You want a nice picture of the irred 2-diml reps of $\mathbb{Z}/2 * \mathbb{Z}/2 = \langle F, \varepsilon \rangle$. ~~Choose basis so that $\varepsilon = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$~~ $H = \begin{pmatrix} H_+ \\ H_- \end{pmatrix}$ where H_+, H_- 1 diml. The other line V if $\neq H_+, H_-$ the graph of $T \in \mathrm{Hom}(H_+, H_-)$

$$\therefore \quad V = \begin{pmatrix} 1 \\ T \end{pmatrix} H_+ , \quad V^\perp = \begin{pmatrix} -T^* \\ 1 \end{pmatrix} H_-$$

$$F(1+X) = (1+X)\varepsilon$$

$$F = \frac{(1+X)}{1-X}\varepsilon = \frac{1+2X+X^2}{1-X^2}\varepsilon$$

Basically the invariant of the situation is the operator T modulo left + right mult by unitaries.

---

what about the contraction viewpoint?

~~So far you've looked at~~

So far you have looked at $H = \begin{pmatrix} H_+ \\ H_- \end{pmatrix}$ polarized equipped with $V \subset H$ (which is the same as an F). In this case there should be some spectral decomposition according to the irreducible reps of $\mathbb{Z}/2 * \mathbb{Z}/2$, which are parametrized by $\omega \in [0, \infty]$ essentially - slight problems with $0, \infty$.

Point: $j_+^* j_+ = j^* e_+ j$ is a self adj op on V

~~[illegible]~~ spectrum $\subset [0,1]$. ~~[illegible]~~

$$V = \bigoplus_{\lambda \in [0,1]} V_\lambda \qquad 1_V = \bigoplus_\lambda \pi_\lambda$$

$$j_+^* j_+ = \bigoplus_\lambda \lambda \pi_\lambda$$

$$\pi_\lambda j_+^* j_+ \pi_\lambda = \lambda \pi_\lambda$$

So you are looking at ~~[illegible]~~ $V_{\oplus} \subset$ ~~[illegible]~~ $\begin{pmatrix} H_+ \\ H_- \end{pmatrix}$

dilation: Given $V \xrightarrow{j} \begin{pmatrix} H_+ \\ H_- \end{pmatrix}$ NO

given c contraction on V.

Recall: ~~[illegible]~~ $X \underset{c^*}{\overset{c}{\rightleftarrows}} Y$ can be dilated to

a unitary.

$$\begin{matrix} X & & Y \\ \oplus & \underset{u^{-1}}{\overset{u}{\rightleftarrows}} & \oplus \\ V & & W \end{matrix}$$

$$\begin{matrix} & W & \\ {}^{\iota}\nearrow & & \searrow^{j} \\ X & \xrightarrow{c} & Y \end{matrix}$$

$$W = \iota X \oplus j^* Y$$

$$\| \iota x + j^* y \|^2 = \|x\|^2 + \underbrace{(\iota x | j^* y)}_{(y|c^* x)} + \underbrace{(j^* y | \iota x)}_{(y|cx)} + \|y\|^2$$

$$= \|y + cx\|^2 + \underbrace{\|x\|^2 - \|c^* x\|^2}_{x^*(1 - c^* c)x} \geq 0$$

various possibilities

$$\begin{matrix} X & & Y \\ a \searrow & & \swarrow b \\ & W & \end{matrix}$$

$a^* a = 1_X$

$b^* b = 1_Y$

$c = b^* a$

$c^* = a^* b$

Review. $X \overset{a}{\underset{b}{\rightrightarrows}} Y \qquad a^*a = b^*b = 1_X$

$V_+ = \mathrm{Ker}(a^*) \qquad V_- = \mathrm{Ker}(b^*)$.

$$Y = aX \oplus V_+ = V_- \oplus bX.$$

Krein space philosophy: $\begin{pmatrix} Y \\ Y \end{pmatrix}$ naturally a Krein space with herm. form

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}^* \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \|y_1\|^2 - \|y_2\|^2$$

Then $\begin{pmatrix} a \\ b \end{pmatrix} X$ is isotropic. Conversely if $X \subset \begin{pmatrix} Y \\ Y \end{pmatrix}$ is isotropic, let a, b be the two maps to Y. Then $\|ax\|^2 = \|bx\|^2$ so a, b extend to the completion of X for this hermitian inner product

$$\left(\begin{pmatrix} a \\ b \end{pmatrix} X\right)^\perp = \left\{ \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \middle| \; a^* y_1 = b^* y_2 \right\}$$

so $$\left(\begin{pmatrix} a \\ b \end{pmatrix} X\right)^\perp \supset \begin{pmatrix} a \\ b \end{pmatrix} X + \begin{pmatrix} V_+ \\ V_- \end{pmatrix}$$

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} a a^* y_1 \\ b b^* y_2 \end{pmatrix} + \begin{pmatrix} (1 - aa^*) y_1 \\ (1 - bb^*) y_2 \end{pmatrix}$$

Next bring in $L_z = \begin{pmatrix} 1 \\ z \end{pmatrix} Y \qquad |z| = 1$.

$$\left(\begin{pmatrix} a \\ b \end{pmatrix} X\right)^\perp \cap L_z = \left\{ \begin{pmatrix} ax + v_+ \\ bx + v_- \end{pmatrix} \middle| \; \begin{array}{l} z(ax + v_+) = (bx + v_-) \\ (za - b)x = -zv_+ + v_- \end{array} \right\}$$

At the moment you have

$X \overset{a}{\underset{b}{\rightrightarrows}} Y$ replaced by $X \overset{a+b}{\underset{a-b}{\rightrightarrows}} Y$. Can you say something about a

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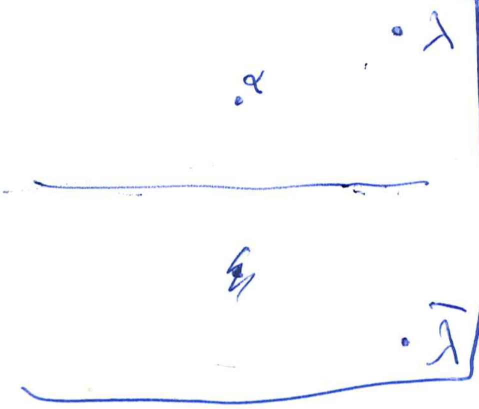
de Branges idea. $|E(\lambda)| < |E(\bar\lambda)|$ $\quad Im(\lambda) > 0$.

For example $E(\lambda) = \lambda - \alpha$ $\quad Im(\alpha) > 0$.



You ought to be able to reconstruct de Branges theory, especially you want to understand how it differs from what you've been doing.

From my viewpoint what's important starting point is a rational function $S(\lambda)$ analytic in the UHP of modulus 1 on the real axis. Such an S has the form

$\dfrac{E(\lambda)}{\overline{E(\bar\lambda)}}$ where is a polynomial with roots in the UHP. How would I find such an $E(\lambda)$? Form line bundles.

First treat z case. $S(z)$ rationals modulus 1 on $|z| = 1$ analytic for $|z| < 1$.

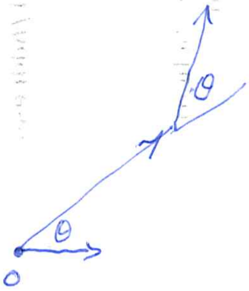
$$S(z) = \frac{p(z)}{\overline{p(z^*)}} \qquad z^* = (\bar z)^{-1} = \frac{1}{\bar z}$$

p polynomial roots in $|z| < 1$.

$$p(z) = \prod (z - \alpha_i) \qquad S(z) = \prod \frac{z - \alpha_i}{z^{-1} - \bar\alpha_i}$$

$$\overline{p(z^*)} = \prod (z^{-1} - \bar\alpha_i) \qquad = z^n \prod \left(\frac{z - \alpha_i}{1 - \bar\alpha_i z}\right) ?$$

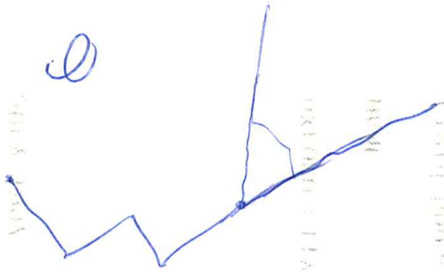
What you need is some understanding of convergence. Convergence is easy for $|z|<1$ and $|z|>1$. So $|z|=1$ is the important situation. An argument in this case might handle a nbd around z. In fact you need to link the D and D^{-1} results. So what technique? I believe broken geodesics will not give a satisfactory tool since they seem to be linked to 0 starting point. But regular polygons give the first insight into convergence.

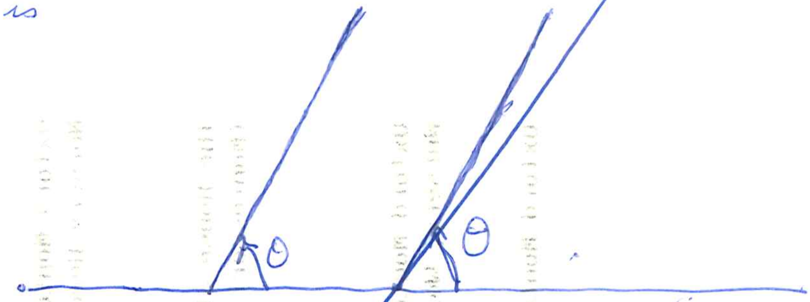


Basic problem: Given $g_1, g_2, \ldots$ and you want convergence of $g_1(\alpha), g_1g_2(\alpha), \ldots$. For what α does this converges and what is the limit? When $g_n = g^n$ the answer is given by eigenvectors. If g loxodromic $|\lambda_1|>1$, then for every α except the expanding fixpt the sequence converges to the contracting fixpt. ~~Concretely~~ If we put the exp. fixpt at ∞, contr at 0, then $g = \lambda_1^{-2}$, $|\lambda_1|>1$. The type of "convergence region" you want to use are maybe ~~circles~~ disks

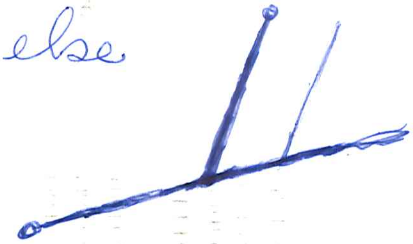
Guess: You basically have the following diagram

better is





Try something else



August 10. 1998 Go back to equations

$$\begin{pmatrix}\psi_0^+\\ \psi_0^-\end{pmatrix} = \underbrace{\begin{pmatrix} a & b \\ b & a\end{pmatrix}\begin{pmatrix} z^{1/2} & 0 \\ 0 & z^{-1/2}\end{pmatrix}}_{\begin{pmatrix} az^{1/2} & bz^{-1/2} \\ bz^{1/2} & az^{-1/2}\end{pmatrix}}\begin{pmatrix}\psi_1^+\\ \psi_1^-\end{pmatrix}$$

$$\lambda^2 - a\overbrace{(z^{1/2}+z^{-1/2})}^{2\cos\theta}\lambda + 1 = 0$$

Expect nice properties at $z = 1$. $\lambda+\lambda^{-1} = a(z^{\frac{1}{2}}+z^{-1/2})$

Define $\psi_0 = g_0\psi_1 = g_0 g_2 \psi_2$

g(0)

~~What you need this~~

First understand constant coeff case. Here you iterate g which is loxodromic in general

Idea: You want to avoid broken geodesics because things should be analytic in z. Also the ~~contr~~ limiting ~~[illegible]~~ situation ~~expan~~ is clear for $|z|>1$

So it seems that a broken geodesic has the form

$$0,\ \underbrace{\begin{pmatrix} z_1^{1/2} & 0 \\ 0 & z_1^{-1/2}\end{pmatrix}\begin{pmatrix} a_1 & b_1 \\ b_1 & a_1\end{pmatrix}}_{g_1}(0),\ \begin{pmatrix} z_t^{1/2} & 0 \\ 0 & z_t^{-1/2}\end{pmatrix}\begin{pmatrix} a_t & b_t \\ b_t & a_t\end{pmatrix}\begin{pmatrix} z_2^{1/2} & 0 \\ 0 & z_2^{-1/2}\end{pmatrix}\begin{pmatrix} a_2 & b_2 \\ b_2 & a_2\end{pmatrix}(0)$$

~~This makes sense~~ This makes sense because any product

$$g_1 g_2 = k_1 a_1 k_1' k_2 a_2 k_2'$$
$$= k_1 a_1 k_1^{-1}(k_1 k_1' k_2)\, a_2\, (k_1 k_1' k_2)^{-1}(k_1 k_1' k_2 k_2')$$

You are using $G = KAK$ then

$$(g_1 g_2 \cdots g_n)(0) = k_1 a_1 k_2 a_2 \cdots k_n a_n k_{n+1}(0)$$

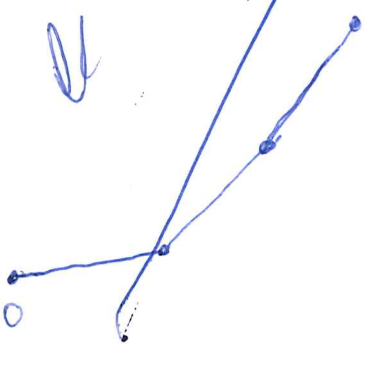
At this point you understand the relation between matrices and broken geodesics. Next you want the end of the broken geodesic to go to the boundary.

The key statement has to be some statement about $a_n \cos\frac{\theta}{2} \to 0$ then you get convergence

4. You want to see what happens when you add something to Maybe you need a triangle. Point.

Somehow you want to estimate the effect of adding another leg.

$\Downarrow$



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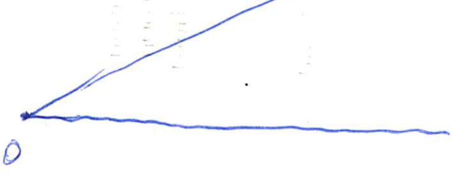


0

Now I try to use the

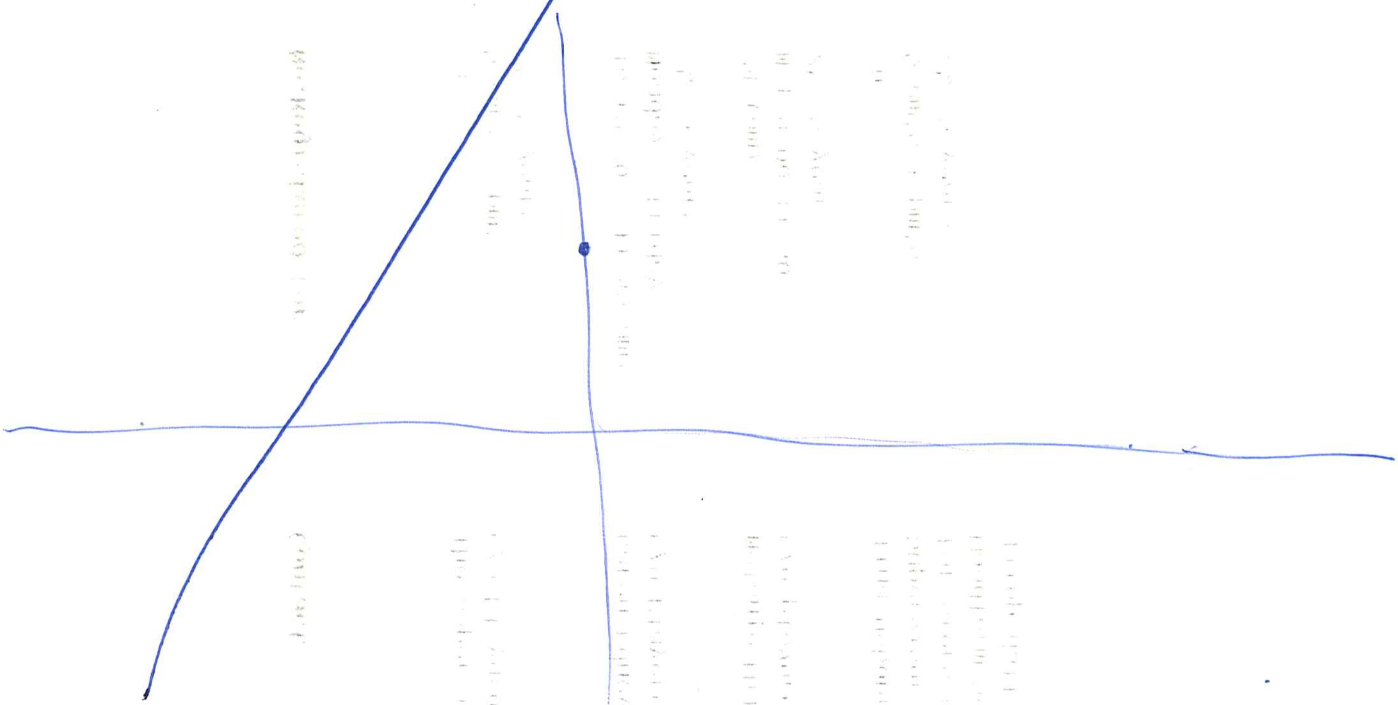
$$z = \frac{1+i\omega}{1-i\omega}$$

$$\begin{pmatrix} a & b \\ b & a \end{pmatrix}\begin{pmatrix} i & 1 \\ -i & 1 \end{pmatrix} = \begin{pmatrix} i & 1 \\ -i & 1 \end{pmatrix}\begin{pmatrix} a-b & \\ & a+b \end{pmatrix}$$

$$\frac{1}{2i}\begin{pmatrix} 1 & -1 \\ i & i \end{pmatrix}\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}\begin{pmatrix} i & 1 \\ -i & 1 \end{pmatrix}$$

$$= \frac{1}{2i}\begin{pmatrix} 1 & -1 \\ i & i \end{pmatrix}\begin{pmatrix} \lambda i & \lambda \\ -\lambda^{-1} i & \lambda^{-1} \end{pmatrix} = \begin{pmatrix} \frac{\lambda+\lambda^{-1}}{2} & \frac{\lambda-\lambda^{-1}}{2i} \\ -\frac{\lambda-\lambda^{-1}}{2i} & \frac{\lambda+\lambda^{-1}}{2} \end{pmatrix} = \begin{pmatrix} \cos\frac{\theta}{2} & \sin\frac{\theta}{2} \\ -\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix}$$

g becomes $\begin{pmatrix} a-b & 0 \\ 0 & a+b \end{pmatrix}\begin{pmatrix} \cos\frac{\theta}{2} & \sin\frac{\theta}{2} \\ -\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix}$



two orth unit vectors $\hat{u}_1$, $\hat{u}_2$

Condition $\quad |\hat{u}_2 \cdot \psi| \geq \varepsilon \|\psi\|$

~~Same~~ equiv.: $\quad |\hat{u}_1 \cdot \psi| \leq (1-\varepsilon^2)^{1/2} \|\psi\|$

Maybe the reason for uniqueness is that once you know

---

check.

~~$w = \frac{-iz+i}{z+1} = \begin{pmatrix} -1 & 1 \\ i & i \end{pmatrix} z$~~

~~[illegible]~~

$$w = \frac{1}{i}\,\frac{z-1}{z+1} = \begin{pmatrix} 1 & -1 \\ i & i \end{pmatrix}$$

$$\frac{1}{2i}\begin{pmatrix} 1 & -1 \\ i & i \end{pmatrix}\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}\begin{pmatrix} i & 1 \\ -i & 1 \end{pmatrix} = \frac{1}{2i}\begin{pmatrix} 1 & -1 \\ i & i \end{pmatrix}\begin{pmatrix} \lambda i & \lambda \\ -\lambda^{-1} i & \lambda^{-1} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{\lambda+\lambda^{-1}}{2} & \frac{\lambda-\lambda^{-1}}{2i} \\ i\frac{\lambda-\lambda^{-1}}{2} & \frac{\lambda+\lambda^{-1}}{2} \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \qquad \lambda = e^{i\theta}$$

$$g = \begin{pmatrix} \lambda & \\ & \lambda^{-1} \end{pmatrix}\begin{pmatrix} \cos\frac{\theta}{2} & \sin\frac{\theta}{2} \\ -\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix}$$

Go over what you are trying to do.

You have g. You understand how g acts on S^1 scrunches from small eigenvector $\sim \begin{pmatrix} -z^{-1/2} \\ z^{1/2} \end{pmatrix}$ to the large one $\sim \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$$g = \frac{1}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix}(a+b)\begin{pmatrix} z^{1/2} & z^{-1/2} \end{pmatrix} + \frac{1}{2}\begin{pmatrix} -1 \\ 1 \end{pmatrix}(a-b)\begin{pmatrix} -z^{1/2} & z^{-1/2} \end{pmatrix}$$

What sort of estimates do you want? You would like to know that ~~You know that~~ ~~ideal slides~~

Is what you're seeking a perturbation. You $g_0 = \begin{pmatrix} a & b \\ b & a \end{pmatrix}$ and the perturbation $g = \begin{pmatrix} a & b \\ b & a \end{pmatrix}\begin{pmatrix} z^{1/2} & 0 \\ 0 & z^{-1/2} \end{pmatrix}$

This takes place inside $SU(1,1)$, and if z ~~is compl~~ moves over S^1 inside $SL(2,\mathbb{C})$. So you a perturbation of something you understand and maybe you can ~~do~~ explain the phenomena this way.

<u>This perturbation idea looks good.</u> But work is needed

$$\frac{z-1}{z+1} = \frac{e^{i\phi}-1}{e^{i\phi}+1} \qquad \frac{i\cos + \sin}{-i\sin + \cos} = i$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

Jean & Dan Quillen

try a little.

$z = 0 \longmapsto w = 0$

$z = -1 \longmapsto w = \infty$

$z = i \longmapsto w = +1.$

$$w = \frac{1}{i}\frac{z-1}{z+1}$$

$$\begin{pmatrix} -i & 1 \\ 1 & 1 \end{pmatrix}$$

z side $\quad z = 1$ contr. fixpt

$z = -1$ exp fixpt.

$$\begin{pmatrix} a & b \\ b & a \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} a+b & \\ & a-b \end{pmatrix}$$

Take ~~[illegible]~~ $\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$

$$\begin{pmatrix} a & b \\ b & a \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} (a+b)w_1 \\ (a-b)w_2 \end{pmatrix}$$

~~$\begin{pmatrix} a & b \\ b & a \end{pmatrix}\begin{pmatrix} +1 & 1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$~~

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$$

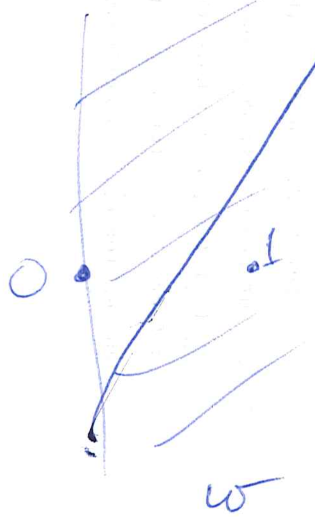
$$z = \frac{-w+1}{w+1}$$

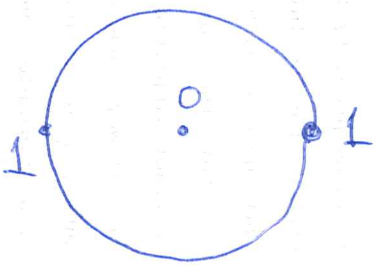
~~$z = 0 \quad w = 1$~~

$w = 0 \quad z = 1$

$w = \infty \quad z = -1$

$w = 1 \quad z = 0$





~~So your point We~~ We can pick any interval $\ni$ $g(I) \subsetneq I$ $\forall$ g satisfying appropriate scrunch. Can maybe control ~~bound~~ the derivative of g on I so that $d(g(\xi), g(\xi')) \leq k\, d(\xi, \xi')$. This should give the fixpt needed.

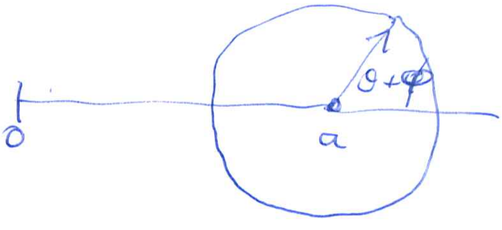
$$g(\xi) = \frac{a\xi + b}{c\xi + d}$$

$$g'(\xi) = \frac{(c\xi + d)(a) - (a\xi + b)(c)}{(c\xi + d)^2} = \frac{1}{(c\xi + d)^2}$$

$$g = \begin{pmatrix} a z^{1/2} & b z^{-1/2} \\ b z^{1/2} & d z^{-1/2} \end{pmatrix} \qquad |b z^{1/2}\xi + d z^{-1/2}|^2$$

$$= |a + b z \xi|^2$$

($z = e^{i\theta}$, $\xi = e^{i\phi}$)



this is minimum when $z\xi = -1$ i.e. $\phi = \pi - \theta$

But for a large

So it seems that this will give convergence result as follows. Assume θ fixed. You want uniqueness somehow. Can I get uniqueness?

Try again $g = \begin{pmatrix} a & b \\ b & a \end{pmatrix}\begin{pmatrix} z^{1/2} & \\ & z^{-1/2} \end{pmatrix}$ $\quad z = e^{i\theta}$

$$g = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} a+b & 0 \\ 0 & a-b \end{pmatrix}\begin{pmatrix} 1 & +1 \\ -1 & +1 \end{pmatrix}\begin{pmatrix} z^{1/2} & 0 \\ 0 & z^{-1/2} \end{pmatrix}$$

The idea $g = \begin{pmatrix} \lambda & \\ & \lambda^{-1} \end{pmatrix} \begin{pmatrix} \cos & \sin \\ -\sin & \cos \end{pmatrix}$

$$g\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \lambda\left(x\cos\frac{\theta}{2} + y\sin\frac{\theta}{2}\right) \\ \lambda^{-1}\left(-x\sin\frac{\theta}{2} + y\cos\frac{\theta}{2}\right) \end{pmatrix} \qquad g\psi = \begin{pmatrix} \lambda & \\ & \lambda^{-1} \end{pmatrix} R_\theta \psi$$

$$\|g\psi\|^2 = \underbrace{\lambda^2}_{\text{small}} \left|x\cos\frac{\theta}{2} + y\sin\frac{\theta}{2}\right|^2 + \underbrace{\lambda^{-2}}_{\text{large}} \left|-x\sin\frac{\theta}{2} + y\cos\frac{\theta}{2}\right|^2$$

$$\|g\psi\|^2 \geq \lambda^{-2} \left| \psi \cdot \begin{pmatrix} -\sin\frac{\theta}{2} \\ \cos\frac{\theta}{2} \end{pmatrix} \right|^2$$

$$\|g\psi\| \geq \lambda^{-1} \left| \underbrace{\begin{pmatrix} -\sin\frac{\theta}{2} \\ \cos\frac{\theta}{2} \end{pmatrix}}_{\hat{u}_2} \cdot \psi \right|$$

You want to restrict ψ to the satisfying $|\hat{u}_2 \cdot \psi| > \varepsilon \|\psi\|$

$$\|\psi\|^2 = \left| \underbrace{\begin{pmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2} \end{pmatrix}}_{\hat{u}_1} \cdot \psi \right|^2 + \left| \underbrace{\begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}}_{\hat{u}_2} \cdot \psi \right|^2$$

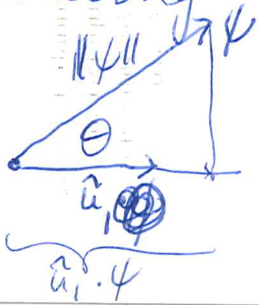
so that $|\hat{u}_2 \cdot \psi| \geq \varepsilon \|\psi\|$ should ~~such~~ be equivalent to

$$|\hat{u}_2 \cdot \psi|^2 \geq \varepsilon^2 \|\psi\|^2$$

$$\Updownarrow$$

$$|\hat{u}_1 \cdot \psi|^2 = \|\psi\|^2 - |\hat{u}_2 \cdot \psi|^2 \leq (1-\varepsilon^2)\|\psi\|^2$$

. Perhaps you are describing a cone ?



$$\cos\theta = \frac{\hat{u}_1 \cdot \psi}{\|\psi\|} \leq \sqrt{1-\varepsilon^2}$$

What am I trying to see? You want to understand

$$\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}\begin{pmatrix} \cos\frac{\theta}{2} & \sin\frac{\theta}{2} \\ -\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix}$$

~~λ ... > 0~~. fixed

$0 < \lambda \leq \lambda_0$

$$g = \begin{pmatrix} \lambda\cos\left(\frac{\theta}{2}\right) & \lambda\sin\left(\frac{\theta}{2}\right) \\ -\lambda^{-1}\sin\left(\frac{\theta}{2}\right) & \lambda^{-1}\cos\left(\frac{\theta}{2}\right) \end{pmatrix}$$

act on a region $|x| \leq 1$

apply g to $\begin{pmatrix} x \\ 1 \end{pmatrix}$ to get

$$\begin{pmatrix} \lambda\left(x\cos\frac{\theta}{2} + \sin\frac{\theta}{2}\right) \\ \lambda^{-1}\left(-x\sin\frac{\theta}{2} + \cos\frac{\theta}{2}\right) \end{pmatrix}$$

ratio is $\lambda^2\left(\dfrac{x\cos\frac{\theta}{2} + \sin\frac{\theta}{2}}{-x\sin\frac{\theta}{2} + \cos\frac{\theta}{2}}\right)$ Is this $|\ \ | < 1$?

for small λ. Yes provided $-x\sin\frac{\theta}{2} + \cos\frac{\theta}{2} \neq 0$

~~Certainly we can arrange Now $|x|$~~

~~Also~~ If false then ~~x tan $x\tan\frac{\theta}{2} = 0$~~

$x\tan\left(\frac{\theta}{2}\right) = 1$ which doesn't happen if $|x| \leq 1$

and $\tan\left(\frac{\theta}{2}\right) < 1$ i.e. $\frac{\theta}{2} < \frac{\pi}{4}$

what about

$$u_2^t g\psi = (-\sin\tfrac{\theta}{2})\lambda u_1^t\psi + (\cos\tfrac{\theta}{2})\lambda^{-1} u_2^t\psi$$

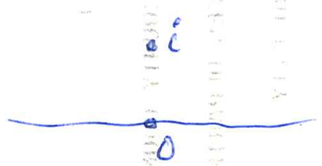
$$\lambda\, u_2^t g\psi = u_2^t(\lambda g\psi) \longrightarrow u_2^t\begin{pmatrix}0\\ u_2^t\psi\end{pmatrix} = \cos\tfrac{\theta}{2}\, u_2^t\psi$$

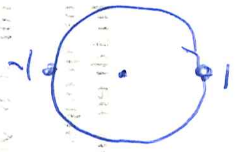
$$\frac{|u_2^t g\psi|}{\|g\psi\|} = \frac{\lambda|u_2^t g\psi|}{\lambda\|g\psi\|}$$

$$\downarrow$$

$$\frac{|\cos\tfrac{\theta}{2}\, u_2^t\psi|}{|u_2^t\psi|}$$

Review. Go back to





$$w = \frac{1}{i}\frac{z-1}{z+1} = \frac{-iz+i}{z+1}$$

$$w = \begin{pmatrix}-i & i\\ 1 & 1\end{pmatrix} z$$

$$z = \begin{pmatrix}1 & -i\\ -1 & -i\end{pmatrix} w$$

$$w = \frac{z-1}{iz+i} = \begin{pmatrix}1 & -1\\ i & i\end{pmatrix} z$$

$$z = \begin{pmatrix}i & 1\\ -i & 1\end{pmatrix} w$$

Idea you had last night, what separate two families i.e.

$$\begin{pmatrix}\lambda_n & \\ & \lambda_n^{-1}\end{pmatrix} \qquad \begin{pmatrix}c_n & s_n\\ -s_n & c_n\end{pmatrix}$$

into $a, b : X \rightrightarrows X'$

Try again: You have

$$g\psi = \begin{pmatrix} \lambda u_1^t \psi \\ \lambda^{-1} u_2^t \psi \end{pmatrix}$$

$$\frac{\|g\psi\|^2}{\|\psi\|^2} = \lambda^2 \underbrace{\frac{|u_1^t\psi|^2}{\|\psi\|^2}}_{\text{small}} + \lambda^{-2} \underbrace{\frac{|u_2^t\psi|^2}{\|\psi\|^2}}_{\text{large}}$$

want $\geq \delta$

basic estimate is

$$\lambda^{-2} \geq \frac{\|g\psi\|^2}{\|\psi\|^2} \geq \lambda^{-2}\frac{|u_2^t\psi|^2}{\|\psi\|^2} \geq \lambda^{-2}\delta^2$$

assuming $\psi \ni \frac{\|u_2^t\psi\|}{\|\psi\|} \geq \delta$

$$\Delta = \left\{ \psi \,\middle|\, \frac{|u_2^t\psi|}{\|\psi\|} \geq \delta \right\}.$$ Is $g(\Delta) < \Delta$

for large λ?

$$\frac{|u_2^t g\psi|}{\|g\psi\|} =$$

$$\underbrace{\frac{|u_2^t g\psi|}{\|g\psi\|}}_{\downarrow} \cdot \underbrace{\frac{\|g\psi\|}{\|\psi\|}}_{\geq \lambda^{-1}\delta}$$

$\sin\frac{\theta}{2} \; \lambda^{-2}$

$$\frac{\lambda g\psi}{\|g\psi\|} \to \begin{pmatrix} 0 \\ u_2^t\psi \end{pmatrix}$$

$$\lambda^2 \frac{\|g\psi\|^2}{\|\psi\|^2} = \lambda^2 \frac{\lambda^2 |u_1^t\psi| + \lambda^{-2}|u_2^t\psi|^2}{|u_1^t\psi|^2 + |u_2^t\psi|^2} \to 1$$

provided $|u_2^t\psi| \neq 0$

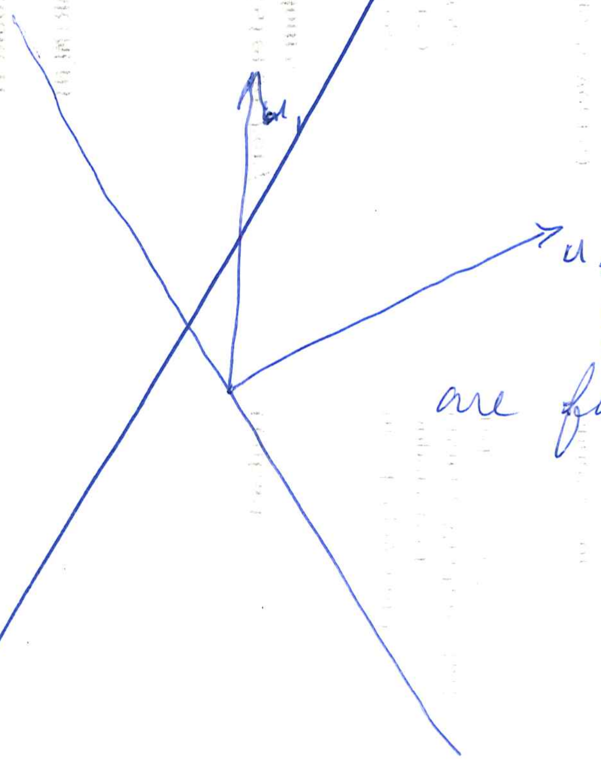
need to know $g\psi > g_2\psi$

$$\cos\theta_2 = \frac{u_2 \cdot \psi}{\|u_2\| \cdot \|\psi\|} \quad \text{assume } \geq 0$$

$$\cos\theta_1 = \frac{u_1 \cdot \psi}{\|u_1\| \cdot \|\psi\|} \quad \geq 0$$

want $\frac{u_2 \cdot \psi}{\|\psi\|} \geq \delta$

i.e. $\theta_2 \leq \frac{\pi}{2} - \delta'$



It seems to be better to ask that $\frac{|u_2 \cdot \psi|}{\|\psi\|} \geq \delta$. Lines gen. by u_2, ψ are far from being $\perp$.

Review the argument $g = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \begin{pmatrix} \cos\frac{\theta}{2} & \sin\frac{\theta}{2} \\ -\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix}$

$z = e^{i\theta}$. θ is fixed but λ varies to solve $\psi_0 = g_1 \psi_1$, $\psi_1 = g_2 \psi_2, \ldots$

find a non zero solution, where $\|\psi_n\|$ decays as $n \to \infty$.

$$g\psi = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \begin{pmatrix} u_1^t \psi \\ u_2^t \psi \end{pmatrix} \qquad \begin{matrix} u_1^t = (\cos \ \sin) \\ u_2^t = (-\sin \ \cos) \end{matrix}$$

$$\|g\psi\|^2 = \lambda^2 |u_1^t \psi|^2 + \lambda^{-2} |u_2^t \psi|^2$$

Look forward to the case where θ is complex

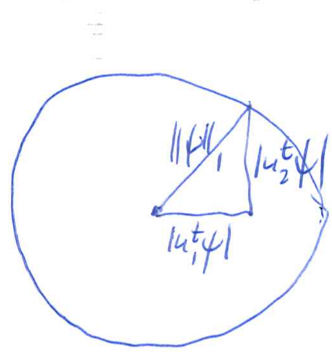
You want $\dfrac{|u_2^t \psi|}{\|\psi\|} \geq \varepsilon > 0$

You have another inner product

$\|\psi\|_1^2 = |u_1^t \psi|^2 + |u_2^t \psi|^2$ so $C^{-1}\|\psi\| \leq \|\psi\|_1 \leq C\|\psi\|$

Thus $\dfrac{|u_2^t \psi|}{\|\psi\|} \geq \dfrac{|u_2^t \psi|}{C\|\psi\|_1} \geq \varepsilon > 0$

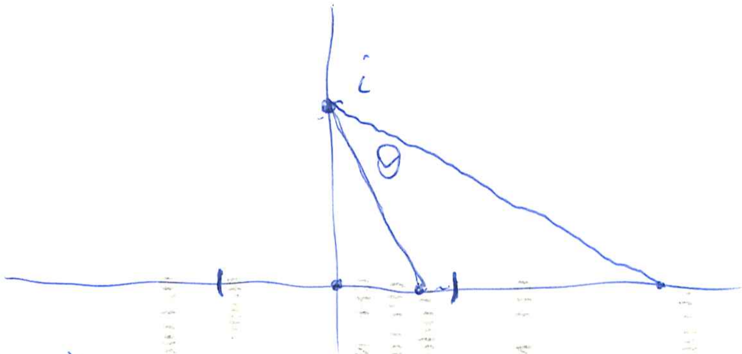
provided $\dfrac{|u_2^t \psi|}{\sqrt{|u_1^t \psi|^2 + |u_2^t \psi|^2}} \geq$ sine $\delta > 0$.



equivalent to $\dfrac{|u_1^t \psi|}{\|\psi\|_1} \leq$ sine $\delta' < 1$.

So continue with the unitary case

I am concerned with iteration. Convexity geometry



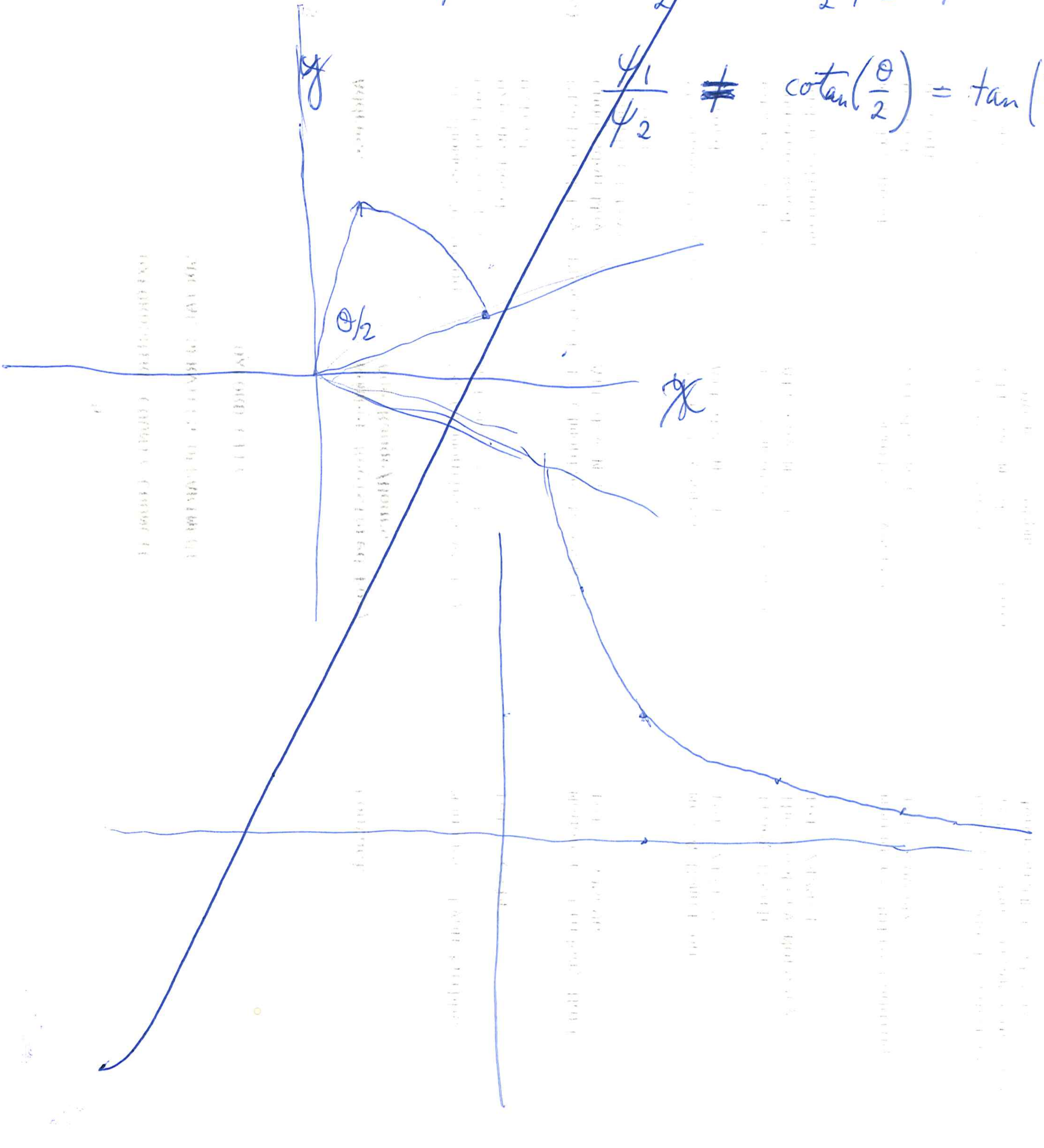
$$\begin{pmatrix}\lambda c_1\\ \lambda^{-1}c_2\end{pmatrix}$$

||

$$\psi = \begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix} \longmapsto \begin{pmatrix}\lambda\cos\frac{\theta}{2}\psi_1+\lambda\sin\frac{\theta}{2}\psi_2\\ -\lambda^{-1}\sin\frac{\theta}{2}\psi_1+\lambda^{-1}\cos\frac{\theta}{2}\psi_2\end{pmatrix} = \begin{pmatrix}\lambda u_1^t\psi\\ \lambda^{-1}u_2^t\psi\end{pmatrix}$$

do [redacted] want $u_2^t\psi = -\sin\frac{\theta}{2}\psi_1+\cos\frac{\theta}{2}\psi_2 \neq 0$

$$\frac{\psi_1}{\psi_2} \neq \cot\left(\frac{\theta}{2}\right) = \tan\left(\frac{\pi-\theta}{2}\right)$$



What's important are the numbers $c_i = u_i^t \psi$

$$g\psi = \begin{pmatrix} \lambda c_1 \\ \lambda^{-1} c_2 \end{pmatrix} \qquad \|g\psi\|^2 = \lambda^2 |c_1|^2 + \lambda^{-2} |c_2|^2$$

$$u_1^t g\psi = \lambda\left(\cos\frac{\theta}{2}\, c_1\right) + \lambda^{-1}\left(\sin\frac{\theta}{2}\, c_2\right)$$

$$|u_1^t g\psi|^2 \leq \left(\lambda^2 \cos^2\frac{\theta}{2} + \lambda^{-1}\sin^2\frac{\theta}{2}\right)\left(c_1^2 + c_2^2\right)$$

Compare $u_1^t g\psi$ and $g\psi$

$$u_1^t g\psi = \lambda c_1 \cos\frac{\theta}{2} + \lambda^{-1} c_2 \sin\frac{\theta}{2}$$

$$\|g\psi\|^2 = \lambda^2 |c_1|^2 + \lambda^{-2} |c_2|^2$$

we are assuming $|u_1^t \psi| \leq \delta \|\psi\|$

$$c_1 \leq \delta \sqrt{c_1^2 + c_2^2}$$

take $c_2 = 1$, ~~then~~ suppose $|c_1| \leq k$

$$\|\psi\|^2 \leq 1 + |c_1|^2$$

$$\|g\psi\|^2 = \lambda^2 k^2 + \lambda^{-2}$$

$$|u_1^t g\psi| \leq \lambda |c_1| \cos\frac{\theta}{2} + \lambda^{-1} \sin\frac{\theta}{2}$$

If $c_1 = 0$ you see that $\delta \nrightarrow$ $|u_1^t g\psi| \leq \delta \|g\psi\|$

$$\lambda^{-1}\sin\left(\frac{\theta}{2}\right) \qquad \delta\lambda^{-1}$$

so δ

$$\psi = \begin{pmatrix} x \\ y \end{pmatrix} \qquad g\psi = \begin{pmatrix} \lambda\, u_1^t \psi \\ \lambda^{-1} u_2^t \psi \end{pmatrix}$$

$u_1^t = \cos \; \sin$
$u_2^t = -\sin \; \cos.$

$$\|g\psi\|^2 = \lambda^2 |u_1^t\psi|^2 + \lambda^{-2}|u_2^t\psi|^2$$

$$\frac{\|g\psi\|^2}{\|\psi\|^2} = \lambda^2 \underbrace{\left(\frac{|u_1^t\psi|}{\|\psi\|}\right)^2}_{\alpha_1} + \lambda^{-2}\underbrace{\left(\frac{|u_2^t\psi|}{\|\psi\|}\right)^2}_{\alpha_2} \qquad \lambda < 1.$$

~~since~~ $\alpha_1^2 + \alpha_2^2 = 1.$

~~$\lambda^2\alpha_1^2 \leq \frac{\|g\psi\|^2}{\|\psi\|^2} = \lambda^2\alpha_1^2 + \lambda^{-2}(1-\alpha_1^2)$~~

Put $\alpha_1^2 = \frac{|u_1^t\psi|^2}{\|\psi\|^2} \qquad \alpha_2^2 = \frac{|u_2^t\psi|^2}{\|\psi\|^2} \qquad \alpha_1^2+\alpha_2^2 = 1.$

$$\lambda^{-2} \geq \left(\frac{\|g\psi\|}{\|\psi\|}\right)^2 = \overset{\text{small}}{\lambda^2\alpha_1^2} + \overset{\text{large}}{\lambda^{-2}\alpha_2^2} \geq \lambda^{-2}(1-\alpha_1^2)$$

Region $\alpha_1^2 = \frac{|u_1^t\psi|^2}{\|\psi\|^2} \leq \delta^2$ Keep the probability that $\psi \approx u_1$ small, then $\|\psi\| \sim |u_2^t\psi|$

Region defined by $|u_1^t\psi| \leq \delta\|\psi\|$
Does g preserve this region.

$$u_1^t g\psi = \lambda\left(\cos\tfrac{\theta}{2}\, u_1^t\psi\right) + \lambda^{-1}\left(\sin\tfrac{\theta}{2}\, u_2^t\psi\right)$$

$$\|g\psi\|^2 = \lambda^2|u_1^t\psi|^2 + \lambda^{-2}|u_2^t\psi|^2 \cong \lambda^2\alpha_1^2 + \lambda^{-2}\alpha_2^2$$

$|u_1^t g\psi|$ should be approx $\lambda^{-1}\sin\frac{\theta}{2}\alpha_2$ $\qquad \|g\psi\|$ approx $\lambda^{-1}\alpha_2$

System. $g = \begin{pmatrix} k & \\ & k^{-1} \end{pmatrix}\begin{pmatrix} 1 & \mu \\ -\mu & 1 \end{pmatrix}$

μ eigenvalue parameter $k > 0$.

$w = \frac{1}{i}\frac{z-1}{z+1} = \begin{pmatrix} 1 & -1 \\ i & i \end{pmatrix} z \qquad z = \begin{pmatrix} i & 1 \\ -i & 1 \end{pmatrix} w$

0	1
∞	-1
i	0

$\frac{1}{2i}\begin{pmatrix} 1 & -1 \\ i & i \end{pmatrix}\begin{pmatrix} t & \\ & t^{-1} \end{pmatrix}\begin{pmatrix} ti & t \\ -t^{-1}i & t^{-1} \end{pmatrix} = \begin{pmatrix} \frac{t+t^{-1}}{2} & \frac{t-t^{-1}}{2} \\ \frac{-t+t^{-1}}{2i} & \frac{t+t^{-1}}{2} \end{pmatrix}$

$\begin{pmatrix} c & s \\ -s & c \end{pmatrix}$ The pair c, s is equivalent to $z^{1/2}$

$z^{1/2} = c + is$

$z^{-1/2} = c - is$

So $\mu = \frac{s}{c} = \frac{1}{i}\frac{z^{1/2} - z^{-1/2}}{z^{1/2} + z^{-1/2}} = \frac{1}{i}\frac{z-1}{z+1}$.

You have a unitary $\mu = u_1 u_2^{-1}$. Then

$(u_1 u_2^{-1} - 1)(u_1 u_2^{-1} + 1)^{-1}$

$(u_1 - u_2)u_2^{-1}\left((u_1 + u_2)u_2^{-1}\right)^{-1}$

$\sqrt{1+\mu^2}$ has problems at $\mu = \pm i$

$= (u_1 - u_2)u_2^{-1}u_2(u_1+u_2)^{-1} = (u_1 - u_2)(u_1 + u_2)^{-1}$

~~$u_1 u_2$~~ $(u_1 - u_2)(u_1 + u_2)^{-1} + (u_1^* + u_2^*)^{-1}(u_1^* - u_2^*) = 0$

$\qquad \| \qquad (u_1^{-1} + u_2^{-1})^{-1}(u_1^{-1} - u_2^{-1})$

$(u_1 u_2^{-1} - 1)(u_1 u_2^{-1} + 1)^{-1} \qquad (1 + u_1 u_2^{-1})^{-1}(1 - u_1 u_2^{-1})$

~~So the problem is clear~~ It seems that given your ladder

Put into words. The "large" eigenvector → $\begin{pmatrix} 1 \\ \frac{\bar{b}}{a} \end{pmatrix}$, the "small" eigenvector → $\begin{pmatrix} -z^{-1/2} \\ \frac{\bar{b}}{a} z^{1/2} \end{pmatrix}$

phase of $\frac{\bar{b}}{a} = \bar{h}$ phase of $\frac{\bar{b}}{a} = \bar{h}$

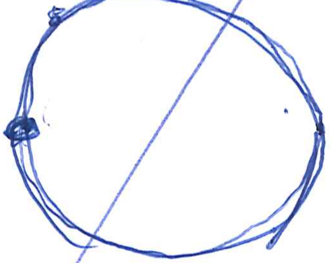
Assume $b > 0$. Limiting eigenvectors are $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} -z^{-1/2} \\ z^{1/2} \end{pmatrix}$

Now we know that $\begin{pmatrix} a & b \\ b & a \end{pmatrix}$ has eigenvector $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ with large eigenvalue $a+b$, and ——— $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$ with small ——— $a-b$. What about $\begin{pmatrix} a & b \\ b & a \end{pmatrix}\begin{pmatrix} z^{1/2} & 0 \\ 0 & z^{-1/2} \end{pmatrix}$ Clearly $\begin{pmatrix} -z^{-1/2} \\ z^{1/2} \end{pmatrix}$ gets rotated by into $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$ whose image under $\begin{pmatrix} a & b \\ b & a \end{pmatrix}$ is small $(a-b)\begin{pmatrix} -1 \\ 1 \end{pmatrix}$. The

---

$$\begin{pmatrix} a & b \\ b & a \end{pmatrix}\begin{pmatrix} z^{1/2} & \\ & z^{-1/2} \end{pmatrix} \qquad \begin{pmatrix} \lambda - a z^{-1/2} \\ b z^{1/2} \end{pmatrix} = \begin{pmatrix} \lambda - a\cos\frac{\theta}{2} + i a \sin\frac{\theta}{2} \\ b \end{pmatrix}$$

$-e^{-i\theta}$



Visualize $\begin{pmatrix} a & b \\ b & a \end{pmatrix}\begin{pmatrix} z^{1/2} & 0 \\ 0 & z^{-1/2} \end{pmatrix}$ and its inverse.

$$\begin{pmatrix} a & b \\ b & a \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = (a+b)\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} a & b \\ b & a \end{pmatrix}\begin{pmatrix} -1 \\ 1 \end{pmatrix} = (a-b)\begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = c\begin{pmatrix} 1 \\ 1 \end{pmatrix} + c'\begin{pmatrix} -1 \\ 1 \end{pmatrix} \qquad \frac{az+b}{bz+a}$$

$$\begin{pmatrix} a & b \\ b & a \end{pmatrix}^n \begin{pmatrix} x \\ y \end{pmatrix} = c(a+b)^n\begin{pmatrix} 1 \\ 1 \end{pmatrix} + c'(a-b)^n\begin{pmatrix} -1 \\ 1 \end{pmatrix} \qquad \begin{pmatrix} a & b \\ b & a \end{pmatrix}$$

moving on a hyperbola. So you had sign wrong.

~~Question~~ Point. ~~Often~~ Given ~~h_n~~ $0 < h_n \nearrow 1$

then $$S(z) = \begin{pmatrix} 1 & h_1 \\ h_1 & 1 \end{pmatrix} \begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & h_2 \\ h_2 & 1 \end{pmatrix} \begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix} \cdots \quad (1)$$

appears to be analytic for $z \neq -1$, including $z = \infty$.
So S is an entire function of $\frac{1}{z+1}$ or $\frac{z-1}{z+1} = 1 - \frac{2}{z+1}$

~~Diff~~ t $$\begin{pmatrix} c & s \\ -s & c \end{pmatrix} = \begin{pmatrix} \frac{z^{1/2}+z^{-1/2}}{2} & \frac{z^{1/2}-z^{-1/2}}{2i} \\ -\frac{z^{1/2}-z^{-1/2}}{2i} & \frac{z^{1/2}+z^{-1/2}}{2} \end{pmatrix}$$

~~Often~~ You want to connect $n \nearrow +\infty$ with $n \searrow -\infty$. Consider the λ_n^{-1} parameters which are positive for $n \geq 0$ and ~~increase~~ tend to ∞. The ~~main~~ next problem is to find ~~the~~ candidates for λ_n which will yield the desired growth.

$$\frac{\Gamma(s)}{n^s} = \int_0^\infty e^{-t} \frac{t^s}{n^s} \frac{dt}{t} = \int_0^\infty e^{-nt} t^s \frac{dt}{t}$$

$$\Gamma(s)\zeta(s) = \int_0^\infty \underbrace{\frac{e^{-t}}{1-e^{-t}}}_{\frac{1}{e^t-1}} t^s \frac{dt}{t}$$ converges for $Re(s) > 1$

decays as $t \to +\infty$
simple pole as $t \to 0$.

$$= \int_0^\infty \left(\frac{1}{e^t-1} - \frac{1}{t}\right) t^s \frac{dt}{t}$$ converges for $0 < Re(s) < 1$

$$\int_0^\infty \underbrace{\left(\frac{1}{e^t-1} - \frac{1}{t}\right)}\, t^{x+iy} \frac{dt}{t}$$

$\sim -\frac{1}{2}$ $\quad t \searrow 0$

$\sim -\frac{1}{t}$ $\quad t \nearrow \infty$

$$\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)n^{-s} = 2\int_0^\infty e^{-\pi t n^2 t^2} t^s \frac{dt}{t}$$

$$\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \int_0^\infty \Big(\overbrace{\sum_{n\in\mathbb{Z}} e^{-\pi n^2 t^2}}^{\theta(t^2)} - 1\Big) t^s \frac{dt}{t} \qquad Re(s)>1$$

$$\theta(t^2) = \frac{1}{t}\theta\left(\frac{1}{t^2}\right) \qquad = \int_0^\infty \left(\theta(t^2)-1-t^{-1}\right)t^s\frac{dt}{t}$$

$$t^{1/2}\theta(t^2) = \left(\frac{1}{t}\right)^{1/2}\theta\left(\left(\frac{1}{t}\right)^2\right) \qquad 0 < Re(s) < 1$$

$$\pi^{-s/2}\Gamma(s/2)\zeta(s) = \int_0^\infty \underbrace{\left(\theta(t^2)-1-t^{-1}\right) t^{\frac{1}{2}+iy}}_{\left(t^{1/2}\theta(t^2)-t^{1/2}-t^{-1/2}\right)} \frac{dt}{t}$$

$$\Gamma(s) = \int_0^\infty e^{-t} t^s \frac{dt}{t} \qquad Re(s)>0.$$

~~YES! What next~~.

$$s\Gamma(s) = \int_0^\infty e^{-t} s t^{s-1} dt = \left[e^{-t}t^s\right]_0^\infty + \int_0^\infty e^{-t}t^s\,dt$$

$$\int_0^\infty e^{-t}t^s\,dt = \Gamma(s+1).$$

$$\Gamma(s) = \frac{1}{s}\Gamma(s+1) = \frac{1}{s(s+1)\cdots(s+n-1)}\Gamma(s+n)$$

~~$s(1+s)$~~

$$\frac{1}{\Gamma(s)} = \frac{s(s+1)\cdots(s+n-1)}{\Gamma(s+n)} = \frac{(n-1)!\,s\,(1+s)\left(1+\frac{s}{2}\right)\cdots\left(1+\frac{s}{n-1}\right)}{\Gamma(s+n)}$$

$$\log\left(1+\frac{s}{k}\right) = 1+\frac{s}{k} \qquad = \frac{s\prod_{k=1}^{n-1} e^{-\frac{s}{k}}\left(1+\frac{s}{k}\right)}{e^{-s\left(1+\frac{1}{2}+\cdots+\frac{1}{n-1}\right)}}$$

$$\frac{1}{\Gamma(s)} = s\prod_{k=1}^{n} e^{-\frac{s}{k}}\left(1+\frac{s}{k}\right) \Bigg/ \frac{\Gamma(s+n+1)\, e^{-s(1+\cdots+\frac{1}{n})}}{n!}$$

~~$-t+(s+n)$~~

$$\Gamma(s+n+1) = \int_0^\infty e^{-t+(s+n)\log t}\, dt$$

$$\approx e^{(s+n)\log(s+n)-(s+n)}$$

$$\left(\int e^{-x^2}dx\right)^2$$

$$= \pi\int_0^\infty e^{-r^2} 2r\,dr = \pi\left[e^{-r^2}\right]_0^\infty = \pi$$

$-t+s\log t$

$-1+\frac{s}{t}$

$-\frac{s}{t^2}\Big|_{t=s} = -\frac{1}{s}$

$$\log(n!) = n\log n - n + \frac{1}{2}\log(2\pi n) \qquad n! = n^{n+\frac{1}{2}}e^{-n}\sqrt{2\pi}$$

$$\Gamma(s+1) = \int_0^\infty e^{-t+s\log t}\,dt = \int_0^\infty e^{-s+s\log s+\frac{1}{2}\left(-\frac{1}{s}\right)(t-s)^2}\,dt$$

$$= e^{s\log s - s}\sqrt{2\pi s}$$

$\log(n) + \log\left(1+\frac{s}{n}\right)$

$$\log\Gamma(s+n+1) = \left(s+n+\tfrac{1}{2}\right)\log(s+n) - s - n$$

$-\log(n!)$ $\quad - ~~\left(n+\frac{1}{2}\right)\log n~~ + K$

$-s\left(1+\cdots+\frac{1}{n}\right)$ $\quad -s\left(1+\cdots+\frac{1}{n}\right)$

$= s\log n$

~~$s\log\left(1+\frac{s}{n}\right)$~~

$\left(n+\frac{1}{2}\right)\log\left(1+\frac{s}{n}\right) \to s$

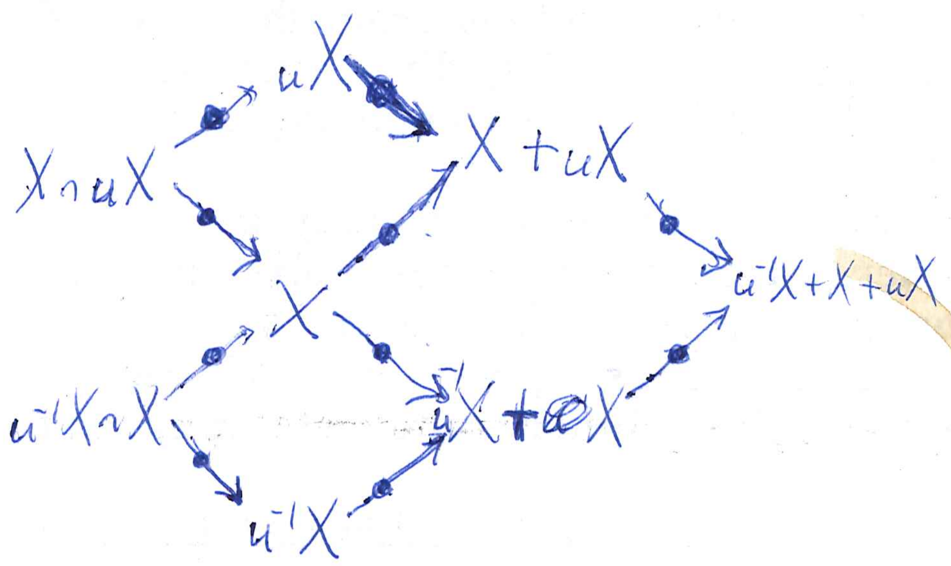
$s\left(\log n - \left(1+\frac{1}{2}+\cdots\frac{1}{n}\right)\right)$

$-\gamma s$

The question - Can you find the skew adjoint operator. You begin with

$$\psi_0 = g_1 \psi_1 \quad \text{etc.} \quad g = \begin{pmatrix} a & b \\ b & a \end{pmatrix}\begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix}$$

which ~~equations~~ equations are transfer forms of the eigenvector equations for a unitary operator.
Review this. Go ~~to~~ back to H, u, X
Assume all layers are of codim 1.



Each vertical column ~~sequence~~ of data ~~represents~~ gives rise to a 2-diml space depending on the eigenvalue ~~Z~~ and ~~the squares~~ each column of squares gives a transfer between the column spaces.

Suppose $\frac{1-u}{1+u}$ = skew-adj operator ξ

$$u = \frac{1-\xi}{1+\xi} = \frac{2-(1+\xi)}{1+\xi} = \frac{2}{1+\xi} - 1$$

$$X + uX = X + (1+\xi)^{-1}X$$

$$u^{-1} = \frac{1+\xi}{1-\xi} = \frac{2-(1-\xi)}{1-\xi} = \frac{2}{1-\xi} - 1$$

$$X + u^{-1}X = X + (1-\xi)^{-1}X$$

~~[illegible] suppose~~ Linear equations

$$\psi_{n-1} = \begin{pmatrix} \lambda_n & 0 \\ 0 & \lambda_n^{-1} \end{pmatrix} \rho \, \psi_n \qquad \rho = \begin{pmatrix} \cos\frac{\theta}{2} & \sin\frac{\theta}{2} \\ -\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix} = \begin{pmatrix} u_1^t \\ u_2^t \end{pmatrix}$$

You ~~[illegible]~~ want to ~~replace~~ write a solutions as a kernel element for some operator

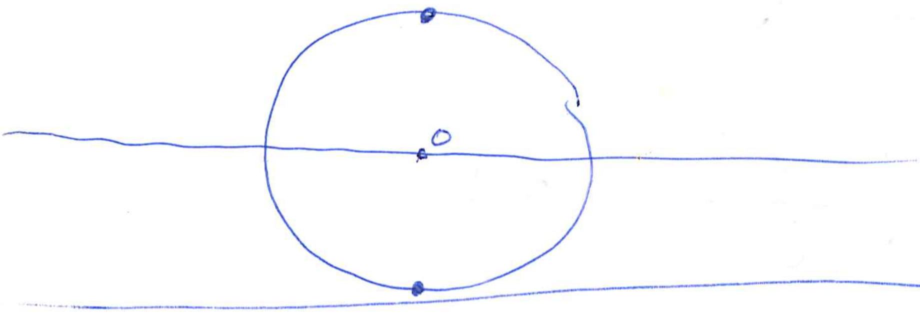
Inhomogeneous equation. Idea: ρ is a perturbation. If $\rho = 1$, then the space of $\psi = (\psi_n)_{n \in \mathbb{Z}}$ splits ~~[illegible]~~ obviously ψ^+ grows as $n \to \infty$, ψ^- decays as $n \to \infty$

$\lambda_n = a_n - b_n$

$$\begin{pmatrix} 1 & \omega \\ -\omega & 1 \end{pmatrix} (\xi) = \frac{\xi + \omega}{1 - \omega\xi} \doteq \xi$$

$$\xi + \omega = \xi - \omega \xi^2 \qquad \xi^2 = -1$$

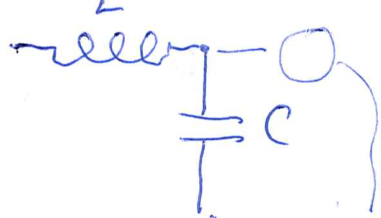
$$\xi = \pm i$$



Idea. Go back to $Z(s) = \sum' a_\omega \frac{s(1+\omega^2)}{s^2+\omega^2} + a_\infty s$

$$\frac{1}{s - i\omega} + \frac{1}{s + i\omega} = \frac{2s}{s^2+\omega^2}. \qquad Z(1) = \sum' a_\omega + a_\infty$$

~~[illegible]~~



$$Z_0 = Ls + \frac{1}{Cs + \frac{1}{Z_1}}$$

~~[illegible]~~

$$E = L \frac{dI}{dt}$$

$Q = CE \qquad I = C\frac{dE}{dt} \qquad \hat{I} = Cs\hat{E}$

$$s = \begin{pmatrix} 1 & s \\ s & 1 \end{pmatrix} R_1 \qquad R_1 = \begin{pmatrix} 1 & s \\ -s & 1 \end{pmatrix} (s)$$

You've made a mistake somewhere.

Given $S_0(z) = \begin{pmatrix} 1 & h_1 \\ h_1 & 1 \end{pmatrix} z S_1 \qquad -1 < h < 1$

$$S_0 = \begin{pmatrix} 1 & h \\ h & 1 \end{pmatrix} \begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix} S_1$$

$$\begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} S_0 = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & h \\ h & 1 \end{pmatrix} \begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} S_1$$

$$R_0 = \underbrace{\begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & h \\ h & 1 \end{pmatrix} \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}} \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -z & z \\ 1 & 1 \end{pmatrix} R_1$$

$$\begin{pmatrix} 1-h & 0 \\ 0 & 1+h \end{pmatrix} \begin{pmatrix} z+1 & -z+1 \\ -z+1 & z+1 \end{pmatrix} R_1$$

$$R_0 = \begin{pmatrix} \frac{1-h}{1+h} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & s \\ s & 1 \end{pmatrix} R_1 \qquad \rho > 0 \qquad s = \frac{1-z}{1+z}$$

Start with $R_0(s) = Ls$. Then

$$S_0 = \frac{1 - Ls}{1 + Ls} = \begin{pmatrix} -L & 1 \\ L & 1 \end{pmatrix} \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} z$$

$$= \begin{pmatrix} L+1 & -L+1 \\ -L+1 & L+1 \end{pmatrix} z$$

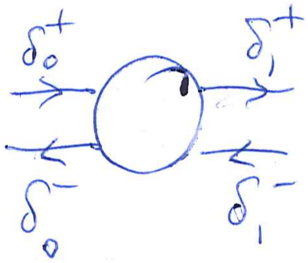
$$R_0 = \begin{pmatrix} L & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & s \\ s & 1 \end{pmatrix} (0)$$

$$\frac{(L-1)s^2 + \frac{1}{C}}{-Ls^2 + 1 - \frac{1}{C}} = \frac{(L-1)s^2 + 1 - L}{-Ls^2 + L} \qquad L + \frac{1}{C} = 1$$

$$s^2 + \frac{1}{C(L-1)}$$

Suppose $\|ax - bx\| = 0$ $\quad ax = bx$

then you have a bound state with eigenvalue $+1$.



$$u(\delta_0^+) = \alpha\,\delta_1^+ + \beta\,\delta_0^-$$

$$u(\delta_1^-) = -\bar\beta\,\delta_1^+ + \alpha\,\delta_0^-$$

$\alpha > 0$

$\alpha = \sqrt{1-|\beta|^2}$

$$\delta_1^+ = \frac{1}{\alpha}u\,\delta_0^+ - \frac{\beta}{\alpha}\delta_0^-$$

$$u(\delta_1^-) = \frac{-\bar\beta}{\alpha}u(\delta_0^+) + \left(\frac{|\beta|^2}{\alpha} + \alpha\right)\delta_0^-$$

$$\begin{pmatrix}\delta_1^+\\ \delta_1^-\end{pmatrix} = \begin{pmatrix}\frac{1}{\alpha}u & -\frac{\beta}{\alpha}\\ -\frac{\bar\beta}{\alpha} & \frac{1}{\alpha}u^{-1}\end{pmatrix}\begin{pmatrix}\delta_0^+\\ \delta_0^-\end{pmatrix}$$

$$\frac{1}{\alpha^2} - \frac{|\beta|^2}{\alpha^2} = \frac{\alpha^2}{\alpha^2} = 1$$

$$\begin{pmatrix}\delta_1^+\\ u\delta_1^-\end{pmatrix} = \frac{1}{k}\begin{pmatrix}1 & -h\\ -\bar h & 1\end{pmatrix}\begin{pmatrix}u\delta_0^+\\ \delta_0^-\end{pmatrix}$$

$$\begin{pmatrix}\delta_1^+\\ u\delta_1^-\end{pmatrix} = \frac{1}{k}\begin{pmatrix}1 & -h\\ -\bar h & 1\end{pmatrix}\begin{pmatrix}u & 0\\ 0 & u^{-1}\end{pmatrix}\begin{pmatrix}\delta_0^+\\ u\delta_0^-\end{pmatrix}$$

$$Z(s) = \begin{pmatrix} 1 & L_1 s \\ & 1 \end{pmatrix} \begin{pmatrix} 1 & \\ C_1 s & 1 \end{pmatrix} \cdots \begin{pmatrix} 1 & 0 \\ C_n s & 1 \end{pmatrix} \begin{pmatrix} \\ \infty \end{pmatrix}$$

Assume $C_n = \infty$, then $\begin{pmatrix} 1 & 0 \\ C_n s & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ C_n s \end{pmatrix} = \begin{pmatrix} 1 \\ \infty \end{pmatrix}$

should be zero, and

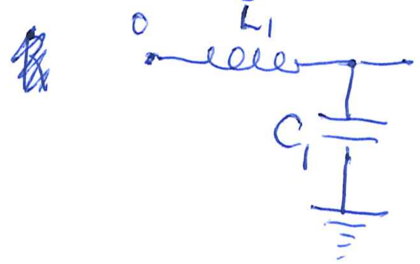
$$Z(s) = \begin{pmatrix} 1 & L_1 s \\ & 1 \end{pmatrix} \cdots \begin{pmatrix} 1 & L_n s \\ & 1 \end{pmatrix} (0)$$

so $Z(0) = 0$. So maybe $Z(s)$ regular

at $s = \infty \iff L_1 = 0$

at $s = 0 \iff C_n = \infty$. ?

Problem that bothers me is that a J-matrix is a s.a. operator bounded so its R-function $R(s)$ should be regular at $s = \infty$.



$$R_0 = L_1 s + \cfrac{1}{C_1 s + \cfrac{1}{R_1}}$$

$$R_0 = \begin{pmatrix} 1 & L_1 s \\ & 1 \end{pmatrix} \begin{pmatrix} 1 & \\ C_1 s & 1 \end{pmatrix} R_1$$

$$R_0 = \begin{pmatrix} L_1 s & 1 \\ 1 & \end{pmatrix} \begin{pmatrix} C_1 s & 1 \\ 1 & \end{pmatrix} (R_1)$$

$L_1 = 0 \iff R_0$ has no pole at ∞.

$$R_0 = L_1 s + \cfrac{1}{C_1 s + \cfrac{1}{L_2 s + \cfrac{1}{C_2 s + \cdots \cfrac{1}{C_n s}}}}$$

$$= \begin{pmatrix} L_1 s & 1 \\ 1 & \end{pmatrix} \begin{pmatrix} C_1 s & 1 \\ 1 & \end{pmatrix} \cdots \begin{pmatrix} L_n s & 1 \\ 1 & \end{pmatrix} \begin{pmatrix} C_n s & 1 \\ 1 & \end{pmatrix} (\infty)$$

If $C_n \neq \infty$, then $R_0(\infty) = \infty$ | $C_n = \infty$ then $R(\infty) = \infty\ C_n s$

$X \overset{a}{\underset{b}{\rightrightarrows}} Y \qquad u = ba^{-1} \quad \xi = \frac{1-u}{1+u} = \frac{1-ba^{-1}}{1+ba^{-1}} = \frac{a-b}{a+b}$

ξ is partially skew-adjoint. Maybe it would help if you first understood partially self-adjoint operators. This should be fairly straightforward. You have

$\mathcal{D}_A \overset{1}{\underset{A}{\rightrightarrows}} H \qquad \|i\xi + A\xi\|^2 = \|\xi\|^2 + \|A\xi\|^2$

so you have isometries $i+A$, $-i+A : X \rightrightarrows H$

where $X = \mathcal{D}_A$ equipped with norm $\|\xi\|^2 + \|A\xi\|^2$.

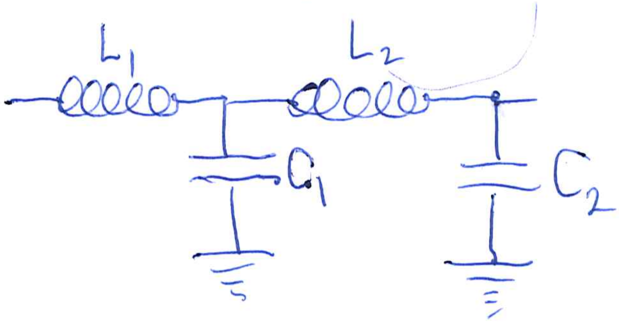
$$\begin{pmatrix} p_{n+1} \\ p_n \end{pmatrix} = \begin{pmatrix} x & -c_n \\ 1 & 0 \end{pmatrix} \begin{pmatrix} p_n \\ p_{n-1} \end{pmatrix} \quad \text{starting w } \begin{pmatrix} p_0 \\ p_{-1} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$\Rightarrow p_n$ monic of degree n.

Here's the puzzle. You consider any

$Z(s) = \sum_{0 \le \omega \le \infty} a_\omega \frac{s(1+\omega^2)}{s^2+\omega^2}$. But only those with

$a_\infty \neq 0$ result from Jacobi matrices.



$$Z = L_1 s + \cfrac{1}{C_1 s + \cfrac{1}{L_2 s + \cfrac{1}{C_2 s}}}$$

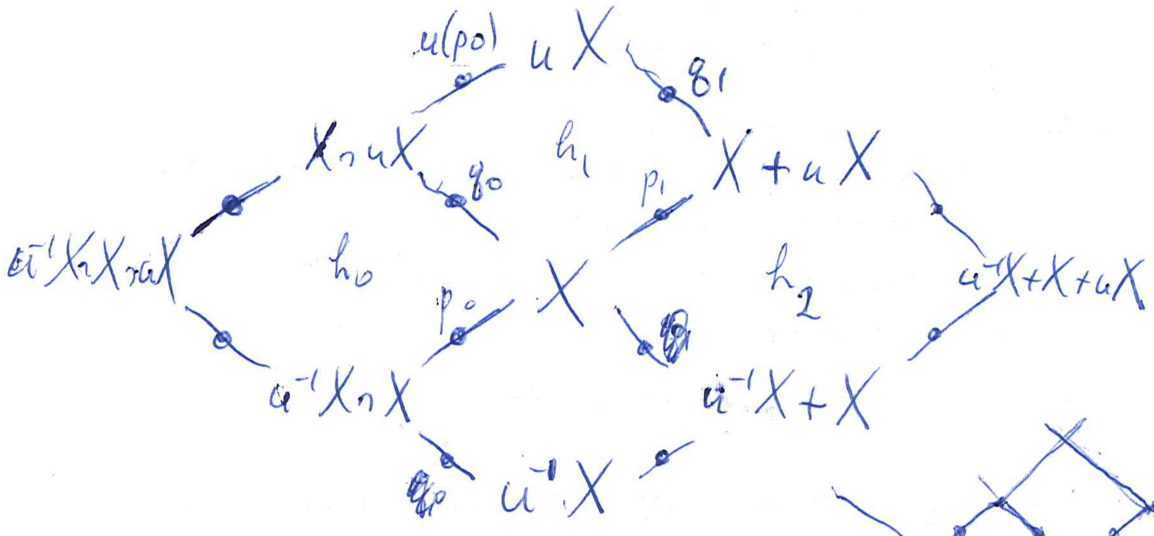
$C_1, L_2 \neq 0$ but L_1 or C_2^{-1} may be zero.

$$Z = \cfrac{1}{C_1 s + \cfrac{1}{L_2 s}} = \frac{L_2 s}{L_2 C_1 s^2 + 1}$$

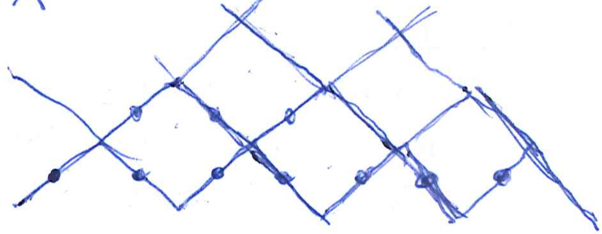
$L_1 = 0$

$\Longleftrightarrow Z$ regular at ∞.

Go back to



natural orthonormal bases are



$$p_1 = \frac{1}{\sqrt{1-|h_1|^2}}\left(u(p_0) + h_1 q_0\right)$$

$$q_1 = \frac{1}{\sqrt{1-|h_1|^2}}\left(\bar{h}_1 u(p_0) + q_0\right)$$

$$\begin{pmatrix} p_1 \\ q_1 \end{pmatrix} = \frac{1}{\sqrt{1-|h_1|^2}} \begin{pmatrix} 1 & h_1 \\ \bar{h}_1 & 1 \end{pmatrix} \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} p_0 \\ q_0 \end{pmatrix}$$

~~In general this becomes unclear~~

Let us see whether it possible to reconstruct partially the Hilbert space + operator from eigenvector equation. General pattern is? ~~You need to have gil~~ The general pattern

Go back to $g = \begin{pmatrix} a & b \\ b & a \end{pmatrix}\begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix}$

$$\frac{1}{2}\begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} g \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} \simeq \begin{pmatrix} a-b & 0 \\ 0 & a+b \end{pmatrix}\begin{pmatrix} \frac{z+1}{2} & \frac{-z+1}{2} \\ \frac{-z+1}{2} & \frac{z+1}{2} \end{pmatrix}$$

$s = \frac{1-z}{1+z}$ | $(1-s^2)^{-1/2}\begin{pmatrix} 1 & s \\ s & 1 \end{pmatrix} = z^{-1/2}\begin{pmatrix} \frac{z+1}{2} & \frac{-z+1}{2} \\ \frac{-z+1}{2} & \frac{z+1}{2} \end{pmatrix}$

$$(1-s^2)^{-1/2} \overset{?}{=} \frac{z^{1/2}+z^{-1/2}}{2}$$

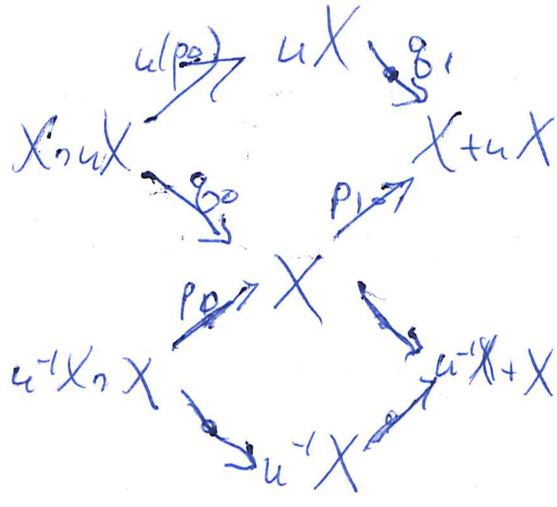
$z = e^{i\theta}$

$\frac{z^{1/2}+z^{-1/2}}{2} = \cos\frac{\theta}{2}$

$$s = \frac{e^{-\frac{i\theta}{2}} - e^{\frac{i\theta}{2}}}{e^{-i\theta/2}+e^{+i\theta/2}} = -1\,\frac{i\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}} = -i\tan\frac{\theta}{2}$$

<u>Go back</u>

Suppose all $h_n = 0$. i.e. everything $\perp$.



~~a(p0)~~

$$\begin{pmatrix} p_1 \\ q_1 \end{pmatrix} = \begin{pmatrix} u p_0 \\ q_0 \end{pmatrix}$$

So the Hilbert space is transparent

Review. Response of an LC circuit. You have a graph edges either of L or C type.

C^1 voltage functions on the edges

C_1 current ————————.

These are natural dual pos definite form.

$V = LsI$

$V = \frac{1}{Cs} I$

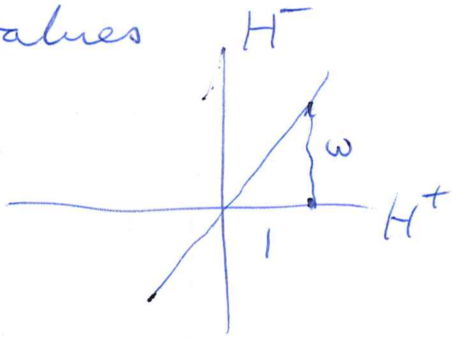
$$\begin{array}{ccc} \bar{C}^0 & \xrightarrow{\ \delta\ } & C^1 = \\ \vdots & & \Big\downarrow\ Z^{-1} = \frac{1}{Ls} \oplus Cs \\ \bar{C}_0 & \xleftarrow{\ \partial\ } & C_1 \end{array}$$

If you equip C^1 with inner product $L^{-1} \oplus C$ you have $C^1 = H = H^+ \oplus H^-$ and you have

$\delta \bar{C}^0 \hookrightarrow H$.

You have $W \overset{i}{\hookrightarrow} H^+ \oplus H^-$. Have decomposition – characteristic values

$W = \bigoplus_\omega W_\omega \qquad i_\omega = \begin{pmatrix} \frac{1}{\sqrt{1+\omega^2}} \\ \frac{\omega}{\sqrt{1+\omega^2}} \end{pmatrix}$



Then

$$i_\omega^* \begin{pmatrix} s & \\ & s^{-1} \end{pmatrix} i_\omega = \begin{pmatrix} \frac{1}{\sqrt{1+\omega^2}} & \frac{\omega}{\sqrt{1+\omega^2}} \end{pmatrix} \begin{pmatrix} s & 0 \\ 0 & s^{-1} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{1+\omega^2}} \\ \frac{\omega}{\sqrt{1+\omega^2}} \end{pmatrix} = \frac{s^2 + \omega^2}{s(1+\omega^2)}$$

and then when you invert.

I also remember ~~that~~

In general you know that a response function

$R(1) = 1.$ $\qquad \begin{pmatrix} 1 & -s \\ -s & 1 \end{pmatrix} R = \dfrac{R-s}{1-sR}$

$R = \dfrac{f(s)}{g(s)}$ $\qquad \dfrac{f(s) - s\,g(s)}{g(s) - s\,f(s)}$ so both num. + denom divisible by s^2-1

$R(1) = 1$

$\Leftrightarrow f(1) = g(1)$ The point is that $R(-s) = -R(s)$

so that $R(1) = 1 \implies R(-1) = -R(1) = -1.$

$$R = \tfrac{1}{2}s + \tfrac{1}{2}s^{-1} = \frac{s^2+1}{2s}$$

$$(s^2+1) - s(2s) = 1 - s^2$$

$$2s - s(s^2+1) = s - s^3$$

$$\frac{\dfrac{s^2+1}{2s} - s}{1 - \dfrac{s^2+1}{2}} = \frac{(1-s^2)/2s}{\dfrac{1-s^2}{2}} = \frac{1}{s}$$

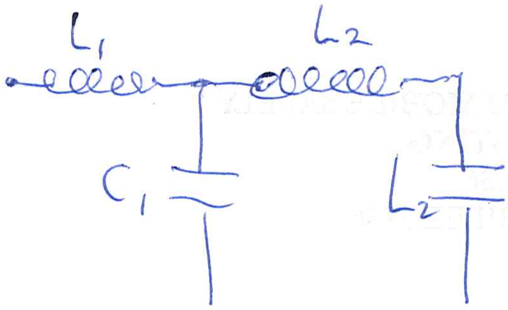
$$\begin{pmatrix} 1 & s \\ s & 1 \end{pmatrix}\begin{pmatrix} 1 & s \\ s & 1 \end{pmatrix}(\infty) = \begin{pmatrix} 1+s^2 & 2s \\ 2s & 1+s^2 \end{pmatrix}(\infty)$$

$$= \frac{1+s^2}{2s}$$

Progress is being made.

To find a Hilbert space interpretation of $R_0 = \begin{pmatrix} 1 & s \\ s & 1 \end{pmatrix} R_1$

continued fraction $\qquad f(s) = \sum a_\omega \frac{s(1+\omega^2)}{s^2+\omega^2}$



$$f_1 = L_1 s + \frac{1}{C_1 s + \frac{1}{L_2 s + \frac{1}{C_2 s + \frac{1}{f_3}}}}$$

$$\begin{pmatrix} 1 & L_1 s \\ & 1 \end{pmatrix} \begin{pmatrix} 1 & \\ Cs & 1 \end{pmatrix} = \begin{pmatrix} 1 + LCs^2 & Ls \\ Cs & 1 \end{pmatrix}$$

A Thought. Because $\omega = \infty$ can occur f seems to be more than a Hilbert space + skew hermitian operator

Actually, what is the link with the moment problem.

$\int x^n d\mu = \mu_n$ odd ones are zero.

$F_n =$ span of $1, x, \ldots, x^n$ $\qquad (p_n, xq)$ " (xp_n, q)

$p_n \in x^n + F_{n-1} \qquad p_n \perp F_{n-1}$

$p_{n+1} - xp_n \in F_n$ and is $\perp F_{n-2}$ $\qquad \begin{pmatrix} 0 & a_0 & \\ 1 & 0 & a_1 \\ & 1 & 0 \end{pmatrix}$

$p_{n+1} = xp_n + c'p_n + c''p_{n-1}$

assume measure even. Then $xp_n = p_{n+1} + a_{n-1}p_{n-1}$

$$xp_n = p_{n+1} + a_{n-1}p_{n-1}$$

$$x = a_n\frac{p_{n+1}}{a_n p_n} + a_{n-1}\frac{p_{n-1}}{p_n}$$

$$\frac{p_{n-1}}{p_n} = \frac{1}{a_{n-1}}x - \frac{1}{a_{n-1}\frac{p_n}{p_{n+1}}}$$

$$q_n = \frac{p_{n+1}}{a_n p_n}$$

$$x = a_n q_n + q_{n-1}^{-1}$$

$$q_{n-1} = \frac{1}{x - a_n q_n} = \begin{pmatrix} 0 & 1 \\ -a_n & x \end{pmatrix} q_n$$

$$p_{n+1} = xp_n - a_{n-1}p_{n-1}$$

$$\begin{pmatrix} p_{n+1} \\ p_n \end{pmatrix} = \begin{pmatrix} x & -a_{n-1} \\ 1 & 0 \end{pmatrix}\begin{pmatrix} p_n \\ p_{n-1} \end{pmatrix}$$

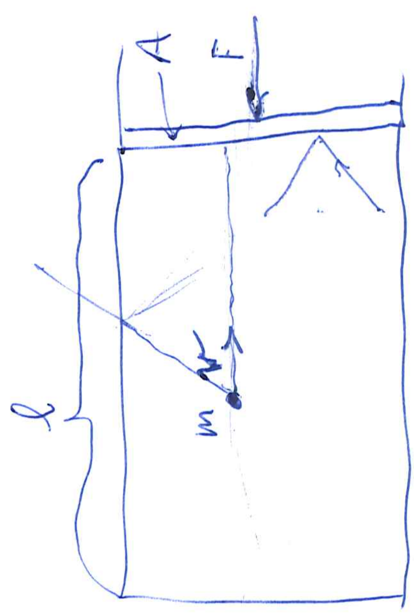
$$\begin{pmatrix} p_n \\ p_{n-1} \end{pmatrix} = \frac{1}{a_{n-1}}\begin{pmatrix} 0 & a_{n-1} \\ -1 & x \end{pmatrix}\begin{pmatrix} p_{n+1} \\ p_n \end{pmatrix}$$

$$\frac{p_n}{p_{n-1}} = \frac{a_n}{x - \frac{p_{n+1}}{p_n}}$$

change

$$\frac{p_1}{p_0} = \frac{a_1}{x-}\;\frac{a_2}{x-}\;\frac{a_3}{x-}$$

$$\begin{pmatrix} 1 & s \\ s & 1 \end{pmatrix}\begin{pmatrix} 1 & s \\ s & 1 \end{pmatrix} = \begin{pmatrix} 1+s^2 & 2s \\ 2s & 1+s^2 \end{pmatrix}$$



$$K.E. = \frac{1}{2}m\sum_{i}(v_{x,i}^2 + v_{y,i}^2 + v_{z,i}^2)$$

$$K.E. = \frac{1}{2}Nm\,c_{RMS}^2$$

$$PV = m\sum_i v_{x,i}^2 = \frac{2}{3}\,K.E.$$

$$F = \frac{d}{dt}(\text{momentum})$$

$$F = pA = \frac{2mv}{\left(\frac{2l}{v}\right)}$$

To find operator interpretation of the expansion

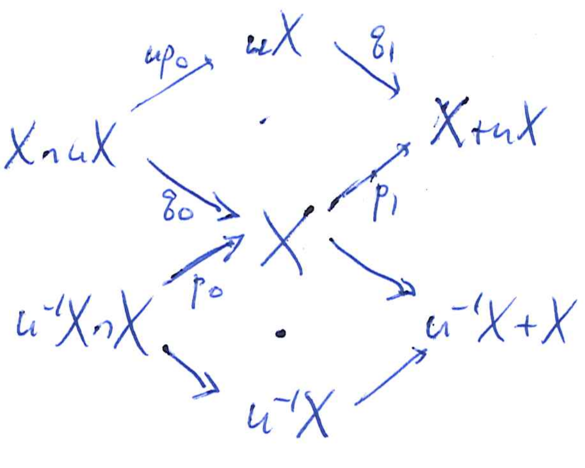
$$R(S) = \begin{pmatrix} \rho_1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & s \\ s & 1 \end{pmatrix} \cdots \cdots \begin{pmatrix} \rho_n & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & s \\ s & 1 \end{pmatrix}(0 \text{ or } \infty)$$

$$\begin{pmatrix} 1 & s \\ s & 1 \end{pmatrix}\begin{pmatrix} p_n \\ q_n \end{pmatrix} = \begin{matrix} p_n + s q_n \\ q_n + s p_n \end{matrix}$$

Go back to partial skew adjoint operator

$$\mathcal{D} \underset{b}{\overset{a}{\rightrightarrows}} H \qquad \xi = b a^{-1}$$

$$X \rightrightarrows Y$$



This diagram remains, but there's a reality condition since the h's are real.

$$\frac{1}{\sqrt{1-|h|^2}}\begin{pmatrix} 1 & h \\ \bar{h} & 1 \end{pmatrix} \qquad \text{need } h \text{ real for } \pm 1 \in S^1$$

to be fixpoints. Changing h to $-h$ is same as taking the inverse.

First task – ~~Suppose~~ Let $Y = X + uX$

Let S_1 be the scattering for $X \underset{u}{\rightrightarrows} Y$ and S_0 the scattering for $u^{-1}X \cap X \rightrightarrows X$. Show

Go back to $X \underset{b}{\overset{a}{\rightrightarrows}} Y \qquad a^*a = b^*b = 1.$

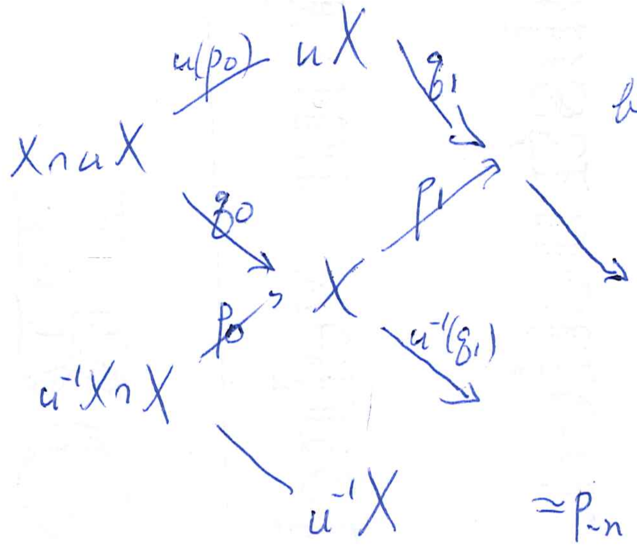
Basic idea is to form $a-b,\ a+b : X \rightrightarrows Y$

we want the partial operator ~~a-b~~ $\dfrac{a-b}{a+b}$ on Y.

---

suppose all $h_n = 0$ ~~Apparently life goes~~ Apparently

⓪



basis $u^n(p_0) = p_n \quad n \in \mathbb{Z}$

$u^n(q_0) = u^n(q_k)$

X spanned by $u^n(q_0)$ for $n \geq 0$ and $u^{-n}(p_0)$

$\simeq p_{-n}$ for $n \geq 0$.

In general you have a doubly inf. sequence of unit vector $u^n(p_k)$ $u^n(q_k)$.

Idea: S is the space of finite vectors, it has the basis $u^n(p_0)$, $u^n(q_0)$ $n \in \mathbb{Z}$. If all $h_n = 0$ then this is an orthonormal basis. ~~Eigenvector equation~~ Eigenfunctions for u are linear functionals on S which kill $(u-z)S$.

$S \underset{u}{\overset{1}{\rightrightarrows}} S \qquad S = \mathbb{C}[u,u^{-1}]^{\oplus 2}$ ~~and we have a few problems at ∞.~~

Cayley transform of u. $S \underset{1-u}{\overset{1+u}{\rightrightarrows}} S$ You want $(1-u)(1+u)^{-1}$ or $(1+u)^{-1}(1-u)$.

Now you know that $1+u : S \longrightarrow (1+u)S \subseteq S$ (codim 2)

OK the Cayley transform

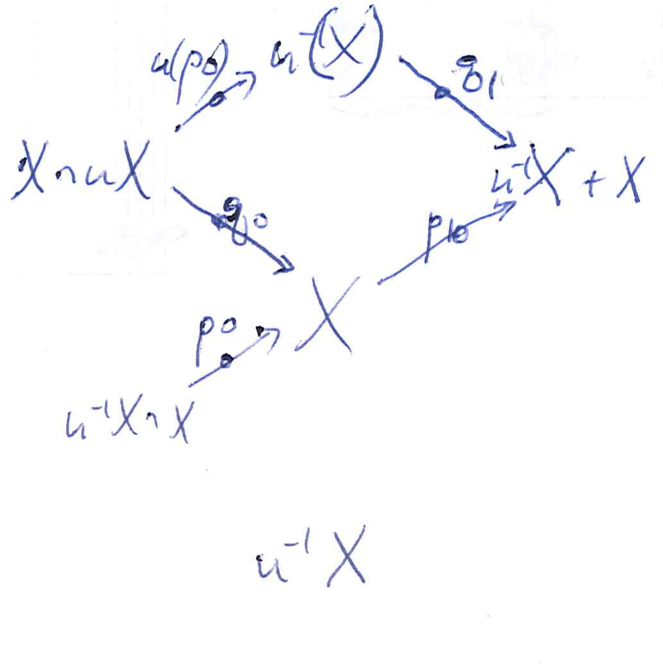
is not defined on the space of finite vectors $\mathcal{S}$ i.e. Laurent polynomials

So when we form $\frac{1-u}{1+u}$ it is not an operator on the space of finite vectors. Further localize so that $\frac{1-u}{1+u}$ is defined.

Focus on the idea $s \neq \pm 1$

~~[illegible]~~ Focus upon the data that gives rise to the eigenfunctions. ~~The eigenfunctions~~ The eigenfunctions are linear functionals on a space $\mathcal{S}$ of "finite" vectors, and the operator is some kind of correspondence $\mathcal{S}' \overset{a}{\underset{b}{\rightrightarrows}} \mathcal{S}$ and then eigenfunctions are linear functionals on $\mathcal{S}/(a\lambda - b)\mathcal{S}'$. ~~[illegible]~~ You start with $\mathcal{S}$ spanned by the unit vectors $u^m(p_n)$, $u^m(q_n)$ linked by

$$\begin{pmatrix} p_{n+1} \\ q_{n+1} \end{pmatrix} = \frac{1}{k_n} \begin{pmatrix} 1 & h_n \\ h_n & 1 \end{pmatrix} \begin{pmatrix} u p_n \\ q_n \end{pmatrix}$$



~~Ho...~~ Consider $\ell = \mathbb{C}[u,u^{-1}]^{\oplus 2}$ operator

$\frac{1-u}{1+u}$ There's a simple way to handle this situation but you haven't found it yet.

$-1<h<1$

$k=\sqrt{1-h^2}$

$$\psi_1 = \frac{1}{k}\begin{pmatrix}1 & h\\ h & 1\end{pmatrix}\begin{pmatrix}z^{1/2} & 0\\ 0 & z^{-1/2}\end{pmatrix}\psi_0$$

$$\phi_1 = \frac{1}{k}\begin{pmatrix}1-h & 0\\ 0 & 1+h\end{pmatrix}\frac{1}{\sqrt{1-s^2}}\begin{pmatrix}1 & s\\ s & 1\end{pmatrix}\phi_0$$

$$s = \frac{-z^{1/2}+z^{-1/2}}{z^{1/2}+z^{-1/2}} = \frac{1-z}{1+z} \qquad \frac{1}{\sqrt{1-s^2}} = \frac{z^{1/2}+z^{-1/2}}{2}$$

Response function for a skew symmetric op. What does it amount to. Op. ξ, have

$D_\xi \overset{1}{\underset{\xi}{\rightrightarrows}} Y$ partial unitary ~~[illegible]~~

$D_\xi \overset{1+\xi}{\underset{1-\xi}{\rightrightarrows}} Y$

Let's do ~~[illegible]~~ bounded hermitian case. H Hilbert space with hermitian x and filtration

Take F_n increasing, ~~[illegible]~~

$F_{n-1}\subset F_n$, $F_n = xF_{n-1}+F_{n-1}$)

$F_{n-1} \overset{\subset}{\underset{x}{\rightrightarrows}} F_n$

Probably want

$$\begin{array}{ccc} F_{n-1} & \hookrightarrow & F_n \\ x\downarrow & & \downarrow x \\ F_n & \hookrightarrow & F_{n+1} \end{array}$$

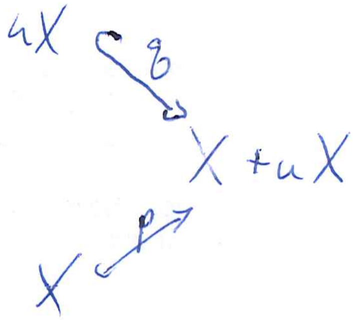
to be bicartesian for all n. P_n

~~#~~ The key point

$$\begin{pmatrix} p_{n+1} \\ p_n \end{pmatrix} = \begin{pmatrix} \frac{x-b_n}{a_n} & -\frac{a_{n-1}}{a_n} \\ 1 & 0 \end{pmatrix} \begin{pmatrix} p_n \\ p_{n-1} \end{pmatrix}$$

Starts $n=0$

$$p_0 = 0 \qquad p_1 = 1.$$



$$p_2 = \frac{x-b_1}{a_1}$$

$$x = \begin{pmatrix} b_1 & a_1 & \\ a_1 & b_2 & \\ & a_2 & \end{pmatrix}$$

$$x p_1 = a_1 p_2 + b_1 p_1$$

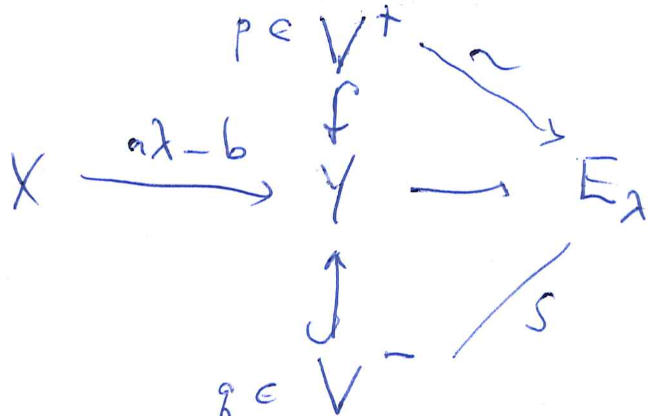
$$x p_2 = a_2 p_3 + b_2 p_2 + a_1 p_1$$

What you need is a bit of understanding about response, namely something simple-minded

~~You need partial fractions.~~

Green's function

Response $X \overset{a}{\underset{b}{\rightrightarrows}} Y$ $\qquad Y = aX \oplus V^+ = V^- \oplus bX$



~~Yes!!! I received it~~

Now take

$$\lambda = \frac{1-s}{1+s}$$

$$a(1-s) - b(1+s) = \begin{matrix} a-b \\ -s(a+b) \end{matrix}$$

$$q = S p \qquad \text{In order to form } \frac{1-S}{1+S} = R$$

you need an isomorphism between V^- and V^+.

anyway H equipped with x hermitian and increasing filtration F_n such that $xF_n \subset F_{n+1}$ $\forall n$ and $x: F_n/F_{n-1} \xrightarrow{\sim} F_{n+1}/F_n$. This is the bicartesian condition. Assume these quotients are all 1-diml. Pick p_0 unit vector $\in F_0 \ominus F_{-1}$, then get unique unit vector $p_n \in F_n \ominus F_{n-1}$ $\ni$ $\underbrace{(xp_n, p_{n+1})}_{a_n} > 0$, whence

$$xp_n = a_n p_{n+1} + b_n p_n + a_{n-1} p_{n-1}$$

$$xp_n \in F_{n+1} \cap (F_{n-2})^\perp \qquad (xp_n, F_{n-2}) = (p_n, xF_{n-2}) \subset (p_n, F_{n-1}) = 0.$$

So we have a nice Hilbert space understanding of a doubly infinite Jacobi matrix. Picture

Eigenvalue equation.

Finite case

$$\begin{matrix} -x & a_1 & & \\ a_1 & -x & a_2 & \\ & a_2 & & \\ & & & a_n \\ & & a_n & \end{matrix}$$

$$a_{n-1}p_{n-1} = (x-b_n)p_n - a_n p_{n+1}$$

$$\frac{a_{n-1}p_{n-1}}{p_n} = x - b_n - \frac{a_n^2 p_{n+1}}{a_n p_n}$$

$$f_n = x - b_n - \frac{a_n^2}{f_{n+1}}$$

Deal with polys.

$$p_{n+1} = \frac{(x-b_n)}{a_n} p_n - \frac{a_{n-1}}{a_n} p_{n-1}$$

$$\begin{pmatrix} p_{n+1} \\ p_n \end{pmatrix} = \begin{pmatrix} \frac{x-b_n}{a_n} & -\frac{a_{n-1}}{a_n} \\ 1 & 0 \end{pmatrix} \begin{pmatrix} p_n \\ p_{n-1} \end{pmatrix}$$