Review yesterday's progress. C = alg gens h_s, $s \in \Gamma$ fin, rel $\sum_\Gamma h_s = 1$, $B = C \rtimes \Gamma$ 189

C, B unital. Given E a firm B-module, factor $h_1 = \beta_1 \alpha_1 : E \longrightarrow E \longrightarrow E$ and E becomes an A module with $p_s = \alpha_1 s \beta_1$. So you get ~~[illegible]~~ restriction of scalar functor from B ~~firm~~ mod. to A-mods. But it depends on choice of $h_1 = \beta_1 \alpha_1$. Two extremes

$$\begin{tikzcd} & E \arrow[dr, "1"] & \\ E \arrow[ur, "h_1"] \arrow[dr, "1"'] & & E \\ & E \arrow[uu, "h_1"] \arrow[ur, "h_1"'] & \end{tikzcd}$$

$\alpha_1 s \beta_1 = h_1 s$

$\alpha_1 s \beta_1 = s h_1$

$$\begin{tikzcd} E \arrow[r, "h_1 s"] & E \\ E \arrow[u, "h_1"] \arrow[r, "s h_1"'] & E \arrow[u, "h_1"'] \end{tikzcd}$$

map between the two A-modules which is a nul isom.

Now you want the firm Morita context behind this Morita context. Now you have two (at least) ~~[illegible]~~ restriction of scalar functions, each gives you an A, B bimodules, call them Q' and Q. Both are $= B$ as B^{op}-module, but $p'_s = s h_1$ on Q', and $p_s = h_1 s$ on Q and one has $h_1 : Q' \longrightarrow Q$

You need to go over general theory for $\begin{pmatrix} A & Q \\ P & B \end{pmatrix}$ strictly idemp. I recall that if $P \otimes_A A \xrightarrow{\sim} P$ then $B \otimes_B P \xrightarrow{\sim} P$. The idea is that

$B \otimes_B P \longrightarrow P$ should be an A^{op} nil isom.

e.g. $PA = PQP = BP$

$$\begin{tikzcd} B \otimes_B P \arrow[r] \arrow[d, "\cdot qp"'] & P \arrow[d, "\cdot qp"] \\ B \otimes_B P \arrow[r] & P \end{tikzcd}$$

$b \otimes p'$ $\quad bp'$

$b \otimes p'qp$ $\quad bp'qp$

$b \otimes p' \longmapsto bp'$

$b \otimes p'q \otimes p$ ( ) $\; bp'q \otimes p$

You've used $\boxed{A = QP \text{ and } B = PQ}$

You need $PQP = P$, $QPQ = Q$ for this argument.

So far $Q = B$ with obvious B^{op}-mult and $p_s \cdot = (h_1 s)\cdot$. By symmetry you expect P to be B with obvious B^{op}-mult and A^{op}-mult given by $\cdot p_s = \cdot(h_1 s)$, or $\cdot(s h_1)$, or possibly with s^{-1} somewhere in place of s. You are reminded of $\alpha = \{\alpha_1 s^{-1}\}$ $\quad \beta = \{s\beta_1\}$

Something useful. You ~~know that~~ know that a first candidate for Q is B with $p_s = h_1 s$ or $s h_1$. But you need $AQ = Q$

Two possible AQ, namely $\sum p_s B = \sum_s h_1 s B = h_1 B$

or $\sum p_s B = \sum_s s h_1 B = \sum_s h_s B \simeq B$

Idea: $\{\alpha_1 s^{-1}\}$ family of q_i $\quad \{s\beta_1\}$ family of p_i

such that $\sum p_i q_i = \sum s\beta_1\alpha_1 s^{-1} = 1 \in B.$

$$\begin{pmatrix} A & Q \\ P & B \end{pmatrix} \qquad \sum p_i q_i = 1.$$

$$Q \xrightarrow{(\cdot p_i)} A^n \subset \tilde{A}^n \xrightarrow{(\cdot q_i)} Q$$

$$q \longmapsto \underset{q p_i}{\langle\ \rangle} \longmapsto \sum_i \underset{(q p_i)}{\langle\ \rangle} q_i = q$$

The situation becomes clearer.
Start with your Morita equivalence

$$\cancel{\begin{pmatrix} A & Y \\ X & B \end{pmatrix} \quad {}_B X_A \text{ contains } x_s = \alpha_1 s^{-1} \quad {}_A Y_B \text{ — } y_s = s\beta_1 \quad \sum y_s x_s = \sum s\beta_1\alpha_1 s^{-1} = 1}$$

$$\begin{pmatrix} A & Y \\ X & B \end{pmatrix} \qquad \sum x_s y_s = 1 \in B$$

$$x_s = s\beta_1 \qquad y_s = \alpha_1 s^{-1}$$

$${}_B X_A \qquad {}_A Y_B$$

$$X = \mathbb{C}\Gamma \otimes A, \quad \overbrace{\langle y_s | x_t \rangle}^{\text{pairing}} = y_s x_t = \alpha_1 s^{-1} t \beta_1 = p_{s^{-1}t}$$

You have to choose $h_1 = \beta_1\alpha_1$, say $h_1 = h_1 1$ or $1 h_1$. $E \mapsto Y \otimes_B E$ Y should be roughly $h_1 B$. Try $y_s = h_1 s^{-1}$. Try

$$\begin{pmatrix} A & h_1 B \\ B & B \end{pmatrix}$$

~~Y contains $y_s = h_1 s^{-1}$~~

~~$\sum y_s B = h_1 B$~~

~~X contains x_s~~

Y contains $y_s = \alpha_1 s^{-1}$, hence $\sum y_s B = \sum_s \alpha_1 s^{-1} B = \alpha_1 B$

X —— $x_s = s\beta_1$, hence $\sum B x_s = B\beta_1$

$$\begin{pmatrix} \alpha_1 B \beta_1 & \alpha_1 B \\ B\beta_1 & B\beta_1\alpha_1 B = B \end{pmatrix}$$

This is a Morita context, it will be a firm Morita context iff $\alpha_1 B \otimes_B B\beta_1 \xrightarrow{\sim} \alpha_1 B \beta_1$

Review: Γ finite C $h_s, s\in\Gamma$, $\sum h_s = 1$

$B = C \rtimes \Gamma$. B module same as a Γ module with of $h_1 \ni \sum s h_1 s^{-1} = 1$. C, B unital. Choose factorization $h_1 = \beta_1\alpha_1$.

$$\begin{pmatrix} A & Y \\ X & B \end{pmatrix}$$

$p_{s^{-1}t} = \alpha_1 s^{-1} t \beta_1$

$1 = \sum s\beta_1\alpha_1 s^{-1}$

$x_s = s\beta_1$, $y_s = \alpha_1 s^{-1}$. ~~Question~~ The grading is not clear.

Let's begin again with B and elements $\alpha_1, \beta_1 \in B$ of degree 1 such that $h_1 = \beta_1\alpha_1$. Consider the Morita context $\begin{pmatrix} B & B \\ B & B \end{pmatrix}$. No.

If E is a B-module (firm), then you get an A-module given by E with of $p_s = \alpha_1 s \beta_1$

Other direction $\quad \mathbb{C}\Gamma \overset{p}{\otimes} A \ni \sum s \otimes f_s \qquad f \in C_c(\Gamma, A)$

$$p(\Gamma\times A) = \left\{ \sum_t t\otimes f_t \;\middle|\; \sum s\otimes p_s \sum t\otimes f_t = \sum u\otimes f_u \right\}$$

$$\boxed{\sum_{st=u} p_s f_t = f_u}$$

$$\sum t\otimes f_t \longmapsto \sum u \otimes \sum_{st=u} p_s f_t = \sum_u u\otimes \sum p_{ut^{-1}} f_t$$

$$\Gamma\times A \xrightarrow{p} \Gamma\times A$$

$$\downarrow$$

$\mathbb{C}\Gamma\otimes A$

$$\sum t^{-1}\otimes f_t \qquad\qquad \sum u^{-1}\otimes \sum_{st=u} p_s f_t$$

$$\sum t\otimes f_t \longmapsto \sum t^{-1}\otimes f_t \longmapsto \sum_{s,t} st^{-1}\otimes p_s f_t \mapsto \sum_{s,t} ts^{-1}\otimes p_s f_t$$

$$\sum_{s,t} ts^{-1}\otimes p_s f_t = \sum_u u\otimes \sum_t p_{u^{-1}t} f_t$$

$u = ts^{-1}$

$us = t \quad s = u^{-1}t$

So if F is an A-module, then the corresp. ~~[illegible]~~ left B-module is

$$\mathbb{C}\Gamma \overset{p}{\otimes} F = \{ f\in C_c(\Gamma, F) \mid f_s = \sum_t p_{s^{-1}t} f_t \}$$

$$u \sum s\otimes f_s = \sum us\otimes f_s = \sum_u u\otimes f_{u^{-1}s}$$

<u>OKAY</u>.

$$\begin{pmatrix} A & Y = (A\times\Gamma)p \\ X = p(\Gamma\times A) & B \end{pmatrix}$$

You have a canon. projector in $A\times\Gamma = \Gamma\times A$

$\mathcal{M}(A) \longrightarrow \mathcal{M}(B)$

$F \longmapsto X\otimes_A F$

You want to write
$X = \mathbb{C}\Gamma \overset{p}{\otimes} A$
to display its B, A bimodule structure

$$\sum_{s\in\Gamma} x_s y_s = \sum_{s\in\Gamma} s\beta_1\alpha_1 s^{-1} = 1 \in B$$

Inside of X you want to find β_1

How do I make progress on these puzzles?

Recall that ~~the~~ one reason for $\mathbb{C}\Gamma \overset{p}{\otimes} A$ is that p $= \sum (\cdot s^{-1})\otimes(p_s \cdot)$ is homogeneous of unit degree, and hence $\mathbb{C}\Gamma \overset{p}{\otimes} A$ is Γ graded.

Look at $\Phi = \Gamma = \mathbb{Z}/2$ again. A in this case is $\mathbb{C}e * \mathbb{C}e$ which should be ~~the~~ the augm. ~~augmentation~~ ideal in $\mathbb{C}\Gamma \overset{\mathbb{C}}{*} \mathbb{C}\Gamma = \mathbb{C}[\text{dihedral group } \mathbb{Z} \rtimes \mathbb{Z}/2]$.

$$B = \mathbb{C}[h_0, h_1]/(h_0+h_1-1) \rtimes \mathbb{Z}/2.$$ here ε interchanges h_0 and h_1

so $\varepsilon\left(h_0 - \frac{1}{2}\right)\varepsilon^{-1} = h_1 - \frac{1}{2} = -\left(h_0 - \frac{1}{2}\right)$ $\qquad h_0 - \frac{1}{2} + h_1 - \frac{1}{2} = 0$

$e^2 = e \iff (1-2e)^2 = 1 \qquad 4e^2 - 4e + 1 = 1$

$$\mathbb{C}e * \mathbb{C}e < \widetilde{\mathbb{C}e} \overset{\mathbb{C}}{*} \widetilde{\mathbb{C}e} = \mathbb{C}[\mathbb{Z}/2 * \mathbb{Z}/2]$$

$$\downarrow$$

$$\mathbb{C}$$

$B = \mathbb{C}[\overset{z}{\overline{h_0 - \frac{1}{2}}}] \rtimes \mathbb{Z}/2 \qquad \varepsilon\left(h_0 - \frac{1}{2}\right) = -\left(h_0 - \frac{1}{2}\right)$

basis $\quad \begin{matrix} 1 & z & z^2 \\ \varepsilon & z\varepsilon & z^2\varepsilon \end{matrix}$

so you can ~~embed~~ embed A into B in four ways

$A = \mathbb{C}e * \mathbb{C}\bar{e}$

e	$\bar{e}e$	e	F	or	$-F$
$\bar{e}$	$e\bar{e}$	$\bar{e}$	$\bar{F}$	or	$-\bar{F}$

A generated by p_0, p_1 rels. $(p_0 \pm p_1)^2 = p_0 \pm p_1$

$\therefore A = \mathbb{C}e * \mathbb{C}\bar{e} \subseteq \mathbb{C}[F] \overset{\mathbb{C}}{*} \mathbb{C}[\bar{F}] \qquad F = 1-2e$

$$\shortparallel$$

$$\mathbb{C}[\underbrace{\mathbb{Z}/2 * \mathbb{Z}/2}_{\text{dihedral group.}}]$$

$B = \mathbb{C}[h_0, h_1]/(h_0+h_1-1) \rtimes \mathbb{Z}/2 \qquad \varepsilon\left(h_0 - \frac{1}{2}\right) = \left(h_1 - \frac{1}{2}\right)$

$= -h_0 + \frac{1}{2}$

Apparently A, B are Morita equivalent

$h_0 = \beta_0 \alpha_0 = h_0 1$

$p_n = \alpha_0 \varepsilon^n \beta_0 = h_0 \varepsilon^n$

$$p_0 = h_0 \qquad p_1 = \varepsilon h_0$$

$$(p_0+p_1)^2 = h_0(1+\varepsilon)h_0(1+\varepsilon)$$

$$= h_0^2 + \underbrace{h_0\varepsilon h_0}_{h_0 h_1 \varepsilon} + h_0^2\varepsilon + \underbrace{h_0\varepsilon h_0\varepsilon}_{h_0 h_1}$$

$$=$$

$$(p_0 \pm p_1)^2 = (h_0 \pm \varepsilon h_0)^2 = h_0^2 \pm h_0\varepsilon h_0 \pm \underbrace{\varepsilon h_0^2}_{h_1 \varepsilon h_0} + \underbrace{\varepsilon h_0 \varepsilon}_{h_1} h_0$$

$$= (h_0+h_1)h_0 \pm (h_0+h_1)\varepsilon h_0 = h_0 \pm \varepsilon h_0 = p_0 \pm p_1$$

$$x_s = s\beta_1 \qquad y_s = \alpha_1 s^{-1} \qquad h_0 = \underset{h_0}{\underset{\shortparallel}{\alpha_0}}\ \underset{1}{\underset{\shortparallel}{\beta_0}}$$

$$x_n = \varepsilon^n \in X, \quad y_n = h_0\varepsilon^{-n} \in Y$$

$$\sum_{\mathbb{Z}/2} x_n y_n = h_0 + h_1 = 1.$$

$$\begin{array}{cc} A & Y \\ X & B \end{array}$$

$$X = \qquad p(\Gamma \times A)$$

$$p = 1\otimes p_0 + \varepsilon\otimes p_1 \qquad f = 1\otimes f_0 + \varepsilon\otimes f_1$$

$$pf = 1\otimes(p_0f_0 + p_1f_1) + \varepsilon\otimes(p_0f_1 + p_1f_0)$$

$$\therefore \quad \begin{aligned} p_0f_0 + p_1f_1 &= f_0 \\ p_0f_1 + p_1f_0 &= f_1 \end{aligned} \qquad \begin{pmatrix} p_0 & p_1 \\ p_1 & p_0 \end{pmatrix}\begin{pmatrix} f_0 \\ f_1 \end{pmatrix} = \begin{pmatrix} f_0 \\ f_1 \end{pmatrix}$$

$$(1\ \varepsilon)\left(\begin{pmatrix} p_0 \\ p_1 \end{pmatrix} f_0 + \begin{pmatrix} p_1 \\ p_0 \end{pmatrix} f_1\right) = (p_0+\varepsilon p_1)f_0 + (p_1+\varepsilon p_0)f_1$$

$$= (p_0+\varepsilon p_1)(f_0+\varepsilon f_1) \qquad f_0, f_1 \in A$$

$$(p_0+\varepsilon p_1)^2 = p_0^2 + \varepsilon p_0 p_1 + \varepsilon p_1 p_0 + \varepsilon^2 p_1^2 = \underbrace{p_0^2+p_1^2}_{p_0} + \varepsilon\underbrace{(p_0p_1+p_1p_0)}_{\varepsilon p_1}$$

You will now do the calculation for $\mathbb{Z}/2 = \{1, \varepsilon\}$

$B = \mathbb{C}[h_0,h_1]/(h_0+h_1-1) \rtimes \mathbb{Z}/2$

$\varepsilon h_0 \varepsilon = h_1 \qquad \varepsilon(h_0 - \frac{1}{2})\varepsilon = h_1 - \frac{1}{2} = -(h_0-\frac{1}{2})$

A generators p_0, p_1 rels. $(p_0 \pm p_1)^2 = p_0 \pm p_1$

$A = \mathbb{C}e * \mathbb{C}\bar{e} \qquad \begin{array}{l} e = p_0+p_1 \\ \bar{e} = p_0 - p_1 \end{array}$

$\tilde{A} = \widetilde{\mathbb{C}e} \overset{\mathbb{C}}{*} \widetilde{\mathbb{C}\bar{e}} \simeq \mathbb{C}[F] \overset{\mathbb{C}}{*} \mathbb{C}[\bar{F}] = \mathbb{C}[\text{dihedral group}]$

$\simeq \mathbb{C}[\underset{u}{\mathbb{Z}}] \rtimes \mathbb{C}[\underset{\varepsilon}{\mathbb{Z}/2}]. \qquad \varepsilon u \varepsilon = u^{-1}.$

$A = \mathbb{C}e * \mathbb{C}\bar{e} \qquad \tilde{A} = \mathbb{C}[e] \overset{\mathbb{C}}{*} \mathbb{C}[\bar{e}] = \mathbb{C}[F] \overset{\mathbb{C}}{*} \mathbb{C}[\bar{F}]$

An A-module is a v.s. V with two projectors $e, \bar{e}$

Since $A = e\tilde{A} + \bar{e}\tilde{A}$ one $AV = eV + \bar{e}V$ &

and ~~since~~ $A = \tilde{A}e + \tilde{A}\bar{e}$, one have ${}_AV = {}_eV \cap {}_{\bar{e}}V$

$= e^{\perp}V \cap \bar{e}^{\perp}V$. Question: Is $e^{\perp}V \cap \bar{e}^{\perp}V$ a complement for $eV + \bar{e}V$?   NO.

$$\begin{tikzcd}[column sep=small]
 & V \arrow[d, dash] & \\
 & eV+\bar{e}V \arrow[dl, dash] \arrow[dr, dash] & \\
eV \arrow[dr, dash] & & \bar{e}V \arrow[dl, dash] \\
 & eV\cap\bar{e}V \arrow[d, dash] & \\
 & 0 &
\end{tikzcd}$$

$$V = eV \oplus e^{\perp}V$$
$$= \bar{e}V \oplus \bar{e}^{\perp}V$$

$$eV \subset eV+\bar{e}V$$
$$eV + e^{\perp}V = V$$

$$eV + (eV+\bar{e}V)\cap\bar{e}^{\perp}V$$
$$= eV+\bar{e}V$$

~~What seem important~~

$A = \mathbb{C}e * \mathbb{C}\bar{e}$, An A-module is a vector space V tog with two idempotent operators $e, \bar{e}$ i.e. two splittings. Since

$$A = e\tilde{A} + \bar{e}\tilde{A} \implies AV = eV+\bar{e}V$$
$$A = \tilde{A}e + \tilde{A}\bar{e} \implies {}_{A}V = \mathrm{Ker}(e) \cap \mathrm{Ker}(\bar{e})$$

Example. $\dim V = 2$, eV $\bar{e}V$ two lines in V with a common complement $\underset{\mathrm{Ker}(e)}{\underset{\parallel}{(1-e)V}} = \underset{\mathrm{Ker}(\bar{e})}{\underset{\parallel}{(1-\bar{e})V}}$. Then

Then $AV = V$ but ${}_{A}V \neq 0$.

$$\begin{tikzcd}[column sep=small]
 & V \arrow[d] & \\
 & W \arrow[dl, "e^{\perp}V\cap W"', "\times" marking] \arrow[dr, "W\cap\bar{e}^{\perp}V", "\times" marking] & \\
eV \arrow[dr, "\times" marking] & & \bar{e}V \arrow[dl, "\times" marking] \\
 & eV\cap\bar{e}V \arrow[d, dash] & \\
 & 0 &
\end{tikzcd}$$

~~[illegible]~~ Can you get a picture of fasin modules? You want $V = eV + \bar{e}V$ and complements $e^{\perp}V$ for eV and $\bar{e}^{\perp}V$ for $\bar{e}V$

~~[illegible]~~ Picture of an A-module is a v.s. V with two splittings. V is round when

$$eV + \bar{e}V = V \quad \text{and} \quad \underbrace{(1-e)V} \cap (1-\bar{e})V = 0, \text{ so}$$

you have $\mathrm{Im}(e) + \mathrm{Im}(\bar{e}) = V$, $\mathrm{Ker}(e) \cap \mathrm{Ker}(\bar{e}) = 0$.

$$V \xrightarrow{\binom{e}{\bar{e}}} \begin{matrix} V \\ \oplus \\ V \end{matrix} \xrightarrow{(e \ \ \bar{e})} V \quad ?$$

Go back to $\mathbb{Z}/2$. $\quad B = \mathbb{C}[h_0, h_1]/(h_0+h_1-1) \rtimes \mathbb{Z}/2$

$$\varepsilon\left(h_0 - \tfrac{1}{2}\right) = \left(h_1 - \tfrac{1}{2}\right) = -\left(h_0 - \tfrac{1}{2}\right), \quad B \text{ is unital}$$

A firm B-module is ~~a vector~~ a vector space E together with involution ε and an odd operator.

A super vector space $\quad E = E_+ \oplus E_-$

$$h_0 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad h_1 = \varepsilon h_0 \varepsilon = \begin{pmatrix} a & -b \\ -c & d \end{pmatrix} \qquad h_0 + h_1 = \begin{pmatrix} 2a & 0 \\ 0 & 2d \end{pmatrix}$$

$$\therefore \ a = \tfrac{1}{2} = d = \tfrac{1}{2}.$$

So you know exactly what firm B-modules are.

$$E = E_0 \leftrightarrows E_-$$

And you know what round $A = \mathbb{C}e * \mathbb{C}\bar{e}$ are

~~[illegible]~~. An A module is a representation of the dihedral group $\mathbb{Z}/2 * \mathbb{Z}/2$. Specify

that $\quad F = +1$ on $\mathrm{Im}\, e \qquad\qquad F = 2e - 1$

$\qquad\quad -1$ on $\mathrm{Ker}(e)$

To understand – fix ε, call other F,

$g = F\varepsilon$ $\qquad \varepsilon g \varepsilon = \varepsilon F = g^{-1}$. So what do you find? Suppose you have $e, \bar{e}$ on V

Look at $\operatorname{Ker}(e) \cap \operatorname{Ker}(\bar{e})$; on this space $F = \bar{F} = -1$

—— $V / eV + \bar{e}V$; —— $F = \bar{F} = +1$

What are you trying to understand? Spectrum. $g = F\varepsilon$ is invertible, its eigenvalues are $z \in \mathbb{C}^\times$ and since $\varepsilon g \varepsilon = g^{-1}$, $z \in \operatorname{Spec} \Rightarrow$ $z^{-1} \in \operatorname{Spec}$. But $+1, -1$ fix pts. ~~Normally the spectrum is~~ Spectrum of the dihedral group = set of irred repns. 1-diml reps : characters. There are 4 homos. $\mathbb{Z}/2 * \mathbb{Z}/2 \longrightarrow \{\pm 1\}$.

You have $g = F\varepsilon = 1$ whence $F = \varepsilon = 1$
or $F = \varepsilon = -1$

The problem: $A = \mathbb{C}e * \mathbb{C}\bar{e}$, $\quad B = \mathbb{C}[h_0 - \frac{1}{2}] \rtimes \mathbb{Z}/2$.
There seem to be homomorphisms $A \longrightarrow B$, and since B is unital, those correspond to unital homos. $\tilde{A} \longrightarrow B$ i.e. pairs of involutions in B. $\quad B = \mathbb{C}[x] \rtimes \{\varepsilon\}$

$$B = \{ f(x) + g(x)\varepsilon \mid f(x), g(x) \in \mathbb{C}[x] \}.$$

$$(f + g\varepsilon)^2 = f^2 + fg\varepsilon + g\varepsilon f + g\varepsilon g \varepsilon$$

$$= \underbrace{f(x)^2 + g(x)g(-x)}_{1} + \underbrace{\big(f(x)g(x) + g(x)f(-x)\big)}_{0}\varepsilon \qquad \therefore f \text{ odd}$$

example $\quad f(x) = +x \qquad g(x) = 1 + x \qquad x^2 + \underbrace{(1+x)(1-x)}_{1-x^2} = 1.$

$f+g\varepsilon \;=\; x+(1+x)\varepsilon$ one involution

$f\varepsilon+g \;=\; x\varepsilon+(1+x)$ should be invertible

with inverse $\varepsilon(f+g\varepsilon) = \varepsilon(x+(1+x)\varepsilon) = -x\varepsilon+(1-x)$

$$\big(x\varepsilon+(1+x)\big)\big(-x\varepsilon+(1-x)\big) = -x\varepsilon x\varepsilon-(1+x)x\varepsilon + x\varepsilon(1-x)+(1-x^2)$$

$$= x^2 - x(1+x)\varepsilon + x(1+x)\varepsilon + (1-x^2) \qquad ?$$

$$= x^2 + [-x-x^2+x+x^2]\varepsilon + 1-x^2 = 1$$

$$\big(x+(1+x)\varepsilon\big)\big(x+(1+x)\varepsilon\big) = x^2+(1+x)(-x)\varepsilon + x(1+x)\varepsilon + (1+x)(1-x)$$

$$= \cancel{x^2}+\{\cancel{-x-x^2+x+x^2}\}\varepsilon + 1-\cancel{x^2} = 1.$$

$$\big(x+(1+x)\varepsilon\big)\varepsilon = x\varepsilon+1+x$$

$$\varepsilon\big(x+(1+x)\varepsilon\big) = \underline{-x\varepsilon+(1-x)}$$

$$= (x\varepsilon)(-x\varepsilon) + x\varepsilon(1-x) + (1+x)(-x\varepsilon) + 1-x^2$$

$$= x^2 + \{x(1+x)-(1+x)x\}\varepsilon + 1-x^2 = 1.$$

So therefore you have $\boxed{\cancel{x+(1+x)\varepsilon}}$

$$g(x)g(-x) = (1+f(x))(1-f(x))$$

invertible element $1+x+x\varepsilon$

with inverse $1-x-x\varepsilon$

$$f(x)^2 + (1+f(x))(1-f(x))$$

In $B = \mathbb{C}[x] \rtimes \mathbb{Z}/2$ $\quad \varepsilon x \varepsilon = -x$, the elt. $f + g\varepsilon$, f and $g \in \mathbb{C}[x]$ is an involution when

$$1 = (f+g\varepsilon)^2 = f(x)^2 + g(x)g(-x) + \big(f(x)g(x) + g(x)f(-x)\big)\varepsilon$$

If $g \neq 0$ then $f(x)$ is odd, ($g=0$ $\ f = \pm 1$ case omitted)

and $g(x)g(-x) = (1+f(x))(1-f(x))$, the obvious choice for $g(x)$ is $1+f(x)$, but already there is also $g(x) = 1 - f(x)$.

Go back to $\quad p_s = \alpha_1 s \beta_1 \qquad h_1 = \beta_1\alpha_1$

$$B = \underbrace{\mathbb{C}[h_0,h_1]/(h_0+h_1-1)}_{\mathbb{C}[x]} \rtimes \mathbb{Z}/2 \qquad x = h_0 - \tfrac{1}{2}$$

$$\mathbb{C}[\mathbb{Z}/2] \otimes A = \{ f_0 + \varepsilon f_1 \mid f_0, f_1 \in A \}. \qquad \varepsilon a = a\varepsilon$$

$$p = p_0 + \varepsilon p_1 \qquad pf = p_0 f_0 + p_1 f_1 + \varepsilon(p_0 f_1 + p_1 f_0)$$

$$p\,(\mathbb{C}[\varepsilon]\otimes A) = (1 \ \ \varepsilon) \cdot \begin{pmatrix} p_0 & p_1 \\ p_1 & p_0 \end{pmatrix} \begin{pmatrix} f_0 \\ f_1 \end{pmatrix}$$

Use the formulas you have.

$$\underbrace{\sum t \otimes f_t}_{\in\ \mathbb{C}\Gamma \otimes A} \xrightarrow{\ \sum s \otimes p_s\ } \sum st \otimes p_s f_t$$

$\mathbb{C}\Gamma \otimes A$

Let's start with a B module E, choose fact.

$$\begin{tikzcd} E \arrow[r, "\alpha_1"] & V \arrow[r, "\beta_1"] \arrow[rr, bend right, "h_1"'] & E \arrow[r, "\alpha_1"] & V \end{tikzcd}$$

$$\begin{tikzcd} E \arrow[r, hook, "\alpha"] \arrow[rr, bend left, "id"] & \mathbb{C}\Gamma\otimes V \arrow[r, "\beta"] \arrow[rr, bend left, "p"] & E \arrow[r, hook, "\alpha"] & \mathbb{C}\Gamma\otimes V \end{tikzcd}$$

$$\xi \longmapsto \sum s\otimes\alpha_1 s^{-1}\xi \longmapsto \sum_t t\beta_1\alpha_1 t^{-1}\xi = \xi$$

$$\sum t\otimes f_t \longmapsto \sum_t t\beta_1 f_t \longrightarrow \sum_s s\otimes \sum_t \underbrace{\alpha_1 s^{-1} t\beta_1}_{p_{s^{-1}t}} f_t$$

~~So you want to take $\mathbb{C}\Gamma\otimes V$~~

Two paths: start V, say $V=\tilde{A}$, form $p(\mathbb{C}\Gamma\otimes V)$ and exhibit how B acts

~~Your aim: Take $V=A$~~

first start with V, then form $p(1\otimes v) = \sum_s s\otimes\alpha_1 s^{-1}\beta_1 v$ $\subset \mathbb{C}\Gamma\otimes V$ and form the Γ-submodule it generates.

$$\begin{aligned} pv &= \alpha_0\beta_0 v + \varepsilon\alpha_0\varepsilon\beta_0 v \\ &= p_0 v + \varepsilon p_1 v \end{aligned}$$

Here V is a vector space with two projections e and $\bar{e}$, in fact $e = p_0+p_1$, $\bar{e} = p_0 - p_1$.

You tensor with $\mathbb{C}\Gamma$

$$\begin{aligned} p(v_0+\varepsilon v_1) &= (p_0+\varepsilon p_1)(v_0+\varepsilon v_1) \\ &= (p_0v_0+p_1v_1) + \varepsilon(p_0v_1+p_1v_2) \end{aligned}$$

the Γ action is mult. by ε. Now you have $E = p(\mathbb{C}\Gamma\otimes V)$ as a Γ subspace of $\mathbb{C}\Gamma\otimes V$, and you need only give $h_0 = \beta_0\alpha_0$

$\Gamma = \mathbb{Z}/2$. $\quad B \simeq \mathbb{C}[h_0, h_1]/(h_0 + h_1 - 1) \rtimes \mathbb{Z}/2$

$A = \mathbb{C}e * \mathbb{C}\bar{e}$, an A module is a V with two splittings, a graded A-module is a ~~super~~ $\mathbb{Z}/2$-graded v.s. $V = V_0 \oplus V_1$ together with an idempotent operator. ?

A generators p_0, p_1 relations $p_0 \pm p_1$ idempotent

$$\begin{cases} p_0^2 + p_1^2 = p_0 \\ p_0 p_1 + p_1 p_0 = p_1 \end{cases}$$

graded A-module $\quad V = V_0 \oplus V_1$

$$\mathrm{End}(V) = \begin{pmatrix} \mathrm{End}(V_0) & \mathrm{Hom}(V_1, V_0) \\ \mathrm{Hom}(V_0, V_1) & \mathrm{End}(V_1) \end{pmatrix}$$

p_0 even operator, p_1 odd op on V.

basic construction $\quad \mathbb{C}\Gamma \otimes V = \{ 1 \otimes f_0 + \varepsilon \otimes f_1 \mid f_0, f_1 \in V \}$.
$= \{ v_0 + \varepsilon v_1 \mid v_0 \text{ and } v_1 \in V \}$. Define $p = 1 \otimes p_0 + \varepsilon \otimes p_1$

$$\boxed{(p_0 + \varepsilon p_1)(v_0 + \varepsilon v_1) = (p_0 v_0 + p_1 v_1) + \varepsilon(p_0 v_1 + p_1 v_0)}$$

ungraded V

Review construction. A gens p_s, $s \in \Gamma$ rels $p_s = \sum_t p_{st^{-1}} p_t$

A is a Γ-graded algebra $\quad A = \bigoplus_s A_s \quad A_s A_t \subset A_{st}$

canonical embedding $\quad A \longrightarrow \mathbb{C}\Gamma \otimes A \quad$ tensor product alg where $sa = as$.

$$p_s \longmapsto s \otimes p_s$$

Preserves Γ-grading where A on the right has unit degree.

Question. What about considering the tensor product $\mathbb{C}\Gamma \otimes A$ as superalgebra.

$\Gamma = \mathbb{Z}/2 \qquad \mathbb{C}\Gamma = \mathbb{C}[\varepsilon]$ You form

$\mathbb{C}\Gamma \otimes A = \{ a_0 + \varepsilon a_1 \}$ ring where ε, A commute, $\varepsilon^2 = 1$.

$p = p_0 + \varepsilon p_1 \in \mathbb{C}\Gamma \otimes A$.

Now $\mathbb{C}\Gamma \otimes A \longrightarrow A \times A \qquad \varepsilon \longrightarrow +1, -1$

so it seems like $\mathbb{C}\Gamma \otimes A$ and the canonical proj $p = p_0 + \varepsilon p_1$ become $A \times A$ with $p = (e, \bar{e})$, so if we apply p we get $eA \times \bar{e}A$

Let's repeat. $\Gamma = \mathbb{Z}/2 \qquad \mathbb{C}\Gamma = \mathbb{C}[\varepsilon] = \mathbb{C} \oplus \mathbb{C}\varepsilon \qquad \varepsilon^2 = 1$.

$$\mathbb{C}[\varepsilon] \otimes A = A \oplus \varepsilon A \qquad \varepsilon a = a\varepsilon, \quad \varepsilon^2 = 1.$$

$$p = \sum_s s \otimes p_s = p_0 + \varepsilon p_1.$$

In our situation

$A = \mathbb{C}e * \mathbb{C}\bar{e} \subset \Omega(\mathbb{C}e)$ with Fedosov product

$$p_0 = e \qquad p_0^2 + p_1^2 = e^2 - \cancel{dede} \cancel{+de^2} = e = p_0$$

$$p_1 = de \qquad p_0 p_1 + p_1 p_0 = ede + dee = d(e^2) = de = p_1$$

So you form $\mathbb{C}[\varepsilon] \otimes A = A \oplus \varepsilon A$. Inside here is the element $p = p_0 + \varepsilon p_1$ which is idempotent (and even). Form $p(A \oplus \varepsilon A)$. Your problem is to understand $p(A \oplus \varepsilon A)$, find a formula for it, you want the Γ action, and the operator h_0

$$\begin{tikzcd}[row sep=small] A \arrow[dr, "\beta_0"] & & A \\ \oplus & E \arrow[ur, "\alpha_0"] \arrow[dr, "\alpha_1"'] & \oplus \\ \varepsilon A \arrow[ur, "\beta_1"'] & & \varepsilon A \end{tikzcd}$$

There should be some organizing principle. You are constructing a fin proj A^{op} module, whose endo ring is the unital ring B.

Look clearly. $\mathbb{C}[\varepsilon]\otimes\tilde{A}$ is a free A^{op} module of rank 2. On this module you have p acting. Actually $M_2(\tilde{A})$ acts – it's the ring of $\mathrm{End}_A(\mathbb{C}[\varepsilon]\otimes\tilde{A})$, and $p = \begin{pmatrix} p_0 & p_1 \\ p_1 & p_0 \end{pmatrix} \in M_2(\tilde{A})$.

Then B should be $p\,M_2(\tilde{A})\,p$.

~~Things should be straightforward!!!!!~~ $\cancel{\tilde{A}}$

$$\mathbb{C}[\varepsilon]\otimes\tilde{A} \xrightarrow{\beta} E \xrightarrow{\alpha} \mathbb{C}[\varepsilon]\otimes\tilde{A} \xrightarrow{\beta} E$$

$$\begin{tikzcd}
E & \mathbb{C}[\varepsilon]\otimes\tilde{A} \arrow[l, "\beta"'] & E \arrow[l, "\alpha"'] \arrow[ll, bend right, "id"'] & \mathbb{C}[\varepsilon]\otimes\tilde{A} \arrow[l, "\beta"'] \arrow[ll, bend left]
\end{tikzcd}$$

$$p = \begin{pmatrix} p_0 & p_1 \\ p_1 & p_0 \end{pmatrix}$$

$$E \xleftarrow{(\beta_0\ \ \beta_1)} \begin{matrix}\tilde{A}\\ \oplus \\ \tilde{A}\end{matrix} \xleftarrow{\begin{pmatrix}\alpha_0\\ \alpha_1\end{pmatrix}} E \xleftarrow{(\beta_0\ \ \beta_1)} \begin{matrix}\tilde{A}\\ \oplus \\ \tilde{A}\end{matrix}$$

$\alpha_1 = \alpha_0\varepsilon \qquad \beta_1 = \varepsilon\beta_0 \qquad$ Check

$$\begin{pmatrix}\alpha_0 \\ \alpha_0\varepsilon\end{pmatrix}\begin{pmatrix}\beta_0 & \varepsilon\beta_0\end{pmatrix} = \begin{pmatrix}\alpha_0\beta_0 & \alpha_0\varepsilon\beta_0 \\ \alpha_0\varepsilon\beta_0 & \alpha_0\beta_0\end{pmatrix}$$

All you should need to do is to go over the formulas enough. In the case of finite Γ ~~[illegible]~~ you start with a fin. gen. free A^{op}-module, namely, $\mathbb{C}\Gamma \otimes \tilde{A}$, you have an idempotent operator $p = \sum s \otimes p_s$ on this module and $E = p(\mathbb{C}\Gamma \otimes \tilde{A})$ is a finite projective A^{op}-module. there's a dual fin proj A-module $E' = \mathrm{Hom}_A(E, \tilde{A})$ whence a Morita context, where $B = E \otimes_A E' = \mathrm{Hom}_{A^{op}}(E,E)$

$$\begin{pmatrix} \tilde{A} & E' \\ E & B = E \otimes_A E' \end{pmatrix}$$

So the only point to be checked, or understood, is why B has the form $B = C \rtimes \Gamma$ $\quad C = \mathbb{C}\langle h_s, s \in \Gamma\rangle / (\sum h_s = 1)$

You need to check that $\langle E', E\rangle = A$, in order to get the desired M.eq. of B, and A.

$$E \xhookrightarrow{\alpha} \mathbb{C}\Gamma \otimes \tilde{A} \xrightarrow{\beta} E \xrightarrow{\alpha} \mathbb{C}\Gamma \otimes \tilde{A}$$

$$\xi \longmapsto \sum_s s \otimes \alpha_1 s^{-1}\xi \longmapsto \sum_s s \otimes \sum_t \underbrace{(\alpha_1 s^{-1}t\beta_1)}_{p_{s^{-1}t}} f_t$$

$$\sum_t t \otimes f_t \longmapsto \sum_t t\beta_1 f_t$$

---

you will have $\quad \alpha_1 s^{-1} \in E' = \mathrm{Hom}_{A^{op}}(E, \tilde{A})$

and $\quad s\beta_1 \in E$

You have to keep on repeating. $\quad \Gamma$ finite

$\mathbb{C}\Gamma \otimes \tilde{A}$ fin. free A^{op}-module

$p(t \otimes a) = \sum_s st \otimes p_s a$ proj. op. $\quad E = p(\mathbb{C}\Gamma \times \tilde{A})$ fin proj A^{op}-module

$$\begin{tikzcd} \sum_t t^{-1}\otimes f_t \arrow[r, "P"] & \sum_{s,t} st^{-1}\otimes p_s f_t \arrow[d] \\ \sum t\otimes f_t \arrow[u] & \sum_{s,t} \underbrace{ts^{-1}}_{u}\otimes p_s f_t \end{tikzcd}$$

$$\sum_u u\otimes \sum_t p_{u^{-1}t} f_t$$

$$\|$$

$$\sum_{s,t} \underbrace{ts^{-1}}_{u}\otimes p_s f_t = \sum_u u\otimes \sum_{ts^{-1}=u} p_s f_t$$

$$\sum t\otimes f_t \longmapsto \sum_s s\otimes \sum_t p_{s^{-1}t} f_t$$

~~Idea Use A is Morita equiv. to M~~

The idea is to break up the M. eq. by going from A to M_nA and from M_nA to $p(M_nA)p$. It might be easier to show that $p(M_nA)p = B$. In fact M_n might arise as $\Gamma \rtimes \hat{\Gamma}$

Maybe you can show A firm by defining and inverse for $A\otimes_A A \longrightarrow A$, sending p_s to $\sum_{t\in\Gamma} p_{st^{-1}}\otimes p_t$. Do for $\mathbb{Z}/2$.

$$p_0 \longmapsto p_0\otimes p_0 + p_1\otimes p_1 \in A\otimes_A A$$

$$p_1 \longmapsto p_0\otimes p_1 + p_1\otimes p_0$$

$$(p_0\otimes p_0 + p_1\otimes p_1)^2 = p_0^2\otimes p_0^2 + p_0p_1\otimes p_0p_1 + p_1p_0\otimes p_1p_0 + p_1^2\otimes p_1^2$$

?

$$(p_0\otimes p_1 + p_1\otimes p_0)^2 = p_0^2\otimes p_1^2 + p_0p_1\otimes p_1p_0 + p_1p_0\otimes p_0p_1 + p_1^2\otimes p_0^2$$

~~so what acts [illegible]~~

Γ finite. Consider $\mathbb{C}\Gamma \otimes \tilde{A}$ free A^{op} module

dual A-module $\quad \mathrm{Hom}_{\tilde{A}}(\mathbb{C}\Gamma\otimes\tilde{A}, \tilde{A}) = \tilde{A}\otimes C(\Gamma)$

$\Gamma = \mathbb{Z}/2.$

$$\begin{array}{c|cc} \tilde{A} & \tilde{A} & \tilde{A} \\ \hline \tilde{A}^* & & \\ & M_2(\tilde{A}) & \\ \tilde{A} & & \end{array}$$

782
102
884
54
938

107

$\mathbb{C}\Gamma\otimes\tilde{A}$ fin. free A^{op} module

$p(\mathbb{C}\Gamma\otimes\tilde{A})$ — proj A^{op} module

8030
.05
401.50

What can you do? $\quad E' = \mathbb{C}\Gamma\otimes\tilde{A}$

Focus on the main difficulty. Given A which comes with p a splitting of $\begin{matrix}\tilde{A}^* \\ \tilde{A}\end{matrix}$

$$E = \begin{pmatrix} p_0 & p_1 \\ p_1 & p_0 \end{pmatrix}\begin{pmatrix}\tilde{A} \\ \oplus \\ \tilde{A}\end{pmatrix} = p\begin{pmatrix}\tilde{A} \\ \oplus \\ \tilde{A}\end{pmatrix} \qquad \tilde{A} \qquad E'$$

$$E' = \mathrm{Hom}_{A^{op}}\left(p\begin{pmatrix}\tilde{A} \\ \oplus \\ \tilde{A}\end{pmatrix}, \tilde{A}\right) = (\tilde{A}\oplus\tilde{A})\,p \qquad E$$

~~[illegible]~~ why is $E^* \otimes_A E' \simeq B$? Put another way, why is $B = \mathrm{Hom}_{A^{op}}(E, E)$?

$$\begin{pmatrix} p_0 & p_1 \\ p_1 & p_0 \end{pmatrix}\begin{pmatrix} \tilde{A} & \tilde{A} \\ \tilde{A} & \tilde{A} \end{pmatrix}\begin{pmatrix} p_0 & p_1 \\ p_1 & p_0 \end{pmatrix}$$

$u = ts^{-1}$
$us = t$
$s = u^{-1}t$

Everything should be contained in what you already know. ~~Start with an A-module V~~ Given an A-module V, you get a Γ-module $\mathbb{C}\Gamma \otimes V = \{\sum_t t \otimes f_t \mid f_t \in V\}$. and a projector on this Γ-module $p \sum_t t \otimes f_t = \sum_{s,t} ts^{-1} \otimes p_s f_t = \sum_u u \otimes \sum_t p_{u^{-1}t} f_t$

so you get the image $p(\mathbb{C}\Gamma \otimes V)$, denoted $E(V)$

better maybe $E(V) = E(\tilde{A}) \otimes_A V$. ~~What do you know Start at~~

Here's an approach. $\mathbb{C}\Gamma \otimes \check{A}$ is a finite free A^{op}-module so $E = p(\mathbb{C}\Gamma \otimes \tilde{A})$ is a finite projective A^{op}-module

$$E \xhookrightarrow{\alpha} \mathbb{C}\Gamma \otimes \tilde{A} \xrightarrow{\beta} E$$

$$\xi \longmapsto \sum s \otimes \alpha_1 s^{-1}\xi \longmapsto \sum \beta_1 \alpha_1 s^{-1}\xi$$

$$\sum t \otimes f_t \longmapsto \sum t \beta_1 f_t$$

<u>Propose</u>: to forget the group, but focus on constructing a finite projective firm A-module. ~~The point is that you have a finite a finite proj~~ What's important is $E \longrightarrow R^n \longrightarrow E$

$$\mathrm{Hom}_{R^{op}}(E,E) \longleftarrow E \otimes_R \mathrm{Hom}_{R^{op}}(E,R)$$

$$\mathrm{id} \qquad \sum_i \xi_i \otimes \lambda_i$$

$$E \xrightarrow{(\lambda_i \cdot)} R^n \xrightarrow{(\xi_i \cdot)} E$$

$$\xi \longmapsto \langle \lambda_i, \xi \rangle \longmapsto \sum_i \xi_i \langle \lambda_i, \xi \rangle = \xi.$$

You have a finite proj R^{op}-module E, whence

$n = 1$.
$$\begin{tikzcd} E \arrow[r, hook, "\lambda"] \arrow[rr, bend right] & R \arrow[r, "\xi_0"] & E \end{tikzcd}$$

$$\xi \longmapsto \xi_0 \langle \lambda, \xi \rangle = \xi.$$

$$E \otimes_R \mathrm{Hom}_R(E,R) \longrightarrow \mathrm{Hom}_R(E,E)$$
$$\xi_0 \otimes \lambda \longmapsto \mathrm{id}$$

$$R \xrightarrow{\xi_0} E \xrightarrow{\lambda} R$$
$$\lambda \longmapsto \xi_0 \lambda \longmapsto \langle \lambda, \xi_0 \lambda \rangle = \underbrace{\langle \lambda, \xi_0 \rangle}_{e} \lambda$$

I think the main problem is how to deal with the rings being universal. Given an A module V you construct a B module $p(\mathbb{C}\Gamma \otimes V)$. Given a B module E you construct an A-module in different ways which are nil isomorphic. Go over this.

An A-module is a V with two idemp. ops $e, \bar{e}$. rather $p_0 + p_1 = e$, $p_0 - p_1 = \bar{e}$. Define p on $\mathbb{C}[\varepsilon] \otimes V$ to be $(p_0 + \varepsilon p_1)(v_0 + \varepsilon v_1) = (p_0 v_0 + p_1 v_1) + \varepsilon(p_0 v_1 + p_1 v_0)$

$$\begin{tikzcd} \mathbb{C}[\varepsilon] \otimes V \arrow[rr, bend left, "p"] \arrow[dr, "\beta"'] & & \mathbb{C}[\varepsilon] \otimes V \\ & X(V) \arrow[ur, "\alpha"'] & \end{tikzcd} \qquad X(V) = p(\mathbb{C}[\varepsilon] \otimes V)$$

a form B module E is a $\mathbb{Z}/2$ module with an operator h_0 such that $h_0 + \varepsilon h_0 \varepsilon = 1$ $\quad (h_0 - \frac{1}{2}) + \varepsilon (h_0 - \frac{1}{2}) = 0$

Choose fact. $h_0 = \beta_0 \alpha_0$ two obvious choices $1 \cdot h_0$ $\quad h_0 \cdot 1$

$$E \xrightarrow{\alpha_0} V \xrightarrow{\beta_0} E \xrightarrow{\alpha_0} V$$

L211

0937 Track 3 am Brugge 10:56

Review the formulas. $\Phi = \Gamma$ finite,

C = Cuntz' E_Γ $\Big\{$ unital alg gens $h_s, s \in \Gamma$
rels $\sum h_s = 1$

$B = C \rtimes \Gamma$, unital alg, modules M are v.s. with Γ action and an op $h_1 \ni \sum_{s\in\Gamma} s h_1 s^{-1} = 1$

A alg gens p_s rel $p_s = \sum_t p_{st^{-1}} p_t$

Let's try to put things into order, to make progress on the remaining steps. Go over what you know. Start with a firm B-module E whence you have Γ, h_1 on E. Point: factor: $h_1 = \beta_1 \alpha_1$, two obvious choices $h_1 = 1 h_1$ or $h_1 = h_1 1$. Then put $p_s = \alpha_1 s \beta_1$ and E becomes an A-module. This amounts to a restriction of scalars for a homom. $A \longrightarrow B$, ~~so you have either~~ provided α_1, β_1 are elements of B. So you get from the above choices

$$p_s \longmapsto h_1 s \qquad \text{or} \qquad p_s \longmapsto s h_1$$

Fri - Sun Antwerp Eden Hotel	4th Ghent – Brugges
Mon + Tues Bruges	5th Thurs
Wed Ostend	
Thurs Ferry home 8:30 am	

Center of Bruges parallel to Zuid Zand Str.

$p_s = h_1 s \qquad \sum_t h_1 s t^{-1} h_1 t = h_1 s \qquad (\beta\alpha)(\xi) = \sum_s t \beta_1 \alpha_1 t^{-1} \xi = \xi$

Go over the formulas again $\qquad h_1 = \beta_1 \alpha_1$

$$\begin{array}{ccccc} E & \xrightarrow{\alpha} & \mathbb{C}\Gamma \otimes V & \xrightarrow{\beta} & E \\ & & \delta_1 \downarrow \uparrow \iota_1 & & \\ & & V & & \end{array} \qquad \begin{aligned} (\alpha\xi)_s &= \alpha_1 s^{-1} \xi \\ \beta f &= \sum_{t\in\Gamma} t \beta_1 f_t \end{aligned}$$

Given A-module V, ie. family $p_s \in \mathcal{L}(V)$ $p_s = \sum_t p_{st^{-1}} p_t$. Then you get p on $\mathbb{C}\Gamma \otimes V$

first version $\qquad \mathbb{C}\Gamma \otimes V \xrightarrow{\sum s \otimes p_s} \mathbb{C}\Gamma \otimes V$

$$\sum_t t \otimes f_t \longmapsto \sum_{s,t} st \otimes p_s f_t$$

Commutes with right mult of Γ on $\mathbb{C}\Gamma$

$$\sum_t t^{-1} \otimes f_t \longmapsto \sum_{s,t} st^{-1} \otimes p_s f_t$$

$u = ts^{-1} \qquad s = u^{-1}t$

second version

$$\sum_t t \otimes f_t \qquad \sum_{s,t} ts^{-1} \otimes p_s f_t = \sum_s s \otimes \sum_t p_{s^{-1}t} f_t$$

get projection operator $\qquad (f_t) \longmapsto (pf)_s = \sum_t p_{s^{-1}t} f_t$

which commutes with left Γ multiplication. Note that the kernel $p_{s^{-1}t}$ is left invariant.

So far you have reviewed the formulas which should yield the desired Morita equivalence, but you are still far from understanding the bimodules. How to get a good grasp of the situation? Look at the two homoms $A \longrightarrow B$ inducing the Morita equivalence

$$p_s \longmapsto \begin{cases} h_1 s \\ s h_1 \end{cases}$$

$$\begin{tikzcd}
\mathrm{Ker}(h_1) \arrow[d, "0"] & E \arrow[r, "h_1"] \arrow[d, dashed, "sh_1"'] & E \arrow[d, "h_1 s"] & E/h_1E \arrow[d, dashed, "0"] \\
\mathrm{Ker}(h_1) & E \arrow[r, "h_1"] & E & E/h_1E
\end{tikzcd}$$

first A-mod $\qquad\qquad$ 2nd A-mod

Review the situation carefully. B is a unital ring defined by a universal mapping property, A similar but non unital. Work with left modules. E a B-module get A module via $\rho_s = h_1 s$ or $s h_1$, V an A-module, get B module $p(\mathbb{C}\Gamma \otimes V)$, an exact functor of V killing nil modules.

Let's review how a M eq gives rise to and is given by a dual pair.

$$\mathcal{M}(A) \underset{Q\otimes_B -}{\overset{P\otimes_A -}{\rightleftarrows}} \mathcal{M}(B) \qquad P = p(\mathbb{C}\Gamma\otimes \tilde{A})$$

You have the unital ring $\mathbb{C}\Gamma\otimes\tilde{A}$ and idemp. p.
Can form $E = p(\mathbb{C}\Gamma\otimes\tilde{A})$, $F = (\mathbb{C}\Gamma\otimes\tilde{A})p$; these are resp right and left A modules, in fact resp B, A and A, B bimodules. Since Γ finite these are fin. proj modules over A which should be dual via a pairing $F\times E \longrightarrow \tilde{A}$.
In this case you know that $E\otimes_A F$ is the ring of endos of E_A

You need to review finite proj modules formalism, maybe even the Vaughan Jones tower, you want better control over the correspondence between a partition of 1 and a projection op

Let E be a R^{op}-module (unital)

$$M\otimes_R \mathrm{Hom}_{R^{op}}(E,R) \longrightarrow \mathrm{Hom}_{R^{op}}(E,M)$$

$$\sum \xi_i \otimes \lambda_i \longmapsto 1$$

Let E be a unital R^{op}-module, canon map

$$E \otimes_R \mathrm{Hom}_{R^{op}}(E, R) \longrightarrow \mathrm{Hom}_{R^{op}}(E, E)$$

$$\sum_s \xi_s \otimes \lambda_s \longmapsto \sum_i \xi_s \lambda_s(-) \underset{\substack{\uparrow \\ \text{assume}}}{=} id_E$$

$$E \xrightarrow{(\lambda_s)} \underset{\substack{\| \\ \mathbb{C}\Gamma \otimes R}}{R^\Gamma} \xrightarrow{(\xi_s)} E.$$ What do you

---

Go over the situation, ~~What is the~~ use new ideas, that $\mathbb{C}\Gamma \otimes \tilde{A}$ is a free fin gen ~~[illegible]~~ $\tilde{A}^{op}$-module hence $E = p(\mathbb{C}\Gamma \otimes \tilde{A})$ is a fin.gen proj $\tilde{A}^{op}$ module $= p(\mathbb{C}\Gamma \otimes A) = p(\mathbb{C}\Gamma \otimes \tilde{A})A$ $\therefore$ firm. ~~similar~~

Also there's a dual fin gen. proj ~~firm~~ A^{op} ~~module~~ F

Automatically $E \otimes_A F \cong \mathrm{End}_{A^{op}}(E) = \mathrm{End}_A(F)^{op}$?

You should be able to construct the pairing

$F \otimes E \longrightarrow A$. $\qquad E = p(\mathbb{C}\Gamma \otimes \tilde{A})$, $F = (\mathbb{C}\Gamma \otimes \tilde{A})p$

want $F \otimes E = (\mathbb{C}\Gamma \otimes \tilde{A})p \otimes p(\mathbb{C}\Gamma \otimes \tilde{A})$

Important to remember that you have explicit direct imbedding

$$E \longrightarrow \mathbb{C}\Gamma \otimes \tilde{A} \longrightarrow E$$

$$F \longleftarrow \tilde{A} \otimes (\mathbb{C}\Gamma)^* \longleftarrow F$$

There is some formalism here to be worked out which doesn't involve Γ as a group, only as set, as index set for a grading

Situation: You have a f g free A^{op}-module $\mathbb{C}\Gamma\otimes\tilde{A}$ with projection operator p, whence a fin. gen proj A^{op} module $E = p(\mathbb{C}\Gamma\otimes\tilde{A})$, a dual module $F = \mathrm{Hom}_{A^{op}}(E,\tilde{A})$, and ring $B = E\otimes_A F \xrightarrow{\sim} \mathrm{Hom}_{A^{op}}(E,E)$.

All this you know to be true.

Better is to start with the intrinsic stuff, namely: $E\in \mathcal{P}(A^{op})$, $F = E^\vee$, $E\otimes_A \check{E} \xrightarrow{\sim} \mathrm{Hom}_{A^{op}}(E,E) = B$ and to identify a split embedding $E \to \mathbb{C}\Gamma \to E$ with maps $\Gamma \xrightarrow{(\xi_s)} E$, $\Gamma \xrightarrow{(\lambda_s)} E^\vee$ sat $\sum \xi_s\lambda_s = id_E$.

There ~~should~~ might be some $\underbrace{\text{category}}_{\text{interesting}}$ of partitions of unity.

In the case Γ = finite group you have an equivariant partition of unity

---

Review. In the case Γ finite you have the proper picture of the desired Morita equivalence, namely, $\begin{pmatrix} A & F \\ E & B\end{pmatrix}$ where B is unital. ~~In this~~

~~The~~ Problem: If you start with B, F_B, ${}_BE$ you seem to have problems with showing your A is $F\otimes_B E$

So it seems better to start with A. ~~The point~~

Over A you have a canonical firm dual pair E_A, ${}_AF$ where $E\in\mathcal{P}(A^{op})$, $F = E^\vee \in \mathcal{P}(A)$ and $\check{E}E \simeq A$.

Actually you still seem to have problems unless you can show A is firm

So how do you get started? Philosophy of partition of unity needs to be worked out.

$$E\otimes_R \mathrm{Hom}_{R^{op}}(E,R) \longrightarrow \mathrm{Hom}_{R^{op}}(E,E)$$
$$\xi\otimes\lambda \longmapsto \left(\xi' \longmapsto \xi\langle\lambda,\xi'\rangle\right)$$

Assume $1_E \in \mathrm{Hom}_{R^{op}}(E, E)$ is nuclear

choose ξ_s, λ_s such that $\sum \xi_s \otimes \lambda_s \longmapsto 1_E$

i.e. $\xi = \sum_{s\in\Gamma} \xi_s \langle \lambda_s, \xi \rangle \quad \forall \xi \in E.$

$$\begin{tikzcd} E \arrow[r, "(\lambda_s)\cdot"] \arrow[rr, bend right, "1"'] & R^\Gamma \arrow[r, "(\xi_s)\cdot"] \arrow[rr, bend right, "p"'] & E \arrow[r, "(\lambda_s)\cdot"] & R^\Gamma \end{tikzcd}$$

$p =$ ~~right~~ left multiplication by $(\lambda_s \xi_t)_{s,t\in\Gamma}$

$p^2 =$ ———————— $\sum_u \lambda_s \xi_u \lambda_u \xi_t = \lambda_s \xi_t$

$\therefore \quad p^2 = p$

Point to understand: Why $\sum \xi_s \lambda_s = 1$ implies

$N \otimes_R E^\vee \xrightarrow{\sim} \mathrm{Hom}_{R^{op}}(E, N)$ for all N.

Surjectivity should be immediate. Given $\varphi : E \to N$ you have $\varphi = \varphi \sum \xi_s \lambda_s = \sum_s \varphi(\xi_s)\lambda_s$ comes from $\sum_s \varphi(\xi_s) \otimes \lambda_s \in N \otimes_R E^\vee$

$$\begin{tikzcd} N \otimes_R E^\vee \arrow[r] \arrow[d] & \mathrm{Hom}_{R^{op}}(E, N) \arrow[d] \arrow[dl] \\ N \otimes_R E^\vee \arrow[r] & \mathrm{Hom}_{R^{op}}(E, N) \end{tikzcd}$$

$n \otimes \lambda \longmapsto (\xi \mapsto n\lambda\xi)$

$\updownarrow$

$\sum_s n\lambda(\xi_s) \otimes \lambda_s$

$\parallel$

$n \otimes \lambda(\sum \xi_s \lambda_s)$

$$N \otimes_R E^\vee \longrightarrow \mathrm{Hom}_{R^{op}}(E, N) \longrightarrow N \otimes_R E^\vee$$

$$n \otimes \lambda \longmapsto (\xi \mapsto n\langle \lambda, \xi \rangle), \quad \varphi \longmapsto \sum_s \varphi(\xi_s) \otimes \lambda_s$$

$$n \otimes \lambda \longmapsto \sum_s n\langle \lambda, \xi_s \rangle \otimes \lambda_s = \sum_s n \otimes \langle \lambda, \xi_s \rangle \lambda_s$$

so you need to know why $\sum_s \langle \lambda, \xi_s \rangle \lambda_s = \lambda$?

i.e. $\underbrace{\sum_s \langle \lambda, \xi_s \rangle \langle \lambda_s, \xi \rangle}_{\langle \lambda, \sum_s \xi_s \langle \lambda_s, \xi \rangle \rangle} = \langle \lambda, \xi \rangle$?

Finally given $\varphi : E \to N$, you want to know

$\xi \longmapsto \sum_s \varphi(\xi_s) \langle \lambda_s, \xi \rangle \stackrel{?}{=} \varphi(\xi)$ clear by applying φ to $\sum \xi_s \langle \lambda_s, \xi \rangle = \xi$

A generators p_s, $s \in \Gamma$ relations $p_s = \sum_{t \in \Gamma} p_{st^{-1}} p_t$

Consider $A \otimes_A A$. Can you use the universal mapping property to construct a homomorphism $A \xrightarrow{\phi} A \otimes_A A$ such that $\phi\mu = \mathrm{id}$. It would seem that you want

$p_s \longmapsto \sum_{t \in \Gamma} p_{st^{-1}} \otimes p_t$. Thus ~~define~~ put $\phi(p_s) = \sum_{s = tu} p_t \otimes p_u$.

Are the relations satisfied?

$$\phi(p_s) = \sum_{s=tu} p_t \otimes p_u = \sum_{s = tu} \sum_{t = t_1 t_2} \sum_{u = u_1 u_2} p_{t_1} p_{t_2} \otimes p_{u_1} p_{u_2}$$

$\| ?$

$$\sum_{s=tu} \phi(p_t)\phi(p_u) = \sum_{s=tu} \sum_{t = t_1 t_2} \underbrace{(p_{t_1} \otimes p_{t_2})(p_{u_1} \otimes p_{u_2})}$$

$$p_{t_1} \otimes p_{t_2} p_{u_1} p_{u_2} = p_{t_1} p_{t_2} \otimes p_{u_1} p_{u_2}$$

Be careful, you seem to be able to show that $A = A_\Gamma$ is a firm ring, by constructing a homom. $\phi : A \longrightarrow A \otimes_A A$ such that $\mu\phi = 1$.

put $\phi(p_s) = \sum_{s=tu} p_t \otimes p_u$ so that $\mu\phi(p_s) = p_s \quad \forall s$

Need to check relations $\quad \phi(p_s) \overset{?}{=} \sum_{s=tu} \phi(p_t)\phi(p_u)$

$$\phi(p_t) \overset{\text{def}}{=} \sum_{t=t_1t_2} p_{t_1} \otimes p_{t_2} \qquad \phi(p_u) = \sum_{u=u_1u_2} p_{u_1} \otimes p_{u_2}$$

$$\phi(p_t)\phi(p_u) = \sum_{t=t_1t_2} \sum_{u=u_1u_2} \underbrace{p_{t_1} \otimes p_{t_2}p_{u_1}p_{u_2}}_{p_{t_1}p_{t_2} \otimes p_{u_1}p_{u_2}}$$

$$= p_t \otimes p_u \quad ?$$

$$\therefore \sum_{s=tu} \phi(p_t)\phi(p_u) = \sum_{s=tu} p_t \otimes p_u = \phi(p_s)$$

So we know now that A is a firm ring, which should complete the proof that $\begin{pmatrix} A & F \\ E & B \end{pmatrix}$ is a firm Morita context. Except you need to check $FE = A$.

$$E \xrightarrow[\text{col}]{(\lambda_s)\cdot} \underset{\text{column vectors}}{\mathbb{C}\Gamma \otimes A} \xrightarrow[\text{row}]{(\xi_t)\cdot} E \xrightarrow[\text{col}]{(\lambda_s)\cdot} \mathbb{C}\Gamma \otimes A$$

FE should be the ideal gen. by $\quad \alpha_1 s^{-1}t \beta_1 = p_{s^{-1}t}$, so it's clear.

What else do you want to understand?

~~[illegible]~~ supports $- A_\Phi$. More control over E as B module infinite case. A gens p_s, $s \in \Gamma$ rels $p_s = 0$ $s \notin \Phi$

$p_s = \sum_{s=tu} p_t p_u$, this sum finite as $\{(t,u) \mid t \in \Phi, u \in \Phi\} = \Phi \times \Phi$ is finite. Firmness should be proved similarly: Define $\phi: A \longrightarrow A \otimes_A A$ by putting $\phi(p_s) = \sum_{s=tu} p_t \otimes p_u$. Check rels.

$$\phi(p_t) \phi(p_u) = \sum_{t=t_1 t_2} (p_{t_1} \otimes p_{t_2}) \sum_{u=u_1 u_2} (p_{u_1} \otimes p_{u_2})$$

$$= \sum_{\substack{t=t_1 t_2 \\ u=u_1 u_2}} p_{t_1 t_2} \otimes p_{u_1} p_{u_2} = p_t \otimes p_u. \text{ So } \sum_{s=tu} \phi(p_t)\phi(p_u) =$$

$$\sum_{s=tu} p_t \otimes p_u = \phi(p_s)$$ showing the main relns holds.

What about Φ support? Assume $s \in \Phi$

Your notation is confusing. Put $\hat{p}_s = \sum_{s=tu} p_t \otimes p_u \in A \otimes_A A$

Then $$\hat{p}_t \hat{p}_u = \sum_{t=t_1 t_2} \sum_{u=u_1 u_2} \overbrace{(p_{t_1} \otimes p_{t_2})(p_{u_1} \otimes p_{u_2})}^{p_{t_1} p_{t_2} \otimes p_{u_1} p_{u_2}} = p_t \otimes p_u$$

$$\therefore \sum_{s=tu} \hat{p}_t \hat{p}_u = \sum_{s=tu} p_t \otimes p_u = \hat{p}_s$$

Prolongation for Φ?

Look at $\Gamma = \mathbb{Z}$ $\Phi = \{-1, 0, 1\}$. Question: Is there some superversion reminiscent of the way the irred reps of sl_2 arise from the fund. repn.

Look at philosophy. $\begin{pmatrix} A & F \\ E & B \end{pmatrix}$. B will not be unital, but will have local units, E and F will be firm over B equiv. $E = BE$, $F = FB$.

Again you should start with A, then E will be a summand of $\mathbb{C}\Gamma \otimes \tilde{A}$, a firm projective A^{op} module, F summand $\tilde{A} \otimes \mathbb{C}\Gamma$ ——— A module and there should be an obvious pairing

$F \otimes_{\mathbb{C}} E \twoheadrightarrow A$. ~~[crossed out]~~ : you have a firm dual pair over A, both modules flat, so the ring $B = E \otimes_A F$ will be both left and right flat, which checks with local units $\exists$ ring.

Point to check: $E \otimes_A F = C \cancel{X} \Gamma$.

You need A to be a firm ring in order that the canonical map $F \otimes_B E \longrightarrow A$ be an isom.

How does the argument go?

You are trying to understand the infinite case where a support Φ, or system of supports, is given. In this case B should have local units

---

Review: Γ finite set, A universal ring equipped with a projector p on the free $\tilde{A}^{op}$-module $\mathbb{C}\Gamma \otimes \tilde{A}$, equiv. p is a matrix $(p_{st})_{s,t\in\Gamma}$ satisfying $p_{su} = \sum_t p_{st} p_{tu}$. Put $E = p(\mathbb{C}\Gamma \otimes \tilde{A})$. There is a dual free $\tilde{A}$-module $\mathrm{Hom}_{A^{op}}(\mathbb{C}\Gamma \otimes \tilde{A}, \tilde{A}) = \tilde{A} \otimes (\mathbb{C}\Gamma)^\vee$ to $\mathbb{C}\Gamma \otimes \tilde{A}$, and a corresponding dual $E^\vee = \mathrm{Hom}_{A^{op}}(E, \tilde{A})$ which should be something like $(\tilde{A} \otimes \mathbb{C}\Gamma^\vee)\,p$ (maybe ${}^t p$?)

$$
\begin{tikzcd}
E \arrow[r, hook, "\alpha"] \arrow[dr, "\alpha_s"'] & \mathbb{C}\Gamma \otimes \tilde{A} \arrow[r, two heads, "\beta = p"] \arrow[d, shift right, "\gamma_s"'] & E \arrow[r, hook, "\alpha"] & \mathbb{C}\Gamma \otimes \tilde{A} \\
 & \tilde{A} \arrow[u, shift right, "\iota_t"'] \arrow[ur, "\beta_t"'] & &
\end{tikzcd}
$$

$$(\alpha\xi)_s = \alpha_s \xi$$

$$\beta\Big(\sum_t t \otimes f_t\Big) = \beta \sum \iota_t f_t = \sum \beta_t f_t$$

simpler might be

$\alpha\xi = \sum$

$$\begin{array}{ccc} \mathbb{C}\Gamma\otimes\tilde{A} & \xrightarrow{(a_{st})} & \mathbb{C}\Gamma\otimes\tilde{A} \\ j_t \downarrow \uparrow \iota_t & & j_s \downarrow \uparrow \iota_s \\ \tilde{A} & & \tilde{A} \end{array} \qquad a_{st} = j_s\, a\, \iota_t$$

Let $f \in \mathbb{C}\Gamma\otimes\tilde{A}$. Then $\boxed{f = \sum_t \iota_t f_t}$ $\quad f_t = j_t f$

$$af = \sum_{s,t} \iota_s j_s a \iota_t j_t = \boxed{\sum_{s,t} \iota_s a_{st} f_t = af}$$

basis ι_t

$(\mathbb{C}\Gamma)\otimes\tilde{A}$ $\qquad\qquad$ $\tilde{A}\otimes(\mathbb{C}\Gamma')$ basis j_t for functions on Γ.

row vectors $\quad j_t \downarrow \uparrow \iota_t \quad \tilde{A}$ $\qquad\qquad$ column vectors.

what's important is the pairing $\quad$ Kronecker δ

$$(\tilde{A}\otimes(\mathbb{C}\Gamma')) \otimes_{\mathbb{Z}} (\mathbb{C}\Gamma\otimes\tilde{A}) \longrightarrow \tilde{A}$$

$$\left\langle \sum_s g_s j_s \,,\, \sum_t \iota_t f_t \right\rangle = \sum g_s \overbrace{(j_s \iota_t)}^{\delta_{st}} f_t$$

What remains? $\qquad \langle j_s, \iota_t \rangle = j_s \iota_t = \begin{cases} 1 & s = t \\ 0 & s \neq t. \end{cases}$

$$\langle g, af \rangle = \sum_s g_s j_s \sum_t \overbrace{\iota_s a j_t}^{a_{st}} \iota_t f_t$$

$$= \sum \quad ?$$

There is something peculiar about this image

diagrams, some angle I am missing

$$g_s j_s a \iota_t j_t f$$

$$\langle g, af \rangle = \sum_s g_s j_s a f = \sum_{s,t} g_s a_{st} f_t$$

$$\langle ga, f \rangle = \sum_{s,t} g_s j_s a \iota_t j_t f$$ should be clear.

What is the goal?

You have a finite set Γ, A = universal ring equipped with an idempotent operator p on the free A^{op}-module $\mathbb{C}\Gamma \otimes \tilde{A}$. This module is characterized by A^{op}-module maps

$$\begin{tikzcd} \mathbb{C}\Gamma \otimes \tilde{A} \arrow[d, shift right, "j_s"'] \\ \tilde{A} \arrow[u, shift right, "\iota_s"'] \end{tikzcd}$$

Satisfying $j_s \iota_t = \delta_{st}$

$$\sum \iota_s j_s = 1$$

Think of $\mathbb{C}\Gamma \otimes \tilde{A}$ as the A^{op}-module of columns vectors over $\tilde{A}$, where Γ indexes the columns. Dual A-module $\tilde{A} \otimes (\mathbb{C}\Gamma)^\wedge$ with basis j_s, think of it as row vectors.

Put $P = \mathbb{C}\Gamma \otimes \tilde{A}$

column vector

$$\begin{tikzcd} P \arrow[d, shift right, "j_s"'] \\ \tilde{A} \arrow[u, shift right, "\iota_s"'] \end{tikzcd}$$

ι_s is the good basis for P

j_s is ~ dual basis for $P^\vee = \tilde{A} \otimes \mathbb{C}\Gamma^\wedge$

$$\phi \in \mathrm{Hom}_{A^{op}}(P, P) = P \otimes_A P^\vee$$

$$\| \qquad \text{elt of } \tilde{A}$$

$$\sum \iota_s (j_s \phi \iota_t) j_t$$

What were you doing? Γ finite set

$E = \mathbb{C}\Gamma \otimes \tilde{A}$ = free A^{op}-module, basis Γ

canon. maps $\tilde{A} \xrightarrow{\iota_s} E \xrightarrow{j_t} \tilde{A}$

satisfying $j_t \iota_s = \delta_{ts}$, $\sum_s \iota_s j_s = 1$.

$F = E^\vee = (\mathbb{C}\Gamma \otimes \tilde{A})^\vee = \tilde{A} \otimes \mathbb{C}\Gamma^*$ free A-module

with basis j_t, $t \in \Gamma$ canon maps $\tilde{A} \xrightarrow{j_t^*} E^\vee \xrightarrow{\iota_s^*} \tilde{A}$

$$\iota_s^* j_t^* = \delta_{st}, \quad \sum j_s^* \iota_s^* = 1$$

Let p be an idemp. op on E.

$$p = \sum_{s,t \in \Gamma} \iota_s \underbrace{j_s p \iota_t}_{= p_{st} \in \tilde{A}} j_t$$

$$p^* = \sum_{s,t} j_t^* \iota_t^*$$

$$E_0 = \mathbb{C}\Gamma \otimes \tilde{A} \underset{\iota_t}{\overset{j_s}{\rightleftarrows}} \tilde{A}$$

$$E_0^\vee = \tilde{A} \otimes (\mathbb{C}\Gamma)^* \underset{\iota_t^*}{\overset{j_s^*}{\leftrightarrows}} \tilde{A}$$

$j_s \iota_t = \delta_{st}$

$\sum \iota_s j_s = 1$

$\iota_t^* j_s^* = \delta_{st}$

$\sum j_s^* \iota_s^* = 1$

$p \in \mathrm{Hom}_{A^{op}}(E_0, E_0)$

$$p = \sum_{s,t} \iota_s \underbrace{(j_s p \iota_t)}_{p_{s,t} \in A} j_t$$

$E = pE_0$

elements ξ of E_0 are $\sum \iota_s \xi_s$ $\quad \xi_s = j_s \xi$

$$E_0 = \left\{ \sum_{s,t} i_s \, p_{s,t} \, \xi_t \;\middle|\; \xi_t \text{ arb.} \right\}.$$

$$E_0 = \left\{ \sum_{s} i_s \, g_s \;\middle|\; g_s = \sum_t p_{s,t} f_t \,,\ f_t \in \tilde{A} \right\}.$$

$$\therefore\ E = p(\overset{\text{col. vectors}}{\mathbb{C}^r} \otimes \tilde{A})$$

$$\begin{tikzcd}
E_0 & E_0 \arrow[l, "p\cdot"'] \\
\times & \times \\
E_0^\vee \arrow[r, "\cdot p"] \arrow[d] & E_0^\vee \arrow[d] \\
\tilde{A} & \tilde{A}
\end{tikzcd}$$

$$\langle \eta, p\xi \rangle = \sum_s \eta_s \, p_{s,t} \, \xi_t$$

$$= \langle \eta p, \xi \rangle$$

$$\xi = \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_n \end{pmatrix} \qquad \eta = (\eta_1 \cdots \eta_n)$$

A acts on right — A acts on left

Question: Do you expect a pairing between pE_0 and $E_0^\vee p$, specifically

$$\langle \eta, \xi \rangle = \sum_s \eta_s \xi_s$$

$$E_0^\vee p \times pE_0 \longrightarrow \tilde{A}$$

$$\langle \eta, p\xi \rangle = \langle \eta p, \xi \rangle$$

depends only on $(\eta p, \ p\xi) \in E_0^\vee p \times pE_0$

Let $\sigma \in E_0^\vee p$, $\tau \in pE_0$

choose $\xi_0 \in E_0$ s.t. $p\xi_0 = \tau$

—— $\eta_0 \in E_0^\vee$ —— $\eta_0 p = \sigma$

Consider

This pairing between the images $E_0^\vee p$ and pE_0 involves inserting p^{-1} ~~[illegible]~~ in some sense and then using the pairing between $E_0^\vee$ and E_0. When $p^2 = p$ the p^{-1} is unnecessary.

So ~~the~~ perhaps there is a canonical pairing

$$Bh_{*} \times h_{*}B \longrightarrow Bh_{*}B$$

$$\langle b_1 h, h b_2 \rangle = (b_1 h) b_2$$

~~[illegible]~~ Go back to Γ finite, start with $B = C \rtimes \Gamma$ $\quad C$: unital gen $h_s, s \in \Gamma$, rel $\sum h_s = 1$

B is unital, get dual pair $(Bh_1, h_1 B)$ with above pairing $\langle bh_1, h_1 b' \rangle = bh_1 b'$. Ideal of lin. comb. of inner products is Bh_1B, which is B. So you have a firm dual pair with B unital, get a Meq

$$\begin{pmatrix} A & h_1 B \\ Bh_1 & B \end{pmatrix}$$

$A = h_1 B \otimes_B Bh_1$ is a firm ring by general theory $Bh_1 \in \mathcal{P}(A^{op})$ $\quad h_1 B \in \mathcal{P}(A)$ are dual f. proj. modules over A.

Note that the mult. map $A = h_1 B \otimes_B Bh_1 \longrightarrow h_1 B h_1$ is not a ring homom, because in the pairing

$Bh_1 \otimes_2 h_1 B \longrightarrow B$ you "divide" by h_1

mult in A Use h for h_1

$$A = hB \otimes_B Bh \longrightarrow hBh$$

$$hb_1 \otimes b_2 h \longmapsto hb_1 b_2 h$$

$$(hb_1 \otimes b_2 h) \overset{*}{\wedge} (hb_3 \otimes b_4 h) \qquad (hb_1 b_2 h) * (hb_3 b_4 h)$$

$$\parallel$$

$$hb_1 \otimes b_2 h b_3 b_4 h \qquad hb_1 b_2 h\, b_3 b_4 h$$

Thus hBh is a ring with product

$$(hb_1 h) \overset{*}{\wedge} (hb_2 h) = hb_1 h b_2 h$$

~~and the A maps onto~~ $\cancel{hBh}$

and then there is a surjective homom.

$$A = hB \otimes_B Bh \twoheadrightarrow hBh$$

with respect to $*$ product

Discuss the situation. I think the above concerns a unital B with $\Gamma \to B^\times$ and h such that $\sum_{s \in \Gamma} s h s^{-1} = 1$. ~~Better~~

Better: You have used only B unital, $h \in B$ $BhB = B$ to get $\begin{pmatrix} B & Bh \\ hB & A = hB \otimes_B Bh \end{pmatrix}$

Back to M.eq. You had this observation that $\langle bh, hb' \rangle = bhb'$ is a well defined pairing between Bh and hB yielding a Morita equiv. between BhB and $hB \otimes_B Bh$. But

---

Let's work as the details using our initial Morita equivalence. $B = C \rtimes \Gamma$ C unital gln h_s $\sum h_s = 1$.

A firm B-module is a Γ-module with op. h satisfying $\sum shs^{-1} = 1$. An A-module is a v.s. V equipped with $p = p^2$ on $\mathbb{C}\Gamma \otimes V$. Functor $E(V) = p(\mathbb{C}\Gamma \otimes V)$

What do you hope for?

Maybe the grading will be important

Go back to a firm B-module E, i.e. a Γ-module equipped with $h = h_1$ such that $\sum_{s \in \Gamma} s h_1 s^{-1} = 1$. Associate to E the image $h_1 E$ which should naturally be an A-module. Why? You have $E \xrightarrow{\alpha_1 = h_1} h_1 E \xrightarrow{\beta_1 = in} E$, and Γ action on E, hence

$$E \xrightarrow{\alpha} \mathbb{C}\Gamma \otimes h_1 E \xrightarrow{\beta} E$$

$$\xi \longmapsto \sum s \otimes \alpha_1 s^{-1} \xi \longmapsto \sum s \beta_1 \alpha_1 s^{-1} \xi = \xi$$

$$\sum s \otimes f_s \stackrel{\beta}{\longmapsto} \sum_{s \in \Gamma} s \beta_1 f_s$$

So what's important is the maps $E \xrightarrow{\alpha_1 = h_1} h_1 E \xrightarrow{\beta_1 = in} E$ then $p_s = \alpha_1 s \beta_1$ on $h_1 E$.

$$\sum_t p_{st^{-1}} p_t = \sum_t \alpha_1 s t^{-1} \beta_1 \alpha_1 t \beta_1 = \alpha_1 s \beta_1 = p_s$$

Can you prove that $h_1 B \otimes_B E \longrightarrow h_1 E$ is an A-mil isom.

$$\begin{tikzcd} B \otimes_B E \arrow[r, "\sim"] \arrow[d, "h_1"] & E \arrow[d, "h_1"] \\ h_1 B \otimes_B E \arrow[r] \arrow[d] & h_1 E \arrow[d, hook] \\ B \otimes_B E \arrow[r, "\sim"] & E \end{tikzcd}$$

$p_s = h_1 s$

Assume $\sum h_1 b_i \otimes \xi_i \longmapsto \sum h_1 b_i \xi_i = 0$

$$\Big\downarrow {\scriptstyle \alpha_1 s \beta_1 \otimes 1}$$

$$\sum_i h_1 s h_1 b_i \otimes \xi_i$$

Clearer. $h_1 B \otimes_B E \longrightarrow h_1 E$ is onto Let $h_1 \xi \in h_1 E$, as $B \otimes_B E \xrightarrow{\sim} E$, you have $\xi = \sum b_i \xi_i$ so $h_1 \xi = \sum_i h_1 b_i \xi_i$ which comes for $\sum_i h_1 b_i \otimes \xi_i \in h_1 B \otimes_B E$. If $\sum h_1 b_i \otimes \xi_i \longmapsto 0$ in $h_1 E$, i.e. $\sum h_1 b_i \xi_i = 0$, then

$\alpha_1 s \beta_1$: $h_1 s \sum_i h_1 b_i \otimes \xi_i = \sum_i h_1 s h_1 b_i \otimes \xi$

now the next step is to go from an A-module V to $E(V) = p(\mathbb{C}\Gamma \otimes V)$.
At some point you have to relate Bh_1 to $E(A)$

I want to finish Γ finite today.

$B = C \rtimes \Gamma$ $\qquad$ C unital alg gens. $h_s,\ s \in \Gamma$ $\qquad$ $t h_s t^{-1} = h_{ts}$

rel $\sum_{s\in\Gamma} h_s = 1$

C, B are unital.

A gens $p_s,\ s \in \Gamma$ $\qquad$ $p_s = \sum_{s=tu} p_t p_u$

Explicit functors.

$$\mathrm{Mod}(B) \rightleftarrows \mathcal{M}(A)$$
$$N \qquad\qquad M$$

$$N \longmapsto h N \qquad \text{with } p_s(h\eta) = h_s h\eta$$

$$\sum_{s=tu} p_t p_u(h\eta) = \sum_{\substack{\cancel{s=tu}\\ t}} h_t h_{t^{-1}s} h\eta = h_s h\eta = p_s(h\eta)$$

Also you ~~know~~ have an A-nil isom., surj.

$$hB \otimes_B N \longrightarrow hN$$

Ass $\qquad \sum_i h b_i \otimes \eta_i \longmapsto \sum h b_i \eta_i = 0.$

then $\qquad \sum_i p_s(hb_i) \otimes \eta_i = \sum_i h_s h b_i \otimes \eta_i = \sum h_s \otimes \underbrace{h b_i \eta_i}_{0}$

Anyway you have ~~[illegible]~~ functors between modules, an explicit Morita equivalence,

Given a (firm) B-module W, you send it to the A-module $h_1 W$, or to $h_1 B \otimes_B W$, and given an A-module V, you send it to $p(\Gamma \times V)$ with suitable B module structure.

Maybe you want an intermediate category of ${}_B W$, ${}_A V$, $W \underset{\beta}{\overset{\alpha}{\rightleftarrows}} V$ such that $\beta\alpha = h$ on W $\quad \alpha s \beta = p_s$ on V

$$D = \begin{pmatrix} A & hB \\ Bh & B \end{pmatrix}$$

You probably want $\begin{pmatrix} A & \alpha B \\ B\beta & B \end{pmatrix}$

$$D = \begin{pmatrix} A & F \\ E & B \end{pmatrix} = \begin{pmatrix} A \\ E \end{pmatrix} \otimes_A \begin{pmatrix} A & F \end{pmatrix}$$

Then $\begin{pmatrix} A \\ E \end{pmatrix} \otimes_A V$ is a typical D-mod.

this should be nil equiv. to $\begin{pmatrix} V \\ E \otimes_A V \end{pmatrix}$

---

what about noncentral GNS.

Review. Last idea was to introduce modules for the Morita context.

$$\begin{pmatrix} A & Q \\ P & B \end{pmatrix} \quad \text{i.e.} \quad \begin{tikzcd} {}_B W \arrow[r, bend left, "\alpha_1"] & {}_A V \arrow[l, bend left, "\beta_1"] \end{tikzcd}$$

If you can concentrate everything should work out. B is the unital alg gen by Γ and $h = h_1$ satisfying $\sum shs^{-1} = 1$. A firm B module W is a Γ-module ($s \in \Gamma$) with such an operator h.

(It seems that there is a GNS_α situation where modules are $\begin{tikzcd} {}_B W \arrow[r, bend left, "\alpha"] & {}_A V \arrow[l, bend left, "\beta"] \end{tikzcd}$

$\beta\alpha = h$ (not 1))

~~Now all you have to do is~~

So introduce the category of $\begin{tikzcd} {}_B W \arrow[r, bend left, "j"] & {}_A V \arrow[l, bend left, "i"] \end{tikzcd}$

satisfying $ij = h$, $jsi = p_s$.

Start again with the cat of $\begin{tikzcd} {}_B W \arrow[r, bend left, "j"] & {}_A V \arrow[l, bend left, "i"] \end{tikzcd}$

$ij = h$, $jsi = p_s$

recall A alg w gens p_s, $s \in \Gamma$ rels

$$p_s = \sum_{s=tu} p_t p_u$$

Suppose given $_B W$, look at the possible extensions (V, i, j). Answer: Any vector space with op. $W \underset{i}{\overset{j}{\rightleftarrows}} V$ sat $ij = h$. Why? because the A-mod structure is determined by $jsi = \rho_s$.

$$
\begin{tikzcd}
W \arrow[rr, "j"] \arrow[dr, two heads] & & V \arrow[rr, "i"] \arrow[dr, two heads] & & W \\
& jW \arrow[ur, hook] \arrow[dr, two heads] & & iW \arrow[ur, hook] & \\
& & hW \arrow[ur, hook] & &
\end{tikzcd}
$$

There is a smallest choice for V namely hW

$i =$ inclusion $hW \hookrightarrow W$

$j = h_\cdot : W \twoheadrightarrow hW$.

$$\rho_s(hw) = jsi(hw) = hshw$$

$s = tu$

$$\rho_t \rho_u (hw) = \rho_t huhw = hthuhw$$

$= u$

$t^{-1}s$

$$\sum_t htht^{-1}shw = hshw = \rho_s(hw)$$

Next given $_A V$ you ~~ask~~ want to understand all $_B W$ equipped with

$$W \xrightarrow{j} V \xrightarrow{i} W$$

$$W \longrightarrow \mathbb{C}\Gamma \otimes V$$

${}_BW \xrightarrow{j} V$ equiv. $W \longrightarrow \mathrm{Hom}(B,V)$

$V \xrightarrow{i} W$ $\sim$ $B \otimes V \longrightarrow W$?

In unital GNS situation you have

$ji = 1$

$${}_AV \underset{j}{\overset{i}{\rightleftarrows}} {}_BW \qquad ji = 1$$

$\rho : B \longrightarrow A$ $\qquad \rho(b)v = jbiv$

It seems as if you want W to be primarily a Γ-module

Basic categories: (1) Γ module ${}_\Gamma W$ equipped with operator h such that $\sum_{s \in \Gamma} s h s^{-1} = 1$

Enlarge to (2) ${}_\Gamma W \underset{i}{\overset{j}{\rightleftarrows}} V$ such that

$$\sum_{s \in \Gamma} s i j s^{-1} = 1_W$$

(adjoining a factorization $h = ij$ of h).

(3) ${}_AV$ i.e. $p_s \in \mathcal{L}(V)$ $s \in \Gamma$

etc.

① ${}_\Gamma W$ equipped with of h sat $\sum shs^{-1} = 1$

obvs

② ${}_\Gamma W \overset{i}{\underset{j}{\rightleftarrows}} V \qquad ij = h \qquad \sum_s s i j s^{-1} = 1$

obvs

③ ${}_A V$ i.e. $p_s \in \mathcal{L}V, s \in \Gamma \qquad p_s = \sum_t p_t p_{t^{-1}s}$

$p_s = jsi$, check $\sum_t j(tijt^{-1})si = jsi = p_s$

You should understand these obvious functors as restriction wrt certain homos.

$$B \hookrightarrow \begin{pmatrix} B & Bi \\ jB & A \end{pmatrix} \hookleftarrow A$$

You expect to get an isom.

$$Bi \otimes_A jB \xrightarrow{\sim} B$$

Actually you have an explicit firm dual pair over B namely ${}_B Bi$, jB_B) with $\langle bi, jb' \rangle = bhb'$

and you have the other pairing

$$jb' \otimes_B bi \longmapsto jb'bi$$

What is jbi? e.g. $jhi = jijij = h^2$

b confusing

situation at hand

$\mathbb{C}\Gamma \oplus \mathbb{C}$.

You have a category of (some kind of modules)

$({}_\Gamma W, V, i, j)$ O.K.

$$\left[\begin{array}{c|c} B & \cdots si \cdots \\ \hline \vdots & \\ jt^{-1} & A \\ \vdots & \end{array}\right]$$

U

$$(\cdots si \cdots)\begin{pmatrix} \vdots \\ js^{-1} \\ \vdots \end{pmatrix} = \sum s i j s^{-1} = 1$$

M

$$\begin{pmatrix} \vdots \\ js^{-1} \\ \vdots \end{pmatrix}(\cdots ti \cdots) = \left(p_{s^{-1}t}\right)$$

---

to describe modules you need a kind of quiver

$${}_\Gamma W \underset{i}{\overset{j}{\rightleftarrows}} W$$

Maybe you should work on writing stuff out as simply as possible. You have this abelian category of $({}_\Gamma W, V, i, j)$ which is a "representation" (?) of an "enlarged version" of the group Γ

The Morita context $\begin{pmatrix} B & Bi \\ jB & A \end{pmatrix}$ should be the enveloping alg of this

You have a category of modules namely ${}_\Gamma W \underset{i}{\overset{j}{\rightleftarrows}} V$ such that $\sum_{s \in \Gamma} s i j s^{-1} = 1_W$

Maybe you want to choose a small projective generator. This should yield the unital ring $\begin{pmatrix} B & Bi \\ jB & \tilde{A} \end{pmatrix}$.

Can you prove this?

OKAY begin with the modules

Basically you have

It looks like you want to consider again the relation $ij = e$, $e^2 = e$. This is the case $\Gamma = \{1\}$.

Claim that the category of $W \underset{i}{\overset{j}{\rightleftarrows}} V$ such that $ij = 1_W$ is the category of modules over the unital ring with generators i, j relation ~~$iji = i$ $jij = j$~~ ?

$(ij)^2 = ij$

On $W \oplus V$ you have operators e, $1-e$, i, j satisfying $je = j = (1-e)j$

modules are $W \underset{i}{\overset{j}{\rightleftarrows}} V$ $\quad ij = 1_W$

Just look at $\Gamma = 1$. $\quad B = \mathbb{C}$

A gen by $p_\phi = j \circ i$ rel.

$$p_1 = \sum_{1 = st} p_s p_t \quad \text{i.e.} \quad p_1^2 = p_1$$

$$\begin{pmatrix} \mathbb{C} & \mathbb{C}i \\ j\mathbb{C} & \mathbb{C}e \end{pmatrix} \qquad \begin{array}{l} ij = 1. \\ e = ji = e^2 \end{array}$$

Finite Γ.

~~curious relations.~~ $\Gamma = 1$

$A = \mathbb{C}p$ the non-unital alg. gen by an idemp. p

$B =$ unital ring $\mathbb{C}$ these seem to be the same.

$1, e, i, j, \quad ij = e = e^2$ $\qquad \left(\begin{array}{l} iji = i \\ jij = j. \end{array} \right.$?

$W \underset{i}{\overset{j}{\rightleftarrows}} V \qquad ij = 1$

$W \qquad jW \oplus \mathrm{Ker}(i)$

Consider the abelian cat of ~~modules~~ diagrams

$$\Gamma \circlearrowright W \underset{i}{\overset{j}{\rightleftarrows}} V \qquad \sum_{s \in \Gamma} s\, i j\, s^{-1} = 1_W$$

Such diagrams should be equivalent to unital modules for a unital ring. The functor sending a diagram to $W \oplus V$ should be represented by a small projective generator P in the category of these diagrams, and the unital ring should be $\mathrm{End}(P)$. There should be a small projective for each vertex.

$$\Gamma = 1. \qquad W \underset{i}{\overset{j}{\rightleftarrows}} V \qquad ij = 1_W$$

have ji on V satisf. $jiji = ji$

better $jij = j$ and $iji = i$. So the ring $\mathrm{End}(P_W \oplus P_V)$ should be $\begin{pmatrix} \mathbb{C} & \mathbb{C}i \\ \mathbb{C}j & \mathbb{C} \oplus \mathbb{C}ji \end{pmatrix}$

Go over simplest case: $W \underset{i}{\overset{j}{\rightleftarrows}} V \qquad ij = 1_W$

Same as modules for the unital ring

$$\begin{pmatrix} \mathbb{C} & \mathbb{C}i \\ j\mathbb{C} & \mathbb{C} + \mathbb{C}e \end{pmatrix} \qquad ij = 1$$

Better look at the nonunital ring generated by i, j rels. $iji = i$, $i^2 = 0$, $j^2 = 0$
$jij = j$

words in the generators ~~which~~ must avoid i^2, j^2

two gen. ij must alt. $\quad i, j, ij, ji$

$i, \; ij, \; iji = i$

$j, \; ji, \; jij = j$

So you get non unital ring $\begin{pmatrix} \mathbb{C}ij & \mathbb{C}i \\ \mathbb{C}j & \mathbb{C}ji \end{pmatrix}$

$(ij)^2 = ij \;,\; (ij)i = i, \;\; j(ij) = j, \; ij = ij$

$(ji)^2 = ji \;,\; (ji)j = j, \;\; i(ji) = i, \; ji = ji$

<u>which happens</u> to be unital.

next might be Toeplitz type algebra. (non comm)

So what is the non comm. Toeplitz algebra again for a vector space T. Alg of left ops. on $T(V)$ generated by left mult. by v, ~~interior~~ contraction with $\lambda \in V^*$.

$$e_v(v_1 \otimes \cdots \otimes v_n) = v \otimes v_1 \otimes \cdots \otimes v_n$$

$$i_\lambda(v_1 \otimes \cdots \otimes v_n) = \lambda(v_1)\, v_2 \otimes \cdots \otimes v_n$$

$i_\lambda e_v = \lambda(v)$. So if e_a are the "creation" and i_a the "annihilation" ops, the relations are $\quad i_a e_b = \delta_{ab} \quad$ and $\sum_a e_a i_a = 1$.

So return to $\quad i_s = s\,b_i^* \quad j_s = j s^{-1}$

and you get $\sum i_s j_s = 1 \qquad j_s i_t = j s^{-1} t i$

You have some insight from the Toeplitz situation, namely the use of $\sum i_s j_s = 1$ to construct an inductive limit, .
But there seems to be other things around. In the Toeplitz case (situation) you are able to replace $i_\lambda e_\mu$ by the scalar $\langle \lambda, \mu \rangle$. In the present situation you have $j_s i_t = j s^{-1} t i \in A$.

Example: $\Gamma = \mathbb{Z}/2$.

In general for Γ finite what happens.
You have generators ti, js^{-1} where $t, s \in \Gamma$.
Relations. $ti\, t'i = 0$. same for $js^{-1} j s_1^{-1} = 0$.

$$\begin{pmatrix} & ti \\ js^{-1} & \end{pmatrix}$$

Then you have the quadratic elements in the generators

$tijs^{-1} = ths^{-1}$ which gives the cross product alg B.

Wait: Forget Γ as a group. Think of Γ as a finite set. You have generators i_s, j_s $s \in \Gamma$ relations $i_s i_{s'} = 0$, $j_s j_{s'} = 0$,

$$\sum_s (i_s j_s) i_t = i_t \qquad \sum_s j_t (i_s j_s) = j_t .$$

$\Gamma = \{1\}$. $\quad i, j \quad i^2 = 0, \quad j^2 = 0,$

$iji = i \qquad jij = j$

Questions to ask. What about the structure just defined? $\sum_s \iota_s \jmath_s = 1$. B is generated by $\iota_s \jmath_t$

$$\left(\begin{array}{c|c} B & \\ \hline & A \end{array}\right) \qquad W \underset{\iota_t}{\overset{\jmath_s}{\rightleftarrows}} V$$

A gen by the elements $\jmath_s \iota_t$

$$\sum_t (\jmath_s \iota_t)(\jmath_t \iota_u) = \jmath_s \iota_u$$

Toeplitz algebra, Cuntz's algebra $\mathcal{O}_n$ generated by $s_1, \dots, s_n$ such that $s_i^* s_j = \delta_{ij}$ and

$$\sum_{i=1}^n s_i s_i^* = 1.$$

orth isometries

So you begin with the left algebra of operators on $T(V)$ namely, mult by x_i and ~~left~~ contraction with y_j, satisfying

$$y_j x_i = \delta_{ji}$$

If $n=1$ you get $yx = 1$.

Can you connect $\mathcal{O}_n$ to your Morita equivalence. Describe latter.

$$\left(\begin{array}{c|c} & \iota_s \\ \hline \jmath_t & \end{array}\right)$$

generators $\iota_s, \jmath_t$ $s, t \in \Gamma$

relations $\sum_{s \in \Gamma} \iota_s \jmath_s = 1_W$ in the sense that $(\sum \iota_s \jmath_s)\iota_t = \iota_t$

and $\jmath_t(\sum \iota_s \jmath_s) = \jmath_t$.

Now you want to impose conditions on j_s, ι_t. Go back to tensor alg $T(V)$ with ~~operators x_i~~ left mult ops x_i and left contraction y_j satisfying $y_j x_i = \delta_{ji}$ and

$$\sum x_i y_i = \begin{cases} 1 & \text{degree} \geqslant 1. \\ 0 & \text{degree} \leqslant 0 \end{cases}$$

Now you should try to understand the relations between these examples.

Assume. $y_j x_i = \delta_{ji}$

Then
$$\sum_j x_j y_j x_i = \sum_i x_j \delta_{ji}$$

$$\left(\sum_j x_j y_j\right) x_i = x_i$$

First example. $W \underset{\iota}{\overset{j}{\rightleftarrows}} V$ or

$(\iota j)\iota = \iota$, $j(\iota j) = j$.