I think I finally see a way forward. First you need to identify $p(\Gamma \times A)p$ with a unital ring containing A as ideal. So try to understand

$$f_s = \sum_{t \in \Gamma} p_{st^{-1}} f_t = \sum_t f_t p_{t^{-1}s}.$$

Now you can also begin with

$$E = p(\Gamma \times A) \quad \text{as } B, A \text{ bimodule}$$
$$F = (\Gamma \times A)p \quad \text{— } A, B \text{ —}$$

Try to construct an isomorphism

$$\boxed{Bh_1 \simeq p(\Gamma \times A)}$$

You have h_1

$\cdot h_1 : B \longrightarrow B$

You have

something FISHY

$$\begin{tikzcd}
p(\Gamma \times A) \arrow[r, hook, "\alpha"] \arrow[dr, "\alpha_1"'] & \Gamma \times A \arrow[r, two heads, "\beta"] \arrow[d, shift right, "j_*"'] & p(\Gamma \times A) \\
& A \arrow[u, shift right, "i_*"'] \arrow[ur, "\beta_1"'] &
\end{tikzcd}$$

Take $B = C \rtimes \Gamma$ or $\Gamma \ltimes C$.

Try $\quad p(\Gamma \times A)p = p(\Gamma \times A) \otimes_A (\Gamma \times A)p$

$$h_1 E = h_1 p(\Gamma \times A) = A \, ?$$

need formalism straight.

$A = A_\Phi$, $\underset{\sum p_s}{A} \longrightarrow \underset{\sum s p_s}{\Gamma \times A}$

$\sum p_s \longmapsto \sum s p_s = p$

so p lies naturally in A, but is not homogeneous. So now you return to the question of Γ grading. That is, A and B are Γ-graded algebras, doesn't this mean that the bimodules ${}_BE_A$, ${}_AF_B$ should be Γ-graded?

You have then a puzzle because although $\Gamma \times A$ is Γ-graded, the projector $p = \sum s p_s$ is not homogeneous.

Is there a Γ-graded module category appropriate to a Γ-graded algebra $A = \bigoplus_{s \in \Gamma} A_s$. Yes. What about Morita equivalence? Is it true that firm modules for $A \rtimes \Gamma$ are the same as firm Γ-graded modules over A.

$$\underbrace{A \rtimes \Gamma}_{J} \longrightarrow \underbrace{\tilde{A} \rtimes \Gamma}_{R \text{ unital}} \longrightarrow \mathbb{C}\Gamma$$

NO you are confusing Γ and $\hat{\Gamma}$. Start again. Let $A = \bigoplus_{s \in \Gamma} A_s$ be Γ-graded let $M = \bigoplus_{s \in \Gamma} M_s$ be a Γ-graded A-module: $A_s M_t \subset M_{st}$. ~~How do~~ $A \rtimes \hat{\Gamma}$? There should be an alg $A \rtimes \hat{\Gamma}$ whose firm modules are Γ-graded A-modules which are firm:

$$\mu : A \otimes_A M \xrightarrow{\sim} M \qquad \mu(a\otimes m) = am$$

There is an algebra $A \rtimes \hat{\Gamma} = A \otimes C_c(\Gamma)$

$$= \bigoplus_{s\in\Gamma} A e_s \qquad (ae_s)(a'e_t) = a\, e_s(a')\, e_s \ ?$$

$$M = \bigoplus_{t\in\Gamma} M_t \qquad e_t \text{ proj on } M_t$$

Let $a \in A_s$ $\quad aM_t \subset M_{st}$

The point is that on M you have operator $a \in A_s$, $s \in \Gamma$ and e_t $t\in\Gamma$ $\quad a\in A_s$, $e_{st} a_s e_t = a_s e_t$

$$e_t a_s = a \ ?$$

$$\begin{tikzcd} M_u \arrow[r, "a_s"] & M_{su} \\ M \arrow[u, "e_u"] \arrow[r, "a_s"] & M \arrow[u, "e_{su}"'] \end{tikzcd}$$

$$\begin{tikzcd} M_t \arrow[r, "a_s"] \arrow[d, hook] & M_{st} \arrow[d, hook] \\ M \arrow[r, "a_s"] & M \end{tikzcd}$$

$$a_s e_u = e_{su} a_s$$

$$j_{su} a_s = a_s j_u$$

$$a_s i_u = i_{su} a_s$$

$$a_s e_u = a_s i_u j_u = i_{su} a_s j_u = i_{su} j_{su} a_s = e_{su} a_s$$

There seems to be an ~~projection~~ alg $\boxed{A \rtimes \hat{\Gamma}}$

$$= \bigoplus_{s,t} A_s e_t \qquad a_s e_t = e_{st} a_s$$

$$a_s e_{s^{-1}t} = e_t a_s$$

It looks like you have another version of $C_c(\Gamma, A)$

$g, f : \Gamma \to A$ of finite support.

• What is $A \rtimes \hat{\Gamma} = A \otimes \hat{\Gamma}$, $\hat{\Gamma} = \bigoplus_{s \in \Gamma} \mathbb{C} e_s = C_c(\Gamma)$

$a_s e_t = e_{st} a_s$ $\qquad \boxed{e_t a_s = a_s e_{s^{-1}t}}$

$A \rtimes \hat{\Gamma} = C_c(\Gamma, A)$ but it is hard to write the product. Is it possible to use the embedding

$$\begin{array}{ccc} A & \hookrightarrow & \hat{\Gamma} \times A \\ \cup & & \cup \\ A_s & \xrightarrow{\sim} & s A_s \end{array} \quad ?$$

$$\begin{tikzcd}
A \otimes M \arrow[r] & (\hat{\Gamma} \times A) \otimes (\hat{\Gamma} \times M) \\
A_s \otimes M_t \arrow[u, phantom, "\cup"] \arrow[r, "\sim"] \arrow[dr, bend right] & s A_s \otimes t M_t \arrow[u, phantom, "\cup"] \arrow[d] \\
& st (A_s \otimes M_t)
\end{tikzcd}$$

~~it looks as if you have~~

V is a Γ-graded v.s. means you have a canonical map $V \xrightarrow{\Delta_V} \mathbb{C}\Gamma \otimes V$ $\ni \left| \begin{array}{l} (\Delta \otimes 1)\Delta_V = (1 \otimes \Delta)\Delta_V \\ \eta \Delta_V = 1. \end{array}\right.$

then can define

$$\begin{tikzcd}
V \otimes W \arrow[r, "\Delta_V \otimes \Delta_W"] \arrow[dr, "\Delta_{V \otimes W}"'] & \overset{s}{\mathbb{C}\Gamma} \otimes \overset{v}{V} \otimes \overset{t}{\mathbb{C}\Gamma} \otimes \overset{w}{W} \arrow[d] \\
& \overset{st}{\mathbb{C}\Gamma} \otimes \overset{v}{V} \otimes \overset{w}{W}
\end{tikzcd}$$

What point are you missing? ~~[illegible]~~

$A = A_\Phi$ gens $p_s, s\in\Gamma$, rels $p_s = 0$ $s\notin\Phi$ $\quad p_s = \sum_t p_{st^{-1}}p_t$

$A \xrightarrow{\text{unique}} \mathbb{C}\Gamma\otimes A = \Gamma\times A$ tensor product alg.

$p_s \longmapsto s\otimes p_s$ where A is degree 1

Then $(\Delta_\Gamma\otimes 1)\Delta_A : p_s \longmapsto s\otimes p_s \longmapsto s\otimes s\otimes p_s$

$(1\otimes\Delta_A)\Delta_A : p_s \longmapsto s\otimes p_s \longmapsto s\otimes s\otimes p_s$.

Observe that have $p = \sum s\otimes p_s \in \Gamma\times A$ $\quad p^2 = p$.

~~[illegible]~~ Next step for $p(\Gamma\times A)$

$\Gamma\times A = \{f\in C_c(\Gamma, A)\}$. $\quad \sum_s s\otimes f_s \sum_t t\otimes g_t = \sum_s s\overbrace{\sum_t f_{st^{-1}}g_t}^{(f*g)_s}$

$(p*f)_s = \sum_t p_{st^{-1}}f_t \qquad \Gamma$ action $\boxed{(R_t f)_s = f_{st}}$

Check: $R_t\left(\sum_\Gamma s\otimes f_s\right) = \sum_{s\in\Gamma} st^{-1}\otimes f_s = \sum_{s\in\Gamma} s\otimes f_{st}$

So $p(\Gamma\times A)$ is naturally a Γ-module.

Also you have

$$\begin{tikzcd}
p(\Gamma\times A) \arrow[r, hook, "\alpha"] \arrow[dr, "\alpha_1"'] & \Gamma\times A \arrow[r, two heads, "\beta"] \arrow[d, shift right, "j_1"'] & p(\Gamma\times A) \\
& A \arrow[u, shift right, "i_1"'] \arrow[ur, "\beta_1"'] &
\end{tikzcd}$$

$\beta i_1 j_1 \alpha = h_1$

$f_s = (p*f)_s = \sum_t p_{st^{-1}}f_t$

$\cancel{(h_1 f)_s = (j_1(p*f))_s = \sum_t p_{t^{-1}}f_t}$

$\alpha_1 f = j_1\alpha f$
$= j_1(p*f) = \sum_t p_{t^{-1}}f_t \in A$

$$(h_1 f)_s = (\underbrace{\beta \iota_1 j_1 \alpha f}_{f_1 \in A})_s = (p * f_1)_s = p_s f_1$$

$$\boxed{(h_1 f)_s = p_s f_1} \qquad h_1 t h_1 f = p_s (t h_1 f)_1 = p_s (h_1 f)_t = p_s p_t f_1$$

---

review the calculation $\qquad h_1 = \beta \iota_1 j_1 \alpha : p(\Gamma\times A) \xrightarrow{\alpha} \Gamma \times A \xrightarrow{\beta} p(\Gamma\times A)$, $\quad j_1 \downarrow \uparrow \iota_1$, A

α inclusion $\beta = p$. $\qquad f \in p(\Gamma\times A)$, $j_1 \alpha f = f_1$, ~~$\beta\iota_1 j_1 \alpha f$~~

then $(\beta \iota_1 j_1 \alpha f)_s = (\beta \iota_1 f_1)_s = (p \iota_1 f_1)_s = \sum_t p(st^{-1}) \underbrace{(\iota_1 f_1)_t}_{0 \text{ for } t \neq 1} = p_s f_1$

$(h_1 t h_1 f)_s = p_s (t h_1 f)_1 = p_s (h_1 f)_t = p_s p_t f_1$ not very clear.

OK. the question is whether $E = p(\Gamma \times A)$ is a Γ-graded ~~bim~~ B, A bimodules. Now you have an action of Γ on E

There's a possibility you may be overlooking, namely the bimodule you seek ~~is is~~ is naturally related to $\Gamma \times A$ ~~...~~ with Γ acting on ~~the~~ the left by mult and A acts on the right by mult. and p works between Γ & A sort of as a tensor product relation. The idea is that

$\Gamma \times A$ consists of $\sum_{s \in \Gamma} s \otimes f_s$ $\qquad p = \sum t \otimes p_t$

apply $p \sum s \otimes f_s = \sum_t st^{-1} \otimes p_t f_s$

$$p(\Gamma \times A) = \{ f \in C_c(\Gamma, A) \mid f_s = (p * f)_s = \sum_t p_{st^{-1}} f_t \}$$

$$\Gamma \times A = \{ \sum_{t \in \Gamma} t \otimes f_t \mid f \in C_c(\Gamma, A) \}.$$

$$\sum_{s \in \Gamma} s \otimes p_s \sum_{t \in \Gamma} t \otimes f_t = \sum_{s,t} st \otimes p_s f_t = \sum_s s \otimes \sum_t p_{st^{-1}} f_t$$

You propose now to ~~replace~~ set $g_t = f_{t^{-1}}$, $g_{t^{-1}} = f_t$

$$\sum_{s \in \Gamma} s \otimes p_s \sum_t t \otimes g_{t^{-1}} = \sum_{\cancel{t}} \cancel{t} \otimes \sum_{\cancel{u}} \cancel{p_{tu^{-1}}} g_{u^{-1}}$$

$$\sum_{s \in \Gamma} s \otimes p_s \sum_t t^{-1} \otimes g_t = \sum_t t^{-1} \otimes \sum_u p_{t^{-1}u^{-1}} g_{u^{-1}}$$

$$= \sum_t t^{-1} \otimes \sum_u p_{t^{-1}u} g_u$$

$$= \sum_s s^{-1} \otimes \sum_t p_{s^{-1}t} g_t$$

notation is worse but somehow we ~~h~~ now have $\Gamma \underset{p}{\times} A = E$. The hope now is that this is naturally Γ graded bimodule, because s^{-1} has been combined with p_s.

Check: You expect E to be $Bh_1 = \Gamma \ltimes Ch_1$ degree 1. so Bh_1 and $h_1 B = h_1 C \rtimes \Gamma$ appear to be Γ-graded.

New idea yesterday: arrange $p(\Gamma \times A)$ into a form $\Gamma \overset{p}{\times} A$ making evident the Γ action on the left and right A action on the right. The map $p(\Gamma \times A) \xrightarrow{\sim} \Gamma \overset{p}{\times} A$ should be induced by inversion on Γ. * and hopefully the Γ grading. ~~So lets see what to do?~~ Note that you are ~~replacing~~ ~~changing~~ running the natural algebra structure on $\Gamma \times A$. So see if it works.

Begin with $\Gamma \times A = \mathbb{C}\Gamma \otimes A = C_c(\Gamma, A)$ and the operator p of left mult. by $p = \sum_{s \in \Gamma} s \otimes p(s)$ which is idempotent. Next transform $p\cdot$ via the isom $\mathbb{C}\Gamma \otimes A \to \mathbb{C}\Gamma \otimes A$, $t \otimes a \mapsto t^{-1} \otimes a$. Thus should yield $\sum_s \cdot s^{-1} \otimes p(s)\cdot$ on $\mathbb{C}\Gamma \otimes A$, which should be an idempotent operator homogeneous of deg 1. respecting the left Γ, right A actions.

So details: $\mathbb{C}\Gamma \otimes A$ left Γ ~~action~~ mult. right A mult. Define p on $\mathbb{C}\Gamma \otimes A$ to be

$$p(t \otimes a) = \sum_{s \in \Gamma} ts^{-1} \otimes p(s)a \quad , \quad \text{define } \Gamma\text{-grading}$$

by

$$(\mathbb{C}\Gamma \otimes A)_s = \bigoplus_{t \in \Gamma} t \otimes A_{t^{-1}s} = \bigoplus_{tu=s} t \otimes A_u$$

Thus $\left|\begin{array}{lll} t \otimes A_u & \text{has degree} & tu \\ ts^{-1} \otimes p(s)A_u & \text{———} & ts^{-1}su = tu. \end{array}\right.$

$$p(u \otimes a) = \sum_{s \in \Gamma} us^{-1} \otimes p(s)a$$ well defined respects Γ. -A

$$p^2(u \otimes a) = \sum_{t \in \Gamma} \sum_{s \in \Gamma} us^{-1}t^{-1} \otimes p(t)p(s)a$$ $t \to ts^{-1}$?

$$p(t \otimes a) = \sum_s ts^{-1} \otimes p(s)a = \sum_s \cancel{tt^{-1}}s^{-1} \otimes p(st)a$$

$$p^2(t \otimes a) = \sum_u \sum_s \underbrace{s^{-1}u^{-1}}_{(us)^{-1}} \otimes p(u)p(st)a$$

$$= \sum_u \sum_s (uu^{-1}s)^{-1} \otimes p(u)p(u^{-1}st)a$$

$$= \sum_s s^{-1} \otimes \underbrace{\sum_u p(u)p(u^{-1}st)a}_{p(s\cancel{t})a} = p(t \otimes a)$$

Can say

$$p(1 \otimes a) = \sum_s s^{-1} \otimes p(s)a$$ degree $p(1 \otimes a)$ = degree a

$$p^2(1 \otimes a) = \sum_s s^{-1}t^{-1} \otimes p(t)p(s)a = \sum_u u^{-1} \otimes \underbrace{\sum_{ts=u} p(t)p(s)a}_{p(u)}$$ $u = ts$

$\mathbb{C}\Gamma \otimes \tilde{A}$ left Γ, right A module

define p on $\mathbb{C}\Gamma \otimes \tilde{A}$ by $p(t \otimes a) = t \sum_s s^{-1} \otimes p_s a$

another thing that's new. Go back to $p(\Gamma \times \tilde{A})$ and $(\Gamma \times \tilde{A})p$. $\Gamma \times \tilde{A}$ is a unital algebra and $p = \sum s \otimes p(s)$ is a projector in this alg.

Keep at it. $\mathbb{C}\Gamma \otimes \tilde{A}$ you want to view this as a Γ, A bimodule free on one generator, and you have an endom. defined by

$$p(1 \otimes 1) = \sum_{s \in \Gamma} s^{-1} \otimes p_s$$

$$p(t \otimes a) = \sum_{s \in \Gamma} ts^{-1} \otimes p_s a$$

$$pp(1 \otimes 1) = \sum_{s \in \Gamma} p(s^{-1} \otimes p_s)$$

$$= \sum_{s \in \Gamma} s^{-1}\Big(\sum_{t \in \Gamma} t^{-1} \otimes p_t\Big) p_s \qquad \text{OK}$$

$$= \sum_{s,t} (ts)^{-1} \otimes p_t p_s = \sum_s s^{-1} \otimes \sum_t p_{ts^{-1}} p_s$$

$$\mathbb{C}\Gamma \otimes \tilde{A} = \bigoplus_{s,t \in \Gamma} s \otimes \tilde{A}_t \quad \text{is } \Gamma \text{ graded} \qquad \text{OKAY.}$$

$s \otimes \tilde{A}_t$ degree st.

So what's important? $\quad f = pf$

$$\sum_t t^{-1} \otimes f_t = \sum_{t,s} t^{-1} s^{-1} \otimes p_s f_t = \sum_{t,s} t^{-1} \otimes p_s f_{s^{-1}t}$$

Go back to $p(\Gamma \times A)$ and $(A \times \Gamma)p$ which are supposed to be a B, A bimodules (resp A, B bimod.). It seems that you want to form $\Gamma \dot{\times}$

How to describe your construction namely $\mathbb{C}\Gamma \underset{p}{\times} A$ which is Γ-graded

~~How to describe this~~ You have a construction $\mathbb{C}\Gamma \times_p A$ which is the subbimod of the $\mathbb{C}\Gamma, A$ bimodule $\mathbb{C}\Gamma \otimes A$ given by the image of the idempotent operator

$$p(t\otimes a) = \sum_s ts^{-1} \otimes p_s a$$

Also $\mathbb{C}\Gamma \otimes A$ is Γ-graded with $t \otimes A_s$ of degree ts. and p preserves the degree, so $\mathbb{C}\Gamma \times_p A$ is naturally Γ graded. It

$$\mathbb{C}\Gamma \times_p A = \Big\{ \sum_{t\in\Gamma} t\otimes f_t \;\Big|\; p\sum_{t\in\Gamma} t\otimes f_t = \sum_{t\in\Gamma} t\otimes f_t \Big\}$$

$$\sum_{t\in\Gamma} t\otimes f_t = \sum_{t,s} \overbrace{ts^{-1}}^{u} \otimes p_s f_t$$

$u = ts^{-1}$
$ut^{-1} = s^{-1}$
$tu^{-1} = s$

$$= \sum_{u,t} u \otimes p\, f_t \qquad ??$$

Review $\mathbb{C}\Gamma \times A \ni \sum t\otimes f_t$

$$pf: \sum_s s\otimes p_s \sum_{u} {}^{s^{-1}}u \otimes f_{s^{-1}u} = \sum_u u \otimes \sum_s p_s f_{s^{-1}u}$$

$$\begin{array}{ccc} t\otimes a & & \sum_s st \otimes p_s a \\ \mathbb{C}\Gamma\otimes A & \xrightarrow{p} & \mathbb{C}\Gamma\otimes A \end{array}$$

apply $t\otimes a \longmapsto t^{-1}\otimes a$.

$$p(t\otimes a) = \sum_s st \otimes p_s a$$

How to ~~sort~~ get this straight

$$\tilde p(t^{-1}\otimes a)$$

$$\Gamma\times A \xrightarrow{p} \Gamma\times A$$

$$t\otimes a \longmapsto \sum_s ts \otimes p_s a$$

$$\begin{tikzcd} t^{-1}\otimes a \arrow[r, mapsto, "p\cdot"] \arrow[ddd, bend right] & \sum_s s t^{-1}\otimes p_s a \\ \Gamma\times A \arrow[r] \arrow[d] & \Gamma\times A \arrow[d] \\ \Gamma\times A & \Gamma\times A \\ t\otimes a \arrow[r, mapsto, "p\cdot"] & \sum_s t s^{-1}\otimes p_s a \end{tikzcd}$$

So what goes on

$$\sum_t t\otimes f_t \overset{p\cdot}{\longmapsto} \sum_t \sum_s \overbrace{ts^{-1}}^{u}\otimes p_s f_t$$

$$= \sum_u u\otimes \sum_{ts^{-1}=u} p_s f_t$$

$ts^{-1}=u$
$t=us$
$s=u^{-1}t.$

$$= \sum_u u\otimes \sum_t p_{u^{-1}t} f_t$$

in this picture $(pf)_s = \sum_t p_{s^{-1}t} f_t$

What's important. Consider $\Gamma\otimes A$ as left Γ, right A bimodule with total degree grading: $t\otimes A_s$ has degree ts.

$$p(t\otimes a) = \sum_{s\in\Gamma} ts^{-1}\otimes p_s a$$

$\therefore$ $\cancel{p(t\otimes A_t)\ldots}$

$$\therefore\ p\Big(\sum_t t\otimes f_t\Big) = \sum_t\sum_s ts^{-1}\otimes p_s f_t$$

$$= \sum_t t\otimes \sum_s p_s f_{ts}$$

It's still not clear whether you want this notation. Anyway

$u=ts$
$s=t^{-1}u$

$$f_t = \sum_s p_s f_{ts} = \sum_s p_{t^{-1}s} f_s$$

$$f_s = \sum_s p_{s^{-1}t} f_t$$

Idea: $\mathbb{C}\Gamma \otimes A$ left Γ right A bimodule, Γ-graded for the total degree

$$(\mathbb{C}\Gamma \otimes A)_u = \bigoplus_{st=u} s \otimes A_t$$

degree 1 part is $\bigoplus_{t\in\Gamma} t^{-1} \otimes A_t$. ~~endom~~ operator p on this bimodule $p(t \otimes a) = t\left(\sum_s s^{-1} \otimes p_s\right) a$

$$\cancel{\sum_s s^{-1} \sum_t t^{-1} \otimes p_t} \qquad p^2(u \otimes a) = \sum_s u s^{-1} \left(\sum_t t^{-1} \otimes p_t\right) p_s a$$

$$\sum_{s,t} s^{-1} t^{-1} \otimes p_t p_s = \sum_u u^{-1} \otimes \sum_{u=ts} p_t p_s$$

So what goes on?

Discuss. ~~details~~ philosophy. $A = A_\Phi$. define by gens + rels. Γ graded: ~~canon~~ homom. $A \longrightarrow \Gamma \times A$ canonical
$A_s \longrightarrow s \otimes A$ $\quad p = \sum s p_s \quad p^2 = p$

A is Γ graded and $\exists$ canonical $p = \sum p_s \in A$, so for any A-module V you have a projection on V

Example $\Gamma = \Phi = \mathbb{Z}/2$. A_Φ in this case is a superalgebra $A = A_0 \oplus A_1$ generated by an idempotent

$$p = p_0 + p_1 = (p_0 + p_1)^2 = (\underbrace{p_0^2 + p_1^2}_{p_0}) + (\underbrace{p_0 p_1 + p_1 p_0}_{p_1})$$

so A should be commutative. Spectrum

Representation on a $\mathbb{Z}/2$ graded vector space.

$V = V_0 \oplus V_1$

It should be simple. Look at representations. A is $\mathbb{Z}/2$ graded, so you look at ~~Z~~ $\mathbb{Z}/2$-graded representations.

Do this, study A in the case of $\Gamma = \Phi = \mathbb{Z}/2$, in a straightforward way. You have two generators p_0, p_1 and two relations $p_0^2 + p_1^2 = p_0$, $p_0 p_1 + p_1 p_0 = p_1$. Another description is two generators $p = p_0 + p_1$, $\bar{p} = p_0 - p_1$ and two relations saying each generator is idempotent. So A is ~~$A * A$~~. $Q(\mathbb{C}e) = \mathbb{C}e * \mathbb{C}e$, whose structure you should know pretty well.

Repeat, $\Phi = \Gamma = \mathbb{Z}/2$, A in this case is the $\mathbb{Z}/2$-graded alg $A = A_0 \oplus A_1$ generated by the components p_0, p_1 of $p = p_0 + p_1$ satisfying $p^2 = p$. Gens p_0, p_1 Rels $\begin{cases} p_0 = p_0^2 + p_1^2 \\ p_1 = p_0 p_1 + p_1 p_0 \end{cases}$

$$\begin{aligned} \varepsilon p &= p_0 + p_1 \\ \bar{\varepsilon} p &= p_0 - p_1 \end{aligned}$$

This A should be Cuntz's $Q(\mathbb{C}e) = \mathbb{C}e * \mathbb{C}e$ ~~and~~ ~~something else~~ Q

Again A is defined by gens. p_0, p_1 rels $\begin{cases} p_0 = p_0^2 + p_1^2 \\ p_1 = p_0 p_1 + p_1 p_0 \end{cases}$

! $\mathbb{Z}/2$ graded structure p_0 even, p_1 odd. You should know that $A = Q(\mathbb{C}e) \cong \Omega\mathbb{C}e$ with Fedosov product.
$\mathbb{C}e * \mathbb{C}e$

Anyway you should have a complete picture of A in this case. Now there's an equivalence between a $\mathbb{Z}/2$-grading and a $\mathbb{Z}_2$ action, which has to be kept straight. There's an alg. homom.

$$\begin{array}{ccc} A & \longrightarrow & \mathbb{C}\Gamma \otimes A \\ \cup & & \cup \\ A_s & \longrightarrow & s \otimes A_s \end{array}$$

Anyway what next?

Also cross product. Can you see the Morita equivalence in this case

So you should be able to see some things. to see concretely A, B, etc in this $\mathbb{Z}/2$ case. How? What about

$$\mathbb{C}\Gamma \underset{p}{\otimes} A \qquad p^a \qquad g^a$$

because $s \mapsto s^{-1}$ is the identity on $\mathbb{Z}/2$ your $\mathbb{C}\Gamma \times_p A$ should be the same as $p(\mathbb{C}\Gamma \otimes A)$

Start again A is "the" superalgebra containing an element $p_0 + p_1 \in A_0 \oplus A_1$ universal idempotent element. $A = \mathbb{C}e * \mathbb{C}e$

$= Q(\mathbb{C}e) = \Omega(\mathbb{C}e)$ with Fedosov product

$$\omega * \eta = \omega\eta - (-1)^{|\omega|} d\omega\, d\eta$$

$$(a_1 + da_1) * (a_2 + da_2) = a_1a_2 - \cancel{da_1da_2} + a_1da_2 + da_1a_2 + \cancel{da_1da_2}$$

$$= a_1a_2 + d(a_1a_2)$$

Now what to do? C gen h_0, h_1

relations $h_0 = (h_0+h_1).h_0 = h_0(h_0+h_1)$

$$h_1 = (h_0+h_1)h_1 = h_1(h_0+h_1)$$

So C is commutative, + unital because $h_0+h_1=1$

$$x = (x+y)x = x(x+y)$$

$$\tilde{\Omega}(\mathbb{C}e) = \mathbb{C} + \mathbb{C}e + \begin{matrix}\mathbb{C}de \\ \mathbb{C}ede\end{matrix}$$

$$e\,de + de\,e = de$$

~~You are interested in~~ ~~You are~~ You want to understand whether there is a M.eq.

So how to proceed

What is B? $C \rtimes \Gamma$ B is not commutative but things are fairly close.

$A = \Omega(\mathbb{C}e)$

go back to $\mathbb{C}\Gamma \overset{p}{\times} A$ image of the operator

$$p(t \otimes a) = t\left(\sum_s s^{-1} \otimes p_s\right) a$$

$$p^2(u\otimes a) = \sum_s p(us^{-1} \otimes p_s a) = \sum_s \sum_t u(s^{-1}t^{-1} \otimes p_t p_s)a$$

$$\sum_{s,t} (ts)^{-1} \otimes p_t p_s = \sum_u u^{-1} \otimes \sum_{ts=u} p_t p_s = p_{ts} = p_u$$

This $\mathbb{C}\Gamma \overset{p}{\times} A$ is a sub bimodule of $\mathbb{C}\Gamma \otimes A$

This should be E. Now look at

$$\begin{tikzcd} \mathbb{C}\Gamma \overset{p}{\otimes} A \arrow[r, hook, "\alpha"] & \mathbb{C}\Gamma \otimes A \arrow[r, two heads, "\beta = p"] \arrow[d, shift right, "\delta_1"'] & \mathbb{C}\Gamma \overset{p}{\otimes} A \\ & A \arrow[u, shift right, "\iota"'] & \end{tikzcd}$$

What is $\mathbb{C}\Gamma \overset{p}{\otimes} A$. Identify $\mathbb{C}\Gamma \otimes A$ with

$\sum_{t\in\Gamma} t^{-1} \otimes f_t$ satisfying $p \sum_t t^{-1} \otimes f_t = \underbrace{\sum_{t,s} t^{-1}s^{-1} \otimes p_s f_t}_{} = \sum u^{-1} \otimes f_u$

$$\sum_{u\in\Gamma} u^{-1} \otimes \sum_{u=st} p_s f_t$$

i.e. $f_u = \sum_{u=st} p_s f_t = \sum_t p_{ut^{-1}} f_t$

maybe then you want $\sum p_{ut^{-1}} f_t$ $\quad$ $u = ts^{-1}$, $u^{-1} = st^{-1}$, $u^{-1}t = s$

$$p \sum_{t\in\Gamma} t \otimes f_t = \sum_{t,s} ts^{-1} \otimes p_s f_t = \sum_u u \otimes \sum_{u=ts^{-1}} p_s f_t$$

What are you trying for? Namely that $\mathbb{C}\Gamma \overset{p}{\otimes} A$

~~can be~~ $= \left\{ \sum_{t\in\Gamma} t^{-1} \otimes f_t \;\middle|\; f_s = \sum_t p_{st^{-1}} f_t \right\}$

Check
$$p\left(\sum_t t^{-1} \otimes f_t\right) = \sum_{t,s} \underbrace{t^{-1}s^{-1}}_{(st)^{-1}} \otimes p_s f_t = \sum_u u^{-1} \otimes \sum_{u=st} p_s f_t$$

You have $\mathbb{C}\Gamma \otimes A$ a left $\mathbb{C}\Gamma$, right A bimodule

Γ-graded with $t \otimes A_s$ of degree ts

a projection of on this bimodule $p(t \otimes a) = \underline{\sum_{s\in\Gamma} ts^{-1} \otimes p_s a}$
preserves degree.

$$p\left(\sum s^{-1} \otimes f_s\right) = \sum_s s^{-1} \otimes \sum_{s=tu} p_t f_u$$

So you should perhaps work with

$$\mathbb{C}\Gamma \overset{p}{\otimes} A = \{ f \in C_c(\Gamma, A) \mid f = p * f \}$$

$$\sum t^{-1} \otimes f_t$$

Now you have not only the Γ grading on $\mathbb{C}\Gamma \otimes A$ but also the Γ action.

What really interests you is deg = 1 i.e. f such that $f_t \in A_t$. ~~Can you determine~~

So now you look at all $\{(f_s)_{s\in\Gamma} \mid f_s \in A_s\}$

and $f_s = \sum_t p_{st^{-1}} f_t$

At this point you want to understand

$\mathbb{C}\Gamma \overset{P}{\otimes} A$ Γ-graded left $\mathbb{C}\Gamma$ right A bimodule

$\|$ You want the degree 1 sector

$\{\sum_t t^{-1}\otimes f_t \mid p*f = f\}$ i.e. $f_t \in A_t \quad \forall t.$

Now take $A = Q(\mathbb{C}e) = \Omega(\mathbb{C}e)$ with Fedosov prod.

$1 \otimes f_0 + \varepsilon \otimes f_1$ $\qquad \sum_t p_{st^{-1}} f_t = p_0 f_0 + p_1 f_1$

$$\begin{pmatrix}(p*f)_0\\(p*f)_1\end{pmatrix} = \begin{pmatrix}p_0 & p_1\\ p_1 & p_0\end{pmatrix}\begin{pmatrix}f_0\\ f_1\end{pmatrix}$$

$f_0 = p_0 f_0 + p_1 f_1$

$f_1 = p_1 f_0 + p_0 f_1$

$A = Q(\mathbb{C}e)$ $\qquad e = p_0 + p_1 = e + de$

$\bar{e} = p_0 - p_1 = e - de$

$(e+de)^*(e+de) = e \cancel{- de^2} + ede + dee = e + de^+$

$\underset{\text{even}}{\omega_0} + \varepsilon \overset{\text{odd}}{\omega_1}$

$e\omega^+ - de\,d\omega^+$

$\omega^+ = e*\omega^+ + de\,\omega^- = e\omega^+ + de(\omega^- - d\omega^+)$

$\omega^- = de\,\omega^+ \cancel{+ e*\omega^-}$

$a = \sum t^{-1}\otimes a_t = a_0 + \varepsilon a_1$

$pa = 1\otimes p_0 a + \varepsilon\otimes p_1 a$

Useful

$$\begin{tikzcd}
t \otimes a \arrow[r, mapsto] & \sum_s st \otimes p_s a \\
\Gamma \times A \arrow[r, "p"] \arrow[d] & \Gamma \times A \arrow[d] \\
\Gamma \times A & \Gamma \times A \\
t^{-1} \otimes a & \sum_s t^{-1} s^{-1} \otimes p_s a = t^{-1}\left(\sum s^{-1} \otimes p_s\right) a
\end{tikzcd}$$

need deg 1

$$\boxed{\sum_{t \in \Gamma} t^{-1} \otimes f_t \qquad p * f = f \qquad f_t \in A_t}$$

$p*f$: $\begin{pmatrix} p_0 & p_1 \\ p_1 & p_0 \end{pmatrix}\begin{pmatrix} f_0 \\ f_1 \end{pmatrix}$

If you ident. $\mathbb{C}\Gamma \otimes A$ with $\begin{pmatrix} f_0 \in A^+ \\ \oplus \\ f_1 \in A^- \end{pmatrix}$ $\quad \begin{pmatrix} p_0 & p_1 \\ p_1 & p_0 \end{pmatrix}$ ''

Then pf is $\begin{pmatrix} p_0 & p_1 \\ p_1 & p_0 \end{pmatrix}\begin{pmatrix} f_0 \\ f_1 \end{pmatrix}$ Check $\begin{pmatrix} p_0 & p_1 \\ p_1 & p_0 \end{pmatrix}\begin{pmatrix} p_0 & p_1 \\ p_1 & p_0 \end{pmatrix}$

Maybe you can find the pairing. The idea is that

$E = p(\Gamma \times A)$ $\qquad F = (\Gamma \times A)p$

Maybe better is $E = \Gamma \overset{\beta}{\otimes} A$ $\qquad F = A \overset{\beta}{\otimes} \Gamma$

these should be Γ-graded. Assume you understand the operator h_1 on E and hence the B-action

It should be true that $E \otimes_A F = \Gamma \overset{\beta}{\otimes} A \overset{\beta}{\otimes} \Gamma$ is essentially equal to B. What is true?

$$(\Gamma \times \tilde{A}) \otimes_A (\tilde{A} \times \Gamma) = \Gamma \times \tilde{A} \times \Gamma$$

$\sum t' \otimes f_t \quad \cdot \ \deg = 1 \in \Gamma \qquad p^* f = f$

Can you get a feeling for

$f \in \Gamma \times A \qquad 1 \otimes f_0 + \varepsilon \otimes f_1$

pf is $\quad 1 \otimes (p_0 f_0 + p_1 f_1) + \varepsilon \otimes (p_0 f_1 + p_1 f_0)$

$p * f \quad \begin{pmatrix} p_0 & p_1 \\ p_1 & p_0 \end{pmatrix} \begin{pmatrix} f_0 \\ f_1 \end{pmatrix} \qquad f_0 + f_1 \in A_0 \oplus A_1 = A$

~~Obvious genera~~ It seems then that $\Gamma \hat{\otimes} A$ consists of $\cancel{(1 \otimes p_0 + \varepsilon \otimes p_1) f_0 + (\varepsilon \otimes p_0 + 1 \otimes p_1) f_1}$

$1 \otimes p_0 f_0 + \varepsilon \otimes p_1 f_0$

Wait $\Gamma \times A = 1 \otimes A + \varepsilon \otimes A$

typical elt is $\quad 1 \otimes f + \varepsilon \otimes g \qquad f, g \in A.$

apply $p \qquad 1 \otimes p_0 f + \varepsilon \otimes p_1 f + \varepsilon \otimes p_0 g + 1 \otimes p_1 g$

Go back over details. Wait A gen. p_s, $s \in \Gamma$ relns. $p_s = 0 \quad s \notin \Phi \qquad p_s = \sum_t p_{st^{-1}} p_t$. In particular you get a projection $\sum_{s \in \Phi} p_s$ in A. Look at

$\{ x \in A \mid Ax = 0 \}$

$C = \mathbb{C}[h_0, h_1] / (h_0 + h_1 = 1) \qquad B = C \rtimes \mathbb{Z}/2 \qquad$ basis $h_0^n, h_0^n \varepsilon$

$\tilde{A} = \tilde{\mathbb{C}}e * \tilde{\mathbb{C}}e = \Omega(\tilde{\mathbb{C}}e)$ with Fed. product.

basis	1	de	ede
	e	ede	ede^2

seems to have same size

How do you make progress? ~~Maybe make progress~~
Think. Try again. ~~Start with~~ B

Go thru steps carefully. Let E be a unitary B-module. $B = C_\Phi \rtimes \Gamma$. See if you can see that C and B are unital. This means returning to the question of whether $h_s = \sum_t h_t h_s$ suffices without the condition $h_s = \sum_t h_s h_t$. Since we deal with left modules?

~~You want to assume that we~~

You want to have $BE = E$ for firmness.

$$\underbrace{C \rtimes \Gamma}_{B} \hookrightarrow \underbrace{\tilde{C} \rtimes \Gamma}_{R} \twoheadrightarrow \underbrace{\mathbb{C}\Gamma}_{R/B}$$

want R/B right flat i.e. $\exists$ left local units

want $\left(\sum_t h_t\right) h_s = h_s$. Then it should be true that a firm B-module is a unitary R module E such that $\sum_t h_t$ is the identity on E. But a unitary R-module E is a Γ-module with a $\mathbb{C}$-linear op. h_1 satisfying $\sum s h_1 s^{-1} = id$

Here you've been assuming Γ finite.

Repeat. Define C to be alg gen by $h_s, s \in \Gamma$ subject to $\left(\sum_{t \in \Gamma} h_t\right) h_s = h_s \quad \forall s$. ~~$C$ is idempotent~~

C has left unit $e = \sum_{t \in \Gamma} h_t$, C is idempotent,

$$C = eC \quad = eCe \oplus eC(1-e)$$

$$C \otimes_C M$$

Assume $C = eC \qquad e^2 = e$

Then C is the semi-direct product of the unital ring Ce and the bimodule over Ce given by $C(1-e)$ unitary on the left and null on the right. Firm left C modules M:

$$\begin{array}{c} C \otimes_C M \xrightarrow{\sim} M \\ \text{\rotatebox{90}{$=$}} \\ Ce \otimes_{Ce} M \end{array}$$

are the same as ~~unitary~~ (firm) Ce modules.

Firm right C modules N:

$$\begin{array}{c} N \otimes_C C \xrightarrow{\sim} N \\ \text{\rotatebox{90}{$=$}} \\ N \otimes_C (Ce \oplus C(1-e)) \\ \text{\rotatebox{90}{$=$}} \\ Ne \oplus N(1-e) \end{array}$$

Note the way to remember What you want is $A \supset I \quad IA = 0$

$$\begin{pmatrix} I & I \\ 0 & 0 \end{pmatrix}\begin{pmatrix} A & A \\ A & A \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} A & A \\ A & A \end{pmatrix}\begin{pmatrix} I & I \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} & \\ & \end{pmatrix}$$

A I ideal in A such that $IA = 0$

$$\begin{array}{c} M \longmapsto M \\ \mathcal{M}(A) \xrightarrow{\ \sim\ } \mathcal{M}(A/I) \\ N \longleftarrow N \end{array}$$

$IM = 0$ for $M = AM$

$$A \otimes_A M \xrightarrow{\sim} M$$

$$IA = 0 \implies IM = 0$$

$$\begin{pmatrix} A & A/I \\ A & A/I \end{pmatrix}$$

$$\begin{pmatrix} A & A \\ A & A \end{pmatrix}\begin{pmatrix} 0 & I \\ 0 & I \end{pmatrix} \subset \begin{pmatrix} 0 & AI \\ 0 & AI \end{pmatrix}$$

$$\begin{pmatrix} 0 & I \\ 0 & I \end{pmatrix}\begin{pmatrix} A & A \\ A & A \end{pmatrix} \subset \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Go back to $C = eC$ $e^2 = e$.

$C = Ce \oplus C(1-e)$

$$\begin{pmatrix} C & C/C(1-e) \\ C & C/C(1-e) \end{pmatrix} = \begin{pmatrix} C & Ce \\ C & Ce \end{pmatrix}$$

what is the point? The point is that given $e = e^2$ inside C, you have a Morita context

$$\begin{pmatrix} C & Ce \\ eC & eCe \end{pmatrix}$$

which yields a m.eq when $C = CeC$.

$$C = eeC \subset CeC \subset C$$

and the Morita equivalence is given by

$M \in \mathcal{M}(C)$ goes to

~~This problem can be solved.~~

C left unit e, then get Morita context

$$\begin{pmatrix} C & Ce \\ eC & eCe \end{pmatrix} = \begin{pmatrix} C & Ce \\ C & Ce \end{pmatrix} \qquad \begin{matrix} C \\ \vee \\ eC \subset CeC \end{matrix}$$

$$M \longmapsto C \otimes_C M = M$$

so get Morita equivalence $\mathcal{M}(C) \sim \mathcal{M}(Ce)$

The point ~~may~~ is that $C = Ce \oplus C(1-e)$

A firm left C module is ~~the~~ a firm (unitary) Ce-module. A firm right C module should be of the form

$$V \otimes_{Ce} C = V \otimes_{Ce} (Ce \oplus C(1-e))$$

Just what should you be trying to say.

The point is that firm left C-modules have the form $Ce \otimes_{Ce} N = N$ $\quad N$ firm Ce-module

firm right C-modules ~~of~~ have the form

$$V \otimes_{Ce} C = V \otimes_{Ce} (Ce \oplus C(1-e))$$
$$= V \oplus V \otimes_{Ce} eC(1-e)$$

It seems to me that what's important is the Morita equivalence, so that C and Ce can be considered equivalent. Be precise

Example. C gen h_s, $s \in \Gamma$ finite relation

$e = \sum_{t \in \Gamma} h_t$ is left unit: $\quad e h_s = h_s \quad \forall s$. But now to put in the extra condition $h_s e = h_s$ should replace C by Ce

Let's continue with the Morita equivalence. Next form $B = C \rtimes \Gamma$, where C has the left unit $\sum_{t \in \Gamma} h_t$.

Actually I remember working out $C = Ce \oplus C(1-e)$

page 947, 944 Your C has gen h_0, h_1 rels $(h_0+h_1)h_i = h_i$, for $i=0,1$. Put $e = h_0+h_1$, $h = h_0$, so that $h_1 = e-h$ and the rels become

$eh = h$, $e(e-h) = e-h$, $\therefore$ $\boxed{eh=h,\ e^2=e.}$

So C gens e,h + $e^2 = e$ and $eh=h$.

A C-module is a v.s. with splitting

$V = eV \oplus (1-e)V$ and $h: V \longrightarrow eV$

$V = V_0 \oplus V_1$ $\qquad$ $h: V \longrightarrow V_0$

$$e = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \qquad h = \begin{pmatrix} * & * \\ 0 & 0 \end{pmatrix} \text{ on } \begin{matrix} V_0 \\ \oplus \\ V_1 \end{matrix}$$

Then
$$he = \begin{pmatrix} * & * \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} * & 0 \\ 0 & 0 \end{pmatrix} \neq h$$
in general. This means that $e^2=e$, $eh=h$ does not imply $he=h$.

can you compute this.

Example to calculate: C = alg gen by h_0, h_1 relns $(h_0+h_1)h_i = h_i$. Put $e = h_0+h_1$, $h = h_0$, can also say C = alg gens e,h relations $e^2=e$, $eh=h$.

A C-module is a V with splitting $V = e_q V \oplus (1-e_q)V$ and an op $h: V \longrightarrow eV$

You want a basis for C, start with the free module $\tilde{C}$, $1, e, h, he, h^2, \ldots$ It looks as if $\tilde{C}$ is the free $\mathbb{C}[h]$-module with generators 1 and e.

Ex: C gens e, h rels $e^2 = e$, $eh = h$

Thus C has the left unit e. Aim to describe C completely. Facts.

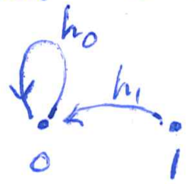
1) A left C-module is a vector space V equipped with a splitting $V = eV \oplus (1-e)V$ and an operator h such that $hV \subset eV$.

2)
$$\begin{tikzcd} 0 \arrow[r] & C \arrow[r] & \tilde{C} \arrow[l, bend right, "e\cdot"'] \arrow[r] & \mathbb{C} \arrow[r] & 0 \end{tikzcd}$$

so
$$\tilde{C} = eC \oplus (1-e)C \quad \text{as right } C \text{ module}$$
$$= C \oplus \mathbb{C}(1-e)$$

3) A C-module V is firm $\iff V = eV$. In this case e is the identity on V, so V is a $\mathbb{C}[h]$-module. So firm C-module are just $\mathbb{C}[h]$-modules.

4) A left C-module consists of two vector spaces V_0, V_1 and a map $h : V_0 \oplus V_1 \to V_0$.

Quiver with 

basis h_0^n, $h_0^n h_1$ $n \geq 0$, and $1-e$

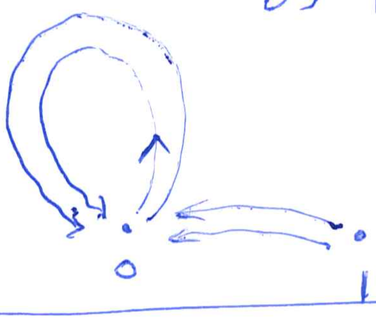
$$h_0 = he, \quad h_1 = h(1-e)$$
$$h_0^n = h^n e, \quad h_0^n h_1 = h^n e h(1-e) = h^{n+1}(1-e)$$

You are amazingly far from an understanding.

$$\left(\sum_{i=0}^{n} x_i\right) x_j = x_j \quad \forall j \qquad \underbrace{e = \sum_{0}^{n} x_i, \; x_1, \dots, x_n}_{\text{new gens.}} \qquad x_0 = e - \sum_{1}^{n} x_i$$

rel $e^2 = e$, $e x_i = x_i$ $\quad i = 1, \dots, n$

left module is V_0, V_1 and $x_1, \dots, x_n : V_0 \oplus V_1 \to V_0$



C gens e, h rels $e^2 = e$ $\; eh = h$. You would like to show that

$$\mathbb{C}[h]^2 \longrightarrow C$$
$$f, g \longmapsto fh + ge$$

is ~~an isomorphism~~ bijective, thus that C is a free rank 2 module over $\mathbb{C}[h] = Ce$. The idea is that ~~C~~ C is generated from ~~by~~ h, e by left mult by e and h, and e is the identity so that you get the elements $h, e, h^2, he, h^3, h^2e, \dots$ spanning C. The method maybe is to ~~make~~ find ~~a vector space M which is to be a model for C with maps M → M~~ a concrete model M for $\tilde{C}$ as left C-module, a concrete C-module M plus module maps $M \to \tilde{C}$ plus an element $1 \in M$ mapping to $1 \in \tilde{C}$, $1 \in M$ should generate M.

$\mathbb{1}, h, h^2, \ldots$ basis for M
$e, he, h^2e, \ldots$

Take $\mathbb{C}[h]^2 = \{(f,g) \mid f,g \in \mathbb{C}[h]\}$. $\quad f(h)h + g(h)e$

Model will be $\mathbb{C} \oplus \mathbb{C}[h] \oplus \mathbb{C}[h]$ $\quad \lambda\mathbb{1} + f(h)h + g(h)e$

~~obvious m~~

$$\hat{h}\begin{pmatrix}\lambda\\ f\\ g\end{pmatrix} = \begin{pmatrix}0\\ \lambda + hf\\ hg\end{pmatrix}$$

$$e(\lambda\mathbb{1} + f(h)h + g(h)e) = e\lambda + f(h)h + g(h)e^2 = 0 + f(h)h + (\lambda + g(h))e$$

$$\hat{e}\begin{pmatrix}\lambda\\ f\\ g\end{pmatrix} = \begin{pmatrix}0\\ f\\ \lambda+g\end{pmatrix} = \begin{pmatrix}0&0&0\\0&1&0\\1&0&1\end{pmatrix}\begin{pmatrix}\lambda\\ f\\ g\end{pmatrix}$$

$$\begin{pmatrix}0&0&0\\0&1&0\\1&0&1\end{pmatrix}\begin{pmatrix}0&0&0\\0&1&0\\1&0&1\end{pmatrix} = \begin{pmatrix}0&0&0\\0&1&0\\1&0&1\end{pmatrix} \qquad \hat{e}^2 = \hat{e}$$

$$\hat{h}\begin{pmatrix}\lambda\\ f\\ g\end{pmatrix} = \begin{pmatrix}0\\ \lambda+hf\\ hg\end{pmatrix} = \begin{pmatrix}0&0&0\\1&h&0\\0&0&h\end{pmatrix}\begin{pmatrix}\lambda\\ f\\ g\end{pmatrix}$$

$$\begin{pmatrix}0&0&0\\0&1&0\\1&0&1\end{pmatrix}\begin{pmatrix}0&0&0\\1&h&0\\0&0&h\end{pmatrix} = \begin{pmatrix}0&0&0\\1&h&0\\0&0&h\end{pmatrix} \qquad \hat{e}\hat{h} = \hat{h}$$

$$\begin{pmatrix}0&0&0\\1&h&0\\0&0&h\end{pmatrix}\begin{pmatrix}0&0&0\\1&h&0\\0&0&h\end{pmatrix} = \begin{pmatrix}0&0&0\\h&h^2&0\\0&0&h^2\end{pmatrix}$$

$$\hat{h}\hat{e} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & h & 0 \\ 0 & 0 & h \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & h & 0 \\ h & 0 & h \end{pmatrix} \quad ?$$

$$\hat{h}^n = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & h^n \\ 0 & 0 & h^n \end{pmatrix} \qquad n \geq 2$$

$$\hat{e} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \qquad \hat{h} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & h & 0 \\ 0 & 0 & h \end{pmatrix}$$

$$\hat{h}^2 = \begin{pmatrix} 0 & 0 & 0 \\ h & h^2 & 0 \\ 0 & 0 & h^2 \end{pmatrix}, \qquad \hat{h}^3 = \begin{pmatrix} 0 & 0 & 0 \\ h^2 & h^3 & 0 \\ 0 & 0 & h^3 \end{pmatrix}$$

$$\hat{h}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & h & 0 \\ 0 & 0 & h \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad ?$$

~~Re~~ Mistake some where

$\tilde{C}$ $\quad 1, e, h, he, h^2, h^2e, h^3, \ldots$

model for $\tilde{C}$ is $\mathbb{C}1 \oplus \mathbb{C}[h]e \oplus \mathbb{C}[h]h$

typical elt is $\lambda 1 + fe + gh$

$$\hat{e}(\lambda 1 + fe + gh) = \lambda e + fe + gh$$

$$\hat{e}\begin{pmatrix} \lambda \\ f \\ g \end{pmatrix} = \begin{pmatrix} 0 \\ \lambda + f \\ g \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} \lambda \\ f \\ g \end{pmatrix}$$

$$\hat{h}(\lambda 1 + fe + gh) = \lambda h + hfe + hgh$$

$$\hat{h}\begin{pmatrix}\lambda\\ f\\ g\end{pmatrix} = \begin{pmatrix}0\\ hf\\ \lambda + hg\end{pmatrix} = \begin{pmatrix}0&0&0\\0&h&0\\1&0&h\end{pmatrix}\begin{pmatrix}\lambda\\ f\\ g\end{pmatrix}$$

$$\hat{h}^n = \begin{pmatrix}0&0&0\\h^{n-1}&h^n&0\\0&0&h^n\end{pmatrix}$$ first version

$$\hat{h}^n\hat{e} = \begin{pmatrix}0&0&0\\h^{n-1}&h^n&0\\0&0&h^n\end{pmatrix}\begin{pmatrix}0&0&0\\0&1&0\\1&0&1\end{pmatrix} = \begin{pmatrix}0&0&0\\0&h^n&0\\h^n&0&h^n\end{pmatrix}$$

apply to $\begin{pmatrix}1\\0\\0\end{pmatrix}$

$$\hat{h}^n\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}0\\h^{n-1}\\0\end{pmatrix}$$ means $h^{n-1}h$

$$\hat{h}^n\hat{e}\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}0\\0\\h^n\end{pmatrix}$$ means $h^n e$

$$\hat{h}^2 = \begin{pmatrix}0&0&0\\0&h&0\\1&0&h\end{pmatrix}\begin{pmatrix}0&0&0\\0&h&0\\1&0&h\end{pmatrix} = \begin{pmatrix}0&0&0\\0&h^2&0\\h&0&h^2\end{pmatrix}$$

$$\hat{h}^{n+1} = \begin{pmatrix}0&0&0\\0&h&0\\1&0&h\end{pmatrix}\begin{pmatrix}0&0&0\\0&h^n&0\\h^{n-1}&0&h^n\end{pmatrix} = \begin{pmatrix}0&0&0\\0&h^{n+1}&0\\h^n&0&h^{n+1}\end{pmatrix}$$

$$\hat{h}^{n+1}\hat{e} = \begin{pmatrix}0&0&0\\0&h^{n+1}&0\\h^n&0&h^{n+1}\end{pmatrix}\begin{pmatrix}0&0&0\\1&1&0\\0&0&1\end{pmatrix} = \Big($$

back to $\Phi = \Gamma = \mathbb{Z}/2$. We have just studied ~~the~~ the alg. gens h_0, h_1 rels. $(h_0+h_1)h_i = h_i$ ~~You want to understand~~ If you require also $h_i(h_0+h_1) = h_i$ then C is unital alg.

$\mathbb{C}[h_0,h_1]/(h_0+h_1-1)$. Return to problem of A_Φ

gen p_0, p_1 rels $(p_0 \pm p_1)^2 = p_0 \pm p_1$. Cuntz's alg

$\mathbb{C}e * \mathbb{C}e = \Omega(\mathbb{C}e)$ Fedov. not unital.

$1 \quad \begin{matrix} e \\ \bar{e} \end{matrix}$ What do you

You A is a superalg. The basic idea is that ~~you~~ $Q(A) \rtimes \mathbb{Z}/2$ will have a canonical proj. p.

$A = \mathbb{C}e * \mathbb{C}e'$ ~~What is the point?~~ You want to ~~che~~ check that there is a Morita equivalence between A and the [illegible] unital ring $B = \mathbb{C}[h_0,h_1]/(h_0+h_1-1) \rtimes \mathbb{Z}/2$

~~You need to check that~~ $(h_0 - \frac{1}{2}) + (h_1 - \frac{1}{2}) = 0$.

So $B = \mathbb{C}[h] \rtimes \mathbb{Z}/2$ action $\varepsilon h \varepsilon = -h$

B is unital so firm module are fairly clear.

How do you, What approach, path seems best. A is a ring generated by two idempotents $p_0 \pm p_1$ or

$e = p_0 + p_1 = e + de$

It's a superalg. $\bar{e} = p_0 - p_1 = e - de$

$(e+de)\cdot(e+de) = e^2 - \cancel{de^2} + ede + dee \cancel{+(de)^2} = e^2 + d(e^2)$

Obvious basis ~~$e \ \bar{e} \ e\bar{e} \ \bar{e}e \ ee$~~

$1 \quad e \quad \bar{e} \quad \bar{e}e \quad e\bar{e} \quad e\bar{e}e \quad \bar{e}e\bar{e}$

A is nonunital $\quad$ B is unital

$\begin{pmatrix} A & Q \\ P & B \end{pmatrix}$ $\quad$ P, Q f.g. projective over A. ~~and~~ dual to each other

and B is the endom. ring of P.

What do we know about $A = \mathbb{C}e \ast \mathbb{C}\bar{e}$

de commutes with $de^2 = de \circ de$

$\cdot (de)^2 \circ e = (de)^2 e - \cancel{d(de)^2 de}$

$e \circ de = e\,de \qquad de \circ e = (de)e = \underbrace{d(e^2)}_{de} - e\,de$

~~So basic A appears as $(de)^2$~~

B is unital, a firm B-module is a vector space E with $\mathbb{Z}/2$ action i.e. operator $\varepsilon \ni \varepsilon^2 = 1$ and an operator h_0 such that $h_0 + \varepsilon h_0 \varepsilon = 1$. Need next to understand $V = h_0 E$

$$\begin{tikzcd} E \arrow[r, "\alpha"] \arrow[dr, "\alpha_1 = h_0"'] & \mathbb{C}[\Gamma] \otimes V \arrow[r, "\beta"] \arrow[d, shift right, "p_0"'] & E \\ & h_0 E = V \arrow[u, shift right, "\iota_0"'] \arrow[ur, hook, "\beta_1 = \mathrm{inc.}"'] & \end{tikzcd}$$

E is a vector space equipped with operators ε, h_0 such that $\varepsilon^2 = 1$, $h_0 + \varepsilon h_0 \varepsilon = 1$. Somehow you can construct E from $h_0 E$ and two maps p_0, p_1 in $\mathcal{L}(h_0 E)$

$$B = \mathbb{C}[h_0, h_1]/(h_0+h_1-1) \rtimes \mathbb{Z}/2$$

unitale ring. $\quad E \quad$ B-module (form)

$$E \xrightarrow{\alpha_0 = h_0} h_0 E \xrightarrow{\beta_0 = \text{inc.}} E \qquad h_0 = \beta_0 \alpha_0$$

$$E \xrightarrow{\alpha} \mathbb{C}\Gamma \otimes h_0 E \xrightarrow{\beta} E \xrightarrow{\alpha} \mathbb{C}\Gamma \otimes h_0 E$$

$$(\alpha\xi)_s = \alpha_1 s^{-1}\xi \qquad \beta\Big(\sum_t t\otimes f_t\Big) = \sum_t t\beta_1 f_t$$

$$\Big(\alpha\beta\Big(\sum t\otimes f_t\Big)\Big)_s = \Big(\alpha\Big(\sum_t t\beta_1 f_t\Big)\Big)_s = \alpha_1 s^{-1}\sum_t t\beta_1 f_t$$

$$= \sum_t \underbrace{\alpha_1 s^{-1} t \beta_1}_{p(s^{-1}t)} f_t$$

$$p_0 = \alpha_0\beta_0 \qquad p_1 = \alpha_0 \varepsilon \beta_0$$

$$p_0 + p_1 = \alpha_0(1+\varepsilon)\beta_0$$

$$(p_0+p_1)^2 = \alpha_0 \underbrace{\underbrace{(1+\varepsilon)h_0(1+\varepsilon)}_{(h_0 + \varepsilon h_0 + h_0\varepsilon + \varepsilon h_0 \varepsilon)}}_{h_0 + h_1 + \varepsilon(h_0+h_1) = (1+\varepsilon)}\beta_0$$

$$p_0^2 + p_1^2 = \alpha_0 h_0 \beta_0 + \alpha_0 \underbrace{\varepsilon h_0 \varepsilon}_{h_1}\beta_0 = \alpha_0\beta_0 = p_0$$

$$p_0p_1 + p_1p_0 = \alpha_0(h_0)\varepsilon\beta_0 + \alpha_0\varepsilon h_0\beta_0$$

$$= \alpha_0(h_0+h_1)\varepsilon\beta_0 = \alpha_0\varepsilon\beta_0 = p_1$$

Since there is no support condition here you can choose $h_0 = \beta_0\alpha_0 : E \xrightarrow{\alpha_0} V \xrightarrow{\beta_0} E$ arbitrarily - shouldn't make a difference, you expect to get nil equivalent A-modules.

So the argument seems to work!!!! It should be clear that $h_0B = h_0C \otimes \mathbb{C}\Gamma$

$A \qquad Q$

$P \qquad B$ unital $\qquad C \rtimes \mathbb{Z}/2$

Bh_0 $\qquad$ Look at $\quad h_0B = h_0C \otimes (\mathbb{C} \oplus \mathbb{C}\varepsilon)$

C is the poly ring gen. by h_0

$$0 \longrightarrow B \xrightarrow{h_0 \cdot} B \longrightarrow \mathbb{C}[\mathbb{Z}/2] \longrightarrow 0$$

Idea is that h_0B is the bimodule Q left A, right B. And Bh_0 is the left B, right A bimodule P.

$$0 \longrightarrow C \xrightarrow{h_0 \cdot} C \longrightarrow \mathbb{C} \longrightarrow 0$$

$$P = \mathbb{C}\Gamma \otimes Ch_0 \qquad Q = h_0C \otimes \mathbb{C}\Gamma$$

$$B = \mathbb{C}\Gamma \otimes C$$

$A \qquad h_0B$

$Bh_0 \qquad B$

In this finite Γ case B is unital. Is it clear that $Bh_0h_0B = B$?

$A \qquad h_0 B$

$Bh_0 \qquad B$

now B is unital, so ~~[illegible]~~ it's necessary in order for a Meq that $Bh_0 h_0 B = B$

Review the situation: Γ finite $B = C \rtimes \Gamma$, where C has gen h_s $s \in \Gamma$ subject to $\sum_{t\in\Gamma} h_t h_s = h_s \sum_{t\in\Gamma} h_t = h_s$ which means that C is unital. ~~it the unital algebra~~ C has gen. h_s, $s\in\Gamma$ subject to $\sum_{s\in\Gamma} h_s = 1$, a non-commutative simplex. $B = C \rtimes \Gamma$ is unital. So return to E a firm B-module, same as a vector space with Γ action and operator h_1 such that $\sum_{t\in\Gamma} t h_1 t^{-1} = 1$. You are now trying

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Anyway ~~[illegible]~~ Bh_0 should be $\mathbb{C}\Gamma \overset{p}{\otimes} A$

Γ finite $\quad \Phi = \Gamma \quad C$ gens $h_s, s\in\Gamma \quad \sum h_s = 1$

$B = C \rtimes \Gamma$ unital. Let E be firm B-module

$$E \xrightarrow{\alpha_1 = h_1} h_1 E \xhookrightarrow{\beta_1 = \mathrm{inc}} E$$

$$E \xrightarrow{\alpha} \mathbb{C}\Gamma \otimes h_1 E \xrightarrow{\beta} E \xrightarrow{\alpha}$$

$$\xi \longmapsto \sum s \otimes \alpha_1 s^{-1} \xi \longmapsto \cancel{\sum_s \ldots}$$

$$\sum t \otimes f_t \longmapsto \sum t \beta_1 f_t$$

$\beta\alpha$: $\quad \xi \longmapsto \sum_s s \otimes \alpha_1 s^{-1}\xi \longmapsto \sum_s s\beta_1 \alpha_1 s^{-1} \xi = \sum h_s \xi = \xi$

$\alpha\beta$: $\quad \sum t \otimes f_t \longmapsto \sum t \beta_1 f_t \longmapsto \sum_s s \otimes \underbrace{\alpha_1 s^{-1} \sum t \beta_1 f_t}_{\sum p(s^{-1}t) f_t}$

where $P_t = \alpha_1 t \beta_1$ What are you trying to prove?

Philosophy B is a unital ring so

Start again $\Phi = \Gamma$ finite C = alg gens h_s, $s \in \Gamma$ rel $\sum h_s = 1$.

$B = C \rtimes \Gamma$. C, B unital. A ferm B module is a ~~vector space~~ vector space E with Γ-action and op. $h_1 \ni$ $\displaystyle\sum_{s \in \Gamma} s h_1 s^{-1} = 1$. Ex. $\Gamma = \mathbb{Z}/2 = \{0, 1\}$

$$C = \mathbb{C}[h_0, h_1]/(h_0 + h_1 - 1) \simeq \mathbb{C}[h_0].$$

A ferm B-module is a $\mathbb{Z}/2$-graded vector space $V = V_+ \oplus V_-$ with an operator h_0 such that

$h_0 + \varepsilon h_0 \varepsilon = 1$ $\quad h_0 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ $\quad h_1 = \varepsilon h \varepsilon = \begin{pmatrix} a & -b \\ -c & d \end{pmatrix}$

$= h_0 - \frac{1}{2} + h_1 - \frac{1}{2}$ $\qquad h = h_0 - \frac{1}{2}$

$h_0 = \frac{1}{2} + h$ $\qquad h_1 = 1 - h_0 = \frac{1}{2} - h$

$h_0 = \begin{pmatrix} 1/2 & b \\ c & 1/2 \end{pmatrix}$ $\qquad b, c$ arb.

Check this carefully

A B-module is a vector space E with $\mathbb{Z}/2$ action ε, equivalently a $\mathbb{Z}/2$-grading $E_+ \oplus E_-$ and an op. h_0 such that $h_0 + \varepsilon h_0 \varepsilon = 1$.

$$h_0 = \begin{pmatrix} a & b \\ c & d \end{pmatrix} + \varepsilon h_0 \varepsilon = \begin{pmatrix} a & -b \\ -c & d \end{pmatrix} = \begin{pmatrix} 2a & 0 \\ 0 & 2d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\therefore \quad h_0 = \begin{pmatrix} 1/2 & b \\ c & 1/2 \end{pmatrix}$$

Now look at $h_0 : E \xrightarrow{\alpha} h_0 E \xhookrightarrow{\beta} E$

Use any factorization of h_0. Simplest is

$$E \xrightarrow{\alpha_0 = h_0} E \xrightarrow{\beta_0 = 1} E$$

What is the corresponding $p_s = \alpha_0 s \beta_0$

$$p_0 = \alpha_0 \beta_0 = h_0$$
$$p_1 = \alpha_0 \varepsilon \beta_0 = h_0 \varepsilon$$

$$(p_0 + p_1)^2 = h_0(1 \pm \varepsilon) h_0 (1 \pm \varepsilon)$$
$$= h_0 h_0 \pm \underbrace{h_0 \varepsilon h_0}_{h_0 h_1 \varepsilon} \pm h_0 h_0 \varepsilon + \underbrace{h_0 \varepsilon h_0 \varepsilon}_{h_0 h_1}$$
$$= h_0(h_0 + h_1) \pm h_0(h_1 + h_0)\varepsilon = h_0 \pm h_0 \varepsilon.$$

So it seems to work. Specifically you have a homomorphism $A \longrightarrow B$ sending $p_0 \longmapsto h_0$ and $p_1 \longmapsto h_1$?

$\Phi = \Gamma$ finite. A = unital alg gens $h_s, s \in \Gamma$ rel $\sum_{s \in \Gamma} h_s = 1$

$B = C \rtimes \Gamma$ unital. A firm B-module is a v.s. E with Γ acting and an op h_1 on E s.t. $\sum_{s \in \Gamma} s h_1 s^{-1} = 1$. Factor $h_1 = \beta_1 \alpha_1$

$$E \xrightarrow{\alpha_1 = h_1} E \xrightarrow{\beta_1 = 1} E \longrightarrow$$

Then E becomes an A-module with $p_s = \alpha_1 s \beta_1 = h_1 s$. $\sum_t p_{st^{-1}} p_t = \sum_t h_1 s t^{-1} h_1 t = h_1 s = p_s$

Another factorization is

$$E \xrightarrow{\alpha_1 = 1} E \xrightarrow{\beta_1 = h_1} E \longrightarrow$$

$$p_s = \alpha_1 s \beta_1 = s h_1 \qquad \sum_t p_{st^{-1}} p_t = \sum_t s t^{-1} h_1 t h_1 = s h_1 = p_s$$

So you have two homom. $A \longrightarrow B$ which are supposed to induce the Morita equivalence.

$$p_s \longmapsto h_1 s \qquad \text{or} \qquad p_s \longmapsto s h_1$$

Go back to factorizations

$$\begin{tikzcd}
E \arrow[rr, "\alpha"] \arrow[dr, two heads] & & V \arrow[rr, "\beta"] \arrow[dr, two heads] & & E \\
& \alpha E \arrow[ur, hook] \arrow[dr, two heads] & & \beta V \arrow[ur, hook] & \\
& & \beta\alpha V \arrow[ur, hook] & &
\end{tikzcd}$$

each object yields a factorization. It seems there is a category of factorizations

$$\begin{tikzcd}
& V \arrow[dr, "\beta'"] & \\
E \arrow[ur, "\alpha'"] \arrow[dr, "\alpha"'] & & E \\
& W \arrow[uu, "\gamma"] \arrow[ur, "\beta"'] &
\end{tikzcd}$$

where a map from $(\beta, \alpha) \longrightarrow (\beta', \alpha')$ is a γ such that $\gamma\alpha = \alpha'$, $\beta = \beta'\gamma$

Try to understand the effect of this

$$p_s = \alpha s \beta = \alpha s \beta' \gamma \qquad \gamma \alpha s \beta' = \alpha' s \beta' = p'_s$$

It seems

$$\begin{tikzcd}
V \arrow[r, "\beta'"] & E \arrow[r, "\alpha'"] \arrow[d, "s"] & V \\
W \arrow[u, "\gamma"] \arrow[r, "\beta"] & E \arrow[r, "\alpha"] & W \arrow[u, "\gamma"']
\end{tikzcd}
\qquad
\begin{tikzcd}
V \arrow[dr, "\beta'"] \arrow[rr, bend left, "\rho'_s"] & & V \\
& E \arrow[ur, "\alpha'"] \arrow[dr, "\alpha"] & \\
W \arrow[uu, "\gamma"] \arrow[ur, "\beta"] \arrow[rr, bend right, "\rho_s"'] & & W \arrow[uu, "\gamma"']
\end{tikzcd}$$

It seems that you just get an A-module map $\gamma : W \longrightarrow V$

$$\gamma \rho_s = \underbrace{\gamma \alpha}_{\alpha'} s \beta \qquad \rho'_s \gamma = \alpha' s \underbrace{\beta' \gamma}_{\beta}$$

Check: Given

$$\begin{tikzcd}
& V \arrow[dr, "\beta"] & \\
E \arrow[ur, "\alpha"] \arrow[dr, "\alpha'"'] & & E \\
& W' \arrow[uu, "\gamma"] \arrow[ur, "\beta'"'] &
\end{tikzcd}
\qquad
\begin{aligned}
\alpha &= \gamma \alpha' \\
\beta' &= \beta \gamma
\end{aligned}$$

$$\begin{tikzcd}
V \arrow[dr, "\beta"] \arrow[rr, bend left, "\alpha s \beta"] & & V \\
& E \arrow[ur, "\alpha"] \arrow[dr, "\alpha'"] & \\
V' \arrow[uu, "\gamma"] \arrow[ur, "\beta'"] \arrow[rr, bend right, "\alpha' s \beta'"'] & & V' \arrow[uu, "\gamma"']
\end{tikzcd}$$

$$\begin{aligned}
\rho'_s &= \alpha' s \beta' \quad \text{on } V' \\
\rho_s &= \alpha s \beta \quad \text{on } V
\end{aligned}
\qquad
\underbrace{\gamma \alpha'}_{\alpha} s \beta' \stackrel{?}{=} \alpha s \underbrace{\beta \gamma}_{\beta'}$$

So you find that a map between factorizations induces a map between corresp. A-modules.

Interesting cases:

$$\begin{tikzcd}
& E \arrow[dr, "id"] & \\
E \arrow[ur, "h_1"] \arrow[dr, "id"'] & & E \\
& E \arrow[uu, "h_1"] \arrow[ur, "h_1"'] &
\end{tikzcd}$$

Interesting case:

$$\begin{tikzcd} & E \arrow[dr, "1=\beta"] & \\ E \arrow[ur, "\alpha=h_1"] \arrow[dr, "\alpha'=1"'] & & E \\ & E \arrow[uu, "h_1"] \arrow[ur, "h_1=\beta'"'] & \end{tikzcd}$$

$$\begin{tikzcd} E \arrow[rr, bend left, "p_s = \alpha s \beta = h_1 s"] \arrow[dr, "1=\beta"'] & & E \\ & E \arrow[ur, "\alpha=h_1"'] \arrow[dr, "\alpha'=1"] & \\ E \arrow[uu, "h_1"] \arrow[ur, "h_1=\beta'"] \arrow[rr, bend right, "p'_s = \alpha' s \beta' = s h_1"'] & & E \arrow[uu, "h_1"'] \end{tikzcd}$$

So you have E considered as an A-module ~~[illegible]~~ in two ways $p_s = h_1 s$, $p'_s = s h_1$ and

$$\begin{tikzcd} E \arrow[r, "p_s"] & E \\ E \arrow[u, "h_1"] \arrow[r, "p'_s"] & E \arrow[u, "h_1"'] \end{tikzcd}$$

intertwines them

Look at p_s on E/h_1E. $p_s = h_1 s$ get 0.
Similarly $p'_s = 0$ on $\mathrm{Ker}(h_1)$.

Things are clearer. Review. C gen h_s, $s \in \Gamma$ rel $\sum h_s = 1$, $B = C \rtimes \Gamma$, firm B-mods $=$ Γ modules tog. with h_1 op $\ni$ $\sum s h_1 s^{-1} = 1$.

$$E \xrightarrow{\alpha_1} E \xrightarrow{\beta_1} E. \qquad p_s = \alpha_1 s \beta_1 \quad \text{either } \begin{matrix} h_1 s \\ s h_1 \end{matrix}$$

~~[illegible]~~ Choice leads to the same A module mod nil modules. So what can you do $\boxed{\text{next???}}$

Philosophy: B is a unital ring, so $\begin{pmatrix} A & Q \\ P & B \end{pmatrix}$

${}_BP$, Q_B are unitary modules equipped with pairing $P \otimes Q \longrightarrow B$ which is onto, so there must be $\sum p_i q_i = 1$. Now you have some idea about Q, namely how it looks up to nil equivalence. It should be B with A acting as $p_s = h_1 s$ or $p'_s = s h_1$.