Go over the flow of ideas

Γ groupoid $\rightsquigarrow$ $\mathbb{C}\Gamma$ coalg yields Γ graded vector spaces, but $\mathbb{C}\Gamma$ also an alg, yielding Γ and Γ^{op} modules, also have $\otimes$ for Γ graded vector spaces, Γ-graded algebras, Γ Grassmannian

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$\Gamma = M_2$

Γ-graded algs are block rings (A_{ij})

Morita contexts. | can adjoin object units.

You seem to get a generalization of functor for a Γ-graded algebra.

Go back to the A-module V, and see if you can understand

$$\mathrm{Img}\left\{ \begin{tikzcd} A \otimes V \arrow[r] \arrow[d] & \mathrm{Hom}(A,V) \\ A \otimes_A V \arrow[r] & \mathrm{Hom}_A(A,V) \arrow[u, hook] \end{tikzcd} \right\}$$

$$a \otimes v \longmapsto (a' \mapsto a'av)$$

This map functor

~~So~~ the point seems to be that

$A \otimes V$ is a quotient of 4 copies of V

$A = \sum p_{ij} A$ You seek the image of

$$A \otimes_A V \longrightarrow \mathrm{Hom}_A(A, V)$$

$$\boxed{AV \longrightarrow V/_AV}$$

now you know that ~~that~~ $p_s = \sum_{s=tu} p_t p_u$

$\therefore\ p_s \in A^2$ all s?

$A = \sum p_s \tilde{A}$

$$A \otimes_A V = \sum p_s A \otimes_A V =$$

$$A = \sum A p_s$$

$$A = \sum p_s \tilde{A}$$

$$\tilde{A}^4 \twoheadrightarrow A$$

$$(\tilde{a}_s) \longmapsto \sum p_s \tilde{a}_s$$

$$V^4 = \tilde{A}^4 \otimes_A V \twoheadrightarrow A \otimes_A V$$

$$(v_s) \longmapsto \sum_s p_s \otimes v_s$$

$$0 \longrightarrow \mathrm{Hom}_A(\mathbb{C}, V) \longrightarrow \mathrm{Hom}_A(\tilde{A}, V) \longrightarrow \mathrm{Hom}_A(A, V)$$

$$\mathrm{Hom}_A(\mathbb{C}, V) = {}_AV \qquad \mathrm{Hom}_A(\tilde{A}, V) = V$$

Here is what you learned. Let ρ be a multiplier of A, let V be an A-module, and

$$\operatorname{Img}\{AV \longrightarrow V/_A V\}$$

the corresponding reduced A-module. Define ρ on this image by $\rho\left(\sum a_i v_i\right) = \sum (\rho a_i) v_i \mod {}_A V$.

To show the right side is independent of the repn $\sum a_i v_i$, it suffices to ~~multiply~~ by any $a' \in A$:

$$a' \sum (\rho a_i) v_i = \sum a'(\rho a_i) v_i = \sum (a'\rho) a_i v_i = (a'\rho) \sum a_i v_i$$

which implies $\sum (\rho a_i) v_i \mod {}_A V$ is a well-defined function of the element $\sum a_i v_i \in AV$.

Note that ρ is a multiplier, two-sided.

This looks strange so for V firm:

$$A \otimes_A V \xrightarrow{\sim} V$$

The whole left mult. ring $\operatorname{Hom}_{A^{op}}(A, A)$ acts on V. But note that for V cofirm:

$$V \xrightarrow{\sim} \operatorname{Hom}_A(A, V)$$

the whole right mult. ring $\operatorname{Hom}_A(A, A)^{op}$ acts on V.

Discussion: ~~Today to finish M_n case.~~

Consider M_n case - connected groupoid, all objects.

You started with an A-module V, ~~and considered~~ ~~considered retract~~ equivalently a retract

$$W \twoheadleftarrow \Lambda \otimes V \hookleftarrow W$$

Repeat. M_n case. You started with a retract situation

$$W \xleftarrow{\beta} \Lambda \otimes V \xleftarrow{\alpha} W$$

equiv. an A-module structure on V. ~~[illegible]~~

But from your topos stuff you real want the free module $\Lambda \otimes V$ to have the form

$$\bigoplus_X \Lambda e_X \otimes V_X$$

But now you ~~[illegible]~~ should know that upon replacing V by $\mathrm{Im}\{AV \longrightarrow V/_A V\}$ this occurs.

$\mathcal{C}$ small cat

$\mathcal{C}$-set $=$ fun: $L: \mathcal{C} \longrightarrow$ sets

$\mathcal{C}^{op}$ set $=$ fun: $R: \mathcal{C}^{op} \longrightarrow$ sets

$$\underset{\text{sets}}{\mathrm{Hom}}(R \times_{\mathcal{C}} L, S) = \underset{\mathcal{C}^{op}\text{-sets}}{\mathrm{Hom}}(R, \mathrm{Hom}(L, S)) \ni \phi$$

$$R \times_{\mathcal{C}} L \longleftarrow \coprod_X R(X) \times L(X) \leftleftarrows \coprod_{Y \leftarrow X} R(Y) \times L(X)$$

$$R \underset{Ob}{\times} L \qquad\qquad R \underset{Ob}{\times} Ar \underset{Ob}{\times} L$$

$$\phi = (\phi_X)_{Ob} \qquad \phi_X \in \mathrm{Hom}(R(X), \mathrm{Hom}(L(X), S))$$

$$\qquad\qquad\qquad \| $$

$$\qquad\qquad\qquad \mathrm{Hom}(R(X) \times L(X), S)$$

$$Y \xleftarrow{f} X$$

$$\begin{array}{ccc} \phi_X : R(X) & \longrightarrow & \mathrm{Hom}(L(X), S) \\ f^* \uparrow & & \uparrow f_*^t \\ \phi_Y : R(Y) & \longrightarrow & \mathrm{Hom}(L(Y), S) \end{array}$$

$$\phi_X f^* : R(Y) \to R(X) \to \mathrm{Hom}(L(X), S)$$

same as

$$\begin{tikzcd} R(Y) \times L(X) \arrow[d, "f^* \times 1"] \arrow[dr] & \\ R(X) \times L(X) \arrow[r, "\phi_X"'] & S \end{tikzcd}$$

$$f_*^t \phi_Y : R(Y) \to \mathrm{Hom}(L(Y), S) \to \mathrm{Hom}(L(X), S)$$

same as

$$\begin{tikzcd} R(Y) \times L(X) \arrow[d, "1 \times f_*"] \arrow[dr] & \\ R(Y) \times L(Y) \arrow[r, "\phi_Y"'] & S \end{tikzcd}$$

Thus it amounts to

$$\begin{tikzcd} R(Y) \times L(X) \arrow[r, "f^* \times 1"] \arrow[d, "1 \times f_*"] & R(X) \times L(Y) \arrow[d, "\phi_X"] \\ R(Y) \times L(Y) \arrow[r, "\phi_Y"'] & S \end{tikzcd}$$

$$\begin{tikzcd} R(Y) \times \overbrace{\mathrm{Hom}(X,Y)}^{Ar(Y,X)} \times L(X) \arrow[r] \arrow[d] & R(X) \times L(X) \arrow[d, "\phi_X"] \\ R(Y) \times L(Y) \arrow[r, "\phi_Y"'] & S \end{tikzcd}$$

so first review action of $\mathrm{Mult}(A)$ on

$$\mathrm{Img}\{A\otimes V \longrightarrow \mathrm{Hom}_A(A,V)\}.$$

$$a_1\otimes v \longmapsto (a_2 \longmapsto a_2 a_1 v)$$

$a_1\otimes v$ $\qquad\qquad$ $(a_2 \longmapsto a_2 v)$

$$\begin{tikzcd} A\otimes V \arrow[r] \arrow[d] & \mathrm{Hom}_A(A,V) \arrow[d] \\ A\otimes V \arrow[r] & \mathrm{Hom}_A(A,V) \end{tikzcd}$$

$\mu a_1\otimes v$ $\qquad\qquad$ $a_2 \longmapsto (a_2\mu)v$

$a_1\otimes v$

$$\cancel{}\;(a_2 \longmapsto (a_2\mu)(a_1 v))$$

$$\mu a_1\otimes v \longmapsto a_2 \longmapsto a_2(\mu a_1)v$$

Better is that $\mathrm{Mult}(A)$ acts on $\mathrm{Img}\{AV \to V/_AV\}$.

Let $\mu\in\mathrm{Mult}(A)$, let $\sum a_i v_i \in AV$.

define $\quad \mu\left(\sum a_i v_i\right) = \sum (\mu a_i) v_i \mod {}_AV$

$$a'\sum(\mu a_i)v_i = \sum a'(\mu a_i)v_i$$
$$= (a'\mu)\Big(\sum_i a_i v_i\Big)$$

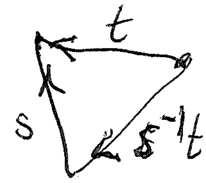
If $\quad \xi = \sum a_i v_i = \sum a'_j v'_j$

Start with A action on V

$$W \overset{\beta}{\twoheadleftarrow} \Lambda \otimes V \overset{\alpha}{\hookleftarrow} W$$

without changing W can replace V by V_{red} ~~How to~~

$\cancel{p(\Lambda \otimes V)}$



$$p\Big(\sum_t t \otimes f(t)\Big) = \sum_s s \otimes \sum_t p(s^{-1}t) f(t)$$

now assume that V is graded wrt objects.
there are four possible $f(t)$.

First use Morita eq.

$$W^\# \xleftarrow{(\beta_1\ \beta_2)} T^* \otimes V \xleftarrow{\binom{\alpha_1}{\alpha_2}} W^\#$$

$\Lambda = T \otimes T^*$

p on $T^* \otimes V$ $\qquad$ $p($

Idea there is a "free" case where $\mathbf{p = 1}$ on $T^* \otimes V$. See if you can analyze.
Assume V reduced and $p =$ identity.

Assume V is a reduced A-module such that $p = 1$ on $\Lambda \otimes V$.

$$W \longleftarrow \Lambda \otimes V \hookleftarrow W$$

$$\cancel{\oplus}\ \begin{matrix} V \\ \oplus \\ V \end{matrix}$$

~~[illegible]~~ Assume V reduced $\quad V = \begin{matrix} V_1 \\ \oplus \\ V_2 \end{matrix}$

$$\begin{matrix} V_1 \\ \oplus \\ V_2 \end{matrix} \xleftarrow{\sim} W^{\#} \xleftarrow{\sim} \begin{matrix} V_1 \\ \oplus \\ V_2 \end{matrix}$$

$$(\quad)$$

Start again with groupoid Γ (finite)

$$\mathrm{Hom}_\Lambda(\Lambda \otimes V, \Lambda \otimes V) = \Lambda^{op} \otimes \mathrm{End}(V) \simeq \Lambda \otimes \mathrm{End}(V)$$

$$p\left(\sum t \otimes f(t)\right) = \sum_{t,u} t u^{-1} \otimes p(u) f(t)$$

$tu^{-1} = su$
$s^{-1}t = u$

$$= \sum_s s \otimes \sum_t p(s^{-1}t) f(t)$$

Assume V is a reduced $\mathbf{A}$-module. ~~A has identity elt ... ex. ... Wait: V is a~~

A is Γ-graded — you know how to adjoin object units to A to make it unital + Γ graded

~~Start~~ Case ~~of~~ $\Gamma = M_2$ $\quad \Lambda = \mathbb{C}\Gamma$

$$\mathrm{End}_\Lambda(\Lambda \otimes V) \hookleftarrow \Lambda^{op} \otimes \mathrm{End}(V)$$
$$\qquad\qquad\qquad\qquad \parallel$$
$$\qquad\qquad\qquad\qquad \Lambda \otimes \mathrm{End}(V)$$

inj in general ~~&~~ iso for Γ fin.

so a $p \in \mathrm{End}_\Lambda(\Lambda \otimes V)$ equiv. to A-mod. st. on V.

$$W \xleftarrow{\beta} \Lambda \otimes V \xhookleftarrow{\alpha} W$$

W ~~[illegible]~~ depends on V mod nil, so you get same ~~nil module~~ W from $\mathrm{Ing}(AV \to V/AV)$

When V red. have $\mathrm{Mult}(A)$ acting, ~~put another ways~~ so can adjoin object idempotents to A to get a unital Γ-graded alg. ~~So~~

V becomes a ~~red.~~ unital module over $A' \simeq \begin{pmatrix} \tilde{A}_{11} & A_{12} \\ A_{21} & \tilde{A}_{22} \end{pmatrix}$

so $V = \begin{pmatrix} e_1 V \\ e_2 V \end{pmatrix}$. Since

$$A'_{ij} = e_i A' e_j \quad \text{one has}$$

~~$A'_{ij} V_k \subset V_i$~~

$$A'_{ij} V_k = e_i A' e_j e_k V \begin{cases} = 0 & j \neq k \\ \subset V_i & j = k \end{cases}$$

You are trying to say that V is graded ~~with~~ wrt set of objects Ob and that A is graded wrt ~~$Ob \times Ob$~~ groupoid M_{Ob}

The next question is what happens to reduced, rather, what is the graded version of reduced. You have $V = AV$

$$V_i = e_i V = e_i A \sum e_j V = \sum_j A_{ij} V_j$$

$$A v = 0 \iff \underset{\substack{\| \\ A_{ij} v_j}}{A_{ij} v} = 0 \quad \forall i, j$$

So V reduced means $V_i = \sum_j A_{ij} V_j$

and $A_{ij} v_j = 0 \ \forall i \implies v_j = 0$

(better: $\forall j$ and $v_j \in V_j$ if $A_{ij} v_j = 0 \ \forall i$ then $v_j = 0$.).

Look at M_2 case

$$\begin{tikzcd} W^{\#} & \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} \arrow[l, "(\beta_1\ \beta_2)"'] & W^{\#} \arrow[l, hook', "\begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}"'] & \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} \arrow[l, "(\beta_1\ \beta_2)"'] \end{tikzcd}$$

~~what does red. mean~~

V reduced should mean that $V_i = \sum_j p_{ij} V_j$ and

$(\forall j)(\forall v_j \in V_j)$, if $p_{ij} v_j = 0 \ \forall i$, then $v_j = 0$

$$V_i = \sum_j \alpha_i \beta_j V_j = \alpha_i (W^{\#})$$

$\alpha_i \beta_1 v_1 = 0 \ \forall i \implies v_1 = 0$

$\therefore \beta_1$ inj. sim β_2

M_2 case $\qquad \Lambda = M_2\mathbb{C}$

Repeat interesting point. Let Γ be a groupoid, $\Lambda = \mathbb{C}\Gamma$, assoc. arrow ring, let A be a Γ-graded algebra, claim $\exists$

Start again. Γ groupoid, $\Lambda = \mathbb{C}\Gamma$ the assoc arrow ring, A a Γ-graded alg, e_X the element of Λ given by 1_X, X an object, claim that e_X determines a multiplier on A

$$e_X\, {}_Yf_Z = \delta_{XY}\, {}_Yf_Z \qquad\qquad {}_Yf_Z \in$$

$$ {}_Yf_Z\, e_X = \delta_{ZX}\, {}_Yf_Z$$

$$\begin{array}{ccc}
(e_X\, {}_Yf_Z)\, {}_Wg_V & \overset{?}{=} & e_X({}_Yf_Z\, {}_Wg_V) \\
\| & & \| \\
\delta_{XY}\, {}_Yf_Z\, {}_Wg_V & & e_X(\delta_{ZW}\, {}_Yfg_V) \\
\| & & \| \\
\delta_{XY}\,\delta_{ZW}\, {}_Yfg_V & & \delta_{ZW}\,\delta_{XY}\, {}_Yfg_Y
\end{array}$$

$$({}_Yf_Z\, e_X)\, {}_Wg_V = \delta_{ZX}\, {}_Yf_Z\, {}_Wg_V = \delta_{ZX}\,\delta_{ZW}\, {}_Zfg_V$$

$$ {}_Yf_Z(e_X\, {}_Wg_V) = {}_Yf_Z\, \delta_{XW}\, {}_Wg_V = \delta_{XW}\,\delta_{ZW}\, {}_Yf\, g_V$$

messy. Other way is to use

$$\Delta : A \longrightarrow \Lambda \otimes A \subset \Lambda \otimes \tilde{A}$$

$$\Delta(a_s) = s \otimes a_s$$

$$e_x \otimes 1$$

$$(e_x \otimes 1)(s \otimes a_s) = \delta_{xy}\, s \otimes a_s$$ where $y = \mathrm{targ}(s)$

$$(s \otimes a_s)(e_x \otimes 1) = \underbrace{s e_x}\otimes a_s$$

$$\delta_{zx}\, s \otimes a_s \qquad z = \mathrm{source}(s)$$

Consider a general Γ suppose finite
A Γ-graded alg. Ask when A can be embedded as ideal in a Γ-graded alg A' which is unital. ~~Basic questn~~

Ask when there exists a Γ-graded alg A which is unital. Take $1 \in A$ and look at its components
$1 = \sum a_s$ particular example of a projection.

Basic question: Given Γ, have A_Γ, have for each A_Γ module V a p on $\Lambda \otimes V$. Assume $p = 1$. What can you conclude? ~~Thus~~

First of all $\underline{V \text{ is reduced}}$, since replacing V by AV will not affect the image of p.

Ex. Γ a group, finite

$$\Lambda \otimes V \ni \sum_t t \otimes f(t) \overset{p}{\longmapsto} \sum_s s \otimes \sum_t p(s^{-1}t) f(t)$$

Thus $p(s^{-1}t) = \delta_{st}$ for p to be the identity

So if A is a Γ-graded algebra which is unital, then what?

Start again with Γ, $\Lambda = \mathbb{C}\Gamma$, A_Γ plays the role of the Grassmannian ~~in a Γ-graded context~~.

Be more concrete.

Let Γ be a group (finite to simplify). Have equivalence between A_Γ-~~modules~~ structures on a v.s. V and retracts of the free Γ-module $\Lambda \otimes V$. Recipe

$$V \text{ tog with } p(s) \longmapsto p(\Lambda \otimes V)$$

$$p\Big(\sum_t t \otimes f(t)\Big) = \sum s \otimes \sum_t p(s^{-1}t) f(t)$$

When is $p = 1$? iff $p(s^{-1}t) = \delta_{st}$

Problem: Given Γ, $\Lambda = \mathbb{C}\Gamma$, A universal alg gen by components of a proj in Γ-gr alg.

What A-modules V yield $p=1$ on $\Lambda \otimes V$?

Claim that V is reduced. Because consider $AV \hookrightarrow V$

$$\cancel{\begin{tikzcd} p(\Lambda\otimes AV) \arrow[r, "\sim"] \arrow[d, shift left] & p(\Lambda\otimes V) \arrow[d, equal] \\ \Lambda\otimes AV \arrow[u, shift left] \arrow[r] & \Lambda\otimes V \end{tikzcd}}$$

$$\cancel{\begin{tikzcd} \Lambda\otimes AV \arrow[r] \arrow[d] & p(\Lambda\otimes AV) \arrow[r, hook] \arrow[d, "\wr"] & \Lambda\otimes AV \arrow[d] \\ \Lambda\otimes V \arrow[r] & p(\Lambda\otimes V) \arrow[r] & \Lambda\otimes V \end{tikzcd}}$$

Better is this. You have $p=1$ on $\Lambda\otimes V$, hence $p=1$ on $\Lambda\otimes AV$, so

$$\begin{tikzcd} p(\Lambda\otimes AV) \arrow[r, "\sim"] \arrow[d, equal] & p(\Lambda\otimes V) \arrow[d, equal] \\ \Lambda\otimes AV \arrow[r, hook, "\neq"] & \Lambda\otimes V \end{tikzcd}$$

~~[illegible]~~ When V is red (at least for Γ a groupoid), V has object grading $V = \bigoplus (e_i V)\, V_i$ and $\sum_j p_{ij} V_j = V_i$

~~you have object~~ $A \subset \tilde{A} = A \oplus \bigoplus_i \mathbb{C} e_i$

It should not be true that $p=1$ because you want to cut $\Lambda\otimes V$ down to $\bigoplus \Lambda e_i \otimes V_i$

Something overlooked: Because

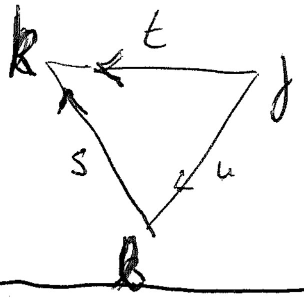
$$\mathrm{End}_\Lambda(\Lambda\otimes V) = \Lambda^{op}\otimes \mathrm{End}(V) = \Lambda\otimes\mathrm{End}(V)$$

for Λ unital and fin. diml, there is a 1-1 correspondence between Λ module structures on V and retracts of $\Lambda\otimes V$. In particular what corresponds to $p=1$ is $\mathbb{1}\otimes\mathbb{1}$ in $\Lambda\otimes\mathrm{End}(V)$

Suppose $\Gamma = M_2$ $\quad \Lambda = M_2\mathbb{C}$, $\quad \mathbb{1} = e_{11}+e_{22}$

so p_{ij} is ? $\qquad p\left(\sum_t t\otimes f(t)\right) = \sum_s s\otimes\sum_t p(s^{-1}t)f(t)$

$f = f(ij)$ $\qquad (pf)(k\ell) = \sum_{ij} \underbrace{p((\ell k)(ij))}_{\delta_{ki}\, p(\ell j)} f(ij)$ $\qquad$ "$\sum_j p(\ell j) f(kj)$



Continue with $\Gamma = M_2$

$$\mathrm{End}_\Lambda(\Lambda\otimes V) = \mathrm{Hom}(V,\Lambda\otimes V) = \Lambda\otimes\mathrm{End}(V) \quad \text{if } \Lambda \text{ f.d.}$$

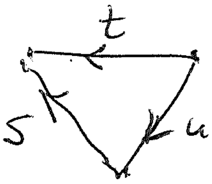
maybe

$$\mathrm{End}_{\Lambda^{op}}(V\otimes\Lambda) = \mathrm{End}(V)\otimes\Lambda$$

is better notation, especially when using [redacted] crossproduct

~~In any case you have yet to~~

$$p\left(\sum_t t\otimes f(t)\right) = \sum_{t,u} tu^{-1}\otimes p(u)f(t)$$



$tu^{-1}=s$

$u=s^{-1}t$

$p(st)=0$ when $st=0$ $\qquad = \sum_s s\otimes\sum_t p(s^{-1}t)f(t)$

$$\Delta(p(s))\Delta(p(t)) = (s\otimes p(s))(t\otimes p(t)) \qquad \Delta(p(s)p(t)) = st\otimes p(s)p(t)$$

So what next? $\Lambda = M_2\mathbb{C}$. Projections p on ~~Λ~~ $\Lambda\otimes V$ resp. Λ-module structure are equiv. to A-module structures on V, i.e. op. p_{ij} on V sat $p_{ik} = \sum_j p_{ij} p_{jk}$ Formula

$$p(\lambda\otimes v) = \sum_u \lambda u^{-1} \otimes p(u) v$$

$$(pf)(s) = \sum_t p(s^{-1}t) f(t)$$

$$(pf)(ij) = \sum_{kl} \underbrace{p((ji)(kl))}_{\delta_{ik}\, p(jl)} f(kl) = \sum_l p(jl) f(il)$$

$$(pf)(ia) = \sum_b p(ab) f(ab)$$

Check $$(p^2 f)(ij) = \sum_a p(ja)\,(pf)(ia)$$

$$= \sum_a p(ja) \sum_b p(ab)\, f(ib)$$

$$= \sum_b \underbrace{\left(\sum_a p(ja) p(ab)\right)}_{p(jb)} f(ib) = (pf)(ij)$$

~~better~~ $$(pf)(ij) = \sum_a p(ja) f(ia)$$

$$= \sum_a \sum_b p(jb) p(ba) f(ia)$$

$$= \sum_b p(jb)\,(pf)(ib) = (p(pf))(ij)$$

$$(pf)(ij) = \sum_a p(ja) f(ia)$$ when is $pf = f$ for all f

$$(pf)(s) = \sum_t p(s^{-1}t) f(t) \qquad p=1 \Leftrightarrow p(s^{-1}t) = \delta_{st}$$

formula $(pf)(ij) = \sum_a p(ja) f(ia)$

when is $pf = f$ for all f.

$$f(ij) = \sum_a p(ja) f(ia)$$

~~Take~~ $f(ia) = \delta_{ix}\delta_{ay}$

$$\delta_{ix}\delta_{jy} = \sum_a p(ja)\,\delta_{ix}\delta_{ay}$$

$\delta_{ix}\delta_{jy} = p(jy)\,\delta_{ix}$ take $x=i \Rightarrow$ $p(jy) = \delta_{jy}$

conversely if $p(ja) = \delta_{ja}$, then

$$f(ij) = \sum_a \delta_{ja} f(ia) = f(ij).$$

$$(pf)(s) = \sum_t p(s^{-1}t) f(t)$$

You need to check $p(s^{-1}t) = 0$ if $s^{-1}t$ not defd.

$$\Delta : A \longrightarrow \Lambda \otimes A$$

$$\Delta(a_s) = s \otimes a_s$$

relns. $p(s)p(t) = 0$ if $st = 0$

$$p(s) = \sum_{s=tu} p(t)\,p(u)$$

$$p = \sum_{s \in \Gamma} p_s \qquad \Delta(p) = \sum s \otimes p_s$$

$$\Delta(p_s p_t) = \Delta(p_s)\Delta(p_t) = (s \otimes p_s)(t \otimes p_t) = st \otimes p_s p_t$$

if $st = 0$, then $\Delta(p_s p_t) = 0$ so $p(s)\,p(t) = 0$

You need to check carefully in using the formula $(pf)(s) = \sum_t p(s^{-1}t)f(t)$ that this sum is take over all $t \in \Gamma$ such that $s^{-1}t \in \Gamma$, which means that s, t have the same target.

$$(pf)(ij) = \sum p(ja)f(ia)$$

Assume $(pf)(ij) = \sum_a p(ja)f(ia) \qquad \forall f : M_2 \to V$

take $f(ij) = \delta_{ix}\delta_{jy} v$

$$\delta_{ix}\delta_{jy} v = \sum_a p(ja)\,\delta_{ix}\delta_{ay} v = \delta_{ix}\, p(jy) v$$

$$\therefore \quad p(jy)v = \delta_{jy} v.$$

$p = 1$ corresp. to $p_{ij} = \delta_{ij}$ and for this A module structure on V, V is clearly reduced. Wait.

$v_{ij} = \sum_a p_{ja} v_{ia}$ for all families v_{ij}

take $v_{ij} = \delta_{ix}\delta_{jy} v$

$$\delta_{ix}\delta_{jy} v = \sum_a p_{ja}\,\delta_{ix}\delta_{ay} v = p_{jy}\,\delta_{ix} v$$

$$\delta_{jy} v = p_{jy} v \quad \forall v$$

So you've been looking at p on

$M_2\mathbb{C}\otimes V \qquad p\left(\sum_t t\otimes f(t)\right) = \sum_s s\otimes \sum_t p(s^{-1}t)f(t)$

$$(pf)_{ij} = \sum_a p(ja)\, f(ia)$$

you find $p=1 \iff p(ij)=\delta_{ij}$. Thus the four generators of A namely $p(ij)$ are the operators

$$p_{11} = 1 = p_{22} \qquad p_{12} = p_{21} = 0 \qquad \text{on } V.$$

Now V is clearly reduced since $V=\sum_{ij} p_{ij}V$ and $\bigcap_{ij} \mathrm{Ker}(p_{ij}) = 0$.

Think of f_{ij} as a matrix values in V

$$f_{ij} \longmapsto \sum_a p_{ja} f_{ia} = \sum_a p_{ja} (f^t)_{ai} = (p f^t)_{ji}$$
$$= \left(p(f^t)\right)^t_{ij}$$

So you have the operator

$$f \longmapsto (p f^t)^t = f p^t$$

$V\otimes\Lambda \qquad s = ut \quad u = st^{-1} \qquad \sum_s \left(\sum_t p(st^{-1})f(t)\right)\otimes s$

$$p\left(\sum_t f(t)\otimes t\right) = \sum_{t,u} p(u) f(t)\otimes ut = \sum_{t,s} p(st^{-1}) f(t)\otimes s$$

$$(pf)(s) = \sum_t p(st^{-1}) f(t)$$

At the moment you have a contradiction somewhere. ~~The form 1 Λ⊗V~~ For each A-mod structure on V you get a proj on the free Λ-mod $\Lambda \otimes V$, given by $(pf)(s) = \sum_t p(s^{-1}t) f(t)$

$$(pf)_{ij} = \sum_{kl} \underbrace{p((ji)(kl))}_{\delta_{ik} p_{jl}} f_{kl} = \sum_l p_{jl} f_{il}$$

here $f : \Lambda \longrightarrow V$ and $p : \Lambda \longrightarrow \mathrm{End}(V)$

Check. $(pf)_{ij} = \sum_a p_{ja} f_{ia}$

$$(p^2 f)_{ij} = \sum_a p_{ja} (pf)_{ia}$$

$$= \sum_a p_{ja} \sum_b p_{ab} f_{ib} = \sum_b \left(\sum_a p_{ja} p_{ab} \right) f_{ib}$$

$$= \sum_b p_{jb} f_{ib}$$

Assume $p = 1$. For all $f : \Lambda \longrightarrow V$ you have

$$f_{ij} = \sum_a p_{ja} f_{ai} \qquad \text{suppose } f_{ij} = \lambda_i \mu_j$$

$$\lambda_i \mu_j = \sum_a p_{ja} \lambda_a \mu_i$$

~~Suppose $V = \mathbb{C}$~~

$\Lambda = M_2\mathbb{C}$ $\quad$ $\mathrm{Hom}_\Lambda(\Lambda\otimes V, \Lambda\otimes V)$

$= \mathrm{Hom}(V, \Lambda\otimes V) = \Lambda\otimes \mathrm{Hom}(V,V)$

because Λ unital and fin. diml.

Given $t\otimes\theta \in \Lambda\otimes\mathrm{Hom}(V,V)$ define

$$(t\otimes\theta)(\lambda\otimes v) = \lambda t\otimes \theta v$$

Then $(t'\otimes\theta')\big((t\otimes\theta)(\lambda\otimes v)\big) = (t'\otimes\theta')(\lambda t\otimes\theta v)$

$$= \lambda t t'\otimes \theta'\theta v$$

$$= (tt'\otimes\theta'\theta)(\lambda\otimes v)$$

$\therefore$ ring structure is $\Lambda^{op}\otimes\mathrm{End}(V)$. But $\Lambda^{op}=\Lambda$ via $\boxed{\text{transpose}}$.

So you are trying to understand $\mathrm{End}_\Lambda(\Lambda\otimes V)$ as $\Lambda\otimes\mathrm{End}(V)$. You should have an isom-

$$\Lambda\otimes\mathrm{End}(V) \xrightarrow{\sim} \mathrm{End}_\Lambda(\Lambda\otimes V)$$

$$\Big(\sum_u u\otimes\theta(u)\Big)\Big(\sum_t t\otimes f(t)\Big) = \sum_{u,t} tu^*\otimes\theta(u)f(t)$$

$$= \sum_s s\otimes\sum_t\theta(s^*t)f(t) \qquad \begin{matrix} s = tu^* \\ u = s^*t\end{matrix}$$

Let $\underset{\sim}{\theta}$ denote ~~a family an elt~~ a family $\theta(u)$ as in $\sum u\otimes\theta(u)$

$$(\underset{\sim}{\theta} f)(s) = \sum_t \theta(s^*t)f(t)$$

$$\left(\underset{\sim}{\theta}'\left(\underset{\sim}{\theta} f\right)\right)(s) = \sum_t \theta'(s^*t)\left(\underset{\sim}{\theta} f\right)(t)$$

$$= \sum_t \theta'(s^*t) \sum_u \theta(t^*u) f(u)$$

$$= \sum_u \underbrace{\left(\sum_t \theta'(s^*t)\theta(t^*u)\right)}_{\text{type of convolution.}} f(u).$$

$$(pf)(s) = \sum_t p(s^{-1}t)\, f(t)$$

$$(pf)(ij) = \sum_{kl} p(\underbrace{e_{ij}^* e_{kl}}_{\delta_{ik} e_{jl}})\, f(kl)$$

$$\boxed{(pf)(ij) = \sum_l p(jl)\, f(il)}$$

$f : \Lambda \longrightarrow V$
$p : \Lambda \longrightarrow \mathrm{End}(V)$

So the identity operator arises from $p(jl) = \delta_{jl}$
$(pf)(ij) = \sum_l \delta_{jl} f(il) = f(ij)$. For this A-module namely V with $p(ij) = \delta_{ij}$, V is clearly red.

A should be a M_2-graded alg

$$A \xrightarrow{\Delta} M_2\mathbb{C} \otimes A$$
$$p(ij) \longmapsto e_{ij} \otimes p(ij)$$

$\Delta(p(ij)\,p(kl))$
$= e_{ij} e_{kl} \otimes p(ij)p(kl)$
$= 0$ if $j \neq k \Rightarrow p_{ij} p_{kl} \underset{j\neq k}{=} 0$

So the problem becomes clear, namely

$p(ij)\,p(k\ell) = 0$ for $j \neq k$ is

inconsistent with $p(ij) = \delta_{ij}$. What goes wrong?

You argued as follows. A is M_2 graded

$$\therefore\ A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \quad \text{and} \quad A = A^2$$

A is an ideal in $\tilde{\tilde{A}} = \begin{pmatrix} \tilde{A}_{11} & A_{12} \\ A_{21} & \tilde{A}_{22} \end{pmatrix}$ unital

so any reduced A-module should have unique $\tilde{\tilde{A}}$ module structure.

Your definition of A involves support conditions: $p_{ij}\,p_{k\ell} = 0 \quad j \neq k$ ~~Besides~~

in addition to the idempotence condition

$$p_{ik} = \sum_j p_{ij}\,p_{jk}$$

---

try dealing with a general connected groupoid Γ.

what should you do to make progress

focus on the good version of $\Lambda \otimes V$

clarify the link between A and Λ

"free" Λ modules; these are like representable functors. Try maybe to find a linearized version of $\mathcal{C}^{op} \longrightarrow \mathcal{C}$-sets

Program: clarify relation between A, Λ especially, to understand free Λ modules.

Γ groupoid, $\Lambda = \mathbb{C}\Gamma$, $A = \mathcal{P}_\Gamma$

A univ. alg generated by p_s $s \in \Gamma$, relations

idemp. $p_s = \sum_{s=tu} p_t p_u$

supp. $p_s p_t = 0$ if $st = 0$.

Claim A is Γ-graded alg:

$$\Delta : A \longrightarrow \Lambda \otimes A \qquad \Delta(p_s) = s \otimes p_s$$

$$\sum_{s=tu} (t \otimes p_t)(u \otimes p_u) = \sum_{s=tu} tu \otimes p_t p_u$$

$$= s \otimes \sum_{s=tu} p_t p_u = s \otimes p_s .$$

$$\Delta(p_s)\Delta(p_t) = (s \otimes p_s)(t \otimes p_t) = st \otimes p_s p_t = 0 \quad \text{if } st=0$$

unclear. Claim $\exists!$ alg map $\Delta : A \longrightarrow \Lambda \otimes A$ s.t.

$\Delta(p_s) = s \otimes p_s$. $\qquad (s \otimes p_s)(t \otimes p_t) = st \otimes p_s p_t = 0$ if $st=0$

You need some understanding!

$$A = \bigoplus_{s \in \Gamma} A_s \qquad A_s A_t \subseteq A_{st} \quad st \neq 0$$
$$= 0 \quad st = 0$$

So you start with an A module V, i.e. of $p_s \in \mathrm{End}(V)$ sat relns. Get retract $p(\Lambda \otimes V) = W$ of the free module $\Lambda \otimes V$. W doesn't change upon replacing V by $\mathrm{Img}\{AV \longrightarrow V/_A V\}$, so assume V reduced. Then V extends to an $\tilde{A}$ mod.

Summarize: Γ finite groupoid (ult. should be groupoid with a finite subset)

$A = P_\Gamma$ is Γ-graded algebra

Aim? Assembly for a groupoid Γ seems to involve something new & interesting. What? From Groth's viewpoint groupoids which are equivalent (as categories) should be mathematically indistinguishable. But assembly produces K-theory objects, Grassmannians describing retracts of "free" Γ-modules, also leading in a natural way to noncommutative partitions of unity. These constructions seem to go beyond the Grothendieck picture.

It seems that there is an interesting noncomm. picture of $B\Gamma$ for Γ a group (or groupoid).

Starting point. Γ groupoid, $\Gamma^{op} \hookrightarrow \Gamma$-sets. If C is a small cat, then a Γ-~~torsor~~ over C should be a functor

$$C\text{-sets} \xleftarrow{f^*} \Gamma\text{-sets}$$

which is right cont and left exact. You know that rt cont $\Longrightarrow$ $f^*(L) = R \times_\Gamma L$, where R is a C set with right Γ action. In fact R should be the composite

$$\Gamma^{op} \hookrightarrow \Gamma\text{-sets} \xrightarrow{f^*} C\text{-sets}$$

Also f^* rt exact $\Longrightarrow$ R representable, Γ^{op} not, at each object of C

So a topos map C-sets $\longrightarrow$ Γ-sets should be given by a functor $C \longrightarrow \Gamma$. When Γ is a category but not a groupoid, then a topos map C-sets $\longrightarrow$ Γ-sets should be given by a functor $C \longrightarrow Pro(\Gamma)$. Reason is that C can be identified with a full subcat of the cat of points in C-sets which is $Pro(C)$. Then for Γ a groupoid $\underline{Pro(\Gamma) = \Gamma.}$

For a groupoid Γ you have

$$\Gamma^{op} \xhookrightarrow{\text{Yoneda}} \Gamma\text{-sets}$$
$$X \longmapsto h^X = \coprod_Y Ar(Y,X)$$
$$Ar = \coprod_{Y,X} Ar(Y,X) = \coprod_X h^X$$

gives left module structure of $\Lambda = \mathbb{C}[Ar]$

$$\Lambda = \mathbb{C}[Ar] = \bigoplus_X \underbrace{\mathbb{C}[h^X]}_{\bigoplus_Y \mathbb{C}[Ar(Y,X)]}$$

Γ groupoid Ob Ar

$\Lambda = \mathbb{C}\Gamma$

$A = \mathcal{P}_\Gamma$ generators p_s $s \in \Gamma$

relations $p_s = \sum_{s=tu} p_t p_u$ | $p_s p_t = 0$ if $st = 0$

Repeat: Γ groupoid (finite), Ob, Ar

$$\Lambda = \mathbb{C}\Gamma = \bigoplus_{Y,X} \mathbb{C}[Ar(Y,X)]$$

$$\begin{array}{ccc} Ar(Z,Y) \times Ar(Y,X) & \longrightarrow & Ar(Z,X) \\ g \qquad\quad f & & gf \end{array}$$

$A = \mathcal{P}_\Gamma$ univ alg $\left|\begin{array}{l} \text{gen by } p_s \quad s \in Ar \\ \text{rels. } \quad p_s p_t = 0 \quad \text{if } st \text{ not defd} \\ \qquad p_s = \sum_{s=tu} p_t p_u \end{array}\right.$

Claim A is a Γ-graded alg. Define $\Delta : A \longrightarrow \Lambda \otimes A$ first on the gen p_s by $\Delta(p_s) = s \otimes p_s$. Check relns sat. $(s\otimes p_s)(t \otimes p_t) = \underset{\overset{\shortparallel}{0}}{st} \otimes p_s p_t$ etc.

$\therefore$ Next step is to construct semidirect product of A by $\bigoplus_{X \in Ob} \mathbb{C}e_X$. Do this: ~~...~~ $e_X \otimes 1 \in \Lambda \otimes \tilde{A}$

$\Delta(A) \subset \Lambda \otimes A \subset \Lambda \otimes \tilde{A}$ $\qquad (s\otimes a)(e_X \otimes 1) = \underset{0}{s\otimes a}$

Suppose Γ is a connected groupoid trivial isotropy, i.e. M_n. Have $\Lambda = M_n\mathbb{C}$, $A = \mathcal{P}_\Gamma$

gen p_{ij} rels $p_{ik} = \sum p_{ij} p_{jk}$, $p_{ij} p_{kl} = 0 \quad j \neq k$

~~$\tilde{A}_+ = A \oplus \mathbb{C}e_i$~~ A is M_n graded

can form $A_+ = A \oplus \bigoplus \mathbb{C}e_i$

$$A = \begin{pmatrix} & e_{ij}\otimes A_{ij} & \end{pmatrix} \oplus \begin{pmatrix} \mathbb{C}e_{11} & & \\ & \ddots & \\ & & \mathbb{C}e_{nn} \end{pmatrix}$$

So where are you? $\begin{pmatrix} \tilde{A}_{11} & A_{12} \\ A_{21} & \tilde{A}_{22} \end{pmatrix}$

Let V be a reduced A-module $V = \begin{pmatrix} V_1 \\ V_2 \end{pmatrix}$

$$A_{ij} V_k \subset \begin{matrix} 0 & j \neq k \\ V_i & j = k \end{matrix}$$

Reduced amounts to $V_i = \sum_j A_{ij} V_j \quad \forall i$

and $A_{ij} v_j = 0 \quad \forall i \implies v_j = 0$.

p_{ij}

$$0 = \sum_{ij} A_{ij} \sum v_k = \sum_{i,j} A_{ij} v_j$$

Now you understand V reduced A module

Better is $V = \sum_{ij} p_{ij} V$

$$\boxed{V_i = \sum_j p_{ij} V_j}$$

Let $v \in V$ sat

$0 = p_{ij} v \quad \forall i \ \forall j$. But $p_{ij} v = p_{ij} v_j$

$\therefore$ condition is $\boxed{(\forall j)(\forall i)\ p_{ij} v_j = 0 \implies v_j = 0}$

M_n The place to begin is with V graded: $V = \bigoplus V_k$, the "free" module should be $\bigoplus_k \Lambda e_{kk} \otimes V_k$ But $\Lambda e_{kk} = \bigoplus_i \mathbb{C} e_{ik}$ is the column vector representation of M_n denoted T, alt: $\Lambda = T \otimes T^*$ $\Lambda e_{kk} = T$.

~~In order to understand~~

So your free module is $\bigoplus_k T \otimes V_k$ with the standard col vector rep of Λ on T. It seem the Γ action separates ~~in some~~

formula for p?

$$\bigoplus_k \Lambda e_{kk} \otimes V_k = \bigoplus_{j,k} \mathbb{C} e_{jk} \otimes V_k$$

$$\Lambda \otimes V = \bigoplus_{i,j,k} \mathbb{C} e_{ij} \otimes V_k$$

$$(pf)(ij) = \sum_a p(ja) f(ia) \qquad ?$$

too hard

Use the M.eq. $T^* \otimes_\Lambda -$, $T \otimes -$

From $W \xleftarrow{\beta} \Lambda \otimes V \xrightarrow{\alpha} W$

get $W^\# \xleftarrow{(\beta_1 \cdots \beta_n)} T \otimes V \xrightarrow{\begin{pmatrix}\alpha_1 \\ \alpha_n\end{pmatrix}} W^\#$ $\qquad \sum \beta_j \alpha_j = 1_{W^\#}$

$$p = \begin{pmatrix}\alpha_1 \\ \alpha_n\end{pmatrix} (\beta_1 \quad \beta_n) \qquad p_{ij} = \alpha_i \beta_j \quad \text{on } V$$

~~V reduced?~~ ~~$V = \sum_{ij} \alpha_i \beta_j V = \sum$~~

~~$p_{ij} p_{kl} = \alpha_i \beta_j \alpha_k \beta_l \quad V = \sum \alpha_i \beta$~~

$$W^\# \xleftarrow{(\beta_1 \ \ \beta_n)} T \overset{*}{\otimes} V \xhookleftarrow{\begin{pmatrix}\alpha_1\\ \alpha_n\end{pmatrix}} W^\# \qquad \sum \beta_j \alpha_j = 1_{W^\#}$$

$$(p_{ij}) = \begin{pmatrix}\alpha_1\\ \\ \alpha_n\end{pmatrix}(\beta_1 \ \ \beta_n)$$

What does V red. mean?

$V = \sum p_{ij} V =$

$$V = \sum_i p_{ij} V = \sum_{ij} \alpha_i \beta_j V = \sum_i \alpha_i W^\#$$

$$\forall_{ij} \ \ \underline{p_{ij} v = \alpha_i \beta_j v = 0} \iff \forall_j \ \ \beta_j v = 0$$

~~Suppose~~ How to use $p_{ij} p_{k\ell} = 0$ for $j \neq k$.

$$\alpha_i (\beta_j \alpha_k) \beta_\ell = 0 \quad \text{for } j \neq k \quad 0$$

$$\implies \beta_j \alpha_k = 0 \qquad j \neq k.$$

Start again with case $\Gamma = M_n$ $\qquad \Lambda = M_n \mathbb{C}$

A gen p_{ij} rels $p_{ij} p_{k\ell} = 0 \quad j \neq k$

$$p_{ik} = \sum_j p_{ij} p_{jk}$$

$p = p^2$ on $\overbrace{\Lambda \otimes V}^{\text{the } \Lambda \text{ mod}}$ ~~equivalent to a A-module structure~~

~~on V~~ is equiv. to $(p_{ij}) \in End(V)$ satis

$p = p^2$ on Λ mod $\Lambda \otimes V$ equivalent to $p = p^2$ on $T \overset{*}{\otimes} V$, because $\Lambda = M_n \mathbb{C}$ is M. eq. to $\mathbb{C}$.

$$W^\# \xleftarrow{(\beta_1 \cdots \beta_n)} T^* \otimes V \xhookleftarrow{\begin{pmatrix}\alpha_1\\ \alpha_n\end{pmatrix}} W^\# \qquad \sum \beta_i \alpha_i = 1$$

$$p_{ij} = \alpha_i \beta_j \qquad \underset{j \neq k}{0 = p_{ij} p_{k\ell}} = \alpha_i \beta_j \alpha_k \beta_j \implies \underset{j \neq k}{\beta_j \alpha_k = 0}$$

$$W^\# \xleftarrow{(\beta_1 \quad \beta_n)} T^*\otimes V \hookleftarrow\!\xrightarrow{\begin{pmatrix}\alpha_1\\ \alpha_n\end{pmatrix}} W^\# \qquad \sum_i \beta_j \alpha_j = 1$$

$$(p_{ij}) = \begin{pmatrix}\alpha_1\\ \\ \alpha_n\end{pmatrix} (\beta_1 \quad \beta_n) \qquad p_{ij} = \alpha_i \beta_j$$

$$j \neq k \Rightarrow 0 = p_{ij}\, p_{k\ell} = \alpha_i (\beta_j \alpha_k) \beta_\ell \qquad \forall i, \ell$$

which implies $\boxed{\beta_j \alpha_k = 0 \text{ on } W^\# \text{ for } j \neq k}$

$$\sum \beta_j \alpha_j = 1$$

Check this carefully.

$$W^\# \xleftarrow{(\beta_1 \cdots \beta_n)} T^*\otimes V \hookleftarrow\!\xrightarrow{\begin{pmatrix}\alpha_1\\ \vdots \\ \alpha_n\end{pmatrix}} W^\#$$

$$\sum \beta_j \alpha_j = 1 \implies p = \begin{pmatrix}\alpha_1\\ \vdots \\ \alpha_n\end{pmatrix} (\beta_1 \cdots \beta_n) \quad \text{sat} \quad p = p^2$$

$p_{ij} = \alpha_i \beta_j$ check idem rel $\sum_j \alpha_i \beta_j \alpha_j \beta_k = \alpha_i \beta_k$

other relation: $p_{ij}\, p_{k\ell} = 0 \qquad j \neq k$

$$\alpha_i \beta_j \alpha_k \beta_\ell = 0 \qquad \forall i, \ell, \; j \neq k$$

$$\Downarrow$$

$$0 = \sum_i \beta_i \alpha_i \beta_j \alpha_k \beta_\ell = \beta_j \alpha_k \beta_\ell \implies 0 = \sum_\ell \beta_j \alpha_k \beta_\ell \alpha_\ell = \beta_j \alpha_k$$

important to understand the meaning of $\beta_j \alpha_k = 0$ for $j \neq k$ (in addition to $\sum \beta_j \alpha_j = 1$)

Consider
$$\begin{tikzcd} W' & V \oplus V \arrow[l, "(\beta_1, \beta_2)"'] & W' \arrow[l, "\begin{pmatrix}\alpha_1\\ \alpha_2\end{pmatrix}"'] \end{tikzcd} \qquad \sum \beta_j \alpha_j = 1_{W'}$$

<u>To understand</u> the condition $\beta_j \alpha_i = 0 \quad j \neq i$

~~Let's [illegible] grafting for W.M.~~

Consider more generally

$$\begin{tikzcd} W' & V_1 \oplus V_2 \arrow[l, "(\beta_1, \beta_2)"'] & W' \arrow[l, "\begin{pmatrix}\alpha_1\\ \alpha_2\end{pmatrix}"'] \end{tikzcd} \qquad \beta_1\alpha_1 + \beta_2\alpha_2 = 1_{W'}$$

β

$$\begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix} = \begin{pmatrix} \alpha_1\beta_1 & \alpha_1\beta_2 \\ \alpha_2\beta_1 & \alpha_2\beta_2 \end{pmatrix}.$$

Here $\beta_1\alpha_2$, $\beta_2\alpha_1$ are alltomatically zero, at least undefined.

<u>Repeat</u>: Begin with a retract of $\Lambda \otimes V$ as Λ module

$$W \xleftarrow{\beta} \Lambda \otimes V \xhookleftarrow{\alpha} W \qquad \beta\alpha = 1_W$$

this should be Morita equivalent to a retract

$$W^\# \xleftarrow{(\beta_1 \cdots \beta_n)} T^* \otimes V \xhookleftarrow{\begin{pmatrix}\alpha_1\\ \vdots \\ \alpha_n\end{pmatrix}} W^\# \qquad \sum \beta_j \alpha_j = 1_{W^\#}$$

get corresp $p = p^2$ on $T^* \otimes V$

$$p = \begin{pmatrix}\alpha_1\\ \vdots \\ \alpha_n\end{pmatrix}(\beta_1 \cdots \beta_n) \qquad p_{ij} = \alpha_i \beta_j \quad \in \mathrm{End}(V)$$

The p_{ij} satisfy the idemp rel $\displaystyle\sum_j p_{ij} p_{jk} = \sum_j \alpha_i \beta_j \alpha_j \beta_k = \alpha_i \beta_k \overset{p_{ik}}{=}$

You think it's reasonable to impose the condition (support)

$p_{ij} p_{kl} = 0 \quad$ for $j \neq k \quad$ so that $A = P_\Gamma$ is Γ graded.

~~If you do this~~ if $p_{ij} p_{kl} = \alpha_i \beta_j \alpha_k \beta_l = 0$ for $\forall_{i,j,k,l}$, $j \neq k$

then apply $\sum_i \beta_i \cdot$ on left, $\sum_l \cdot \alpha_l$ on the right to

get $\beta_j \alpha_k = 0$ for $j \neq k$.

What have you done? You have described the Λ-module $W = p(\Lambda \otimes V)$ corresp to an A-module V as $T \otimes W^{\#}$ where

$$W^{\#} \xleftarrow{(\beta_1\ \beta_2)} \begin{matrix} V \\ \oplus \\ V \end{matrix} \xleftarrow{\begin{pmatrix}\alpha_1 \\ \alpha_2\end{pmatrix}} W^{\#}$$

is a retract of $\begin{matrix} V \\ \oplus \\ V \end{matrix}$, i.e. $\sum_j \beta_j \alpha_j = 1$, such that $\beta_j \alpha_i = 0$ for $i \neq j$. So far you have not assumed V is reduced. However the two copies of V are independent vectors spaces, not related by any operator.

Suppose now V is reduced. $V = \sum p_{ij} V = \sum_{i,j} \alpha_i \beta_j V = \alpha_1 W^{\#} + \alpha_2 W^{\#}$. Other condition

Suppose $\alpha_i \beta_j v = 0$ $\forall i, j$. Then $\beta_j v = 0$

$$V = \alpha_1 W^{\#} + \alpha_2 W^{\#}$$

$$\beta_1 V = \beta_1 \alpha_1 W^{\#} \qquad \beta_2 V = \beta_2 \alpha_2 W^{\#}$$

Start again. $\Gamma = M_n$, $\Lambda = M_n \mathbb{C}$

$$\mathrm{End}_\Lambda(\Lambda \otimes V) = \Lambda^{op} \otimes \mathrm{End}(V)$$

$$\mathrm{End}_{\mathbb{C}}(T^* \otimes V) = \mathrm{End}(T^*) \otimes \mathrm{End}(V)$$

$$W' \xleftarrow{(\beta_1 \cdots \beta_n)} T^* \otimes V \xleftarrow{\begin{pmatrix}\alpha_1 \\ \vdots \\ \alpha_n\end{pmatrix}} W' \qquad \sum \beta_j \alpha_j = 1$$

$p_{ij} = \alpha_i \beta_j$ impose $p_{ij} p_{kl} = \alpha_i \beta_j \alpha_k \beta_l = 0$ $j \neq k$

$\implies \boxed{\beta_j \alpha_k = 0 \quad j \neq k.}$

$$W' \xleftarrow{(\beta_1,\dots,\beta_n)} \bigoplus V_j \xleftarrow{\begin{pmatrix}\alpha_1\\ \alpha_n\end{pmatrix}} W' \qquad \sum \beta_j \alpha_j = 1_{W'}$$

$$p_{ij} = \alpha_i \beta_j \in \mathrm{Hom}(V_i \leftarrow V_j) \qquad V_i \leftarrow V_j .$$

$$p_{k\ell} = \qquad \in \mathrm{Hom}(V_k \leftarrow V_\ell) \qquad V_k \leftarrow V_\ell$$

is what you get

Method : $\Gamma \quad \Lambda \quad A = \tilde{P}_\Gamma \subset A \tilde{\times} \mathcal{O}b$

So you have this nice M_n-graded unital alg. $A \tilde{\times} \mathcal{O}b$ containing A as ideal.

next ~~that~~ consider V over $A \tilde{\times} \mathcal{O}b$ which are reduced

$$W' \xleftarrow{(\beta_1 \cdots \beta_n)} \bigoplus V_j \xleftarrow{\begin{pmatrix}\alpha_1\\ \vdots \\ \alpha_n\end{pmatrix}} W'$$

$$V_i = \sum_j p_{ij} V_j = \alpha_i W'$$

$$p_{ij} v_j = \alpha_i(\beta_j v_j) = 0 \qquad \forall i$$

$$\Longrightarrow \beta_j v_j = 0$$

$\therefore$ reduced means each β_j inj., each α_i surj. means you can recover V_i as image of $\beta_i \alpha_i$

$\Gamma = M_{n}$ $\quad A = P_\Gamma$, $\quad A \check{\oplus} \mathrm{Ob}$

[illegible] Consider the category. Objects are vector spaces graded wrt $\mathrm{Ob} = \{1 \le i \le n\}$, $V = \bigoplus_i V_i$ together with operators $p_{ij} : V_i \leftarrow V_j$ satisfying relations $p_{ik} = \sum_j p_{ij} p_{jk}$, $p_{ij} p_{kl} = 0$ $\quad j \neq k$

Maps are obvious ones resp. grading + operators.

~~Equivalently~~ Form: $p = (p_{ij})$ on $\bigoplus_i V_i$

get retract $\quad W' \xleftarrow{(\beta_1 \cdots \beta_n)} \bigoplus_i V_i \xleftarrow{\begin{pmatrix}\alpha_1 \\ \vdots \\ \alpha_n\end{pmatrix}} W'$ $\qquad \alpha_i = e_i \alpha$, $\quad \beta_j = \beta e_j$

$\sum \beta_j \alpha_j = 1_{W'}$ $\qquad p_{ij} = \alpha_i \beta_j$

since $0 = p_{ij} p_{kl} = \alpha_i \beta_j \alpha_k \beta_l$ for $k \neq l$ $\Rightarrow$ $\beta_j \alpha_k = 0$

Let $p(v_j) = \sum_j p_{ij} v_j$ $\qquad p^2 = p$

~~Start~~ at a different place. You must describe the modules carefully. $A \oplus' \mathbb{C}[\mathrm{Ob}]$ unital modules over

~~Attack~~ $n = 2$

$$\begin{pmatrix} \tilde{A}_{11} & A_{12} \\ A_{21} & \breve{A}_{22} \end{pmatrix}$$

Find words to describe.

A unitary module V over $A \check{\oplus} \mathrm{Ob}$ should ~~consist of~~ be a vector space equipped with a grading $V = \bigoplus V_i$ and ~~an A~~ operators $p_{ij} \in \mathrm{Hom}(V_i \leftarrow V_j)$ satisfying the relations

Start again. You need the appropriate object, kind of module. Try:

$$W' \overset{\beta}{\twoheadleftarrow} \bigoplus_{i=1}^{n} V_i \overset{\alpha}{\hookleftarrow} W' \qquad \beta\alpha = 1.$$

thus a module consists of a retract W' of a a vector space V equipped with a splitting (action of $\bigoplus_i \mathbb{C}e_i$).

$$1 = \sum_{i=1}^{n} \underbrace{(\beta e_i)}_{\beta_i} \overbrace{(e_i\alpha)}^{\alpha_i}$$

$$\boxed{\begin{array}{ll} \sum_j \beta_j\alpha_j = 1 & \\ \beta_j\alpha_k = 0 & j \neq k \end{array}}$$

so it seems that W' ~~...~~ is equipped with operators $h_j = \beta_j\alpha_j$ satisfying $\sum_{j=1}^{n} h_j = 1$

On V you have the operators $e_1, \dots, e_n$ and $p = \alpha\beta$

$p_{ij} = e_i\alpha\beta e_j = \alpha_i\beta_j$. $\qquad \sum_j p_{ij}p_{jk} = \sum_j \alpha_i\beta_j\alpha_j\beta_k = \alpha_i\beta_k = p_{ik}$

$p_{ij}p_{k\ell} = \alpha_i\beta_j\alpha_k\beta_\ell = 0$ for $j \neq k$.

You should be able to replace V by its reduced version without ~~altering~~ changing W'.

Reduced should mean $\quad V_i = \sum_j p_{ij}V_j = \sum_j \alpha_i\beta_j V_j = \alpha_i W'$

$\forall i \quad \alpha_i\beta_j v_j = 0 \iff \beta_j v_j = 0 \implies v_j = 0.$

$\therefore$ Reduced should mean α_i surjective, β_j injective if so then you can recover V_j as the image of $h_j = \beta_j\alpha_j$.

Let's reverse the process - let W' be a vector space equipped with operators $h_1, \dots, h_n$ satisfying $\sum_{i=1}^n h_i = 1$. Let $V_i = \mathrm{Img}\{h_i : W' \to W'\}$

let $W' \xrightarrow{\alpha_i = h_i} V_i \subset \xrightarrow{\beta_i = \text{the inclusion}} W'$ $\quad \therefore \beta_i \alpha_i = h_i$ with α_i surj + β_i inj

Then $\sum \beta_i \alpha_i = 1_{W'}$, so you have

$$W' \xleftarrow{(\beta_1 \cdots \beta_n)} \bigoplus V_i \xleftarrow{\begin{pmatrix}\alpha_1 \\ \vdots \\ \alpha_n\end{pmatrix}} W'$$

Therefore it seems

Review. $\Gamma = M_n$ $\quad$ A univ ~~alg~~ gen p_{ij} 2 rels

$\Delta : \dot{A} \longrightarrow \Lambda \otimes A$ $\quad$ Γ graded ring, can adjoin idemp. e_i to obtain A_+ Γ graded unital ring. Reduced A-mod. same as $V = \bigoplus V_i$ $\quad p_{ij} V_k = 0 \quad j \neq k$

$\sum_j p_{ij} V_j = V_i$, $\quad p_{ij} v_j = 0 \ \forall i \implies v_j = 0$.

This is the ~~[illegible]~~ module cat on A side

Next. B universal unital ring gen h_i $i = 1, \dots, n$ $\sum h_i = 1$.

$V_i = h_i W$ $\qquad W \xleftarrow{\beta} V \xleftarrow{\alpha} W$

$h_i = \beta_i \alpha_i$ can fact of $\quad W \xleftarrow{\beta_i} V_i \xleftarrow{\alpha_i} W$

When you write $V = \bigoplus V_i$ you have in mind $\quad \eta_j \varepsilon_i = \delta_{ji} \qquad \sum \varepsilon_i \eta_i = 1$

$$\alpha = \sum_i \varepsilon_i \alpha_i \qquad \beta = \sum_j \beta_j \eta_j \qquad \alpha = \sum \varepsilon_i \alpha_i$$

$$\beta_j \eta_j \, \varepsilon_k \alpha_k = \begin{cases} 0 & j \neq k \\ \beta_j \alpha_j = h_j & j = k \end{cases}$$

$$W \xleftarrow{(\beta_1 \cdots \beta_n)} V \xleftarrow{\begin{pmatrix}\alpha_1 \\ \alpha_n\end{pmatrix}} W \xleftarrow{\beta} V$$

so far you take W and construct V

Define $p = \alpha\beta =$??

Try again $h_i = \beta_i \alpha_i$ $\quad W \xleftarrow{\beta_i} V_i \xleftarrow{\alpha_i} W$

$$W \xleftarrow{\beta} V \xleftarrow{\alpha} W \qquad \alpha = \sum_i \varepsilon_i \alpha_i$$

$$\beta\alpha = \sum_{j\,i} \beta_j \overbrace{\eta_j \varepsilon_i}^{\delta_{ji}} \alpha_i = \sum h_i \qquad \beta = \sum_j \beta_j \eta_j$$

$$p = \alpha\beta = \sum_{ij} \underbrace{\varepsilon_i \alpha_i \beta_j \eta_j}_{p_{ij}} \quad \text{operator from } V_i \leftarrow V_j$$

This defines an A module structure on V.
Moreover it makes V a reduced A-module

(i) $\sum p_{ij} V = V$? $\iff \forall i \ \sum_j p_{ij} V_j = V_i$

$$\sum_j \alpha_i \beta_j V_j = \alpha_i W$$

$p_{ij} v = 0 \quad \forall i, j$

$p_{ij} v_j = \alpha_i \beta_j v_j \quad \forall i \Rightarrow \beta_j v_j = 0$

$\therefore v_j = 0.$

Conversely suppose V reduced A-module. Then unique extension to unital $\tilde{A}$ module, yielding $V = \oplus V_i$
with $p_{ij} V_k \subset \begin{cases} 0 & j \neq k \\ V_i & j = k \end{cases}$ also have $\sum_j p_{ij} V_j = V_i$

$\forall i, j \ p_{ij} v = 0$ ~~mean~~ $\alpha_i \beta_j v_j = 0 \ \forall i j$ $\qquad \alpha_i : W \to V_i$ onto what do you know? V reduced A-mod

$V = \bigoplus V_i$ with $p_{ij} V_k \subset \begin{cases} 0 & j \neq k \\ V_i & j = k \end{cases}$

$\sum_{i,j} p_{ij} V = V \iff \sum_j p_{ij} V_j = V_i$

$\forall i j \ p_{ij} v = 0 \iff \forall i \ p_{ij} v_j = 0$
$\Rightarrow v = 0 \qquad \Rightarrow v_j = 0$

Define $W \xleftarrow{\beta} V \xleftarrow{\alpha} W$ $\quad \alpha\beta = p$, $\beta\alpha = 1$.

W as retract of V corresp to $(pv)_i = \sum_j p_{ij} v_j$

if $\alpha = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix}$ $\quad \beta = (\beta_1 \cdots \beta_n)$ $\quad \sum \beta_i \alpha_i = 1_W$

$p_{ij} = \alpha_i \beta_j$ $\qquad \beta_j \alpha_k = 0 \quad j \neq k.$

~~[illegible]~~ Also you have

$\forall i \ 0 = p_{ij} v_j = \alpha_i \beta_j v_j \Longrightarrow \beta_j v_j = 0$

---

to get it clearer. Claim $\exists$ equivs

(red A-mod V) $\qquad$ (red B mods. W)

$p_{ij} \longleftarrow W \xleftarrow{(\beta_1 \cdots \beta_n)} \bigoplus_{i=1}^n V_i \xleftarrow{\begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix}} W \qquad \sum \beta_j \alpha_j = 1_W$

$W \xrightarrow{\alpha_i} V_i \xrightarrow{\beta_i} W$

Morita context should be simple.

$hBh \qquad hB \qquad \langle bh, hb' \rangle = bhb'$

$Bh \qquad B$

$$\begin{pmatrix} h_i B h_j & h_i B \\ B h_j & B \end{pmatrix}$$

Given a B module W (unital) you associate $V = \bigoplus_{i=1}^{n} h_i W$ which is an A-module. How? Factor $h_i = \beta_i \alpha_i : W \overset{\beta_i}{\hookleftarrow} h_i W \overset{\alpha_i = h_i}{\twoheadleftarrow} W$,

then define $p_{ij} : h_i W \xleftarrow{\alpha_i} W \xleftarrow{\beta_j} h_j W$, in other notation

~~$p_{ij} h_k w = 0$ for $j \neq k$ and $p_{ij} h_j w = h_i h_j w$??~~

$p_{ij} h_j w = \alpha_i \beta_j h_j w = \alpha_i h_j w = h_i h_j w \qquad p_{ij} h_k w = h_i w \, \delta_{jk}$

Check that the relations hold.

$p_{ij} p_{kl} h_m w = p_{ij} h_k h_l w \, \delta_{lm} =$

$h_i h_j h_k h_l w \, \delta_{jk} \delta_{lm}$

$\therefore p_{ij} p_{kl} = 0$ for $j \neq k$

$p_{ij} p_{jl} h_m w = h_i h_j$

$\sum_j p_{ij} p_{jl} h_m w = h_i w$

$V = \bigoplus_i h_i W \qquad$ factor $\quad h_i = \beta_i \alpha_i : W \overset{\beta_i = \mathrm{inc.}}{\hookleftarrow} h_i W \overset{\alpha_i = h_i}{\twoheadleftarrow} W$

$$\bigoplus V_i \xleftarrow{\begin{pmatrix}\alpha_1 \\ \vdots \\ \alpha_n\end{pmatrix}} W \xleftarrow{(\beta_1 \cdots \beta_n)} \bigoplus V_j$$

$p_{ij} : V_i \longleftarrow V_j$

$p_{ij} : V_i \xleftarrow{\alpha_i} W \hookleftarrow{\beta_j} V_j$

$p_{ij}(h_j w) = \alpha_i (h_j w) = h_i h_j w$

$p_{ij}(h_k w) = 0 \qquad j \neq k.$

$p_{ij} p_{kl} V_m = 0$ unless $l = m$ and $j = k$

$\sum_j \underbrace{p_{ij} p_{jl} (h_l w)}_{} = p_{ij} h_j h_l w = h_i h_j h_l w$

$= h_i h_l w = p_{il}(h_l w).$

After the Morita context B gen. by h_i $1 \le i \le n$, $\sum h_i = 1$. You are going to construct over B some sort of dual pair.

Given a unital B-module W let $V_i = h_i W$, let $h_i = \beta_i \alpha_i : W \xleftarrow{\beta_i} V_i \xleftarrow{\alpha_i} W$ be the canon. fact. of h_i. Let $V = \bigoplus V_i$

let $$W \xleftarrow{\beta} V \xleftarrow{\alpha} W \qquad \beta = (\beta_1 \cdots \beta_n),\ \alpha = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix}$$

so that $\beta\alpha = \sum \beta_j \alpha_j = \sum h_j = 1_W$, and then

let $p = \alpha\beta \qquad p_{ij} = \alpha_i \beta_j \in \mathrm{Hom}(V_i \leftarrow V_j)$

You want to construct a Morita context

$$\begin{array}{c|c} & j \\ \hline i & \left[\begin{array}{ccc|c} & & & \alpha_1 \\ & \alpha_i\beta_j & & \vdots \\ & & & \alpha_n \\ \hline \beta_1 & \cdots & \beta_n & B \end{array}\right] \end{array} \qquad \begin{bmatrix} V_1 \\ \vdots \\ V_n \\ W \end{bmatrix}$$

$$\begin{bmatrix} A & Y \\ X & B \end{bmatrix}$$

X is a B mod gen by $\beta_1 \cdots \beta_n$

Y —— B^{op} —— $\alpha_1 \cdots \alpha_n$

$$\langle \beta_j, \alpha_k \rangle = h_j \delta_{jk}$$

~~Define~~ B univ. $\overset{\text{unital}}{\text{alg}}$ gen. by elts $h_1, \dots, h_n$ satis $\sum_{1}^{n} h_i = 1$

W unital B-module given, let $V_i = h_i W$

$$h_i = \beta_i \alpha_i : W \xleftarrow{\beta_i = \text{inc}} V_i \xleftarrow{\alpha_i = h_i} W$$

Define $V = \bigoplus_{i=1}^{n} V_i$

$$W \xleftarrow{\beta = (\beta_1 \cdots \beta_n)} \bigoplus_1^n V_i \xleftarrow{\alpha = \begin{pmatrix}\alpha_1 \\ \alpha_n\end{pmatrix}} W$$

$\beta_i \alpha = \alpha_i$

$$V = \bigoplus_{i=1}^{n} V_i \qquad V_i \underset{\varepsilon_i}{\overset{\eta_i}{\rightleftarrows}} V \qquad \eta_j \varepsilon_i = \delta_{ij} \, id_{V_i}$$

$$\sum_i \varepsilon_i \eta_i = id_V$$

$$\beta = \beta \sum_i \varepsilon_i \eta_i = \sum (\beta \varepsilon_i) \eta_i = \sum \beta_i \eta_i$$

$$\alpha = \sum_i \varepsilon_i \eta_i \alpha = \sum_i \varepsilon_i \alpha_i \qquad \boxed{\alpha_i = \eta_i \alpha} \boxed{\beta_i = \beta \varepsilon_i}$$

$$1_W = \sum_i \beta_i \alpha_i = \sum_i \beta \varepsilon_i \eta_i \alpha = \beta \alpha$$

$$\beta_j \alpha_k = \beta \varepsilon_j \eta_k \alpha = \delta_{jk} h_k$$

So what ~~are~~ is your aim? From W you get V_i and α_i, β_i and α, β. ~~The catego~~

Type of module you get W with h_i, V with e_i, $W \xleftarrow{\beta} V \xleftarrow{\alpha} W$ such that $\beta_j = \beta e_j$ $\alpha_k = e_k \alpha$ satisf $\beta_j \alpha_k = \delta_{jk}$?

~~So what? You are trying to~~

from (W, h_i) you get (V, e_i) a dilation $h_i = \beta e_i \alpha$.

dilation process from (W, h_i) to (V, e_i)

is this an RA situation? $A = \bigoplus_{i=1}^{n} \mathbb{C}e_i$

unital alg map $u : RA \longrightarrow B$ same as $\rho : A \longrightarrow B$ linear such that $\rho(1) = 1$. If $A = \bigoplus_{i=1}^{n} \mathbb{C}e_i$, then ρ is the same as elements $\rho(e_i) = h_i$ in B such that $1 = \rho(1) = \rho(e_1 + \cdots + e_n) = h_1 + \cdots + h_n$.

So what happens? You still need the Morita equivalence.

Idea that GNS has this interesting (idempotent) case

Abelian group representations W with a generating subspace V are given by a positive hermitian-valued measure on the dual, special case is where the pos. herm. matrices are projections.

So look at ring $\Lambda = \bigoplus_{1}^{n} \mathbb{C}e_i$ functions on $\{1, \ldots, n\}$

W retract of $V = \bigoplus e_i V$

$$W \xleftarrow{(\beta_1 \cdots \beta_n)} \bigoplus V_i \xleftarrow{\begin{pmatrix}\alpha_1 \\ \alpha_n\end{pmatrix}} W \qquad \sum \beta_i \alpha_i = 1$$

$$\beta_j \alpha_k = 0 \quad j \neq k$$

$e_i \alpha = \alpha_i \qquad \beta e_j = \beta_j$

$\beta_j \alpha_k = \beta e_j e_k \alpha = 0 \quad j \neq k$

$\sum_j \beta_j \alpha_j = \sum_j \beta e_j \alpha = \beta\alpha = 1. \qquad p_{ij} = \alpha_i \beta_j = e_i p e_j$

A retract W of the Λ mod V is equiv to family p_{ij} making V an A-module. V red means $V_i = \sum_j p_{ij} V_j \Leftrightarrow V_i = \alpha_i W$ and $\forall i$ $p_{ij} v_j = 0 \Leftrightarrow \beta_j v_j = 0 \ \forall v_j \Leftrightarrow \beta_j$ inj.

Do you understand? $\Lambda = \bigoplus_{i=1}^{n} \mathbb{C} e_i$

Let's look at the Hilbert space ~~situation~~.

W Hilbert space with hermitian $h_i \geq 0$ & $\sum h_i = 1$.

$V_i = \overline{h_i^{1/2} W} \subset W$ $\qquad$ V_i = completion of W wrt $\|h_i^{1/2}\omega\|^2 = (\omega, h_i\omega)$

$$W \xrightarrow{\alpha_i = h_i^{1/2}} V_i \hookrightarrow{\beta_i = h_i^{1/2}} W$$

It should be true that $\alpha_i^* = \beta_i^*$. Then $\alpha^* = \beta^*$

$$W \xleftarrow{\beta} \oplus V_i \xleftarrow{\alpha} W \qquad \alpha^*\alpha = \beta\alpha = 1_W$$

Basically you have the C^* alg Λ and the Hilbert space repn $V = \oplus V_i$. Next you have closed subspace W of V generating the repn V. i.e $\overline{\sum e_i W} = V$. $\qquad W \underset{\alpha}{\overset{\beta}{\rightleftarrows}} V$

α = inclusion, β = proj onto W. On W you have the operators $\beta e_i \alpha = h_i \geq 0$ $\qquad \sum h_i = 1$.

So the h_i give a completely pos. ~~function on W~~ map $\Lambda \longrightarrow \mathcal{L}(W)$, and you can reconstruct V from it. Of course what's special is that $\sum_i h_i = 1$

Spend some time on the Morita context.

$$\begin{array}{cc|c} & & \alpha_{1,n+1} \\ & & \vdots \\ & & \alpha_{n,n+1} \\ \hline \beta_{n+1,1} & \cdots\ \beta_{n+1,n} & B \end{array} \qquad \begin{pmatrix} V_1 \\ \vdots \\ V_n \\ W \end{pmatrix} \qquad \begin{bmatrix} A & Y \\ X & B \end{bmatrix}$$

Can you define X, Y in $\begin{bmatrix} A & Y \\ X & B \end{bmatrix}$?

It seems that $X = \sum_j B\beta_j$, $Y = \sum_j \alpha_k B$

and the obvious thing to try, is to ~~set~~ the pairing $X \otimes Y \longrightarrow B$ non degenerate. Thus

$$\begin{tikzcd} B^n \arrow[d, two heads] & (B^n)^\vee \\ X \arrow[r] & \operatorname{Hom}_{B^{op}}(Y, B) \arrow[u, hook] \end{tikzcd}$$

So your idea is take the pairing

$$\left\langle \sum b_j \beta_j, \sum_k \alpha_k b_k' \right\rangle = \sum_{j=1}^n b_j h_j b_j'$$

~~In fact you might find that the quotient of B^n arising is $\sum_j B h_j$~~

between B^n and $(B^{op})^n$. ~~You don't know~~

When should $\sum b_j \beta_j$ be zero? iff

$$\left\langle \sum_j b_j \beta_j, \sum_k \alpha_k b_k' \right\rangle = \sum_j b_j h_j b_j'$$

for all (b_j'), i.e. iff $b_j h_j = 0 \quad \forall j$

$b_j \in B_{h_j}$ right annihilator. So

$X = \bigoplus_j B/B_{h_j} = \bigoplus_j B h_j$ YES clearly.

Construction of the Morita context:

$$\begin{bmatrix} & & \alpha_1 \\ & & \vdots \\ & & \alpha_n \\ \beta_1 & \cdots \ \beta_n & B \end{bmatrix} \begin{bmatrix} V_1 \\ \vdots \\ V_n \\ W \end{bmatrix} \qquad W \overset{\beta_i}{\longleftrightarrow} V_i \xleftarrow{\alpha_i} W \qquad \left[\begin{array}{c|c} A & Y \\ \hline X & B \end{array}\right]$$

Idea: $X = \sum_j B\beta_j = B^n/R_X$, $Y = \sum_k \alpha_k B = (B^{op})^n/R_Y$

the pairing $\left\langle \sum_j b_j \beta_j , \sum_k \alpha_k b'_k \right\rangle = \sum_j b_j h_j b'_j$

$R_X = \{(b_j) \mid \sum_j b_j h_j b'_j = 0 \quad \forall (b'_j)\}$ now take

$b'_j = \delta_{ji}$ get $b_i h_i = 0$ $\forall i$ i.e. $b_i \in B_{h_i}$

$\therefore$ $X = \bigoplus_i B/B_{h_i} = \bigoplus_i B\overset{\alpha_i}{h_i}$

Similarly $Y = \bigoplus_i B/{}_{h_i}B = \bigoplus_i h_i \overset{\beta_i}{B}$

10 hour flight
9 h 21 min

Now if $X = \oplus Bh_i$, $Y = \oplus h_i B$ then

$Y \otimes_B X \simeq \bigoplus_{i,j} h_i B \otimes_B Bh_j$ — should contain $\alpha_i \beta_j = p_{ij}$

In any case $Y \otimes_B X$ is be the firm version of A *
and the reduced version of A should be $\bigoplus_{i,j} h_i B h_j$

* because Y, X is a firm dual pair over B

At some point universal properties became important. Let's first look at the Morita context

$$\begin{bmatrix} hBh & hB \\ Bh & B \end{bmatrix}$$

Better: ~~Consider~~ Start with the dual pair (Bh, hB) where $\langle bh, hb' \rangle = bhb'$

To see what you need for a clear picture. The result is that two rings are Morita equivalent. Begin by describing the reduced module categories. For the ring B ~~you have v.s.~~ a red. module is a v.s. W tog. w. operators $h_i \in End(V)$ $\ni$ $\sum h_i = 1$. These are unital modules over the universal unital ring with gens $h_1, \ldots, h_n$ and reln. $\sum^n_1 h_i = 1$.

For the ring A a red. module is ~~an~~ n-tuple of vector spaces $(V_1, \ldots, V_n)$ together with operators $p_{ij}: V_j \to V_i$ $\forall i, j$ sat: (i) $p_{ik} = \sum_j p_{ij} p_{jk}$

(ii) $\forall i$, $V_i = \sum_j p_{ij} V_j$, (iii) $\forall v_j \in V_j$ s.t. $(\forall i)$ $p_{ij} v_j = 0$, have $v_j = 0$

How to go from W to $(V_1, \ldots, V_n)$ is easy namely put $V_i = h_i W$, let $\left\{ W \xleftarrow{x_i} V_i \xleftarrow{y_i} W \right.$ $h_i = x_i y_i$

Then put $p_{ij} = y_i x_j : V_i \xleftarrow{y_i} W \xleftarrow{x_j} V_j$

Check conditions: (i) $\sum_j p_{ij} p_{jk} = \sum_j y_i x_j y_j x_k = y_i x_k = p_{ik}$

(ii) $\sum_j p_{ij} V_j = \sum y_i x_j V_j = y_i \left(\sum_j x_j V_j\right) = y_i \left(\sum_j x_j y_j W\right) = y_i W = V_i$

(iii) ass $p_{ij} v_j = y_i x_j v_j = 0$ $\forall i$, $\Rightarrow \sum_i x_i y_i x_j v_j = x_j v_j = 0$ $\Rightarrow v_j = 0$ as x_j injective.

Not clear. Certainly going from W to V is easily. Probably it's ~~case~~ worthwhile ~~making~~ using the module category for the Morita context. ~~Philosophy~~ Answer should be simple. Module should be a sequence of vector spaces $(V_1, \dots, V_n, W)$ and maps $W \xleftarrow{x_i} V_i \xleftarrow{y_i} W$ satisf $\sum_1^n x_i y_i = 1_W$, x_i surj, y_i injective. Now the equivalence of this with a B-module is easy. Given W Take the ~~canonical~~ fact of $h_i = x_i y_i$: $W \xleftarrow{x_i} V_i \xleftarrow{y_i} W$ for each i.

Go back to the type of module: a sequence of vs $(V_1, \dots, V_n, W)$ and operators $W \xleftarrow{x_i} V_i \xleftarrow{y_i} W$ s.t. $\sum_1^n x_i y_i = 1_W$, x_i inj, y_i surj

$e_1 \quad y_1 \quad \dots \quad e_n \quad y_n$

~~Alt description~~

unital M_{n+1} gr alg gen. by $x_1, \dots, x_n$ ~~&~~ B $\qquad x_i y_j = 0$, $i \neq j$, $\sum x_i y_i = 1$

how about M_{n+1} gr alg gen by x_i of degree $(i, n+1)$, y_i of deg $(n+1, i)$, $1 \le i \le n$, still not clear.

Maybe you should ~~retreat~~ return to the group Γ case. basic module type: consists of a Γ-module W, a vector space V and maps $W \xleftarrow{x} V \xleftarrow{y} W$ such that

$$\sum_{s \in \Gamma} s(xy)s^{-1} = 1_W, \quad x \text{ inj}, \quad y \text{ surj}$$

Look at question of whether any idempotent ring is Morita equivalent to a ring with local units. First example would be the ring of sequences $(f_n)_{n\geq 0}$ with $f_n \in \mathbb{C}$ such that $f_n \to 0$ all $n \to \infty$. This example is a C^*-alg.

Now you think you can prove that two commutative idempotent rings A,B which are Morita equivalent are in fact isomorphic, the argument being that given $\begin{pmatrix} A & Q \\ P & B \end{pmatrix}$ giving the Morita equivalence, one has

$$A\otimes_A \simeq Q\otimes_B P\otimes_A \simeq P\otimes_A Q\otimes_B = B\otimes_B$$

canon. isomorphisms, hence because A and B are comm. one has canon additive isomorphism $A \simeq B$ $\left(A\otimes_A = A/[A,A] = A \text{ if } A \text{ comm.}\right)$

Therefore you have to look for a non comm.

B. Look for a dual pair

To write up the stuff about M. eq. Outline the steps. Begin with M_n case. First discuss the Morita equivalence as an equivalence between certain module systems, or module types

The combined "module type" in the M_n case consists of a sequence of vector spaces $(V_1, \dots, V_n, W)$ together with operators $W \xleftarrow{\beta_i} V_i \xleftarrow{\alpha_i} W$ for each i satisfying: $\sum \beta_i \alpha_i = 1_W$, α_i surj, β_i inj work out the details of the M eq.

Case Γ group: Working "module" category

"module" ~~[illegible]~~ V vector space, W Γ-module

$W \xleftarrow{\beta_1} V \xleftarrow{\alpha_1} W$ | $\sum_s s\beta_1\alpha_1 s^{-1} = 1_W$, $\alpha_1 s\beta_1 = 0$, $s \neq e$

β_1 inj, α_1 surj

Ideas about ~~[illegible]~~ the problem of whether any idempotent ring is Mor. eq. to a ring with local units. Go back to the proof of Roos thm. ~~[illegible]~~ Ultimately you have to choose both a generator Y for $\mathcal{M}(A)$ and one X for $\mathcal{M}(A^{op})$, and a suitable pairing. ~~[illegible]~~ This choice should be the dual pair ~~[illegible]~~ giving the required morita equivalence. If $B = X \otimes_A Y$ has a local unit, what does this say about X? B has left local unit means $\exists$ net b_α such that

$\lim_\alpha b_\alpha b = b$ for any $b \in B$, then since $X = BX$ one has $b_\alpha \sum b_i x_i \longrightarrow \sum_i b_\alpha b_i x_i \longrightarrow \sum b_i x_i$

$\therefore$ B has local left unit $(b_\alpha) \Rightarrow$ $\underline{X}$ has local left unit ?

Go back to see if ~~[illegible]~~ $\forall a \ \exists a' \ni a(1-a') = 0$ can be carried out effectively. What does this mean for operators if a' is a function of a.

$0 = \lambda(1-\lambda f(\lambda)) \Rightarrow \lambda = 0$ or $\lambda \neq 0$ so $\lambda f(\lambda) = 1$

Back to M. eq. details. The idea ~~[illegible]~~ is to first introduce certain module categories, ~~and~~ show they are equivalent, then later identify these categories with the reduced module categories for A, C, B; $C = \begin{pmatrix} A & Y \\ X & B \end{pmatrix}$.

Case M_n: The ~~combined module~~ C-type of modules are defined to be sequences of v.s. $(V_1, \ldots, V_n, W)$ ~~equipped~~ together with operators $W \xleftarrow{\beta_i} V \xleftarrow{\alpha_i} W$ such that $\sum_i \beta_i \alpha_i = 1_W$, α_i surj, β_i inj.

The B-type of modules are v.s. W equipped with operators $h_1, \ldots, h_n$ s.t. $\sum h_i = 1_W$.

There is a functor from C type to B type which send $(V_i, W, \alpha_i, \beta_i)$ to W and $h_i = \beta_i \alpha_i$. There is a functor going the other way sending (W, h_i) to $(V_i, W, \alpha_i, \beta_i)$ where $W \xleftarrow{\beta_i} V_i \xleftarrow{\alpha_i} W$ is the canonical factorization of h_i into a surjection followed by injection. Clearly the two functors are inverse up to canon. isom.

The graded A-type of module is a sequence of v.s. $V_1, \ldots, V_n$ together with ops $p_{ij} : V_j \longrightarrow V_i$ $\forall i, j$ satis:

$$p_{ik} = \sum_j p_{ij} p_{jk}, \quad (\forall i)\ V_i = \sum_j p_{ij} V_j,$$

$$(\forall j)(\forall v_j \in V_j)\left[(\forall i)\ p_{ij} v_j = 0 \implies v_j = 0\right]$$

There is a functor from C-type modules to graded A-type mod. sending $(V_i, W, \alpha_i, \beta_i)$ to

(V_i, p_{ij}) with $p_{ij} = \alpha_i \beta_j : V_i \leftarrow W \leftarrow V_j$,, p_{ik}

Check well defined $\sum_j p_{ij} p_{jk} = \sum_j \alpha_i \beta_j \alpha_j \beta_k = \alpha_i \beta_k$

Given $v_i \in V_i$ choose $w \ni v_i = \alpha_i w$, then

$v_i = \alpha_i \sum_j \beta_j \alpha_j w \subset \sum_{ij} p_{ij} V_j$ surj of the α_i

Given $v_j \in V_j$ s.t. $0 = p_{ij} v_j$ for all i

Then $0 = \underline{\sum_i \beta_i (\alpha_i \beta_j v_j)} = \beta_j v_j \Rightarrow v_j = 0$ inj of v_j.

There is a functor from graded A-type modules to C-type modules sending (V_i, p_{ij}) as follows. Let $V = \bigoplus_{i=1}^n V_i$, e_i = proj onto V_i, let p be the operator on V given by

$$p(v_j) = \left(\sum_j p_{ij} v_j\right)$$

let $W = pV$. Then $p^2 = p$ so that W is a retract of V, i.e. the linear maps

$$W \xleftarrow{\beta} V \xleftarrow{\alpha} W \qquad \begin{array}{l}\beta = \text{inclusion} \\ \alpha = p \quad \text{surj.}\end{array}$$

~~such that~~ satisfy $\beta\alpha = 1_W$ and $\alpha\beta = p$. Let $\alpha_i = e_i \alpha$, $\beta_i = \beta e_i$.

Check $(V_i, W, \alpha_i, \beta_i)$ is a C-type mod have $\sum_i \beta_i \alpha_i = \sum_i \beta e_i e_i \alpha = \beta\alpha = 1_W$

$$\alpha_i \beta_j = e_i \alpha \beta e_j = e_i p e_j = p_{ij}$$

need to prove α_i surj, β_j inj. by ass.

$V_i = \sum_j p_{ij} V_j = \alpha_i \sum_j \beta_j V_j = \alpha_i \beta \sum_j e_j V_j = \alpha_i \beta V = \alpha_i W$

let $\beta_j v_j = 0$. Then $0 = \alpha_i \beta_j v_j = p_{ij} v_j$ for all i, so $v_j = 0$. $\therefore \beta_j$ inj.

Next to see that

$$\text{C type} \rightleftarrows \text{graded A type}$$

$$(V_i, W, \alpha_i, \beta_i) \longmapsto (V_i, p_{ij}) \quad \longleftarrow$$

are inverses up to canonical isom.

Still something's missing. You also want to have description of C-type modules as (V_i, W, α, β)

where

$$W \xleftarrow{\beta} \bigoplus V_i \xleftarrow{\alpha} W \qquad \beta = (\beta_1, \dots, \beta_n) \quad \alpha = \begin{pmatrix}\alpha_1 \\ \vdots \\ \alpha_n\end{pmatrix}$$

sat $\beta\alpha = 1_W$, α_i surj
β_i inj.

ungraded A type modules are v.s. V tog w. p_{ij} such that $p_{ik} = \sum_j p_{ij} p_{jk}$,

$p_{ij} p_{k\ell} = 0 \quad j \neq k, \qquad V = \sum_{ij} p_{ij} V, \quad \bigcap_{ij} \operatorname{Ker}(p_{ij} \text{ on } V) = 0$

Let A be the universal alg gen. by p_{ij} sat above two relations. Show A is M_n graded

$$\Delta : A \longrightarrow M_n\mathbb{C} \otimes A \qquad \Delta(p_{ij}) = e_{ij} \otimes p_{ij}$$

Then A can be enlarged by adjoining $e_{ii} = e_i$

$$\tilde{A} = A \oplus \bigoplus_1^n \mathbb{C} e_i \qquad \left(\begin{array}{lll} e_i p_{jk} = 0 & i \neq j \\ p_{jk} e_i = 0 & k \neq i \end{array} \right.$$

Next you want idemp rings ~~graded~~ yielding these cats.

C type module $= (V_i, W, \alpha_i, \beta_i)$ $\qquad \sum \beta_i \alpha_i = 1_W$

form ~~[illegible]~~ $\begin{array}{c} V_1 \\ \oplus \\ \vdots \\ \oplus \\ V_n \\ \oplus \\ W \end{array}$

Your problem is to identify your module types with reduced modules ~~[illegible]~~ over certain rings

C-module type $(V_1, \dots, V_n, W, \underleftarrow{\alpha_i, \beta_i})$ $\quad W \xleftarrow{\beta_i} V_i \xleftarrow{\alpha_i} W$

sat. $\sum_i \beta_i \alpha_i = 1_W$, α_i surj, β_i inj.

C $\overbrace{\text{unital}}$ M_{n+1}-graded ring, gen ~~$\alpha_i \in C_{ni}$, $\beta_i \in C$~~

$\beta_i \in C_{n+1,i}$, $\alpha_i \in C_i$, relation $\sum \beta_i \alpha_i = e_{n+1,n+1}$

When is an M_n graded alg $\overset{A}{\wedge}$ unital?

$1 = \sum_{ij} t_{ij}$ $\qquad 1^2 = 1 \iff t_{ik} = \sum_1^n t_{ij} t_{jk}$

~~$\sum_{ij} t_{ij} = 1 \implies t_{kl} = \sum_{kj} t_{kl} t_{ij} = \sum_i t_{ki} t_{ij}$~~

$\sum_{i,j=1} t_{ij} = 1 \implies$ ~~$t_{i} = \sum_{ij} \underbrace{t_{ij} t_{kl}}_{0\ j \neq k} = \sum_{ij} t_{ij} t_{jl}$~~

$\Downarrow$

$t_{kl} = \sum_{ij} t_{kl} t_{ij} = \sum_{\substack{ij \\ l=i}} t_{kl} t_{lj} = \sum_{ij} t_{ki} t_{ij}$

$t_{kl} t_{ij} = t_{ki} t_{ij} \delta_{il}$

$t_{kl} = \sum_{ij} t_{ki} t_{ij} \delta_{il} = \sum_j t_{kl} t_{lj}$

A M_n graded. $A = \bigoplus_{ij} A_{ij}$

~~Let~~ Suppose A unital, let $1 = \sum_{ij} t_{ij}$ $\quad t_{ij} \in A_{ij}$

$$t_{kl} = \sum_{ij} t_{kl} t_{ij} = \sum_j t_{kl} t_{lj}$$

$$t_{kl}\Big(1 - \sum_j t_{lj}\Big) = 0 \qquad \text{Put } e_l = \sum_j t_{lj}$$

$$\boxed{t_{kl} = t_{kl} e_l}$$

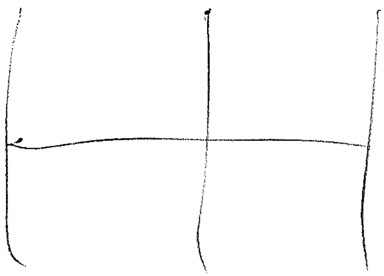
$$t_{kl} = \sum_{ij} t_{ij} t_{kl} = \sum_i t_{ik} t_{kl} \qquad \text{put } e'_k = \sum_i t_{ik}$$

$$\boxed{t_{kl} = e'_k t_{kl}}$$

$$t_{kl} = \sum_{ij} t_{kl} t_{ij} = \sum_{ij} \delta_{li} t_{kl} t_{ij}$$

Set $\cancel{e_i = \sum_k t_{ij}}$ $\quad e_i = \sum_k t_{ik}$

$$\cancel{e_i e_j = \sum_{k,l} t_{ik} t_{jl}}$$



$$\sum_{ij} t_{ij} = 1 \qquad \text{set} \qquad e'_i = \sum_j t_{ij}$$

$$e''_\ell = \sum_k t_{k\ell}$$

$$e'_i e''_\ell = \sum_{j,k} t_{ij} t_{k\ell}$$

$$e'_i t_{ab} e''_\ell = \sum_{j,k} t_{ij} t_{ab} t_{k\ell} = t_{ia} t_{ab} t_{b\ell}$$

$$\sum_{ij} t_{ij} = 1 \quad \text{in } A$$

$$t_{ab} = \sum_{ij} t_{ab} t_{ij} = \sum_j t_{ab} t_{bj}$$

$$t_{ab} = t_{ab} \sum_j t_{bj}$$

$$t_{ab} \underbrace{\sum_j t_{ij}}_{e'_i} = \begin{cases} 0 & b \neq i \\ t_{ab} & b = i \end{cases}$$

$$\therefore (e'_i)^2 = e'_i$$

Check this. $\sum_{i,j} t_{ij} = 1 \implies t_{ab} = \sum_{ij} \overbrace{t_{ab} t_{ij}}^{\delta_{bi}\, t_{ab} t_{bj}}$

$$= \sum_j t_{ab} t_{bj} = t_{ab} e'_b$$

This should be easy. A is unital and M_n-graded. Let $1 = \sum_{ij} t_{ij}$ $\quad t_{ij} \in A_{ij}$

Then $\sum_{ij} t_{ij} t_{k\ell} = \sum_{ij} t_{ij} \delta_{jk} t_{k\ell} = \sum_i t_{ik} t_{k\ell}$ $\quad$ $t_{k\ell} =$

$\therefore \sum_i t_{ik} t$

$$1 = \sum_{i,j} t_{ij} \qquad t_{k\ell} = \sum_{ij} t_{k\ell} t_{ij} = \sum_{ij} \delta_{\ell i} t_{k\ell} t_{ij}$$

$$= \sum_j t_{k\ell} t_{\ell j}$$

$$t_{ab} t_{cd} = \sum_{ij} t_{ab} t_{ij} t_{cd} = t_{ab} t_{bc} t_{cd}$$

$$t_{ab} = \sum_{ij} t_{ab} t_{ij} = \sum_j t_{ab} t_{bj} = t_{ab} \Big(\sum_j t_{bj} \Big)$$

$$t_{ab} = \sum_{ij} t_{ij} t_{ab} = \sum_i t_{ia} t_{ab} = \Big(\sum_i t_{ia} \Big) t_{ab}$$

$$\rho_b = \sum_j t_{bj} \qquad \lambda_a = \sum_i t_{ia}$$

$$\lambda_a \rho_b = \sum_{ij} t_{ia} t_{bj} = \begin{cases} 0 & a \neq b \\ & \end{cases}$$

$$\rho_b \rho_c = \sum_j t_{bj} \sum_k t_{ck} = \sum_k t_{bc} t_{ck} = t_{bc} \rho_c$$

$$\begin{pmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{pmatrix} \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix} = \begin{pmatrix} t_{11}u_{11} + t_{12}u_{21} & \\ t_{21}u_{11} + t_{22}u_{21} & \end{pmatrix}$$

$$t_{11}u_{11} + t_{12}u_{21} = u_{11} \implies t_{11} = 1,\ t_{12} = 0$$

$$t_{21}u_{11} + t_{22}u_{22} = u_{22}$$

$$\sum_{ij} t_{ij} \sum_{kl} u_{kl} = \sum_{ijl} t_{ij} u_{jl} = \sum_{kl} u_{kl}$$

take $u_{jl} = 0$ for $l \neq 1$. Then

$$\sum_{ij} t_{ij} u_{j1} = \sum_{k} u_{k1}$$

Given $A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$ a M_2 graded ring

such that $\begin{pmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{pmatrix}$ is an identity element

means ~~[illegible]~~ it's a left identity

$$\begin{pmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

There has to be a mult. arg.

$$A \xrightarrow{\Delta} M_2\mathbb{C} \otimes A$$

$$a_{ij} \longmapsto e_{ij} \otimes a_{ij}$$

you need to start with the general viewpoint, and check it. If C small cat, ~~then~~ let

$\Lambda =$ arrow ring $= \mathbb{C}[Ar]$, $Ar = \coprod_{X_0, X_1} Ar(X_0 \leftarrow X_1)$

Any $f \in Ar$ gives

C small cat. Ob, Ar, target, source, comp., unit

$$\begin{tikzcd} Ob \arrow[r, shift left] \arrow[r, shift right] & Ar \arrow[l, bend left] & Ar \times_{Ob} Ar \arrow[l] \arrow[l, shift left] \arrow[l, shift right] \end{tikzcd}$$

---

stop wasting time. $\Lambda = M_n \mathbb{C}$

A is a M_n graded alg $\qquad A = \bigoplus_{ij \in M_n} A_{ij}$

$\Delta : a_{ij} \longmapsto e_{ij} \otimes a_{ij}$ ring hom. $A \xrightarrow{\Delta} \Lambda \otimes A \subset \Lambda \otimes \tilde{A}$

let $e_{ii} \otimes 1 \in \Lambda \otimes \tilde{A}$

claim left & right mult by $e_{ii} \otimes 1$ preserves ΔA, defining a multiplier e_i on A

$$\Delta(e_i a) = (e_{ii} \otimes 1) \Delta$$

$$e_i a = \eta (e_{ii} \otimes 1) \Delta a$$

$$e_i a_{jk} = \eta (e_{ii} \otimes 1)(e_{jk} \otimes a_{jk})$$

$$= \eta \begin{cases} 0 & i \neq j \\ e_{jk} \otimes a_{jk} & i = j \end{cases} = \begin{cases} 0 & i \neq j \\ a_{jk} & i = j \end{cases}$$

$$\sum_i e_i a_{jk} = e_j a_{jk} = a_{jk}$$

A is M_2 graded and unital.

λ, ρ left + right ident. $a\rho = (\lambda a)\rho = \lambda(a\rho) = \lambda a$?

A is a ring, $\lambda \in A$ is a left identity: $\lambda a = a \ \forall a$

$\rho \in A$ is a right identity $a\rho = a \ \forall a$. Then

$\lambda = \lambda\rho = \rho$. You have this $1 = \sum_{ij} t_{ij} \in A$

which is an identity, and you've constructed multipliers e_i on A ~~such that~~ $\cancel{A_{ij}}$ which are annihilating idemps. ~~Sum~~ 1 +

$e_i A e_j = A_{ij}$. In particular $t_{ij} = e_i 1 e_j = \delta_{ij} e_j$

$$1 = \sum t_{ij} \in A \qquad e_i = e_i 1 = \sum_j t_{ij}$$

$$e_j = 1 e_j = \sum_k t_{jk}$$

$$e_i = \sum_j t_{ij} \qquad e_i' = \sum_j t_{ji}$$

$$e_b e_a' = \sum_{jk} t_{bj} t_{ka} = \sum_j t_{bj} t_{ja} = t_{ba}$$

$$\sum e_i = \sum e_i' = 1$$

$$\sum_a e_a' \sum_b e_b = \sum_{ab} e_a' e_b$$

$$e_a' e_b = \sum_j t_{ja} \sum_k t_{bk} = 0 \qquad a \neq b$$

$$\sum_{ij} t_{ij} = 1 \qquad e_a = \sum_j t_{aj} \qquad e_b' = \sum_i t_{ib}$$

$$e_b' e_a = \begin{cases} 0 & b \neq a \\ \sum_{ij} t_{ib} t_{bj} & b = a \end{cases}$$

$$\sum_b e_b' e_b = \sum_{ij} t_{ij} = 1$$

$$e_a e_b' = \sum_j t_{aj} t_{jb} = t_{ab}$$

$$\sum t_{ij} = 1, \quad t_{ab} = \sum_{ij} t_{ij} t_{ab} = \underbrace{\left(\sum_i t_{ia}\right)}_{e_a'} t_{ab}$$

$$t_{ab} = \sum_{ij} t_{ab} t_{ij} = \sum_j t_{ab} t_{bj} = t_{ab} \underbrace{\left(\sum_j t_{bj}\right)}_{e_b}$$

Def $e_a' = \sum_i t_{ia}$. Then

$$e_a' t_{jk} = \sum_i t_{ia} t_{jk} = \delta_{aj} \overbrace{\sum_i t_{ia}}^{e_a'} t_{jk}$$

$$e_a' t_{jk} = \sum_i t_{ia} t_{jk} = \begin{cases} 0 & a \neq j \\ e_j' t_{jk} & \text{if } a = j \end{cases}$$

$$\sum_{ij} t_{ij} = 1 \qquad e'_a = \sum_i t_{ia}\,, \quad e_a = \sum_i t_{ai}$$

$$t_{ab} = \sum_{ij} t_{ab} t_{ij} = \sum_j t_{ab} t_{bj} = t_{ab} e_b$$

$$t_{ab} = \sum_{ij} t_{ij} t_{ab} = \sum_i t_{ia} t_{ab} = e'_a t_{ab}$$

$$t_{ij} e_b = \sum t_{ij} \sum_k t_{bk} = 0 \quad \text{for } j \neq b.$$

$$e'_a t_{ij} = \sum_k t_{ka} t_{ij} = 0 \quad \text{for } a \neq i$$

~~Better would have been~~

$$t_{ij} e_b = \begin{cases} 0 & j \neq b \\ t_{ij} & j = b \end{cases} \qquad e'_a t_{ij} = \begin{cases} 0 & a \neq i \\ t_{ij} & a = i \end{cases}$$

Start again $\quad A = \bigoplus_{ij} A_{ij} \quad$ M_n-graded alg

unital $\quad 1 = \sum_{ij} t_{ij}\,, \quad t_{ij} \in A_{ij}$.

$$t_{ab} = \sum_{ij} t_{ab} t_{ij} = \sum_j t_{ab} t_{bj} = t_{ab} \overbrace{\sum_j t_{bj}}^{e_b}$$

$$t_{ab} = \sum_{ij} t_{ij} t_{ab} = \overbrace{\sum_i t_{ia}}^{e'_a} t_{ab}$$

$$e'_a t_{ij} = \begin{cases} 0 & a \neq i \\ t_{ij} & a = i \end{cases}$$

$$t_{ij} e_b = \begin{cases} 0 & j \neq b \\ t_{ij} & j = b \end{cases}$$

$$e'_a t_{ij} = \begin{cases} 0 & a \neq i \\ t_{ij} & a = i \end{cases}$$

therefore $e'_a \cdot$ is the projection $= 1$ on the a-th row $= \sum_j \mathbb{C} t_{aj}$ and 0 on other rows.

It follows that

$e'_a e'_b = 0 \quad a \neq b \quad$ and $\quad \sum_a e'_a = 1$

Similarly $\quad t_{ij} e_a = \begin{cases} 0 & j \neq a \\ t_{ij} & j = a \end{cases}$ so right mult $\cdot e_a$ is $= 1$ on the a-th column $\bigoplus_i \mathbb{C}[t_{ia}]$ and $= 0$ on other columns.

$\therefore \; e_a e_b = 0$ for $a \neq b$ and $\sum e_a = 1$

(Cleaner ~~is that~~ to say that $\sum_a e_a = 1$ and that $e_a e_b$ ~~...~~ $= 0$ for $a \neq b$.)

You still need to show $t_{ij} = 0 \quad i \neq j$

$t_{ik} = \sum_j t_{ij} t_{jk}$.

$$e'_a e_b = e'_a \left(\sum t_{ij} \right) e_b = t_{ab}$$

$$e_b e'_a = \sum_{ij} t_{bj} t_{ia} = \sum_j t_{bj} t_{ja} = t_{ba}$$

A M_n-graded and unital ~~...~~

$1 = \sum t_{ij}$

A is M_n-graded: $A = \bigoplus_{ij} A_{ij}$ where

$A_{ij}A_{k\ell} = 0$ $j \neq k$, $A_{ij}A_{j\ell} \subset A_{i\ell}$

Assume A unital, let $\underset{t}{1} = \sum_{ij} t_{ij}$ with $t_{ij} \in A_{ij}$.

~~Define~~ $A^n \longleftarrow \quad \xleftarrow{\beta}$

This looks good. You have $A = \bigoplus_{ij} A_{ij}$ an M_n-graded ring which is unital whence $1 = \sum_{ij} t_{ij}$ with $t_{ij} \in A_{ij}$. Now use the matrix $(t_{ij}) \in M_n(A)$ to define an A^{op}-module map

$$A^n \xleftarrow{\ t\ } A^n$$

Repeat $A = \bigoplus A_{ij}$ is an M_n-graded alg which happens to be unital, say $\sum_{ij} t_{ij}$ is the identity. You want to prove that $t_{ij} = 0$ for $i \neq j$. So use $\Delta : A \longrightarrow \Lambda \otimes A$ alg hom. $\ni$ $\Delta(a_{ij}) = e_{ij} \otimes a_{ij}$. Now ~~the question is~~ ~~whether~~ $\Lambda \otimes A$ is unital, so the question to ask is whether Δ preserves ident.

$$\Delta(1) = \sum_{ij} e_{ij} \otimes t_{ij}$$

$$\Delta(1)\Delta(1) = \sum_{ijk\ell} e_{ij}e_{k\ell} \otimes t_{ij}t_{k\ell} = \sum_{i\ell} e_{i\ell} \otimes \sum_j t_{ij}t_{j\ell}$$

Repeat: A ~~is~~ M_n-graded, $A = \bigoplus_{ij} A_{ij}$:

$$A_{ij} A_{kl} = 0 \quad j \neq k \ , \quad A_{ij} A_{jk} \subset A_{ik} .$$

Assume A is unital with identity $1 = \sum_{ij} t_{ij}$ with $t_{ij} \in A_{ij}$.

A M_n-graded alg means $\exists\ \Delta : A \longrightarrow \Lambda \otimes A$ alg homom. given by $\Delta(a_{ij}) = e_{ij} \otimes a_{ij}$. Better to say A is a M_n-graded alg means that it is M_n graded where you have $\Delta : A \longrightarrow \Lambda \otimes A$, $\Delta(a_{ij}) = e_{ij} \otimes a_{ij}$, is a comodule map, then require Δ to be alg map.

Repeat: Let A be an M_n-graded alg. This means A is an M_n-graded v.s. $A = \bigoplus_{ij} A_{ij}$ such that the map $\Delta : A \longrightarrow M_n\mathbb{C} \otimes A$, where

$\Delta(a_{ij}) = e_{ij} \otimes a_{ij}$ for $a_{ij} \in A_{ij}$, is an alg. morph.,

equival. $\left| \begin{array}{ll} A_{ij} A_{kl} = 0 & \text{for } j \neq k \\ A_{ij} A_{jk} \subset A_{ik} & \end{array} \right.$

Now suppose A is unital, let the identity be $1 = \sum t_{ij}$. To show $t_{ij} = 0$ for $i \neq j$

~~Then~~ $\Delta(1) = \Delta\left(\sum_{ij} t_{ij}\right) = \sum_{ij} e_{ij} \otimes t_{ij}$

$\cancel{\sum e_{kk} \otimes t_{ij}}$

$$\sum_{ij} t_{ij} t_{ab} = \sum_{i} t_{ia} t_{ab} = \boxed{e'_a t_{ab} = t_{ab}}$$

$$\sum_{ij} t_{ab} t_{ij} = \sum_{j} t_{ab} t_{bj} = \boxed{t_{ab} e_b = t_{ab}}$$

$$\boxed{e'_i t_{ab} = \begin{cases} 0 & i \neq a \\ t_{ab} & i = a \end{cases}} \qquad \boxed{t_{ab} e_j = \begin{cases} 0 & b \neq j \\ t_{ab} & b = j \end{cases}}$$

$$e'_a = \sum_i t_{ia} \qquad e_b = \sum_j t_{bj} \qquad e'_a e_b = \begin{cases} 0 & a \neq b \\ \end{cases}$$

$$\cancel{\sum_a e'_a e_a = \sum t_{ia} t_{aj}}$$

$$\sum_a e'_a e_a = \sum_{i,j,a} t_{ia} t_{aj} = \sum_{ij} t_{ij} = 1$$

$$\boxed{e'_a e_b = \delta_{ab}}$$

$$\boxed{e_b e'_a = t_{ba}}$$

$$e_b e'_a = \sum_{ji} t_{bj} t_{ia} = \sum_i t_{bi} t_{ia} = t_{ba}$$

$$\boxed{e'_a e'_a = e'_a} \qquad \boxed{e_b e_b = e_b}$$

$$\boxed{\sum e'_a = 1} \qquad \boxed{\sum e_b = 1}$$

$$\cancel{e'_a e_b e'_c = \delta_{ab} e'_c} \qquad \cancel{e'_a t_{ac} = t_{ac}}$$

$$e'_a e_b e'_c = e'_a t_{bc} = \begin{cases} 0 & a \neq b \\ t_{bc} & a = b \end{cases}$$

$$\|$$

$$\delta_{ab} e'_c \qquad e'_c = t_{ac}$$

$$\Delta : A \longrightarrow M_n\mathbb{C}\otimes A \overset{\Delta\otimes 1}{\underset{1\otimes\Delta}{\rightrightarrows}} M_n\mathbb{C}\otimes M_n\mathbb{C}\otimes A$$

$$\Delta(a_{ij}) = e_{ij}\otimes a_{ij}$$

~~Suppose A unital i.e. 1 = ...~~

The method which should work is the "normalizer" of $\Delta(A)$. Consider $e_{kk}\otimes 1 \in M_n\mathbb{C}\otimes A$. Then

$$\Delta(a_{ij})(e_{kk}\otimes 1) = (e_{ij}\otimes a_{ij})(e_{kk}\otimes 1)$$

$$= e_{ij}e_{kk}\otimes a_{ij} = \begin{cases} 0 & j\neq k \\ e_{ij}\otimes a_{ij} & j=k \end{cases}$$

$$(e_{kk}\otimes 1)\Delta(a_{ij}) = (e_{kk}\otimes 1)(e_{ij}\otimes a_{ij})$$

$$= e_{kk}e_{ij}\otimes a_{ij} = \begin{cases} 0 & k\neq i \\ e_{ij}\otimes a_{ij} & k=i \end{cases}$$

$$\Delta(a_{ij})(e_{kk}\otimes 1) = \begin{cases} 0 & j\neq k \\ \Delta(a_{ij}) & j=k \end{cases}$$

Actually you want to identify $\Delta(1) = \sum_{ij} e_{ij}\otimes t_{ij}$ with $\sum e_{kk}\otimes 1$ inside $\Lambda\otimes A$.

A unital $1_A = \sum_{ij} t_{ij}$

$$\begin{pmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{pmatrix}\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} t_{11}a_{11}+t_{12}a_{21} & \cdot \\ t_{21}a_{11}+t_{22}a_{22} & \end{pmatrix}$$

$$= \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

$\Delta : A \longrightarrow \Lambda \otimes A$ $\qquad \Delta(a_{ij}) = e_{ij} \otimes a_{ij}$ for all $a_{ij} \in A_{ij}$

~~Δ alg map~~

$$\cancel{\begin{array}{ccc} a_{ij}\,a_{kl} & \xmapsto{\Delta} & (e_{ij}\otimes a_{ij})(e_{kl}\otimes a_{kl}) \\ \downarrow \text{prod} & & \downarrow \text{prod} \\ \delta_{jk}\,a_{ij}\,a_{kl} & & \delta_{jk}\,e_{il} \otimes a_{ij}a_{kl} \end{array}}$$

$$\begin{array}{ccc} a_{ij},\ a_{kl} & \xmapsto{\ \Delta\ } & e_{ij}\otimes a_{ij},\ e_{kl}\otimes a_{kl} \\ \downarrow \text{prod} & & \downarrow \text{prod} \\ & & e_{ij}e_{kl} \otimes a_{ij}a_{kl} \\ & & \shortparallel \\ \delta_{jk}\,a_{ij}a_{kl} & \xmapsto{\ \Delta\ } & \underbrace{\delta_{jk}\,e_{ij}e_{kl}}_{\delta_{jk}\,e_{il}} \otimes a_{ij}a_{kl} \end{array}$$

Look: Assume Δ alg map, then

$$\Delta(a_{ij}a_{kl}) = \delta_{jk}\,e_{il} \otimes a_{ij}a_{kl}$$

$j=k \implies a_{ij}a_{kl} \in A_{il}$

$j=k \implies \Delta(a_{ij}a_{kl}) = 0 \implies a_{ij}a_{kl} = 0.$

Some screwy problem to resolve.

$$\Delta : A \longrightarrow \Lambda \otimes A \qquad\qquad \underline{A \oplus \begin{pmatrix} \mathbb{C} & 0 \\ 0 & \mathbb{C} \end{pmatrix}}$$

First point $\quad A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{2} \end{pmatrix} \subset \begin{pmatrix} \tilde{A}_{11} & A_{12} \\ A_{21} & \tilde{A}_{22} \end{pmatrix}$

$\exists$ this embedding as an ideal in a unital ring. Suppose A is unital with $1 = \begin{pmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{pmatrix}$

$$0 \longrightarrow A \longrightarrow A \oplus \begin{pmatrix} \mathbb{C} & 0 \\ 0 & \mathbb{C} \end{pmatrix} \longrightarrow \begin{pmatrix} \mathbb{C} & 0 \\ 0 & \mathbb{C} \end{pmatrix} \longrightarrow 0$$

$$0 \longrightarrow A \longrightarrow R \longrightarrow R/A \longrightarrow 0$$

assume A unital with identity element e

Then $e \in A$ and $ea = a = ae \quad \forall a \in A$.

Is e central in R? Let $r \in R$, then since A ~~$re, er \in A$~~ is an ideal in R one has

$re, er \in A$ so $re = ere = er$

$$\begin{pmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{pmatrix} = \begin{pmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{pmatrix}\begin{pmatrix} t_{11} & t_{12} \\ 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} t_{11}^2 & t_{11}t_{12} \\ t_{21}t_{11} & t_{21}t_{22} \end{pmatrix}$$

$$\begin{pmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{pmatrix} = \begin{pmatrix} t_{11} & t_{12} \\ 0 & 0 \end{pmatrix}$$

$$\| $$

$$\begin{pmatrix} t_{11} & 0 \\ t_{21} & 0 \end{pmatrix} \qquad \therefore \quad t_{12} = t_{21} = 0$$

$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$ is assumed unital, call ident $e = \begin{pmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{pmatrix}$

A is an ideal in $\begin{pmatrix} \tilde{A}_{11} & A_{12} \\ A_{21} & \tilde{A}_{22} \end{pmatrix}$

with quotient $\begin{pmatrix} \mathbb{C} & 0 \\ 0 & \mathbb{C} \end{pmatrix}$.

$$e\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = e\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}e = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}e$$

$$\| $$

$$\begin{pmatrix} t_{11} & 0 \\ t_{21} & 0 \end{pmatrix} \qquad\qquad \begin{pmatrix} t_{11} & t_{12} \\ 0 & 0 \end{pmatrix}$$

⌈ Return to your program, ~~that~~ which is to understand Cuntz's Morita equivalence for M_n

B-type module: $(W, h_1, \dots, h_n)$ $\quad \sum_{i=1}^{n} h_i = 1$

C-type module $(W, V_1, \dots, V_n, \alpha_i, \beta_i)$ $\quad W \xleftarrow{\beta_i} V_i \xleftarrow{\alpha_i} W$

$\sum \beta_i \alpha_i = 1_W$, α_i surj, β_i inj

A-type module $(V_1, \dots, V_n, p_{ij})$ $\quad V_i \xleftarrow{p_{ij}} V_j$

$p_{ik} = \sum_j p_{ij} p_{jk}$, $\quad V_i = \sum_j p_{ij} V_j$, $\quad p_{ij} v_j = 0 \ \forall i \Rightarrow v_j = 0$

from C to B Given ($W, V_1, \dots, V_n, \alpha_i\, \beta_i$) 674

take W with $h_i = \alpha_i \beta_i$ $V_i \xleftarrow{\alpha_i} W \xleftarrow{\beta_j} V_j$

from C to A, take $V_1, \dots, V_n$ with $p_{ij} = \alpha_i \beta_j$

idemp. ✓ $\sum p_{ij} V_j = \alpha_i \sum \beta_j V_j = \alpha_i W = V_i$ α_i surj

$w = \sum_j \beta_j \alpha_j w \subset \sum_j \beta_j V_j$ sum $p_{ij} v_j = \alpha_i(\beta_j v_j) = 0 \ \forall i$

$\Rightarrow \beta_j v_j = 0 \Rightarrow v_j = 0$ as β_j inj.

---

the ring A: gens p_{ij}, rels $\left| \begin{array}{ll} p_{ij} p_{k\ell} = 0 & j \neq k \\ p_{ik} = \sum_j p_{ij} p_{jk} & \end{array} \right.$

A is naturally M_n graded, define $\Delta : A \longrightarrow \Lambda \otimes A$ by $\Delta(p_{ij}) = e_{ij} \otimes p_{ij}$. Do the $\Delta(p_{ij})$ satisfy the relations defining A? Better, ask whether the elements $\tilde{p}_{ij} = e_{ij} \otimes p_{ij} \in \Lambda \otimes A$ satisfy the relations defining A. $\tilde{p}_{ij} \tilde{p}_{k\ell} = \underbrace{e_{ij} e_{k\ell}}_{0 \text{ if } j \neq k} \otimes p_{ij} p_{k\ell}$

if $j = k$ $\sum_j \tilde{p}_{ij} \tilde{p}_{j\ell} = e_{i\ell} \otimes \sum_j p_{ij} p_{j\ell} = e_{i\ell} \otimes p_{i\ell} = \tilde{p}_{i\ell}$.

Define A by gens p_{ij}
rels $p_{ik} = \sum_j p_{ij} p_{jk}$

Next form $\Lambda \otimes A$, $\tilde{p}_{ij} = e_{ij} \otimes p_{ij}$. These elts satisfies the relns $\Rightarrow \exists!\ \Delta : A \longrightarrow \Lambda \otimes A$ $\Delta(p_{ij}) = \tilde{p}_{ij}$.

compose with counit;

$$\begin{array}{ccc} A & \longrightarrow & \Lambda \otimes A \\ & \searrow & \downarrow \eta \otimes 1 \\ & & A \end{array}$$

Then $\tilde{p}_{ij} \tilde{p}_{k\ell} = 0 \Longrightarrow p_{ij} p_{k\ell} = 0$ $j \neq k$.

Because of this M_n grading you can adjoin an identity to A

Go over difficulties before. V a reduced A module. Actually this is probably unnecessary for the Morita equiv.

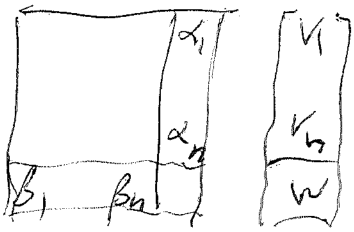
C type mod. $(V_1, \ldots, V_n, W, \; W \xleftarrow{\beta_i} V_i \xleftarrow{\alpha_i} W)$ $\quad \sum \beta_i \alpha_i = 1_W$

define C to be the ring gen by elements β_i, α_i sat $\sum \beta_i \alpha_i = 1$. Is this ?

C generators α_i, β_i subject to

$$\left(\sum \beta_i \alpha_i - 1\right) \beta_j = 0$$

$$\alpha_j \left(\sum \beta_i \alpha_i - 1\right) = 0$$



Show C is M_{n+1} graded with

$$\tilde{\beta}_j = e_{n+1, j} \otimes \beta_j$$

$$\tilde{\alpha}_i = e_{i, n+1} \otimes \alpha_i$$

$$\sum_i \tilde{\beta}_i \tilde{\alpha}_i = e_{n+1, n+1} \otimes \sum_i \beta_i \alpha_i \quad 1$$

this resembles your construction of the e_i

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Idea yesterday where the object identity e_{n+1} ~~[illegible]~~ to be adjoined to the M_{n+1}-graded algebra C appears naturally as $e_{n+1,n+1} \otimes 1$ in $\Lambda \otimes \tilde{C}$.

$$\begin{bmatrix} & & & \alpha_1 \\ & & & \vdots \\ & & & \alpha_n \\ \beta_1 & \cdots & \beta_n & \end{bmatrix} \qquad \begin{bmatrix} V_1 \\ \vdots \\ V_n \\ W \end{bmatrix}$$

C-type module
$(V_1, \ldots, V_n, W, \; W \xleftarrow{\beta_i} V_i \xleftarrow{\alpha_i} W)$

C is defined by generators α_i, β_i and rels $\sum \beta_i \alpha_i = 1_W$. For this to make sense you need 1_W. Method: $\left(\sum \beta_i \alpha_i - 1\right)\beta_j = 0 \quad \forall j$

need finite supp condition of $\alpha_i \beta_j$ $\qquad \alpha_j\left(\sum \beta_i \alpha_i - 1\right) = 0 \quad \forall j$

which allows for $n = \infty$. So you get a ring C which is idempotent. Check it's M_{n+1}-graded

$$\tilde{\alpha}_i = e_{i,n+1} \otimes \alpha_i \qquad \text{in } M_{n+1}\mathbb{C} \otimes C$$
$$\tilde{\beta}_j = e_{n+1,j} \otimes \beta_j$$

$$\sum_i \tilde{\beta}_i \tilde{\alpha}_i = e_{n+1,n+1} \otimes \sum_i \beta_i \alpha_i$$

$$\tilde{\alpha}_j \sum_i \tilde{\beta}_i \tilde{\alpha}_i = e_{j,n+1} \otimes \sum_i \alpha_j \beta_i \alpha_i = e_{j,n+1} \otimes \alpha_j = \tilde{\alpha}_j \qquad \text{etc.}$$

so you should get $\Delta : C \longrightarrow M_{n+1}\mathbb{C} \otimes C$, so C is M_{n+1} graded. This implies $\beta_j \alpha_k = 0 \quad j \neq k$

Now look at adjoining object identities

$B = C_{n+1,n+1}$ ~~spanned by~~ is gen. by $\beta_j \alpha_k$ in fact just $\beta_i \alpha_i = h_i$ are needed + ~~they~~ satisf $h_j(\sum h_i - 1) = (\sum h_i - 1) h_j = 0$
$\therefore$ B is unital.

Something new. Go back to a principal Γ-bundle $E \xrightarrow{\pi} B$, a bundle of Γ-torsors parametrized by $b \in B$ linearize to a bundle of $\mathbb{C}\Gamma^{op}$-lines, which you embed as a retract of trivial bundle with fibre a free $\mathbb{C}\Gamma^{op}$-module over B.

Some things to clarify: sections of the $\mathbb{C}\Gamma$ line bundle should be cont. functions on E with compact support, $C_c(E)$. Γ acts on this ~~on~~ functions

Maybe the comm ring structure is ~~causing~~ confusing the situation. Let's try to find what's essential. You have a Γ^{op}module, in fact many Γ^{op}-modules, given as spaces of sections, and locally the bundle is trivial, resulting in a ~~[illegible]~~ Γ grading compatible with Γ action.

~~[illegible]~~

Basic construction ~~is to~~ goes from a principal Γ bundle $E \to B$, B compact (nice), to a retract of a trivial bundle over B with fibre a fin. generated free Γ^{op}-module

Actually it seems that the groupoid you want to study is $M_n \times \Gamma$, M_n providing the partition of unity and Γ the autom

~~Quiet~~ Review Γ group case, C-type module is a $(W, V, W \xleftarrow{\beta_1} V_1 \xleftarrow{\alpha_1} W)$, W a Γ module V a v.s. st $\sum_{s\in\Gamma} s\beta_1\alpha_1 s^{-1} = 1_W$, α_1 surj, β_1 inj.

$$\alpha_1 s\beta_1 = 0 \qquad s \notin \Phi.$$

$$\left[\begin{array}{c|c} & y_s \\ \hline x_s & \end{array}\right] \begin{bmatrix} V \\ \tilde{W} \end{bmatrix}$$

Define C: gens. x_s, y_s $s \in \Gamma$

rels $y_s x_t$ $(= \alpha_1 s^{-1} t \beta_1)$ depends only on $s^{-1}t$

$y_s x_t = 0$ $\qquad s^{-1}t \notin \Phi$

$$\left(\sum x_s y_s - 1\right) x_t = 0, \quad y_s\left(\sum x_t y_t - 1\right) = 0$$

You want to define Γ action on C and a Γ grading. More, [redacted] to yield W a Γ-module, V a vector space you expect the Mult ring of C to to contain $\left(\begin{array}{c|c} \mathbb{C} & 0 \\ \hline 0 & \mathbb{C}\Gamma \end{array}\right)$. ~~Thesis~~ You have a way to do this using the local identity $\sum_s x_s y_s$

~~Hmm~~ The idea is that Mult(C) = Mult alg of the dual pair X, Y over A, which is gen. by $y_s x_t$. ~~...~~ Point is that $\sum x_s y_s$ is a local id in B, and a local left identity on X, local right identity on Y.

$$X = \sum x_s A \qquad x_s = s\alpha_1$$

$$u \sum_s x_s y_s = \sum x_{us} y_s$$

Define $u(\xi) = \sum x_{us} y_s \xi$ $\qquad \xi = x_t a$

$$u(x_t a) = \sum_s x_{us} y_s x_t a = \sum_s x_{us} y_{us} x_{ut} a = x_{ut} a$$

You have Γ grading $\quad x_t$ degree t

y_s —— s^{-1}

$$\tilde{x}_t = t \otimes x_t \qquad \tilde{y}_s = s^{-1} \otimes y_s$$

$$\tilde{y}_s \tilde{x}_t = s^{-1}t \otimes y_s x_t \quad \text{depends only on } s^{-1}t.$$

$$= 0 \quad \text{for} \quad s^{-1}t \notin \Phi$$

$$\tilde{x}_s \tilde{y}_s = 1 \otimes x_s y_s \qquad \tilde{x}_s \tilde{y}_s \tilde{x}_t = t \otimes x_s y_s x_t$$

$$\sum_s \tilde{x}_s \tilde{y}_s \tilde{x}_t = t \otimes \sum_s x_s y_s x_t$$

$$= t \otimes x_t = \tilde{x}_t$$

So C is Γ graded, also B, B is generated by $x_t y_s$ of degree ts^{-1} $\quad t\alpha_1\beta_1 s^{-1} = t\alpha_1\beta_1 t^{-1}_{ts^{-1}}$

$$x_t y_s = ts^{-1}(x_s y_s) = x_t y_t \, ts^{-1}$$

So it seems that B is the crossproduct of Γ acting on the "simplex" gen. by $x_s y_s$ subject to $x_s y_s x_t y_t = 0 \quad s^{-1}t \notin \Phi$

$$\left(\sum x_s y_s\right) x_t y_t = x_t y_t \quad \text{etc.}$$

At some point you should look at a finite covering, $B = U_1 \cup U_2$

So where to start?

Take a principal bundle for a group Γ, $E \longrightarrow B$ and assume $B = U_1 \cup U_2$ with E trivial over U_i. Remember that you are interested in Serre's thm.

Start again: $\Gamma \longrightarrow E \longrightarrow B$ principal Γ bundle, form vector bundle $E \times_\Gamma \mathbb{C}\Gamma$ look at module of continuous sections, you know this is the space of continuous $\mathbb{C}$-valued functions on E having compact support: $C_c(E)$. ~~Now you have~~ This is a comm. algebra, but the algebra structure is mysterious, it's related to the Γ-grading which arises from a local section of the principal bundle.

What do you want? You want to embed $C_c(E)$ as a retract of $C(B) \otimes \mathbb{C}\Gamma$, the space of cont. sections w comp. supp of $B \times \Gamma$. Move Γ to left.

$$W \xleftarrow{\ \beta\ } \mathbb{C}\Gamma \otimes C(B) \xhookleftarrow{\ \alpha\ } W$$

What do you need to get α and β?

$W = C_c(E)$. $\quad B = B_1 \cup B_2 \quad E_i = \pi^{-1} B_i \simeq B_i \times \Gamma$

$$W \longleftarrow \mathbb{C}\Gamma \otimes \begin{pmatrix} C(B_1) \\ C(B_2) \end{pmatrix} \longleftarrow W$$

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$$W \xleftarrow{\beta} \begin{pmatrix} C_c(E_1) \\ C_c(E_2) \end{pmatrix} \xleftarrow{\alpha} W$$

$$\wr\!\mathsf{S}$$

$$\mathbb{C}\Gamma \otimes \begin{pmatrix} C(B_1) \\ C(B_2) \end{pmatrix}$$

So far this makes sense for $B = B_1 \cup \cdots \cup B_n$

$B = B_1 \cup \cdots \cup B_n$ over each B_i the principal bundle is trivial. The covering + partition of unity express $C_c(E)$ as a retract of $\bigoplus_{i=1}^{n} C_c(E_i)$.

What is your aim? Understand the M.eq. in the case of the groupoid $M_n \times \Gamma$, having n objects $1, 2, \ldots, n$

Suppose $\mathcal{G}$ is a connected groupoid, choose a basepoint and suppose isot. gp is a group Γ.

So back over the M_n case. Let $\Lambda = \mathbb{C}[M_n] = M_n\mathbb{C}$

Object to consider is ~~a free~~ a retract of a free Λ-module. Normally by a free Λ-module one means $\Lambda \otimes V$ for some v.s. V. Now $\Lambda = T \otimes T^*$, so another meaning of free might be $T \otimes V'$. Latter definition seems more general and seems better from the viewpoint of Karoubi envelope, and Morita equivalence ~~Look at a retract of $T\otimes$~~ A Λ-module retract of $T\otimes V$ is the same as a retract of V ??

Start again. Λ module retract of $\Lambda\otimes V$ is the same by M. equiv. as a v.s. retract of $T^*\otimes V$. T^* comes with a basis, so $T^*\otimes V = \begin{matrix} V \\ \oplus \\ V \end{matrix}$

$$W \xleftarrow{\ \beta\ } \begin{matrix} V \\ \oplus \\ V \end{matrix} \xleftarrow{\ \alpha\ } W \qquad \beta\alpha = 1_W$$

~~The~~ A retract ~~of~~ is equivalent to a proj. $\begin{matrix} V \\ \oplus \\ V \end{matrix} \xleftarrow{\ p=\alpha\beta\ } \begin{matrix} V \\ \oplus \\ V \end{matrix}$

$$\alpha = \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} \qquad \beta = \begin{pmatrix} \beta_1 & \beta_2 \end{pmatrix}$$

$$p = \begin{pmatrix} \alpha_1\beta_1 & \alpha_1\beta_2 \\ \alpha_2\beta_1 & \alpha_2\beta_2 \end{pmatrix} \qquad p_{ij} = \alpha_i\beta_j$$

V is a module over the ring A : $\left|\begin{matrix} \text{gen} & p_{ij} \\ \text{rel} & p_{ik} = \sum_j p_{ij}p_{jk} \end{matrix}\right.$

~~xxx~~ A is M_2 graded. $\tilde{p}_{ij} = e_{ij}\otimes p_{ij} \in \Lambda\otimes A$

and you find $p_{ij}\, p_{kl} = 0 \qquad j\neq k$.

~~xxx~~ W does not change as we reduced V.

~~xxx~~ For V reduced, V is a module over $A \oplus \mathbb{C}e_1 \oplus \mathbb{C}e_2$

i.e. $V = V_1 \oplus V_2 \qquad p_{ij} : V_i \leftarrow V_j$

V reduced means $\quad V = \alpha_1 W + \alpha_2 W$?
$\qquad \operatorname{Ker}\beta_1 \cap \operatorname{Ker}\beta_2 = 0$

$$V = \sum_{ij} \alpha_i \beta_j V \qquad V\oplus V \hookleftarrow W \twoheadleftarrow V \oplus V$$

$$= \alpha_1 W + \alpha_2 W$$

$0 = \alpha_i \beta_j v \quad \forall i,j \implies \beta_j v = 0 \;\forall j$

Go back to

$$W \xleftarrow{(\beta_1,\beta_2)} V \oplus V \xleftarrow{\binom{\alpha_1}{\alpha_2}} W \xleftarrow{(\beta_1,\beta_2)} V \oplus V$$

V reduced means $\quad V = \alpha_1 W + \alpha_2 W$

$\beta_1 v = \beta_2 v = 0 \implies v = 0$

$p_{ij} p_{kl} = \alpha_i (\beta_j \alpha_k) \beta_l = 0 \qquad \forall i, j, k, l$ st $j \neq k$

You know that $\beta_j \alpha_k = 0$ for $j \neq k$

$V = \alpha_1 W + \alpha_2 W \qquad \beta_1 V = \beta_1 \alpha_1 W$

---

you want to prove that V

$v = \alpha_1 w_1 + \alpha_2 w_2 \qquad \beta_1 v = \beta_1 \alpha_1 w_1$

$\qquad \beta_2 v = \beta_2 \alpha_2 w_2$

Repeat. Begin with

$$W \xleftarrow{(\beta_1\ \beta_2)} \begin{matrix} V \\ \oplus \\ V \end{matrix} \xleftarrow{\begin{pmatrix}\alpha_1 \\ \alpha_2\end{pmatrix}} W \qquad \beta_1\alpha_1 + \beta_2\alpha_2 = 1_W$$

get $p = \begin{pmatrix}\alpha_1 \\ \alpha_2\end{pmatrix}(\beta_1\ \beta_2) = (\alpha_i\beta_j)$ on $V \ni \sum_j p_{ij}p_{jk} = p_{ik}$

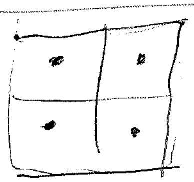
$\underline{V}$ becomes a module [redacted] over A : gens p_{ij} rels

$$\begin{tikzcd} V & V \arrow[l, "\alpha_1\beta_1"'] \arrow[dl, "\alpha_2\beta_1"] \\ \oplus & \oplus \\ V & V \arrow[l, "\alpha_2\beta_2"] \arrow[ul, "\alpha_1\beta_2"'] \end{tikzcd}$$

~~$\alpha_1 W \oplus \alpha_2 W$~~

$$\begin{matrix} \alpha_1 W & & V/\mathrm{Ker}\,\beta_1 \\ \oplus & & \\ \alpha_2 W & & V/\mathrm{Ker}\,\beta_2 \end{matrix}$$

$$\begin{tikzcd} \alpha_1 W & & \beta_1 V \\ \oplus & W \arrow[ul] \arrow[dl] \arrow[ur, leftrightarrow] \arrow[dr, leftrightarrow] & \oplus \\ \alpha_2 W & & \beta_2 V \end{tikzcd}$$



$$\begin{matrix} V \\ \oplus \\ V \end{matrix} \supset \begin{matrix} \alpha_1 W \\ \oplus \\ \alpha_2 W \end{matrix} \xleftarrow{\begin{pmatrix}\alpha_1 \\ \alpha_2\end{pmatrix}} W \xleftarrow{(\beta_1\ \ \beta_2)} \begin{matrix} V/\mathrm{Ker}\,\beta_1 \\ \oplus \\ V/\mathrm{Ker}\,\beta_2 \end{matrix} \twoheadleftarrow \begin{matrix} V \\ \oplus \\ V \end{matrix}$$

factorization of [redacted] p on $\begin{matrix} V \\ \oplus \\ V \end{matrix}$

$$\begin{matrix} \beta_1 V \xleftarrow{\sim} V/\mathrm{Ker}\,\beta_1 \\ \oplus \\ \beta_2 V \xleftarrow{\sim} V/\mathrm{Ker}\,\beta_2 \end{matrix} \twoheadleftarrow \begin{matrix} V \\ \oplus \\ V \end{matrix} \hookleftarrow \begin{matrix} \alpha_1 W \\ \oplus \\ \alpha_2 W \end{matrix}$$

Start again. $\Lambda = M_2\mathbb{C} = T\otimes T^*$

Consider a Λ-module retract of $\Lambda\otimes V$.

By M.eq ~~this~~ is the same as a v.s. retract of $T^*\otimes V = \begin{matrix} V \\ \oplus \\ V \end{matrix}$

$$W \xleftarrow{(\beta_1\ \beta_2)} \begin{matrix} V \\ \oplus \\ V \end{matrix} \xleftarrow{\binom{\alpha_1}{\alpha_2}} W$$

retract ~~compo~~ equiv. to projection $p = \begin{pmatrix}\alpha_1 \\ \alpha_2\end{pmatrix}(\beta_1\ \ \beta_2)$

~~a projection~~ $p_{ij} = \alpha_i\beta_j$ $\qquad \sum_j \underbrace{\alpha_i\beta_j}_{p_{ij}}\underbrace{\alpha_j\beta_k}_{p_{jk}} = \underbrace{\alpha_i\beta_k}_{p_{ik}}$

So V becomes a mod [redacted] over A: gens p_{ij}

rels $p_{ik} = \sum_j p_{ij}p_{jk}$

Claim A is M_2-graded

$$\tilde p_{ij} = e_{ij}\otimes p_{ij} \in M_2\mathbb{C}\otimes A$$

$$\sum_j \tilde p_{ij}\tilde p_{jk} = \sum_j \underbrace{e_{ij}e_{jk}}_{e_{ik}}\otimes p_{ij}p_{jk} = e_{ik}\otimes p_{ik} = \tilde p_{ik}$$

Get alg ~~hom~~ map $\quad A \xrightarrow{\Delta} M_2\mathbb{C}\otimes A$

$$a_{ij} \longmapsto e_{ij}\otimes a_{ij}$$

$$\Delta(p_{ij}p_{kl}) = (e_{ij}\otimes p_{ij})(e_{kl}\otimes p_{kl}) = \overbrace{e_{ij}e_{kl}}^{0 \text{ if } j\neq k}\otimes p_{ij}p_{kl}$$

Δ injective $\Longrightarrow$ $p_{ij}p_{kl} = 0 \quad j\neq k$

Now A $\quad M_2$

Consider a retract

$$W \xleftarrow{(\beta_1\ \beta_2)} \begin{matrix} V \\ \oplus \\ V \end{matrix} \xleftarrow{\begin{pmatrix}\alpha_1\\ \alpha_2\end{pmatrix}} W \qquad \beta_1\alpha_1 + \beta_2\alpha_2 = 1_W$$

equivalently a projection $p = \begin{pmatrix}\alpha_1\\ \alpha_2\end{pmatrix}(\beta_1\ \beta_2) : \begin{matrix} V \\ \oplus \\ V \end{matrix} \longleftarrow \begin{matrix} V \\ \oplus \\ V \end{matrix}$

equivalently $\begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix}\begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix} = \begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix}$ $\quad p_{ik} = \sum_j p_{ij} p_{jk}$

$\underline{A}$ gen p_{ij}, rel $\quad$ so A is idempotent

Assume V reduced: $AV = V = V/_A V$

~~AV~~ $\quad AV = \sum_{ij} p_{ij} V = \sum_{ij} \alpha_i \beta_j V = \sum_i \alpha_i W$

$V = \alpha_1 W + \alpha_2 W \qquad p_{ij} v = \alpha_i \beta_j v = 0 \quad \forall i,j$

${}_{\beta_1}V = {}_{\beta_2}V = 0 \qquad \Rightarrow \beta_1 v = \beta_2 v = 0$

Express V as an image

$$\underline{A \otimes_A V \longrightarrow \mathrm{Hom}_A(A, V)}$$

$$\begin{matrix} W \\ \oplus \\ W \end{matrix} \xleftarrow{\begin{pmatrix}\beta_1\\ \beta_2\end{pmatrix}} V \xleftarrow{(\alpha_1, \alpha_2)} \begin{matrix} W \\ \oplus \\ W \end{matrix} \qquad W \xleftarrow{(\beta_1\ \beta_2)} \begin{matrix} V \\ \oplus \\ V \end{matrix}, \quad W \xleftarrow{(\beta_1\ \beta_2)} \begin{matrix} V \\ \oplus \\ V \end{matrix}$$

What should happen is that

$$\begin{pmatrix}\beta_1\\ \beta_2\end{pmatrix}(\alpha_1\ \ \alpha_2) = \begin{pmatrix} \beta_1\alpha_1 & 0 \\ 0 & \beta_2\alpha_2 \end{pmatrix}$$

so that V splits into $\alpha_1 W \oplus \alpha_2 W$.

$$\begin{tikzcd}
A \otimes V \arrow[r] & \mathrm{Hom}(A, V) \\
a \otimes v \arrow[r, mapsto] \arrow[d, mapsto, "\mu"'] & (a' \mapsto (a'a)v) \arrow[d, mapsto, "\mu"] \\
\mu a \otimes v \arrow[dr, bend right, mapsto] & (a' \mapsto (a'\mu)(av)) \\
& (a' \mapsto a'((\mu a)v))
\end{tikzcd}$$

$(a'a)v = a'(av)$

$(a'\mu)a$

$a'(\mu a)$

So how to see this works.

$$W \xleftarrow{(\beta_1\ \beta_2)} \begin{matrix} V \\ \oplus \\ V \end{matrix} \xleftarrow{\begin{pmatrix}\alpha_1 \\ \alpha_2\end{pmatrix}} W \qquad\qquad \begin{matrix} W \\ \oplus \\ W \end{matrix} \xleftarrow{\begin{pmatrix}\beta_1 \\ \beta_2\end{pmatrix}} V \xleftarrow{(\alpha_1\ \alpha_2)} \begin{matrix} W \\ \oplus \\ W \end{matrix}$$

V reduced iff $(\alpha_1\ \alpha_2)$ surj + $\begin{pmatrix}\beta_1 \\ \beta_2\end{pmatrix}$ inj.

Question: What is $\begin{pmatrix}\beta_1 \\ \beta_2\end{pmatrix}(\alpha_1\ \alpha_2)$?

Go back to $p_{ij} = \alpha_i \beta_j$: $\begin{matrix} V \\ \oplus \\ V \end{matrix} \leftarrow W \leftarrow \begin{matrix} V \\ \oplus \\ V \end{matrix}$

You showed that A is M_2-graded, hence

$p_{ij} p_{k\ell} = \alpha_i \beta_j \alpha_k \beta_\ell = 0$ for $\forall\, i, j, k, \ell$ with $j \neq k$

$$\Downarrow$$

~~Hence~~

$$0 = \sum_{i\,\ell} \beta_i \alpha_i \beta_j \alpha_k \beta_\ell \alpha_\ell = \beta_j \alpha_k$$

A' gen p_{ij} no relations imposed

$\emptyset$ $M_2\mathbb{C} \otimes A'$ $\qquad \tilde{p}_{ij} = e_{ij} \otimes p_{ij}$

$$A' \xrightarrow{\Delta} M_2\mathbb{C} \otimes A'$$

A' free alg w. gens p_{ij}, $i,j = 1,2$

Define $\Delta : A' \longrightarrow M_2\mathbb{C} \otimes A'$ to be unique alg map sending p_{ij} to $e_{ij} \otimes p_{ij}$. Let I = kernel of Δ.

What happens maybe is that there is a free algebra generated by the p_{ij}, which has for basis all words in the generators. You use the monoid $\Gamma_+ = M_2 \cup \{0\}$ to assign a degree in Γ_+ for each word.

You have to go over Γ-grading again, Γ_+ semigroup with basepoint 0 absorbing, look at Γ modules M, ~~[illegible]~~ i.e. $\Gamma_+ \to \mathrm{End}(V)$

$\mathbb{C}[\Gamma_+]/\mathbb{C}[*]$ check that $\mathbb{C}\{*\}$ is an ideal

$s\{*\} = \{*\}s = \{*\}$

$$\begin{array}{c} \mathbb{C}[\Gamma_+] \longrightarrow \mathbb{C}[\Gamma_+] \otimes \mathbb{C}[\Gamma_+] \\ \downarrow \\ \mathbb{C}[\Gamma_+]/\mathbb{C}\{*\} \otimes \mathbb{C}[\Gamma_+]/\mathbb{C}\{*\} \end{array}$$

so $\mathbb{C}[\Gamma]$ is a Hopf alg. It's counital comodules are $V = \bigoplus_{s\in\Gamma} V_s$. Given another $W = \bigoplus_t W_t$ then $V \otimes W = \bigoplus_{s,t\in\Gamma} V_s \otimes W_t = \bigoplus_{u\in\Gamma} \bigoplus_{u=st} V_s \otimes W_u$

amounts to using $\mu : \mathbb{C}[\Gamma] \otimes \mathbb{C}[\Gamma] \longrightarrow \mathbb{C}[\Gamma]$

Go back to A gen rels P_{ij}

do in general. ~~So let V be~~ Let $V = \bigoplus_{s\in\Gamma} V_s$ be Γ-graded v.s. Then can form $T(V) = V \oplus (V \hat{\otimes} V) \oplus (V \hat{\otimes} V \hat{\otimes} V) +$

What's important is that $V \hat{\otimes} V = V \otimes V / (V \otimes V)_0$

So $T(V)$ ~~be~~ will be graded with respect to words $s_1, \dots, s_k \in \Gamma$ with non-zero product.

Exit 63, take 72 east

head left (North) L.B. Blvd.

into Surf City where hotel

check in at 3:00 pm.

Sandcastle Hotel

North Beach

Harvey Cedars

Loveladies (right after

Big Restaur + Liquor

Owltree

HCH Realty

Seashell Lane.