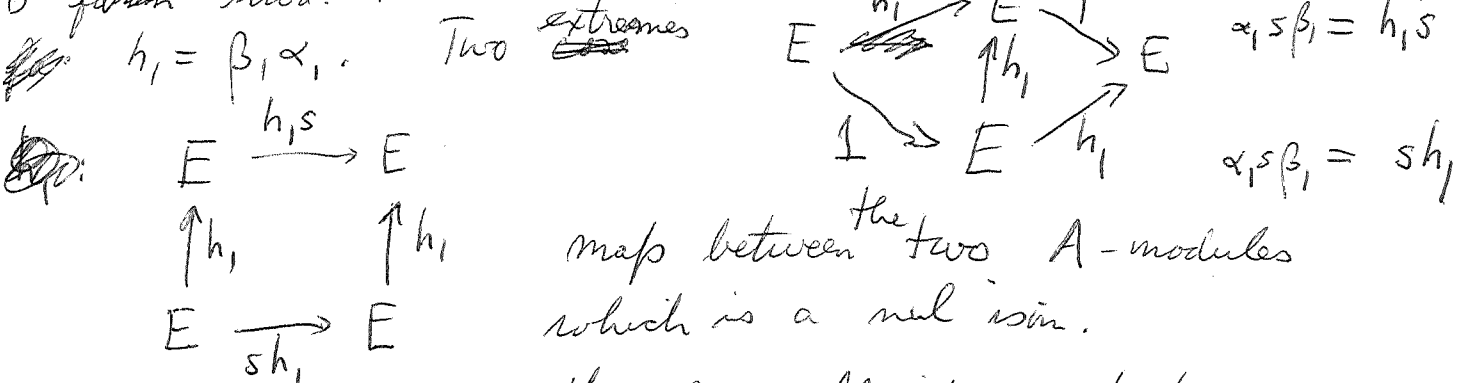


Preview yesterday's progress. $C = \text{alg gens } h_s, s \in \Gamma \text{ fin } 189$
 $\text{rel } \sum h_s = 1 \quad B = C \times \Gamma$

C, B unital. Given E a firm B -module, factor $h_1 = \beta_1 \alpha_1 : E \rightarrow E \rightarrow E$
 and E becomes an A module with $p_s = \alpha_1 s \beta_1$. So
 you get ~~restriction~~ restriction of scalar functor from
 B firm mod. to A -mods. But it depends on choice of

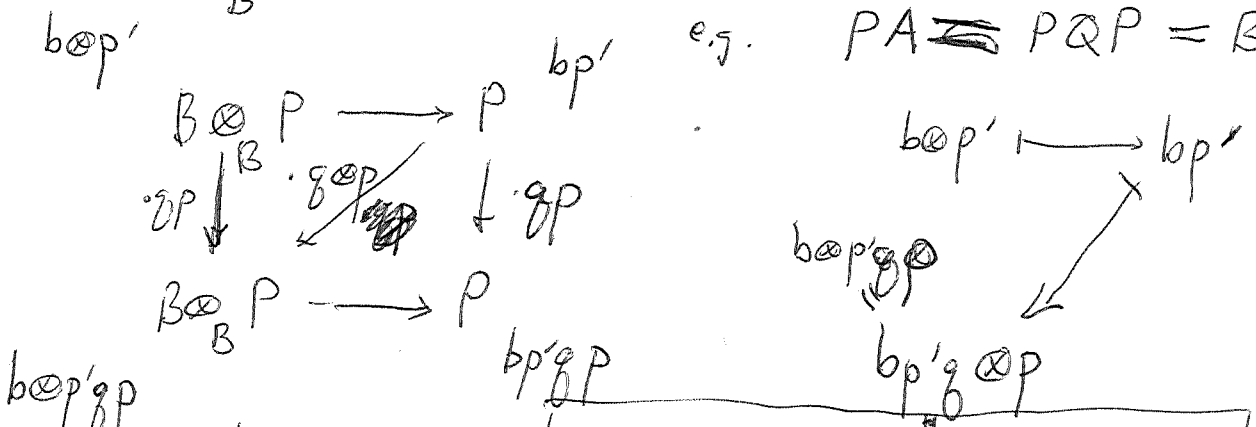


Now you want the firm Morita context behind this Morita context. Now you have two ~~functors~~ (at least) restriction of scalar functors, each gives you an A, B bimodules, call them Q' and Q . Both are $= B$ as B^{op} -module, but $p'_s = s h_1$ on Q' , and $p_s = h_1 s$ on Q and one has $h_1 : Q' \rightarrow Q$

You need to go over general theory for $\begin{pmatrix} A & Q \\ P & B \end{pmatrix}$

strictly idemp. I recall that if $P \otimes_A A \xrightarrow{\sim} P$ then $B \otimes_B P \xrightarrow{\sim} P$. The idea is that

$B \otimes_B P \rightarrow P$ should be an A^{op} -nil ism.



You've used $A = QP$ and $B = PQ$

You need $PQP = P$, $QPQ = Q$ ~~for this argument~~ 190

~~for this argument~~ for this argument.

So far $Q = B$ with obvious $B^{\circ P}$ -mult and $p_s = (h, s)$. By symmetry you expect P to be B with obvious $B^{\circ P}$ -mult and $A^{\circ P}$ -mult given by $p_s = (h, s)$, or $(sh, 1)$, or possibly with s^{-1} somewhere in place of s . You are reminded of $\alpha = \{\alpha_i s^{-1}\}$ $\beta = \{\beta_i s\}$

Something useful. You ~~know that~~ know that a first candidate for Q is B with $p_s = h, s$ or $sh, 1$. But you need $AQ = Q$

Two possible AQ , namely $\sum p_s B = \sum_s h, s B = h, B$
 or $\sum p_s B = \sum_s sh, 1 B = \sum_s h, s B = B$

Idea: $\{\alpha_i s^{-1}\}$ family of g_i $\{\beta_i s\}$ family of p_i
 such that $\sum p_i g_i = \sum \beta_i \alpha_i s^{-1} = 1 \in B$.

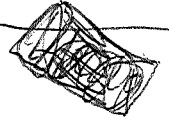
$\begin{pmatrix} A & Q \\ P & B \end{pmatrix}$

~~$\sum p_i g_i = 1$~~
 $Q \xrightarrow{(\circ p_i)} A^n \subset \tilde{A}^n \xrightarrow{(\circ g_i)} Q$

$g \longmapsto \sum_i g_i = g$
 $\quad \quad \quad \beta p_i \quad \quad \quad (\beta p_i)$

$$\begin{pmatrix} \alpha_1 B \beta_1 & \alpha_1 B \\ B \beta_1 & B \beta_1 \alpha_1 B = B \end{pmatrix}$$

This is a Morita context, it will be a firm Morita context iff $\alpha_1 B \otimes_B B \beta_1 \xrightarrow{\sim} \alpha_1 B \beta_1$



Review: Γ finite C $h_s, s \in \Gamma, \sum h_s = 1$

$B = C \rtimes \Gamma$. B module same as C module with h_s C, B unital. ~~Choose~~ factorization

$h_1 = \beta_1 \alpha_1$. ~~scribble~~ $\begin{pmatrix} A & Y \\ X & B \end{pmatrix}$ $p_{s,t} = \alpha_1 s^{-1} t \beta_1$
 $\downarrow = \sum s \beta_1 \alpha_1 s^{-1}$

$x_s = s \beta_1, y_s = \alpha_1 s^{-1}$. ~~Question~~ The grading is not clear.

Let's begin again with B and elements $\alpha_1, \beta_1 \in B$ of degree 1 such that $h_1 = \beta_1 \alpha_1$. Consider the Morita context $\begin{pmatrix} B & B \\ B & B \end{pmatrix}$. No.

~~Basic~~ If E is a B -module (firm), then you get an A -module given by E with op ~~ps~~ $p_s = \alpha_1 s \beta_1$
 Other direction $C \overset{p}{\otimes} A \ni \sum s \otimes t_s \quad f \in C_c(\Gamma, A)$

$$p(\Gamma \times A) = \left\{ \sum_t t \otimes f_t \mid \sum_s s \otimes p_s \sum_t t \otimes f_t = \sum_u u \otimes f_u \right\}$$

$$\boxed{\sum_{st=u} p_s t = f_u}$$

$$\sum_t t \otimes f_t \xrightarrow{\Gamma \times A \xrightarrow{p} \Gamma \times A} \sum_u u \otimes \sum_{st=u} p_s t = \sum_u u \otimes \sum_{ut} p_{ut} t$$

$$\sum_t t' \otimes f_t \quad C \otimes A$$

$$\sum_u u^{-1} \otimes \sum_{st=u} p_s t$$

$$\sum t \otimes f_t \mapsto \sum t^{-1} \otimes f_t \mapsto \sum_{s,t} s t^{-1} \otimes p_s f_t \mapsto \sum_{s,t} t s^{-1} \otimes p_s f_t$$

$$\sum_{s,t} t s^{-1} \otimes p_s f_t = \sum_u u \otimes \sum_t p_{u^{-1}t} f_t$$

$$u = t s^{-1}$$

$$u s = t \quad s = u^{-1} t$$

So if F is an A -module, then the corresp.

~~By using the isomorphism~~ left B -module is

$$\mathbb{C}\Gamma \otimes F = \left\{ f \in C_c(\Gamma, F) \mid f_s = \sum_t p_{s^{-1}t} f_t \right\}$$

$$u \sum s \otimes f_s = \sum u s \otimes f_s = \sum_u u \otimes f_{u^{-1}s}$$

OKAY.

$$\left(\begin{array}{cc} A & Y = (A \times \Gamma) p \\ X = p(\Gamma \times A) & B \end{array} \right)$$

You have a canon. projector in $A \times \Gamma = \Gamma \times A$

$$M(A) \longrightarrow M(B)$$

$$F \longmapsto X \otimes_A F$$

You want to write

$$X = \mathbb{C}\Gamma \overset{p}{\otimes} A$$

to display its B, A bimodule structure

$$\sum_{s \in \Gamma} x_s y_s = \sum_{s \in \Gamma} s \beta_1 \alpha_1 s^{-1} = 1 \in B$$

Inside of X you want to find β_1

How do I make progress on these puzzles?

Recall that ~~the~~^{one} reason for $\mathbb{C}\Gamma \overset{p}{\otimes} A$ is that p

$= \sum (\cdot s^{-1}) \otimes (p_s \cdot)$ is homogeneous of unit degree, and hence

$\mathbb{C}\Gamma \overset{p}{\otimes} A$ is Γ graded.

Look at $\mathbb{F} = \Gamma = \mathbb{Z}/2$ again. A in this case is $\mathbb{C} \times \mathbb{C}$ which should be ~~the~~ ^{the augm.} ~~augmentation~~ ideal in $\mathbb{C}\Gamma \rtimes \mathbb{C}\Gamma = \mathbb{C}[\text{dihedral group } \mathbb{Z} \times \mathbb{Z}/2]$.

$B = \mathbb{C}[h_0, h_1] / (h_0 + h_1 - 1) \rtimes \mathbb{Z}/2$. here ε interchanges h_0 and h_1 .

so $\varepsilon(h_0 - \frac{1}{2})\varepsilon^{-1} = h_1 - \frac{1}{2} = -(h_0 - \frac{1}{2})$ $h_0 - \frac{1}{2} + h_1 - \frac{1}{2} = 0$

$e^2 = e \iff (\frac{1-2e}{2})^2 = 1$ $4e^2 - 4e + 1 = 1$

$\mathbb{C}e \times \mathbb{C}e \subset \tilde{\mathbb{C}}e \rtimes \tilde{\mathbb{C}}e = \mathbb{C}[\mathbb{Z}/2 \times \mathbb{Z}/2]$



$B = \mathbb{C}[h_0 - \frac{1}{2}] \rtimes \mathbb{Z}/2$ $\varepsilon(h_0 - \frac{1}{2}) = -(h_0 - \frac{1}{2})$

basis

1	z	z^2
ε	$z\varepsilon$	$z^2\varepsilon$

so you can embed

A into B in four ways

$A = \mathbb{C}e \times \mathbb{C}\bar{e}$

e	$\bar{e}e$
\bar{e}	$e\bar{e}$

e	F	or	\bar{F}
\bar{e}	\bar{F}	or	$-F$

A generated by p_0, p_1 rels. $(p_0 \pm p_1)^2 = p_0 \pm p_1$

$\therefore A = \mathbb{C}e \times \mathbb{C}\bar{e} \subseteq \mathbb{C}[F] \rtimes \mathbb{C}[F]$ $F = 1 - 2e$

$\mathbb{C}[\mathbb{Z}/2 \times \mathbb{Z}/2]$
dihedral group.

$B = \mathbb{C}[h_0, h_1] / (h_0 + h_1 - 1) \rtimes \mathbb{Z}/2$ $\varepsilon(h_0 - \frac{1}{2}) = (h_1 - \frac{1}{2})$

Apparently A, B are Morita equivalent

~~$\mathbb{C}[h_0, h_1]$~~
 $-h_0 + \frac{1}{2}$

$h_0 = \beta_0 \alpha_0 = h_0 \perp$

$p_n = \alpha_0 \varepsilon^n \beta_0 = h_0 \varepsilon^n$

$$p_0 = h_0 \quad p_1 = \varepsilon h_0$$

$$\begin{aligned} (p_0 + p_1)^2 &= h_0(1+\varepsilon)h_0(1+\varepsilon) \\ &= h_0^2 + \underbrace{h_0\varepsilon h_0}_{h_0 h_1 \varepsilon} + h_0^2 \varepsilon + \underbrace{h_0\varepsilon h_0 \varepsilon}_{h_0 h_1} \\ &= \end{aligned}$$

$$\begin{aligned} (p_0 \pm p_1)^2 &= (h_0 \pm \varepsilon h_0)^2 = h_0^2 \pm h_0 \varepsilon h_0 \pm \underbrace{\varepsilon h_0^2}_{h_1 \varepsilon h_0} + \underbrace{\varepsilon h_0 \varepsilon h_0}_{h_1} \\ &= (h_0 + h_1)h_0 \pm (h_0 + h_1)\varepsilon h_0 = h_0 \pm \varepsilon h_0 = p_0 \pm p_1 \end{aligned}$$

$$x_s = s\beta_1$$

$$y_s = \alpha_1 s^{-1}$$

$$h_0 = \begin{matrix} \alpha_0 & \beta_0 \\ \hline h_0 & 1 \end{matrix}$$

$$x_n = \varepsilon^n \in X, \quad y_n = h_0 \varepsilon^{-n} \in Y$$

$$\sum_{\mathbb{Z}/2} x_n y_n = h_0 + h_1 = 1.$$

A Y

~~scribble~~

X B

$$X = \begin{matrix} \text{scribble} \\ \text{scribble} \\ \text{scribble} \end{matrix} \quad p(\Gamma \times A)$$

$$p = 1 \otimes p_0 + \varepsilon \otimes p_1 \quad f = 1 \otimes f_0 + \varepsilon \otimes f_1$$

$$pf = 1 \otimes (p_0 f_0 + p_1 f_1) + \varepsilon \otimes (p_0 f_1 + p_1 f_0)$$

$$\begin{aligned} \therefore p_0 f_0 + p_1 f_1 &= f_0 \\ p_0 f_1 + p_1 f_0 &= f_1 \end{aligned}$$

$$\begin{pmatrix} p_0 & p_1 \\ p_1 & p_0 \end{pmatrix} \begin{pmatrix} f_0 \\ f_1 \end{pmatrix} = \begin{pmatrix} f_0 \\ f_1 \end{pmatrix}$$

$$(1 \varepsilon) \begin{pmatrix} p_0 \\ p_1 \end{pmatrix} f_0 + \begin{pmatrix} p_1 \\ p_0 \end{pmatrix} f_1 = (p_0 + \varepsilon p_1) f_0 + (p_1 + \varepsilon p_0) f_1$$

$$= (p_0 + \varepsilon p_1)(f_0 + \varepsilon f_1) \quad f_0, f_1 \in A$$

$$(p_0 + \varepsilon p_1)^2 = p_0^2 + \varepsilon p_0 p_1 + \varepsilon p_1 p_0 + \varepsilon^2 p_1^2 = \underbrace{p_0^2 + p_1^2}_{p_0} + \varepsilon \underbrace{(p_0 p_1 + p_1 p_0)}_{\varepsilon p_1}$$

You will now do the calculation for $\mathbb{Z}/2 = \{1, \varepsilon\}$

$$B = \mathbb{C}[h_0, h_1] / (h_0 + h_1 - 1) \times \mathbb{Z}/2$$

$$\varepsilon h_0 \varepsilon = h_1 \quad \varepsilon (h_0 - \frac{1}{2}) \varepsilon = h_1 - \frac{1}{2} = -(h_0 - \frac{1}{2})$$

A generators p_0, p_1 rels. $(p_0 \pm p_1)^2 = p_0 \pm p_1$

$$A = \mathbb{C}e * \mathbb{C}\bar{e} \quad \begin{cases} e = p_0 + p_1 \\ \bar{e} = p_0 - p_1 \end{cases}$$

$$\tilde{A} = \tilde{\mathbb{C}}e * \tilde{\mathbb{C}}\bar{e} \simeq \mathbb{C}[F] * \mathbb{C}[\bar{F}] = \mathbb{C}[\text{dihedral group}]$$

$$\simeq \mathbb{C}[Z] * \mathbb{C}[Z/2] \quad \varepsilon u \varepsilon = u^{-1}$$

$$A = \mathbb{C}e * \mathbb{C}\bar{e} \quad \tilde{A} = \mathbb{C}[e] * \mathbb{C}[\bar{e}] = \mathbb{C}[F] * \mathbb{C}[\bar{F}]$$

An A -module is a v.s. V with two projectors e, \bar{e}

since $A = e\tilde{A} + \bar{e}\tilde{A}$ one $AV = eV + \bar{e}V$

and ~~since~~ $A = \tilde{A}e + \tilde{A}\bar{e}$, one have $AV = eV \cap \bar{e}V$

$= e^\perp V \cap \bar{e}^\perp V$. Question: Is $e^\perp V \cap \bar{e}^\perp V$ a complement for $eV + \bar{e}V$? No.

$$V = eV \oplus e^\perp V$$

$$= \bar{e}V \oplus \bar{e}^\perp V$$

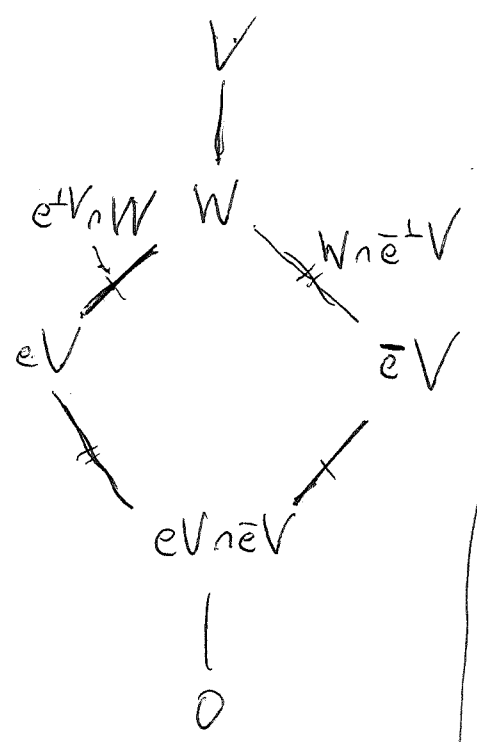


$$eV \subset (eV + \bar{e}V)$$

$$eV + e^\perp V = V$$

$$eV + (eV + \bar{e}V) n \bar{e}V = eV + \bar{e}V$$

~~WTA seems unimportant~~



$A = \mathbb{C}e * \mathbb{C}\bar{e}$, An A -module is a vector space V with two idempotent operators e, \bar{e} i.e. two splittings. Since $A = e\tilde{A} + \bar{e}\tilde{A} \Rightarrow AV = eV + \bar{e}V$
 $A = \tilde{A}e + \tilde{A}\bar{e} \Rightarrow AV = \text{Ker}(e) \oplus \text{Ker}(\bar{e})$

Example. $\dim V = 2$, $eV, \bar{e}V$ two lines in V with a common complement $(1-e)V = (1-\bar{e})V$. Then $\text{Ker}(e) = \text{Ker}(\bar{e})$.

then $AV = V$ but $AV \neq 0$.

~~Can you get a picture of finit modules?~~ Can you get a picture of finit modules?

You want $V = eV + \bar{e}V$ and complements $e^\perp V$ for eV and $\bar{e}^\perp V$ for $\bar{e}V$

~~Picture of an A-module~~ Picture of an A-module is a v.s. V with two splittings. V is round when $eV + \bar{e}V = V$ and $\underbrace{(1-e)V \cap (1-\bar{e})V}_V = 0$, so you have $\text{Im}(e) + \text{Im}(\bar{e}) = V$, $\text{Ker}(e) \cap \text{Ker}(\bar{e}) = 0$.

$$V \xrightarrow{\begin{pmatrix} e \\ \bar{e} \end{pmatrix}} \bigoplus \begin{matrix} V \\ V \end{matrix} \xrightarrow{\begin{pmatrix} e & \bar{e} \end{pmatrix}} V \quad ?$$

Go back to $\mathbb{Z}/2$. $B = \mathbb{C}[h_0, h_1] / (h_0 + h_1 - 1) \rtimes \mathbb{Z}/2$
 $\varepsilon(h_0 - \frac{1}{2}) = (h_1 - \frac{1}{2}) = -(h_0 - \frac{1}{2})$, B is unital

A firm B-module is ~~a vector~~ a vector space E together with involution ε and an odd operator.

A super vector space $E = E_+ \oplus E_-$
 $h_0 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $h_1 = \varepsilon h_0 \varepsilon = \begin{pmatrix} a & -b \\ -c & d \end{pmatrix}$ $h_0 + h_1 = \begin{pmatrix} 2a & 0 \\ 0 & 2d \end{pmatrix}$

$\therefore a = \frac{1}{2}$ ~~$a = d$~~ $d = \frac{1}{2}$.

So you know exactly what firm B-modules are.

$\bar{E} = E_0 \rightleftarrows E_-$

And you know what round $A = \mathbb{C}e \rtimes \mathbb{C}\bar{e}$ are

~~Representation~~ An A module is a representation of the dihedral group $\mathbb{Z}/2 \times \mathbb{Z}/2$. Specify

that $F = +1$ on $\text{Im}(e)$ $F = -1$ on $\text{Ker}(e)$

To understand - fix ε , call other F ,

$g = F\varepsilon$ $\varepsilon g \varepsilon = \varepsilon F = g^{-1}$. So what do you find?
 Suppose you have e, \bar{e} in V

Look at $\text{Ker}(e) \cap \text{Ker}(\bar{e})$; on this space $F = \bar{F} = -1$
 $\text{--- } V/eV + \bar{e}V$; $\text{--- } F = \bar{F} = +1$

What are you trying to understand?

Spectrum. $g = F\varepsilon$ is invertible, its eigenvalues are $z \in \mathbb{C}^\times$ and since $\varepsilon g \varepsilon = g^{-1}$, $z \in \text{Spec} \implies z^{-1} \in \text{Spec}$. But $+1, -1$ fix pts. ~~Normally the~~

~~spectrum is~~ Spectrum of the dihedral group = set of irred reps. 1-diml reps : characters.

There are 4 homos. $\mathbb{Z}/2 \times \mathbb{Z}/2 \longrightarrow \{ \pm 1 \}$

You have $g = F\varepsilon = 1$ whence $F = \varepsilon = 1$
 or $F = \varepsilon = -1$



The problem: $A = \mathbb{C}e * \mathbb{C}\bar{e}$, $B = \mathbb{C}[h_0^{-1/2}] \rtimes \mathbb{Z}/2$.

There seem to be homomorphisms $A \rightarrow B$, and since B is unital, these correspond to unital homos. $\tilde{A} \rightarrow B$ i.e. pairs of involutions in B . $B = \mathbb{C}[x] \rtimes \{ \varepsilon \}$

$B = \{ f(x) + g(x)\varepsilon \mid f(x), g(x) \in \mathbb{C}[x] \}$

$$(f + g\varepsilon)^2 = f^2 + fg\varepsilon + g\varepsilon f + g\varepsilon g\varepsilon$$

$$= \underbrace{f(x)^2 + g(x)g(-x)}_1 + \underbrace{(f(x)g(x) + g(x)f(-x))}_0 \varepsilon$$

example $f(x) = +x$ $g(x) = 1+x$ $x^2 + \frac{(1+x)(1-x)}{1-x^2} = 1$. $\therefore f$ odd

$$f+g\varepsilon = x+(1+x)\varepsilon \quad \text{one involution}$$

$$f\varepsilon+g = x\varepsilon+(1+x) \quad \text{should be invertible}$$

$$\text{with inverse} \quad \varepsilon(f+g\varepsilon) = \varepsilon(x+(1+x)\varepsilon) = -x\varepsilon+(1-x)$$

$$(x\varepsilon+(1+x))(-x\varepsilon+(1-x)) = -x\varepsilon x\varepsilon - (1+x)x\varepsilon + x\varepsilon(1-x) + (1-x)^2$$

$$= x^2 - x(1+x)\varepsilon + x(1+x)\varepsilon + (1-x)^2$$

$$= x^2 + [-x + x + x^2]\varepsilon + (1-x)^2 = 1$$

$$(x+(1+x)\varepsilon)(x+(1+x)\varepsilon) = x^2 + (1+x)(-x)\varepsilon + x(1+x)\varepsilon + (1+x)(1-x)$$

$$= x^2 + \{-x - x^2 + x + x^2\}\varepsilon + 1 - x^2 = 1$$

$$(x+(1+x)\varepsilon)\varepsilon = x\varepsilon + 1+x$$

$$\varepsilon(x+(1+x)\varepsilon) = -x\varepsilon + (1-x)$$

$$= x\varepsilon(-x\varepsilon) + x\varepsilon(1-x) + (1+x)(-x\varepsilon) + 1-x^2$$

$$= x^2 + \{x(1+x) - (1+x)x\}\varepsilon + 1-x^2 = 1$$

So therefore you have

~~$x+(1+x)\varepsilon$~~
 ~~$x\varepsilon+(1+x)$~~

$$g(x)g(-x) = (1+f(x))(1-f(x))$$

invertible element $1+x+x\varepsilon$

with inverse $1-x-x\varepsilon$

$$f(x)^2 + (1+f(x))(1-f(x))$$

In $B = \mathbb{C}[x] \rtimes \mathbb{Z}/2$ $\varepsilon x \varepsilon = -x$, the elt. 201

$f + g\varepsilon$, f and $g \in \mathbb{C}[x]$ is an involution when

$$1 = (f + g\varepsilon)^2 = f(x)^2 + g(x)g(-x) + (f(x)g(x) + g(x)f(-x))\varepsilon$$

If $g \neq 0$ then $f(x)$ is odd, ($g=0$ $f=\pm 1$ case omitted)

and $g(x)g(-x) = (1+f(x))(1-f(x))$, the obvious choice for $g(x)$ is $1+f(x)$, but already there is also $g(x) = 1-f(x)$.

Go back to $p_s = \alpha_1 s \beta_1$ $h_1 = \beta_1 \alpha_1$

$$B = \underbrace{\mathbb{C}[h_0, h_1]}_{\mathbb{C}[x]} / (h_0 + h_1 - 1) \rtimes \mathbb{Z}/2$$

$x = h_0 - \frac{1}{2}$

$$\mathbb{C}[\mathbb{Z}/2] \otimes A = \{f_0 + \varepsilon f_1 \mid f_0, f_1 \in A\}. \quad \varepsilon \varepsilon = 1$$

$$p = p_0 + \varepsilon p_1 \quad pf = p_0 f_0 + p_1 f_1 + \varepsilon(p_0 f_1 + p_1 f_0)$$

$$p \cdot (\mathbb{C}[\varepsilon] \otimes A) = (1 \ \varepsilon) \cdot \begin{pmatrix} p_0 & p_1 \\ p_1 & p_0 \end{pmatrix} \begin{pmatrix} f_0 \\ f_1 \end{pmatrix}$$

Use the formulas you have.

$$\underbrace{\sum t \otimes f_t}_{\in \mathbb{C} \otimes A} \xrightarrow{\sum s \otimes p_s} \sum st \otimes p_s f_t$$

$$\mathbb{C} \otimes A$$

Let's start with a B module E , choose fact.

$$E \xrightarrow{\alpha_1} \mathbb{C} \xrightarrow{\beta_1} E \xrightarrow{\alpha_1} V$$

$\underbrace{\hspace{10em}}_{h_1}$

$$E \xrightarrow{\alpha} \mathbb{C}\Gamma \otimes V \xrightarrow{\beta} E \xrightarrow{\alpha} \mathbb{C}\Gamma \otimes V$$

$$\{1\} \mapsto \sum_s s \otimes \alpha_s^{-1} \mapsto \sum_t t \beta_t \alpha_t^{-1} = \{1\}$$

$$\sum_t t \otimes f_t \mapsto \sum_t t \beta_t f_t \mapsto \sum_s \sum_t s \otimes \alpha_s^{-1} t \beta_t f_t$$

~~So you want to take $\mathbb{C}\Gamma \otimes V$~~

Two paths: start V , say $V = \tilde{A}$, form $p(\mathbb{C}\Gamma \otimes V)$ and exhibit how B acts

~~Your aim: Take $V = A$~~

first start with V , then form $p(1 \otimes v) = \sum_s s \otimes \alpha_s^{-1} \beta_s v \subset \mathbb{C}\Gamma \otimes V$ and form the Γ -submodule it generates.

$$p v = \alpha_0 \beta_0 v + \epsilon \alpha_0 \epsilon \beta_0 v = p_0 v + \epsilon p_1 v$$

Here V is a vector space with two ~~the~~ projections e and \bar{e} , in fact $e = p_0 + p_1$, $\bar{e} = p_0 - p_1$

~~You tensor with $\mathbb{C}\Gamma$~~ You tensor with $\mathbb{C}\Gamma$

$$p(v_0 + \epsilon v_1) = (p_0 + \epsilon p_1)(v_0 + \epsilon v_1) = (p_0 v_0 + p_1 v_1) + \epsilon(p_0 v_1 + p_1 v_2)$$

the Γ action is mult. by ϵ . Now you have $E = p(\mathbb{C}\Gamma \otimes V)$ as a Γ subspace of $\mathbb{C}\Gamma \otimes V$, and you need only give $h_0 = \beta_0 \alpha_0$

$\Gamma = \mathbb{Z}/2$. $B = \mathbb{C}[h_0, h_1] / (h_0^2, h_1^2) \rtimes \mathbb{Z}/2$

$A = \mathbb{C}e \times \mathbb{C}\bar{e}$, an A module is a V with two splittings, a graded A -module is a ~~super~~ $\mathbb{Z}/2$ -graded v.s. $V = V_0 \oplus V_1$, together with an idempotent operator. ~~?~~ ?

A generators p_0, p_1 relations $p_0 \pm p_1$ idempotent

$$\begin{cases} p_0^2 + p_1^2 = p_0 \\ p_0 p_1 + p_1 p_0 = p_1 \end{cases}$$

graded A -module $V = V_0 \oplus V_1$

~~End(V)~~ $End(V) = \begin{pmatrix} End(V_0) & Hom(V_1, V_0) \\ Hom(V_0, V_1) & End(V_1) \end{pmatrix}$

p_0 even operator, p_1 odd op on V .

basic construction $\mathbb{C}\Gamma \otimes V = \{ \cancel{1} \otimes f_0 + \epsilon \otimes f_1 \mid f_0, f_1 \in V \}$
 $= \{ v_0 + \epsilon v_1 \mid v_0 \text{ and } v_1 \in V \}$. Define $p = 1 \otimes p_0 + \epsilon \otimes p_1$

$$(p_0 + \epsilon p_1)(v_0 + \epsilon v_1) = (p_0 v_0 + p_1 v_1) + \epsilon(p_0 v_1 + p_1 v_0)$$

ungraded V

Review construction, A gens $p_s, s \in \Gamma$ rels $p_s = \sum_t p_{st} p_t$

A is Γ -graded algebra $A = \bigoplus_s A_s$ $A_s A_t \subset A_{st}$

canonical embedding $A \rightarrow \mathbb{C}\Gamma \otimes A$ tensor product alg where $sa = as$.

$$p_s \mapsto s \otimes p_s$$

~~?~~ Preserves Γ -grading where ~~?~~ A on the right has unit degree.

Question. What about considering the tensor product $\mathbb{C}\Gamma \otimes A$ as superalgebra.

$\Gamma = \mathbb{Z}/2$ $\mathbb{C}\Gamma = \mathbb{C}[\varepsilon]$ You form

~~$\mathbb{C}\Gamma \otimes A$~~ $= \{ a_0 + \varepsilon a_1 \}$ ring where ε, A commute $\varepsilon^2 = 1$.

$p = p_0 + \varepsilon p_1 \in \mathbb{C}\Gamma \otimes A$.

Now $\mathbb{C}\Gamma \otimes A \longrightarrow A \times A$ $\varepsilon \longrightarrow +1, -1$

So it seems like $\mathbb{C}\Gamma \otimes A$ and the canonical proj $p = p_0 + \varepsilon p_1$ become $A \times A$ with $p = (e, \bar{e})$, so if we apply p we get $eA \times \bar{e}A$

Let's repeat. $\Gamma = \mathbb{Z}/2$ $\mathbb{C}\Gamma = \mathbb{C}[\varepsilon] = \mathbb{C} \oplus \mathbb{C}\varepsilon$ $\varepsilon^2 = 1$.

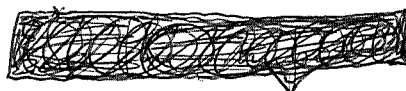
$\mathbb{C}[\varepsilon] \otimes A = A \oplus \varepsilon A$ $\varepsilon a = a\varepsilon, \varepsilon^2 = 1$.

$p = \sum_s s \otimes p_s = p_0 + \varepsilon p_1$. In our situation

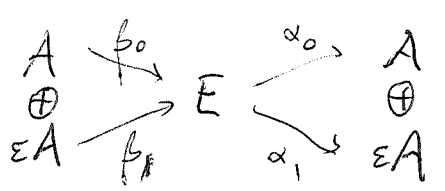
$A = \mathbb{C}e * \mathbb{C}\bar{e} \subset \Omega(\mathbb{C}e)$ with Fedosov product

$p_0 = e$
 $p_1 = de$

$p_0^2 + p_1^2 = e^2 - de de + de^2 = e = p_0$
 $p_0 p_1 + p_1 p_0 = ede + dee = d(e^2) = de = p_1$



So you form $\mathbb{C}[\varepsilon] \otimes A = A \oplus \varepsilon A$ Inside here is the element $p = p_0 + \varepsilon p_1$ which is idempotent, Form (and even) $p(A \oplus \varepsilon A)$. Your problem is to understand $p(A \oplus \varepsilon A)$, find a formula for it, you want the Γ action, and the operator h_0



There should be some organizing principle. & You

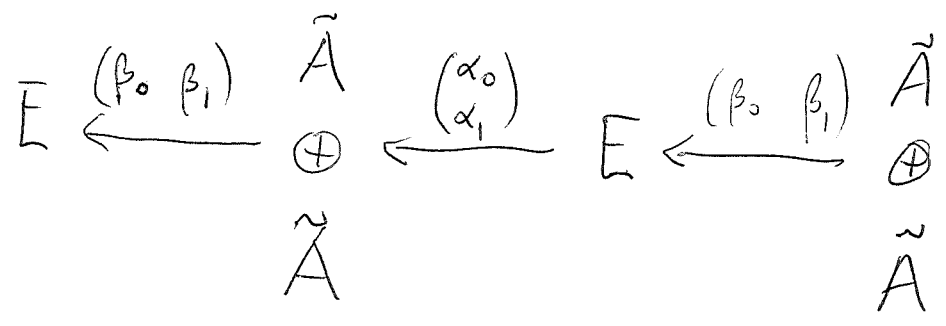
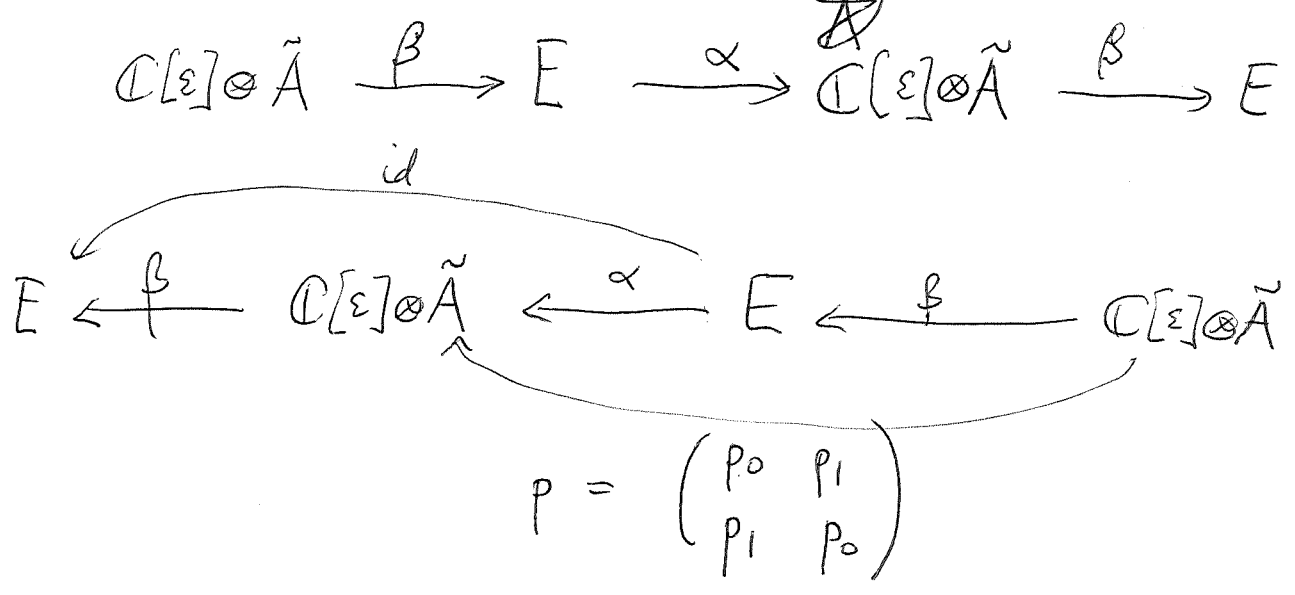
are constructing a fin proj $A^0 p$ module, whose endo ring is the unital ring B .

Look clearly. $\mathbb{C}[\varepsilon] \otimes \tilde{A}$ is a free A^{op} module of rank 2. ~~On~~ On this module you have ρ acting. Actually $M_2(\tilde{A})$ acts - it's the ring of $\text{End}_A(\mathbb{C}[\varepsilon] \otimes \tilde{A})$, and $\rho = \begin{pmatrix} p_0 & p_1 \\ p_1 & p_0 \end{pmatrix} \in M_2(\tilde{A})$.

Then B should be $\rho M_2(\tilde{A}) \rho$.

~~Things~~

~~Things should be straightforward!!!!~~



$\alpha_1 = \alpha_0 \varepsilon$ $\beta_1 = \varepsilon \beta_0$ Check

$$\begin{pmatrix} \alpha_0 \\ \alpha_0 \varepsilon \end{pmatrix} \begin{pmatrix} \beta_0 & \varepsilon \beta_0 \end{pmatrix} = \begin{pmatrix} \alpha_0 \beta_0 & \alpha_0 \varepsilon \beta_0 \\ \alpha_0 \varepsilon \beta_0 & \alpha_0 \beta_0 \end{pmatrix}$$

All you should need to do is to go over the formulas enough. ~~you~~ In the case of finite Γ ~~you start with a fin. gen. free A^{op} -module, namely, $\mathbb{C}\Gamma \otimes \tilde{A}$~~ , you start with a fin. gen. free A^{op} -module, namely, $\mathbb{C}\Gamma \otimes \tilde{A}$, you have an idempotent operator $p = \sum s \otimes p_s$ on this module and $E = p(\mathbb{C}\Gamma \otimes \tilde{A})$ is a finite projective A^{op} -module.

There's a dual fin. proj. A -module $E' = \text{Hom}_A(E, \tilde{A})$ whence a Morita context, where $B = E \otimes_A E' = \text{Hom}_{A^{\text{op}}}(E, E)$

$$\begin{pmatrix} \tilde{A} & E' \\ E & B = E \otimes_A E' \end{pmatrix}$$

So the only point ~~to~~ to be checked, or understood, is why B has the form $B = \mathbb{C} \rtimes \Gamma$ $\mathbb{C} = \mathbb{C}\langle h_s, s \in \Gamma \rangle / (\sum h_s = 1)$

You need to check that $\langle E', E \rangle = A$, in order to get the desired M. eq. of B and A .

$$\begin{aligned} E &\xrightarrow{\alpha} \mathbb{C}\Gamma \otimes \tilde{A} \xrightarrow{\beta} E \xrightarrow{\gamma} \mathbb{C}\Gamma \otimes \tilde{A} \\ \{ &\longmapsto \sum_s s \otimes \alpha_s^{-1} \} \longmapsto \sum_s s \otimes \sum_t \underbrace{(\alpha_s^{-1} t \beta_t)}_{p_s^{-1} t} f_t \\ &\sum_t t \otimes f_t \longmapsto \sum_t t \beta_t f_t \end{aligned}$$

you will have $\alpha_s^{-1} \in E' = \text{Hom}_{A^{\text{op}}}(E, \tilde{A})$
and $s \beta_t \in E$

You have to keep on repeating. Γ finite
 $\mathbb{C}\Gamma \otimes \tilde{A}$ fin. free A^{op} -module
 $p(t \otimes a) = \sum_s s t \otimes p_s a$ proj. op. $E = p(\mathbb{C}\Gamma \otimes \tilde{A})$
 fin. proj. A^{op} -module

$$\sum_t t^{-1} \otimes f_t \xrightarrow{P} \sum_{s,t} st^{-1} \otimes p_s f_t$$

$$\sum_u u \otimes \sum_t p_{ut} f_t$$

$$\sum t \otimes f_t$$

$$\sum_{s,t} \underbrace{ts^{-1}}_u \otimes p_s f_t = \sum_u u \otimes \sum_{ts^{-1}=u} p_s f_t$$

$$\sum t \otimes f_t \mapsto \sum_s s \otimes \sum_t \overset{p_{s^{-1}t}}{\cancel{f_t}} f_t$$

~~The map A is Morita equiv. to M~~

The idea is to break up the M. eq. by going from A to $M_n A$ and \otimes from $M_n A$ to $p(M_n A)p$. It might be easier to show that $p(M_n A)p = B$. In fact M_n might arise as $\Gamma \rtimes \hat{\Gamma}$

Maybe you can show A finite by defining an inverse for $A \otimes_A A \rightarrow A$, sending p_s to $\sum_{t \in \Gamma} p_{st^{-1}} \otimes p_t$. Do for $\mathbb{Z}/2$.

$$p_0 \longmapsto p_0 \otimes p_0 + p_1 \otimes p_1 \in A \otimes_A A$$

$$p_1 \longmapsto p_0 \otimes p_1 + p_1 \otimes p_0$$

$$(p_0 \otimes p_0 + p_1 \otimes p_1)^2 = p_0^2 \otimes p_0^2 + p_0 p_1 \otimes p_0 p_1 + p_1 p_0 \otimes p_1 p_0 + p_1^2 \otimes p_1^2$$

$$(p_0 \otimes p_1 + p_1 \otimes p_0)^2 = p_0^2 \otimes p_1^2 + p_0 p_1 \otimes p_1 p_0 + p_1 p_0 \otimes p_0 p_1 + p_1^2 \otimes p_0^2$$

~~So what about \tilde{A} ?~~

Γ finite. Consider $\mathbb{C}\Gamma \otimes \tilde{A}$ free A^{op} module
dual A -module $\text{Hom}_{\tilde{A}}(\mathbb{C}\Gamma \otimes \tilde{A}, \tilde{A}) = \tilde{A} \otimes \mathbb{C}(\Gamma)$

$\Gamma = \mathbb{Z}/2$.

\tilde{A}	\tilde{A}	\tilde{A}
\tilde{A}	$M_2(\tilde{A})$	
\tilde{A}		

782
102
884
54
938

107

$\mathbb{C}\Gamma \otimes \tilde{A}$ fin. free A^{op} module
 $p(\mathbb{C}\Gamma \otimes \tilde{A})$ — proj A^{op} module

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401.50

What can you do? $E' = \mathbb{C}\Gamma \otimes \tilde{A}$

Focus on the main difficulty. Given A which comes with p a splitting of $\tilde{A}^{\oplus 2}$

$$E = \begin{pmatrix} p_0 & p_1 \\ p_1 & p_0 \end{pmatrix} \begin{pmatrix} \tilde{A} \\ \tilde{A} \end{pmatrix} = p \begin{pmatrix} \tilde{A} \\ \tilde{A} \end{pmatrix} \quad \tilde{A} \quad E'$$

$$E' = \text{Hom}_{A^{\text{op}}}\left(p \begin{pmatrix} \tilde{A} \\ \tilde{A} \end{pmatrix}, \tilde{A}\right) = (\tilde{A} \oplus \tilde{A}) p \quad E$$

why is $E \otimes_A E' \simeq B$? Put another way, why is $B = \text{Hom}_{A^{\text{op}}}(E, E)$?

$$\begin{pmatrix} p_0 & p_1 \\ p_1 & p_0 \end{pmatrix} \begin{pmatrix} \tilde{A} & \tilde{A} \\ \tilde{A} & \tilde{A} \end{pmatrix} \begin{pmatrix} p_0 & p_1 \\ p_1 & p_0 \end{pmatrix}$$

Everything should be contained in what you already know. ~~Start with an A-module~~ 209

Given an A -module V , you ~~can show~~ get a Γ -module $\mathbb{C}\Gamma \otimes V = \{ \sum_t t \otimes f_t \mid f_t \in V \}$ and a projector

$$\begin{aligned} u &= ts^{-1} \\ us &= t \\ s &= u^{-1}t \end{aligned}$$

on this Γ -module $p \sum_t t \otimes f_t = \sum_{st} ts^{-1} \otimes p_s f_t = \sum_u u \otimes \sum_t p_{u^{-1}t} f_t$

so you get the image of $p(\mathbb{C}\Gamma \otimes V)$, denoted $E(V)$

better maybe ~~$E(V)$~~ $E(V) = E(\tilde{A}) \otimes_A V$. ~~What do you~~

~~know~~ ~~start at~~

Here's an approach. $\mathbb{C}\Gamma \otimes \tilde{A}$ is a finite free A^{op} -module so $E = p(\mathbb{C}\Gamma \otimes \tilde{A})$ is a finite projective A^{op} -module

$$E \xleftarrow{\alpha} \mathbb{C}\Gamma \otimes \tilde{A} \xrightarrow{\beta} E$$

$$\xi \longmapsto \{ s \otimes \alpha_s^{-1} \} \longmapsto \{ \beta_s \alpha_s^{-1} \}$$

$$\sum_t t \otimes f_t \longmapsto \sum_t t \beta_t f_t$$

Propose: to forget the group, but focus on constructing a finite projective firm A -module. ~~The point is~~ ~~that you have a finite~~ a finite proj. What's important

$$E \longrightarrow R^n \longrightarrow E$$

$$\text{Hom}_{R^{\text{op}}}(E, E) \xleftarrow{\text{id}} E \otimes_R \text{Hom}_{R^{\text{op}}}(E, R) \xrightarrow{\sum_i \xi_i \otimes \lambda_i}$$

$$E \xrightarrow{(\lambda_i \circ)} R^n \xrightarrow{(\xi_i \circ)} E$$

$$\xi \longmapsto \langle \lambda_i, \xi \rangle \longmapsto \sum_i \xi_i \langle \lambda_i, \xi \rangle = \xi.$$

You have a finite proj R^{op} -module E , whence

~~Hom~~ $n=1$.
$$E \xleftarrow{\lambda} R \xrightarrow{\xi_0} E$$

$$\xi \mapsto \langle \xi, \xi \rangle = \xi.$$

$$E \otimes_R \text{Hom}_R(E, R) \longrightarrow \text{Hom}_R(E, E)$$

$$\xi_0 \otimes \lambda \longmapsto \text{id}$$

$$R \xrightarrow{\xi_0} E \xrightarrow{\lambda} R \xrightarrow{e} R$$

$$\mathbb{Z} \longmapsto \xi_0 \mathbb{Z} \longmapsto \langle \mathbb{Z}, \xi_0 \rangle = \langle \lambda, \xi_0 \rangle \mathbb{Z}$$

I think the main problem is how to deal with the rings being universal. Given an A module V you construct a B module $p(\mathbb{C} \otimes V)$. Given a B module E you construct an A -module in different ways which are all isomorphic. Go over this.

An A -module is a V with two idemp. ops e, \bar{e} .
 rather $p_0 + p_1 = e$, $p_0 - p_1 = \bar{e}$. Define p on $\mathbb{C}[\varepsilon] \otimes V$ to be $(p_0 + \varepsilon p_1)(v_0 + \varepsilon v_1) = (p_0 v_0 + p_1 v_1) + \varepsilon(p_0 v_1 + p_1 v_0)$

$$\mathbb{C}[\varepsilon] \otimes V \xrightarrow{p} \mathbb{C}[\varepsilon] \otimes V$$

$$\beta \searrow \xrightarrow{X(V)} \swarrow \alpha$$

$$X(V) = p(\mathbb{C}[\varepsilon] \otimes V)$$

A form B module E is a $\mathbb{Z}/2$ module with an operator h_0 such that $h_0 + \varepsilon h_0 \varepsilon = 1$ $(h_0 - \frac{1}{2}) + \varepsilon(h_0 - \frac{1}{2}) = 0$

Choose fact. $h_0 = \beta_0 \alpha_0$ two obvious choices $1 \cdot h_0$ $h_0 \cdot 1$

$$E \xrightarrow{\alpha_0} V \xrightarrow{\beta_0} E \xrightarrow{\alpha_0} V$$

Review the formulas. $\Phi = \Gamma$ finite,
 $C = \text{Cuntz}' E_\Gamma$ unital alg gens $h_s, s \in \Gamma$
 rels $\sum h_s = 1$

$B = C \rtimes \Gamma$ unital alg, modules M are v.s.
 with Γ action and an op $h_1, \exists \sum_{s \in \Gamma} s h_s^{-1} = 1$

A alg gens p_s rel $p_s = \sum_t p_{st^{-1}} p_t$

Let's try to put things into order, to make progress on the remaining steps. Go over what you know: ~~start~~ start with a firm B -module E whence you have Γ, h_1 on E . Point: factor: $h_1 = \beta_1 \alpha_1$, two obvious choices $h_1 = 1 h_1$ or $h_1 = h_1 1$. Then put $p_s = \alpha_1 s \beta_1$ and E becomes an A -module. This amounts to a restriction of scalars for a homom. $A \rightarrow B$, ~~so you have~~ provided α_1, β_1 are ~~elements~~ elements of B . So you ~~also~~ get from the above choices

$$p_s \mapsto h_1 s \quad \text{or} \quad p_s \mapsto s h_1$$

Fri - Sun Antwerp Eden Hotel
 Mon + Tues Bruges
 Wed Ostend
 Thurs Ferry home

8:30 am

~~Wtda~~
 4th Ghent - Bruges
 5th Thurs

Center of Bruges parallel to Zuid Zand Str.

$$p_s = h_1 s \quad \sum_t h_1 s t^{-1} h_1 t = h_1 s \quad (\beta \alpha)(\xi) = \sum_s t \beta_1 \alpha_1 t^{-1} \xi = \xi$$

Go over the formulas again

$$h_1 = \beta_1 \alpha_1$$

$$E \xrightarrow{\alpha} C\Gamma \otimes V \xrightarrow{\beta} E$$

$\begin{array}{c} \uparrow \\ \beta \downarrow \\ V \end{array}$

$$(\alpha \xi)_s = \alpha_1 s^{-1} \xi$$

$$\beta \xi = \sum_{t \in \Gamma} t \beta_1 f_t$$

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 10937 am Brugg
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 10:56

Given A -module V , i.e. family $p_s \in \mathcal{L}(V)$

$$p_s = \sum_t p_s t^{-1} p_t. \quad \text{Then you get } p \text{ on } \mathcal{L} \otimes V$$

first version

$$\mathcal{L} \otimes V \xrightarrow{\sum s \otimes p_s} \mathcal{L} \otimes V$$

$$\sum_t t \otimes f_t \longmapsto \sum_{s,t} st \otimes p_s f_t$$

Commutates with right mult of Γ on \mathcal{L}

$$\sum_t t^{-1} \otimes f_t \longmapsto \sum_{s,t} st^{-1} \otimes p_s f_t$$

second version

$$\sum_t t \otimes f_t \xrightarrow{\quad} \sum_{s,t} ts^{-1} \otimes p_s f_t = \sum_s s \otimes \sum_t p_{s^{-1}t} f_t$$

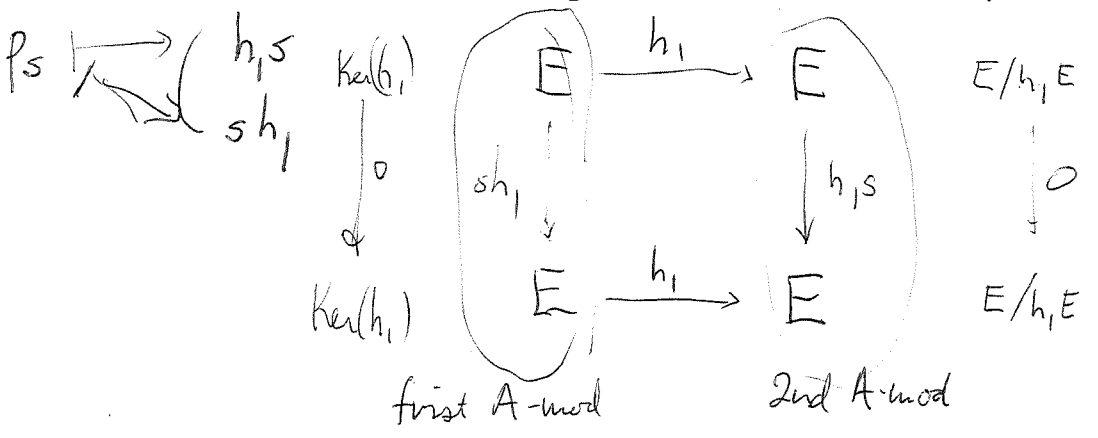
$u = ts^{-1} \quad s = u^{-1}t$

get projection operator

$$(f_t) \longmapsto (pf)_s = \sum_t p_{s^{-1}t} f_t$$

which commutes with left Γ multiplication. Note that the kernel $p_{s^{-1}t}$ is left invariant.

So far you have reviewed the formulas which should yield the desired Morita equivalence, but you are still far from understanding the bimodules. How to get a good grasp of the situation? Look at the ^{two} homoms $A \rightarrow B$ inducing the Morita equivalence



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Review the situation carefully. B is a unital ring defined by ϵ universal mapping property, A similar but non unital. Work with left modules. E a B -module get A module via $p_s = h, s$ or s, h , V an A -module, get B module ~~the~~ $p(\Gamma \otimes V)$, an exact functor of V killing nil modules.

Let's review how a M eq ^{gives rise to and} is given by a dual pair.

$$M(A) \begin{matrix} \xrightarrow{P \circ_A -} \\ \xleftarrow{Q \otimes_B -} \end{matrix} M(B) \quad P = p(\Gamma \otimes \tilde{A})$$

You have the ^{unital} ring $\Gamma \otimes \tilde{A}$ and idemp. p . Can form $E = p(\Gamma \otimes \tilde{A})$, $F = (\Gamma \otimes \tilde{A})p$; these are resp right and left A modules, in fact B, A and A, B bimodules. Since Γ finite these ~~are~~ are fin. proj ~~modules~~ modules over A which should be dual via a pairing $F \times E \rightarrow \tilde{A}$. In this case you know that $E \otimes_A F$ is the ring of endos of E_A

You need to review finite proj modules formalism, maybe ~~even~~ the Vaughan Jones tower, you want better control over the correspondence between a partition of 1 and a projection op

Let E be a R^{op} -module (unital)

$$\mathbb{N} \otimes_R \text{Hom}_{R^{op}}(E, R) \rightarrow \text{Hom}_{R^{op}}(E, \mathbb{N})$$

$$\sum \lambda_i \otimes \lambda_i \longmapsto 1$$

Let E be a unital R^{op} -module, canon map

$$E \otimes_R \text{Hom}_{R^{\text{op}}}(E, R) \longrightarrow \text{Hom}_{R^{\text{op}}}(E, E)$$

$$\sum_s \xi_s \otimes \lambda_s \longmapsto \sum_i \xi_s \lambda_s(-) \stackrel{\text{assume}}{=} \text{id}_E$$

$$E \xrightarrow{(\lambda_s)} R^\Gamma \xrightarrow{(\xi_s)} E. \quad \text{What do you}$$

CTOR

Go over the situations, ~~that is~~ use new ideas,
 that $\text{CT} \otimes \tilde{A}$ is a free ~~fin gen~~ \tilde{A}^{op} -module
 hence $E = p(\text{CT} \otimes \tilde{A})$ is a fin gen proj \tilde{A}^{op} module
 $= p(\text{CT} \otimes A) = p(\text{CT} \otimes \tilde{A})A \therefore$ fin. ~~fin~~

Also there's a dual fin gen ^{proj} fin A^{\otimes} module F
 Automatically $E \otimes_A F \cong \text{End}_{A^{\text{op}}}(E) = \text{End}_A(F)^{\text{op}}$?

You should be able to construct the pairing
 $F \otimes E \longrightarrow A. \quad E = p(\text{CT} \otimes \tilde{A}), F = (\text{CT} \otimes \tilde{A})p$

want $F \otimes E = (\text{CT} \otimes \tilde{A})p \otimes p(\text{CT} \otimes \tilde{A})$

Important to remember that you have explicit direct
 embedding

$$E \longrightarrow \text{CT} \otimes \tilde{A} \longrightarrow E$$

$$F \longleftarrow \tilde{A} \otimes (\text{CT})^* \longleftarrow F$$

There is some formalism here to be worked out
 which doesn't involve Γ as a group, only as set,
 as index set for a grading

Situation: You have a f.g free A^{op} -module $\Gamma \otimes \tilde{A}$ with ~~the~~ projection operator p , whence a fin. gen. proj A^{op} module $E = p(\Gamma \otimes \tilde{A})$, a dual module $F = \text{Hom}_{A^{op}}(E, \tilde{A})$ and ring $B = E \otimes_A F \xrightarrow{\sim} \text{Hom}_{A^{op}}(E, E)$.

All ~~this~~ this you know to be true.



Better is to start with the intrinsic stuff, namely: $E \in \mathcal{P}(A^{op})$, $F = E^\vee$, $E \otimes_A E^\vee \xrightarrow{\sim} \text{Hom}_{A^{op}}(E, E) = B$ and to identify a split embedding $E \rightarrow \Gamma \rightarrow E$ with maps $\Gamma \xrightarrow{\{\lambda_s\}} E$, $\Gamma \xrightarrow{\{\lambda_s\}} E^\vee$ sat $\sum \lambda_s \lambda_s = \text{id}_E$. There ~~might~~ ^{might} be some ^{interesting} category of partitions of unity. In the case $\Gamma = \text{finite group}$ you have an equivariant partition of unity

Review. In the case Γ finite you have the proper picture of the desired Morita equivalence, namely, $\begin{pmatrix} A & F \\ E & B \end{pmatrix}$ where B is unital. ~~In this~~

~~The~~ Problem: If you start with B, F_B, B^E you seem to have problems with showing your A is $F \otimes_B E$. So it seems better to start with A . The point

Over A you have a canonical ^{form} dual pair $E_{A, A}^F$ where $E \in \mathcal{P}(A^{op})$, $F = E^\vee \in \mathcal{P}(A)$ and $E^\vee E = A$.

Actually you ~~still~~ still seem to have problems unless you can show A is firm

So how do you get started? Philosophy of partition of unity needs to be ~~worked~~ worked out.

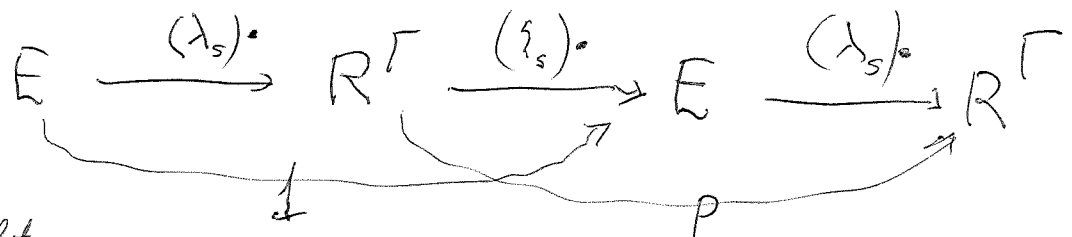
$$E \otimes_R \text{Hom}_{R^{op}}(E, R) \longrightarrow \text{Hom}_{R^{op}}(E, E)$$

$$\{\lambda\} \otimes \lambda \longmapsto (\lambda' \longmapsto \{\langle \lambda, \lambda' \rangle\})$$

Assume $\mathbb{1}_E \in \text{Hom}_{R^{\text{op}}}(E, E)$ is nuclear

choose ξ_s, λ_s such that $\sum \xi_s \otimes \lambda_s \mapsto \mathbb{1}_E$

i.e. $\xi = \sum_{s \in \Gamma} \xi_s \langle \lambda_s, \xi \rangle \quad \forall \xi \in E.$



$\rho =$ ~~right~~ left multiplication by $(\lambda_s \xi_t)_{s,t \in \Gamma}$

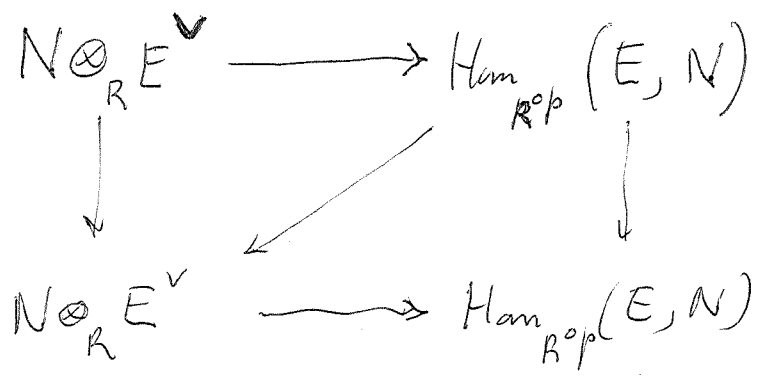
$\rho^2 = \sum_u \lambda_s \xi_u \lambda_u \xi_t = \lambda_s \xi_t$
 $\therefore \rho^2 = \rho$

Point to understand: Why $\sum_s \xi_s \lambda_s = \mathbb{1}$ implies $N \otimes_R E^V \xrightarrow{\sim} \text{Hom}_{R^{\text{op}}}(E, N)$ for all N .

Surjectivity should be immediate. Given $\varphi: E \rightarrow N$

you have $\varphi = \varphi \sum_s \xi_s \lambda_s = \sum_s \varphi(\xi_s) \lambda_s$ comes

from $\sum_s \varphi(\xi_s) \otimes \lambda_s \in N \otimes_R E^V$



$n \otimes \lambda \mapsto (\xi \mapsto n \otimes \lambda \xi)$
 $\sum_s n \otimes \lambda(\xi_s) \otimes \lambda_s$
 \parallel
 $n \otimes \lambda(\sum_s \xi_s \lambda_s)$

$$N \otimes_R E^V \longrightarrow \text{Hom}_{\text{Rop}}(E, N) \longrightarrow N \otimes_R E^V$$

$$n \otimes \lambda \longmapsto (\xi \mapsto n \langle \lambda, \xi \rangle), \quad \varphi \longmapsto \sum_s \varphi(\xi_s) \otimes \lambda_s$$

$$n \otimes \lambda \longmapsto \sum_s n \langle \lambda, \xi_s \rangle \otimes \lambda_s = \sum_s n \otimes \langle \lambda, \xi_s \rangle \otimes \lambda_s$$

so you need to know why $\sum \langle \lambda, \xi_s \rangle \lambda_s = \lambda$?

i.e. $\sum_s \langle \lambda, \xi_s \rangle \langle \lambda_s, \xi \rangle = \langle \lambda, \xi \rangle$?

$$\langle \lambda, \sum_s \xi_s \langle \lambda_s, \xi \rangle \rangle$$

Finally given $\varphi: E \rightarrow N$, you want to know

$$\xi \mapsto \sum_s \varphi(\xi_s) \langle \lambda_s, \xi \rangle \stackrel{?}{=} \varphi(\xi) \quad \text{clear by applying}$$

$$\varphi \text{ to } \sum_s \xi_s \langle \lambda_s, \xi \rangle = \xi$$

A generators p_s , $s \in \Gamma$ relations $p_s = \sum_{t \in \Gamma} p_{st^{-1}} p_t$

Consider $A \otimes_A A$. Can you use the universal mapping property to construct a homomorphism $A \rightarrow A \otimes_A A$ such that $\phi \mu = \text{id}$. It would seem that you want

$$p_s \mapsto \sum_{t \in \Gamma} p_{st^{-1}} \otimes p_t. \quad \text{Thus } \phi(p_s) = \sum_{s=t_1} p_{t_1} \otimes p_{u_1}$$

Are the relations satisfied?

~~the relations are not satisfied~~

$$\phi(p_s) = \sum_{s=t_1} p_{t_1} \otimes p_{u_1} = \sum_{s=t_1} \sum_{t_2} \sum_{u_2} p_{t_1} p_{t_2} \otimes p_{u_1} p_{u_2}$$

||?

$$\sum_{s=t_1} \phi(p_{t_1} p_{u_1}) = \sum_{s=t_1} \sum_{t_2} (p_{t_1} \otimes p_{t_2}) (p_{u_1} \otimes p_{u_2})$$

$$p_{t_1} \otimes p_{t_2} p_{u_1} p_{u_2} = p_{t_1} p_{t_2} \otimes p_{u_1} p_{u_2}$$

Be careful, you seem to be able to show that $A = A_\Gamma$ is a firm ring, by constructing a homom. $\phi: A \rightarrow A \otimes_A A$ such that $\mu \phi = 1$.

put $\phi(p_s) = \sum_{s=tu} p_t \otimes p_u$ so that $\mu \phi(p_s) = p_s \quad \forall s$

Need to check relations

$$\phi(p_s) \stackrel{?}{=} \sum_{s=tu} \phi(p_t) \phi(p_u)$$

$$\phi(p_t) \stackrel{\text{def}}{=} \sum_{t=t_1 t_2} p_{t_1} \otimes p_{t_2}$$

$$\phi(p_u) = \sum_{u=u_1 u_2} p_{u_1} \otimes p_{u_2}$$

$$\begin{aligned} \phi(p_t) \phi(p_u) &= \sum_{t=t_1 t_2} \sum_{u=u_1 u_2} p_{t_1} \otimes p_{t_2} p_{u_1} p_{u_2} \\ &= \sum_{t=t_1 t_2} \sum_{u=u_1 u_2} p_{t_1} p_{t_2} \otimes p_{u_1} p_{u_2} \\ &= \sum_{t=t_1 t_2} p_t \otimes p_u \quad ? \end{aligned}$$

$$\therefore \sum_{s=tu} \phi(p_t) \phi(p_u) = \sum_{s=tu} p_t \otimes p_u = \phi(p_s)$$

So we know now that A is a firm ring, which should complete the proof that $\begin{pmatrix} A & F \\ E & B \end{pmatrix}$ is a firm Morita context. Except you need to check $FE = A$.

$$E \xrightarrow{\begin{smallmatrix} \text{col} \\ (\lambda_s) \end{smallmatrix}} \begin{smallmatrix} \text{row} \\ \Gamma \otimes A \end{smallmatrix} \xrightarrow{\begin{smallmatrix} \text{col} \\ (\xi_t) \end{smallmatrix}} E \xrightarrow{\begin{smallmatrix} \text{col} \\ (\lambda_s) \end{smallmatrix}} \begin{smallmatrix} \text{row} \\ \Gamma \otimes A \end{smallmatrix}$$

column vectors

FE should be the ideal gen. by $\alpha_i s^{-1} t \beta_i = p_{s^{-1}t}$

so it's clear.

What else do you want to understand?

~~Supports~~ supports - A_Γ . More control over E as B module

infinite case. A plus $p_s, s \in \Gamma$ rels $p_s = 0 \quad s \notin \Phi$

$p_s = \sum_{s=tu} p_t p_u$, this sum finite as $\{(t,u) \mid t \in \Phi, u \in \Phi\} = \Phi \times \Phi$

is finite. Firmness should be proved similarly: Define

$\phi: A \rightarrow A \otimes_A A$ by putting $\phi(p_s) = \sum_{s=tu} p_t \otimes p_u$. Check

rels. $\phi(p_t) \otimes \phi(p_u) = \sum_{t=t_1 t_2} (p_{t_1} \otimes p_{t_2}) \otimes \sum_{u=u_1 u_2} (p_{u_1} \otimes p_{u_2})$

$= \sum_{\substack{t=t_1 t_2 \\ u=u_1 u_2}} p_{t_1 t_2} \otimes p_{u_1 u_2} = p_t \otimes p_u$. So $\sum_{s=tu} \phi(p_t) \phi(p_u) =$

$\sum_{s=tu} p_t \otimes p_u = \phi(p_s)$ showing the main rels holds.

What about Φ support? Assume $s \in \Phi$

Your notation is confusing. Put $\hat{p}_s = \sum_{s=tu} p_t \otimes p_u \in A \otimes_A A$

Then $\hat{p}_t \hat{p}_u = \sum_{t=t_1 t_2} \sum_{u=u_1 u_2} (p_{t_1} \otimes p_{t_2}) (p_{u_1} \otimes p_{u_2}) = p_t \otimes p_u$

$\therefore \sum_{s=tu} \hat{p}_t \hat{p}_u = \sum_{s=tu} p_t \otimes p_u = \hat{p}_s$

Prolongation for Φ ?

Look at $\Gamma = \mathbb{Z} \quad \Phi = \{-1, 0, 1\}$. Question: Is there some ~~super~~ supervision reminiscent of the way the irred reps of ~~the~~ sl_2 arise from the fund. repr.

~~Look at philosophy.~~ $\begin{pmatrix} A & F \\ E & B \end{pmatrix}$. B will not be unital, but ~~the~~ will have local units, E and F will be firm over B equiv. $E = BE, F = FB$.

~~Again~~ Again you should start with A, then E will be a summand of $\Gamma \otimes \tilde{A}$, a firm projective A^{ϕ} module, F summand $\tilde{A} \otimes \Gamma$ A module and there should be an obvious pairing

$F \otimes E \rightarrow A$. ~~you have a dual pair~~ ~~over A~~ ~~both modules flat~~ \therefore you have a dual pair ~~over A~~ over A , both modules flat, so the ring $B = E \otimes_A F$ will be both left and right flat, which checks with local units [ring].

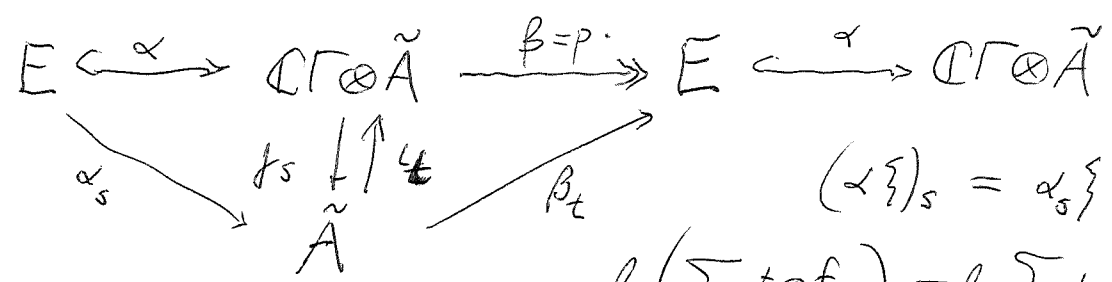
Point to check: ~~that~~ $E \otimes_A F = C \times I$.

You need A ~~to~~ to be a firm ring in order that the canonical map $F \otimes_B E \rightarrow A$ be an isom.

How does the argument go?

You are trying to understand the ~~infinite~~ infinite case where a support Φ , or system of supports, is given. In this case B should have local units

Review: Γ finite set, A universal ring ~~with~~ equipped with a projector ~~$p \in A$~~ p on the free A^{op} -module $\mathbb{C}\Gamma \otimes \tilde{A}$, equiv. p is a matrix $(p_{st})_{s,t \in \Gamma}$ satisfying $p_{su} = \sum_t p_{st} p_{tu}$. Put $E = p(\mathbb{C}\Gamma \otimes \tilde{A})$. There is a dual free A -module $\text{Hom}_{A^{\text{op}}}(\mathbb{C}\Gamma \otimes \tilde{A}, \tilde{A}) = \tilde{A} \otimes (\mathbb{C}\Gamma)^\vee$ to $\mathbb{C}\Gamma \otimes \tilde{A}$, and a corresponding dual $E^\vee = \text{Hom}_{A^{\text{op}}}(E, \tilde{A})$ which should be something like $(\tilde{A} \otimes (\mathbb{C}\Gamma)^\vee)_p$ (maybe ${}^t p$?)



$$\beta \left(\sum_t {}^t \circ f_t \right) = \beta \sum_t \eta f_t = \sum_t \beta_t f_t$$

simpler might be $\alpha_s = \sum$

$$\begin{array}{ccc} \mathbb{C}\Gamma \otimes \tilde{A} & \xrightarrow{(a_{st})} & \mathbb{C}\Gamma \otimes \tilde{A} \\ \downarrow \uparrow \iota_t & & \downarrow \uparrow \iota_s \\ \tilde{A} & & \tilde{A} \end{array}$$

$$a_{st} = f_s a \iota_t$$

Let $f \in \mathbb{C}\Gamma \otimes \tilde{A}$. Then

$$f = \sum_t i_t f_t$$

$$f_t = \iota_t f$$

$$af = \sum_{s,t} \iota_s f_s a \iota_t f_t = \sum_{s,t} i_s a_{st} f_t = af$$

basis ι_t

$$\mathbb{C}\Gamma \otimes \tilde{A}$$

row vectors

$$\downarrow \uparrow \iota_t \\ \tilde{A}$$

$$\tilde{A} \otimes \mathbb{C}\Gamma$$

basis f_t for functions on Γ .

column vectors.

what's important is the pairing Kronecker δ

$$(\tilde{A} \otimes \mathbb{C}\Gamma) \otimes (\mathbb{C}\Gamma \otimes \tilde{A}) \longrightarrow \tilde{A}$$

$$\left\langle \sum_s g_s f_s, \sum_t \iota_t f_t \right\rangle = \sum g_s \delta_{st} f_t$$

What remains?

$$\langle f_s, \iota_t \rangle = f_s \iota_t = \begin{cases} 1 & s=t \\ 0 & s \neq t \end{cases}$$

$$\begin{aligned} \langle g, af \rangle &= \sum_s g_s f_s \sum_t \overbrace{f_t a \iota_t}^{a_{st}} \iota_t f_t \\ &= \sum \quad ? \end{aligned}$$

There is something peculiar about this image (perhaps), some angle of an image



$$\langle g, af \rangle = \sum_s g_s f_s a f = \sum_{s,t} g_s a_{st} f_t$$

$$\langle g^a, f \rangle = \sum_{s,t} g_s f_s a_{st} f_t \quad \text{should be clear.}$$

What is the goal?

You have a finite set Γ , $A =$ universal ring equipped with an idempotent operator p on the free A^{op} -module $\mathbb{C}\Gamma \otimes \tilde{A}$. This module is characterized by A^{op} -module maps

$$\begin{array}{c} \mathbb{C}\Gamma \otimes \tilde{A} \\ \downarrow f_s \quad \uparrow l_s \\ \tilde{A} \end{array}$$

Satisfying $f_s l_t = \delta_{st}$

$$\sum l_s f_s = \mathbb{1}$$

Think of $\mathbb{C}\Gamma \otimes \tilde{A}$ as the A^{op} -module of column vectors over \tilde{A} , where Γ indexes the columns.

Dual A -module $\tilde{A} \otimes (\mathbb{C}\Gamma)^n$ with basis f_s , think of it as row vectors.

Put $P = \mathbb{C}\Gamma \otimes \tilde{A}$
column vector

$$\begin{array}{c} P \\ \downarrow f_s \quad \uparrow l_s \\ \tilde{A} \end{array}$$

l_s is the ^{good} basis for P

f_s is a dual basis for $P^\vee = \tilde{A} \otimes (\mathbb{C}\Gamma)^n$

$$\phi \in \text{Hom}_{A^{op}}(P, P) = P \otimes_A P^\vee$$

$$\sum l_s f_s \phi l_t f_t$$

~~What~~ What were you doing? Γ finite set 223

$E = \mathbb{C}\Gamma \otimes \tilde{A} =$ free A^{op} -module, basis Γ

canon. maps $\tilde{A} \xrightarrow{\iota_s} E \xrightarrow{f_t} \tilde{A}$

satisfying $f_t \iota_s = \delta_{ts}$, $\sum_s \iota_s f_s = 1$

$F = E^\vee = (\mathbb{C}\Gamma \otimes \tilde{A})^\vee = \tilde{A} \otimes \mathbb{C}\Gamma^\vee$ free A -module

with basis $f_t, t \in \Gamma$ canon maps $\tilde{A} \xrightarrow{f_t^*} E^\vee \xrightarrow{\iota_s^*} \tilde{A}$

$\iota_s^* f_t^* = \delta_{st}$, $\sum_s f_s^* \iota_s^* = 1$

Let p be an idemp. of an E .

$$p = \sum_{s,t \in \Gamma} \iota_s f_s p \iota_t f_t$$

$\underbrace{\iota_s f_s p \iota_t f_t}_{p_{st} \in \tilde{A}}$

$$p^* = \sum_{s,t} f_t^* \iota_t^*$$

~~UN~~

$$E_0 = \mathbb{C}\Gamma \otimes \tilde{A} \begin{array}{c} \xrightarrow{f_s} \\ \xleftarrow{\iota_t} \end{array} \tilde{A}$$

$$E_0^\vee = \tilde{A} \otimes (\mathbb{C}\Gamma)^\vee \begin{array}{c} \xleftarrow{f_s^*} \\ \xrightarrow{\iota_t^*} \end{array} \tilde{A}$$

$$f_s \iota_t = \delta_{st}$$

$$\sum_s \iota_s f_s = 1$$

$$\iota_t^* f_s^* = \delta_{st}$$

$$\sum_s f_s^* \iota_s^* = 1$$

$p \in \text{Hom}_{A^{\text{op}}}(E_0, E_0)$

$$E = pE_0$$

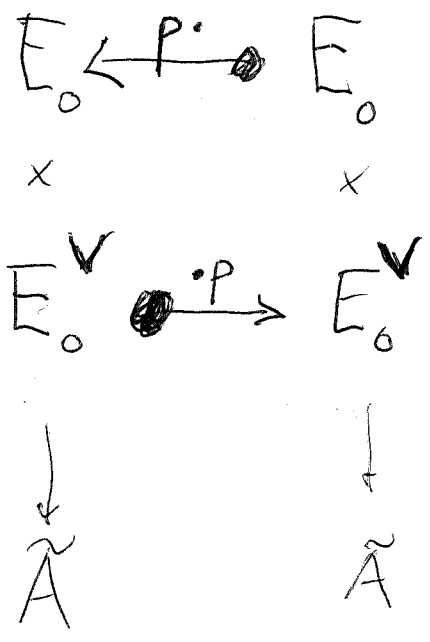
$$p = \sum_{s,t} \iota_s \underbrace{(f_s p \iota_t)}_{p_{st} \in A} f_t$$

~~are~~ elements of E_0
are $\sum_s \iota_s \{f_s\}$ $\{f_s = f_s\}$

$$E_0 = \left\{ \sum_{s,t} l_s p_{s,t} \xi_t \mid \xi_t \text{ arb.} \right\}$$

$$E_0 = \left\{ \sum_s i_s g_s \mid g_s = \sum_t p_{s,t} f_t, f_t \in \tilde{A} \right\}$$

$$E = p(\text{col. vectors} \otimes \tilde{A})$$



$$\langle \eta, p \xi \rangle = \sum_s \eta_s p_{s,t} \xi_t$$

$$= \langle \eta p, \xi \rangle$$

$$\xi = \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_n \end{pmatrix} \quad \eta = (\eta_1 \dots \eta_n)$$

A acts on right A acts on left

Question: Do you expect a pairing between pE_0 and $E_0^v p$, specifically

$$\langle \eta, \xi \rangle = \sum_s \eta_s \xi_s$$

$$E_0^v p \times p E_0 \longrightarrow \tilde{A}$$

$$\langle \eta, p \xi \rangle = \langle \eta p, \xi \rangle$$

depends only on $(\eta p, p \xi) \in E_0^v p \times p E_0$

Let ~~...~~ $\sigma \in E_0^v p, \tau \in p E_0$

choose $\xi_0 \in E_0$ s.t. $p \xi_0 = \tau$

$\eta_0 \in E_0^v$ $\eta_0 p = \sigma$

Consider

This pairing between the images E_{op}^v and pE_0 225 involves inserting p^{-1} ~~in some sense~~ in some sense and ^{then} using the pairing between E_0^v and E_0 . When $p^2 = p$ the p^{-1} is unnecessary.

so ~~perhaps~~ perhaps there is a canonical pairing $Bh_0 \times h_0 B \longrightarrow Bh_0 B$

$$\langle b_1 h_1, h_1 b_2 \rangle = (b_1 h_1) b_2$$

~~with~~ Go back to Γ finite, start with $B = C \rtimes \Gamma$ C : ^{unital} gen $h_s, s \in \Gamma$, rel $\sum h_s = 1$

B is unital, get dual pair $(Bh_1, h_1 B)$ with above pairing $\langle b h_1, h_1 b' \rangle = b h_1 b'$. Ideal

of lin. comb. of ^{such} inner products is $Bh_1 B$, which is B . So you have a firm dual pair with B unital, get a Meg $\begin{pmatrix} A & h_1 B \\ Bh_1 & B \end{pmatrix}$

$A = h_1 B \otimes_B Bh_1$ is a firm ring by general theory $Bh_1 \in \mathcal{P}(A)$ $h_1 B \in \mathcal{P}(A)$ are dual f. proj. modules over A .

Note that the ^{mult.} map $A = h_1 B \otimes_B Bh_1 \longrightarrow h_1 B h_1$ is not a ring homom, because in the pairing $Bh_1 \otimes_2 h_1 B \longrightarrow B$ you "divide" by h_1

mult in A ~~Use h for h~~ Use h for h,

$$A = hB \otimes_B Bh \longrightarrow hBh$$

$$hb_1 \otimes b_2 h \longmapsto hb_1 b_2 h$$

$$(hb_1 \otimes b_2 h) \underset{\wedge}{*} (hb_3 \otimes b_4 h) \quad (hb_1 b_2 h) * (hb_3 b_4 h)$$

||

$$hb_1 \otimes b_2 h b_3 b_4 h \quad hb_1 b_2 h b_3 b_4 h$$

Thus hBh is a ring with product

$$(hb_1 h) \underset{\wedge}{*} (hb_2 h) = hb_1 h b_2 h$$

~~and the A maps onto hBh~~

and then there is a surjective homom.

$$A = hB \otimes_B Bh \longrightarrow hBh$$

with respect to * product

Discuss the situation. I think the above concerns a unital B with $\Gamma \rightarrow B^*$ and h such that $\sum_{s \in \Gamma} s h s^{-1} = 1$.

Better: You have used only B unital, $h \in B$

$$BhB = B \text{ to get } \begin{pmatrix} B & Bh \\ hB & A = hB \otimes_B Bh \end{pmatrix}$$

Back to M. eq. You had this observation that $\langle bh, hb' \rangle = bhb'$ is a well defined pairing between Bh and hbB yielding a Morita equiv. between BhB and $hbB \otimes_B Bh$. But

Let's work on the details using our initial Morita equivalence. $B = C \rtimes \Gamma$ C unital h_s $\sum h_s = 1$.

A B -module is a Γ -module with op. h satisfying $\sum s h s^{-1} = 1$. An A -module is a v.s. V equipped with $p = p^2$ on $C\Gamma \otimes V$. Functor $E(V) = p(C\Gamma \otimes V)$

What do you hope for?
 Maybe the grading will be important

Go back to a finit B -module E , i.e. a Γ -module equipped with $h = h_s$ such that $\sum_{s \in \Gamma} s h_s s^{-1} = 1$. Associate to E the image $h_1 E$ which should naturally be an A -module. Why? You have $E \xrightarrow{\alpha_1 = h_1} h_1 E \xleftarrow{\beta_1 = \text{id}} E$, and Γ action on E , hence

$$\begin{array}{ccc}
 E & \xrightarrow{\alpha} & C\Gamma \otimes h_1 E \xrightarrow{\beta} E \\
 \{ & \longmapsto & \sum s \otimes \alpha_s s^{-1} \} \longmapsto \sum s \beta_s \alpha_s s^{-1} \} = \{ \\
 & & \sum s \otimes f_s \xrightarrow{\beta} \sum_{s \in \Gamma} s \beta_s f_s
 \end{array}$$

So what's important is the maps $E \xrightarrow{\alpha_1 = h_1} h_1 E \xleftarrow{\beta_1 = \text{id}} E$ then $p_s = \alpha_1 s \beta_1$ on $h_1 E$.

$$\sum_t p_s \otimes t^{-1} p_t = \sum_t \alpha_1 s t^{-1} \beta_1 \alpha_1 t \beta_1 = \alpha_1 s \beta_1 = p_s$$

Can you prove that
is an A -mod isom.

$$h_1 B \otimes_B E \longrightarrow h_1 E$$

$$\begin{array}{ccc}
 B \otimes_B E & \xrightarrow{\sim} & E \\
 h_1 \downarrow & & h_1 \downarrow \\
 h_1 B \otimes_B E & \xrightarrow{\quad} & h_1 E \\
 \downarrow & & \updownarrow \\
 B \otimes_B E & \xrightarrow{\sim} & E
 \end{array}$$

$p_s = h_1 s$

Assume $\sum h_1 b_i \otimes \xi_i \mapsto \sum h_1 b_i \xi_i = 0$

$\alpha, \beta \parallel \downarrow$

$$\sum_i h_1 s h_1 b_i \otimes \xi_i$$

Clearer. ~~is~~ $h_1 B \otimes_B E \longrightarrow h_1 E$

is onto. Let $h_1 \xi \in h_1 E$, as $B \otimes_B E \xrightarrow{\sim} E$, you have $\xi = \sum b_i \otimes \xi_i$ so $h_1 \xi = \sum h_1 b_i \otimes \xi_i$ which comes from $\sum h_1 b_i \otimes \xi_i \in h_1 B \otimes_B E$. If $\sum h_1 b_i \otimes \xi_i \mapsto 0$ in $h_1 E$, i.e. $\sum h_1 b_i \xi_i = 0$, then

$$\alpha, \beta \parallel \downarrow \quad h_1 s \sum_i h_1 b_i \otimes \xi_i = \sum_i h_1 s h_1 b_i \otimes \xi_i$$

now the next step is to go from an A-module V to $E(V) = p(\mathbb{C}\Gamma \otimes V)$.

At some point you have to relate Bh_s to $E(A)$

I want to finish Γ finite today.

$$B = \mathbb{C} \rtimes \Gamma \quad \begin{array}{l} C \text{ unital alg} \\ \text{gens. } h_s, s \in \Gamma \\ \text{rel } \sum_{s \in \Gamma} h_s = 1 \end{array} \quad t h_s t^{-1} = h_{ts}$$

C, B are unital.

$$A \text{ gens } p_s, s \in \Gamma \quad p_s = \sum_{s=ts} p_t p_u$$

~~Explicit~~ Explicit functors.

$$\text{Mod}(B) \rightleftharpoons \mathfrak{m}(A)$$

$$N \quad M$$

$$N \longmapsto h_s N \quad \text{with } p_s(h_s N) = h_s h_s N$$

$$\sum_{s=ts} p_t p_u (h_s \eta) = \sum_{\substack{s=ts \\ t}} h_t h_u h_s \eta = h_s h_s \eta = p_s(h_s \eta)$$

Also you ~~have~~ have an A-nil isom., surj.

$$hB \otimes_B N \longrightarrow hN$$

Ass $\sum_i h b_i \otimes \eta_i \longmapsto \sum_i h b_i \eta_i = 0.$

then $\sum_i p_s(h b_i) \otimes \eta_i = \sum_i h_s h b_i \otimes \eta_i = \sum_i h_s \otimes h b_i \eta_i$

Anyway you have ~~functors~~ functors between modules, an explicit Morita equivalence,

Given a (firm) B-module W, you send it to the A-module $h_! W$, or to $h_! B \otimes_B W$, and given an A-module V, you send it to $p_!(\Gamma \times V)$ with suitable B module structure.

Maybe you want an intermediate category of ${}_B W, A V$, $W \begin{matrix} \xrightarrow{\alpha} \\ \beta \end{matrix} V$ such

that $\beta \alpha = h$ on W $\alpha \beta = p_s$ on V

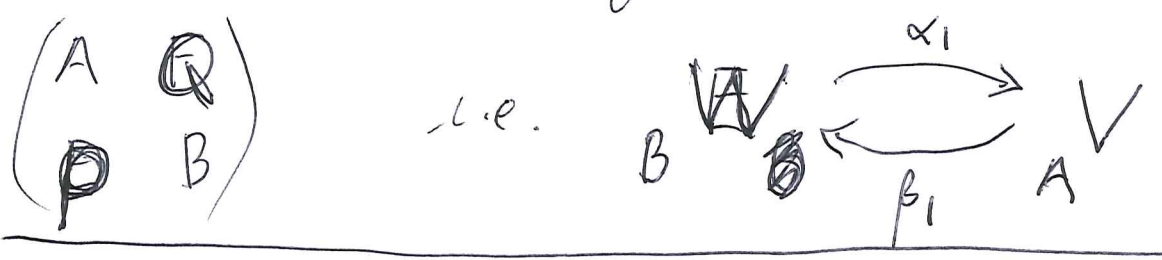
$$D = \begin{pmatrix} A & hB \\ B & B \end{pmatrix}$$

You probably want $\begin{pmatrix} A & \alpha B \\ B & B \end{pmatrix}$

$$D = \begin{pmatrix} A & F \\ E & B \end{pmatrix} = \begin{pmatrix} A \\ E \end{pmatrix} \otimes_A \begin{pmatrix} A & F \end{pmatrix}$$

~~Then~~ Then $\begin{pmatrix} A \\ E \end{pmatrix} \otimes_A V$ is a typical D-mod.
 this ~~should~~ should be nil equiv. to $\begin{pmatrix} V \\ E \otimes_A V \end{pmatrix}$
 what ~~about~~ about moncentral GNS.

Review. ~~Last idea~~ was to introduce modules for the Morita context.



~~work out.~~ If you can concentrate everything should work out. B is the unital alg gen by Γ and $h = h_1$ satisfying $\sum s h s^{-1} = 1$. A finit B module W is a Γ -module with such an operator h .

(It seems that there is a GNS situation where modules are $B \overset{\alpha}{\rightleftarrows} A$ with $\beta \alpha = h$ (not 1))

~~Now all you have to do is~~

So introduce the category of $B \overset{*}{\rightleftarrows} A$

satisfying $i j = h, j s i = p_s$.

~~Given A, B~~

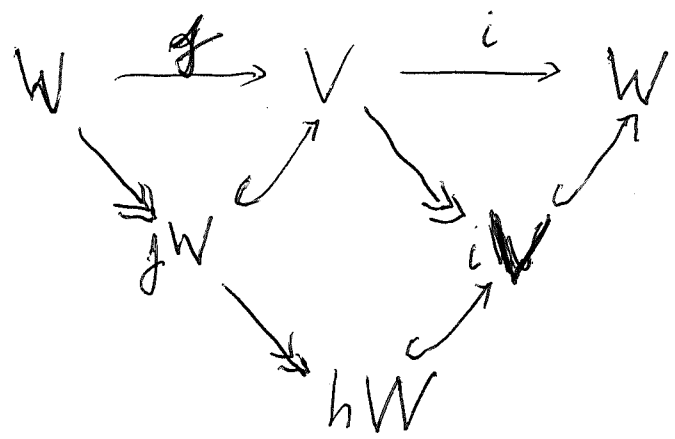
Start again with the cat of $B \overset{j}{\rightleftarrows} A$
 $i j = h, j s i = p_s$

recall A alg w gens $p_s, s \in \Gamma$ rels
 $p_s = \sum_{t=1}^n p_t p_u$

~~Suppose given B^W , look at the possible extensions (V, ι, j) .~~

Suppose given B^W ,

look at the possible ~~extensions~~ extensions (V, ι, j) . ~~Answer:~~ Answer: Any vector space with op. $W \xrightleftharpoons[\iota]{f} V$ sat $if = h$. Why? because the A -mod structure is determined by $jsi = p_s$.



There is a smallest choice for V namely hW .
 $i =$ inclusion $hW \hookrightarrow W$
 $f = h: W \twoheadrightarrow hW$.

~~$p_s(hw) = js i(hw) = hshw$~~

$s = tu$
 ~~$t^s = u$~~
 $t^{-1}s$

$p_t p_u(hw) = p_t h u h w = h t h u h w$

$\sum_t h t h t^{-1} s h w = h s h w = p_s(hw)$

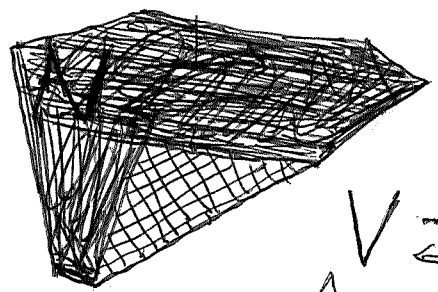
Next given A^V you ~~are~~ want to understand all B^W equipped with

$W \xrightarrow{j} V \xrightarrow{\iota} W$
 $W \rightarrow \mathcal{C}T \otimes V$

$${}_B W \xrightarrow{f} V \quad \text{equiv.} \quad W \rightarrow \text{Hom}(B, V)$$

$$V \xrightarrow{i} W \quad \sim \quad B \otimes V \rightarrow W \quad ?$$

In initial GNS situation you have



$$f^i = 1$$

$${}_A V \xrightleftharpoons[f]{i} {}_B W$$

$$f^i = 1$$

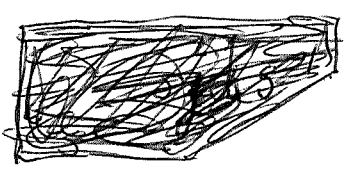
$$p: B \rightarrow A$$

$$p(b) \omega = f b \omega$$

It seems as if you want W to be primarily a Γ -module

Basic categories: ① Γ module, ${}_A W$ equipped with ~~an~~ operator h such that $\sum_{s \in \Gamma} s h s^{-1} = 1$

Enlarge to ② ${}_A W \xrightleftharpoons[i]{f} V$ such that



$$\sum_{s \in \Gamma} s i f s^{-1} = 1_W$$

(adjoining a factorization $h = ij$ of h).

③ ${}_A V$ i.e. $p_s \in \mathcal{L}(V)$ $s \in \Gamma$ etc.

- ① ΓW equipped with ϕ h sat $\sum_s s h s^{-1} = 1$
 - ② $\Gamma W \xrightleftharpoons[\gamma]{\phi^i} V$ $ij = h$ $\sum_s s i \gamma s^{-1} = 1$
 - ③ A^V i.e. $p_s \in LV, s \in \Gamma$ $p_s = \sum_t P_t P_t^{-1} s$
- $p_s = \gamma s i$, check $\sum_t \gamma (t i \gamma^{-1} t^{-1}) s i = \gamma s i = p_s$

You should understand these ^{obvious} functors as restriction wrt certain homs.

$$B \hookrightarrow \begin{pmatrix} B & B i \\ \gamma B & A \end{pmatrix} \longleftrightarrow A$$

You ~~should~~ expect to get an isom.

$$B i \otimes_A \gamma B \xrightarrow{\sim} B$$

Actually you have an explicit ^{form} dual pair over B namely ${}_B B i, \gamma B_B$, with $\langle b i, \gamma b' \rangle = b h b'$

and you have the other pairing

$$\gamma b' \otimes_B b i \longmapsto \gamma b' b i$$

What is $\gamma b i$? e.g. ~~$\gamma h i$~~ $\gamma h i = \gamma \gamma i = h^2$
 b confusing

~~Module~~ You have a category of modules
 namely $\Gamma W \xrightleftharpoons[i]{f} V$ such that $\sum_{S \in T} s c f s^{-1} = 1_W$

Maybe you want to choose a small unital projective generator. This should yield the ring

$\begin{pmatrix} B & B_i \\ jB & \tilde{A} \end{pmatrix}$. Can you prove this?

OKAY begin with the modules

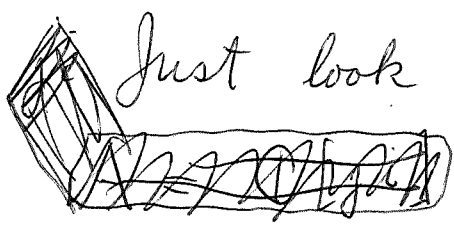
Basically you have

It looks like you want to consider again the relation $ij = e$, $e^2 = e$. This is the case $\Gamma = \{i, j\}$.

~~Module~~ Claim that the category of $W \xrightleftharpoons[i]{f} V$ such that $ij = 1_W$ is the category of modules over the unital ring with generator i, j relation ~~$ij = i$ $ji = j$~~ ?
 $(ij)^2 = ij$

On $W \oplus V$ you have operators $e, 1-e, i, j$ satisfying ~~$ij = i$ $ji = j$~~ $f^e = f = (1-e)f$

modules are $W \begin{matrix} \xrightarrow{j} \\ \xleftarrow{i} \end{matrix} V$ $uj = I_W$



Just look at $\Gamma = 1$. $B = \mathbb{C}$
 A gen by $p_i = j \circ i$ rel.

$$p_i = \sum_{l=st} p_s p_t \quad \text{i.e.} \quad p_i^2 = p_i$$

$$\begin{pmatrix} \mathbb{C} & \mathbb{C}i \\ j\mathbb{C} & \mathbb{C}e \end{pmatrix}$$

$$uj = 1, \quad e = ji = e^2$$

Finite Γ .

~~curious relations.~~ $\Gamma = 1$
 $A = \mathbb{C}p$ the non-unital ^{alg.} gen by an idemp. p
 $B =$ unital ring \mathbb{C} these seem to be
~~the same.~~

$$1, e, uj, \quad uj = e = e^2$$

$$\begin{pmatrix} yi = i \\ j \circ y = j \end{pmatrix}$$

$$W \begin{matrix} \xrightarrow{j} \\ \xleftarrow{i} \end{matrix} V \quad uj = 1$$

$$W \quad j(W \oplus \text{Ker } i)$$

Consider the abelian cat of ~~modules~~ diagrams

$$i: W \rightleftarrows V \quad \sum_{s \in \Gamma} s \circ j \circ s^{-1} = 1_W$$

Such diagrams should be equivalent to unital modules for a unital ring. ~~modules~~ The functor sending a diagram to $W \oplus V$ should be represented by a small projective generator P in the category of these diagrams, and the unital ring should be $\text{End}(P)$. There should be a ~~small~~ small projective for each vertex.

$$\Gamma = 1. \quad W \rightleftarrows V \quad \circ j = 1_W$$

have f_i on V satisf. $f_i \circ f_i = f_i$

~~the~~ better $f_i \circ j = j$ and $\circ f_i = i$. so the ring $\text{End}(P_W \oplus P_V)$ should be $\begin{pmatrix} \mathbb{C} & \mathbb{C}i \\ \mathbb{C}j & \mathbb{C} \oplus \mathbb{C}j \end{pmatrix}$

~~the~~ ~~same~~ ~~as~~ ~~modules~~ ~~for~~ ~~the~~ ~~unital~~ ~~ring~~

Go over simplest case: $W \rightleftarrows V \quad \circ j = 1_W$

~~the~~ same as modules for the unital ring

$$\begin{pmatrix} \mathbb{C} & \mathbb{C}i \\ j\mathbb{C} & \mathbb{C} \oplus \mathbb{C}j \end{pmatrix} \quad \circ j = 1$$

better look at the nonunital ring generated by

i, j rels. $\circ j = 1, i^2 = 0, j^2 = 0, j \circ i = j$

words in the generators ~~which~~ must avoid i^2, j^2
two gen. ij must alt. i, j, ij, j^2

$$i, ij, (ij)^2 = i$$

$$j, j^2, (j^2)^2 = j$$

So you get non unital ring $\begin{pmatrix} \mathbb{C} & \mathbb{C} \\ \mathbb{C} & \mathbb{C} \end{pmatrix}$

$$(ij)^2 = ij, (ij)^3 = i, j(ij) = j, ij = ij$$

$$(j^2)^2 = j^2, (j^2)^3 = j, i(j^2) = i, j^2 = j^2$$

which happens to be unital.

next might be Toeplitz type algebra. (non comm.)

So what is the Toeplitz algebra again for a vector space T . Alg of left ops on $T(V)$ generated by left mult. by v , ~~interior~~ contraction with $\lambda \in V^*$

$$e_v(v_1 \otimes \dots \otimes v_n) = v \otimes v_1 \otimes \dots \otimes v_n$$

$$L_\lambda(v_1 \otimes \dots \otimes v_n) = \lambda(v_1) v_2 \otimes \dots \otimes v_n$$

$L_\lambda e_v = \lambda(v)$. So if e_a are the "creation" and l_a the "annihilation" ops, the relations are $l_a e_b = \delta_{ab}$ and $\sum_a e_a l_a = 1$.

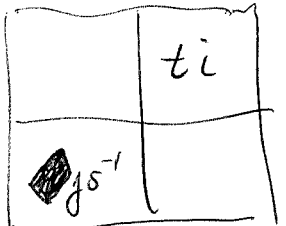
So return to $l_s = s l_s^*$ $f_s = f_s^{-1}$

and you get $\sum_s l_s f_s = 1$ $f_s l_t = f_s^{-1} l_t$

You have some insight from the Toeplitz situation, namely the use of $\sum_i \iota_s j_s = 1$ to construct an inductive limit, but there seems to be other things ~~around~~ around. In the Toeplitz ~~situation~~ case you are able to ~~replace~~ ~~replace~~ ~~replace~~ $\iota_\lambda e_\mu$ by the scalar $\langle \lambda, \mu \rangle$. In the present situation you have $j_s \iota_t = j_s^{-1} i_t \in A$.

Example: $\Gamma = \mathbb{Z}/2$.

In general for Γ finite what happens. You have ~~generators~~ ι_i, j_s^{-1} where $t, s \in \Gamma$, relations. $\iota_i \iota_i = 0$. same for $j_s^{-1} j_s^{-1} = 0$.



Then you have the quadratic elements in the generators $\iota_t j_s^{-1} = \iota_t h_s^{-1}$ which gives

the cross product alg B.

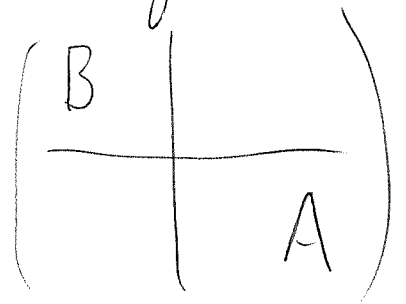
Wait: Forget Γ as a group. Think of Γ as a finite set. You have generators ι_s, j_s $s \in \Gamma$ relations $\iota_s \iota_{s'} = 0, j_s j_{s'} = 0,$

$$\sum_s (\iota_s j_s) \iota_t = \iota_t \quad \sum_s j_t (\iota_s j_s) = j_t$$

$$\Gamma = \{1\} \quad \iota, j \quad \iota^2 = 0, j^2 = 0, \\ \iota j \iota = i \quad j \iota j = j$$

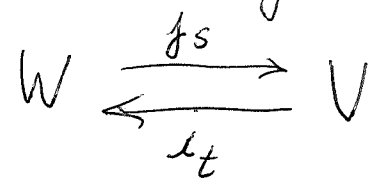
Questions to ask. just defined?

What about the structure



$\sum_s \iota_s \jmath_s = 1$. B is generated

by $\iota_s \jmath_t$



A^{gen} by the elements $\jmath_s \iota_t$

$$\sum_t (\jmath_s \iota_t) (\jmath_t \iota_u) = \jmath_s \iota_u$$

Toeplitz algebra, Cuntz's algebra \mathcal{O}_n generated by s_1, \dots, s_n such that $s_i^* s_j = \delta_{ij}$ and

$$\sum_{i=1}^n s_i s_i^* = 1.$$

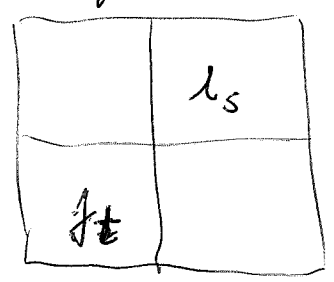
orth isometries

~~So you begin with the algebra~~ So you begin with the algebra of operators on $T(V)$ namely, mult by x_i and left contraction with y_j satisfying

$$y_j x_i = \delta_{ji}$$

If $n=1$ you get $y x = 1$.

~~Month~~ Can you connect \mathcal{O}_n to your Morita equivalence. Describe latter.



generators ι_s, \jmath_t $s, t \in \Gamma$

relations $\sum_{s \in \Gamma} \iota_s \jmath_s = 1_W$ in the sense that $(\sum \iota_s \jmath_s) \iota_t = \iota_t$

and $\jmath_t (\sum \iota_s \jmath_s) = \jmath_t$.

Now you want to impose conditions on

Is t . Go back to tensor alg $T(V)$ with ~~operators~~ x_i left mult ops x_i and left ~~contraction~~ y_j satisfying $y_j x_i = \delta_{ji}$ and

$$\sum x_i y_i = \begin{cases} 1 & \text{degree} \geq 1 \\ 0 & \text{degree} \leq 0 \end{cases}$$

Now you ~~can~~ should try to understand the relations between these examples.

Assume. $y_j x_i = \delta_{ji}$ ~~xxxxxxxxxxxxxxxx~~

Then $\sum_j x_j y_j x_i = \sum_i \cancel{\dots} x_j \delta_{ji}$

~~xxxxxxxxxxxxxxxx~~

$$\left(\sum_j x_j y_j \right) x_i = x_i$$

First example. $W \xrightleftharpoons[i]{f} V$ or

$$f \circ g = \text{id}_W, \quad g \circ f = \text{id}_V$$

~~xxxxxxxx~~