9'''

$$V = \begin{pmatrix} \mathbb{R} \\ \mathbb{R} \end{pmatrix} \qquad \varepsilon = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad W = \begin{pmatrix} x \\ y \end{pmatrix}\mathbb{R}$$

$t = \frac{y}{x}$

$$1 + X = \begin{pmatrix} 1 & -t \\ t & 1 \end{pmatrix}$$

$$g = F\varepsilon = \frac{1+2X+X^2}{1-X^2} = \begin{pmatrix} 1-t^2 & -2t \\ 2t & 1-t^2 \end{pmatrix} \frac{1}{1+t^2}$$

$$= \cancel{\begin{pmatrix} 1 & \frac{-2t}{1+t^2} \\ \frac{2t}{1+t^2} & 1 \end{pmatrix}} \qquad F = g\varepsilon = \underbrace{\begin{pmatrix} \frac{1-t^2}{1+t^2} & \frac{+2t}{1+t^2} \\ \frac{2t}{1+t^2} & -\frac{1-t^2}{1+t^2} \end{pmatrix}}_{\begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}}$$

$$-\frac{4t^2}{(1+t^2)^2} - \frac{1-2t^2+t^4}{(1+t^2)^2} = -1$$

$$\frac{1-\tan^2(2\theta)}{1+\tan^2} = \frac{\cos^2 - \sin^2}{1} = \cos(\theta)$$

$$\frac{2\tan}{1+\tan^2} = \frac{2\sin\cos}{1} = \sin(\theta)$$

Discuss the situation. Let's start with

$$W \overset{i}{\hookrightarrow} \begin{pmatrix} V_+ \\ V_- \end{pmatrix} \hookleftarrow W^\perp \qquad \text{simplest case}$$

where these four spaces have dim 1, also real

$$V = \begin{pmatrix} \mathbb{R} \\ \mathbb{R} \end{pmatrix} \qquad \varepsilon = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad W = \begin{pmatrix} x \\ y \end{pmatrix}\mathbb{R} \subset V$$

$$W^\perp = \begin{pmatrix} -y \\ x \end{pmatrix}\mathbb{R} \subset V$$

$$F\begin{pmatrix} x & -y \\ y & x \end{pmatrix} = \begin{pmatrix} x & -y \\ y & x \end{pmatrix}\varepsilon \qquad \text{assume } x^2+y^2=1$$

$$F = \begin{pmatrix} x & -y \\ y & x \end{pmatrix}\varepsilon\begin{pmatrix} x & y \\ -y & x \end{pmatrix} = \begin{pmatrix} x & -y \\ y & x \end{pmatrix}\begin{pmatrix} x & -y \\ y & x \end{pmatrix}\varepsilon = \begin{pmatrix} x^2-y^2 & -2xy \\ 2xy & x^2-y^2 \end{pmatrix}\varepsilon$$

b''' Can generalize $\cancel{W = \theta}$

$$W \xrightarrow{\binom{x}{y}} \begin{pmatrix} V_+ \\ V_- \end{pmatrix} \longrightarrow W \quad \Big| \quad \begin{pmatrix} x \\ y \end{pmatrix}^* \begin{pmatrix} x \\ y \end{pmatrix} = x^* x + y^* y = 1$$

$$F \begin{pmatrix} x & -y^* \\ y & x^* \end{pmatrix} \qquad \begin{pmatrix} x \\ y \end{pmatrix}^* \begin{pmatrix} -y^* \\ x^* \end{pmatrix} = (\cancel{x^*} \; y^*) \begin{pmatrix} -y^* \\ x^* \end{pmatrix}$$

$$W = \begin{pmatrix} 1 \\ T \end{pmatrix} V_+ \qquad \begin{pmatrix} 1 \\ T \end{pmatrix}^* \begin{pmatrix} 1 \\ T \end{pmatrix} = 1 + T^* T$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ T \end{pmatrix} (1 + T^* T)^{-1/2} \quad \Big| \quad \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} -T^* \\ 1 \end{pmatrix} (1 + T T^*)^{-1/2}$$

$$\begin{pmatrix} x & u \\ y & v \end{pmatrix} = (1 + X)(1 - X^2)^{-1/2} = \frac{1 + X}{\sqrt{1 - X^2}} = g^{1/2}$$

~~and understand~~ $F = g\varepsilon = g^{1/2} \varepsilon g^{-1/2}$

$$F g^{1/2} = g^{1/2} \varepsilon$$

?

$$F =$$

First $\quad W \hookrightarrow \begin{pmatrix} V_+ \\ V_- \end{pmatrix} \longleftrightarrow W^\perp \qquad$ all 1-dim'l $/\mathbb{R}$

$$W = \begin{pmatrix} x \\ y \end{pmatrix} \mathbb{R}, \quad x^2 + y^2 = 1, \quad W^\perp = \begin{pmatrix} -y \\ x \end{pmatrix} \mathbb{R}$$

$$F = \begin{pmatrix} x \\ y \end{pmatrix} (x \; y) - \begin{pmatrix} -y \\ x \end{pmatrix} (-y \; x) = \begin{pmatrix} x^2 & xy \\ yx & y^2 \end{pmatrix} - \begin{pmatrix} y^2 & -xy \\ -xy & x^2 \end{pmatrix}$$

$$= \begin{pmatrix} x^2 - y^2 & 2xy \\ 2xy & -x^2 + y^2 \end{pmatrix} \qquad g = \begin{pmatrix} x^2 - y^2 & -2xy \\ 2xy & x^2 - y^2 \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

c''' What is the good point of view?

complete picture

Starting with $\begin{pmatrix} W \\ W^\perp \end{pmatrix} \simeq \begin{pmatrix} V_+ \\ V_- \end{pmatrix}$

$$W \xrightarrow{\begin{pmatrix} \alpha_+ \\ \alpha_- \end{pmatrix}} \begin{pmatrix} V_+ \\ V_- \end{pmatrix} \qquad \begin{pmatrix} \alpha_+ \\ \alpha_- \end{pmatrix}^* \begin{pmatrix} \alpha_+ \\ \alpha_- \end{pmatrix} = \alpha_+^* \alpha_+ + \alpha_-^* \alpha_- = 1_W$$

You

$$\begin{pmatrix} X \\ Y \end{pmatrix} \quad \begin{pmatrix} V_+ \\ V_- \end{pmatrix} \quad \begin{pmatrix} \cancel{X} \\ \cancel{Y} \end{pmatrix}$$

$$\begin{pmatrix} X \\ Y \end{pmatrix} \xleftarrow{\begin{pmatrix} a & b \\ c & d \end{pmatrix}} \begin{pmatrix} U \\ V \end{pmatrix} \xleftarrow{\begin{pmatrix} a^* & c^* \\ b^* & d^* \end{pmatrix}} \begin{pmatrix} X \\ Y \end{pmatrix} \xleftarrow{\begin{pmatrix} a & b \\ c & d \end{pmatrix}} \begin{pmatrix} U \\ V \end{pmatrix}$$

$aa^* + bb^* = 1_X$	$a^*a + c^*c = 1_U$ ✓
$ac^* + bd^* = 0$	$a^*b + c^*d = 0$ ✓
$ca^* + db^* = 0$	$b^*a + d^*c = 0$ ✓
$cc^* + dd^* = 1_Y$	$b^*b + d^*d = 1_V$ ✓

another view:

$$U \underset{\begin{pmatrix} a \\ c \end{pmatrix}}{\overset{(a^* \ c^*)}{\rightleftarrows}} \begin{pmatrix} X \\ Y \end{pmatrix} \underset{(b^* \ d^*)}{\overset{\begin{pmatrix} b \\ d \end{pmatrix}}{\leftrightarrows}} V$$

$$\begin{pmatrix} a \\ c \end{pmatrix}(a^* \ c^*) + \begin{pmatrix} b \\ d \end{pmatrix}(b^* \ d^*) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

d''' Idea the irred 2 dim'l real orthog reps of $\langle F, \varepsilon \rangle$ are given by

$$F = \begin{pmatrix} \cos\theta & +\sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix} \qquad \varepsilon = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{on} \begin{pmatrix} \mathbb{R} \\ \mathbb{R} \end{pmatrix}$$

Probably for $0 < \theta < \pi$.

Note that

$$\cancel{\begin{pmatrix} \cos\frac{\theta}{2} & -\sin\frac{\theta}{2} \\ \sin\theta/2 & \cos\frac{\theta}{2} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \cos\frac{\theta}{2} & \sin\frac{\theta}{2} \\ +\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix}}$$

$$\cancel{= \begin{pmatrix} c^2 - s^2 & 2sc \\ 2sc & s^2 - c^2 \end{pmatrix} \left(\frac{\theta}{2}\right) \begin{pmatrix} \cos\theta \end{pmatrix}}$$

$$g = F\varepsilon = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \qquad g^{1/2} = \begin{pmatrix} \cos\frac{\theta}{2} & -\sin\frac{\theta}{2} \\ \sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix}$$

$$g^{1/2}\varepsilon g^{-1/2} = \begin{pmatrix} \cos\frac{\theta}{2} & \sin\frac{\theta}{2} \\ \sin\frac{\theta}{2} & -\cos\frac{\theta}{2} \end{pmatrix} \begin{pmatrix} \cos\frac{\theta}{2} & \sin\frac{\theta}{2} \\ -\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix} = \begin{pmatrix} c^2 - s^2 & 2cs \\ 2cs & s^2 - c^2 \end{pmatrix}$$

$$= \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix} = F.$$

Consider again $W \xrightarrow{i} \begin{pmatrix} V_+ \\ V_- \end{pmatrix} \xleftarrow{f} W^{\perp}$

Consider a f.d. Euclidean space V with two orthog inv F, ε. You want a complete picture of this. You want to split into types

e''' Recall that you want the Grassmannian situation:

$$\begin{tikzcd} W \arrow[r, "i = \binom{\iota_+}{\iota_-}"] & \begin{pmatrix} V_+ \\ V_- \end{pmatrix} & W^\perp \arrow[l, "j = \binom{j_+}{j_-}"'] \end{tikzcd}$$

for LC ~~networks~~. ~~What you need~~ You need the quadratic form $s|\xi_+|^2 + s^{-1}|\xi_-|^2$ on V pulled back to W and also pushed forward to $W^\perp$.

The former is $\langle w | (s\,\iota_+^*\iota_+ + s^{-1}\iota_-^*\iota_-)\,w\rangle = s\|\iota_+ w\|^2 + s^{-1}\|\iota_- w\|^2$

The latter should involve pulling back via j and then inverting: $(s\,j_+^*j_+ + s^{-1}j_-^*j_-)^{-1}$

Review.

$$\begin{tikzcd} W \arrow[r, hook, "\iota"] \arrow[d, "\iota^* A_s \iota"'] & V \arrow[r, "j^*"] \arrow[d, shift right, "A_s"'] & W^\perp \\ W & V \arrow[l, "\iota^*"'] \arrow[u, shift right, "A_s^{-1}"'] & W^\perp \arrow[l, hook', "j"] \end{tikzcd}$$

$A_s = \begin{pmatrix} s & 0 \\ 0 & s^{-1} \end{pmatrix}$ on V

Look again at quadratic forms Euclidean space

$$\begin{tikzcd} W \arrow[r, hook, "\iota"] & V \arrow[r, two heads, "j^*"] \arrow[d, shift right, "A"'] & Q \\ W^* & V^* \arrow[l, two heads, "\iota^*"'] \arrow[u, shift right, "A^{-1}"'] & Q^* \arrow[l, hook', "j"] \end{tikzcd}$$

Given $q \in Q$ lift to $v \ni j^* v = q$

Consider $A(v + iw)$ for $w \in W$

look for stationary point. Vary w to $w + \delta w$

$(v + iw, Av + Aiw)$

f''' $(v+iw, A(v+iw)) = (v,Av) + (\iota w, Av) + $
$(v, A\iota w) + (\iota w, A\iota w)$

if v is the stationary point then $\forall \delta w$

$(\delta w, i^* A v) = 0 \qquad i^* A v = 0$

Converse also true.

Now take any v and form $v - \underbrace{i(\iota^* A i)^{-1} \iota^* A v}_{\in iW}$ killed by $\iota^* A$ vanishes on iW

So the induced quadratic form on $Q = V/iW$ is $j^* v \longmapsto v - i(\iota^* A i)^{-1} \iota^* A v$

$$W \xhookrightarrow{i} V \twoheadrightarrow Q$$

(v, Av) on V nondeg.

restricts to $(\iota w, A \iota w) = (w, \iota^* A i w)$

---

idea Gaussian manipulations where F.T. converts A to A^{-1}. Maybe you should look at the Legendre transform. $L = \xi x - x^t A x \qquad \frac{\partial L}{\partial x} = \xi - Ax$

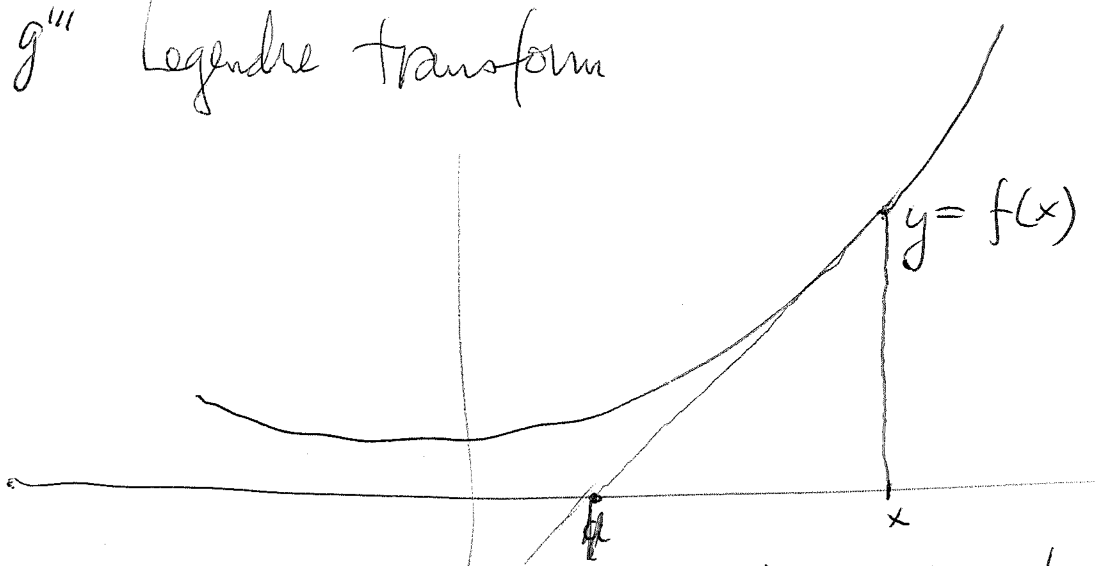
$H = p\dot{q} - L(\dot{q})$, view $H = H(p)$

Try L.T.

$$\hat{L}(s) = \int e^{-st + L(t)} dt \qquad -s + L'(t) = 0$$

$$\sim e^{-L'(t)t + L(t)}$$

g''' Legendre transform



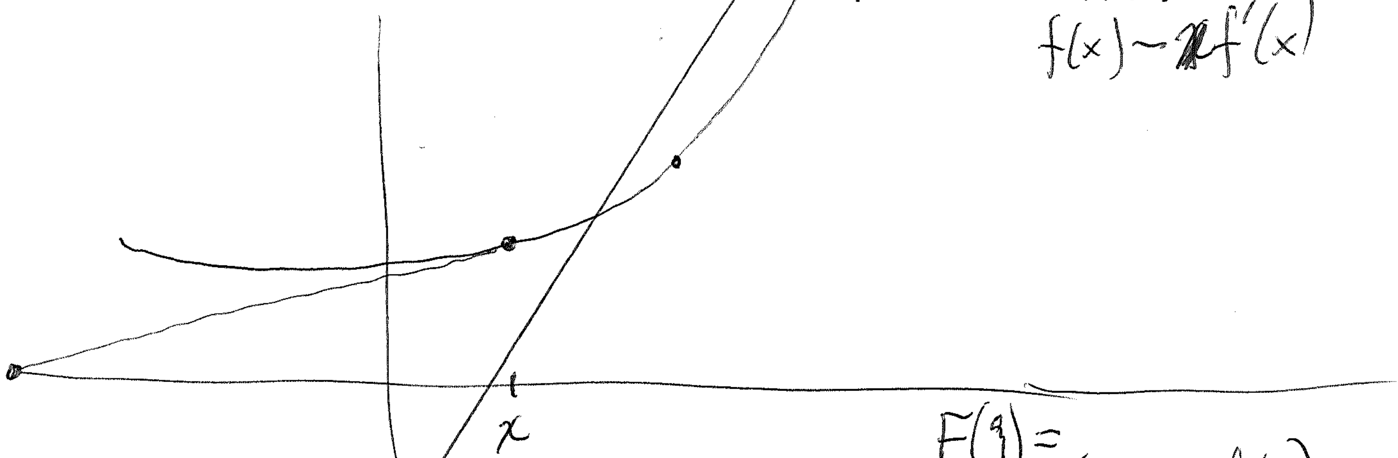
you want to change indep vbl to $\xi = f'(x)$

tangent line is $y_t = \xi x$

$$\frac{f(x) - b}{x} = f'(x)$$

$$f(x) - b = x f'(x)$$

$$b = f(x) - x f'(x)$$



$$F(\xi) = \xi x - f(x)$$

$$F = \xi x - f(x)$$

ξ

$$H = p\dot{q} - L(q, \dot{q})$$

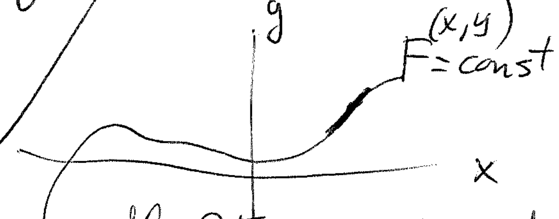
$$H(p,q) = p\dot{q} - L(q, \dot{q})$$

$$F(x,\xi) = \xi x - f(x)$$

$$\partial_\xi F = x$$

x

$$F(x,\xi) = \xi x - f(x)$$



if $\partial_y F \neq 0 \Rightarrow y$ fn of x

$\partial_x F \neq 0 \Rightarrow x$ fn of y

h''' Lag eq. $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = \frac{\partial L}{\partial q}$

$$\text{Action} = \int_a^b L(q, \dot{q}, t)\, dt$$

$$\delta A = \int_a^b \left(\frac{\partial L}{\partial q} \delta q + \frac{\partial L}{\partial \dot{q}} \delta \dot{q} \right) dt$$

$$= \int_a^b \left[\frac{\partial L}{\partial q} \delta q + \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}} \delta q \right) - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}} \right) \delta q \right] dt$$

$$= \int_a^b \left(\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}} \right) \right) \delta q\, dt + \left[\frac{\partial L}{\partial \dot{q}} \delta q \right]_a^b$$

$$H = p\dot{q} - L$$ H dep on $p, q, \dot{q}, t$

$$F(x, \xi) = \xi x - f(x)$$

You want x to be a fn of ξ

need $\underset{\substack{\| \\ x}}{\partial_\xi F} \neq 0$

$\xi x - f(x)$ const.

$x + \xi \frac{dx}{d\xi} - f'(x) \frac{dx}{d\xi}$

$$0 = dF = (\xi - f'(x))\, dx + x\, d\xi$$

$$\Rightarrow (\xi - f'(x)) \frac{dx}{d\xi} + x = 0$$

i''' $\displaystyle\int e^{-st+F(t)}\,dt \approx e^{G(s)}$

$0 = \partial_t\left(-st+F(t)\right) = -s + F'(t)$

So you have stationary point where $s = F'(t)$.

You seem to get $e^{F(t) - tF'(t)}$

Ex, $\displaystyle\Gamma(s) = \int_0^\infty e^{-t}\, t^s \frac{dt}{t} = \int_0^\infty e^{-t+s\log t}\,\frac{dt}{t}$

$0 = \partial_t\left(-t + s\log t\right) = -1 + \frac{s}{t}$ $\therefore t = s$. so you get asymptotic behavior $e^{-s+s\log s}$

Back to Lag. $\quad L = L(q, \dot q, t)$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot q}\right) = \frac{\partial L}{\partial q}$$

forget time dep.

$$\boxed{\begin{gathered} L = L(q,\dot q) \\ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot q}\right) = \frac{\partial L}{\partial q}\end{gathered}}$$

$m\ddot q = -kq$

$L = \frac{1}{2}m\dot q^2 - \frac{1}{2}kq^2$

$\frac{\partial L}{\partial \dot q} = m\dot q \quad \frac{\partial L}{\partial q} = -kq$

$$H = p\dot q - L$$

where $p = \frac{\partial L}{\partial \dot q}$ and

$p = \frac{\partial L}{\partial \dot q}(q, \dot q) \qquad \frac{\partial p}{\partial \dot q} = \frac{\partial^2 L}{\partial \dot q^2} \neq 0.$

$L = \frac{1}{2}m\dot q^2 - \frac{1}{2}kq^2$

$p = \frac{\partial L}{\partial \dot q} = m\dot q$

$\dot q = \frac{p}{m}$

$$H = \frac{p^2}{m} - \frac{1}{2}m\frac{p^2}{m^2} + \frac{1}{2}kq^2 = \frac{p^2}{2m} + \frac{k}{2}q^2$$

$\int^{(1)}$ $\quad p = \frac{\partial L}{\partial \dot q}(q,\dot q)$ allows $\dot q = \dot q(p,q)$.

$$H(p,q) = p\dot q - L(q,\dot q) = \frac{\partial L}{\partial \dot q}\dot q - L(q,\dot q)$$

$$\frac{\partial H}{\partial p} = \dot q + p\frac{\partial \dot q}{\partial p} - \frac{\partial L}{\partial \dot q}\frac{\partial \dot q}{\partial p} = \dot q$$

$$\cancel{\frac{\partial H}{\partial q} = p\frac{\partial \dot q}{\partial q} - \frac{\partial L}{\partial q} \quad \frac{\partial L}{\partial q} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot q}\right)}$$

$$H(p,q) = p\,\dot q(p,q) - L(q,\dot q(p,q))$$

$$\partial_q H(p,q) = \cancel{p\frac{\partial \dot q}{\partial q}(p,q)} - \underbrace{\frac{\partial L}{\partial q}(q,\dot q(p,q))}_{\frac{dp}{dt}} - \cancel{\frac{\partial L}{\partial \dot q}\frac{\partial \dot q}{\partial q}}$$

The way to make this clear is perhaps to list dep. variables.

$\dot q$ is a fn of p, q via $p = \frac{\partial L}{\partial \dot q}(q,\dot q)$

$$H(p,q) = p\dot q - L(q,\dot q)$$

$$\frac{\partial H}{\partial p} = \dot q + \cancel{p\frac{\partial \dot q}{\partial p}} - \cancel{\frac{\partial L}{\partial \dot q}\frac{\partial \dot q}{\partial p}} = \dot q$$

$$\frac{\partial H}{\partial q} = \cancel{p\frac{\partial \dot q}{\partial q}} - \frac{\partial L}{\partial q} - \cancel{\frac{\partial L}{\partial \dot q}\frac{\partial \dot q}{\partial q}} = -\frac{d}{dt}p$$

---

didn't understand Legendre transf.

$-st + F(t)$ ~~stationary~~ critical pt at $-s + F'(t) = 0$.

$-F'(t)t + F(t)$ ~~[illegible]~~ but written as a fn of s.

k'''

$\cancel{F = \frac{1}{2}at^2}$ $\qquad F'(t) = \boxed{at = s}$

$\cancel{F = \frac{1}{2}\frac{s^2}{a}}$, $\qquad \cancel{F = -st + \frac{1}{2}at^2 = -}$

$\cancel{F(t) = \frac{1}{2}at^2 \qquad -st + F(t) = -s}$

$$-st + \frac{1}{2}at^2 \qquad s = F'(t) = at$$

$$-\frac{s^2}{a} + \frac{a}{2}\frac{s^2}{a^2} = -\frac{1}{2}\frac{s^2}{a}$$

So the next thing to work on is A_s

$$W \hookrightarrow V \twoheadrightarrow Q = W^\perp$$

$$A_s$$

there are induced quadratic forms on W, Q. The push forward to Q is the most interesting since it leads to A_s^{-1}.

Consider ~~s.s. of Euclidean~~

$$W \xhookrightarrow{i} V \xrightarrow{j} Q \qquad A \text{ quad form on } V$$

can restrict A to i^*Ai on W. If i^*Ai is non deg, then there should be a complementary subspace to V which is A-orthogonal to W. Introduce duals.

$$\begin{array}{ccccc} W & \xhookrightarrow{i} & V & \xrightarrow{j} & Q \\ \cong\downarrow i^*Ai & & \downarrow A & & \\ W^* & \xleftarrow{i^*} & V^* & \hookleftarrow & Q^* \end{array}$$

$v \in V$

kills iW

$$v - i(i^*Ai)^{-1}i^*Av$$

this be a projection operator on V with kernel containing iW. Image is killed by i^*A

ℓ''' Start with $W \xrightarrow{\iota} V \xrightarrow{j} Q$ and non deg quad form A on V, means $A: V \to V^*$ symm & isom. Assume $\iota^* A \iota$ nondeg on W i.e. $\forall w\ \iota^* A \iota w = 0 \Rightarrow w = 0$. Want $\forall q \in Q$ a

---

V Euclidean, W subspace, $W^\perp$

A symm ~~non deg~~ invertible, $\iota^* A \iota$ assumed invertible.

$V = V_+ \oplus V_-$

$$C = \left(\begin{array}{c|c} A & X \\ \hline X & B \end{array}\right)$$

Choose a notation

$$V = \begin{pmatrix} V_+ \\ V_- \end{pmatrix} \qquad A$$

$V = W \oplus U$

$$W \xrightarrow{\iota} V \xleftarrow{j} U$$

$$A\begin{pmatrix} \iota w \\ j u \end{pmatrix}$$

$$\underbrace{\begin{pmatrix} x \\ y \end{pmatrix}^t \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}} \qquad V = \begin{pmatrix} X \\ Y \end{pmatrix}$$

fix y look for stationary point for x

$$x^t a x + x^t b y + y^t c x + y^t d y$$

$$\delta x^t a x + x^t a \delta x + \delta x^t b y + y^t c \delta x = 0$$

$$\delta x^t (2ax + 2by) = 0$$

$ax + by = 0$ equiv. to x stationary point.

$x = -a^{-1} b y$

$$\begin{pmatrix} -y^t b^t a^{-1} & y^t \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} -a^{-1} b y \\ y \end{pmatrix}$$

m'''

$$(-y^t c\, a^{-1} \quad y^t)\begin{pmatrix} a & b \\ c & d \end{pmatrix} = (\underbrace{\cancel{-y^t c + y^t c}}_{0} \quad -y^t c a^{-1} b + y^t d)$$

stationary value is

$$(0 \quad y^t(d - ca^{-1}b))\begin{pmatrix} -a^{-1} b y \\ y \end{pmatrix} = y^t(d - ca^{-1}b)y$$

$$\begin{tikzcd} X \arrow[r, hook, "\iota"] \arrow[d, "{}^t\iota A \iota"'] & V \arrow[r, two heads, "j"] \arrow[d, "A"] & Y \\ X^* & V^* \arrow[l, "\iota^t"'] & Y^* \arrow[l, "j^*"'] \end{tikzcd}$$

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$\begin{pmatrix} a^{-1} & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & a^{-1}b \\ c & d \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ -c & 1 \end{pmatrix}\begin{pmatrix} 1 & a^{-1}b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & a^{-1}b \\ 0 & d - ca^{-1}b \end{pmatrix}$$

h'''

$$\cancel{\begin{pmatrix} 1 & 0 \\ -c & 1 \end{pmatrix}\begin{pmatrix} a^{-1} & 0 \\ 0 & 1 \end{pmatrix}\Big(}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} ? = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ -ca^{-1} & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} ? = \begin{pmatrix} 1 & 0 \\ -ca^{-1} & 1 \end{pmatrix}$$

$$\begin{pmatrix} a & b \\ 0 & d-ca^{-1}b \end{pmatrix} ? = \begin{pmatrix} 1 & 0 \\ -ca^{-1} & 1 \end{pmatrix}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$\begin{pmatrix} a^{-1} & 0 \\ -ca^{-1} & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & a^{-1}b \\ 0 & -ca^{-1}b+d \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 \\ 0 & d-ca^{-1}b \end{pmatrix}$$

$$\begin{pmatrix} a^{-1} & 0 \\ -ca^{-1} & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & -a^{-1}b \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & a^{-1}b \\ 0 & d-ca^{-1}b \end{pmatrix}\begin{pmatrix} 1 & -a^{-1}b \\ & 1 \end{pmatrix}$$

~~Identity is~~

$$\begin{pmatrix} 1 & 0 \\ -ca^{-1} & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & -a^{-1}b \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ -ca^{-1} & 1 \end{pmatrix}\begin{pmatrix} a & 0 \\ c & d-ca^{-1}b \end{pmatrix}$$

$$\begin{pmatrix} a & 0 \\ 0 & d-ca^{-1}b \end{pmatrix} \qquad \begin{pmatrix} 0 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\underbrace{\begin{pmatrix} 1 & 0 \\ -ca^{-1} & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}}\begin{pmatrix} 1 & -a^{-1}b \\ 0 & 1 \end{pmatrix}$$

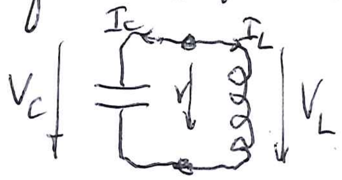
$$\begin{pmatrix} a & b \\ 0 & d-ca^{-1}b \end{pmatrix}\begin{pmatrix} 1 & -a^{-1}b \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & d-ca^{-1}b \end{pmatrix}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 0 \\ ca^{-1} & 1 \end{pmatrix}\begin{pmatrix} a & 0 \\ 0 & d-ca^{-1}b \end{pmatrix}}\begin{pmatrix} 1 & a^{-1}b \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} a & 0 \\ c & d-ca^{-1}b \end{pmatrix}\begin{pmatrix} 1 & a^{-1}b \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \begin{pmatrix} 1 & -a^{-1}b \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a^{-1} & 0 \\ 0 & (d-ca^{-1}b)^{-1} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -ca^{-1} & 1 \end{pmatrix}$$

q''' sho example.



4 variables V_C, V_L, I_C, I_L

4 eqns. $V_C = V_L, \quad I_C + I_L = 0$

$C s V_C = I_C, \quad L s I_L = V_L$

These equations give the free motion, or normal modes. In Laplace Transform theory you get the free motion from the Initial Value Problem. The "state" variables have initial values at $t=0$ which yield inhomogeneous terms on integrating from 0 to ∞.

$$\hat{I}_C = \int_0^\infty e^{-st} C \partial_t V_C \, dt = Cs\hat{V}_C - CV_C(0)$$

$$\hat{V}_L = Ls\hat{I}_L - LI_L(0)$$

So the 4 equations are

$$\hat{V}_C = \hat{V}_L \qquad \hat{I}_C + \hat{I}_L = 0$$

$$Cs\hat{V}_C - \hat{I}_C = CV_C(0)$$

$$Ls\hat{I}_L - \hat{V}_L = LI_L(0).$$

$$Cs\hat{V}_C + \hat{I}_L = CV_C(0)$$

$$-\hat{V}_C + Ls\hat{I}_L = LI_L(0)$$

$$\begin{pmatrix} Cs & 1 \\ -1 & Ls \end{pmatrix} \begin{pmatrix} \hat{V}_C \\ \hat{I}_L \end{pmatrix} = \begin{pmatrix} CV_C(0) \\ LI_L(0) \end{pmatrix}$$

p''' A quadratic form on $\begin{pmatrix} X \\ Y \end{pmatrix} = V$

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad a = a^t \quad d = d^t \qquad b = c^t$$

You would like to get straight the stuff about

$$X \longrightarrow V \longrightarrow Y$$

$$\begin{pmatrix} x \\ y \end{pmatrix}^t \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

fix y vary x

look for stationary point

$$\begin{pmatrix} \delta x \\ 0 \end{pmatrix}^t \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} x \\ y \end{pmatrix}^t \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \delta x \\ 0 \end{pmatrix}$$

$$= 2\,\delta x\,(ax + by).$$

Stationary condition is $ax + by = 0 \implies x = -a^{-1}by$

stationary value is

$$\begin{pmatrix} -a^{-1}by \\ y \end{pmatrix}^t \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} -a^{-1}by \\ y \end{pmatrix}$$

$$y^t \underbrace{\begin{pmatrix} \underbrace{-b^t a^{-1}}_{-ca^{-1}} & 1 \end{pmatrix}}_{\begin{pmatrix} -ca^{-1} & 1 \end{pmatrix}} \underbrace{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} -a^{-1}b \\ 1 \end{pmatrix}}_{\begin{pmatrix} 0 \\ d - ca^{-1}b \end{pmatrix}} y$$

$$= y^t (d - ca^{-1}b) y$$

$$\int_0^\infty e^{-st} \partial_t x \, dt = \int_0^\infty \left(\partial_t [e^{-st} x] + s e^{-st} x \right) dt = -x(0) + s\hat{x}$$

r'''

$$\begin{pmatrix} \hat{V}_C \\ \hat{I}_L \end{pmatrix} = \frac{1}{LCs^2+1} \begin{pmatrix} Ls & -1 \\ 1 & Cs \end{pmatrix} \begin{pmatrix} CV_C(0) \\ LI_L(0) \end{pmatrix}$$

$$\bar{C}^0 \hookrightarrow \underset{\{\binom{V_C}{V_L}\}}{\overset{}{C^1}} \twoheadrightarrow H^1$$

$V \mapsto \binom{V}{V}$, $\left\{\binom{V_C}{V_L}\right\}$

$$\begin{pmatrix} V_C \\ V_L \end{pmatrix}^t \begin{pmatrix} Cs & 0 \\ 0 & \frac{1}{Ls} \end{pmatrix} \begin{pmatrix} V_C \\ V_L \end{pmatrix} = CsV_C^2 + \frac{1}{Ls}V_L^2$$

orth. comp. to $X = \bar{C}^0$ is defined by $\begin{pmatrix} 1 \\ 1 \end{pmatrix}^t \begin{pmatrix} Cs & 0 \\ 0 & \frac{1}{Ls} \end{pmatrix} \begin{pmatrix} V_C \\ V_L \end{pmatrix} = 0$

i.e. $\underbrace{CsV_C + \frac{1}{Ls}V_L = 0}$

This defines a line in C^1. Pick a vector in this line. Simplest seems $V_C = \frac{1}{Cs}$ $V_L = -Ls$

$$\begin{pmatrix} \frac{1}{Cs} \\ -Ls \end{pmatrix}^t \begin{pmatrix} Cs & 0 \\ 0 & \frac{1}{Ls} \end{pmatrix} \begin{pmatrix} \frac{1}{Cs} \\ -Ls \end{pmatrix} = \frac{1}{Cs} + Ls$$

$$\bar{C}^0 \hookrightarrow C^1 \twoheadrightarrow H^1$$

$V \mapsto \binom{V}{V} \in \left\{\binom{V_C}{V_L}\right\}$ On C^1 you have the quadratic form A_s: $\begin{pmatrix} V_C \\ V_L \end{pmatrix}^t \begin{pmatrix} Cs & 0 \\ 0 & \frac{1}{Ls} \end{pmatrix} \begin{pmatrix} V_C \\ V_L \end{pmatrix} = CsV_C^2 + \frac{1}{Ls}V_L^2$

Restriction to $\bar{C}^0$ is $\begin{pmatrix} V \\ V \end{pmatrix}^t \begin{pmatrix} Cs & 0 \\ 0 & \frac{1}{Ls} \end{pmatrix} \begin{pmatrix} V \\ V \end{pmatrix} = V\left(Cs + \frac{1}{Ls}\right)V$

If nonsing (say $Re(s) \neq 0$), then you get an induced quadratic form on H^1 as follows. Orthogonal space to $\bar{C}^0$ is $\left\{\binom{V_C}{V_L} \,\middle|\, \binom{1}{1}^t \begin{pmatrix} Cs & 0 \\ 0 & \frac{1}{Ls} \end{pmatrix} \binom{V_C}{V_L} = CsV_C + \frac{1}{Ls}V_L = 0\right\}$

5'''   Question: Given a fd real v.s. Y equipped with a quadratic form A_s which is generically invertible. Do you get some sort of dynamics on Y? Example: $A_s =$ [redacted] $\dfrac{s(1+\omega^2)}{s^2+\omega^2}$.

$$\frac{1}{s+i\omega} + \frac{1}{s-i\omega} = \frac{2s}{s^2+\omega^2}$$

inverse L.T. should yield $e^{-i\omega t}$, $e^{i\omega t}$

Take $\begin{pmatrix} V_C \\ V_L \end{pmatrix} \in \bar{C}^{\circ\perp}$, means $CsV_C + \frac{1}{Ls}V_L = 0$

gen $\begin{pmatrix} V_C \\ V_L \end{pmatrix} = \begin{pmatrix} 1 \\ -LCs^2 \end{pmatrix}$ restrict q.f. to get

$$Cs1^2 + \frac{1}{Ls}(-LCs^2)^2 = Cs^2 + \frac{L^2C^2s^4}{Ls} \quad ?$$

$$= Cs^2 + LC^2s^3 = Cs^2(1+$$

$$CsV_C + \frac{1}{Ls}V_L = 0 \quad \Big| \quad V_L = -LCs^2V_C$$

$$\begin{pmatrix} V_C \\ V_L \end{pmatrix} = \begin{pmatrix} 1 \\ -LCs^2 \end{pmatrix}$$

$$\begin{pmatrix} 1 \\ -LCs^2 \end{pmatrix}^t \begin{pmatrix} Cs & \\ & \frac{1}{Ls} \end{pmatrix} \begin{pmatrix} 1 \\ -LCs^2 \end{pmatrix} = \begin{pmatrix} 1 \\ -LCs^2 \end{pmatrix}^t \begin{pmatrix} Cs \\ -Cs \end{pmatrix}$$

$$= \begin{pmatrix} 1 & -LCs^2 \end{pmatrix} \begin{pmatrix} Cs \\ -Cs \end{pmatrix} = Cs + LC^2s^3 = Cs(1+LCs^2)$$

t''

Repeat the simple LC oscillator

$$\bar{C}^0 \hookrightarrow \underset{\Psi}{C^1} \twoheadrightarrow H^1$$

$$V \mapsto \begin{pmatrix} V \\ V \end{pmatrix} \in \left\{ \begin{pmatrix} V_C \\ V_L \end{pmatrix} \right\}$$

$$CsV_C^2 + \frac{L}{Ls} V_L^2 = \begin{pmatrix} V_C \\ V_L \end{pmatrix}^t \begin{pmatrix} Cs & 0 \\ 0 & \frac{1}{Ls} \end{pmatrix} \begin{pmatrix} V_C \\ V_L \end{pmatrix}$$

General theory should tell us that the quadratic form on H^1 should be the inverse of the quadratic form on H_1 induced by restricting A_s^{-1}

$$A^0 \ni 1 \mapsto \begin{pmatrix} V^0 \\ V^0 \end{pmatrix}$$

$$\begin{pmatrix} V_C \\ V_L \end{pmatrix} \mapsto V_C - V_L$$

$$\bar{C}^0 \xrightarrow{\binom{1}{1}} C^1 \xrightarrow{(1\ -1)} H^1$$

$$\bar{C}_0 \twoheadleftarrow C_1 \leftrightarrow H_1$$

$$\begin{pmatrix} I_C \\ I_L \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} I_\lambda \leftarrow I_\lambda$$

$$\bar{C}^0 \xrightarrow{\binom{1}{1}} C^1 \xrightarrow{(1\ -1)} H^1$$

$$\bar{C}_0 \xleftarrow{(1\ 1)} C_1 \xleftarrow{\binom{1}{-1}} H_1$$

$$\begin{pmatrix} I_\lambda \\ -I_\lambda \end{pmatrix} \leftarrow I_\lambda$$

$$I_C + I_L \leftarrow \begin{pmatrix} I_C \\ I_L \end{pmatrix}$$

restricted quad form on H_1 is

$$\underbrace{(1\ -1)\begin{pmatrix} (Cs)^{-1} & \\ & Ls \end{pmatrix}\begin{pmatrix} I_\lambda \\ -I_\lambda \end{pmatrix}} = I_\lambda\left(\frac{1}{Cs} + Ls\right) I_\lambda$$

need to invert this

u'''

$$\mathbb{C} \xrightarrow{\begin{pmatrix}1\\1\end{pmatrix}} \mathbb{C}^2 \xrightarrow{(1\ -1)} \mathbb{C}$$

$$\mathbb{C} \xleftarrow{(1\ \ 1)} \mathbb{C}^2 \xleftarrow{\begin{pmatrix}1\\-1\end{pmatrix}} \mathbb{C}$$

$$I_0 = I_C + I_L \longleftarrow \begin{pmatrix}I_C\\I_L\end{pmatrix}, \quad \begin{pmatrix}I_\lambda\\-I_\lambda\end{pmatrix} \longleftarrow I_\lambda$$

$$\bar{C}^0 \hookrightarrow C^1 \twoheadrightarrow H^1$$

$$\bar{C}_0 \twoheadleftarrow C_1 \hookleftarrow H_1$$

~~what you~~ You think the inverse of

$$I_\lambda \longmapsto \underbrace{I_\lambda \underbrace{\left(\frac{1}{Cs} + Ls\right) I_\lambda}_{V_\lambda}}_{V_\lambda} \quad \text{is}$$

$$V_\lambda \frac{1}{\frac{1}{Cs} + Ls} V_\lambda$$

$$\begin{array}{ccccc} \bar{C}^0 & \xrightarrow{\iota} & C^1 & \xrightarrow{j} & H^1 \\ & & \downarrow A & & \downarrow (jA^{-1}j^t)^{-1} \\ \bar{C}_0 & \xleftarrow{\iota^t} & C_1 & \xleftarrow{j^t} & H_1 \end{array}$$

If $\iota^t A v = 0$

then $Av = j^t \gamma$
unique

so $v = A^{-1} j^t \gamma$

$jv = (jA^{-1}j^t)\gamma$

$\mathrm{Ker}(\iota^t A)$ = orthog space to $\bar{C}^0$

$\therefore$ jv and γ are related by $jA^{-1}j^t$

V'''

When you did this before

$$X \xrightarrow{\binom{1}{0}} \left(\cancel{\;}\right) \xrightarrow{(0\ 1)} Y \qquad \begin{pmatrix}x\\y\end{pmatrix}^t A \begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}x\\y\end{pmatrix}^t \begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}$$

orth space $X^\perp$: $\begin{pmatrix}1\\0\end{pmatrix}^t \begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} = \underbrace{ax+by}=0$

$\Rightarrow x = -a^{-1}by$

$$\therefore X^\perp = \begin{pmatrix}-a^{-1}b\\ 1\end{pmatrix} Y$$

$$y^t \begin{pmatrix}-a^{-1}b\\ 1\end{pmatrix}^t \underbrace{\begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}-a^{-1}b\\ 1\end{pmatrix}} y$$

$$y^t\left(-b^t a^{-t}\ \ 1\right)\begin{pmatrix}0\\ d-ca^{-1}b\end{pmatrix} y \quad = y^t(d-ca^{-1}b)y$$

$$\begin{tikzcd}
V_0 \arrow[r, mapsto] & \begin{pmatrix}V_C = V_0\\ V_L = V_0\end{pmatrix} \arrow[d, "A_s"] & \begin{pmatrix}V_C = \frac{1}{Cs}I_\lambda\\ V_L = -LsI_\lambda\end{pmatrix} \arrow[r, mapsto] & \left(\frac{1}{Cs}+Ls\right)I_\lambda = V_\lambda \\
\left(Cs+\frac{1}{Ls}\right)V_0 & \begin{pmatrix}CsV_0\\ \frac{1}{Ls}V_0\end{pmatrix} \arrow[l, mapsto] & \begin{pmatrix}I_C = I_\lambda\\ I_L = -I_\lambda\end{pmatrix} \arrow[u, "A_s^{-1}"] & I_\lambda \arrow[l, mapsto]
\end{tikzcd}$$

x'''

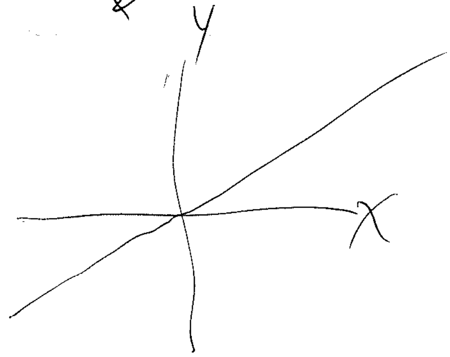
$$X \xrightarrow{\binom{1}{0}} \begin{pmatrix} X \\ Y \end{pmatrix} \xrightarrow{(0\ 1)} Y$$

$$\begin{pmatrix}x\\y\end{pmatrix}^t A \begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}x\\y\end{pmatrix}^t \begin{pmatrix}a&b\\c&d\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}$$

Find $X^\perp$ wrt A.

$$x^t(1\ 0)A\begin{pmatrix}x\\y\end{pmatrix} = x^t(1\ 0)\begin{pmatrix}a&b\\c&d\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} = x^t(ax+by)$$

$$X^\perp = \left\{\begin{pmatrix}x\\y\end{pmatrix} \,\middle|\, ax+by=0\right\}.$$



$X^\perp$ should project isom. on Y.

$$X^\perp \subset \begin{pmatrix}X\\Y\end{pmatrix} \xrightarrow{(0\ 1)} Y$$

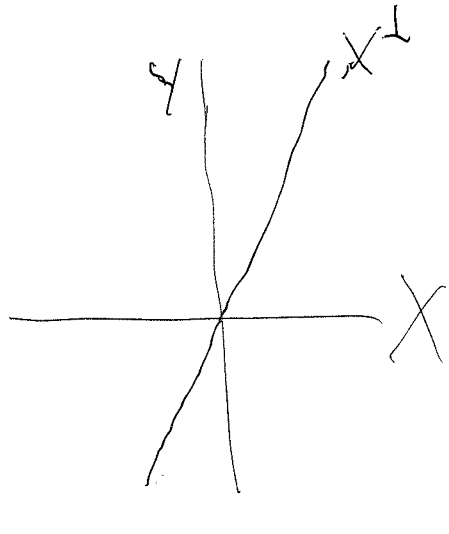
$$\|$$

$$\left\{\begin{pmatrix}x\\y\end{pmatrix} \,\middle|\, \begin{matrix}ax+by=0\\ x=-a^{-1}by\end{matrix}\right\} = \begin{pmatrix}-a^{-1}b\\1\end{pmatrix}Y$$

You need to understand this much better.

Construct

$$\begin{pmatrix}X\\X^\perp\end{pmatrix} \xleftarrow{\begin{pmatrix}1&a^{-1}b\\0&1\end{pmatrix}} \begin{pmatrix}X\\Y\end{pmatrix} \quad ?$$



$$\begin{tikzcd} X & \begin{pmatrix}X\\Y\end{pmatrix} \arrow[l, "(a\ b)"'] \arrow[r, bend left, "\binom{a^{-1}b}{1}(0\ 1)"] \arrow[d, shift right, "(0\ 1)"'] & X^\perp \arrow[l] \\ & Y \arrow[u, shift right, "\binom{a^{-1}b}{1}"'] & \end{tikzcd}$$

y''' Let's try an intrinsic viewpoint.

$$\begin{tikzcd} X \arrow[r, hook, "i"] & V \arrow[r, two heads, "j"] \arrow[d, "A"] & Y \\ X^* & V^* \arrow[l, two heads, "i^t"] & Y^* \arrow[l, hook', "j^t"'] \end{tikzcd}$$

the $\perp$ space to X in V is $\{v \mid i^tAv = 0\}$.

Given such a v $\exists!\ \gamma \in Y^*$ s.t. $j^t\gamma = Av$

$\Rightarrow$ $A^{-1}j^t\gamma = v$

$(jA^{-1}j^t)\gamma = jv$

prove $\operatorname{Ker}(i^tA) = \operatorname{Im}(A^{-1}j^t)$

Note $A = A_s$ is invertible for $s \neq 0, \infty$

$$i^tAv = 0 \iff Av = j^t\gamma \iff v = A^{-1}j^t\gamma$$

$$\begin{tikzcd} X \arrow[r, hook, "i"] & V \arrow[r, two heads, "j"] \arrow[d, shift right, "A"'] & Y \\ X^* & V^* \arrow[l, two heads, "i^t"'] \arrow[u, shift right, "A^{-1}"'] & Y^* \arrow[l, hook', "j^t"'] \end{tikzcd}$$

discuss the good case

the good case should be when i^tAi and $jA^{-1}j^t$ are invertible.

---

try
$$X \xhookrightarrow{i} V \xrightarrow{j} Y \qquad X^\perp = \operatorname{Ker} i^tA$$
invertible quad.

You would like to do everything on the level of space with quad. form.

$$X \xhookrightarrow{i} V \xrightarrow{j} Y$$

$$(v+ix,\ A(v+ix)) = (v, Av) + (ix, Av) + (v, Aix) + O(x^2)$$

Somehow what you want to do should be Gaussian. There should be a Gaussian argument ~~which~~ ~~is~~ missing.

Z''' ~~$\boxed{}$~~ On $\begin{pmatrix} X \\ Y \end{pmatrix}$ have $\begin{pmatrix} x \\ y \end{pmatrix}^t \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$

$$X^\perp = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \middle| \, ax + by = 0 \right\} = \left\{ \begin{pmatrix} -a^{-1}by \\ y \end{pmatrix} \middle| \, y \in Y \right\}.$$

Then
$$\begin{pmatrix} -a^{-1}by \\ y \end{pmatrix}^t \underbrace{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} -a^{-1}by \\ y \end{pmatrix}}_{\begin{pmatrix} 0 \\ (d-ca^{-1}b)y \end{pmatrix}} = y^t (d - ca^{-1}b) y$$

Repeat.

$$\begin{tikzcd} X \arrow[r, hook, "\iota"] & V \arrow[r, two heads, "g"] \arrow[d, shift right, "A"'] & Y \\ X^* & V^* \arrow[l, "\iota^t"'] \arrow[u, shift right, "A^{-1}"'] & Y^* \arrow[l, hook', "g^t"'] \end{tikzcd}$$

Assume A, $\iota^* A \iota$ invertible

$\iota^t A \iota$ inv. $\Rightarrow$ $V = X \oplus X^\perp$

$$v = \iota(\iota^t A \iota)^{-1} \iota^t A v \oplus (v - \iota(\iota^t A \iota)^{-1} \iota^t A v)$$

~~Usse~~ You would like to see the splitting clearly still too complicated.

$$X \xhookrightarrow{\iota} V \xrightarrow{g} Y \qquad \text{symm. inv. } A : V \to V^*$$

def $X^\perp = \{ v \mid \forall x \ (\iota x, Av) = 0 \}$.

construct $V = X \oplus X^\perp$