How to straighten things out. Go back to your example, where $E = \mathbb{C}[\Gamma]\otimes W$ h_1 proj onto $\delta_1\otimes W$, ~~and you~~ and you have an arbitrary fact of h_1.

$$\begin{tikzcd}
\mathbb{C}[\Gamma]\otimes W \arrow[r, "\alpha_1"] \arrow[dr, two heads] & V \arrow[r, "\beta_1"] & \mathbb{C}[\Gamma]\otimes W \arrow[r, "\alpha_1"] \arrow[d] & {} \\
W \arrow[u, dashed] & W \arrow[ur, hook] & W &
\end{tikzcd}$$

You think that W should split off everything leaving

$$\mathbb{C}[\Gamma-\{1\}]\otimes W \xrightarrow{\alpha_1^\perp} V^\perp \xrightarrow{\beta_1^\perp} \mathbb{C}[\Gamma-\{1\}]\otimes W$$

$\underline{C_c(\Gamma)}$

Start again. $E = \mathbb{C}[\Gamma]\otimes W$ h_s = projection on $\delta_s\otimes W$. Take an arb. fact $h_1 = \beta_1\alpha_1$

$$\begin{tikzcd}
C_c(\Gamma)\otimes W \arrow[r, "\alpha_1"] \arrow[dr, two heads, "pr_1"] & V \arrow[r, "\beta_1"] & C_c(\Gamma)\otimes W \arrow[d, "pr_1"] \\
W_1 \arrow[u, hook, "in_1"] \arrow[r, phantom, "="] & W \arrow[ur, hook, "in_1"] \arrow[r, phantom, "="] & W
\end{tikzcd}$$

Remove W pass to the kernel of the maps pr_1, $pr_1\beta_1$

You seem to be looking at a fact of the 0 map.

$$\begin{tikzcd}
E \arrow[rr, "\alpha_1"] \arrow[dr, two heads] & & V \arrow[rr, "\beta_1"] \arrow[dr, two heads] & & E \\
& \alpha_1 E \arrow[ur, hook] \arrow[rr, "0"] \arrow[dr, two heads] & & \beta_1 V \arrow[ur, hook] & \\
& & \beta_1\alpha_1 E \arrow[ur, hook] & &
\end{tikzcd}$$

Next see what happens when you enlarge

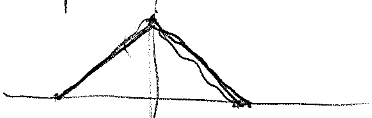
$$\begin{tikzcd}
E \arrow[r, "\alpha"] \arrow[rr, bend right, "1"'] & C_c(\Gamma)\otimes V \arrow[r, "\beta"] & E \arrow[r, "\alpha"] & C_c(\Gamma)\otimes V
\end{tikzcd}$$

α β γ δ, ε ζ η Θ, ι κ λ μ, ν ξ ο π, ρ σ τ υ, φ χ ψ ω

OLD SHEET 93a

Let's state the problem arising. You want to look at $C_c(\mathbb{R})$ as a $B = C_\Phi \rtimes \mathbb{Z}$-module. You believe in a Morita equivalence between B modules and P_Φ modules. So $C_c(\mathbb{R})$ should corresponds to a P_Φ-module, you calculated this to be $C([-1,1])$ $p_0 = h_0$, $p_1 = h_1 h_0 u$, $p_{-1} = h_{-1} h_0 u^{-1}$ $u =$ shift. $p_n = \alpha_0 u^n \beta_0$

$h_0 = \beta_0 \alpha_0$

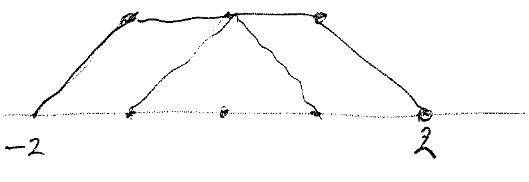


$\beta_0 : C([-1,1]) \to C_o(\mathbb{R})$ mult by h_0, $\alpha_0 : C_a(\mathbb{R}) \to C([-1,1])$ mult by $h_{-1} + h_0 + h_1$ + restriction to $[-1,1]$. This seems amazingly concrete. ~~So in fact~~

You look at $h_0 : C_c(\mathbb{R}) \xrightarrow{res} C([-1,1]) \xhookrightarrow{h_0} C_c(\mathbb{R})$

$$\begin{tikzcd}
C_c(\mathbb{R}) \arrow[rr, "\overset{k_0 = h_{-1}+h_0+h_1}{\overset{\shortparallel}{\alpha_0}}"] \arrow[dr, "k_0"] & & C_c(\mathbb{R}) \arrow[rr, "\beta_0 = h_0"] \arrow[dr, "h_0"] & & C_c(\mathbb{R}) \\
& k_0 C_c(\mathbb{R}) \arrow[ur, hook] \arrow[dl, equal] \arrow[dr] & & h_0 C_c(\mathbb{R}) \arrow[ur, hook] & \\
C([-2,2]) \arrow[drr, "res"] & & h_0 C_c(\mathbb{R}) \arrow[ur, hook] \arrow[d, equal] & & \\
& & C([-1,1]) & &
\end{tikzcd}$$

$$\begin{tikzcd}
C_c(\mathbb{R}) \arrow[rr, "\alpha_0 = k_0"] \arrow[dr, two heads, "res"'] & & C_c(\mathbb{R}) \arrow[rr, "\beta_0 = h_0"] \arrow[dr, two heads, "res"] & & C_c(\mathbb{R}) \\
& C([-2,2]) \arrow[ur, hook, "k_0"] \arrow[rr, two heads, "\cancel{res}"] \arrow[dr, two heads, "res"'] & & C([-1,1]) \arrow[ur, hook, "h_0"] \arrow[dl, equal] & \\
& & C([-1,1]) & &
\end{tikzcd}$$



Consider C_Φ in the case Γ arb., $\Phi = \{1\}$

$C_\Phi = C_c(\Gamma)$ under mult.

$= \bigoplus_{s \in \Gamma} \mathbb{C} h_s \qquad h_s h_t = \delta_{st} h_t$

Any finin B-mod has form $C_c(\Gamma) \otimes W = \mathbb{C}[\Gamma] \otimes W$ and h_s projects onto the s component $\simeq \bigoplus_{s \in \Gamma} s \otimes W$

~~Now choose a fact. $h_1 = \beta_1 \alpha_1$~~ :

Take $E = C_c(\Gamma) \otimes W$, choose a fact $h_1 = \beta_1 \alpha_1 : E \to V \to E$

$$\begin{tikzcd}
\mathbb{C}[\Gamma] \otimes W \arrow[r, "\alpha_1"] \arrow[d, shift right] & V \arrow[r, "\beta_1"] & \mathbb{C}[\Gamma] \otimes W \arrow[d, shift right] \\
W \arrow[u, shift right] \arrow[rr, equal] & & W \arrow[u, shift right]
\end{tikzcd}$$

~~What happens is that~~ W is naturally a summand of E, V preserved by α_1, β_1.

~~category of v. spaces over W~~

cat of X over S with section

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, shift left, "g"] & X' \arrow[d, shift left, "g'"] \\
S \arrow[u, shift left, "i"] \arrow[r, equal] & S \arrow[u, shift left, "i'"]
\end{tikzcd}
\qquad
\begin{aligned} g'f &= g \\ fi &= i' \end{aligned}$$

$$\begin{tikzcd}
X \arrow[r] & X' \arrow[dl] \\
S \arrow[u, bend left] &
\end{tikzcd}
\qquad \circlearrowleft 1_S$$

$EW = \mathbb{C}[\Gamma] \otimes W$ $\quad$ h_1 = proj onto component $1 \otimes W$

$h_1 = \iota_1 \gamma_1 : \mathbb{C}[\Gamma]\otimes W \twoheadrightarrow W \hookrightarrow \mathbb{C}[\Gamma]\otimes W$

Choose a factorization of h_1

$$EW \xrightarrow{\alpha_1} V \xrightarrow{\beta_1} EW$$

$$EW \xrightarrow{\gamma_1} W \xhookrightarrow{\iota_1} EW$$

In the general case you have

$$\begin{tikzcd}
E \arrow[r, "\alpha_1"] \arrow[dr, two heads, "\alpha_1 E"'] & V \arrow[r, "\beta_1"] & E \\
& h_1 E \arrow[ur, hook, "\beta_1 V"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
E \arrow[d, two heads] \arrow[dr, "\alpha_1"] & & \\
\alpha_1 E \arrow[r, hook] \arrow[d, two heads] & V \arrow[d] \arrow[dr, "\beta_1"] & \\
\beta_1\alpha_1 E \arrow[r, hook] & \beta_1 V \arrow[r, hook] & E
\end{tikzcd}$$

Is there an obvious way to make factorizations of a map into a category? Yes.

$$\begin{tikzcd}
E \arrow[r, "\alpha"] \arrow[d, "\alpha'"'] & V \arrow[d, "\beta"] \\
V' \arrow[ur] \arrow[r, "\beta'"'] & F
\end{tikzcd}$$

Start again.

You need to properly understand the M.eq between $B = C_\Phi \rtimes \Gamma$ and A_Φ. Let E be a firm B-module i.e. $\sum_{s \in \Gamma} h_s = 1$ on E. Choose a fact $h_1 = \beta_1 \alpha_1 : E \xrightarrow{\alpha_1} V \xrightarrow{\beta_1} E$, then you get

$$\begin{tikzcd}
E \arrow[r, "\alpha"] \arrow[rr, bend right, "1"'] & \mathbb{C}[\Gamma]\otimes V \arrow[r, "\beta"] \arrow[rr, bend left, "p"] & E \arrow[r, "\alpha"] & \mathbb{C}[\Gamma]\otimes V
\end{tikzcd}$$

where $p = p^2$ in $\mathbb{C}[\Gamma] \otimes \mathcal{L}(V)$.

$$(pf)(s) = \sum \underbrace{(\alpha_1 s^{-1} t \beta_1)}_{p(s^{-1}t)} f(t)$$

But there's a problem with the support, namely $h_s h_t = 0 \iff h_1 s^{-1} t h_1 = 0 \iff \beta_1 (\alpha_1 s^{-1} t \beta_1) \alpha_1 = 0$

so you need β_1 inj, α_1 surj ~~to~~ conclude $\alpha_1 s^{-1} t \beta_1 = 0$. You feel there should be a way to ~~control this behavior~~ circumvent this problem. To find it you look first at the case $\Phi = \{1\}$, whence $\{h_s\}_{s \in \Gamma}$ are mutually annihilating idempotents, and we have $E = \mathbb{C}[\Gamma] \otimes W$:

with $h_1 = \iota_1 j_1 : \mathbb{C}[\Gamma] \otimes W \xrightarrow[j_1]{} W \xrightarrow[\iota_1]{} \mathbb{C}[\Gamma] \otimes W$.

Put $EW = \mathbb{C}[\Gamma] \otimes W$ and choose the factorization

$$\begin{tikzcd}
EW \arrow[r, "\alpha_1"] \arrow[d, "j_1"'] & V \arrow[d, "\beta_1"] & \\
W \arrow[r, hook, "\iota_1"'] & EW \arrow[r, "\alpha_1"] \arrow[d, "j_1"'] & V \arrow[d, "\beta_1"] \\
& W \arrow[r, hook, "\iota_1"] & EW
\end{tikzcd}$$

So what is important?

Each factorization leads to a Γ-graded projection in $\mathcal{L}(V)$. At the moment you have no control over the support. But ~~what is~~ you can join any factorization to the minimal one

$$\begin{tikzcd} EW \arrow[d, "j_1"'] & \\ W \arrow[r, hook, "\iota_1"] & EW \end{tikzcd}$$

So what can I try? Review the problem.
Given Γ, Φ you get $B = C_\Phi \rtimes \Gamma$, $A = P_\Phi$ which are Morita equivalent. Given a firm B-mod E to get the corresp A-module, you factor $h_1 : E \xrightarrow{\alpha_1} V \xrightarrow{\beta_1} E$, then get Γ-mod maps

$$\begin{tikzcd} E \arrow[r, "\alpha"] & \mathbb{C}[\Gamma]\otimes V \arrow[r, "\beta"] \arrow[d, shift right, "j_1"'] & E \\ & V \arrow[u, shift right, hook, "\iota_1"'] & \end{tikzcd}$$

$$\begin{cases} \alpha_1 = j_1 \alpha \\ \beta_1 = \beta \iota_1 \end{cases}$$

such that $\beta\alpha = 1_E$ hence $p = \alpha\beta$ is a projector on $\mathbb{C}[\Gamma]\otimes V$.

$$(pf)(s) = (\alpha\beta f)(s) = \alpha_1 s^{-1} \sum_t t \beta_1 f(t)$$
$$= \sum_t p(s^{-1}t) f(t) \qquad p(s) = \alpha_1 s \beta_1$$

The main problem is with the support of $p(s)$. You assume $h_1 s h_1 = 0$ of E for $s \notin \Phi$.

i.e. $\beta_1(\alpha_1 s \beta_1)\alpha_1 = 0 = \beta_1 p(s) \alpha_1 \qquad s \notin \Phi$

In the case of the canonical factorization thru $\mathrm{Im}(h_1)$ α_1 is surj, β_1 is inj $\therefore$ $p(s) = 0$ for $s \notin \Phi$.

Start again. $B = C \rtimes \Gamma$ $\quad C = C_\Phi$

$E = BE$ $\quad$ E firm, let $h_1 = \beta_1\alpha_1 : E \xrightarrow{\alpha_1} V \xrightarrow{\beta_1} E$

$$\begin{tikzcd}
E \arrow[r, "\alpha"] \arrow[rr, bend right, "\underline{1}"'] & \mathbb{C}[\Gamma]\otimes V \arrow[r, "\beta"] \arrow[rr, bend right] & E \arrow[r, "\alpha"] & \mathbb{C}[\Gamma]\otimes V
\end{tikzcd}$$

$$f \in C_c(\Gamma, V) \longmapsto (pf)(\sigma) = \sum_t p(s^{-1}t)\, f(t)$$

where $p(\sigma) = \alpha_1 s \beta_1$. Supp. cond: $h_1 s h_1 = 0$ $\quad s \notin \Phi$ means $\beta_1(\alpha_1 s \beta_1)\alpha_1 = \beta_1 p(s) \alpha_1 = 0$ $\quad s \notin \Phi$. This $\Rightarrow$ $p(s) = 0$ when α_1 surj, β_1 inj. You want to know what happens, when α_1, β_1 do not have these props.

$$\begin{tikzcd}
E \arrow[r, two heads, "\alpha_1"] & V \arrow[r, "\beta_1"] \arrow[d, two heads] & E \\
& h_1E \arrow[ur, hook] &
\end{tikzcd}$$

minimal case α_1 surj
β_1 inj

To keep things simple suppose $\Phi = \{1\}$, ~~so that~~ in which case $E = EW = C_c(\Gamma, W)$ and h_1 is $\iota_1 j_1$. You have maps

$$\begin{tikzcd}
& EW \arrow[r, two heads, "\alpha_1"] \arrow[dr, two heads, "j_1"] & V \arrow[r, "\beta_1"] \arrow[d, "\theta"] & EW \arrow[dr, "j_1"] & \\
W \arrow[ur, "\iota"] \arrow[rr, phantom, "\simeq"] & & W \arrow[ur, hook, "\iota"] \arrow[rr, phantom, "="] & & W
\end{tikzcd}$$

~~Since $\beta_1\alpha_1 = \iota_1 j_1$ you know that ?~~

Assume α_1 surjective. As $\beta_1\alpha_1(EW) = h_1(EW) = W$, you have $\beta_1 V = \iota_1 W$. Simpler: If α_1 surj, then $\beta_1 V = h_1(EW)$ $\iota_1 W$, $\exists!\ \theta : V \to W$ s.t. $\beta_1 = \iota_1\theta$, and then ι_1 inj $\Rightarrow$ $\theta\alpha_1 = j_1$. So V should split

$V = \alpha_1 \iota_1 W \oplus \operatorname{Ker}\theta$, also

$$\begin{tikzcd}
EV \arrow[rr, bend right, "\theta"'] & = & EW \arrow[ll, bend right, "\alpha"'] & \oplus & E(\operatorname{Ker}\theta)
\end{tikzcd}$$

Clear might be

$$\begin{tikzcd}
 & E(\operatorname{Ker}\theta) \arrow[d] \arrow[dr, "0"] & \\
EW \arrow[r, "\alpha"] \arrow[dr, equal] & EV \arrow[r, "\beta"] \arrow[d, "E(\theta)"] & EW \arrow[dl, equal] \\
 & EW &
\end{tikzcd}$$

so it seems that α splits into the identity on EW and an arb. Γ-module map $\alpha' : EW \longrightarrow E(\operatorname{Ker}\theta)$, which arises from $\underline{\alpha'_1 : EW \longrightarrow}$ $\operatorname{Ker}(\theta)$

Dually suppose β_1 injective

$$\begin{tikzcd}
 & EW \arrow[r, "\alpha_1"] \arrow[dr, "j_1"] & V \arrow[r, hook, "\beta_1"] & EW \arrow[d, "j_1"] \\
W \arrow[ur, "\iota_1"] \arrow[rr, equal] & & W \arrow[u, "\theta"] \arrow[ur, hook, "\iota_1"] \arrow[r, equal] & W
\end{tikzcd}$$

$\theta j_1 = \alpha_1$

$\beta_1 \theta = \iota_1$

$$\begin{tikzcd}
 & \text{assume} \arrow[d] & \\
EW \arrow[r, "\alpha_1"] \arrow[dr, "j_1"'] & V \arrow[r, hook, two heads, "\beta_1"] \arrow[d, shift right, "j_1\beta_1"'] & EW \arrow[d, "j_1"] \\
 & W \arrow[u, shift right, "\varphi"'] \arrow[ur, hook, "\iota_1"] \arrow[r, equal] & W
\end{tikzcd}$$

$\operatorname{Ker}(\alpha_1) = \operatorname{Ker}(j_1)$ so you get φ inj s.t. $\varphi j_1 = \alpha_1$
$\beta_1 \varphi = \iota_1$

Go back to α_1 surj

$$\begin{tikzcd}
EW \arrow[r, two heads, "\alpha_1"] \arrow[dr, "j_1"] & V \arrow[r, "\beta_1"] \arrow[d, "\theta"] & EW \arrow[d, "j_1"] \\
W \arrow[u, "\iota_1"] \arrow[r, equal] & W \arrow[ur, hook, "\iota_1"] \arrow[r, equal] & W
\end{tikzcd}$$

$\mathrm{Ker}(\alpha_1) \subset \mathrm{Ker}(j_1) \Rightarrow \exists!\ \theta \ni \theta\alpha_1 = j_1$

$\alpha_1(\xi) = 0 \Rightarrow j_1(\xi) = 0$ | $\iota_1\theta\alpha_1 = \iota_1 j_1 = \beta_1 \underbrace{\alpha_1}_{\text{surj.}} \Rightarrow \iota_1\theta = \beta_1$

Repeat the calculation. Given

$$\begin{tikzcd}
EW \arrow[r, "\alpha_1"] \arrow[dr, "j_1"] & V \arrow[r, hook, "\beta_1"] & EW \arrow[d, "j_1"] \\
W \arrow[u, "\iota_1"] \arrow[r, equal] & W \arrow[u, hook, "\varphi"] \arrow[ur, hook, "\iota_1"] \arrow[r, equal] & W
\end{tikzcd}$$

Ass β_1 inj. $\mathrm{Ker}(\alpha_1) = \mathrm{Ker}(\beta_1\alpha_1) = \mathrm{Ker}(\iota_1 j_1) = \mathrm{Ker}(j_1)$

so $\exists!\ \varphi : W \to V$ s.t. $\varphi j_1 = \alpha_1$, $\beta_1\varphi = \iota_1$

$\Downarrow$ $\Uparrow$

$\beta_1\varphi j_1 = \beta_1\alpha_1 = \iota_1 j_1$ and j_1 surj

---

Ass α_1 surj,

$$\begin{tikzcd}
EW \arrow[r, two heads, "\alpha_1"] \arrow[dr, "j_1"] & V \arrow[r, "\beta_1"] \arrow[d, "\theta"] & EW \arrow[d, "j_1"] \\
W \arrow[u, "\iota_1"] \arrow[r, equal] & W \arrow[ur, hook, "\iota_1"] \arrow[r, equal] & W
\end{tikzcd}$$

$\mathrm{Ker}(\alpha_1) \subset \mathrm{Ker}(\beta_1\alpha_1)$
$= \mathrm{Ker}(\iota_1 j_1) = \mathrm{Ker}(j_1)$

so $\exists!\ \theta : V \to W$ s.t. $\theta\alpha_1 = j_1$, then $i_1\theta\alpha_1 = \iota_1 j_1 = \beta_1\alpha_1$ and α_1 surj $\Rightarrow \iota_1\theta = \beta_1$

The next point ~~[illegible]~~ should be to find a way to control α, β. Ass β_1 injective. What choices are there for V? $V(=\beta_1 V)$ can be an subspace of EW containing $W(=\iota_1 W)$, φ is the inclusion of $\iota_1 W$ in $\beta_1 V$ and α_1 is the composition φj_1. What happens? You will have a splitting $\beta_1 V = \iota_1 W \oplus \mathrm{Ker}(j_1\beta_1)$.

simplify notation.

$\overline{EW} = \mathrm{Ker}(j_1 : EW \to W)$

$\bar{V} = \mathrm{Ker}(j_1\beta_1 : V \to W)$

$$\begin{tikzcd}
\overline{EW} \arrow[r, "0"] & \bar{V} \arrow[r, hook] & \overline{EW} \\
\oplus & \oplus & \oplus \\
W \arrow[r, phantom, "="] & W \arrow[r, phantom, "="] & W
\end{tikzcd}$$

IDEA: $EW = \mathbb{C}[\Gamma]\otimes W$ has also the norm map $\eta\otimes 1$ to W, usual augmentations map. So you have to be careful.

Begin with $V \hookrightarrow EW$

$$\begin{tikzcd}
EW \arrow[r, "j_1"] \arrow[rr, bend left, "\alpha_1"] & W \arrow[r, hook, "\varphi"] \arrow[rr, bend right, "\iota_1"'] & V \arrow[r, hook, "\beta_1"] & EW
\end{tikzcd}$$

$$\begin{tikzcd}
\mathrm{Hom}_\Gamma(M, EW) \arrow[d, "(j_1)_*"'] \\
\mathrm{Hom}(M, W)
\end{tikzcd}$$
is roughly an isom.

$$EW \qquad EV \xrightarrow{\beta} EW$$

$$\begin{tikzcd}
 & W \arrow[dd, hook, "\varphi"] \arrow[dr, squiggly, "i_1"] & \\
EW \arrow[ur, "j_1"] \arrow[dr, "\alpha_1"'] & & EW \\
 & V \arrow[ur, hook, "\beta_1"'] &
\end{tikzcd}$$

ℂ-linear maps

actually the vertical direction is backward

l

$$\begin{tikzcd}
 & EW \arrow[dd, "E\varphi"] \arrow[dr, "1"] & \\
EW \arrow[ur, "1"] \arrow[dr, "\alpha"'] & & EW \\
 & EV \arrow[ur, "\beta"'] &
\end{tikzcd}$$

---

repeat Looking at $EW \xrightarrow{\alpha_1} V \xhookrightarrow{\beta_1} EW$ β_1 inj.

$$\begin{tikzcd}
EW \arrow[r, "j_1"'] & W \arrow[r, hook, "i_1"'] & EW
\end{tikzcd}$$

so you have

$$\begin{tikzcd}
 & V \arrow[dr, hook, "\beta"] & \\
EW \arrow[dr, "j_1"'] & & EW \\
 & W \arrow[uu, hook] \arrow[ur, hook, "i_1"'] &
\end{tikzcd}$$

~~this is a fancy way of saying that if β_1 is inj. then α_1 β is fix~~

$$\begin{tikzcd}
 & V \arrow[dr, hook, "\beta_1"] & \\
EW \arrow[ur, "\alpha_1"] \arrow[dr, "j_1"'] & & EW \\
 & W \arrow[uu, hook, "\varphi"] \arrow[ur, hook, "i_1"'] &
\end{tikzcd}$$

ℂ linear picture

you

$$\begin{tikzcd}
 & EV \arrow[dr, "\beta"] & \\
EW \arrow[ur, "\alpha"] \arrow[dr, equal] & & EW \\
 & EW \arrow[uu, "E(\varphi)"] \arrow[ur, equal] &
\end{tikzcd}$$

But $V = \varphi W \oplus \mathrm{Ker}(j_1\beta_1)$

~~So it seems that β is made~~

$$\begin{tikzcd}
EV = E(\varphi W) \arrow[d, "\beta"] \arrow[d, phantom] & \oplus\, E\,\mathrm{Ker}(j_1\beta_1) \arrow[dl, "\text{arbitrary injective when rest. to } \mathrm{Ker}(j_1\beta_1)"] \\
EW &
\end{tikzcd}$$

obvious

S

$$\begin{tikzcd} & EV' \arrow[rd, "\beta'"] & \\ EW \arrow[r, "\begin{pmatrix}0\\1\end{pmatrix}"] & \oplus \; EW \arrow[r, "(\beta' \ 1)"'] & EW \end{tikzcd}$$

comp is id. $\begin{pmatrix}0\\1\end{pmatrix}(\beta' \quad 1) = \begin{pmatrix}0 & 0\\ \beta' & 1\end{pmatrix}$

α_1 surj case.

$$\begin{tikzcd} & & V \arrow[d, "\theta"] \arrow[rd, "\beta_1"] & \\ & EW \arrow[ru, two heads, "\alpha_1"] \arrow[r, "j_1"'] & W \arrow[r, hook, "\iota_1"'] & EW \\ W \arrow[ru, "\iota_1"] \arrow[rr, equal] & & W & \end{tikzcd}$$

~~$V = \mathrm{Im}(\alpha_1\iota_1) = \alpha_1\iota_1 W$~~

$V = \underbrace{\alpha_1\iota_1 W}_{W} \oplus \underbrace{\mathrm{Ker}(\theta)}_{V'}$

$\beta_1(V') = \iota_1\theta V' = 0$ in EW.

EV

$$\begin{tikzcd} EW \arrow[rd, equal] & & EW \arrow[ld, equal] \\ & EW & \end{tikzcd}$$

$$\begin{tikzcd} EV = E(\alpha_1\iota_1 W) \arrow[d, "\beta"] \arrow[d, shift left=4, "1"] & \oplus \; E(V') \arrow[ld, "0"] \\ EW & \end{tikzcd}$$

So it seems you have

$$\begin{tikzcd} EW \arrow[r, "\begin{pmatrix}\alpha'\\1\end{pmatrix}"] & EV' \oplus EW \arrow[r, "(0 \ 1)"] & EW \end{tikzcd} \qquad ?$$

~~Interesting~~ part has to be with α_1. ~~[illegible] is~~ ~~[illegible]~~ V' is ~~any~~ roughly any ~~quotient~~ of EW/W just as before V' is roughly any subspace of $\mathrm{Ker}(j_1 : EW \to W)$

General
$$\begin{tikzcd} EW \arrow[rr, "\alpha_1"] \arrow[dr, two heads] & & V \arrow[rr, "\beta_1"] & & EW \\ & \cdot \arrow[ur, hook] \arrow[dr, two heads] & & \cdot \arrow[ur, hook] & \\ & & W \arrow[ur, hook] & & \end{tikzcd}$$

~~[illegible] over preceding. Start with general $\Sigma \subset \Gamma$~~

What about $p(s) = \alpha_1 s \beta_1$. First case β_1 injective.

$$\begin{tikzcd} EW \arrow[r, "\alpha_1"] \arrow[dr, "j_1"'] & V \arrow[r, hook, "\beta_1"] & EW \\ & W \arrow[u, "\varphi"] \arrow[ur, hook, "\iota_1"'] & \end{tikzcd}$$

Note this picture has a meaning even when h_1 is not idempotent

But maybe something works on a category level, with the appropriate category notion

YES

$$\begin{tikzcd} & V \arrow[dr, "\beta_1"] & \\ E \arrow[ur, "\alpha_1"] \arrow[dr, "j_1"'] & & E \\ & W \arrow[u, "\varphi"] \arrow[ur, "\iota_1"'] & \end{tikzcd}$$

$$\begin{tikzcd} & \Gamma V \arrow[dr, "\beta"] \arrow[rr, bend left, "p=\alpha\beta"] & & \Gamma V \\ E \arrow[ur, "\alpha"] \arrow[dr, "j"'] & & E \arrow[ur, "\alpha"] \arrow[dr, "j"'] & \\ & \Gamma W \arrow[u, "\tilde\varphi"] \arrow[ur, "i"] \arrow[rr, bend right, "e=ji"'] & & \Gamma W \arrow[uu, "\tilde\varphi"'] \end{tikzcd}$$

$$p\tilde\varphi = \alpha\beta\tilde\varphi = \alpha i$$

$$\tilde\varphi e = \tilde\varphi j i = \alpha i$$

$$X^5 = \begin{pmatrix} 0 & \beta\alpha\beta\alpha\beta \\ \alpha\beta\alpha\beta\alpha & 0 \end{pmatrix}$$

$$X^3 = \begin{pmatrix} 0 & \beta\alpha\beta \\ \alpha\beta\alpha & 0 \end{pmatrix}$$

$$X^4 = ($$

Is this picture related to your study of $E \underset{\beta}{\overset{\alpha}{\rightleftarrows}} V$ satisfying $p^2 = p$ for $p = \alpha\beta$?

$$X = \begin{pmatrix} & \beta \\ \alpha & \end{pmatrix} \text{ on } \begin{pmatrix} E \\ V \end{pmatrix}$$

$$Y = X^2 =$$

If you have $E \underset{\beta}{\overset{\alpha}{\rightleftarrows}} V$ $p = \alpha\beta$ $p^2 = p$?

$$\begin{tikzcd}
E \arrow[r, "\alpha"] \arrow[rr, bend right, "\beta\alpha = 1"'] & V \arrow[r, "\beta"] \arrow[rr, bend right, "p = \alpha\beta"'] & E \arrow[r, "\alpha"] & V
\end{tikzcd}$$

point $\beta\alpha = 1 \Longrightarrow (\alpha\beta)^2 = (\alpha\beta)$ i.e. $p^2 = p$

conversely if $p^2 = p$ then $(\beta\alpha)^3 = \beta p^2 \alpha = \beta p \alpha = (\beta\alpha)^2$

so V splits into $p = 1$ eigenspace and $\mathrm{Ker}(p^2)$

Go over yesterday's stuff, where you began to understand the case $\Phi \cong 1$. firm β-modules $EW = \mathbb{C}[\Gamma] \otimes W$ $h_1 = \iota_1 j_1$

$$\begin{tikzcd}
 & V \arrow[rd, "\beta_1"] \arrow[dd, dash, dotted] & \\
EW \arrow[ru, "\alpha_1"] \arrow[rd, two heads, "j_1"] & & EW \arrow[d, "j_1"] \\
W \arrow[u, "\iota_1"] \arrow[r, equal] & W \arrow[ru, hook, "\iota_1"] \arrow[r, equal] & W
\end{tikzcd}$$

Case β_1 inj, get $\begin{tikzcd} V \\ W \arrow[u, hook, "\varphi"'] \end{tikzcd}$

case α_1 surj, get $\begin{tikzcd} V \arrow[d, "\theta"] \\ W \end{tikzcd}$

$$\begin{tikzcd}
EW \arrow[rr, "\alpha_1"] \arrow[rd] & & V \arrow[rr, "\beta_1"] \arrow[rd] & & EW \\
 & \mathrm{Im}(\alpha_1) \arrow[ru, hook] \arrow[rd, two heads] & & \mathrm{Im}(\beta_1) \arrow[ru, hook] & \\
 & & W \arrow[ru, hook] & &
\end{tikzcd}$$

Go over factorization stuff again. Do general situation Φ finite $\subset \Gamma$. $B = C_\Phi \rtimes \Gamma$

E firm B-module, $h_1 = \beta_1\alpha_1 : E \xrightarrow{\alpha_1} V \xrightarrow{\beta_1} E$

get

$$\begin{tikzcd}
E \arrow[r, "\alpha"] \arrow[rr, bend right, "1"'] & \mathbb{C}[\Gamma]\otimes V \arrow[r, "\beta"] \arrow[rr, bend right, "p"'] & E \arrow[r, "\alpha"] & \mathbb{C}[\Gamma]\otimes V
\end{tikzcd}$$

Problem: Condition $h_1 s h_1 = 0$ $s \notin \Phi$ does not imply that $\alpha_1 s \beta_1 = 0$ for $s \notin \Phi$ unless β_1 inj, α_1 surj. in which case $V = h_1 E$

Go over factorization again in the general case Φ finite $\subset \Gamma$. E a firm $B = C_\Phi \rtimes \Gamma$ - module: $\sum_{s\in\Gamma} s h_1 s^{-1} = 1$ on E. Factor $h_1 = \beta_1\alpha_1$

$E \xrightarrow{\alpha_1} V \xrightarrow{\beta_1} E$ this yields

$$\begin{tikzcd}
E \arrow[r, "\alpha"] \arrow[rr, bend right, "1"'] & \mathbb{C}[\Gamma]\otimes V \arrow[r, "\beta"] \arrow[rr, bend right, "p"'] & E \arrow[r, "\alpha"] & \mathbb{C}[\Gamma]\otimes V
\end{tikzcd}$$

$p^2 = p$

Problem $h_1 s h_1 = 0$ for $s \notin \Phi$ does not imply that $\alpha_1 s h_1 = 0$ for $s \notin \Phi$ unles α_1 surj, β_1 inj in which case $E \xrightarrow{\alpha_1 = h_1} V = h_1 E \xrightarrow{\beta_1 = \mathrm{inc}} E$.

other cases worth examining $V = E$ $\alpha_1 = h_1$, $\beta_1 = 1$
or $\alpha_1 = 1$, $\beta_1 = h_1$

You need some support condition to proceed; possibly larger. Look at the case $V = E$ $\alpha_1 = 1$, $\beta_1 = h_1$ $E \xrightarrow{1} E \xrightarrow{h_1} E$

$\mathbb{C}[\Gamma]\otimes E$

So you find something overlooked, namely, α_1 has to yield a ~~map~~ function $s \longmapsto \alpha_1 s^{-1}\xi$ from Γ to V of finite support for each $\xi \in E$; ~~this~~ it should be enough for $\xi = h_1\eta$. ~~It should be enough.~~

How to proceed? Condition $\{s \mid \alpha_1 s^{-1} h_1\eta \neq 0\}$ is finite $\forall \eta \in V$, perhaps $\{s \mid \alpha_1 s^{-1}\beta_1\alpha_1$??

Try the following. You want to consider all factorizations of h_1 if possible

$E = \sum_{t\in\Gamma} t\beta_1 V$ you need $\overset{\forall\xi\in E}{\{s \mid \alpha_1 s^{-1}\xi \neq 0\}}$ finite

in order that $(\alpha\xi)(s) = \sum_{s\in\Gamma} \alpha_1 s^{-1}\xi \quad \in \mathbb{C}[\Gamma]\otimes V$

be defined on E. $\therefore$ Need $\sum_{s\in\Gamma} \alpha_1 s^{-1}\beta_1\sigma$ finite sum.

Summary: You need $\boxed{(\forall\sigma) \quad \{s \mid \alpha_1 s\beta_1\sigma \neq 0\} \text{ finite}}$

and a slightly stronger condition is $\boxed{\{s \mid \alpha_1 s\beta_1 \neq 0\} \text{ finite}}$

So you want to consider factorizations $h_1 = \beta_1\alpha_1$

$E \xrightarrow{\alpha_1} V \xrightarrow{\beta_1} E$ such that $\{s\in\Gamma \mid \alpha_1 s\beta_1 \neq 0\}$ is finite. You start with $\{s \mid h_1 s h_1 \neq 0\}$ finite.

Yesterday you encountered the condition on ~~the~~ a factorization $h_1 = \beta_1\alpha_1 : E \longrightarrow V \longrightarrow E$ ~~saying that~~ ~~stating~~ that α_1 induces $\alpha : E \longrightarrow \mathbb{C}\Gamma\otimes V$. Now $(\alpha\xi)(s) = \alpha_1 s^{-1}\xi$ ∴ you want to have fin. supp for any $\xi \in E$, enough for $\xi = \beta_1 v$ $\therefore$ condition is

$$\boxed{\forall v \quad \{s \mid \alpha_1 s \beta_1 v \neq 0\} \text{ is finite}}$$

slightly stronger : $\boxed{\{s \mid \alpha_1 s \beta_1 \neq 0\} \text{ is finite}}$

Case where this is clear is when β_1 injective and α_1 surjective, since $\{s \mid \underset{\beta(\alpha_1 s \beta)\alpha_1}{\underset{\shortparallel}{h_1 s h_1}} \neq 0\}$ is finite

Go back to your

$$\begin{tikzcd}
E \arrow[rr, "\alpha_1"] \arrow[dr] & & V \arrow[rr, "\beta_1"] \arrow[dr] & & E \arrow[rr, "\alpha_1"] \arrow[dr] & & V \\
& \alpha_1 E \arrow[ur] \arrow[dr] & & \beta_1 V \arrow[ur] & & \alpha_1 E \arrow[ur] & \\
& & h_1 E \arrow[ur] & & h_1 E & &
\end{tikzcd}$$

Your problem: Given $p(s) = \alpha_1 s \beta_1$

$\dim(W) = 1$

$$\begin{tikzcd}
EW \arrow[rr, "\alpha_1"] \arrow[dr, "\gamma_1"'] & & V \arrow[rr, "\beta_1"] & & EW \\
& W \arrow[urrr, "\iota_1"'] & & &
\end{tikzcd}
\qquad
\begin{tikzcd}
EW \arrow[r, "\alpha"] \arrow[dr, equal] & EV \arrow[r, "\beta"] & EW \\
& EW &
\end{tikzcd}$$

You (now) have a very simple situation

$$\mathbb{C}[\mathbb{Z}] \longrightarrow \mathbb{C}[\mathbb{Z}] \otimes V \longrightarrow \mathbb{C}[\mathbb{Z}]$$

~~So what a~~

$$\mathbb{C}[\mathbb{Z}] \longrightarrow \mathbb{C}[\mathbb{Z}] \otimes \mathbb{C}[\mathbb{Z}] \quad ?$$

$$\mathrm{Hom}_R(M, \mathrm{Hom}_{\mathbb{C}}(R, V)) = \mathrm{Hom}_{\mathbb{C}}(R \otimes_R M, V) = \mathrm{Hom}_{\mathbb{C}}(M, V)$$

$$\begin{tikzcd}
EW \arrow[r, "\alpha_1"] \arrow[dr, "\gamma_1"'] & V \arrow[r, "\beta_1"] & EW \\
& W \arrow[ur, "i_1"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
& EV \arrow[d, "\gamma_1"] & & \\
EW \arrow[ur, "\alpha"] \arrow[r, "\alpha_1"] & V \arrow[r, "\beta_1"] \arrow[dr, "i_1"'] & EW \\
& & EV \arrow[u, "\beta"']
\end{tikzcd}$$

$$\begin{tikzcd}
& EV \arrow[dr, "\gamma_1"] & & EV \arrow[dr, "\beta"] & \\
EW \arrow[ur, "\alpha"] \arrow[rr, "\alpha_1"] & & V \arrow[ur, "i_1"] \arrow[rr, "\beta_1"] & & EW
\end{tikzcd}$$

What's intriguing is how Γ-module maps arise.

Given $\alpha_1 : \overset{M}{\boxed{EW}} \longrightarrow V$ and $\beta_1 : V \longrightarrow \overset{N}{\boxed{EW}}$ you get canonical Γ-module maps:

$$\begin{tikzcd}
M \arrow[r, "\alpha_1^{\#}"] \arrow[rr, bend right, "\alpha"'] & Map(\Gamma, V) & \mathbb{C}\Gamma \otimes V \arrow[l, hook'] \arrow[r, "\overset{\beta}{\beta_1^{\#}}"] & N
\end{tikzcd}$$

$(\alpha_1^{\#} m)(s) = \alpha_1 s^{-1} m$. The good case is where the image of $\alpha_1^{\#}$ lies in $\mathbb{C}\Gamma \otimes V$.

Assume $M = \mathbb{C}\Gamma$ and $\alpha_1 : \mathbb{C}\Gamma \longrightarrow V = \mathbb{C}$ is a linear fnl. $(\alpha_1^{\#} t)(s) = \alpha_1 s^{-1} t$, so you need $\{s \mid \alpha_1 s^{-1} t \neq 0\}$ to be finite

Repeat $M = \mathbb{C}\Gamma$ and $\alpha_1 : \mathbb{C}\Gamma \longrightarrow V = \mathbb{C}$ is a linear functional of $\mathbb{C}\Gamma$ i.e. a function on Γ values in $\mathbb{C}$. You have $\alpha_1^{\#} : \mathbb{C}\Gamma \longrightarrow \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}\Gamma, \mathbb{C}) = Map(\Gamma, \mathbb{C})$ defined by $\alpha_1^{\#}(t)(s) = \alpha_1(st)$. For each t you want $\alpha_1^{\#}(t)$ to have fin supp i.e. $\{s \mid \alpha_1(st) \neq 0\}$ is finite, indep of t.

E, F Γ-modules, V vector space

$$\begin{tikzcd}
E \arrow[r, "\alpha_1"] \arrow[dr, "\alpha_1^{\#}"'] & V \arrow[rr, "\beta_1"] & & F \\
& \mathrm{Hom}(\mathbb{C}\Gamma, V) & \mathbb{C}\Gamma \otimes V \arrow[l, "\theta"'] \arrow[ur] &
\end{tikzcd}$$

$$\mathrm{Hom}_\Gamma(E, \mathrm{Hom}(\mathbb{C}\Gamma, V)) = \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}\Gamma \otimes_\Gamma E, V)$$

$$\| $$

$$\mathrm{Hom}_{\mathbb{C}}(E, V)$$

let $\xi \in E$, $r \in \mathbb{C}\Gamma$ then $\alpha_1^{\#}(\xi) = (r \mapsto \alpha_1 r \xi)$

$$(\alpha_1^{\#}\xi)(s) = \alpha_1 s \xi \qquad (\alpha_1^{\#} t\xi)(s) = \alpha_1^{\#}(st\xi) = (\alpha_1^{\#}\xi)(st)$$

~~use map~~ $\alpha_1^{\#}\xi \in \mathrm{Map}(\Gamma, V)$ left Γ module by

$$\| \qquad (tf)(s) = f(st)$$

$$s \mapsto \alpha_1 s \xi$$

You want

$$\begin{tikzcd}
\mathrm{Map}(\mathbb{C}\Gamma, V) \arrow[dr, "ev_1"'] & & \mathbb{C}\Gamma \otimes V \arrow[ll] \arrow[dl, "j_1"] \\
& V &
\end{tikzcd}$$

to commute

wait. $\mathrm{Hom}_\Gamma(\overset{\xi}{\mathbb{C}\Gamma \otimes V}, \mathrm{Map}(\Gamma, V))$ ~~$\mathrm{Hom}_{\mathbb{C}}(V, \mathrm{Map}(\Gamma, V))$~~

$$= \mathrm{Hom}(\mathbb{C}\Gamma \otimes V, V) \ni j_1$$

$$\left(j_1^{\#}(t \otimes v)\right)(s) = j_1(ts \otimes v) = \delta_{t^{-1}}(s) v$$

$$\mathbb{C}\Gamma \otimes V \longrightarrow \mathrm{Hom}(\mathbb{C}\Gamma, V)$$

$$t \otimes v \longmapsto \left(s \longmapsto \underbrace{j_1^{\#}(st) v}_{\delta_{t^{-1}}(s) v}\right)$$

function v at $t = s^{-1}$
0 otherwise

$$\mathbb{C}\Gamma\otimes V \xrightarrow{j_1^{\#}} \mathrm{Hom}(\mathbb{C}\Gamma, V)$$

$$t^{-1}\otimes v \longmapsto \left(s \mapsto \delta_t(s)\,v\right)$$

$$(\alpha_1^{\#}\xi)(s) = \alpha_1(s\xi)$$

$$\sum_t t^{-1}\otimes\alpha_1 t\xi$$

$$\|$$

$$\sum_s s\otimes\alpha_1 s^{-1}\xi.$$

$$E \xrightarrow{\alpha_1} V$$

somehow what's important is that $C_c(\Gamma)$ and $\mathbb{C}\Gamma$ are identified by $f \mapsto \sum s\otimes f(s^{-1})$

$\mathbb{C}\Gamma\otimes$

~~Basically~~ Organizing principle should be GNS.

$A = \mathbb{C}$ $\quad B = \mathbb{C}\Gamma$

$$\begin{tikzcd} V \arrow[r, shift left, "\iota=\beta_1"] & E \arrow[l, shift left, "j=\alpha_1"] \end{tikzcd}$$

$$\rho(b) = jb\iota = \alpha_1 b\beta_1$$

Then you have $\mathbb{C}\Gamma\otimes V \xrightarrow{\beta_1^{\flat}} E \xrightarrow{\alpha_1^{\#}} \mathrm{Hom}(\mathbb{C}\Gamma, V)$

$$\sum t\otimes v_t \longmapsto \sum t\beta_1 v_t \longmapsto \Big(s \mapsto \sum_t \underbrace{\alpha_1 s^{-1}t\beta_1}_{\rho(s^{-1}t).} v_t\Big)$$

Can you improve this to a nice form?

Important point is that $\forall\xi\in E$ you have $\{s \mid \alpha_1 s^{-1}\xi \neq 0\}$ is finite.

Given

$$\begin{tikzcd} E \arrow[r, "\alpha_1"] \arrow[dr, "\alpha"'] & V \arrow[r, "\beta_1"] \arrow[d, shift left, "\iota_1"] & E \\ & \mathbb{C}\Gamma\otimes V \arrow[u, shift left, "j_1"] \arrow[ur, "\beta"'] & \end{tikzcd}$$

$$\mathbb{C}\Gamma\otimes V = \Big\{\sum_{t\in\Gamma} t\otimes f_t \;\Big|\; \underset{\text{fin supp}}{f(t)}\Big\}$$

$$\beta\Big(\sum_t t\otimes f(t)\Big) = \sum_t t\,\beta_1 f(t).$$

[illegible]

Let $\alpha\xi = \sum_t t\otimes f(t)$

then $\alpha s^{-1}\xi = \sum_t s^{-1}t\otimes f(t)$

$\alpha_1 s^{-1}\xi = j_1\alpha s^{-1}\xi = f(s)$

You have been over the formulas for the nth time.

Summary: Given $E \xrightarrow{\alpha_1} V \xrightarrow{\beta_1} E$ where E is a Γ-module and V a vector space. ~~What can you~~

Assume $\forall \xi \in E$ $\{s \in \Gamma \mid \alpha_1 s^{-1}\xi \neq 0\}$ is finite, then

~~[illegible] α_1 extends uniquely to a Γ-mod. map α~~

$\exists!$ Γ-module maps $E \xrightarrow{\alpha} \mathbb{C}\Gamma \otimes V \xrightarrow{\beta} E$ such that $j_1\alpha = \alpha_1$, $\beta\iota_1 = \beta_1$. Formula

$$\mathbb{C}\Gamma \otimes V = \Big\{ \sum_{t\in\Gamma} t \otimes f(t) \;\Big|\; f \in \underbrace{C_c(\Gamma, V)}_{\mathbb{C}\Gamma\otimes V} \Big\}.$$

$$(\alpha\xi)(s) = \alpha_1 s^{-1}\xi$$

$$\beta\Big(\sum_t t \otimes f(t)\Big) = \sum_t t\beta_1 f(t).$$

$$\therefore \quad (\beta\alpha)(\xi) = \beta\Big(\sum_t t \otimes \alpha_1 t^{-1}\xi\Big) = \sum_t t\beta_1\alpha_1 t^{-1}\xi$$

$$(\alpha\beta f)(s) = \alpha(\beta f)(s) = \alpha_1 s^{-1}\beta f(s)$$

$$= \alpha_1 s^{-1}\sum_t t\beta_1 f(t) = \sum_t (\alpha_1 s^{-1} t \beta_1) f(t)$$

What do you want to do? To make progress ~~[illegible]~~ start with a fin B_Φ module E, choose a suitable factorization $h_1 = \beta_1\alpha_1$ and you get an A_Φ-module. No if you choose the minimal fact. then you get an A_Φ-module

idea: $\mathbb{C}\Gamma \xrightarrow{\alpha_1} \mathbb{C}$ linear functional

then corres

$$\begin{tikzcd} \mathbb{C}\Gamma \arrow[r, "\alpha_1"] \arrow[dr, "\alpha"'] & \mathbb{C} \\ & \mathbb{C}\Gamma \arrow[u, "j_1"'] \end{tikzcd}$$

$(\alpha\xi)(s) = \alpha_1 s^{-1}\xi$ gives coeff of s

new idea that $\alpha_1 : E \longrightarrow V$ should be nuclear in a strong sense. Better: Let ~~[illegible]~~ To see if any of this makes sense. Go back to $EW = \mathbb{C}\Gamma \otimes W$ and $EW \xrightarrow{\alpha_1} V$. You have taken α_1 to be any linear map, but you have learned that ~~[illegible]~~ in order for α_1 to ~~[illegible]~~ give rise to $\alpha : EW \longrightarrow EV$, you need $\forall \xi \in EW$ that $\{s \mid \alpha_1 s^{-1}\xi \neq 0\}$ is finite.

This is not said well. Suppose V finite dimensional. Then α_1 amounts to ~~[illegible]~~ ~~[illegible]~~ a finite set of linear functionals. Look at one $\bigoplus_{s \in \Gamma} s \otimes W \xrightarrow{\alpha_1} \mathbb{C}$. You need a symmetry, namely, you want a framework in which

$$\begin{array}{lll} \alpha_1 : E \longrightarrow V & \text{induces} & \alpha : E \longrightarrow EV \\ \beta_1 : V \longrightarrow E & \text{——} & \beta : EV \longrightarrow E \end{array}$$

You want these on the same footing. ~~In general~~

Study the special case $E = \mathbb{C}\Gamma$, $V = \mathbb{C}$.

~~[illegible]~~ Let E be any Γ-module and $\alpha_1 : E \longrightarrow \mathbb{C}$ linear. Then $\exists!\ \alpha : E \longrightarrow C(\Gamma)$ such that 1) comp with Γ action 2) $\delta_1 \alpha = \alpha_1$.

$(\alpha\xi)(s) = \alpha_1 s^{-1}\xi$. You want finite supp. How

$\alpha_1 : \mathbb{C}\Gamma \longrightarrow \mathbb{C}$

So what you need to understand is something ~~peculiar~~ special to group rings, ~~or really~~ ~~[illegible] cyclic work~~.

$\mathbb{C}\Gamma \longrightarrow \mathrm{Hom}(\mathbb{C}\Gamma, \mathbb{C})$ two obvious linear functls on $\mathbb{C}\Gamma$ namely $\int = \sum_s$ and $\int_1$.

idea that ~~seems to~~ might work. Look for module maps which admit adjoints

~~That~~ $T: \mathbb{C}\Gamma \otimes W \longrightarrow \mathbb{C}\Gamma \otimes V$

$T^t: \mathbb{C}\Gamma \otimes V^* \longrightarrow \mathbb{C}\Gamma \otimes W^*$

Setup the duality. You have $\mathbb{C}\Gamma$ and

$(\mathbb{C}\Gamma)^* = \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}\Gamma, \mathbb{C}) = \mathrm{Map}(\Gamma, \mathbb{C})$ $\quad \mathbb{C}\Gamma \,\|\, C_c(\Gamma)$

basic pairing seems to be obtained from a Γ-module map

$$\mathbb{C}\Gamma \longrightarrow \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}\Gamma, \mathbb{C})$$

which is equivalent to a map $\int_1$

$$\begin{tikzcd}
\mathbb{C}\Gamma \otimes_{\mathbb{C}\Gamma} \mathbb{C}\Gamma \arrow[r] \arrow[d, "\text{mult. } \mu"', "\wr"] & \mathbb{C} \\
\mathbb{C}\Gamma \arrow[ur, "\int_1"'] &
\end{tikzcd}$$

It seems to me that $\mathbb{C}\Gamma \longrightarrow \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}\Gamma, \mathbb{C})$ is the Γ-map sending 1 to $\int_1$.

thus $t \longmapsto (s \longmapsto \int_1(st))$. $\int_1$ is a trace

$\mathbb{C}\Gamma /$

Need to find the basic idea. You will have f.g. free Γ-modules $\mathbb{C}\Gamma \otimes V$ $\dim(V) < \infty$ and some sort of ~~[illegible]~~ morphisms

What kind of module maps from $\mathbb{C}[\Gamma]$ to itself? $\mathrm{Hom}_{\Gamma}(\mathbb{C}\Gamma\otimes W, \mathbb{C}\Gamma\otimes V) = \mathrm{Hom}(W, \mathbb{C}\Gamma\otimes V)$?

Start somewhere to straighten out ideas! How.

$\alpha_1 : \mathbb{C}\Gamma \longrightarrow \mathbb{C}$ leads to unique

$\alpha : \mathbb{C}\Gamma \longrightarrow \mathrm{Hom}(\mathbb{C}\Gamma, \mathbb{C})$ - Γ-mod. map

$$\alpha(t) = (s \longmapsto \alpha(st))$$

when α_1 has finite support then you get.

$$\alpha : \mathbb{C}\Gamma \longrightarrow \mathbb{C}\Gamma \quad ?$$
$$s \longmapsto \alpha_1(s^{-1}) \quad ,$$

$$\mathrm{Hom}_{\mathbb{C}\Gamma}(E, \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}\Gamma, V))$$
$$= \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}\Gamma\otimes_{\mathbb{C}\Gamma} E, V) = \mathrm{Hom}_{\mathbb{C}}(E, V)$$

Given $\varphi \in \mathrm{Hom}_{\mathbb{C}}(E,V)$ the corresp $\tilde\varphi : E \longrightarrow \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}\Gamma, V)$ is $(\tilde\varphi\xi)(r) = \varphi(r\xi)$, from function viewpt.

$(\tilde\varphi\xi)(s) = \varphi(s\xi)$. $\quad \therefore \varphi(\xi) = (\tilde\varphi\xi)(1) = j_1\tilde\varphi(\xi)$.

$\therefore$ given $E \xrightarrow{\alpha_1} V$ get $\tilde\alpha_1 : E \longrightarrow \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}\Gamma, V)$ the unique Γ-equiv. map such that $j_1\tilde\alpha_1 = \alpha_1$ ~~where~~ given by $(\tilde\alpha_1\xi)(s) = \alpha_1(s\xi)$. The good case now is when $s \longmapsto \alpha_1(s\xi)$ has finite support. $\forall \xi \in E$, in which case you get $\alpha : E \longrightarrow \mathbb{C}\Gamma\otimes V$

given by $\alpha\xi = \sum_{s\in\Gamma} s\otimes\alpha_1 s^{-1}\xi$. Check this

$$\alpha t\xi = \sum_{s\in\Gamma} s\otimes\alpha_1 s^{-1}t\xi = \sum_{s} ts\otimes\alpha_1 (ts)^{-1}t\xi$$

$$= t\sum_{s} s\otimes\alpha_1 s^{-1}\xi = t\alpha\xi$$

Important condition is that $\forall \xi\in E$ $\{s \mid \alpha_1 s^{-1}\xi \neq 0\}$ is finite. If true for ξ_1, ξ_2 true for $t_1\xi_1 + t_2\xi_2$, enough to check for gen. of the Γ-module E.

Look at this for $E = \mathbb{C}\Gamma$ $V = \mathbb{C}$. Given $\alpha_1 : \mathbb{C}\Gamma \longrightarrow \mathbb{C}$ gen 1 $\{s \mid \alpha_1(s^{-1}1) \neq 0\}$ finite ∴ α_1 must have finite support.

Next take $\alpha_1 : \mathbb{C}\Gamma\otimes W \longrightarrow V$, gen W.

so need $\{s \mid \alpha_1(s^{-1}\otimes w)\neq 0\}$ finite. Now if you say W fin. dim. this means that α_1 detects on finitely many sW. If I stick to V, W finite dimensional, then the linear functionals on $\mathbb{C}\Gamma\otimes W$ which arise from Γ-module maps $\mathbb{C}\Gamma\otimes W \longrightarrow \mathbb{C}\Gamma$ are those supported on $\bigoplus_{s\in\Phi} sW$ for some finite subset $\Phi\subset\Gamma$

Something funny. $B = C_\Phi \rtimes \Gamma$. You can factor h_1 using $h_K = \sum_{t \in K} h_t$ K finite $\subset \Gamma$. namely $h_1 = \sum_t h_1 h_t = \sum_t h_t h_1$. where the sums are finite $\therefore$ $h_1 = h_1 h_K = h_K h_1$ for K large enough. Now if E is a firm B-module, then you ~~know that~~ should have a corresponding P_Φ module. What will ~~the~~ be the best way to proceed?

Let's review what you believe..

$B_\Phi = C_\Phi \rtimes \Gamma$ $\quad C_\Phi$ gens h_s $s \in \Gamma$

rels $h_s h_t = 0$ $\quad s^{-1}t \notin \Phi$

$\mathbb{C}$

$$h_s = \sum_{t \in s\Phi} h_s h_t = \sum_{t \in s\Phi^{-1}} h_t h_s$$

B_Φ is a kind of alg of ~~kernels~~ kernels support in a nbd of the diagonal.

OKAY

$$0 \longrightarrow \underbrace{C_\Phi \rtimes \Gamma}_{B} \longrightarrow \underbrace{\tilde{C}_\Phi \rtimes \Gamma}_{R} \longrightarrow \underset{R/B}{\mathbb{C}\Gamma} \longrightarrow 0$$

B has local units $\Longrightarrow$ R/B flat over R

To try once more, then Michener.

$$\to B = C_\Phi \rtimes \Gamma \longrightarrow R = \tilde{C}_\Phi \rtimes \Gamma \longrightarrow \mathbb{C}[\Gamma] \longrightarrow 0$$

B has local left units $\Longleftrightarrow$ R/B flat R^{op} module

firm modules E are those $\ni \forall \xi \in E \ \exists b \in B$ $b\xi = \xi$.

Set up Morita equivalence. Given E you factor $h_1 = \beta_1\alpha_1$:

$$\begin{tikzcd} E \arrow[r, two heads, "\alpha_1"] \arrow[dr, "h_1"'] & V \arrow[r, hook, "\beta_1"] \arrow[d, phantom, "\parallel"] & E \\ & h_1 E \arrow[ur, "h_K = \beta_1"'] & \end{tikzcd}$$

. This is a very simple choice. Then you get

$$\begin{tikzcd} E \arrow[r, "\alpha"] \arrow[rr, bend right, "1"'] & \mathbb{C}\Gamma\otimes V \arrow[r, "\beta"] \arrow[rr, bend right, "p = \alpha\beta"'] & E \arrow[r, "\alpha"] & \mathbb{C}\Gamma\otimes V \end{tikzcd}$$

you know

$$\alpha\xi = \sum s\otimes\alpha_1 s^{-1}\xi$$

$$\beta\Big(\sum_t t\otimes f(t)\Big) = \sum t\beta_1 f(t)$$

$$\beta\alpha\xi = \beta\sum_s s\otimes\alpha_1 s^{-1}\xi = \sum_s \overbrace{s\beta_1\alpha_1 s^{-1}}^{h_s}\xi = \xi$$

$$\alpha\beta\Big(\sum_t t\otimes f(t)\Big) = \alpha\sum_t t\beta_1 f(t) = \sum_{s,t} s\otimes \underbrace{\boxed{\alpha_1 s^{-1}t\beta_1}}_{p(s^{-1}t)} f(t)$$

$$= \sum_s s\otimes(pf)(s)$$

$$\boxed{(pf)(s) = \sum_t p(s^{-1}t)f(t)}$$

You propose to define functor E to h_1E which is naturally an $A = P_\Phi$ - modules. When applied to $B = C_\Phi \rtimes \Gamma$ you get $h_1B = h_1C_\Phi \rtimes \Gamma$

$$\boxed{\begin{matrix} B = P\otimes_A Q & P \\ Q & A \end{matrix}}$$

What do you know about $h_1 C_\Phi$??

F right B module (finim)

$$\begin{tikzcd} F \arrow[r, "\alpha_1 = \cdot h_1"] & \underset{W}{\boxed{Fh_1}} \arrow[r, hook, "\beta_1 = \cdot h_K"] & F \end{tikzcd}$$

$$\begin{tikzcd}
F \arrow[r, two heads, "\gamma_1"] & W \arrow[r, hook, "\delta_1"] & F \\
\phi \arrow[r, mapsto] & \phi h_1 &
\end{tikzcd}$$

$$\begin{tikzcd}
F \arrow[r] & W \otimes \mathbb{C}[\Gamma] \\
\phi \arrow[r, mapsto] & \sum_s \phi s \gamma_1 \otimes s
\end{tikzcd}$$

$$\begin{tikzcd}
 & \mathrm{Hom}(F, W) \arrow[d, equal] \\
\mathrm{Hom}_{\mathbb{C}\Gamma^{op}}(F, \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}\Gamma, W)) \arrow[r, phantom, "\approx"] \arrow[d, equal] & \mathrm{Hom}_{\mathbb{C}}(F \otimes_{\Gamma} \mathbb{C}\Gamma, W) \arrow[d, equal] \\
\mathrm{Hom}_{\mathbb{C}\Gamma^{op}}(F, \mathrm{Map}(\Gamma, W)) \arrow[r, "(\gamma_1)_*"] & \mathrm{Hom}_{\mathbb{C}}(F, W) \\
\mathrm{Hom}_{\mathbb{C}\Gamma^{op}}(F, W \otimes \mathbb{C}\Gamma) \arrow[u, hook] &
\end{tikzcd}$$

$$\phi\left(\overset{s}{s^{-1}}\right)\gamma_1$$

$$\phi \xmapsto{\gamma_1^{\#}} \sum_{s\in\Gamma} \phi \boxed{\gamma_s} \otimes s$$

$$\phi t \longmapsto \sum_{s\in\Gamma} \phi t \gamma_s \otimes \overset{s}{\cancel{st}} = \sum_{s\in\Gamma} \phi \gamma_s \otimes st$$

$$\phi t \gamma_{st} \otimes st = \phi \gamma_s \otimes st$$

$$\therefore \quad t\gamma_{st} = \gamma_s \qquad t\gamma_t = \gamma_1 \qquad \gamma_t = t^{-1}\gamma_1$$

$$\phi \xmapsto{\gamma_1^{\#}} \sum_s \phi s^{-1}\gamma_1 \otimes s$$

check $$\sum_s \phi t s^{-1}\gamma_1 \otimes s = \sum_s \phi t (st)^{-1}\gamma_1 \otimes st = \left(\sum \phi s^{-1}\gamma_1 \otimes s\right)t$$

So $\gamma_1 : F \xrightarrow{\gamma_1} W \xrightarrow{\delta_1} F$ extends to

$\gamma^{\#} : F \longrightarrow W\otimes\mathbb{C}[\Gamma]$ $\qquad \delta : W\otimes\mathbb{C}[\Gamma] \longrightarrow F$

$$\phi\gamma^{\#} = \sum_{s\in\Gamma} \phi s^{-1}\gamma_1 \otimes s$$

provided $\{s \mid \phi s^{-1}\gamma_1 \neq 0\}$ is finite $\forall \phi\in F$.

$$F \xrightarrow{\gamma} W\otimes\mathbb{C}\Gamma \xrightarrow{\delta} F$$

$$\phi \longmapsto \sum_{s\in\Gamma} \phi s^{-1}\gamma_1 \otimes s$$

$$\sum_{t\in\Gamma} w(t)\otimes t \longmapsto \sum_t w(t)\delta_1 t$$

$$\phi \longmapsto \sum_t \phi t^{-1}\overbrace{\gamma_1\delta_1}^{h_1} t = \phi .$$

So it seems that you get

$$\begin{tikzcd}
F \arrow[r, "\gamma"] \arrow[rr, bend right, "1"'] & W\otimes\mathbb{C}\Gamma \arrow[r, "\delta"] \arrow[rr, bend right, "p"'] & F \arrow[r, "\gamma"] & W\otimes\mathbb{C}\Gamma
\end{tikzcd}$$

$$\sum_{s\in\Gamma} w(s)\otimes s \xmapsto{\delta} \sum_{t\in\Gamma} w(t)\delta_1 t \xmapsto{\gamma} \sum_{s\in\Gamma}\sum_{t\in\Gamma} w(t)\delta_1 t s^{-1}\gamma_1 \otimes s$$

$$(wp)(s) = \sum_t w(t)\left(\delta_1 t s^{-1}\gamma_1\right)$$

$$\begin{tikzcd}
W \arrow[r, hook, "\delta_1"] \arrow[rr, bend right] & F \arrow[loop above, "s"] \arrow[r, "\gamma_1"] \arrow[rr, bend left, "h_1"] & W \arrow[r, "\delta_1"] & F
\end{tikzcd}$$

Does sur $\delta_1 s\gamma_1$ satisfy ing support cond?

$h_1 s h_1 = \enclose{circle}{\gamma_1}\,\delta_1 s\gamma_1\,\enclose{circle}{\delta_1}$ YES

So what's happening. You seem to have a M equiv. between firm B_Φ modules and A_Φ modules

$E \longmapsto \quad p(s) = \alpha_1 s \underset{\substack{\parallel \\ \text{inclusion}}}{\beta_1}$ on h_1E $\qquad \alpha_1 = h_1 : E \twoheadrightarrow h_1E$

inclusion $h_1E \hookrightarrow E$

$p(s) \in \mathcal{L}(h_1E)$

Go back. $B_\Phi = C_\Phi \rtimes \Gamma$ consider as left B_Φ-module, the corresponding A_Φ module is $h_1B_\Phi = h_1C_\Phi \rtimes \Gamma$. Explain how?

In general $p(s) \in \mathcal{L}(h_1E)$ is $\alpha_1 s \beta_1$ where

$h_1E \xrightarrow{\beta_1 = \text{inc.}} E \xrightarrow{\alpha_1 = h_1} h_1E$ $\qquad \beta_1$ is the inclusion of h_1E in E

α_1 is the map $h_1 : E \twoheadrightarrow h_1E$

We know that $\underbrace{h_1 s h_1}_{} = 0^{\text{in } B}$ for $s \notin \Phi$ in E

$h_1 s h_1 = \underset{\text{inj}}{\beta_1} \alpha_1 s \beta_1 \underset{\text{surj}}{\alpha_1} \Longrightarrow \alpha_1 s \beta_1 = 0$ for $s \notin \Phi$

So what happens is that the functor you are looking at from firm B-modules to round A-modules is $E \longmapsto h_1E$ equipped with the operators $p(s)$

Go over the round business. The question is whether

$$\left(\begin{array}{l} A h_1E = h_1E \\ {}_A(h_1E) = 0 \end{array}\right. \qquad \begin{array}{l} \sum p(s) h_1E = h_1E \\ \bigcap_s \operatorname{Ker}\{p(s) \text{ on } h_1E\} = 0 \end{array}$$

$p(s) = \alpha_1 s \beta_1$ $\quad$ suppose $\forall s$ $\alpha_1 s \beta_1 v = 0$. $v \in h_1E$

say $v = h_1\xi$, then $\beta_1 v = h_1\xi$ and $\forall s$ $\alpha_1 s h_1 \xi = 0$

$\Rightarrow \forall s \quad s^{-1} \underbrace{\beta_1 \alpha_1}_{h_1} s h_1 \xi = 0 \Longrightarrow h_1\xi = 0$

So what do you know, learn?

E firm B-module $\rightsquigarrow$ $h_1 E$ round A-module

and you have an inverse function which takes an A-module V to $E(V) = \underbrace{\operatorname{Im} p}_{\underset{\operatorname{Ker}(1-p)}{\parallel}}$ on $\mathbb{C}\Gamma \otimes V$

$$p\Big(\sum_s s \otimes f(s)\Big) = \sum_s s \otimes \sum_t p(s^{-1}t) f(t)$$

$$\parallel$$

$$\sum_s s \otimes (pf)(s)$$

~~Still a bit hard~~

$$E = B \otimes_B E \qquad h_1 E \longleftarrow h_1 B \otimes_B E$$

Review things with care. $B = C \rtimes \Gamma$

$C = C_\Phi$ gens. h_s $s \in \Gamma$

rels $h_s h_t = 0$ if $s^{-1}t \notin \Phi$ $\Big|$ $h_s = \sum_{t \in s\Phi} h_s h_t = \sum_{t \in s\Phi^{-1}} h_t h_s$

A_Φ gens $p(s)$ $s \in \Gamma$

relns. $p(s) = 0$ $s \notin \Phi$

$$p(s) = \sum_{\{(t,u) \mid s = tu\}} p(t) p(u)$$

$$= \sum_{u \in \Gamma} p(su^{-1}) p(u)$$

Given an A_Φ module V, means a v.s. with ops $p(s) \in \mathcal{L}(V)$ satisfying these rels. Form Γ-module

$$\mathbb{C}\Gamma \otimes V = \Big\{ \sum_{s \in \Gamma} s \otimes f(s) \;\Big|\; \underset{\text{fin. support}}{f : \Gamma \to V} \Big\}$$

$$t\Big(\sum_{s \in \Gamma} s \otimes f(s)\Big) = \sum_{s \in \Gamma} ts \otimes f(s) = \sum_{s \in \Gamma} s \otimes f(t^{-1}s)$$

~~$(pf)(s)$~~ $$\beta \sum_{t\in\Gamma} t\otimes f(t) = \sum_{t\in\Gamma} t\,\beta_1 f(t)$$

$$\alpha\beta\Big(\sum_{t\in\Gamma} t\otimes f(t)\Big) = \sum_{s,t\in\Gamma} s\otimes \underbrace{\alpha_1 s^{-1}t\,\beta_1}_{p(s^{-1}t).} f(t)$$

So define p on $\mathbb{C}\Gamma\otimes V$ by

$$p\Big(\sum_{t\in\Gamma} t\otimes f(t)\Big) = \sum_{s\in\Gamma} s\otimes \sum_{t} p(s^{-1}t)\, f(t)$$

If you identify $\mathbb{C}\Gamma\otimes V = C_c(\Gamma; V)$

$$\sum_{s\in\Gamma} s\otimes f(s) \longleftrightarrow f$$

$$(pf)(s) = \sum_{t\in\Gamma} p(s^{-1}t)\, f(t)$$

$p^2 = p$

$pL_u = L_u p$

$$E = \begin{cases} \operatorname{Im} p \text{ on } \mathbb{C}\Gamma\otimes V \\ \operatorname{Ker}(1-p) \end{cases} \qquad p(\mathbb{C}\Gamma\otimes V)$$

Look at A_Φ carefully.

Wait. Usual approach – start with E firm B-module then define A_Φ-module $h_1 E$ which is round. Get an equivalence between firm B_Φ-modules and round A_Φ-modules. But in fact maybe firm B-modules are round? Local left units?

$$0 \longrightarrow B\otimes_B \cancel{A} \longrightarrow E \longrightarrow E/BE \longrightarrow 0$$

Is it possible to have an element of E killed by B?

Can $\exists\ \xi \in E$ such that $B\xi = 0$. No
because $h_s \xi = 0 \quad \forall s \implies \xi = \sum h_s \xi = 0$.

~~Next let~~ Consider $h_1 B \otimes_B E \longrightarrow h_1 E$

$h_1 B = h_1 C \rtimes \Gamma \overset{?}{\implies} h_1 B \otimes_B E = h_1 C \otimes_C E$?

Is it possible that $h_1 C \otimes_C E \longrightarrow h_1 E$ is an isomorphism of A_Φ modules?

Is $h_1 C$ a flat C^{op}-module?

What is the ideal situation?

C, B have local left + right units.

A_Φ gens $p(s)$ $s \in \Gamma$

rels $p(s) = 0$ $s \notin \Phi$

$$p(s) = \sum_{\{(t,u) \mid tu = s\}} p(t)p(u) = \sum_u p(tu^{-1})p(u)$$

$$/\!/ \quad \sum_t p(t)p(t^{-1}s)$$

$$\mathbb{A} \times \Gamma \simeq \mathbb{A} \otimes \mathbb{C}\Gamma = \bigoplus_{s \in \Gamma} \mathbb{A}s$$

$$p = \sum_{s \in \Gamma} p(s)s \qquad p^2 = \sum_{s,t} p(s)p(t)\, st$$

Yes. $A_\Phi = A_\Phi^2$ So A_Φ is idempotent

Given $A_\Phi \longrightarrow \mathcal{L}V$ i.e. an A_Φ-module V

Then get P_V on $\mathbb{C}\Gamma \otimes V$ $\qquad p = \sum_{s \in \Gamma} s \otimes p(s)$

$p^2 = p$ $\qquad E(V) = p(\mathbb{C}\Gamma \otimes V)$ exact functor of V.

So you have an exact functor

$$\mathrm{Mod}(\tilde{A}_\Phi) \longrightarrow \mathcal{M}(B_\Phi)$$

which kills nil A_Φ modules i.e. $N \ni p(s)N = 0 \ \forall s$.

---

details of geometric case $\Gamma \to X \to Y$

$$\begin{tikzcd} X\times_Y X \arrow[r, "pr_2"] \arrow[d, "pr_1"'] & X \arrow[d, "\pi"] \\ X \arrow[r, "\pi"] & Y \end{tikzcd} \qquad X\times_Y X \simeq \Delta X \times \Gamma$$

first point is that $k \in C_c(X\times_Y X)$ gives rise to an operator on $C_c(X)$, namely

$$kf = (pr_1)_!(k\, pr_2^* f) \qquad (kf)(x) = \sum_{x' \in \pi^{-1}\pi x} k(x,x')f(x').$$

Moreover you have $C_c(X) \otimes_{C_c(Y)} C_c(X) \xrightarrow{\sim} C_c(X\times_Y X)$

$$p \otimes g \longmapsto (pr_1^* p)(pr_2^* g)$$

$(pr_1^*(p)\, pr_2^*(g))(x,x') = p(x)\, g(x')$. Now observe that if $k = pr_1^*(p)\, pr_2^*(g)$, then

$$kf = \sum_{x' \in \pi^{-1}\pi x} p(x)\, q(x') f(x') = p\langle q, f\rangle$$

This is all pretty clear. The point is that you have isom. $C_c(X) \otimes_{C_c(Y)} C_c(X) \xrightarrow{\sim} C_c(X\times_Y X)$

$$p(x) \otimes q(x') \longmapsto p(x)\, q(x')$$

and this ~~is an~~ respects the algebra structures ~~isomorphism~~ ~~[illegible]~~ where on the left you have the mult. defined via $\langle g, p\rangle = \pi_!(gp)$ and on the right composition of kernels.

Now what? $\Gamma \supset \Phi$ finite

A_Φ gens. $p(s)$ $s \in \Gamma$

rels $p(s) = 0$ $s \notin \Phi$, $\boxed{p(s) = \sum_{t\in\Gamma} p(t)\, p(t^{-1}s)}$

A_Φ clearly idempotent. $A \subset A^2$.

Let V be an A_Φ-module: $p(s) \in \mathcal{L}V$

$$\mathbb{C}\Gamma \otimes V = C_c(\Gamma; V)$$

$$\begin{tikzcd}
\sum_s s \otimes f(s) \arrow[r, leftrightarrow] \arrow[d, mapsto] & (s \mapsto f(s)) \arrow[dd, mapsto, "L_t"] \\
\sum_s ts \otimes f(s) \arrow[d, phantom, "\parallel"] & \\
\sum_s s \otimes f(t^{-1}s) \arrow[r, leftrightarrow] & (s \mapsto f(t^{-1}s))
\end{tikzcd}$$

$$(pf)(s) = \sum_t p(s^{-1}t) f(t)$$

$$\left(\sum_{s\in\Gamma} s \otimes p(s^{-1})\right)\left(\sum_{t\in\Gamma} t \otimes f(t)\right)$$

$$= \sum_{s,t\in\Gamma} st \otimes p(s^{-1}) f(t) = \sum_u u \otimes \sum_t p(ut^{-1}) f(t)$$

$st = u$ $s = ut^{-1}$ $s^{-1} = tu^{-1}$

$$\mathrm{Hom}_\Gamma(\mathbb{C}\Gamma \otimes V, \mathbb{C}\Gamma \otimes W) = \mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C}\Gamma \otimes W)$$

$$\cup$$

$$\mathbb{C}\Gamma \otimes \mathrm{Hom}_{\mathbb{C}}(V, W)$$

Go back to $E \xrightarrow[h_1]{\alpha_1} h_1E \xrightarrow[\text{incl.}]{\beta_1} E$

$$E \xrightarrow{\alpha} \mathbb{C}\Gamma\otimes E \xrightarrow{\beta} E$$

$$\xi \longmapsto \sum_{s\in\Gamma} s\otimes\alpha_1 s^{-1}\xi \longmapsto \sum_s \overbrace{s\beta_1\alpha_1 s^{-1}}^{h_s}\xi = \xi.$$

$$\sum t\otimes f(t) \longmapsto \sum_t t\beta_1 f(t)$$

$$p\sum_t t\otimes f(t) = \alpha\beta\sum_t t\otimes f(t) = \alpha\sum t\beta_1 f(t)$$

$$= \sum_s s\otimes \underbrace{\boxed{\alpha_1 s^{-1}t\beta_1}}_{p(s^{-1}t)} f(t)$$

Important is that when we use

$$\mathbb{C}\Gamma\otimes\mathrm{Hom}_{\mathbb{C}}(V,W) \hookrightarrow \mathrm{Hom}_{\mathbb{C}\Gamma}(\mathbb{C}\Gamma\otimes V, \mathbb{C}\Gamma\otimes W)$$

that $\underset{t}{\mathbb{C}\Gamma} \longrightarrow \mathrm{Hom}_{\mathbb{C}\Gamma}(\mathbb{C}\Gamma,\mathbb{C}\Gamma)$ is right mult by t^{-1}!

So maybe you get better formulas using $E\otimes\mathbb{C}\Gamma$.

$$(t\otimes\theta)(s\otimes v) = st^{-1}\otimes\theta v$$

$$\left(\sum_{s\in\Gamma} s\otimes p(s)\right)\left(\sum_{t\in\Gamma} t\otimes f(t)\right)$$

$$= \sum_{s,t\in\Gamma\times\Gamma} \overbrace{ts^{-1}}^{u}\otimes p(s)f(t)$$

$u = ts^{-1}$
$us = t \quad s = u^{-1}t$

$$= \sum_u u\otimes \sum_t p(u^{-1}t)f(t)$$

So the point seems to be that given an A_{Φ} module V with ops $p(s)$ $s\in\Gamma$, you do something, form p on $\mathbb{C}\Gamma\otimes V$.

$$p\sum_t t\otimes f(t) = \sum_{s,t} \overset{u}{\overbrace{ts^{-1}}}\otimes p(s)f(t)$$

$u = ts^{-1}$
$us = t$
$s = u^{-1}t$

$$= \sum_u u\otimes\sum_t p(u^{-1}t)f(t)$$

Maybe something can be said for using $u = ts^{-1}$, $s = u^{-1}t$

$$p\sum_t t\otimes g(t^{-1}) = \sum_{s,t}\boxed{ts^{-1}}\otimes p(s)g(t^{-1})$$

$$= \sum_u u\otimes\sum_t p(u^{-1}t)g(t^{-1}).$$

$$(pg)(u^{-1}) = \sum_t p(u^{-1}t)g(t^{-1})$$

What to do next???? YES!!!

$$B_{\Phi} = C_{\Phi}\rtimes\Gamma$$

exact functor from A_{Φ}-modules to firm B_{Φ}-modules.

$$V\mapsto E(V) = p(\mathbb{C}\Gamma\otimes V)$$

Important case is where $V = A_{\Phi}$. Get left + right straight

$$\begin{tikzcd} M(B) & M(A) \arrow[l] \\ P & A \end{tikzcd}$$

so $P = p(\mathbb{C}\Gamma\otimes A)$

$$\mathbb{C}\Gamma\otimes A\xrightarrow{\beta}\overset{E(A)}{\overset{\parallel}{P}}\xrightarrow{\alpha}\mathbb{C}\Gamma\otimes A$$

You ~~are~~ should be looking at the ring $\mathbb{C}\Gamma \otimes A$ ~~[illegible]~~
To get your act together. ~~A~~

Still puzzled about the notation. Perhaps because you have started with B.

Maybe you should start with A, try to keep track of the Γ-grading. ~~[illegible]~~ Try to keep track of Γ grading.

$$A \longrightarrow \mathbb{C}\Gamma \otimes A = \bigoplus_{s\in\Gamma} A$$

No discipline. Repeat.

Γ- Graded $\mathbb{C}$-modules ~~modules~~ $(= \hat{\Gamma}$ modules$)$ $\quad V = \bigoplus_{s\in\Gamma} V_s$

same as firm modules over $\bigoplus_{s\in\Gamma} \mathbb{C}e_s$

So what can I do?

Facts $\mathbb{C}\Gamma$ is a comm. counital coalg

~~[illegible]~~ For the moment leave out comodule

Def. Γ set Γ-graded $\mathbb{C}$-module $(= \hat{\Gamma}$-module$)$ $\quad V = \bigoplus_{s\in\Gamma} V_s$

Same as firm module over $\left[\begin{matrix} \bigoplus_{s\in\Gamma} \mathbb{C}e_s \\ \| \\ C_c\Gamma \end{matrix}\right.$ $\quad e_s e_t = \begin{cases} 0 & s \neq t \\ e_t & s = t \end{cases}$

comp. supp functions on Γ

~~adjoint~~ adjunction props.

$\mathbb{C}$-Mod $\qquad \hat{\Gamma}$-mod

$$\bigoplus_{s\in\Gamma} V_s \xleftarrow{\ F\ } (V_s)_{s\in\Gamma}$$

$$W \xmapsto{\ G\ } (W)_{s\in\Gamma}$$

$$\mathrm{Hom}\Big(\bigoplus_{s\in\Gamma} V_s, W\Big) = \prod_s \mathrm{Hom}_{\mathbb{C}}(V_s, W)$$
$$= \mathrm{Hom}_{\hat{\Gamma}}\big((V_s), (W)_{s\in\Gamma}\big)$$

adj. $\quad FG^W \to W$, $\quad \bigoplus_s W \to W$

$(V_s) \qquad GFV_s$

$1 \longrightarrow GF$

$$(V_s) \xrightarrow{(\iota_s)} \Big(\bigoplus_t V_t\Big)_s$$

$$\bigoplus V_s \longrightarrow \bigoplus_s \bigoplus_{t\ldots} V_t$$

Γ group $$\left(\bigoplus_s V_s\right)\otimes\left(\bigoplus_t W_t\right) = \bigoplus_{s,t} V_s\otimes W_t$$

$$= \bigoplus_{u\in\Gamma}\ \bigoplus_{\{(s,t)\mid st=u\}} V_s\otimes W_t$$

$$= \bigoplus_{u\in\Gamma}\ \bigoplus_{s\in\Gamma} V_s\otimes W_{s^{-1}u}$$

$\hat\Gamma$ alg $\quad A=\bigoplus_{s\in\Gamma} A_s \qquad A_sA_t\subset A_{st}$

canonical ~~[illegible]~~ $\hat\Gamma$ alg homom.

~~[illegible]~~

$$A \longrightarrow \mathbb{C}\Gamma\otimes A = \bigoplus_{s\in\Gamma} A$$

Yesterday I started to review $\hat\Gamma$ algs. I'm hoping to get ~~less awkward formulas~~ a better understanding of the M eq starting from A_Φ.

Recall def. $\hat\Gamma$-module (Γ a set) = ℂ-module with grading wrt Γ: $V=\bigoplus_{s\in\Gamma} V_s$, same as firm module over $C_c(\Gamma)=\bigoplus_{s\in\Gamma}\mathbb{C}e_s \qquad e_se_t = 0 \quad s\neq t, \quad = e_t \quad s=t.$

Maybe a better definition ~~is~~ of $\hat\Gamma$-module is a family of v.s. $(V_s)_{s\in\Gamma}$ indexed by Γ.

$$\begin{tikzcd} \mathbb{C}\text{-mod} & \hat\Gamma\text{-mod} \\ \bigoplus_{s\in\Gamma} V_s & (V_s)_{s\in\Gamma} \arrow[l, maps to, "F"'] \\ W \arrow[r, maps to, "G"] & (W)_{s\in\Gamma} \end{tikzcd}$$

$$\mathrm{Hom}_{\mathbb{C}}\left(\bigoplus V_s, W\right) = \prod_s \mathrm{Hom}(V_s, W) = \mathrm{Hom}_{\hat\Gamma}\left((V_s)_{s\in\Gamma}, (W)_{s\in\Gamma}\right)$$

Canon. adj. $\quad W = \bigoplus_{t\in\Gamma} W$

You want to study $\hat{\Gamma}$ algs and $\hat{\Gamma}$-modules

$$A_s M_t \subset M_{st}$$

~~Canon. A_s $\bigoplus$~~ canon. $A \longrightarrow A \times \hat{\Gamma}$

What's the point? Given $(A_s)_{s\in\Gamma}$ and $(M_t)_{t\in\Gamma}$ you have mult. $A_s M_t \subset M_{st}$ and canon structural maps

$\forall s \quad \iota_s : A_s \subset \bigoplus_{u\in\Gamma} A_u \quad , \quad \iota_t : M_t \hookrightarrow \bigoplus_{u\in\Gamma} M_u$

$$\boxed{\bigoplus \iota_s : \bigoplus_s A_s \hookrightarrow \bigoplus_s \bigoplus_u A_u \quad , \quad \bigoplus \iota_t : \bigoplus_t M_t \hookrightarrow \bigoplus_t \bigoplus_u M_u}$$

$$\begin{tikzcd}
A_s \otimes M_t \arrow[r] & \bigoplus_{u\in\Gamma} A_u \otimes \bigoplus_{u\in\Gamma} M_u \arrow[d] \\
 & \bigoplus_{u\in\Gamma} /
\end{tikzcd}$$

You want to see the compatible canon. map $A \longrightarrow A \otimes \mathbb{C}\Gamma$ Z

$A = \bigoplus_{s\in\Gamma} A_s$ is a Γ-graded algebra, you can regard it as an ungraded algebra i.e. Γ-graded where all elements have degree 1, [illegible] and then form $A' = \bigoplus_{t\in\Gamma} A$ the constant family: $A'_t = A \quad \forall t \in \Gamma$.

Let's do this properly. Let $A = \bigoplus_{s\in\Gamma} A_s$ be a Γ graded algebra, i.e. $A_s A_t \subset A_{st}$. Form the tensor product algebra of the underlying algebra A with the group alg $\mathbb{C}\Gamma$; write this $A\times\Gamma$. Its elements are finite linear comb. $\sum_{s\in\Gamma} a_s s$, better might be, where $(a_s)_{s\in\Gamma}$ is a family of elements of A indexed by Γ of finite support, mult is such that $sa = as$. $\therefore \left(\sum_{s\in\Gamma} a_s s\right)\left(\sum_{s\in\Gamma} a'_s s\right)$

$$= \sum_{s\in\Gamma}\sum_{t\in\Gamma} a_s a'_t st = \sum_{u\in\Gamma}\left(\sum_{t\in\Gamma} a_{ut^{-1}} a'_t\right) u$$

Principle: A Γ grading on V can be defined as a counital comodule structure for $\mathbb{C}\Gamma$.

$$\begin{tikzcd}
V \arrow[r] & \mathbb{C}\Gamma\otimes V \\
V_s \arrow[u, phantom, "\cup"] \arrow[r] & s\otimes V \arrow[u, phantom, "\cup"]
\end{tikzcd}$$

$$v \longmapsto \sum s\otimes e_s(v)$$

$$\boxed{\sum e_s(v) = v}$$

tensor product $V\otimes W \longrightarrow \mathbb{C}\Gamma\otimes V \otimes \mathbb{C}\Gamma\otimes W$,

$$\hookrightarrow \mathbb{C}\Gamma\otimes\mathbb{C}\Gamma\otimes V\otimes W \xrightarrow{\mu\otimes 1\otimes 1} \mathbb{C}\Gamma\otimes V\otimes W.$$

$$v\otimes w \longmapsto \sum_{t\in\Gamma} t\otimes e_t(v) \sum_{u\in\Gamma} u\otimes e_u(w)$$

$$= \sum_s s\otimes \underbrace{\sum_{s=tu} e_t(v)\, e_u(w)}_{e_s(v\otimes w)}$$

Put into words the idea that a Γ graded v.s. V sits naturally inside $\mathbb{C}\Gamma\otimes V$, which is naturally Γ graded with $s\otimes v$ of degree s.

It should be clear that given

$$\begin{tikzcd} A \arrow[r] & \mathbb{C}\Gamma \otimes A \\ A_s \arrow[r] \arrow[u, phantom, "\cup"] & s \otimes A_s \arrow[u, phantom, "\cup"] \end{tikzcd}$$

$$\left(\begin{tikzcd} V \arrow[r] & \mathbb{C}\Gamma \otimes V \\ V_s \arrow[r] \arrow[u, phantom, "\cup"] & s \otimes V_s \arrow[u, phantom, "\cup"] \end{tikzcd} \right) \otimes \left(\begin{tikzcd} W \arrow[r] & \mathbb{C}\Gamma \otimes W \\ W_s \arrow[r] \arrow[u, phantom, "\cup"] & s \otimes W_s \arrow[u, phantom, "\cup"] \end{tikzcd} \right) \quad ?$$

You want consistency in

$$\begin{tikzcd} V \otimes W \arrow[r, "\Delta\otimes\Delta"] & (\mathbb{C}\Gamma \otimes V) \otimes (\mathbb{C}\Gamma \otimes W) \arrow[d, phantom, "\cong"] \\ & \mathbb{C}\Gamma \otimes \mathbb{C}\Gamma \otimes V \otimes W \arrow[d, "\mu \otimes 1_{V\otimes W}"] \\ V \otimes W \arrow[r, "\sum_s s \otimes \sum_{s=tu} e_t \otimes e_u"'] & \mathbb{C}\Gamma \otimes (V \otimes W) \end{tikzcd} \quad ?$$

$$\begin{tikzcd} A \arrow[r, hook] & \mathbb{C}\Gamma \otimes A \end{tikzcd}$$

$\boxed{P(s)}$ degree(s) $\qquad s \otimes \boxed{P(s)}$ degree 0.

Suppose A

$A = \bigoplus A_\Phi$ gens $p(s)$ relative...

A_Φ

A_Φ is an algebra defined by gens $p(s)$, $s\in\Gamma$ are relations.

An alg homom. $A_\Phi \xrightarrow{u} B$ is equiv. to $p_s \in B$, $s\in\Gamma$ satisfying the rels. This is the same as an element $p\in\mathbb{C}\Gamma\otimes B$, $p=\sum_s s\otimes p(s)$, sat. $p^2=p$ and $\operatorname{Supp}(p)\subset\Phi$. Now $\mathbb{C}\Gamma\otimes B$ is automatically a Γ-graded alg for any alg B ??

Better approach: A_Φ is an alg defined by gen $p(s)$ $s\in\Gamma$ and rels ~~—~~. Let $B=\bigoplus_{s\in\Gamma} B_s$ be any Γ-graded alg.

An alg. homom. $A_\Phi \longrightarrow B$ is the same as an element $p=\sum p(s)\in B$ satisf. $p^2=p$, $p(s)=0$ for $s\notin\Phi$

To show A_Φ has a unique Γ-grading $\ni$ $\deg p_s = s$. Use $\mathbb{C}\Gamma\otimes A$ naturally Γ-graded for any alg A. Then $p=\sum s\otimes p_s \in \mathbb{C}\Gamma\otimes A_\Phi$ satisf $p^2=p$, $\operatorname{Supp}(p)\subset\Phi$. so there is ! alg. homom. $A_\Phi \xrightarrow{\Delta} \mathbb{C}\Gamma\otimes A_\Phi$ sending p_s to $s\otimes p_s$. Comodule props.

~~Γ-graded~~

$$\begin{tikzcd}
A_\Phi \arrow[r, "\Delta_A"] & \mathbb{C}\Gamma\otimes A_\Phi \arrow[r, shift left, "\Delta_\Gamma\otimes 1"] \arrow[r, shift right, "1\otimes\Delta_A"'] \arrow[d, "\eta"'] & \mathbb{C}\Gamma\otimes\mathbb{C}\Gamma\otimes A_\Phi \\
 & A_\Phi &
\end{tikzcd}$$

$$\begin{tikzcd}
p_s \arrow[r, mapsto] & s\otimes p_s \arrow[r, shift left, mapsto] \arrow[r, shift right, mapsto] \arrow[d, mapsto, "\eta"'] & s\otimes s\otimes p_s \;\overset{\shortparallel}{}\; s\otimes s\otimes p_s \\
 & p_s &
\end{tikzcd}$$

Now where are you? You know A_Φ is a Γ-graded alg in a natural way, the canonical $A_\Phi \xrightarrow{\Delta_A} \mathbb{C}\Gamma \otimes A_\Phi$ corresp. to a canon. $p = \sum_{s\in\Gamma} s \otimes p_s$ $\quad p^2 = p \quad$ Supp$(p) \subset \Phi$.

Now $\mathbb{C}\Gamma \otimes A_\Phi$ is Γ-graded with A_Φ ~~of~~ sitting in degree 1.

and you have $p \in \mathbb{C}\Gamma \otimes A_\Phi$, $p^2 = p$.

$p = \sum_{s\in\Gamma} s \otimes p_s$ not homogeneous

What can you do? $(\mathbb{C}\Gamma \otimes A_\Phi)p$ ~~This should~~ This should be a left A module. So you ~~maybe~~ probably want to look at $p(\mathbb{C}\Gamma \otimes A)$ which ~~should~~ is be a right [illegible] $\mathbb{C}\Gamma \otimes A$ module. The Γ action [illegible] can be converted to a left action, and you ~~can~~ should be able to find h_1.

Hopefully your funny formulas ~~sh~~:

$$(pf)(s) = \sum_t p(s^{-1}t) f(t)$$

~~should~~ will emerge! Look at $\mathbb{C}\Gamma \otimes A$. An element has the form $\sum t \otimes f(t)$ with $f \in C_c(\Gamma; A)$

right mult. by u^{-1} $\downarrow$

$$\sum_t tu^{-1} \otimes f(t) = \sum_{t\in\Gamma} t \otimes f(tu)$$

gives a left action of Γ on ~~Ss~~ $f \in C_c(\Gamma; A)$. ~~Left~~ Left mult

$$p = \sum_{s\in\Gamma} s \otimes p(s) : \sum_{t\in\Gamma} t \otimes f(t) \longmapsto \sum_{s,t\in\Gamma} st \otimes p(s) f(t)$$

$$pf = \sum_{\substack{s\in\Gamma \\ t\in\Gamma}} st \otimes p(s)f(t)$$

$u = st$
$s = ut^{-1}$

$$= \sum_{u\in\Gamma} u \otimes \sum_{t\in\Gamma} p(ut^{-1})f(t) \qquad ?$$

$$p = \sum_{s\in\Gamma} s \otimes p(s) \in \mathbb{C}\Gamma \otimes A \quad \ni f = \sum t \otimes \overset{C_c(\Gamma, A)}{\overline{f(t)}}$$

$$pf = \sum_{s\in\Gamma} s \otimes \sum_{t\in\Gamma} p(t)f(t^{-1}s)$$

Let's try to work a new notation.

$$\mathbb{C}\Gamma \otimes A = \Big\{\sum_{t\in\Gamma} t f(t) \;\Big|\; f \in C_c(\Gamma, A)\Big\}.$$

$$\sum_s s f(s) \sum_t t g(t) = \sum_s s \sum_t f(st^{-1}) g(t)$$

$p = \sum s p(s)$ acting on $f = \sum s f(s)$

is $pf = \sum_s s \sum_t p(st^{-1}) f(t)$ ord. convolution

Alternative.

$$\sum_{s\in\Gamma} s^{-1} p(s) \sum_{t\in\Gamma} t^{-1} f(t)$$

$$= \sum_{s,t\in\Gamma} \underset{\underset{(ts)^{-1}}{\|}}{\boxed{s^{-1}t^{-1}}}\, p(s)f(t) = \sum_{u\in\Gamma} u^{-1} \sum_t p(t^{-1}u) f(t)$$

$u = ts$

$$\sum_{s\in\Gamma} s\,p(s) \sum_{t\in\Gamma} t^{-1} f(t) = \sum_{s,t\in\Gamma} s t^{-1} p(s) f(t)$$

$$= \sum_{u,t} u \; p(\overset{ut}{[illegible]}) \, f(t)$$

$u = st^{-1}$ $\quad s = ut$

$$= \sum_{u} u \sum_{t} p(ut) f(t)$$

$$= \sum_{u} u^{-1} \sum_{t} p(u^{-1}t) f(t)$$

So the point seems to be that if you use $p = \sum s\,p(s)$ and $f = \sum t^{-1} f(t)$. then you get funny formula for the action of $p \in \mathbb{C}\Gamma \otimes A$ on $\mathbb{C}\Gamma \otimes A$

Need new notation.

$$p f = \sum_{s} s\,p(s) \sum_{t} t\, f(t) = \sum_{s} s \sum_{t} \underbrace{p(st^{-1})} f(t)$$

this kernel is inv. under $s \mapsto su$, $t \mapsto tu$

Next you need α, β.

Lesson: When using $p = \sum s\,p(s) = \sum p(s) s \in \mathbb{C}\Gamma \otimes A$ to form $p(\mathbb{C}\Gamma \otimes A)$ ~~you need to convert the~~ the left p mult. commutes with right Γ-module, so the kernel of $p\cdot$ has the form $p(st^{-1})$. ~~Operato~~

$$\begin{tikzcd} E(A) \arrow[r, "\alpha_1 = j\alpha"] & A \arrow[r, "\beta_1 = \beta\iota_1"] & E(A) \end{tikzcd}$$

Next operators on $$\begin{tikzcd} p(\mathbb{C}\Gamma \otimes A) \arrow[r, hook, "\alpha"] & \mathbb{C}\Gamma \otimes A \arrow[r, "\beta"] & p(\mathbb{C}\Gamma \otimes A) \end{tikzcd}$$

Given $\Gamma \supset \Phi$ finite, define A_Φ to be the alg defined by gens. $p(s), s \in \Gamma$, rels $p(s) = 0, s \in \Phi$ $p(s) = \sum_t p(st^{-1})p(t)$. An alg homom $A_\Phi \longrightarrow A$ same as an element $p = \sum p(s)s$ of the ~~A~~ Γ-graded alg $A\Gamma = \bigoplus_{s\in\Gamma} As$ satisf. $\mathrm{Supp}(p) \subset \Phi$, $p = p^2$. In particular you get canonical hom. $A_\Phi \xrightarrow{\phi} A_\Phi\Gamma$, $p(s) \longmapsto p(s)s$. ~~Since deg (p(s)s) = s~~ Note that

$$\begin{tikzcd} A_\Phi \arrow[r, "\phi"] \arrow[rr, bend right, "1"'] & A_\Phi\Gamma \arrow[r, "\eta"] & A_\Phi \end{tikzcd} \qquad \eta(as) = a$$

so ϕ identifies A_Φ with a subalg of $A_\Phi\Gamma$. Since $\deg(p(s)s) = s$, ϕA_Φ is a Γ-graded subalgebra. $\therefore$ A_Φ is Γ-graded alg

So you have this canonical $p \in A_\Phi\Gamma$. Put $A = A_\Phi$.

Morita context
$$\begin{pmatrix} A\Gamma & (A\Gamma)p \\ p(A\Gamma) & p(A\Gamma)p \end{pmatrix}$$
you know $(A\Gamma)^2 = A\Gamma$

you probably have to replace $A\Gamma$ by $A\Gamma p A\Gamma$. Conjectures are that $A\Gamma p A\Gamma \simeq B = B_\Phi = C_\Phi \rtimes \Gamma$ and that $p(A\Gamma)p = A$.

$$p = \sum_{s\in\Gamma} p_s s \in A\Gamma = \bigoplus_{s\in\Gamma} As$$

$$A\Gamma = \mathbb{C}\Gamma \otimes A = A \otimes \mathbb{C}\Gamma$$

$pA\Gamma$ is naturally a right $A\Gamma$ module.

$$p(A\Gamma) \overset{\alpha}{\hookrightarrow} \mathbb{C}\Gamma \otimes A \xrightarrow{\beta} p(A\Gamma) \xrightarrow{\alpha} \overset{A\Gamma}{\overrightarrow{\mathbb{C}\Gamma \otimes A}}$$

your idea is that $p(A\Gamma)$ should be canon. isom to

? $p(A\Gamma)$ should be the (B, A) bimodule $P = E(A)$

~~[illegible]~~ A $\mathbf{F} = (A\Gamma)p$

$\mathbf{E} = p(A\Gamma)$ B

So $\mathbf{F}$ is the left A-module corresp to the $\overset{\text{left}}{}$ B-module B which means $\mathbf{F} = h_1 B = h_1 C \otimes \mathbf{C}\Gamma$. Now both A and B are Γ-graded. Shouldn't this mean $\mathbf{E}, \mathbf{F}$ also are Γ-graded?

$B = C \rtimes \Gamma$?

Ultimately you will go from $B \in \mathcal{M}(B)$ to $h_1 B$ which is the A, B bimodule $\mathbf{F}$ and from $B \in \mathcal{M}(B^{op})$ to Bh_1, which is the B, A bimodule $\mathbf{E}$.

$$\therefore \boxed{Bh_1 = p(\Gamma \times A)\,?}$$

$B = C \rtimes \Gamma = \bigoplus_{s \in \Gamma} Cs$

$C = C_\Phi$: gens $h_s, s \in \Gamma$; rels $h_s h_t = 0$ $s^{-1}t \in \Phi$

$h_s = \sum_t h_s h_t = \sum_t h_t h_s$

$\mathbf{E} =$

Review notation $A = A_\Phi \hookrightarrow \underset{A \times \Gamma}{\overset{\shortparallel}{A\Gamma}}$, $p = \sum_{s \in \Gamma} p(s)s$

$B = C_\Phi \rtimes \Gamma$

You have a specific M.eq. between A and B, which you first described by going from a firm B-module E to the round A-module $h_1 E$; in the opposite direction you take any A-module V to p1 ?? You need to get this straight. V is a vector space representation of A, i.e. you have a family of operators $p(s) \in \mathcal{L}(V)$ satisfying the relations. $p(s) = 0$ if $s \notin \Phi$

$$p(s) = \sum_{t \in \Gamma} p(st^{-1})\, p(t)$$

~~Now you want to interpre~~ You want to ~~interpret~~ produce from such a family a projection on Γ-graded vector space. Preliminaries. ~~[illegible]~~ You want to interpret $A\Gamma$ as operating on $V\Gamma$ in the obvious way $(as)(vt) = avst$, then $p = \sum p(s)s \in A\Gamma$ yields the operator ~~[illegible]~~ $at \longmapsto \sum_s p(s)a\, st$

$$\Big(\sum p_s s\Big)\Big(\sum a_t t\Big) = \sum_{s,t \in \Gamma\times\Gamma} p_s a_t\, st = \sum_u \Big(\sum_t^s p_{u^{-1}t}\, a_t\Big) st$$

So we get a functor $V \longmapsto p(V\Gamma)$ from A-modules V to firm B-modules (hopefully), which is exact and kills nil A-modules. ~~You should note~~ Call this functor $V \longmapsto E(V)$. You know there's a canon. isom $E(V) \xleftarrow{\sim} E(\tilde{A}) \otimes_A V$

also $E(\tilde{A}) \otimes_A \mathbb{C}_0 = 0$: $E(\tilde{A})A = E(\tilde{A})$.

Also

$$\begin{tikzcd} & E(\tilde{A}) \otimes_A A \arrow[r, "\sim"] & E(A) \\ E(A) \otimes_A A \arrow[ur, "\sim"] & & \end{tikzcd}$$

so $E(A)$ is A^{op} firm flat.

$E(A) = p(\Gamma \times A)$ natural right $\Gamma \times A$ module

Point to be understood: Why $E(V)$ is ~~[illegible]~~ a firm B-module is a natural way for any A-module V. Somehow, you need to use B gen. by Γ and h, Points B has local units, hence B is left and right flat. $B = E \otimes_A F$ and Mog th. saying that

$$\left(\begin{array}{lcr} B \in \mathcal{M}(B) & \text{corresp to} & F \in \mathcal{M}(A) \\ B \in \mathcal{M}(B^{op}) & \text{———} & E \in \mathcal{M}(A^{op}) \end{array} \right)$$

This checks with what we've done. Now

$$B = E(F) \quad E(A) \otimes_A F$$

---

start again. $\underset{\Phi}{A} = A_{\Phi} \longrightarrow \mathbb{C}\Gamma \otimes A$

$$p_s \longmapsto s \otimes p_s \qquad \sum_{t \in \Gamma} p_t p_{t^{-1}s}$$

A gens $p_s, s \in \Gamma$ rels $p_s = 0$ for $s \notin \Phi$, $p_s = \sum_t p_{st^{-1}} p_t$

$p = \sum p_s s \in A\Gamma$ $\quad p^2 = p \quad \text{Supp}(p) \subset \Phi$.

$A\Gamma$ or $\Gamma \times A$ acts on $\Gamma \times V$ for any A-module V.

$$\left(\sum_s s p_s\right)\left(\sum_t t v_t\right) = \sum_s s \sum_t p_{st^{-1}} v_t$$

So you have the canon. $p = \sum p_s s$ in the Γ-graded alg $A \times \Gamma$ and p acts on $\Gamma \times V$ for any A-module V. also on $\Gamma \times W$ for any A^{op}-mod W $(\Gamma \times W)p$. What to do.

$$\begin{pmatrix} A & {}_A F_B \\ {}_B E_A & B \end{pmatrix}$$

$$\begin{pmatrix} A & \underline{(\Gamma \times A)p} \\ p(\Gamma \times A) & p(\Gamma \times A)p \end{pmatrix}$$

$$_B E(A)_A = p(A \times \Gamma)_A$$

$\underbrace{p(A\Gamma)_A}_{E_A}$ $\underbrace{{}_A(A\Gamma)p}_{{}_AF}$ $\langle F,E\rangle \cong (A\Gamma)p(A\Gamma)$

$$E\otimes_A F = \underbrace{p(A\Gamma)\otimes_A(A\Gamma)p}_{\text{should be } B}$$

~~[illegible]~~

Anyway. $p(\Gamma\times(A\otimes_A A)\times\Gamma)p \xrightarrow[\text{to find}]{} B$

Actually ~~[illegible]~~ when you form $p(\Gamma\times A)$ is this some sort of tensor product $\mathbb{C}\Gamma\otimes_{p?} A$? ~~[illegible]~~

Let's try to put this on a firm footing.

$$A\times\Gamma \cong \Gamma\times A = \bigoplus_{s\in\Gamma} As$$

There is an equivalence relation on $\mathbb{C}\Gamma\otimes V = \bigoplus_{s\in\Gamma} sV$

$p = \sum s\otimes p_s$ applied to $\sum_{t\in\Gamma} t\otimes v_t$ yields $\sum_s s\otimes\sum_t p_{st^{-1}}v_t$

so $p(\mathbb{C}\Gamma\otimes V) = \left\{(v_s)_{s\in\Gamma} \;\middle|\; v_s = \sum_t p_{st^{-1}}v_t\right\}$

You can look at the ~~stup~~ category consisting of elt of $s\in\Gamma$ objects and $\exists$ arrow $s\to t$ when $st^{-1}\in\Phi$

$$p(\mathbb{C}\Gamma\otimes A) = \{f\in C_c(\Gamma,A) \mid f = p*f\}$$

$$f(s) = \sum_t p(st^{-1})f(t)$$

define Γ action by $(uf)(s) = f(su)$

Then $\underset{\displaystyle\overset{\shortparallel}{f(su)}}{(uf)(s)} \overset{?}{=} \sum_t p(st^{-1}) \underset{\displaystyle\overset{\shortparallel}{}}{\underline{(uf)(t)}}$

$$f(su) \overset{\checkmark}{=} \sum_t p(st^{-1}) \cdot f(tu)$$

define h_1 on $p(\mathbb{C}\Gamma \otimes A)$ ~~to be~~ starting from $u_1 f_1$ on $\mathbb{C}\Gamma \otimes A$.

$$\begin{tikzcd} \overset{f}{\mathbb{C}\Gamma\otimes A} \arrow[r, two heads, "\beta"] & \overset{p*f}{p(\mathbb{C}\Gamma\otimes A)} \arrow[r, hook, "\alpha"] & \overset{p*f}{\mathbb{C}\Gamma\otimes A} \end{tikzcd}$$

Review the formula. $A = A_\Phi$, there's a canonical $p = \sum s p_s$ in $\mathbb{C}\Gamma \otimes A$ $\quad p^2 = p \quad \mathrm{Supp}(p) \subset \Phi$, corresp. to the Γ-graded alg homom. $A \longrightarrow \Gamma \times A$ assoc. to the Γ-grading on A

$$p_s \rightsquigarrow s \otimes p_s$$

$$\Gamma \times A = \Big\{ \sum_{s\in\Gamma} s f_s \;\Big|\; f \in C_c(\Gamma, A) \Big\} \qquad pf = \sum_s s p_s \sum_t t f_t$$

$$pf = p * f \quad \text{convolution} \qquad = \sum_s s \sum_t p_{st^{-1}} f_t$$

$$p(\Gamma\times A) = \{ f \mid f = p*f \}$$

$$\Gamma\times A \xrightarrow{p} p(\Gamma\times A) \xrightarrow{p} \Gamma\times A$$

Next comes the action of Γ. You have right mult. by Γ on $p(\Gamma\times A)$ namely $\Big(\sum_s s f_s\Big) u = \sum_s su\, f_s = \sum s f_{su^{-1}}$ becomes left action $\Big(\sum_s s f_s\Big) u^{-1} = \sum s f_{su}$

$$p(\Gamma\times A) = \{ f \in C_c(\Gamma, A) \mid p*f = f \}$$

$(uf)_s = f_{su}$ $\quad$ action of Γ.

$$\begin{tikzcd}
p(\Gamma\times A) \arrow[r, hook, "\alpha = inc\circ p\cdot"] \arrow[dr, "\alpha_1"'] & \Gamma\times A \arrow[r, "\beta = p\cdot"] \arrow[d, shift right, "j_1"'] & p(\Gamma\times A) \\
& A \arrow[u, shift right, "\iota_1"'] \arrow[ur, "\beta_1"'] &
\end{tikzcd}$$

$$j_1\left(\sum s f_s\right) = f_1 \qquad \iota_1 a = 1a = \sum_s s\,(a\,\delta_1(s))$$

$$(\beta_1 a)_s = \sum_t p_{st^{-1}}\,\delta_1(t)\,a = p_s a$$

$$(h_1 f)_s = (\beta_1\alpha_1 f)_s = (\beta_1 f_1)_s = p_s f_1$$

$$\left(h_1(t^{-1}f)\right)_s = p_s\,(t^{-1}f)_1 = p_s f_{t^{-1}}$$

$$\left(t(h_1(t^{-1}f))\right)_s = \left(h_1(t^{-1}f)\right)_{st} = p_{st} f_{t^{-1}}$$

and $\sum_t p_{st} f_{t^{-1}} = \sum p_{st^{-1}} f_t = (p*f)_s = f_s$

$$p(\Gamma\times A) \simeq \{f\in C_c(\Gamma, A) \mid p*f = f\}$$

$$(p*f)(s) = \sum_t p_{st^{-1}} f_t \qquad (tf)_s = f_{st}$$

$$j_1 f = f_1 \qquad \iota_1 a = a\,\delta_{1,s}$$

$$\beta_1 a = p\,\iota_1 a = p*(a\delta_1) = \qquad (\beta_1\alpha_1 f)_s = p_s f_1$$

$$(\beta_1 a)_s = \sum_t p_{st^{-1}}\,a\delta_{1,t} = p_s a \qquad \therefore (h_1 f)_s = p_s f_1$$

So you should have a B action on $p(\Gamma\times A)$ from Γ action $(tf)_s = f_{st}$ and the operator $(h_1 f)_s = p_s f_1$. Check relations

$$\left((h_1 t h_1)f\right)_s = p_s (th_1 f)_1 = p_s (h_1 f)_t = p_s p_t f_1$$

If $t \notin \Phi$, then $p_t = 0 \Longrightarrow h_1 t h_1 = 0$. ~~Suppose~~

$$\left(t^{-1} h_1 t f\right)_s = (h_1 t f)_{st^{-1}} = p_{st^{-1}} (tf)_1 = p_{st^{-1}} f_t$$

$$\therefore \left(\sum_t t^{-1} h_1 t f\right)_s = \sum_t p_{st^{-1}} f_t = (p * f)_s = f_s .$$

At this point you understand $E = p(\Gamma\times A)$ which should be a B,A bimodule firm on both sides ~~flat~~ projective as right A-module. Why: because

$$p(\Gamma\times\tilde{A}) \xrightarrow{\sim} \underbrace{p(\Gamma\times\tilde{A})}_{\text{projective } \Gamma\times\tilde{A}\text{-module}}$$

wait. $\Gamma\times A$ is an ideal in $\Gamma\times\tilde{A}$ which is unital, p is a projection in the alg $\Gamma\times A$, hence also in $\Gamma\times\tilde{A}$, so $p(\Gamma\times\tilde{A})$ is a summand of $\Gamma\times\tilde{A}$ as right $\Gamma\times\tilde{A}$-module, hence $p(\Gamma\times\tilde{A})$ is a summand of the free right $\tilde{A}$ module $\Gamma\times\tilde{A}$ so it seems that $p(\Gamma\times A) = p(\Gamma\times\tilde{A})$ is a firm projective A^{op}-module.

$$\begin{tikzcd}
p(\Gamma\times\tilde{A}) \arrow[r, hook, "\alpha = m"] \arrow[dr, Rightarrow, "\alpha_1"'] & \Gamma\times\tilde{A} \arrow[r, "\beta = p"] \arrow[d, shift right, "\pi_1"'] & p(\Gamma\times A) \\
& \tilde{A} \arrow[u, shift right, "\iota_1"'] \arrow[ur, "\beta_1"'] &
\end{tikzcd}$$

$$(\beta_1 a)_s = (\beta\,\iota_1 a)_s = (p\,a\delta_1(s)) = \sum_t p(st^{-1})\,a\delta_1(t) = p_s a$$

It looks as though ~~p(Γ×A)~~ the identity map on $p(\Gamma\times A)$ is 'nuclear' in a sense new to you!!!

Summarize. $p(\Gamma\times A)$ is a f.g. projective A^{op}-module on which B acts.

Take $a = 1 \in \tilde{A}$, then $(\beta_1 a)_s = p_s \in C_c(\Gamma, A)$ the family p_s corresp to $p = \sum_{s\in\Gamma} s p_s$ of $p(\Gamma\times A)$.

So you learn that the obvious element $p = \sum s p_s \in \Gamma\times A$ is in $p(\Gamma\times A)$ since $p^2 = p$. If $f = (f_s) \in C_c(\Gamma, A)$, then $(tf)_s = f_{st}$, so $\underset{\underset{B}{\cap}}{t}p = \sum_{s\in\Gamma} s p_{st} \in p(\Gamma\times A)$

$$\sum_u u p_u \sum_s s p_{st} =$$

$$\sum_{u,s} us\, p_u p_{st} = \sum_u u \sum_s p_{us^{-1}} p_{st} = \sum u p_{ut}$$

not too clear.

What next? Learn $\beta_1 = \beta\iota_1 : A \longrightarrow p(\Gamma\times A)$ is $a \longmapsto pa$

and $\alpha_1 = j_1\alpha$ sends $f \longmapsto f_1$

$$\alpha_1\left(\sum s f_s\right) = f_1 \qquad \beta_1\alpha_1\left(\sum s f_s\right) = \sum s p_s f_1 = p f_1$$

so h_1 appears to be $f \longmapsto f_1^{\in A} \longmapsto p\cdot f_1$ right mult by $f_1 \in A$

Is α_1 surjective

$$p(\Gamma \times A) = \{ f \in C_c(\Gamma, A) \mid f = p * f \}$$

and you need to understand $f_1 = \sum_{t\in\Gamma} p_{t^{-1}} f_t$

$$\begin{tikzcd} p(\Gamma\times A) \arrow[r, hook, "\alpha"] & \Gamma\times A \arrow[r, two heads, "\beta"] \arrow[d, shift right, "j_1"'] & p(\Gamma\times A) \\ & A \arrow[u, shift right, "i_1"'] & \end{tikzcd}$$

these maps are all A^{op}-module maps, so the image of $\alpha_1 = j_1\alpha$ is a right ideal in A X, and the kernel of $\beta_1 \underset{\sim}{=} \beta i_1$ is a right ideal in A

Maybe put $\tilde{A}$ in for A and then tensor with $\tilde{A}/\mathrm{Im}\,\alpha_1$. Then $\alpha_1 = 0$ so h_1 is 0 so

$$p\left(\Gamma \times \left(\tilde{A}/\mathrm{Im}\,\alpha_1\right)\right) = 0$$

---

look at $p(\Gamma\times A)p = \{ f \in C_c(\Gamma, A) \mid f = pf = fp \}$.

$$f_s = \sum_t p_{st^{-1}} f_t = \sum_t f_t p_{t^{-1}s}$$

but you get a Meq between ideals.

in general pCp is unital with unit p

$p \in C \quad p^2 = p$

$p(pcp) = pcp = (pcp)p$