^m\rangle & \langle z^0, \ldots, z^{n-m}\rangle \end{pmatrix}$$

To simplify you consider ~~...~~ $h_n = 0$ $|n| \gg 0$. Then $T = T_{\infty,-\infty} = T_{n,m}$ has the above form.

$$T_{n,m} = \begin{pmatrix} \bar d & \bar c \\ c & d \end{pmatrix} \qquad c \in \langle z^n, \ldots, z^m\rangle \quad d \in \langle z^0, \ldots, z^{n-m}\rangle$$

$1 = |d|^2 - |c|^2$ $\therefore$ $|d|^2 = 1 + |c|^2$ d is the poly $\ni$ $|d|^2 = 1 + |c|^2$ and ~~such the~~ zeroes outside S^1.

Scattering matrix ~~from~~ + the transfer matrix?



$$\begin{pmatrix} \xi^r_{out} \\ \xi^r_{in} \end{pmatrix} = \begin{pmatrix} \bar d & \bar c \\ c & d \end{pmatrix}\begin{pmatrix} \xi^l_{in} \\ \xi^l_{out} \end{pmatrix}$$

$$\frac{1}{d}\xi^r_{in} - \frac{c}{d}\xi^l_{in} = \xi^l_{out}$$

~~$\xi^r_{out} = (\bar d)\xi^l_{in} + (\ )\xi^r_{in}$~~

~~$\xi^r_{out} = \bar d\left(-\frac{c}{d}\right)\xi^l_{in} + \bar c\,\xi^l$~~

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$$\boxed{\begin{aligned} \xi^r_{out} &= \frac{1}{d}\xi^l_{in} + \frac{\bar c}{d}\xi^r_{in} \\ \xi^l_{out} &= -\frac{c}{d}\xi^l_{in} + \frac{1}{d}\xi^r_{in} \end{aligned}}$$

$$\left(\bar d - \frac{|c|^2}{d}\right) = \frac{1}{d}$$

$$\xi^r_{out} = \bar d\,\xi^l_{in} + \bar c \underbrace{\left(\xi^l_{out}\right)}_{\left(-\frac{c}{d}\xi^l_{in} + \frac{1}{d}\xi^r_{in}\right)}$$

~~Describe picture? You have H, u~~

You get a scattering picture H, u, $\xi^{l,r}_{in,out}$. The key point I think is that everything depends upon the reflection coeff $\frac{\bar c}{d}$

$$\begin{pmatrix} t & r \\ -\frac{t}{\bar t}\bar r & t \end{pmatrix}$$

Problem: You want to construct p_0, q_0 in terms of the bases at ∞. Thus you have the inc orth bases $z^k\xi^l_{in}$, $z^k\xi^r_{in}$ and you can ~~[illegible]~~ try to express p_0, q_0 in terms of this. What happens.

q_0 lin. comb. of $u^{\geq 0}\xi^l_{in}$, $u^{\geq 0}\xi^r_{out}$

p_0 ———— $u^{\geq 0}\xi^l_{in}$, $u^{\geq 0}\xi^r_{in}$