Joachim did understand the Case M_n. He says that the alg A generated by the components of an M_n graded projection is Morita equivalent to functions on a simplex. Look at A generators p_{ij} satisfying

$$\begin{cases} p_{ik} = \sum_j p_{ij} p_{jk} & \text{idemp.} \\ p_{ij} p_{k\ell} = 0 & j \neq k. \end{cases}$$

Yes.

$$A \xrightarrow{\Delta} \mathbb{C}[M_n] \otimes A$$

$$p_{ij} \longmapsto e_{ij} \otimes p_{ij}$$

You want to find B which ~~should~~ might be a cross product

Γ a set, let $\Gamma_+ = \Gamma \cup \{0\}$

a generalization of the group ring $\mathbb{C}\Gamma$ assoc. to a group.

suppose given $\mu : \Gamma_+ \times \Gamma_+ \to \Gamma_+$ assoc. s.t. 0 is absorbing. semi group ring $\mathbb{C}\Gamma_+$ is a Hopf alg. $\Delta(s) = s \otimes s$

Quillen Quillen

~~Ideal $\mathbb{C}\langle 0 \rangle$ in $\mathbb{C}\Gamma_+$ $\mathbb{C}\langle 0 \rangle$ closed under μ~~

$\Delta\ \mathbb{C}\langle 0 \rangle \subset \mathbb{C}\langle 0 \rangle \otimes \mathbb{C}\langle 0 \rangle$

Quillen

~~$\mathbb{C}\Gamma \cdot \mathbb{C}[0]$, $\mathbb{C}[0] \cdot \mathbb{C}[\Gamma] \subset \mathbb{C}\Gamma$~~

$\mathbb{C}\Gamma \cdot \mathbb{C}[0]$, $\mathbb{C}[0] \cdot \mathbb{C}\Gamma \subset \mathbb{C}[0]$ $\Rightarrow$ quotient ring $\mathbb{C}\Gamma_+ / \mathbb{C}[0] = \mathbb{C}\Gamma$

$\Delta : \mathbb{C}[0] \longrightarrow \mathbb{C}[0] \otimes \mathbb{C}[0]$

$$\Gamma_+ \times \Gamma_+ \xrightarrow{\mu} \Gamma_+ \qquad \text{semigp with } 0 \text{ abs.}$$

semigp ring $\mathbb{C}\Gamma_+$ in which $\mathbb{C}[0]$ is ideal so get ring structure on $\mathbb{C}\Gamma = \mathbb{C}\Gamma_+/\mathbb{C}[0]$ s.t.

$$\begin{aligned} [s][t] &= [st] && \text{if } st \in \Gamma \\ &= 0 && \text{if not} \end{aligned}$$

~~But~~ $\mathbb{C}\Gamma_+$ coalg ~~structure~~ $\Delta s = s \otimes s$, get $\mathbb{C}\Gamma$ ~~[illegible]~~ bialgebra.

Γ-graded alg $A = \bigoplus_{s \in \Gamma} A_s$ s.t. $A_s A_t \subset \begin{cases} A_{st} & st \neq 0 \\ 0 & \text{otherw.} \end{cases}$

$$A \xrightarrow{\Delta} \mathbb{C}\Gamma \otimes A$$

$$a_s \longmapsto s \otimes a_s$$

Now you want to understand what you missed. This time you ~~have~~ to start ~~with~~ at the A end which is harder. So maybe ~~[illegible]~~ you try ~~[illegible]~~ special cases:

Γ groupoid, Γ semigroup

Simplest semigroup is pt. which is a gp.

semigroup structure on Γ_+ = two points $0, \alpha$

$0, \alpha$, $\alpha 0 = 0 \alpha = 0$, $\alpha^2 = 0$. Γ graded alg.

$$A = A_0 \oplus A_\alpha \qquad 0 \text{ mult.}$$

$\Gamma_+ = \{\alpha, 0\}$ $\qquad A = A_\alpha$ ~~$\oplus A_0$~~ $\qquad A^2 = 0$

$\Gamma_+ = \{\alpha, \alpha^2, 0\}$ $\qquad A = A_\alpha \oplus A_{\alpha^2}$ ~~[illegible]~~

So it looks like you want to restrict attention to groupoids. ~~Let A be~~ In other words, what is assembly for a groupoid? You want the basic idea. ~~You need the~~ Go back to Serre's thm. idea - trying to construct a vector bundle on the classifying space $B\Gamma$. You have some background from Haefliger theory as to what the classifying space of a groupoid ~~doll~~ should be. But you probably don't understand sufficiently the group case. ~~Example~~

Look at group case. Principal bundle $X \to Y$ with fibre Γ. Open covering of Y over which the bundle becomes trivial

Groupoid case? You should know how to do this. Semi simplicially there is a classifying map to the nerve of the groupoid.

~~The full grou~~ In the case of M_F (F finite) where the objects are elements of F

Model: Given Y, an open covering of Y, you have the nerve of the coverings, which is a simp sp.

$$
\begin{tikzcd}
Y & \mathcal{U} \arrow[l] \arrow[d] & \mathcal{U}\times_Y\mathcal{U} \arrow[l, shift left] \arrow[l, shift right] \arrow[d] & \mathcal{U}\times_Y\mathcal{U}\times_Y\mathcal{U} \arrow[l, shift left=2] \arrow[l] \arrow[l, shift right=2] \arrow[d] \\
 & F & F\times F \arrow[l, shift left] \arrow[l, shift right] & F\times F\times F \arrow[l, shift left=2] \arrow[l] \arrow[l, shift right=2]
\end{tikzcd}
$$

~~Note~~ ~~This leads to a simplicial cx.~~

Lots of ideas to be ~~assum~~ reviewed.

Sheaf version. The simplest ~~situation~~ where things work well is the case of a topological groupoid where the source and target arrows are etales. This is the groupoid generalization of a discrete group.

Ex: Action of a discrete group on a top space F Nerve of groupoid is

$$E \leftleftarrows E\times\Gamma \;\substack{\leftarrow\\ \leftarrow\\ \leftarrow}\; E\times\Gamma\times\Gamma \cdots$$

Point is that there is a nice topos picture of what should be a principal bundle for such a groupoid. It's a sheaf over the base B acted on by the groupoid.

The stalk at any point of B ~~is going to~~ should be a functor ~~on the~~ from ?

Special case of Γ disc. acting on F space

Top. group class. space $E\Gamma\times^{\Gamma}F$

$$\begin{tikzcd} E \arrow[r] \arrow[d] & E\Gamma\times F \arrow[d] \\ B \arrow[r] & E\Gamma\times^{\Gamma}F \end{tikzcd}$$

~~So the first~~

The first thing to do is to analyze carefully the case of the etale groupoid arising from Γ disc acting on a space F. The classical space for this groupoid should be $E\Gamma \times^{\Gamma} F$. Over a base space B the principal bundles associated to this groupoid are obtained by pullback:

$$\begin{tikzcd} E \arrow[r] \arrow[d] & E\Gamma \times F \arrow[d] \\ B \arrow[r] & E\Gamma \times^{\Gamma} F \end{tikzcd}$$

Equivalently a princ. bundle over B assoc. to the groupoid is the same as a principal Γ-bundle E over B equipped with an equivariant map

$$E \longrightarrow F.$$

$\Gamma_+ = \{g, 0\}$ $\qquad g^2 = 0g = g0 = 0 = 0^2$

Then a Γ-graded A is just A_g such that $A_g^2 = 0$.

$\Gamma_+ = \{e, 0\}$ $\qquad e^2 = e,\ e0 = 0e = 0$

$A = A_e$ s.t. $A_e A_e \subset A_e$

Program for today: Understand assembly for a groupoid, yes. First you have to go over past ideas, Haefliger, etale groupoid.

~~first~~ Find examples. First take disc group Γ acting on a space F. This gives an etale groupoid. F is the space of objects, $F\times\Gamma$ is the space of arrows (right action notation) $\begin{cases} \text{source } (\xi, s) = \xi \\ \text{target } (\xi, s) = \xi s \end{cases}$, the nerve of this groupoid is

$$F \leftleftarrows F\times\Gamma \;\substack{\leftarrow\\\leftarrow\\\leftarrow}\; F\times\Gamma\times\Gamma$$
$$\xi \leftarrow\!\shortmid (\xi, s)$$
$$\xi s \leftarrow\!\shortmid$$

$\mathrm{Nerve}(F/\!/\Gamma) = F\times^{\Gamma} E\Gamma$ for the ~~Milnor~~ semi-simplicial $E\Gamma \to B\Gamma$.

2nd there is the sheaf picture for torsors associated to an etale groupoid. Actually, this idea should precede. What is a torsor over B assoc. to $(F/\!/\Gamma)$? Answer it is a Γ-torsor $E \to B$ and an equiv. map from E to F, ~~should be same as the fibre bundle over B with~~. START AGAIN with the first example, namely the etale groupoid F_Γ given by Γ disc acting on space F. You have nerve

$$\substack{\to\\\to\\\to} F\times\Gamma\times\Gamma \rightrightarrows F\times\Gamma \longrightarrow F \qquad \mathrm{Nerve}(F/\!/\Gamma)$$

whose realization is $F\times^{\Gamma}E\Gamma$, the associated fibre bundle over $B\Gamma$ ~~associated to~~ with fibre the Γ space Γ.

A map $B \longrightarrow F \times^{\Gamma} E\Gamma = F_\Gamma$ should yield first of all a map $B \longrightarrow B\Gamma$ i.e. a princ bdle $\Gamma \longrightarrow E \longrightarrow B$ over B and also an equivariant map $E \longrightarrow F$. In fact any $B \to F_\Gamma$ should be equivalent to such a pair by pull-back

$$\begin{tikzcd}
 & F \times E\Gamma \arrow[r] \arrow[d, "\Gamma"] \arrow[dr, phantom, "\text{cart.}"] & E\Gamma \arrow[d, "\Gamma"] \\
B \arrow[r] & F_\Gamma \arrow[r] & B_\Gamma
\end{tikzcd}$$

not yet clear.

Given a Γ-space F, there is $F_\Gamma = E\Gamma \times^{\Gamma} F$, the total space of the fibre bundle

$$F \longrightarrow F_\Gamma \longrightarrow B\Gamma$$

over $B\Gamma$ assoc. to the Γ-space F. A map $B \longrightarrow F_\Gamma$ yields a map $B \longrightarrow B\Gamma$, whence a principle Γ bundle$_F$ $E =_F B \times_{B\Gamma} E\Gamma$

$$\begin{tikzcd}
\Gamma & E\Gamma \times F \arrow[r] \arrow[d] & F_\Gamma \arrow[d] \\
\Gamma & E\Gamma \arrow[r] & B\Gamma
\end{tikzcd}$$

Start again. F space$_\Gamma$ acted on by Γ disc.

Can form

$$\begin{tikzcd} F \arrow[r] \arrow[d, equal] & E\Gamma \times F \arrow[r] \arrow[d] & E\Gamma \arrow[d] \\ F \arrow[r] & F_\Gamma \arrow[r] & B\Gamma \end{tikzcd}$$

Borel mixing diagram$_\Gamma$. A map$_\Gamma$ from B to $B\Gamma$ yields by pullback a principal Γ-bundle over B

$$\begin{tikzcd} E \arrow[r] \arrow[d] & E\Gamma \arrow[d] \\ B \arrow[r, "\phi"'] & B\Gamma \end{tikzcd}$$

$$\begin{tikzcd} E \arrow[r] \arrow[d, "\Gamma"] & E\Gamma \times F \arrow[r] \arrow[d, "\Gamma"] & E\Gamma \arrow[d, "\Gamma"] \\ B \arrow[r] & F_\Gamma \arrow[r] & B\Gamma \end{tikzcd}$$

It seems that a map from B to F_Γ ~~[illegible]~~ amounts to a map $B \to B\Gamma$, that is essentially ~~[illegible]~~ a principal Γ-bundle $E \to B$, and together$_\Gamma$ with a Γ-map$_\Gamma$ $E \to_\Gamma F$. ~~Still~~ confused

$$\begin{tikzcd} E \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{cart}"] & E\Gamma \times F \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{cart}"] & E\Gamma \arrow[d] \\ B \arrow[r] & F_\Gamma \arrow[r] & B\Gamma \end{tikzcd}$$

Simpler version.

$$\begin{tikzcd} E \arrow[r] \arrow[d] & E\Gamma \times F \arrow[d, "\Gamma"] \\ B \arrow[r, "f"] & F_\Gamma \end{tikzcd}$$

a map $f: B \longrightarrow F_\Gamma$ yields a principal Γ bundle $E \longrightarrow B$ by pullback and Γ-map $\tilde{f}: E \rightarrow E\Gamma \times F$ covering f ?.

Maybe you should work ~~[illegible]~~ in the category of principal Γ-bundles, i.e. free Γ-spaces and equivariant maps. Given such an E, then $B = E/\Gamma$, and $\exists!$ up to htpy Γ-map $E \longrightarrow E\Gamma$.

Still confusing!!!

Program: To understand what $E\Gamma \times^\Gamma F$ classifies. ~~[illegible]~~ First answer is a pair consisting of a principal Γ-bundle $E \rightarrow B$ and a Γ-map $E \longrightarrow \Gamma$. Then you must adjust for homotopy.

Review. You are trying to recall classifying spaces for etale groupoids, such as ~~the~~ Haefliger's classifying spaces. ~~A~~ You think there is a good ~~topos~~ classifying topos, ~~that is, a good~~ analogous to the category of Γ-sets for a discrete group Γ. How to understand?

The idea I think is that given an etale groupoid, you have ~~~~ sheaves (of sets) on the space $\mathcal{O}$ of objects and it makes sense to ask for the arrows to act on such sheaves. Think of etale spaces.

$$\Gamma_0 \underset{\text{target}}{\overset{\text{source}}{\leftleftarrows}} \Gamma_1$$

Have sheaf $\mathcal{F}_0$ over Γ_0 together with an iso $s^*\mathcal{F}_0 \simeq t^*\mathcal{F}_0$

So you need to handle a groupoid. You want to handle You still lack control of the simplest case. Anyway, list what you know.

The easiest example to understand is the ~~etale~~ topological groupoid arising from a disc. gp Γ acting on a space F. The nicest case is when $F \longrightarrow F/\Gamma$ is a principal Γ-bundle. In this case you can form the Mischenko line bundle, associated fibre bundle over F/Γ with fibre $\mathbb{C}\Gamma$.

Missing Point to work on. Review the Groth topos picture to understand what should be a torsor for a simple groupoid such as M_2. In fact M_2 can be described as $\mathbb{Z}/2$ acting on itself by translation. Guess that a torsor for a (F, Γ) over a space B should be a principal Γ bundle together with a map $E \longrightarrow F$. When F is a finite set, this means what.

So consider $\Gamma = \mathbb{Z}/2$ acting by translation on $F = \mathbb{Z}/2$. The classifying space for (F, Γ) should be $E\Gamma \times_\Gamma F$. A map

$B \longrightarrow E\Gamma \times_\Gamma F$ should yield a principal Γ bundle E over B together with an equivariant map $E \longrightarrow F$. When $F = \Gamma$ an equivariant map $E \longrightarrow \Gamma$ should trivialize ?

Another idea is the cross product algebra.

If G discrete group, then you get a topos consisting of G sets. Suppose $\mathcal{G}$ is a groupoid. then you should have have a topos consisting of functors from $\mathcal{G}$ to Sets. ~~Also Stake~~ What can you say about such functors. ~~Enough for $\mathcal{G}$ to be coarse~~ !

G group, G-sets, G-ab, $\emptyset$

Suppose you take a groupoid.

Pairing between right and left G sets.

You want Groth viewpoint. $\mathcal{G}$ groupoid, objects and maps. ~~$\mathcal{G}$-sets~~ $\mathcal{G}$-set = functors from $\mathcal{G}$ to Sets

If C is a small cat, then

Today you want to understand classifying space for a groupoid.

Let's begin with Grothendieck's approach. He ~~does~~ treats the case of a discrete group G, by associating to G the ~~category~~ topos of G-sets, i.e. functors from ~~pt, G~~ the category (pt, G) to Sets.

Basic result is that for any space B (eventually any topos), then a map of topoi from Sh_B = sheaves of sets on B to $\{G\text{-sets}\}$ is equivalent to a G torsor over B. Why? In $\{G\text{-sets}\}$ one has the object G_r given by the set G with ~~right~~ G action by right multiplication, So a map $f : Sh_B \longrightarrow \{G\text{-sets}\}$ ~~should yield~~ yields $f^* G_r$ which should be a G-torsor over B.

~~Important point~~

B space, let's ~~understand~~ understand why a principal G-torsor over B yields a morphism of topoi

$$Sh_B \underset{f_*}{\overset{f^*}{\rightleftarrows}} \{G\text{-sets}\}$$

This should be easy, namely a G-torsor over B is an etale space $\pi : E \longrightarrow B$ together with a right G action on E such that locally one has an isom $E \simeq B \times G$. If S is a G-set one can twist: $E \times^G S$ to get a sheaf over B functorial in S. This gives f^* which is clearly right cont. (resps arb $\varinjlim$'s) and resp. fin. $\varprojlim$'s. What is f_*?

$$\cancel{\mathrm{Hom}(\mathcal{F}, f^*S)} \qquad \mathrm{Hom}_{Sh_B}(f^*S, \mathcal{F})$$

$$= \mathrm{Hom}_{Sh_B}(E \times^G S, \mathcal{F}) \qquad \text{restrict to } U \subset B$$

$$= \mathrm{Hom}_{Sh_U}(U \times S, \mathcal{F}) =$$

Better might be $\mathrm{Hom}_{Sh_B}(E \times^G S, \mathcal{F}) = \mathrm{Hom}_{G, Sh_B}(E \times S, \mathcal{F})$

$$= \mathrm{Hom}_{G}(S, \mathrm{Hom}_{Sh_B}(E, \mathcal{F}))$$

$$\begin{tikzcd} E \times S \arrow[d] & \\ E \times^G S \arrow[r] \arrow[d] & \mathcal{F} \arrow[d] \\ B \arrow[r, equal] & B \end{tikzcd}$$

seems correct although needs clarification. Better might be

$$\mathrm{Hom}_{Sh/B}(E \times^G S, \mathcal{F}) = \mathrm{Hom}_{Sh/B}(E \times S, \mathcal{F})^G$$

$$= \mathrm{Hom}(S, \mathrm{Hom}_{Sh/B}(E, \mathcal{F}))^G \qquad \boxed{f_*(\mathcal{F}) = \mathrm{Hom}_{Sh/B}(E, \mathcal{F})}$$

Something here reminds you of $EG \times^G S$

A map $Sh/B \xrightarrow{f} \{G\text{-sets}\}$ is equivalent to the G-torsor E over B given by $E = f^*(G_r)$ and then $f^*(S) = f^*(G_r \times^G S) = E \times^G S$

$$\mathrm{Hom}_{G\text{-sets}}(S, f_*(\mathcal{F})) = \mathrm{Hom}_{Sh/B}(f^*(S), \mathcal{F}) = \mathrm{Hom}_{Sh/B}(E \times^G S, \mathcal{F})$$

$$= \mathrm{Hom}_G(S, \mathrm{Hom}_{Sh/B}(E, \mathcal{F}))$$

~~So you find the same idea namely that the what you get over B~~

Now you want to go to groupoids, say discrete. $\mathcal{G}$ groupoid $\{\mathcal{G}\text{-sets}\} = \{\text{functors from } \mathcal{G} \text{ to Sets}\}$. This is a topos. $\mathcal{G}$ decomposes into connected components (inside which any two objects are isomorphic), and a connected groupoid is equivalent to a group once a basepoint is chosen. ~~...~~ ~~Sets fix~~ Restrict to $\mathcal{G}$ connected. You want to know what a map of topoi f from $Sh_{/B}$ to $\{\mathcal{G}\text{-sets}\}$ looks like.

$\mathcal{G}$ groupoid $\{\mathcal{G}\text{-sets}\} = \underline{\mathrm{Hom}}_{cat}(\mathcal{G}, \text{Sets})$, this is a topos, ~~...~~ topos coproduct (disjoint union) of $\{\mathcal{G}_\alpha\text{-Sets}\}$ where $\mathcal{G}_\alpha$ are the components of $\mathcal{G}$, $\{\mathcal{G}_\alpha\text{-Sets}\}$ equiv to G_x-Sets where $x \in \mathrm{Ob}\,\mathcal{G}_\alpha$, and $G_x = \mathrm{Aut}_{\mathcal{G}}(x)$. ~~You have seen~~ In the case of G a G-torsor over a space B is an etale space $E \to B$ with a right action of G ~~of~~ on E ~~which is~~ over B, which is free. You might view E as ~~an ... object with free G^{op} action in the~~ a G^{op} object in $Sh_{/B}$

Philosophy here is that topoi are the good generalization of the category Sets, so that ~~construction in Sets~~ whatever you do ~~with~~ in Sets should carry over to any topos. G-torsor E is an object with right G-action such that $E \times G \longrightarrow E \times E$ is injective.

It should now be possible to define $\mathcal{G}$-torsor. But first you need to understand $\mathcal{G}$-torsor in Sets. This should be a functor from $\mathcal{G}$ to Sets of some sort.

Try the following viewpoint. You have the groupoid $\mathcal{G}$ and the topos Fun($\mathcal{G}$, Sets) of covariant functors. This topos should be the classifying topos for $\mathcal{G}$-torsors. So you should analyze what is a map ~~from~~ of topoi from any $\mathcal{T}$ to Fun($\mathcal{G}$, Sets). In particular what is a map of topoi from Sets to Fun($\mathcal{G}$, Sets). More generally you can take a small category $\mathcal{C}$ and the topos Fun($\mathcal{C}$, Sets). What is a map of topoi

$$\text{Sets} \underset{f_*}{\overset{f^*}{\rightleftarrows}} \text{Fun}(\mathcal{C}, \text{Sets}).$$

The important condition is that f^* respect finite lim's.

Example. ~~If $\mathcal{C}$ and $\mathcal{D}$~~ $f: \mathcal{C} \longrightarrow \mathcal{D}$ is a functor, then it induces

$$\text{Fun}(\mathcal{C}, \text{Sets}) \underset{f_*}{\overset{f^*}{\rightleftarrows}} \text{Fun}(\mathcal{D}, \text{Sets})$$

Given $f: \mathcal{C} \longrightarrow \mathcal{D}$ you have always

$$\text{Fun}(\mathcal{C}, \text{Sets}) \begin{array}{c} \xrightarrow{\phi_!} \\ \xleftarrow{\phi^*} \\ \xrightarrow[\phi_*]{} \end{array} \text{Fun}(\mathcal{D}, \text{Sets})$$

set $f_* F = F \circ f$

$\mathcal{C}$ category, topos $\mathrm{Fun}(\mathcal{C}, \mathrm{sets})$

$u : \mathcal{C}' \longrightarrow \mathcal{C}$ functor

$$\mathrm{Fun}(\mathcal{C}', \mathrm{sets}) \underset{u_*}{\overset{u_!}{\underset{u^*}{\rightleftarrows}}} \mathrm{Fun}(\mathcal{C}, \mathrm{sets})$$

$$(u^*F)(X) = F(uX)$$

$$(u_!F')(X) = \varinjlim_{(X', uX' \to X)} F'(X') \qquad \text{evidently a fun of } X.$$

$$\xi \in \mathrm{Hom}\Big(\varinjlim_{X', uX' \to X} F'(X'),\ F(X)\Big)$$

ξ consists of maps $F'(X') \longrightarrow F(X)$ $\quad \forall X', uX' \to X$

Anyway do this later. The idea is that the pair $(u_!, u^*)$ constitutes a map of topoi when $u_!$ is exact which should be equivalent to $\forall X$, the category of $(X', uX' \to X)$ is filtering.

look at the case $\mathcal{C}' = pt$. So $\exists ! X'$.

Start again. $\mathcal{C}$ small cat, $\mathrm{Fun}(\mathcal{C}, \mathrm{sets})$ is a topos. Look at morphisms of topos

$$f \quad \mathrm{sets} \longrightarrow \mathrm{Fun}(\mathcal{C}, \mathrm{sets})$$

geometrically this is a map from a point to a space, actually it is a pair of adjoint functors (f^*, f_*) such that f^* resp fin $\varprojlim$ s. Example: X obj of $\mathcal{C}$ and you have $\quad f^*(F) = F(X)$

$$\text{Sets} \underset{f_*}{\overset{f^*}{\rightleftarrows}} \text{Fun}(\mathcal{C}, \text{Sets})$$

$$f^*(F) = F(X)$$

$\mathcal{C}$ category, $\hat{\mathcal{C}} = \text{Fun}(\mathcal{C}^{op}, \text{Sets})$ is a topos

Use contra funs to agree with $\mathcal{C}$ = open sets + presheaves

$u : \mathcal{C} \longrightarrow \mathcal{D}$ $\qquad$ $\hat{\mathcal{C}} \; \overset{u_!}{\longrightarrow}, \; \overset{u^*}{\longleftarrow}, \; \underset{u_*}{\longrightarrow} \; \hat{\mathcal{D}}$

First point: (u^*, u_*) is a morph of topoi from $\hat{\mathcal{C}}$ to $\hat{\mathcal{D}}$

If $u_!$ respects fin. $\varprojlim$s, then you get a diff morphism $(u_!, u^*)$ from $\hat{\mathcal{D}}$ to $\hat{\mathcal{C}}$

Review $\quad (u_* G)(C) = \varprojlim\limits_{u(C) \xrightarrow{f} D} G(D)$

$$(u_! G)(C) = \varinjlim_{D \to u(C)} G(D)$$

Given $F \in \text{Fun}(\mathcal{C}, \text{Sets}) = \hat{\mathcal{C}}$ $\qquad$ $G \in \hat{\mathcal{D}}$ ~~and~~

$(u^* G)(C) = G(uC)$. Given $F \xrightarrow{\theta} u^* G = Gu$

i.e. $\theta_C : F(C) \longrightarrow G(u(C))$ $\quad \forall C$. Then

given $D \longrightarrow u(C)$ get $F(C) \longrightarrow G(u(C)) \longrightarrow G(D)$

Thus given $u : \mathcal{C} \longrightarrow \mathcal{D}$ there is this intermediate category consisting of $(C, D, D \to u(C))$ cofibred over $\mathcal{C}$, fibred over $\mathcal{D}$

left and right fibres, try to recall the notation

$u : \mathcal{C} \to \mathcal{D}$

given $u : \mathcal{C} \to \mathcal{D}$ and $D \in \mathcal{D}$, then you have the categories whose objects are

$$(C, \; uC \to D) \qquad , \qquad (C, \; D \to uC)$$

$$u/D \qquad\qquad\qquad\qquad D\backslash u$$

so what happens is you factor the functor.

$$\mathcal{C} \longrightarrow (\mathcal{C}, u, \mathcal{D}) \longrightarrow \mathcal{D}$$

$$C, D, uC \to D$$

Over $\mathcal{D}$ you have the cofibred cat of $(C, uC \to D)$

$$\mathcal{C} \longrightarrow (C, D, uC \to D) \longrightarrow \mathcal{D}$$

$$C \longmapsto (C, uC, id_{uC})$$

$\mathcal{C}$ cat, get topos $\hat{\mathcal{C}} = \mathrm{Fun}(\mathcal{C}^{o}, \mathrm{Sets})$

$\mathcal{C} \xrightarrow[f]{} \mathcal{D}$ fun, get ~~morph of topoi~~ ~~adjoint funs $\hat{\mathcal{D}} \leftrightarrows \hat{\mathcal{C}}$~~

$$\hat{\mathcal{C}} \; \begin{matrix} \xrightarrow{f_!} \\ \xleftarrow{f^*} \\ \xrightarrow[f_*]{} \end{matrix} \; \hat{\mathcal{D}}$$

get map of topoi $\hat{\mathcal{C}} \underset{u_*}{\overset{u^*}{\leftrightarrows}} \hat{\mathcal{D}}$ with $u^* = f^*$, $u_* = f_*$

When $f_!$ respects finite lim's you also have a map of topoi $\hat{\mathcal{D}} \underset{v_*}{\overset{v^*}{\leftrightarrows}} \hat{\mathcal{C}}$ $\quad v^* = f_!$, $v_* = f^*$

Important to remember that the ~~functor to focus~~ inverse image functor is the important one, and it determines the other.

Recall that a ~~[illegible]~~ topos map from $\mathcal{T}$ to $\mathcal{T}'$ can be defined as a functor $\mathcal{T} \xleftarrow{u^*} \mathcal{T}'$ respecting $\begin{cases} \text{arb } \varinjlim\text{s} \\ \text{fin } \varprojlim\text{s} \end{cases}$

~~[illegible]~~ The first condition implies (~~[illegible]~~ using $\exists$ ~~[illegible]~~ set of generators, a site, for a topos) the existence of the adjoint functor u_*

Ex. $\mathcal{T} = \text{Sets} = Sh_{pt}$, ~~[illegible]~~ $\mathcal{T}' = \hat{\mathcal{C}}$

as fun ~~[illegible]~~ $pt \xrightarrow{f} \mathcal{C}$ given by the object $f(pt) = X$. Special case of

$$\mathcal{D} \xrightarrow{f} \mathcal{C} \qquad \hat{\mathcal{D}} \underset{u_* = f_*}{\overset{u^* = f^*}{\rightleftarrows}} \hat{\mathcal{C}}$$

So an object X of $\mathcal{C}$ gives $i_X : pt \longrightarrow \mathcal{C}$ whence map of topoi $\text{Sets} \xleftarrow{i_X^*} \hat{\mathcal{C}}$. ~~So you first~~ You get other "points" of $\hat{\mathcal{C}}$ ~~[illegible]~~ by taking filtered ~~[illegible]~~ limit (~~[illegible]~~ ind or pro object, depending on whether $\hat{\mathcal{C}}$ is covar or contrav.)

$$\mathcal{C}^{op} \longrightarrow \hat{\mathcal{C}}$$

$$X \longmapsto h^X = (Y \longmapsto \operatorname{Hom}_{\mathcal{C}}(X, Y))$$

Thus you get $\quad \operatorname{Pro}(\mathcal{C}) \hookrightarrow \hat{\mathcal{C}}$

This is ~~[illegible]~~ fully faithful, as image consists the left exact funs (resp fin. $\varprojlim$s).

$$\operatorname{Pro}(\mathcal{C}) = \text{cat of points in } \hat{\mathcal{C}}.$$

So now should be the time to understand what is a $\mathcal{G}$ torsor over a space B.

It should be a map of topoi ~~[illegible]~~ ($\mathcal{G}$ groupoid)

$$Sh_B \xleftarrow{u^*} \mathrm{Fun}(\mathcal{G}, \mathrm{Sets})$$

For **each point** of B you get a prorepresentable functor. But for a groupoid prorepresentable = representable. Thus for a groupoid you want

$$\mathcal{G}^{op} \longrightarrow \mathrm{Fun}(\mathcal{G}, \mathrm{Sets})$$

$$X \longmapsto \left(X' \mapsto h^X(X') = \mathrm{Hom}(X, X') \right)$$

What do you learn? ~~want to understand torsors in an arbitrary topos~~ By defn. a $\mathcal{G}$ torsor in a topos $\mathcal{T}$ is a ~~topos~~ map

$$\mathcal{T} \xleftarrow{f^*} \mathrm{Fun}(\mathcal{G}, \mathrm{Sets}) \qquad f^* \left| \begin{array}{l} \text{rt cont} \\ \text{left exact} \end{array} \right.$$

What does this mean for $\mathcal{T} = \mathrm{Sets}$. ~~Answer~~
Assume $\mathcal{G}$ conn. $\mathrm{Fun}(\mathcal{G}, \mathrm{Sets}) \simeq \{G\text{-sets}\}$

Canon map $\quad f^*(S) \longleftarrow f^*(G) \times^G S \quad$ which is an isom.

Next ask when its left exact - must amount to G^{op} acting freely on $f^*(G)$.

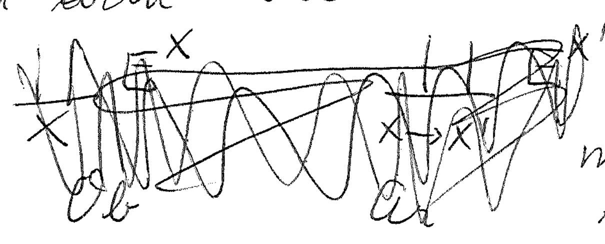
So what next?

Here seems to be the idea. You want to describe a topos map

$$\begin{tikzcd} Sh_B & \mathrm{Fun}(\mathcal{G}, \mathrm{Sets}) = \widehat{\mathcal{G}} \arrow[l, "f^*"'] \\ & \mathcal{G}^{op} \arrow[u, hook, "\text{(Yoneda)}"'] \end{tikzcd}$$

the composition is a functor from $\mathcal{G}^{op}$ to Sh_B, so first of all you have ~~[illegible] $\mathcal{G}^{op}$~~ an object in $\mathrm{Fun}(\mathcal{G}^{op}, Sh_B)$, which means a family of sheaves $E^X \in Sh_B$ assoc. to each $X \in \mathcal{G}$ and ~~morphisms~~ sheaf maps $\tilde{g} : E^X \longrightarrow E^{X'}$ in Sh_B assoc. to each map $g : X \longrightarrow X'$ in $\mathcal{G}$, (~~subject to~~ compatible with comp.) all this amounts to a $\mathcal{G}^{op}$-sheaf over B, the generalization of a $\mathcal{G}^{op}$-set from sets to Sh_B. So far you have described a $\boxed{\mathcal{G}^{op}\text{-sheaf over } B}$, but next you want the action to be free, which means ~~that~~ for each point of B that that the fiber $\mathcal{G}^{op}$-set is representable.

This should simplify. Again you want $\mathcal{G}^{op} \longrightarrow Sh_B$, i.e. $X \mapsto E^X$, $(X \to X') \mapsto (E^X \to E^{X'})$ compat with assoc. & id.



Now you want to recover your past understanding motivated by Graeme's simp. stuff.

$\mathcal{G}$ groupoid has a nerve which is a s. set.

First example. $\underset{N_2}{}$ G acting on F.

$$\begin{array}{ccccc} N_2 & & N_1 & & N_0 \\ G\times G\times F & \rightrightarrows & G\times F & \rightrightarrows & F \end{array}$$

Let's try to construct the topos map. $F \in \mathrm{Fun}(\mathcal{G}, \mathrm{Sets})$
$E \in \mathrm{Fun}(\mathcal{G}^{op}, \mathrm{Sets})$. There will be some sort of $\otimes$

$$\mathrm{Coker}\left\{ E \times_{\mathcal{O}} \mathcal{A} \times_{\mathcal{O}} F \rightrightarrows E \times_{\mathcal{O}} F \right\} = E \times^{\mathcal{G}} F$$

$$\coprod E_x \times \mathcal{A}_{xy} \times F_y$$

$$\boxed{\coprod_{x \leftarrow y} E_x \times F_y \rightrightarrows \coprod_x E_x \times F_x \longrightarrow E \times^{\mathcal{G}} F \;?}$$

$$\mathcal{G}^{op} \hookrightarrow \hat{\mathcal{G}} = \mathrm{Hom}(\mathcal{G}, \mathrm{Sets}) \xrightarrow{f^*} Sh_B$$

$$\begin{array}{ccccc} X & \longmapsto & h^X & \longmapsto & E^X \\ \downarrow & & \uparrow & & \uparrow \\ X' & & h^{X'} & & E^{X'} \end{array}$$

~~The~~ you get a functor $\mathcal{G}^{op} \longrightarrow Sh_B$ with a freeness property namely

$$E^X \times \mathrm{Hom}_{\mathcal{G}}(X, X') \longrightarrow \cancel{\mathrm{Hom}_{Sh_B}(E^{X'}, E^X)}\; E^{X'}$$

$$E \times_{\mathcal{O}} \mathcal{A} \longrightarrow$$

$\mathcal{G}$ groupoid, $\hat{\mathcal{G}} = \text{Fun}(\mathcal{G}, \text{Sets})$

Take $\mathcal{G}$ = groupoid $id_1 \circlearrowright 1 \rightleftarrows 2 \circlearrowleft id_2$

i.e. $\text{Hom}_{\mathcal{G}}(x, y) = pt$ for $x, y \in \{1, 2\}$.

~~Consider~~ A functor ϕ from $\mathcal{G}$ to any cat C consists of two objects $\phi(1)$, $\phi(2)$ and an isom. $\phi(1) \xrightarrow{\sim} \phi(2)$. An M_2 set consists of two sets and an isomorphism between them. There are two ~~representable M_2 sets~~ objects but they are isom, so the two rep functors are isom. There is one represen

---

Look at $\mathbb{Z}/2$ acting on $\mathbb{Z}/2$ by translation

In general look at G acting on a set S. The answer is $EG \times^G S$. A map from B to $EG \times^G S$ is essentially equivalent to a G torsor E over B together with an equivariant map $E \longrightarrow S$. Borel mixing

$$\begin{array}{ccccc} EG & \longleftarrow & EG \times S & & S \\ \downarrow & & \downarrow & & \\ BG & \longleftarrow & EG \times^G S & & \end{array}$$

Start again: You want disc G acting on F ~~to~~ give examples of étale groupoids.

~~the~~ groupoid (Γ, F) is top cat so has a classifying space by Graeme's theory: geom. real of nerve. ~~the~~

$$F \leftleftarrows \Gamma \times F \;\substack{\longleftarrow\\ \longleftarrow\\ \longleftarrow}\; \Gamma \times \Gamma \times F$$

$$pt \leftleftarrows \Gamma \;\substack{\longleftarrow\\ \longleftarrow\\ \longleftarrow}\; \Gamma \times \Gamma$$

yields then $E\Gamma \times^{\Gamma} F$ for the classifying space.

~~[illegible] discrete so that the groupoid is [illegible]~~

What does this classifying space classify?

$$\begin{array}{ccccc} E & \longrightarrow & E\Gamma \times F & \longrightarrow & F \\ \downarrow & & \downarrow & & \\ B & \longrightarrow & E\Gamma \times^{\Gamma} F & & \end{array}$$

a Γ-torsor over B tog. w. a Γ-map $E \longrightarrow F$.

Problem: Assembly map. If F is a point:

Go over it carefully - themes: Look at $\underline{\mathbb{R} \xrightarrow{\pi} \mathbb{R}/\mathbb{Z}}$.

Recall ~~the link with locally compact spaces~~ $\underbrace{\pi_! \, C_c(\mathbb{R})}$ use of sheaves of continuous functions, especially $\pi_! \mathcal{O}$.

What did you learn? $\mathbb{R} \xrightarrow{\pi} \mathbb{R}/\mathbb{Z}$, $\mathbb{R}$ is the total space of the principal bundle. You found that $\Gamma(\mathbb{R}/\mathbb{Z}, \pi_! \mathcal{O}) = C_c(\mathbb{R})$. Ideas occurring:

$$\begin{array}{ccc} \mathbb{R} \times \mathbb{Z} & \longrightarrow & \mathbb{R} \\ \downarrow & \text{cart} & \downarrow \\ \mathbb{R} & \longrightarrow & \mathbb{R}/\mathbb{Z} \end{array}$$

$C_c(\mathbb{R}) \rtimes \mathbb{Z}$ operates on $C_c(\mathbb{R})$

You remember getting a fin gen proj module over the 2-torus as well as over the crossproduct. mult. alg?

~~[illegible]~~ Dioceses philosophy.

Given a principal Γ-bundle $E \to B$, in other words a locally trivial family of Γ-torsors parametrized by points of B. You want to move from this fibre bundle to a kind of vector bundles, why, motivation from N.C. At this point maybe review N.C. Closed (orientable) manifold with fund. gp. π, Too hard, the NC affirms that the numbers obtained by pairing the L classes with cohomology from $B\pi$ are htpy invariants.

~~See if you can~~ Try to find assembly in the case of the groupoid (Γ, F) with classifying space $E\Gamma \times^{\Gamma} F$. First review the case $F = pt$.

Given
$$\begin{array}{c} E \\ \pi \downarrow \Gamma \\ B \end{array}$$
then there is an assoc. fibre bundle with fibre $\mathbb{C}\Gamma$ considered as free $\mathbb{C}\Gamma^{op}$ module with one generator. So over B you have a locally trivial fibre bundle with fibre the Γ^{op} module $\mathbb{C}\Gamma$. Now you wish to apply the Serre thm argument that this fibre bundle (when B has a finite partition of 1 over members of which the bundle is trivial) of Γ^{op} module is a retract of a f.g. free trivial Γ^{op}-module bundle.

A partition of unity involves cont. fns. on B. Ask first what a retract of $B \times \mathbb{C}\Gamma^{\oplus n} \to B$ should look like. It should be given by a an idempotent operator on the trivial Γ^{op} module bundle $B \times \mathbb{C}\Gamma^{\oplus n} \longrightarrow B$, that is an idempotent ~~element~~ section of $B \times M_n(\mathbb{C}\Gamma) \longrightarrow B$, i.e. an idempotent $p \in C(B, M_n(\mathbb{C}\Gamma))$

Serre thm argument. You start with a geometric situation, namely, a principal Γ-bundle $\pi: E \longrightarrow B$. You form the associated fibre bundle L with fibre the Γ^{op}-module $\mathbb{C}\Gamma$. There's a problem with topology on $\mathbb{C}\Gamma$ - maybe this is where the fine topology enters.

What exactly do you have: Locally on B you have isos $L \xleftarrow{\sim} B \times \mathbb{C}\Gamma$ which are related by left multiplication via a group element. So any topology on $\mathbb{C}\Gamma$ preserved by left mult by Γ should be OKAY. Next you want continuous sections of E, really a suitable space $C(L)$ of sections of L over B such that $C(L)$ is a $C(B)$ module. It seems that algebraically there is only one candidate; relative to a trivial.

$L \simeq B \times \mathbb{C}\Gamma$ a cont section $s(b) = \sum_\gamma f_\gamma(b)\gamma$

sum finite over compact subsets, coeffs are cont. fns. on B.

Repeat. You begin with principal Γ bundle $E \xrightarrow{\pi} B$ with B compact. Look at $C_c(E)$. This should be a module over $C(B) \otimes \mathbb{C}\Gamma$

So next look at Γ operating on F.

A torsor for (Γ, F) should be a Γ-torsor E/B tog. w. Γ-map $E \longrightarrow F$.

$$\begin{tikzcd} E \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{cart}"] & E\Gamma \times F \arrow[r] \arrow[d] & F \\ B \arrow[r] & E\Gamma \times^{\Gamma} F & \end{tikzcd}$$

Your idea is to involve $C(F)$ or $C_c(F)$

Really you should work only w.

$$\begin{tikzcd} E \arrow[r, "\phi"] \arrow[d] & F \\ B & \end{tikzcd}$$

~~The~~ Obvious idea is that cross product of Γ and $C_c(F)$ should be relevant, ~~should~~ ~~assembly the~~ assembly should yield

---

In the $\begin{smallmatrix} E \\ \downarrow \\ B \end{smallmatrix}$ case you get a bimodule for $C(B)$ and $\mathbb{C}\Gamma$. You actually construct ~~a retract~~ a retract of the trivial bundle with base B and fibre $\mathbb{C}\Gamma^{\oplus n}$ as Γ^{op} module. The retract is given a proj $p \in C(B, M_n(\mathbb{C}\Gamma))$

endo ring of the Γ^{op} module $\mathbb{C}\Gamma^{\oplus n}$

Next generalize to include F

Next to include F. Given

$$\begin{array}{ccc} E & \xrightarrow{\phi} & F \\ \downarrow & & \\ B & & \end{array} \qquad \phi(\xi g) = g^{-1}\phi(\xi)$$

You expect to have ~~B~~ a bimodule B with B operating on the left and $\Gamma \ltimes C_c(F)$ on the right. Take a $b \in B$, get

$$\Gamma \xrightarrow{\sim} E_b \longrightarrow F$$

Γ disc acting on space F, get groupoid whose torsors over a space B are Γ-torsors $E \to B$ tog w a Γ-map $E \xrightarrow{\phi} F$. ~~What you absolutely need~~ What you should look at maybe is fibres of the map

$$\begin{array}{c} E \times F \\ \downarrow \\ E \times^{\Gamma} F \end{array} \qquad \begin{array}{c} E \\ \downarrow \\ B \end{array} \qquad \begin{array}{c} E \times F \\ \downarrow \\ E \times^{\Gamma} F \end{array}$$

Start again: You are given space B, a Γ torsor E/B and a Γ-map $\phi : E \longrightarrow F$.

$$\begin{tikzcd} E \arrow[r, shift left] \arrow[d] & E \times F \arrow[l, shift left] \arrow[d, no head] \\ B \arrow[r, shift left] & E \times^{\Gamma} F \arrow[l, shift left] \end{tikzcd}$$

Without ϕ you can form $E \times^{\Gamma} F$ the assoc. fibre bundle with fibre F.

You don't seem to get somewhere. Go back to the groupoid $\mathcal{G} = M_2$ two objects objects $\{1, 2\} = \mathcal{O}$ map $\mathcal{O} \times \mathcal{O}$. You have topos.

Let's ~~[illegible]~~ go over again what you did for the groupoid M_2. You started with Groth topos viewpoint. If $\mathcal{G}$ is a groupoid then $\mathrm{Fun}(\mathcal{G}, \mathrm{Sets})$ ($\mathcal{G}$-sets) is a topos and a ~~[illegible]~~ topos map

$$Sh_B \xleftarrow{f^*} \mathrm{Fun}(\mathcal{G}, \mathrm{Sets})$$

should be ~~[illegible]~~ equivalent to a $\mathcal{G}$-torsor in Sh_B. ~~[illegible]~~ ~~[illegible]~~ suitably defined.

Picture to keep in mind: $(G\text{-Sets}) = \mathrm{Fun}(G, \mathrm{Sets})$

~~[illegible]~~ A stcart functor

$$\mathrm{Sets} \xleftarrow{f^*} G\text{-Sets}$$

should have the form $f^*(G) \times^G S \longleftarrow\!\shortmid S$, but $f^*(G)$ is a G torsor iff G^{op} action has one orbit trivial isot. gps. So a topos map $\mathrm{Sets} \xleftarrow{f^*} \{G\text{-sets}\} \Leftrightarrow f^*(G)$ is a G torsor.

Another ingredient is that you have the Yoneda embedding

$$\mathcal{G}^{op} \hookrightarrow \mathrm{Fun}(\mathcal{G}, \mathrm{Sets}) = \hat{\mathcal{G}}$$

$$X \longmapsto h^X = (Y \mapsto \mathrm{Hom}_{\mathcal{G}}(X, Y))$$

The points in $\hat{\mathcal{G}}$ should be the representable functors. Yes

Let's work out exactly how a $\mathcal{G}$ torsor ~~[illegible]~~ looks.

You want also to link with Graeme classifying space.

$$Sh_B \xleftarrow{f^*} \mathrm{Fun}(\mathcal{G}, \text{sets})$$

A topos map as above should be roughly the same as assigning to each point of B a point in $\mathrm{Fun}(\mathcal{G}, \text{sets})$ i.e. an object of $\mathcal{G}^{op}$.

You will need some examples.

~~This is hard but not~~

Try to guess what the structure should be

First do $\text{sets} \xleftarrow{f^*} \mathrm{Fun}(\mathcal{G}, \text{sets})$

Let f^* be a right cont. fun. from $\hat{\mathcal{G}}$ to sets.

Let $\mathcal{C}$ be a small category, you have basic pairing

$$\mathcal{C}^{op} \times \mathcal{C} \longrightarrow \text{sets}$$
$$(X, Y) \longmapsto \mathrm{Hom}(X, Y)$$

~~There should be an analog of dual~~

Let $\hat{\mathcal{C}} = \mathrm{Fun}(\mathcal{C}, \text{sets})$, and let $G : \hat{\mathcal{C}} \longrightarrow \text{sets}$ be a functor. You have

$$\mathcal{C}^{op} \xrightarrow{h} \mathrm{Fun}(\mathcal{C}, \text{sets}) \xrightarrow{G} \text{sets}$$
$$X \longmapsto \left(h^X : Y \to \mathrm{Hom}(X,Y)\right) \longmapsto G \circ h^X$$

This gives a $\mathcal{C}^{op}$-set. Need now a tensor product operation between the $\mathcal{C}^{op}$ set $G \circ h$ and $\mathcal{C}$-set F

$$G(h^X), \quad F(Y)$$

Start again. $\mathcal{C}$ small category, you have left and right $\mathcal{C}$-sets. Is there a kind of tensor product?? $R \otimes_{\mathcal{C}} L$

It should be constructed from $R(X) \times L(Y)$, disjoint union modulo equiv. relation

~~[illegible]~~ Obvious guess is

$$\coprod_{X \to Y} R(Y) \times L(X) \rightrightarrows \coprod_{X} R(X) \times L(X)$$

$$R(Y) \times L(X) \to R(X) \times L(X), \quad R(Y) \times L(X) \to R(Y) \times L(Y)$$

$$\coprod_{X \to Y \to Z}$$

$$\coprod_{X \to Y} R(Y) \times L(X) \rightrightarrows \coprod_{X} R(X) \times L(X)$$

~~[illegible]~~ What is the basic ~~[illegible]~~ idea? Any R in $\widehat{\mathcal{C}^{op}}$ is a colim of representable fems.

$$\coprod_{X \xrightarrow{u} Y,\ \eta \in R(Y)} h^Y \longrightarrow \coprod_{(X,\ \xi \in R(X))} h^X \longrightarrow R$$

$$\begin{array}{ccc} h^Y & \xrightarrow{u^*} & h^X \\ & \searrow^{\eta} & \downarrow{\scriptstyle u^*\eta} \\ & & R \end{array}$$

$$\operatorname{Hom}_{\operatorname{Fun}(\mathcal{C}^{op},\ \text{sets})}(R, T) =$$

to given $\phi: R \longrightarrow T$ you ~~[illegible]~~ give $\forall X,\ \phi_X : R(X) \longrightarrow T(X)$ such that $\forall\ X \xrightarrow{u} Y$ you have

$$\begin{array}{ccc} R(X) & \xrightarrow{\phi_X} & T(X) \\ {\scriptstyle u^*}\uparrow & & \uparrow{\scriptstyle u^*} \\ R(Y) & \xrightarrow{\phi_Y} & T(Y) \end{array}$$

commutes ~~[illegible]~~ ie $\forall \eta \in R(Y)$

$$\phi_X u^*_R \eta = u^*_T \phi_Y \eta$$

Start again. $R : \mathcal{C}^{op} \to$ sets, $L : \mathcal{C} \to$ sets.
Presentation of R by h?

?

$$\coprod_{X,\xi} h^X \longrightarrow R$$

?

$$\Vert$$

$$\coprod_X R(X) \times h^X \longrightarrow R$$

$R : \mathcal{C}^{op} \longrightarrow$ sets. You want presentation of R via representable functors $h_X(-) = \mathrm{Hom}(-, X)$

$$\cancel{\coprod_{(X,\xi \in R(X))} h_X \longrightarrow R}$$

you could form $\mathcal{C}/R$ whose obj are (X, ξ) with X in $\mathcal{C}$ and $\xi \in R(X)$, whose maps $(X,\xi) \longrightarrow (Y,\eta)$ are $u : X \to Y$ s.t. $\xi = u^*\eta$

You've forgotten so much.

$$\mathcal{C} \hookrightarrow \mathrm{Fun}(\mathcal{C}^{op}, \text{sets}) \longleftarrow$$

$$X \longmapsto h_X$$

so given $R \in$

you can talk about $\mathcal{C}/R$ i.e. $\mathrm{Obj}(\mathcal{C}/R)$ = pairs (X,ξ) $X \in \mathrm{Ob}\,\mathcal{C}$ $\xi \in R(X)$.

$$\coprod_X \underbrace{h_X(Y) \times R(X)}_{\mathrm{Hom}(Y,X) \times R(X)} \qquad R(Y)$$

$R : \mathcal{C}^{op} \to$ Sets. Claim that R is the ind. limit of the functor from $\mathcal{C}/R$ to $\widehat{\mathcal{C}^{op}}$ sending (X, ξ) to h_X equipped with the map $h_X \to R$ corresp to ξ.

$$\underbrace{\varinjlim_{\mathcal{C}/R} \left((X,\xi) \longmapsto h_X \right)}\ \xrightarrow{\sim}\ R$$

map given by $h_X \to R$ corresp to ξ in (X,ξ)

$$\mathrm{Hom}\left(\qquad , T \right) = \varprojlim_{\mathcal{C}/R} \left((X,\xi) \longmapsto T(X) \right)$$

$$\mathrm{Hom}(R, T)$$

given $\phi : R \to T$ and $X, \xi \in R(X)$

$\xi \in R(X) \to T(X)$

An element of $\varprojlim_{\mathcal{C}/R} \left((X,\xi) \longmapsto T(X) \right)$

is $\forall X \in \mathrm{Ob}\,\mathcal{C}$, $\xi \in R(X)$ an element $\alpha_{(X,\xi)} \in T(X)$ such that $\forall$ ~~[illegible]~~ $u : Y \to X$ one has

$u^* \alpha(X,\xi) = \alpha(Y, u^*\xi)$ ~~[illegible]~~

Define ~~[illegible]~~ $\alpha_X : R(X) \xrightarrow{\alpha_X} T(X)$ by $\alpha_X(\xi) = \alpha(X,\xi)$

Then given $u : Y \to X$

$$\begin{array}{ccc} R(X) & \xrightarrow{\alpha_X} & T(X) \\ \downarrow u^* & & \downarrow u^* \\ \alpha_Y : R(Y) & \xrightarrow{\alpha_Y} & T(Y) \end{array}$$

$u^* \alpha_X \xi = \alpha_Y u^* \xi$

~~So~~ now you know that given

$R \in \widehat{\mathcal{C}^{op}}$ then $\varinjlim_{\mathcal{C}/R} \left((X, \xi) \mapsto h_X \right) \xrightarrow{\sim} R$

Let now $L \in \hat{\mathcal{C}}$

Recall what you want to do. You are trying to find the analogue of the fact that a right continuous functor $Mod(R) \xrightarrow{F} Ab$ is given by $F(M) = F(R) \otimes_R M$.

~~[illegible]~~ You want to describe all rt cont. funs. $F : Fun(\mathcal{C}, Sets) \longrightarrow Sets$. Discuss examples. $\mathcal{C} =$ group G $\quad F : G\text{-sets} \to Sets$

$$F(G) \times^G S \longrightarrow F(S)$$

defined $(\xi, s) = F(g \mapsto gs)(\xi)$

here use $S = Hom_G(G, S)$.

$\mathcal{C}$ small cat. Construct tensor product, a set $R \times^{\mathcal{C}} L$ where R $\mathcal{C}^{op}$-set, L $\mathcal{C}$ set.

Idea is any R, L can be expressed as indlim of rep. funs

$$R = \varinjlim_{\mathcal{C}/R} h_X \qquad L = \varinjlim_{\substack{(Y,\ \xi \in L(Y)) \\ \xi : h^Y \to L}} h^Y \quad \mathcal{C}^{op}/L$$

$$R \times^{\mathcal{C}} L = \left(\varinjlim_{\mathcal{C}/R} h_X \right) \times^{\mathcal{C}} \left(\varinjlim_{\mathcal{C}^{op}/L} h^Y \right) = \varinjlim_{(X,Y) \in (\mathcal{C}/R) \times (\mathcal{C}^{op}/L)} Hom_{\mathcal{C}}(Y, X)$$

$$R \times^{\mathcal{C}} L = \varinjlim_{(X,Y)\in \underbrace{\mathcal{C}\times\mathcal{C}^{op}/(R\times L)}_{(\mathcal{C}/R)\times(\mathcal{C}^{op}/L)}} \mathrm{Hom}(Y,X)$$

$$\varinjlim \Big\{ \underbrace{(X, \overbrace{\xi : h_X \to R}^{\xi\in R(X)})}_{\text{cat } \mathcal{C}/R} \longmapsto h_X \Big\} = R$$

$$\varinjlim_{(X,\,\xi\in R(X))} \{(X,\xi) \longmapsto h_X\} = R$$

$$\varinjlim_{X,}$$

---

do it as follows. $\mathcal{C}/R$ cat with Ob $= (X,\xi)$ $X \in \mathrm{Ob}\,\mathcal{C}$, $\xi \in R(X)$ equiv $\xi : h_X \to R$ in $\mathcal{C}^{op}$-sets, Claim $\forall Z \in \mathrm{Ob}\,\mathcal{C}$

$$\varinjlim_{(X,\xi)\in\mathcal{C}/R} \mathrm{Hom}_{\mathcal{C}}(Z,X) \xrightarrow{\ \sim\ } R(Z)$$

$$u \longmapsto u^*(\xi)$$

$\xi = v^*\eta$

$\begin{tikzcd} X \arrow[rr, "v"] \arrow[dr, "\xi"'] & & Y \arrow[dl, "\eta"] \\ & R & \end{tikzcd}$

$\begin{tikzcd} \mathrm{Hom}_{\mathcal{C}}(Z,X) \arrow[rr, "v_*"] \arrow[dr] & & \mathrm{Hom}(Z,Y) \arrow[dl] \\ & R(Z) & \end{tikzcd}$

$u \mapsto u^*\xi$ $\qquad$ $u' \mapsto u'^*\eta$

$$u \longmapsto v_*u = vu \longmapsto (vu)^*\eta = u^*v^*\eta$$
$$= u^*\xi$$

Start again, C small cat. have

$$C^{op}\text{-sets} = \mathrm{Fun}(C^{op}, \text{sets}) \xleftarrow{h.} C$$

$$C\text{-sets} = \mathrm{Fun}(C, \text{sets}) \xleftarrow{h^{\cdot}} C^{op}$$

Let R in $\widehat{C^{op}}$, L in $\hat{C}$

A basic idea is that the representable functors are generators. Thus $\widehat{C^{op}}$ has gen $h_Y = \mathrm{Hom}(-, Y)$ for $Y \in \mathrm{Ob}\, C$. Similarly $\hat{C}$ has gen $h^Y = \mathrm{Hom}(Y, -)$ as generators.

To define $R \times^C L$. This is a set defined by universal "bilinearity property.

$\phi \in \mathrm{Hom}_{sets}(R \times^C L, T)$ should be a family of map $\phi_X : R(X) \times L(X) \longrightarrow T$ $\forall$ Ob X such that $\forall\ f : X \to Y$ one has

$$\begin{tikzcd}
 & R(X) \times L(X) \arrow[dr, "\phi_X"] & \\
R(Y) \times L(X) \arrow[ur, "f^* \times 1"] \arrow[dr, "1 \times f_*"'] & & T \\
 & R(Y) \times L(Y) \arrow[ur, "\phi_Y"'] &
\end{tikzcd}$$

so T should be the quotient of $\coprod_X R(X) \times L(X)$ gen. by the relns.

$$(f^*(\rho_Y), \lambda_X) = (\rho_Y, f_*(\lambda_X))$$

What does this mean. Suppose

$R = h_A$ $\qquad$ $L = h^B$

$$\mathrm{Hom}(R \times^{\mathcal{C}} L, T) \overset{?}{\simeq} \mathrm{Hom}_{\widehat{\mathcal{C}^{op}}}(R, \mathrm{Hom}_{\mathcal{C}}(L,T))$$

Let $\phi : R \longrightarrow \mathrm{Hom}_{\mathcal{C}}(L,T)$ be a map in $\widehat{\mathcal{C}^{op}}$

$f : X \to Y$

$$\begin{array}{ccc} \phi_X : R(X) & \longrightarrow & \mathrm{Hom}(L(X), T) \\ f^* \uparrow & & \uparrow \text{transp of } f_* \text{ on } L \\ \phi_Y : R(Y) & \longrightarrow & \mathrm{Hom}(L(Y), T) \end{array}$$

contrav. in X.

$$\begin{tikzcd}
 & R(X) \times L(X) \arrow[r, "\phi_X"] & T \\
R(Y) \times L(X) \arrow[ur, "f^* \times 1"] \arrow[rru, "\phi_X(f^* \times 1)" description] \arrow[dr, "1 \times f_*"'] & & \\
 & R(Y) \times L(Y) \arrow[uur, "\phi_Y"'] &
\end{tikzcd}$$

$\phi_Y(1 \times f_*)$

So this seems to work. ~~Thus must get~~

$- \times^{\mathcal{C}} L$ resp $\varinjlim$'s

$R \times^{\mathcal{C}} -$ " "

$$\mathrm{Hom}(h_A \times^{\mathcal{C}} L, T) = \mathrm{Hom}(h_A, \mathrm{Hom}_{\mathcal{C}}(L,T))$$

$$= \mathrm{Hom}_{\mathcal{C}}(L(A), T)$$

So progress is being made. But what does it all mean? What would be Graeme's viewpoint?

Go back to groupoid M_2

Go back over the ideas

Summarize events. From Cuntz you learned that ~~...~~ for $\Gamma = M_n$ there is an analogue of the universal alg A gen. by the components of a projection in a Γ graded algebra, and ~~it~~ A is a non-commutative n-simplex. You tried to study the question for a general Γ (ie. $\Gamma \amalg \{0\}$ has assoc. mult. with 0 absorbing) but ~~...~~ it seems you want Γ to be a groupoid. Also you know assembly exists for groupoids.

Then arises the question of ~~...~~ the classifying space for a groupoid $\mathcal{G}$. Your idea: to use Groth topos picture (at least when $\mathcal{G}$ is etale eg. a space with discrete gp acting.

You've now understood Groth classifying topos. for at least a discrete groupoid $\mathcal{G}$, but ~~...~~ ~~...~~ there's nothing simplicial about it.

Take $\mathcal{G} = M_2$ Ob has 2 elts, $Ar = Ob \times Ob$ with source + target given by projections. What does a $\mathcal{G}$ torsor look like over a space B. A $\mathcal{G}$ torsor is equivalent to a topos map

$$Sh_B \xleftarrow{f^*} \hat{\mathcal{G}} = \mathrm{Fun}(\mathcal{G}, \text{sets})$$

$$\bigcup \quad \text{Yoneda embedding}$$

$$\mathcal{G}^{op}$$

So your $\mathcal{G}$-torsor should amount to a ~~...~~ contra fun from $\mathcal{G}$ to Sh_B. $\forall \cancel{x} \in \mathcal{G}$ get

How should you picture a torsor for the discrete groupoid $\mathcal{G}$ over the space B? It should be a $\mathcal{G}^{op}$-sheaf, i.e. functor from $\mathcal{G}^{op}$ into Sh_B. So this means that you will have sheaves E_x over B, where x runs over $Ob\,\mathcal{G}$, and for each $x \xrightarrow{g} y$ you are given $g^*: E_y \longrightarrow E_x$.

Torsor

You are still stuck on the assembly stuff for a groupoid. Maybe because you are not paying enough attention to localization. Take a disc. groupoid $\mathcal{G}$ and define in sheaf terms what a $\mathcal{G}$-torsor over a space B is. The answer should be it is a functor $\mathcal{G}^{op} \longrightarrow Sh_B$ which is locally representable in a suitable ~~suitable~~ sense.

Suppose B = point. What is a functor $\mathcal{G}^{op} \longrightarrow$ sets.

$$\forall x \in Ob \quad \text{set } E_x$$
$$\forall x \to y \in Ar \quad E_y \longrightarrow E_x$$

$$\coprod_x E_x \leftleftarrows \coprod_{x \to y} E_y \;\substack{\longleftarrow\\ \longleftarrow\\ \longleftarrow}\; \coprod_{x \to y \to z} E_z$$

$$\coprod_x pt \leftleftarrows \coprod_{x \to y} pt \;\substack{\longleftarrow\\ \longleftarrow\\ \longleftarrow}\; \coprod_{x \to y \to z} pt$$

$$\| $$

$$Ob \underset{s}{\overset{t}{\leftleftarrows}} Ar \;\substack{\xleftarrow{pr_1}\\ \xleftarrow{\circ}\\ \xleftarrow[pr_2]{}}\; Ar\, {}_t\!\times_s Ar$$

The problem is ~~the~~ to understand assembly for an etale groupoid. First case is ~~for~~ a discrete groupoid $\mathcal{G}$. $\mathcal{G}$ consists of a set of objects and a set of arrows, denoted by $\mathcal{G}_0$, $\mathcal{G}_1$ say. together with various arrows

$$\begin{tikzcd} \mathcal{G}_0 \arrow[r, "id"', bend right] & \mathcal{G}_1 \arrow[l, shift right, "s"'] \arrow[l, shift left, "t"] & \mathcal{G}_1 \times_{(s,t)} \mathcal{G}_1 \arrow[l, "\circ"'] \end{tikzcd}$$

~~Upon~~ Maybe you should straighten this out so that you can deal with left and right $\mathcal{G}$ sets.

left $\mathcal{G}$-set is ~~a set F~~ is a set F over $\mathcal{G}_0$ $p: F \to \mathcal{G}_0$ tog. with $\mathcal{G}_1 \times_{(s,p)} F \longrightarrow F$ which is assoc + ident. comp. ~~So~~ any cat $\mathcal{C}$ leads to a arrow ring.

Again: Begin with $\mathcal{G}$ a discrete groupoid $(\mathcal{G}_0, \mathcal{G}_1, s, t, id, \circ, inv)$. Category $\mathrm{Fun}(\mathcal{G}, \mathrm{Sets})$ of $\mathcal{G}$ sets. ~~[illegible]~~ $\mathrm{Fun}(\mathcal{G}, \mathrm{Sets})$ is a topos.

~~[illegible]~~ Problem: Describe $\mathcal{G}$ torsor over ~~B~~ a space B. Possible approaches: topos map $Sh_B \xleftarrow{f^*} \{\mathcal{G}\text{-sets}\}$. If B a point, then you get a point in $\{\mathcal{G}\text{-sets}\}$, which should be the same as ~~[illegible]~~ an object of $\mathcal{G}$. The category of points in the topos $\{\mathcal{G}\text{-sets}\}$, namely ~~Fun(Sets, $\mathcal{G}$ sets)~~ the category of functors sets $\longleftarrow$ $\mathcal{G}$-sets which are right cont. and left exact should ~~be~~ the full subcat of representable functors: $\mathcal{G}^{op} \overset{h}{\hookrightarrow} \hat{\mathcal{G}}$

~~What makes sense~~ You have reached the viewpoint that a $\mathcal{G}$-torsor over a space B is some sort of map from B to $\mathcal{G}^{op}$, or from B to the categ of representable fun in $\hat{\mathcal{G}}$. Let's explore this idea, try to find a precise version. Take $\mathcal{G}$ to be the groupoid given by a group G, ~~i.e. one object~~ the category with one object $*$ whose self maps are elts of G.

If $\mathcal{G} = G$, then $\hat{\mathcal{G}} = G$-sets and the Yoneda embed $\mathcal{G}^{op} \longrightarrow \hat{\mathcal{G}}$ sends $*$ to the G set given by G operating on itself by left mult

~~The rough idea of a $\mathcal{G}$ torsor over B is a family~~

At some spot here it should become clear that to regard a $\mathcal{G}$-torsor as a "map" from B to $\mathcal{G}^{op}$, i.e. to regard a $\mathcal{G}$-torsor over B as a family param. by $b \in B$ of objects of $\mathcal{G}^{op}$, is not going to work.

Maybe shift to a covering viewpoint, i.e. give an open covering of B, say $B = U \cup V$, over U you give an object of $\mathcal{G}$

V ———

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begin again B top space $\mathcal{G}$ discrete groupoid You must define a $\mathcal{G}$-torsor over B. Idea: Such a torsor should provide a topos map

$$Sh_B \xleftarrow{f^*} \hat{\mathcal{G}} = Fun(\mathcal{G}, sets)$$

f^* is right continuous and left exact. Right cont should imply that f^* is given by twisting wrt a $\mathcal{G}^{op}$-sheaf over B, means a functor $\mathcal{G}^{op} \longrightarrow Sh_B$. The way this functor should arise is via the Yoneda embedding

$$\mathcal{G}^{op} \hookrightarrow \hat{\mathcal{G}} = \mathrm{Fun}(\mathcal{G}, \text{sets})$$
$$X \longmapsto h^X(Y) = \mathrm{Hom}_{\mathcal{G}}(X,Y)$$

This is clear because $\mathcal{G}^{op}$ is "dense" in $\hat{\mathcal{G}}$.

So your ~~torsor~~ should be

$$\mathcal{G}^{op} \xhookrightarrow{h} \hat{\mathcal{G}} \xrightarrow{f^*} Sh_B$$
$$\hat{\mathcal{G}} \ni L$$

$$\varinjlim h^X \xrightarrow{\sim} L$$

cat of $(X, h^X \xrightarrow{\xi} L)$ same as $\xi \in L(X)$

$$\Longrightarrow \quad f^*(L) \xleftarrow{\sim} \varinjlim_{X,\, \xi : h^X \to L} f^*(h^X) = (f^* \circ h) \times^{\mathcal{G}} L$$

Review: Problem: Understand $\mathcal{G}$ torsor over a space B, where $\mathcal{G}$ is a discrete groupoid

First def. A topos map: $Sh_B \xleftarrow{f^*} \hat{\mathcal{G}} \overset{def}{=} \mathrm{Fun}(\mathcal{G}, \text{sets})$ means f^* rt cont + left exact.

Examine $\hat{\mathcal{G}}$. Yoneda $\mathcal{G}^{op} \xhookrightarrow{h} \hat{\mathcal{G}}$, $X \mapsto h^X$

Given $f^* : \hat{\mathcal{G}} \longrightarrow Sh_B$, get $f^*h : \mathcal{G}^{op} \longrightarrow Sh_B$

whence a [redacted] fun. $\hat{\mathcal{G}} \longrightarrow Sh_B$, $L \mapsto f^*h \times^{\mathcal{G}} L$

and a map of funs. $f^*h \times^{\mathcal{G}} L \longrightarrow f^*L$. In fact

$$h^X \times^{\mathcal{G}} L \xrightarrow{\sim} L(X)$$

$$\mathrm{Hom}_{sets}(h^X \times^{\mathcal{G}} L, S) = \mathrm{Hom}_{\mathcal{G}^{op}}(h^X, \mathrm{Hom}_{sets}(L,S)) = \mathrm{Hom}_{sets}(L(X), S).$$

Still not clear. You first have to study $R \times^{\mathcal{G}} L$ for R a $\mathcal{G}^{op}$-set, L a $\mathcal{G}$-set.

$$R \times^{\mathcal{G}} L = \operatorname{Coker}\left\{ \coprod_{X \xrightarrow{f} Y} R(Y) \times L(X) \overset{f^* \times 1}{\underset{1 \times f_*}{\rightrightarrows}} \coprod_X R(X) \times L(X) \right\}$$

$$\operatorname{Hom}_{\text{sets}}(R \times^{\mathcal{G}} L, S) = \operatorname{Hom}_{\widehat{\mathcal{G}^{op}}}(R, \operatorname{Hom}_{\text{sets}}(L, S))$$
$$= \operatorname{Hom}_{\widehat{\mathcal{G}}}(L, \operatorname{Hom}_{\text{sets}}(R, S))$$

$f^* : \widehat{\mathcal{G}} \longrightarrow Sh_B$ given, compose with $\mathcal{G}^{op} \overset{h}{\hookrightarrow} \widehat{\mathcal{G}}$

$f^*h : X \longmapsto f^*(h^X)$ get fun $\mathcal{G}^{op} \longrightarrow Sh_B$ (generalization of $(\mathcal{G}^{op})^\wedge$)

$f^*h \times^{\mathcal{G}} L$ twisting of $\mathcal{G}^{op}$-sheaf f^*h by $\mathcal{G}$-set L

This should be a rtexact functor of L. Should there be a canonical map

$$f^*h \times^{\mathcal{G}} L \longrightarrow f^*L \; ?$$

Suffices to consider $L = h_X$ by using

$L = \varinjlim_{(X, \xi \in L(X))} h^X$. ~~Note~~ In effect

$$\exists \; \varinjlim_{(X,\xi)} f^*(h^X) \xrightarrow[\text{canon}]{} f^*(L) \quad \text{and}$$

$$\begin{array}{l} \varinjlim f^*h \times^{\mathcal{G}} h^X = (f^*h)(X) = \varinjlim f^*(h^X) \xrightarrow[\text{canon}]{} f^*(L) \\ \qquad \| \\ f^*h \times^{\mathcal{G}} L \end{array}$$

Repeat: Problem is to understand a $\mathcal{G}$ torsor over a space B, starting from the Groth defn of it as a topos map $Sh_B \xleftarrow{f^*} \hat{\mathcal{G}}$ where topos map means f^* rtcont and left exact means resp. finite $\underleftarrow{\lim}$'s (suffices to respect $-\times-$, kernels)

~~Now you have seen that f^* rtcont~~

Have Yoneda $h : \mathcal{G}^{op} \hookrightarrow \hat{\mathcal{G}}$, $X \mapsto h^X(Y) = \mathrm{Hom}(X,Y)$

so get $\mathcal{G}^{op} \xrightarrow{h} \hat{\mathcal{G}} \xrightarrow{f^*} Sh_B$, i.e. you have a $\mathcal{G}^{op}$ sheaf over B (gen. of $\mathcal{G}$-set), namely $f^* \circ h$. So the twisting $(f^* \circ h) \times^{\mathcal{G}} L \in Sh_B$ is defined, and is a rtcont functor $\hat{\mathcal{G}} \longrightarrow Sh_B$. You have constructed a canon ~~functor~~ map of functors

$$(f^* \circ h) \times^{\mathcal{G}} L \longrightarrow f^* L$$

which is an isom. $\iff$ f^* is rtcont.

(All this pertains to small cat $\mathcal{C}$)

Next remains meaning of f^* left exact.

You have this $\mathcal{G}^{op}$ sheaf in Sh_B in other words $\forall X$ a sheaf E_X and $\forall X \xrightarrow{f} Y$ a sheaf map $E_X \longrightarrow E_Y$, $\xi \longmapsto \xi f$ such that $(\xi f)g = \xi(fg)$ etc. ~~The left exactness~~

Now use what you know about groupoids. $\mathcal{G}$ splits into a disj union of conn. groupoids.

This analysis should suffice over any point of B. So to understand when $f^* L = (f^* \circ h) \times^{\mathcal{G}} L$ is left exact can suppose $B = pt$, so that $f^* \circ h$ is a $\mathcal{G}^{op}$ set.

So you problem is to understand when a $\mathcal{G}^{op}$-set R has the property that $L \longmapsto R \times^{\mathcal{G}} L$ is left exact.

example $R = h^X$, then $h^X \times^{\mathcal{G}} L = L(X)$ respects arb. $\varprojlim$'s.

example $R = h^X \amalg h^Y$. Then get

$$R \times^{\mathcal{G}} L = L(X) \amalg L(Y)$$

not compatible with products.

$$(L \times L)(X) \amalg (L \times L)(Y)$$

$$\big(L(X) \amalg L(Y)\big) \times \big(L(X) \amalg L(Y)\big)$$

$$(R_1 \amalg R_2) \times^{\mathcal{G}} (L \times L)$$

$$R_1 \times^{\mathcal{G}} (L \times L) \amalg R_2 \times^{\mathcal{G}} (L \times L)$$

$$R \times^{\mathcal{G}} (L \times L) \longrightarrow (R \times^{\mathcal{G}} L) \times (R \times^{\mathcal{G}} L)$$

~~getting too hard~~

You should first understand when a $\mathcal{G}^{op}$-set R is a point i.e. $L \longmapsto R \times^{\mathcal{G}} L$ is left exact. This happens if R is representable, ie of the form h_Y for some object Y.

[redacted] Let's take up the next step.
what does a $\mathcal{G}$-torsor over B really look like.

So it should be a $\mathcal{G}^{op}$-sheaf, i.e. contrav. functor from $\mathcal{G}$ to Sh_B such that each stalk is a representable contrav. functor. [redacted] You need now to come to grips with open sets.

Look at group case $\mathcal{G} = G$. ~~See how to~~

You have over B a sheaf E ~~of sets~~, an etale space with transitive free G^{op}-action. ~~no sections of E mea~~

Sheaf properties yield local triviality: Pick a point ξ of E over $b \in B$. By etaleness there is a ~~local~~ section s of E over a nbd U of b with $s(b) = \xi$, now move this around by G

How do I handle a groupoid $\mathcal{G}$?

Cont. with G.
$$\begin{array}{c} E \\ \downarrow \pi \\ B \end{array}$$
local homeom.

means $\forall \xi \in E$, $\exists$ open nbd U of ξ and an open nbd. V of $\pi(\xi)$, such π restricted to U is a homeom of U with V.

Alt. $\forall \xi \in E$ there is an open V containing $\pi(\xi)$ and a continuous $s : V \longrightarrow E$ $\pi s = id_V$ and $s(V)$ is open in E.

$$\begin{tikzcd} s(V) \arrow[r, phantom, "\overset{\text{open}}{\subset}"] \arrow[d, "\cong"] & E \arrow[d, "\pi"] \\ V \arrow[r, phantom, "\underset{\text{open}}{\subset}"] & B \end{tikzcd} \qquad \begin{tikzcd} U \arrow[r, hook, "\text{cont.}"'] \arrow[d, "\cong"] & E \arrow[d, "\pi"] \\ V \arrow[r, hook] & B \end{tikzcd}$$

At this point you have a functor

$\mathcal{G}^{op} \longrightarrow Sh_B$. First understand $B = pt$.

$R : \mathcal{G}^{op} \longrightarrow sets$ contrav functor call it R

$\mathcal{G}_0$ = object set $\mathcal{G}_1$ = arrow set.

$$\begin{tikzcd} R_0 \arrow[d] & & R_1 \arrow[d] \\ \mathcal{G}_0 & & \mathcal{G}_1 \arrow[ll, shift left] \arrow[ll, shift right] \end{tikzcd} \qquad \begin{tikzcd} X & Y \arrow[l, "f"'] & Z \arrow[l, "g"'] \arrow[ll, bend right, "fg"'] \end{tikzcd}$$

$\forall X \in \mathcal{G}_0$ get $\overset{set}{R(X)}$

$\forall \underbrace{f \in \mathcal{G}_1}_{f: X \to X}$ get $f^* : R(X) \longleftarrow R(Y)$

$\forall$

$\forall X \in \mathcal{G}_0$ have set $R(X)$

$\forall X \xleftarrow{f} Y \in \mathcal{G}_1$ have map $R(X) \xrightarrow{f^*} R(Y)$

$\forall X \xleftarrow{f} Y \xleftarrow{g} Z$ have $R(X) \xrightarrow{f^*} R(Y) \xrightarrow{g^*} R(Z)$

$(fg)^* = g^* f^*$

$$\coprod_{X \in \mathcal{G}_0} R(X) \leftleftarrows \coprod_{X \leftarrow Y \in \mathcal{G}_1} R(X) \qquad \coprod_{X \leftarrow Y \leftarrow Z} R(X)$$

$R(X) \quad X \leftarrow Y \quad Y \leftarrow Z$

$$\coprod_{X \in \mathcal{G}_0} R(X) \qquad \coprod R(X) \times_{(\pi,t)} \mathcal{G}_1 \qquad \left(\coprod R(X)\right) \times_{(\pi,t)} \mathcal{G}_1 \times_{(s,t)} \mathcal{G}_1$$

So you learn the **following**, that a functor $\mathcal{C}^{op} \xrightarrow{R}$ Sets yields a nerve

$$\coprod_{X_0} R(X_0) \Leftarrow \coprod_{X_0 \leftarrow X_1} R(X_0) \Lleftarrow \coprod_{X_0 \leftarrow X_1 \leftarrow X_2} R(X_0)$$

You have written down the nerve for the category whose objects are ~~element of~~ $(X, \xi \in R(X))$ and whose maps $(Y, \eta) \longleftarrow (X, \xi)$ are $Y \xleftarrow{f} X$ such that $f^*\eta = \xi$, seems to get cat $\mathcal{C}/R$

i.e. objects are $(X, h_X \xrightarrow{\xi} R)$ maps

$$\begin{array}{ccc} X, & h_X \xrightarrow{\xi} & R \\ f\downarrow & \downarrow & \| \\ Y, & h_Y \xrightarrow[\eta]{} & R \end{array}$$

$$\begin{array}{ccc} u & & u^*\xi \\ \mathrm{Hom}(Z,X) & \xrightarrow{\xi} & R(Z) \\ f_* \downarrow & & \| \\ \mathrm{Hom}(Z,Y) & \xrightarrow{\eta} & R(Z) \\ fu & \longmapsto & (fu)^*\eta \\ & & \| \\ & & u^* f^*\eta = u^*\xi. \end{array}$$

fibred cat over $\mathcal{C}$ given by the contravariant functor R

$R(X)$ $R(Y)$

$X \xrightarrow{f} Y$

$\mathcal{G}^{op} \hookrightarrow \hat{\mathcal{G}} \xrightarrow{f^*} Sh_B$ gives analog of a $\mathcal{G}^{op}$-set but in the topos Sh_B. ~~Look at case~~ At each point of B you get a representable fun R isom to h_X for some Ob X.

You have a contravariant functor R from $\mathcal{G}$ to Sh_B, to sets, when B is replaced by a point.

$\forall X$ ~~have~~ set $R(X)$

$\forall u : X \to Y$ have map $u^* : R(Y) \to R(X)$ assoc id

$\exists Z, \xi \in R(Z)$ such that $\mathrm{Hom}(X, Z) \xrightarrow{\sim} R(X)$

$u \longmapsto u^*\xi$

how to keep direction of arrows straight. Think of

$$R \times^{\mathcal{C}} L = \mathrm{Coker}\left\{ \coprod_X R(X) \times L(X) \leftleftarrows \coprod_{u : X \to Y} R(Y) \times L(X) \right\}$$

$$\begin{array}{ccc} R(Y) \times L(X) & \longrightarrow & R(X) \times L(X) \\ \downarrow & & \downarrow \\ R(Y) \times L(Y) & \longrightarrow & R \times^{\mathcal{C}} L \end{array}$$

$\coprod_X R(X) \times L(X)$ $\qquad$ $\coprod_{X,Y} R(Y) \times \overset{u}{\mathrm{Hom}}(X, Y) \times L(X)$

$(u^*\rho, \lambda) = (\rho, u_*\lambda)$ $\qquad$ (ρ, u, λ) $\quad$ This shouldn't be important

~~$R \times_{\mathcal{G}_0} \mathcal{G}_1$ the correct~~

How do you get the ~~correct~~ best notation

$$R \times L \leftleftarrows R \times G \times L \Lleftarrow R \times G \times G \times L$$

$$R \times_{\mathcal{G}_0} L \leftleftarrows R \times_{\mathcal{G}_0} \mathcal{G}_1 \times_{\mathcal{G}_0} L \Lleftarrow R \times_{\mathcal{G}_0} \mathcal{G}_1 \times_{\mathcal{G}_0} \mathcal{G}_1 \times_{\mathcal{G}_0} L$$

You should now be able to finish this off, namely, to handle a $\mathcal{G}$ torsor over a space B using an open covering. Suppose given a $\mathcal{G}$-torsor R over B i.e. a functor $\mathcal{G}^{op} \to Sh_B$ such that ~~each stalk~~ the stalk at each point of B is a representable contrav. functor on $\mathcal{G}$.

R is a sheaf on sets over B that is an etale space $\pi : R \to B$ over B. It comes with $R \to B \times \mathcal{G}_0$ which means $R = \coprod_{x \in \mathcal{G}_0} R_x$

where each R_x is an etale space over B. R is partitioned according to the objects of $\mathcal{G}$. Then you have a right action

$$\begin{tikzcd} R \arrow[dr, "s"'] & \times & \mathcal{G}_1 \arrow[dl, "t"] \arrow[rr, "s"', bend right] & \longrightarrow R \arrow[dr, "s"] & \\ & \mathcal{G}_0 & & & \mathcal{G}_0 \end{tikzcd}$$

Ask what R representable means at a point b

Idea: $\mathcal{G}^{op}/R$ is a category, there should be an equivalence between R representable and $\mathcal{G}^{op}/R$ having a final object

$$\begin{array}{ll} & (X, \xi : X \to Y) \\ & \quad \downarrow \xi \\ \exists! & (Y, id : Y \to Y) \end{array}$$

form a fibred cat over $\mathcal{G}$

latest idea is that $\mathcal{C}/R$ has a final object iff R is representable.

Here $R \in \mathrm{Fun}(\mathcal{C}^{op}, \text{sets})$ and $\mathcal{C}/R$ is the fibred category over $\mathcal{C}$ with fibre $R(X)$ over X:

$$Ob(\mathcal{C}/R) = \{(X,\xi) \mid X \in Ob\,\mathcal{C},\ \xi \in R(X)\}$$

$$Hom_{\mathcal{C}/R}\big((X,\xi),(X',\xi')\big) = \{f \in Hom_{\mathcal{C}}(X,X') \mid f^*\xi' = \xi\}.$$

Suppose $R(X) = h_Y(X) = Hom_{\mathcal{C}}(X,Y)$

$$Hom_{\mathcal{C}/R}\big((X,\xi : X \to Y),(X', \xi' : X' \to Y)\big)$$

$$= \{f \in Hom_{\mathcal{C}}(X,X') \mid \underbrace{f^*\xi'}_{\xi' f} = \xi\}$$

$$Hom_{\mathcal{C}/R}\big((X,\xi : X \to Y),(Y, id_Y : Y \to Y)\big)$$

$$= \{f : X \to Y \mid \underset{f}{\underset{\parallel}{id_Y f}} = \xi\}$$

$$= \{\xi : X \to Y\}.$$

$\mathcal{C}/R$ cat of $\{X, h_X \xrightarrow{\xi} R\}$, maps

$$\begin{tikzcd} h_X \arrow[rr, "f"] \arrow[dr, "\xi"'] & & h_{X'} \arrow[dl, "\xi'"] \\ & R & \end{tikzcd}$$

cat of $(X, h_X \xrightarrow{\xi} h_Y)$

obvious final elt $(Y, h_Y \xrightarrow{id} h_Y)$