

Try very hard to clean this up. How?

~~Begin by constructing the Montu context~~

Construct a Montu context. ~~Start with~~

$Y = jB$  ~~(Q)~~ Can you adjoin  $i, j$

Consider the Montu context ~~(D)~~

$X = Bx$   $B$  (=  $M_2$  graded alg) generated by  $B$  in degree 22, an element  $i$  degree 21 an element  $j$  degree 12, satisfying relations

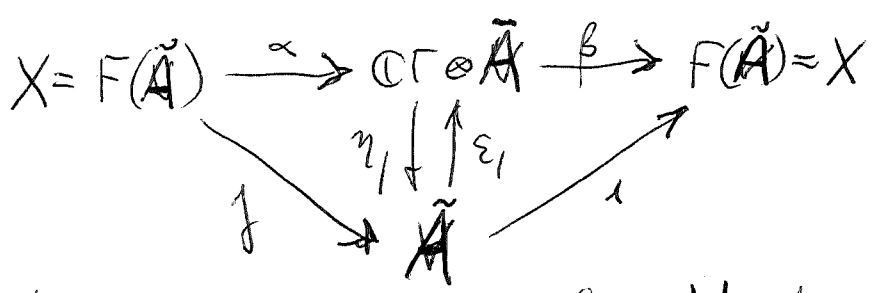
$ij = h$  ~~(D)~~  $\sum_t h_t i = i$   $\sum_s j h_s = j$

Start again ~~(D)~~ Let's begin with  $A$ , construct  $(X, Y, \langle x, y \rangle)$  as dual pair over  $A$ .

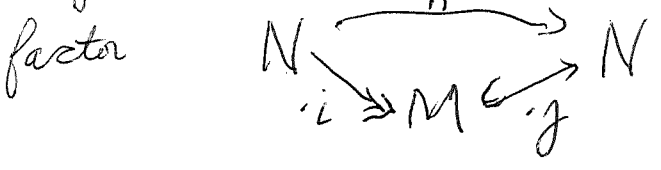
~~This is the question about dual~~ ~~to~~ ~~This is~~

$V \mapsto F(V) = p(\mathbb{C}\Gamma \otimes V)$

$p(s)j = jsh$



right module version Let  $N$  be a right red  $B$ -mod.



$p(s) = jsc$

or  $p(s) = nhsc$

$s \notin \mathbb{I}, 0 = hsh = c(jsc) \underset{\text{surj}}{\neq} \underset{\text{inj}}{}$

$\iota p(s) = hsc$
$p(s)j = jsh$

Right module picture.  $N$   $B^{\circ}P$ -module  
 such that  $NB = N$ , equiv.  $N$  is a  $\Gamma^{\circ}P$  module  
 equipped with an operator  $h: n \mapsto nh$  sat  
 $hsh = 0$  for  $s \in \underline{P}$ ,  $\sum nshs^{-1} = n \quad \forall n \in N$ .

Put  $M = \text{Im} \{ M \xrightarrow{h^s} M \}$ , whence  
 canon maps  $N \xrightarrow{\cdot \iota} M \xleftarrow{\cdot \iota} N \quad h = \iota$

(Observe: When you form  $(M \otimes \mathbb{C}\Gamma)_p$ , the  
 image of the projection  $p$ , ~~at~~ you can say let

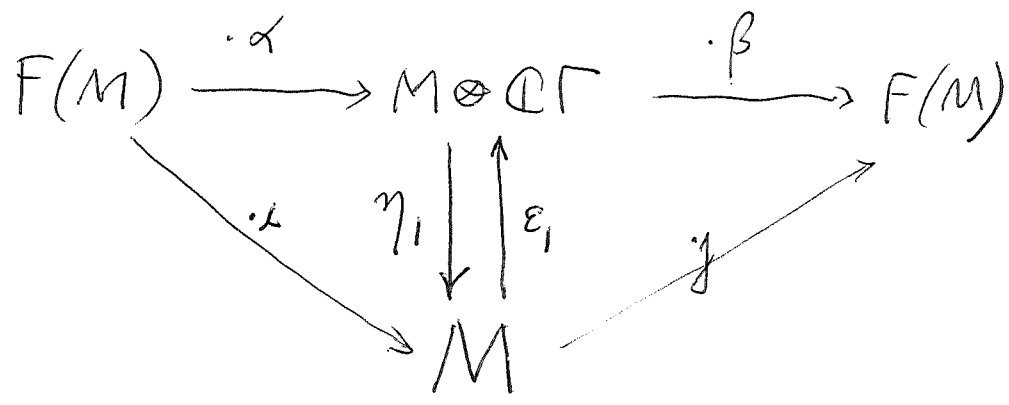
$$F(M) = \text{Im} \{ M \otimes \mathbb{C}\Gamma \xrightarrow{p} M \otimes \mathbb{C}\Gamma \}$$

whence there are canonical maps of  $B^{\circ}P$ -modules

$$F(M) \xrightarrow{\cdot \alpha} M \otimes \mathbb{C}\Gamma \xrightarrow{\cdot \beta} F(M)$$

such that  $\alpha\beta = \iota_{F(M)}$ ,  $\beta\alpha = p$

Next: the diagram



$$M \otimes \mathbb{C}\Gamma = \left\{ \sum_s m(s) \otimes s \mid m: \Gamma \rightarrow M \text{ fin. supp} \right\}$$

$$\left( \sum_{\underline{t}} m(\underline{t}) \otimes \underline{t} \right) u = \sum_s m(\underline{t}u^{-1}) \otimes \underline{t}$$

$$m j = m \varepsilon_1 \beta = (m \otimes 1) \beta$$

$$\begin{array}{ccccc}
 F(M) & \xrightarrow{\alpha} & M \otimes \mathbb{C}\Gamma & \xrightarrow{\beta} & F(M) \\
 & \searrow \scriptstyle \iota = \alpha \eta_1 & \eta_1 \downarrow \uparrow \varepsilon_1 & \nearrow \scriptstyle \jmath = \varepsilon_1 \beta & \\
 & & M & & 
 \end{array}$$

$$\left( \sum_t m(t) \otimes t \right) \beta = \left( \sum_t m(t) \varepsilon_1 t \right) \beta = \sum_t m(t) \jmath t$$

Let  $n\alpha = \sum_t m(t) \otimes t \Rightarrow n\alpha t^{-1} \eta_1 = m(t) = \overline{nt^{-1} \iota}$

$$n\alpha = \sum_t nt^{-1} \iota \otimes t$$

$$\left( \sum_t m(t) \otimes t \right) \beta = \sum_t m(t) \jmath t$$

$$n\alpha\beta = \sum_t nt^{-1} \jmath t = n$$

$$\left( \sum_s m(s) \otimes s \right) \beta \alpha = \left( \sum_s m(s) \jmath s \right) \alpha = \sum_{s,t} m(s) \overbrace{\jmath s t^{-1} \iota}^{p(st^{-1})} \otimes t$$

So there is the formula for  $p$ .

$$\sum_s m(s) \otimes s \xrightarrow{p} \sum_{s,t} m(s) p(st^{-1}) \otimes t$$

action of  $(\mu u)(s) = m(su^{-1})$

Recap. Given  $A$ -module  $V$  ~~and~~ you get  $\Gamma$ -inv idemp  $p$  on  $\mathbb{C}\Gamma \otimes V$  and  $F(V) = \text{Im} \{ p: \mathbb{C}\Gamma \otimes V \rightarrow \mathbb{C}\Gamma \otimes V \}$ , where canonical maps

$$\begin{array}{ccccc}
 F(V) & \xrightarrow{\alpha} & \mathbb{C}\Gamma \otimes V & \xrightarrow{\beta} & F(V) \\
 & \searrow \scriptstyle i & \eta_1 \downarrow \uparrow \varepsilon_1 & \nearrow \scriptstyle \jmath & \\
 & & V & & 
 \end{array}
 \quad \left( \begin{array}{l} \beta\alpha = 1_{F(V)} \\ \alpha\beta = p \end{array} \right)$$

such that  $F(V)$  becomes a  $B$ -mod. with  $h = \jmath$   
 $hsh = \iota p(s) \jmath$

~~More~~ You are beginning with  $A$ , more precisely with left  $A$ -modules  $V$  and right  $A$ -modules  $M$ , then you have a  $\left\{ \begin{array}{l} p: \Gamma\text{-compatible idemp on } \mathbb{C}\Gamma \otimes V \\ \cdot p = \Gamma^{op} \text{ " " " } M \otimes \mathbb{C}\Gamma \end{array} \right.$

$$p\left(\sum_t t \otimes v(t)\right) = \sum_s s \otimes \sum_t p(s^{-1}t)v(t)$$

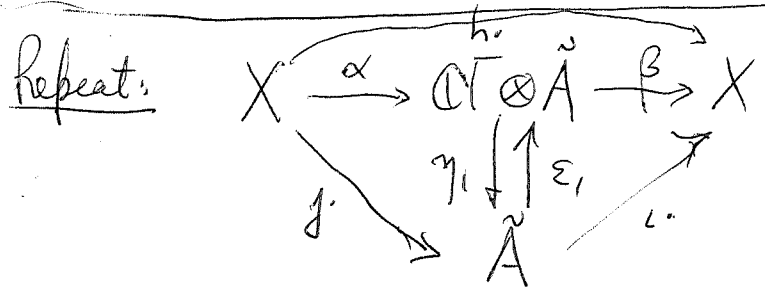
$$\left(\sum_s m(s) \otimes \text{[scribble]}\right) p = \sum_t \left(\sum_s m(s) p(s \circ t^{-1})\right) \otimes t$$

So you have left  $B$ -module  $p(\mathbb{C}\Gamma \otimes V)$   
 right  $\text{---}$   $(M \otimes \mathbb{C}\Gamma)p$

Important cases.

$B, A^{op}$ bimodule	$p(\mathbb{C}\Gamma \otimes A) = X$
$A, B^{op}$ $\text{---}$	$(A \otimes \mathbb{C}\Gamma)p = Y$

Now you want to construct pairings  $Y \times X \rightarrow A$  and  $X \times Y \rightarrow B$



$$\alpha \{ = \sum_s s \otimes f s^{-1} \alpha$$

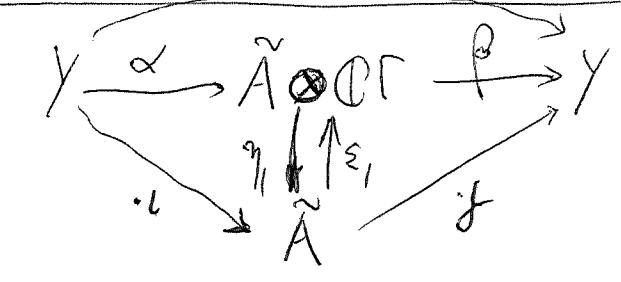
$$\beta\left(\sum_t t \otimes a(t)\right) = \sum_t t c a(t)$$

$$f \alpha \{ = \sum_s s y s^{-1} \alpha = \alpha$$

$$\alpha \beta \left(\sum_t t \otimes a(t)\right) = \alpha\left(\sum_t t c a(t)\right)$$

$$= \sum_s s \otimes \sum_t f s^{-1} t c a(t)$$

$$(p \alpha)(\{) = \sum_t p(s^{-1}t) a(t)$$



$$\eta \alpha = \sum_t \eta t^{-1} c \otimes t$$

$$\left(\sum_s a(s) \otimes s\right) \beta = \sum_s a(s) f s$$

$$\eta \alpha \beta = \sum_s \eta t^{-1} c f t = \eta$$

$$\left(\sum_s a(s) \otimes s\right) \beta \alpha = \sum_t \sum_s a(s) f s t^{-1} c \otimes t$$

$$(a \beta)(\{) = \sum_s a(s) p(st^{-1})$$

$$\begin{pmatrix} A & Y \\ X & B \end{pmatrix}$$

You know that any Morita equivalence corresponds to a firm Morita context. Go back to

$$M(A) \begin{matrix} \xrightarrow{F} \\ \xleftarrow{G} \end{matrix} M(B)$$



$\mathcal{M}$

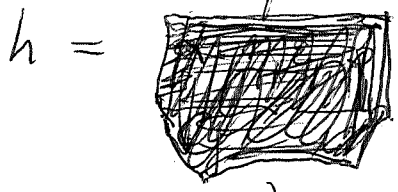
$$F(V) = p(\underbrace{\mathbb{C}\Gamma \otimes V}_X) = p(\mathbb{C}\Gamma \otimes \tilde{A}) \otimes_A V$$

$$G(W) = hW$$

There seems to be a viewpoint where it is unimportant to worry about the type of module, whether firm or reduced etc.

$F(V) = p(\mathbb{C}\Gamma \otimes V)$  equipped with  $B$ -module structure obtained from  $\Gamma$  action and operator

where



$$h = \underbrace{\beta \varepsilon_i \eta_j \alpha}_{ij} = \underbrace{ij}$$

$$\alpha\beta = p$$

$$p\left(\sum_t t \otimes v(t)\right) = \sum_s s \otimes \sum_t p(s^{-1}t) v(t)$$

$$h \underbrace{h}_{ij} = \underbrace{ij}_{ij} = \beta \varepsilon_i \eta_j \alpha \underbrace{\beta \varepsilon_i \eta_j \alpha}_{ij} = p(1 \otimes v) = \sum_s s \otimes p(s^{-1}) v$$

$$\eta_i \underbrace{\alpha \beta \varepsilon_i}_p v = \eta_i \underbrace{p}_{ij} \sum 1 \otimes v$$

$$= \eta_i \underbrace{p}_{ij} \sum_s s \otimes \underbrace{\beta \varepsilon_i \eta_j \alpha}_{ij} p(s^{-1}) v = p(t) v$$

Given a left B-module

W put  $V = hW$  with

$p(s)hw = hshw$ . Then  $p(s) = 0$  for  $s \notin \Phi$

as  $hsh = 0$ , and  $\sum_t p(st^{-1})p(t)hw = \sum_t hst^{-1}ht hw = hshw = p(s)hw$ .

Similarly given a right B module N put

$M = Nh$  with  $nhp(s) = nhsh$ . Again  $p(s) = 0$

if  $s \notin \Phi$  and  $nh \sum_t p(st^{-1})p(t) = \sum_t nhst^{-1}ht h = nhsh = (nh)p(s)$

Back to left modules,  $W = \sum_s shW \Rightarrow$

$hW = \sum_s hshW = \sum_s p(s)hW = {}_A hW$ . And ~~if~~

$p(s)hw = hshw = 0 \quad (\forall s)$ , ~~then~~

~~then~~  $hw = \sum_s shshw = 0$ . So

${}_A hW = 0$ , and  $hW^S$  is reduced.

Right picture is the same, namely

$$N = \sum_s Nhs \quad \therefore Nh = Nhsh$$

$$N = \sum_s Ns^{-1}hs = \sum_s Nhs$$

$$Nh = \sum_s Nhsh = \sum_s Nh p(s)$$

if  $nhp(s) = nhsh = 0 \quad \forall s$ , then

$$0 = \sum_s nhshs^{-1} = nh$$

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$$X \otimes_A M \quad (V \otimes_A Y) \otimes_B (X \otimes_A M) = V \otimes_A M$$

$$\begin{pmatrix} A & Y \\ X & B \end{pmatrix}$$

$$M \otimes_A Y \otimes_B W^X$$

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$M$   $A^e$ -mod

Yesterday, made a simplification, namely  
 $W \mapsto hW$  with  $p(s)hw = hshw$

~~also~~  $W = \sum s h W \Rightarrow \sum h W = \sum_s h s h W = \sum_s p(s) h W$   
 also  $hw = \sum_s s^{-1} h s h w = \sum_s s^{-1} p(s) h w$  shows  
 $p(s) \frac{hw}{hw} = 0 \Rightarrow hw = 0. \therefore hW$  is a red  $A$ -module

~~Not given~~ ~~Approximate~~ This looks good as  
 far as constructing ~~the~~ Morita context  $\begin{pmatrix} hBh & hB \\ Bh & B \end{pmatrix}$

~~Also you have to do it to~~ ~~you have~~  $Bh$  is  
~~a~~  $B, A$  bimodule  ~~$Bh$~~

$Bh$   $B, A$  bimod  
 $hB$   $A, B$  bimod  
 $hBh$   $A$  bimodule, there's an  $A$ -bimodule surj  
 $hB \otimes_B Bh \rightarrow hBh$

You want to identify  $hBh$  with  $A_{red}$

Idea:  $\begin{pmatrix} hBh & hB \\ Bh & B \end{pmatrix} = \begin{pmatrix} gBh & gB \\ B_h & B \end{pmatrix}$

what's important is that  $Bh$  is both a ~~submodule~~ <sup>subspace</sup>  
 of  $B$  and a quotient space.

Put into words:  $Bh$  is a left  $B$ -module  
 generated by the element  $h$ , i.e.  $B/B_h$  ??

~~Bh~~  $Bh = \{bh \in B \mid b \in B\} \simeq B/B_h$   
 where  $B_h = \{b' \in B \mid b'h = 0\}$ .  $Bh$  is  
 a principal  $B$ -module. Now when you  
 factor  $\cdot h = \cdot \iota_j : B \xrightarrow{\iota} Bh \xrightarrow{j} B$  what does  
 ~~$B_h$~~  mean?  
 $Bh = B \iota_j$

$$B \xrightarrow{\cdot h = \iota} Bh \xrightarrow{\text{inc} = j} B$$

$Bh = B \iota$  means  $\{bh \mid b \in B\} \longrightarrow \underbrace{(B)\iota}_{\text{image of } B \text{ under } \iota}$

Similarly  ~~$B_h$~~   
 $h \cdot = \cdot j : B \xrightarrow{h \cdot = j} hB \xrightarrow{\text{inc} = \iota} B$

$$\{hb \mid b \in B\} = hB = j(B) \quad \text{IDEA.}$$

Introduce the obvious module for the Morita  
 context.

$$\left( \begin{array}{cc} jB & jB \\ B & B \end{array} \right) \left( \begin{array}{c} hB \\ B \end{array} \right)$$

Your problem is this: You have the equivalence  
 $M(A) \simeq M(B)$  for left modules, ~~you translate~~  
 ~~$M(A)$~~  want the corresp. M.C.  $\begin{pmatrix} A & Y \\ X & B \end{pmatrix}$ , you  
~~know~~ know what <sup>good</sup> modules for this MC are.

If you're starting with  $B$ , then good modules  
 have the form  $\begin{pmatrix} hW \\ W \end{pmatrix}$



So what to do? You have  $M(A) \simeq M(B)$  <sup>457</sup>  
 and want the corresp. M.C.  $D = \begin{pmatrix} A & Y \\ X & B \end{pmatrix}$ . What  
 are the good D-modules? Answer  $\begin{pmatrix} hW \\ W \end{pmatrix}$ ,  
 with  $W$  a good  $B$ -module. You've seen that such  
 a pair  $\begin{pmatrix} hW \\ W \end{pmatrix}$  comes equipped with ~~maps~~ <sup>canon</sup> maps  
 $\iota: hW \hookrightarrow W$  and  $j: W \twoheadrightarrow hW$  such that  
 $h = \iota j$ . You expect  $D$  to be  
 $D = \begin{pmatrix} A & jB \\ Bi & B \end{pmatrix}$  ~~where  $B$  is  $B$~~   
 $jB = \boxed{hB}$

By symmetry  $h: B \xrightarrow{\iota} Bh \xrightarrow{j} B$   $h: B \xrightarrow{j} hB \xrightarrow{\iota} B$   
 so that  $B\iota = Bh$  so that  $jB = hB$

~~Ultimately I think you want to form~~

Repeat. You have  $M(A)$  equiv. to  $M(B)$ , to  
 find the assoc. M. cont.  $D = \begin{pmatrix} A & Y \\ X & B \end{pmatrix}$ . What are good  
 D-modules: Answer  $\begin{pmatrix} hW \\ W \end{pmatrix}$ ,  $W$  a good  $B$  module,  
~~where this pair is~~ <sup>where this pair is</sup> equipped with the factorization  
 $h = (\iota \circ j): W \xrightarrow{j} hW \xrightarrow{\iota} W$ .  $jW = hW, j^2W = h^2W$

~~Our~~ Our idea for  $D$  is  $\begin{pmatrix} jBi & jB \\ Bi & B \end{pmatrix} = \begin{pmatrix} hBh & hB \\ Bh & B \end{pmatrix}$

What's going on? Mainly you have the functor  
 $W \mapsto hW$  with  $p(s)hw = hshw$ . Another  
 notation  $W \mapsto hW$  with  $p(s)jw = i(jsi)jw$

meaning is ~~not~~ not yet clear. Write

$$D = \begin{pmatrix} jB_l & jB \\ B_l & B \end{pmatrix}$$

think in this ↑ way

~~the same way~~

$$D = \begin{pmatrix} hBh & hB \\ Bh & B \end{pmatrix}$$

work with

$$(b_i)(hb_2) \stackrel{df}{=} b_ihb_2$$

$$(b_1)(jb_2) = b_1hb_2$$

Another ~~version~~ version you have a dual pair over B given by hB, Bh, and  $\langle b_i, h, hb_2 \rangle = b_ihb_2$

Then get Morita context  $D = \begin{pmatrix} hB \otimes_B Bh & hB \\ Bh & B \end{pmatrix}$

(degrees. Let  $(X, Y, \langle y, x \rangle)$  be a dual pair over A. Use  $M_2$  grading to construct a M context.

$$\begin{array}{ccc} D & \xrightarrow{\Delta} & M_2 \otimes D \\ a & \longmapsto & e_{11} \otimes a \\ x & \longmapsto & e_{21} \otimes x \\ y & \longmapsto & e_{12} \otimes y \end{array}$$

When you give the dual pair you use 4 of the products  $a_1a_2, ay, xa, yx$

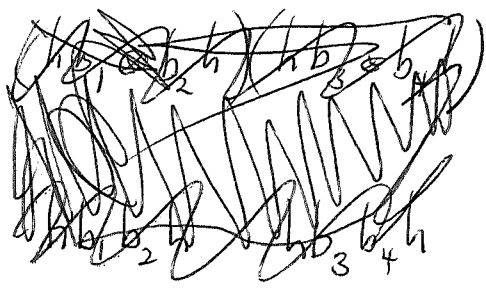
subject to ~~the relations~~ requiring that the products in D are the specified ones, and you want  $XX=0, YY=0$ .

Back to  $D = \begin{pmatrix} hBh & hB \\ Bh & B \end{pmatrix}$ . Check this

is a ~~Morita~~ Morita context. ~~(M, h)~~

You have dual pair over  $B$  given by  $hB, Bh$ ,  
 and the pairing  $\langle b_1h, hb_2 \rangle = b_1hb_2$ . Associated to  
 this dual pair is an algebra  $hB \otimes_B Bh$  with  
 product given by  $(hb_1 \otimes b_2h)(hb_3 \otimes b_4h)$   
 $= hb_1 \otimes (b_2hb_3)b_4h = hb_1b_2hb_3 \otimes b_4h$

Consider  $hB \otimes_B Bh \longrightarrow hBh$   $hb_1 \otimes b_2h \longmapsto hb_1b_2h$



Define ~~the~~ product  
 in  $hBh$  by  
 $(hb_1h)(hb_2h) = hb_1hb_2h$

$$(hb_1 \otimes b_2h, hb_3 \otimes b_4h) \longrightarrow hb_1 \otimes b_2hb_3b_4h$$

$$\downarrow \qquad \qquad \qquad \downarrow$$

$$(hb_1b_2h, hb_3b_4h) \longrightarrow hb_1b_2hb_3b_4h$$

~~What~~ What should be true in this ~~stupid~~  
 situation?

What been accomplished. dual pair  $hB, Bh$ ,  $\langle b_1h, hb_2 \rangle = b_1hb_2$   
 yield M. cont.

$$\begin{pmatrix} hB \otimes_B Bh & hB \\ Bh & B \end{pmatrix} \longrightarrow \begin{pmatrix} hBh & hB \\ Bh & B \end{pmatrix}$$

$(hb_1 \otimes b_2h)hb_3 = hb_1b_2hb_3$  you should know  
 $\parallel$   
 $hb_1b_2hb_3$  that  $hB \otimes_B Bh \rightarrow hBh$   
 is surjective kernel killed by the ring  
 $(hb_1 \otimes b_2h)b_3h$   
 $hb_1b_2hb_3h$

~~Step 1~~. Repeat again. Given a ring  $B$  and an element  $h \in B$ , you <sup>get</sup> ~~can define~~ a Morita context

$$\begin{pmatrix} hBh & hB \\ Bh & B \end{pmatrix}$$

~~Step 2~~ How: Have dual pair over  $B$  given by  $hB, Bh$ ,  $\langle b_1h, hb_2 \rangle = b_1hb_2$ , whence a Morita context

$$\begin{pmatrix} hB \otimes_B Bh & hB \\ Bh & B \end{pmatrix}$$

~~and then you notice~~, a surjective (assuming  $B^2 = B$  which is true when  $B = BhB$ ) ~~alg map~~  $hB \otimes_B Bh \rightarrow hBh$

$$\begin{array}{ccc} (hb_1 \otimes b_2h) \cdot (hb_3 \otimes b_4h) & = & (hb_1 \otimes b_2hb_3b_4h) \\ \downarrow & & \downarrow \\ (hb_1b_2h) \cdot (hb_3b_4h) & = & hb_1b_2hb_3b_4h \end{array}$$

Last step is to consider  $X \otimes_A Y \rightarrow B$  i.e.

$$Bh \otimes_{hBh} hB \rightarrow B$$

$$b_1h \otimes_A hb_2 \mapsto b_1hb_2$$

$$\begin{array}{ccc} (b_1h \otimes hb_2) \cdot (b_3h \otimes hb_4) & = & b_1h \otimes hb_2b_3hb_4 \\ \downarrow & & \downarrow \\ (b_1hb_2) \cdot (b_3hb_4) & = & b_1hb_2b_3hb_4 \end{array}$$

~~the~~ In the  $\Gamma$ -situation, the map

$$Bh \otimes_{hBh} hB \rightarrow B \text{ should be an isomorphism}$$

because of  $\sum_s shs^{-1}b = b$ . Given  $\sum b_i h \otimes h b_i'$

in  $Bh \otimes_{hBh} hB$  ~~with image~~ with image  $0 = \sum_i b_i h b_i'$

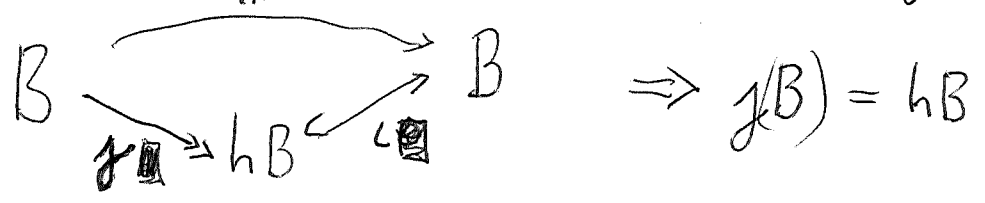
Then 
$$\sum_s \sum_i shs^{-1} b_i h \otimes h b_i' = \sum_i b_i h \otimes h b_i'$$

$$s \otimes \sum_i h s^{-1} b_i h b_i' = 0$$

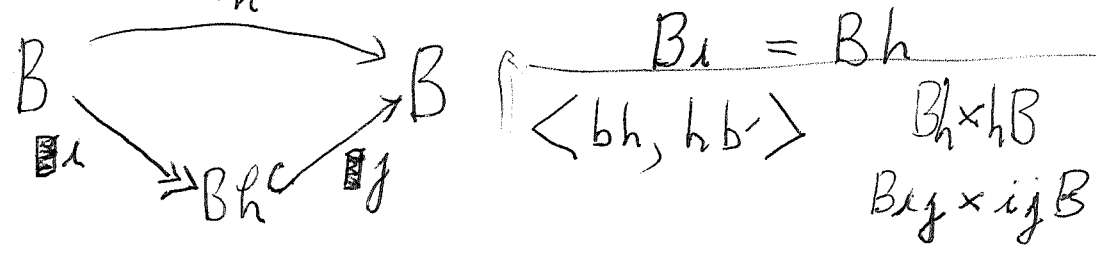
What else happens?  $\begin{pmatrix} hBh & hB \\ Bh & B \end{pmatrix}$

$$\langle b_i h, h b_i' \rangle = b_i h b_i'$$

next maybe ~~you~~ you want to understand role of  $h = \iota \circ j$ . You think that in order to control  $hB$  you need "the" canonical fact of  $h$ .



Similarly to control  $Bh$  you need  $\cdot h$



logic  $\square$   $hB = \iota j B \simeq j B$   
 $Bh = B_i j \simeq B_i$

because  $\iota$  is an inclusion  
 because  $j$  is an inclusion

So exactly what remains to understand.

One thing would be an explicit identification of  $Bh$  with  $p(\mathbb{C}\Gamma \otimes A)$ .

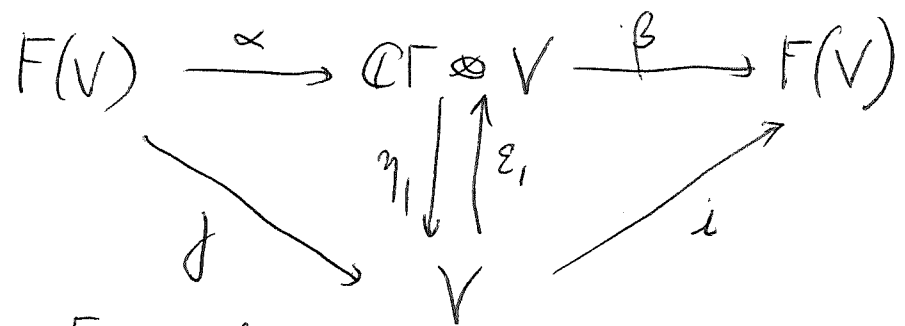
~~At the moment~~ ~~How to proceed?~~ The idea: The basic Morita equivalence for left modules is

$$\begin{array}{ccc} W \longrightarrow hW \longleftarrow hB \otimes_B W \\ m(B) \longrightarrow m(A) \end{array}$$

$p(\mathbb{C}\Gamma \otimes_A \tilde{A}) \otimes_A p(\mathbb{C}\Gamma \otimes V) \longleftarrow V$ . So certainly you have a construction  $V \mapsto F(V) = p(\mathbb{C}\Gamma \otimes V)$ , as this is right cont & exact in  $V$  you get canon. isom

$$F(\tilde{A}) \otimes_A V \xrightarrow{\sim} F(V)$$

where  $F(\tilde{A})$  is flat <sup>firm</sup> over  $A^{\text{op}}$  Diagram



$\alpha, \beta$   $\Gamma$ -module maps.

$$\beta \alpha = \text{id}_{F(V)}$$

$\alpha \beta = p$  where

$$\alpha \omega = \sum_{s \in \Gamma} s \otimes \gamma s^{-1} \omega$$

$$\beta \left( \sum_t t \otimes v(t) \right) = \sum_t t i v(t)$$

$$\beta \alpha \omega = \sum_s s \gamma s^{-1} \omega = \omega$$

$$\alpha \beta \left( \sum_t t \otimes v(t) \right) = \sum_s s \otimes \sum_t \frac{p(s^{-1}t) v(t)}{\gamma s^{-1}t i}$$

Go over the Mor. equiv.

- $A = \text{red. } A\text{-modules } V \text{ with } p(s) \text{ operators}$  (supp cond, idemp)
- $B = \text{--- } B\text{-modules} = \Gamma\text{-mods with } h$  (supp cond,  $hsh=0$ , part. of 1)
- $\mathcal{D} = \text{cat of } (V, W, \iota, j)$ ,  $V \text{ v.s.}, W \Gamma\text{-module}$   
 $\iota: V \hookrightarrow W, j: W \twoheadrightarrow V$  (supp:  $\sum s_i = 0 \quad s \notin \Phi$ , part:  $\sum_s s \iota s^{-1} w = w$ )

$\mathcal{D} \rightarrow B$  sends  $(V, W, \iota, j)$  into the  $\Gamma\text{-mod } W$  equipped with  $h = \iota j$  on  $W$ . But  $\iota j = h$ ,  $\iota j = h$ ,  $\sum s_j = 0 \Rightarrow$  can't win  $V = hW$ ,  $\iota = \text{inc}$ ,  $j = h$

$\therefore \mathcal{D} \rightarrow B$  equivalence of categories.

Next. define  $\mathcal{D} \rightarrow A$  by  $(V, W, \iota, j) \mapsto V$  eq. w.  $p(s) = \sum s_i$   
 supp cond.  $\sum s_i = 0 \rightarrow hsh = 0$   
 idemp.  $\sum_t j s t^{-1} \iota j t = \sum s_i$   ~~$\sum p(s) V = \sum j s_i V$~~

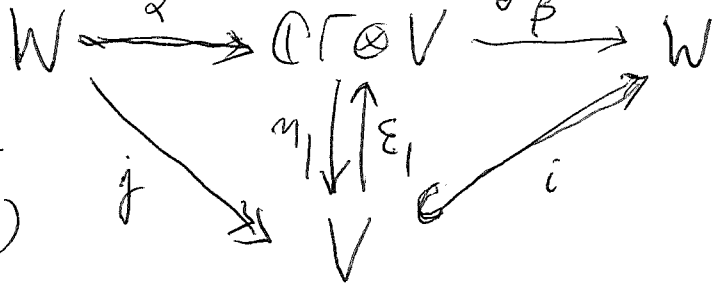
$$\sum p(s) V = \sum j s_i j W = j \sum_{shs^{-1}W} s h W = j W = V$$

~~$\sum p(s) h w = \sum s h w = h s h w$  vs.  $\sum_s s h w = \sum_s s h w$~~

Assume  $p(s) h w = 0 \quad \forall s$ . then  $\sum p(s) h w = \sum s h w$

$$\Rightarrow h w = \sum_s s^{-1} h s h w = 0. \quad \therefore V \text{ reduced.}$$

Next. - show  $W$  determined uniquely by  $V$  /  $i$  extends to a  $\Gamma$ -map



$$\beta \sum t \otimes v(t) = \sum_t t \iota v(t)$$

$$\epsilon_1(v) = 1 \otimes v$$

$$\eta_1(\sum t \otimes v(t)) = v(1)$$

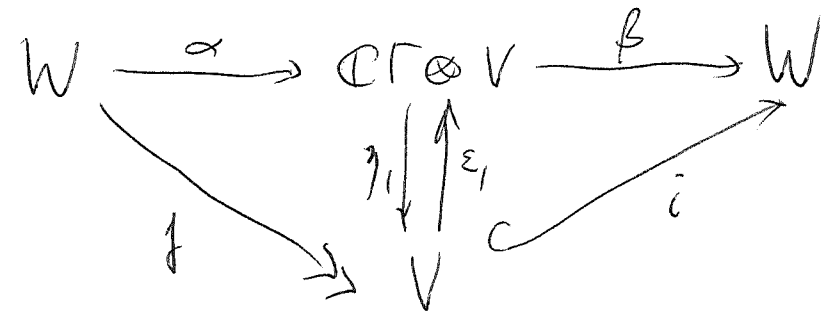
$$\alpha w = \sum_s s \otimes \sum_j s^{-1} w$$

$$\beta \alpha w = \sum_s s \iota \sum_j s^{-1} w = w$$

$$\alpha \beta \sum_t t \otimes v(t) = \alpha \sum_t t \iota v(t) = \sum_s s \otimes \sum_t \underbrace{j s^{-1} t \iota v(t)}_{p(s^{-1}t)}$$

Are the details of the Mor. eq. clear?

~~any given  $(V, W, \iota, j)$~~  You showed that given  $(V, W, \iota, j)$  in  $\mathcal{D}$  there is a diagram.



where  $\alpha, \beta$   $\Gamma$ -maps,  $\beta\alpha = \text{id}_W$ ,  $\alpha\beta = p$  on  $\mathbb{C}\Gamma \otimes V$   
 $f = \eta_1 \alpha$  and  $i = \beta \varepsilon_1$ . Thus  $\alpha: W \cong \text{Im}(p)$ ,  
*canonical*

What should you be trying to say?

Given  $(V, W, \iota, j)$  in  $\mathcal{D}$ , then there are canonical  $\Gamma$ -maps  $W \xrightarrow{\alpha} \mathbb{C}\Gamma \otimes V \xrightarrow{\beta} W$  such that  $\beta\alpha = \text{id}_W$ ,  $\alpha\beta = p$  on  $\mathbb{C}\Gamma \otimes V$ ,  $\iota = \beta \varepsilon_1$ ,  $f = \eta_1 \alpha$

~~any given  $(V, W, \iota, j)$~~  Given  $(V, W, \iota, j)$  in  $\mathcal{D}$

Define  $W \xrightarrow{\alpha} \mathbb{C}\Gamma \otimes V \xrightarrow{\beta} W$  by  
 $\alpha w = \sum_s s \otimes j s^{-1} w$        $\beta(\sum_t t \otimes v(t)) = \sum_t t \iota v(t)$ .

Then  $\beta\alpha w = \sum_s s \iota j s^{-1} w = w$   
 $\alpha\beta(\sum_t t \otimes v(t)) = \sum_s s \otimes \sum_t \underbrace{j s^{-1} t \iota}_{p(s^{-1}t)} v(t)$

Conclude  $\alpha, \beta$  identify  $W$  with the retract of  $\mathbb{C}\Gamma \otimes V$  corresp to the proj op  $p$

$p(s) = j s \iota$  ;  $p(s) = 0 \Rightarrow h s h = 0$   
 but if you want to go from  $\mathbb{B}$ -mod  $W$  to  $V$  *in general and want  $\sum_s j s \iota j = 0 \Rightarrow j s \iota = 0$*   
*you need  $\iota$  very  $j$  very.*



You want to start with A module V

~~My problem~~ Having reviewed the Mor. equivo. you now want to understand why  $Bh = p(\mathbb{C}\Gamma \otimes \tilde{A})$  better, why  $Bh = p(\mathbb{C}\Gamma \otimes A_n)$  where  $A_n = A/\{a | a_n = aA = 0\}$ .

Notice there is a difficulty.  $p(\mathbb{C}\Gamma \otimes V)$ ?

~~Start again. Giving  $V = \begin{pmatrix} A & Y \\ X & B \end{pmatrix}$  you know~~

Start with left modules. An A-module V determines a B-module  $X(V) = p(\mathbb{C}\Gamma \otimes V)$  which has the form  $X \otimes_A V$  where  $X = p(\mathbb{C}\Gamma \otimes \tilde{A})$ . The problem is that ~~you~~ you can't recover V from  $X(V) = p(\mathbb{C}\Gamma \otimes \tilde{A})$ , unless you specify that V is reduced (or firm). Now you can recover the reduced version of V by applying h. So there should be a canon isom.  $hX(V)$  where V is reduced:

$A \cdot V = V, A \cdot V = 0$ . So take  $V =$  ~~left A~~ reduced left A-module version of  $\tilde{A} \supset A \rightarrow A/A$ . It seems that if ~~you~~  $X(A) = Bh$ , then

$hX(A)$  ~~should~~ should be ~~both~~ both  $hBh$  and  $A/A$

$$X(V) = Bh \otimes_A V \qquad X(\tilde{A}) = Bh$$

$$\left( \begin{array}{l} hB \\ Bh \quad B \end{array} \right)$$

Review yesterday's idea about <sup>the algebra</sup>  $A = hBh$  being reduced as both left and right  $A$ -module.

(Put down this morning's idea ~~about~~ about strictly reduced Morita context)

Begin with  $B$  which has  $\Gamma$  <sup>left + right</sup> action ~~and~~ giving rise to  $\sum sh_s s^{-1}$  on either side. Form  $hB$  with  $p(s) = hs$  acting on the left

$$(B = \sum shB \Rightarrow hB = \sum hshB = \sum p(s)hB$$

~~hb~~  $p(s)hb = \cancel{hs}hb = hshb = 0, \forall s$  ~~hb~~

~~hb~~  $hb = \sum s^{-1}hshb \Rightarrow hb = 0$ .) Thus

~~hb~~ Similarly  $A = hBh$  should be reduced left  $A$ -module.  $Bh = \sum shs^{-1}Bh = \sum shBh$

$$\Rightarrow hBh = \sum_s hshBh = \sum_s p(s)hBh$$

Also if  $p(s)hbh = hshbh = 0 \forall s$ , then

$$hbh = \sum_s s^{-1}hshbh = 0.$$

~~So you learn that~~  
~~hbh~~ Different notation

Let  $p(s) = f s i$  meaning

$$f s i (hbh) = hshbh$$

$$bh g s i = bhsh$$

$$g s i hb = hshb$$

Check again you have



$$D = \begin{pmatrix} hBh & hB \\ Bh & B \end{pmatrix}$$

Let  $A = hBh$  with product  $(h b_1 h)(h b_2 h) = h b_1 h b_2 h$

assume  $BhB = B$ . Then

⊗  $AY = (hBh)(hB) = hBhB = hB = Y$

$XY = (Bh)(hB) = BhB = B$

$YX = hBBh \subset hBh$

$hBBh \supset hBhBh = hBh$



$YBX = hBh Bh$

start again with a ring  $B$  an elt  $h \in B$  assume  $BhB = B$ . Then get

⊗  $BX = BBh$

~~$YX = YX$   
 $YX = YX$   
 $YX = YX$   
 $YX = YX$   
 $YX = YX$~~

$$D = \begin{pmatrix} hBh & hB \\ Bh & B \end{pmatrix}$$

a Morita context (strictly idempotent)

~~$BhB = B$   
 $BhB = B$   
 $BhB = B$   
 $BhB = B$   
 $BhB = B$~~

Can do most of this via dual pair  $hB, Bh, \#$

$(b_1 h) \times (h b_2) = b_1 h b_2$ .  $B = BhB \subset B^3$

Next what? #

Keep on going: Is  $D$  reduced?

$$BX = BBh \supset B^3h \supset BhBh = Bh = X$$

$$YB = hBB \supset hBhB = hB = Y.$$

Look at  $A = hBh$  acting via  $*$  as  $\overset{hB}{\text{}} = Y$

Let  $y = \overset{hB}{\text{}} hb'$  be such that  $hbhb'h = 0$  for all  $b$ . ~~So you find the~~

You would like to ~~take~~ take  $a' = hb'h$  such that  $\forall a = hbh \quad aa' = hbhb'h = 0$ , ~~and~~ and conclude that  $a' = hb'h = 0$

So why does this work in the  $\Gamma$  context? Something involving  $hb'h = \sum s^{-1} hshb'h$

At some point you use  $hb' = \sum s^{-1} hshb'$

~~BB~~ You seem to be using the partition of unity. Any individual element ~~is~~  $hb' \in hB$  can be reconstructed from the  $p(s)hb'$ , in fact from  $\sum_s s^{-1} p(s) = \sum_s s^{-1} y s$  ???

~~Start~~ Start again with  $\begin{pmatrix} A & Y \\ X & B \end{pmatrix} = \begin{pmatrix} hBh & hB \\ Bh & B \end{pmatrix}$

The Morita context is strictly idemp when  $BhB = B$

$$XY = BhB = B \quad YB = hBB \supset hBhB = hB = Y$$

$$BX = BBh \supset BhBh = Bh = X$$

$$B = B^2 < B^3 \Rightarrow B = B^2$$

$$YX = hBBh = hBh = A$$

$$XA = (Bh) * (hBh) = BhBh = Bh = X$$

$$AY = (hBh) * (hB) = hBhB = hB = Y$$

~~Observation:~~ Observation: ~~Can h be a multiplier~~ Can h be a multiplier of B?

Continue with the example: ~~start with hBh~~  
 $A = hBh$        $A^2 = hBhBh = hBh = A$

Thus  $BhB = B \Rightarrow A^2 = A$

Now what about

$A = \{ hb'h \mid \forall b \quad hbhb'h = 0 \}$ ? What happens in our example? ~~hBh~~

You first do  $W = B$ ,  $A$ -module  $hB$  with  $p(s)hb' = hshb'$ . To see  $hB$  is reduced you use  $hb' = \sum s^{-1}hshb'$ . Similarly you take case  $W = Bh$ ,  $A$ -module  $hBh$  with  $p(s)hb'h = hshb'h$ .

Then  $p(s)hb'h = 0 \quad \forall s \Rightarrow hshb'h = 0 \quad \forall s \Rightarrow hb'h = \sum s^{-1}hshb'h = 0$   
 So what? Look at  $V = hB$   $p(s)hb' = hshb'$ , seems to involve  $\sum_s s^{-1}p(s)hb' = \sum s^{-1}hshb' = hb'$ .

Something new here with  $\sum s^{-1}p(s)$ . Reminds me of ~~the~~ graded algebras.

$$A \xrightarrow{\Delta} \mathbb{C}\Gamma \otimes A$$

$$A_s \ni a_s \longmapsto s \otimes a_s$$

~~What about~~

$$V \xrightarrow{\Delta} \mathbb{C}\Gamma \otimes V$$

? Maybe this will become clearer with repetition.

$$p(s)v \longmapsto s \otimes p(s)v$$

Try again to show  $hB$  as left  $A = hBh$  is reduced.

Take  $hb' \in hB \ni p(s)hb' = hshb' = 0 \quad \forall s$

$$\text{Then } 0 = \sum_s s^{-1}p(s)hb' = \sum_s s^{-1}hs hb' = hb'$$

$BhB = B, \{b' \mid hbhb' = 0 \quad \forall b'\}$ . You want

this to be zero, sort of a nondegenerate pairing  
 $hb \times hb \rightarrow hb$  better  $hBh \times hBh \rightarrow hBh$   
 $A \times A \rightarrow A$

~~works because of partition of~~ Leave alone.

Back to the example

$$\begin{pmatrix} A & Y \\ X & B \end{pmatrix} = \begin{pmatrix} hBh & hB \\ Bh & B \end{pmatrix}$$

$Y = hB$  and  $A = hBh$   
are left  $A$ -reduced.  
 $X = Bh$  and  $A = hBh$   
are right  $A$ -reduced

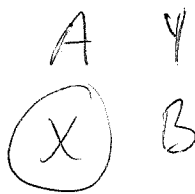
Why.  $(hbh) * (hb') = hbhb' = 0$  for all  $b$

take  $b = s$  get  $hshb' = 0 \quad \forall s$

$$0 = \sum_s s^{-1}hshb' = hb'$$

let's check this:

Remaining problem: To see clearly why  $Bh \simeq p(\mathbb{C}\Gamma \otimes A)$ .



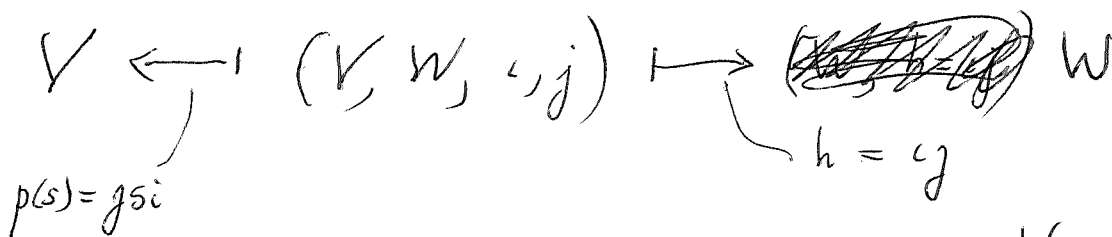
You feel that this should follow easily from the Morita equivalence

$$A \leftarrow D \rightarrow B$$

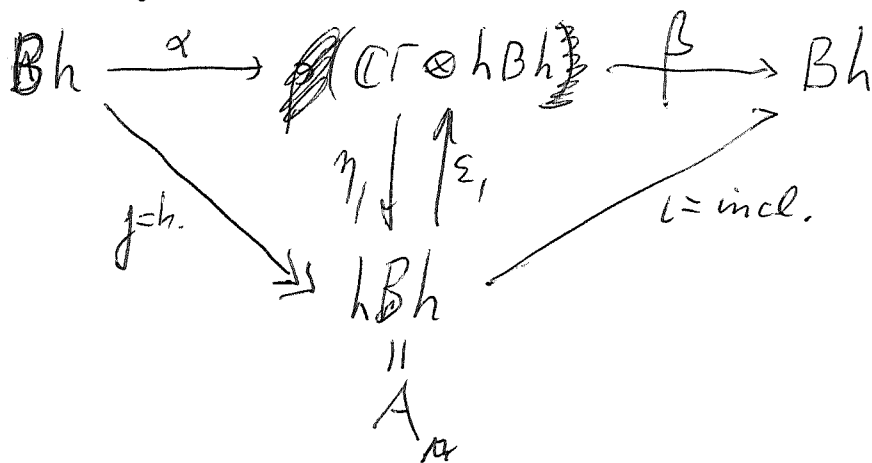
of left modules. ~~On the other~~

$$V \mapsto p(\mathbb{C}\Gamma \otimes V)$$

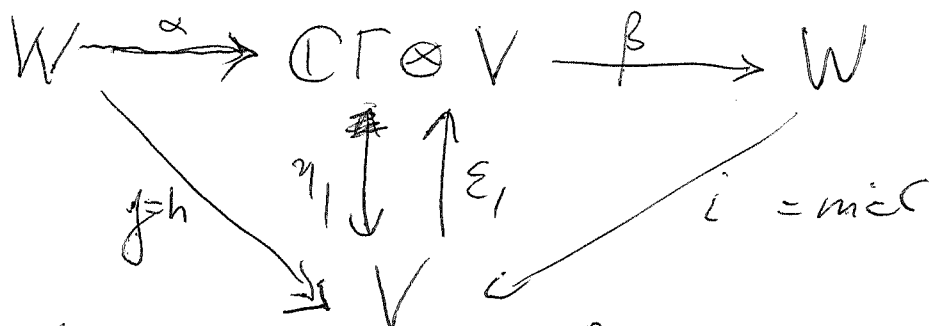
$$hW \leftarrow W$$



This is all very simple, so start with  $W = Bh$  whence the diagram



You need to believe in this diagram.



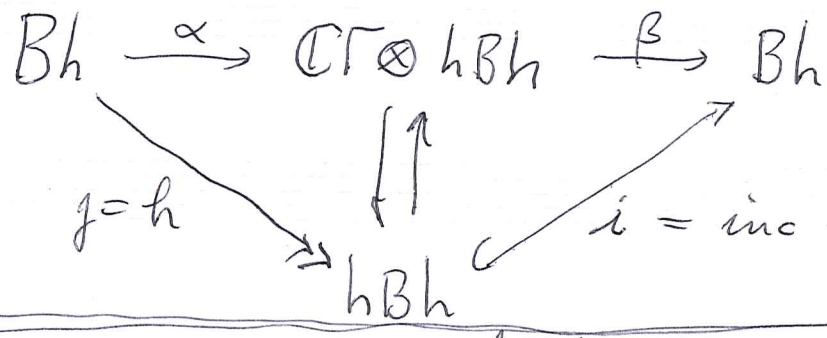
maybe the notation  $\mathbb{C}\Gamma \otimes V$

**Question**

projection in a  $M$  context.

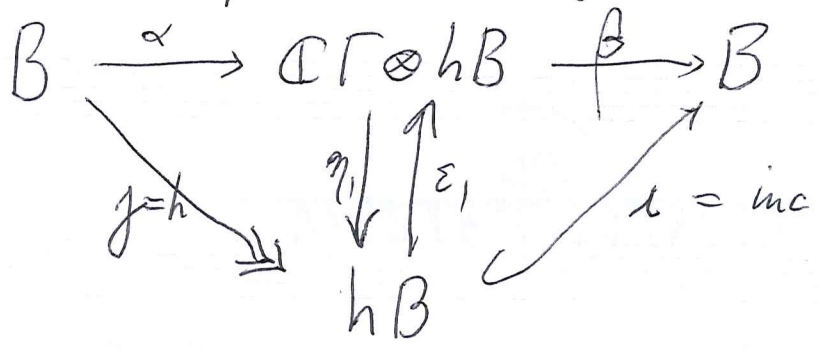
leads to an alg generated by  $p_{ij}$  ( $1 \leq i, j \leq 2$ )  
 subject to ~~satisfying~~ the relations  $p_{ik} = \sum_j p_{ij} p_{jk}$

~~Ques~~ Problem: canon. isom.  $Bh \cong \mathbb{C}\Gamma \otimes hBh$   
 should ~~be~~ follow from the diagram

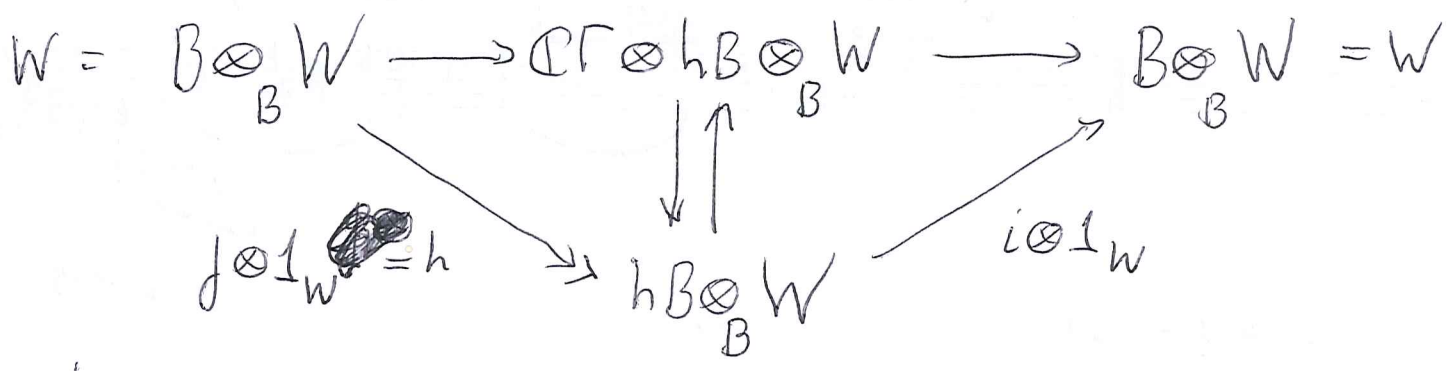


Big Hope is to <sup>develop</sup> somehow ~~to~~ the ~~the~~ techniques found recently for dealing with the image of an operator  $h: W \rightarrow W$  as if it were idempotent. Hope to crack Volodin

In the above replace  $Bh$  by  $B$ .



$$\begin{aligned}
 \beta\alpha &= id_B \\
 \alpha\beta &= p
 \end{aligned}$$



pres



~~Review~~ Review the argument that  $A = hBh$  467

satisfies  $A^2 = 0$  and  $A = 0$ .

$A'$  is <sup>the only</sup> ~~the~~ gen. by <sup>the elements</sup>  $p(s) = hsh$  for  $s \in \Gamma$  subject to the relns.  $p(s) = 0$  for  $s \notin \Gamma$ , OK and

$$\sum_s p(s) * p(s^{-1}t) = \sum_s hshs^{-1}th = hth = p(t) \quad ?$$

Start with  $B, \Gamma, h$  as usual, then form M. context  $\begin{pmatrix} A & Y \\ X & B \end{pmatrix} = \begin{pmatrix} hBh & hB \\ Bh & B \end{pmatrix}$  with the

product defined as if  $h$  were idempotent

**IDEA** Can this be linked to

~~the~~ the algebra ~~with~~  $A \langle D \rangle$  you once tried to understand, especially the role of  $D^2$ ?

~~So~~ So  $A$  has element  $hb_1h$  with product  $(hb_1h)(hb_2h) = hb_1hb_2h$ . So in

$A$  you have elements  $p(s) = hsh$  for  $s \in \Gamma$  satisfying the relations. Do they generate  $hBh = A$ ?

You know  $B = \Gamma \rtimes E$  so every element of  $B$  is a linear comb. of elts.  $t h_{s_1} \dots h_{s_n} =$

$$t s_1 h s_1^{-1} (s_2 h s_2^{-1}) \dots (s_{n-1} h s_{n-1}^{-1}) s_n h s_n^{-1}$$

$$p(t s_1) p(s_1^{-1} s_2) \dots$$

$$p(s_1) \cdots p(s_n)$$

$$= hs_1 hs_2 \cdots hs_n h \in hBh$$

$$= h(s_1 h s_1^{-1})(s_2 h s_2^{-1} s_1^{-1})(s_3 h s_3^{-1} s_2^{-1} s_1^{-1}) \cdots s_1 s_2 s_3 \cdots h$$

$$h h_{t_1} h_{t_2} \cdots h_{t_3}$$

$$p(s_1) \cdots p(s_n) = (hs_1 h) * (hs_2 h) * \cdots * (hs_n h)$$

$$= hs_1 hs_2 \cdots hs_n h$$

$$hs_1 hs_2 h = h(s_1 h s_1^{-1}) s_2 h$$

You want to show that any elt of  $hBh = A$  is a linear ~~and~~ comb. of products  $p(s_1) \cdots p(s_n)$

i.e.  $hs_1 hs_2 h$

$$h(s_1 h s_1^{-1}) s_2 h \quad \text{take } s_2 = s_1^{-1} \quad h h_{s_1} h$$

$$\frac{h(s_1 h s_1^{-1})(s_2 h s_2^{-1}) s_3 h}{\text{span } B}$$

seems obvious: take  $h h_{s_1} \cdots h_{s_n} t h$

$$h s_1 h (s_1^{-1} s_2) h s_2^{-1} s_3 h \cdots h s_n^{-1} t h$$

$$h t h_{s_1} h_{s_2} h = h t s_1 h s_1^{-1} s_2 h s_2^{-1} h$$

So conclude that the  $p(s) = hsh$  generate  $hBh$  for  $*$  product.  $\&$  satisfy the relns.

especially  You should write up something

$$\begin{pmatrix} hBh & hB \\ Bh & BhB \end{pmatrix}$$

with products defined as if  $h$  were idempotent

discrete case  $B = \Gamma \backslash X \mathbb{E}$

$\mathbb{E}$  generators  $h_s$  s.t.

$$\begin{cases} \sum_s h_s h_t = h_t \\ \sum_t h_s h_t = h_s \end{cases}$$

$$hb_1 \otimes b_2 h \quad \begin{pmatrix} hB \otimes_B Bh & hB \\ Bh & B \end{pmatrix}$$

$$(hb_1 \otimes b_2 h) * hb_3 = hb_1 \otimes b_2 hb_3$$

$$hb_1 b_2 h * hb_3 = hb_1 b_2 hb_3$$

You want to put the construction in perspective, a good framework. How to proceed?

There is something related to factoring  $h = y$

Start with  $B$ , then

construct  $hB$ ,  $Bh$  and the pairing  $Bh \times hB \rightarrow B$

What is  significant is the left  $B$  module  $X = Bh$ , right module  $Y = hB$  and the pairing  $Y \otimes_2 X \rightarrow B$  being  surjective. It's perhaps not important that there are inclusions

$$X = Bh \hookrightarrow B \quad \text{or} \quad Y = hB \hookrightarrow B$$

In fact you ~~don't~~ have seen that you don't want to think of both  $X=Bh$  and  $Y=hB$  being contained in  $B$ , rather you would like ~~one~~  <sup>$B$  submodules</sup> one to be a ~~subspace~~ and the other a  $B$  quotient ~~module~~ module.

Given  $(V, W, i, j)$  with  $h=i: W \xrightarrow{j} V \xrightarrow{i} W$   
 If you  $W=B$ , then you get  $B \xrightarrow{j=h} hB \xrightarrow{i=m} B$

$$\begin{pmatrix} jB = hB \\ B i = B h & B \end{pmatrix}$$

$$\begin{array}{c} \parallel \\ Y \\ B \xrightarrow{i} B h \xrightarrow{j} B \end{array}$$

Examine the pairing  $Bh \times hB \longrightarrow BhB$

to find  $\langle b_1 h, h b_2 \rangle = b_1 h b_2$ , you do

either, ~~lift~~ lift  $x$  to  $b_1$  and apply to  $y$  getting  $b_1 y$   
 or lift  $y$  to  $b_2$  and apply to  $x$  getting  $x b_2$   
 $x b_2 = b_1 h b_2$

Again: To define  ~~$x * y$~~   <sup>$x * y$</sup>  where  $x = b_1 h$  and  $y = h b_2$  you lift  $x$  to  $b_1$  and <sup>right</sup> mult by  $y$  to get  $x * y = b_1 y = b_1 h b_2$ , or you lift  $y$  to  $b_2$  and left mult by  $x$  to get  $x * y = x b_2 = b_1 h b_2$

Is it possible to understand better modules over the Morita context  $\begin{pmatrix} hBh & hB \\ Bh & B \end{pmatrix}$  ?

$$\begin{pmatrix} hBh & hB \\ Bh & B \end{pmatrix} \begin{pmatrix} V \\ W \end{pmatrix}$$

I think that you want to ~~replace~~

~~replace~~ focus on factorization. So

replace  $\begin{pmatrix} hBh & hB \\ Bh & B \end{pmatrix}$  by  $\begin{pmatrix} jBj & jB \\ Bi & B \end{pmatrix}$ , which

you understand as ~~a dual pair~~ as associated to the dual pair  $jB, Bi, Bi \times jB \rightarrow B$ .

---

Let us start with  $B, h \in B$  such that  $BhB = B$ . Then define a dual pair by

$X = B$  as ~~right~~  $B^{\text{op}}$  module,  $X = B$  as  $B$ -module, and  $\langle x, y \rangle = xhy$ . Then you have a <sup>surjective</sup> map of dual pairs over  $B$  given by

$$\begin{pmatrix} B & B \\ B & B \end{pmatrix} \longrightarrow \begin{pmatrix} hB & hB \\ Bh & B \end{pmatrix}$$

$$\begin{pmatrix} b_1 \in X, b_2 \in X \\ \langle b_1, b_2 \rangle \\ \parallel \\ b_2 h b_1 \end{pmatrix} \longmapsto \begin{pmatrix} \text{[scribbled out]} \\ (hb_1 e hB, b_2 h e Bh) \\ \text{[scribbled out]} \\ b_2 h * h b_1 = b_2 h b_1 \end{pmatrix}$$

Repeat what you learned namely that your dual pair  $(hB, Bh, \langle b_2h, hb_1 \rangle = b_2hb_1)$  is a quotient of the dual pair  $(B, B, \langle b_2, b_1 \rangle = b_2hb_1)$

Thus ~~we~~ by factoring  $B \times B \longrightarrow Bh \times hB \longrightarrow B$   
 $(b_2, b_1) \longmapsto b_2h \times hb_1 = b_2hb_1$ ,  
 you make the pairing less degenerate.

Is this discussion relevant to the ~~fields~~  $\Gamma$  situation?  
~~no~~

Yesterday you demystified the Morita context  $\begin{pmatrix} hBh & hB \\ Bh & B \end{pmatrix}$ , which you hoped might serve to treat any element of a ring as an idempotent. Idea is ~~that~~ that the dual pair  $(hB, Bh, \langle b_2h, hb_1 \rangle = b_2hb_1)$  is a quotient of  $(B, B, \langle b_2, b_1 \rangle = b_2hb_1)$ . ~~this~~

~~Thus~~ So you get the following picture

$$\begin{pmatrix} hB \otimes_B Bh & hB \\ Bh & B \end{pmatrix} = \begin{pmatrix} B \otimes_B B & B/hB \\ B/hB & B \end{pmatrix}$$

But notice that ~~this~~ this looks different from  $\begin{pmatrix} & hB \\ B & B \end{pmatrix}$

You think of  $f$  as injective  
 $g$  — surjective  $h = g$

~~Given~~ Given  $(U, W, W \xrightarrow{f} V \xrightarrow{g} W)$  more?

What is  $B_i$ ? It is  $B/\{b | bi=0\}$ .  
 $bi=0 \iff b_{ij}=0$  when  $f$  surj.

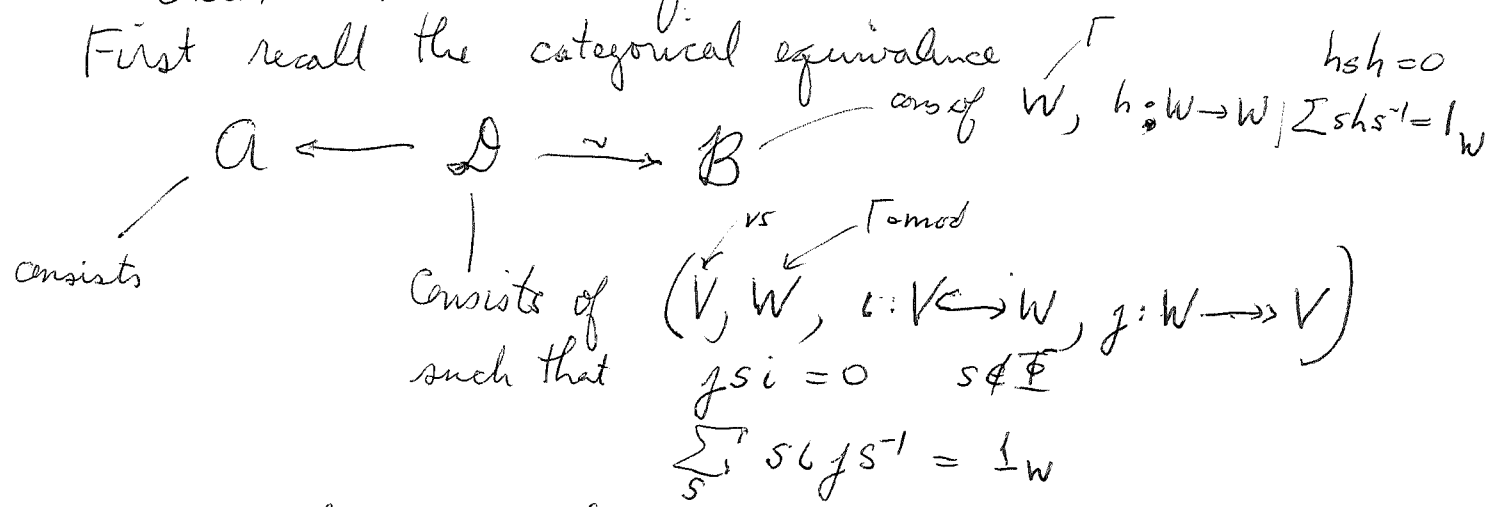
So  $B_i = B/B_h$ .  $gB = B/\{b | gb=0\}$ .

$gb=0 \iff igb=0$  when  $i$  injective

$\therefore gB = B/B_h$

To find what to do next? Let's go over what happens when  $B = \Gamma \ltimes \mathcal{E}$   $\mathcal{E}$  has generators  $h_s$   $s \in \Gamma$  subject to relations  $h_s h_t = 0$   $s^{-1}t \in \mathbb{F}$

and  $\sum_s h_s h_t = h_t$ ,  $\sum_t h_s h_t = h_s$ . At this point today you want ~~to~~ to complete your understanding of the Morita context & Morita equivalence. How?



Notice that the  $D, B$  equivalence is easy, just like constructing  $D$  from  $B, h$ . You may be able now to deal with adjoining  $i, g$  so that  $ig=h$

What to do: Essentially look again at the generators + relations construction of something like D.

Recall the def: D is the Morita context ( $M_2$ -graded alg) with gens  $x_t$  of degree 21 and  $y_s$  of degree 12,  $t, s \in \Gamma$

subject to the relations  $y_s x_t = y_{us} x_{ut} (= 0 \text{ if } s^{-1}t \notin \Phi)$

$$\sum_s x_s y_s x_t = x_t, \quad \sum_t y_s x_t y_t = y_s$$

~~What you want~~ You want to fit this into the  $\begin{pmatrix} hB & hB \\ B_h & B \end{pmatrix}$  pattern. Is it possible to get  $X = B_h, Y = hB$  using  $\Gamma_l = \{x_t | t \in \Gamma\}$  and  $\Gamma_r = \{y_s | s \in \Gamma\}$ ?

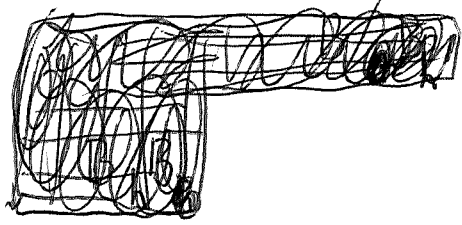
You were hoping for some version of  $X = B/B_h$ . What happens?  $B = \mathbb{C}\Gamma \otimes \mathcal{E} \Rightarrow B_h = \mathbb{C}\Gamma \otimes \mathcal{E}_h$ . What about  $\mathcal{E}_h$ ? Recall  $\mathcal{E}$  has generators  $h_s = s h s^{-1}$  and  $h_s h = s h s^{-1} h = 0$  for  $s^{-1} \notin \Phi$ .

~~What you have~~ You have B defined nicely with  $\Gamma$  acting as multipliers. Same for D

Idea: You have this M-cont  $\begin{pmatrix} hB_h & hB \\ B_h & B_h B \end{pmatrix}$

assume  $B_h B = B$  so that  $B \subset B_h B \subset B^3 \subset B^2$ .

Then there's a Morita equivalence around which ~~might~~ be easy to describe as  $(V, W \dots)$  etc.



$$\begin{pmatrix} B/B_h & \\ B/B_h & B \end{pmatrix} = \begin{pmatrix} hB & \\ B_h & B \end{pmatrix}$$



so 
$$\begin{pmatrix} B/hB & B/hB \\ B/B_h & B \end{pmatrix} = \begin{pmatrix} hB & hB \\ Bh & B \end{pmatrix}$$

Put  $A' = Y \otimes_B X = hB \otimes_B Bh$   
 $= (B/hB) \otimes_B (B/B_h) = \frac{B \otimes B}{hB \otimes B + B \otimes Bh}$

~~that~~ You have <sup>ring</sup> surjection  $A' \rightarrow hBh = A$   
~~kernel~~ kernel killed by  $A$  on left + right

Can you interpret this Morita context ~~via~~ via modules? First idea is ~~that~~ to use  $\begin{pmatrix} hBh & hB \\ Bh & B \end{pmatrix}$   
~~Modules should be~~  $(A, B)$  to obtain the M.eq.

$$W \longmapsto hB \otimes_B W \longrightarrow hW$$

$$W = \left( Bh \otimes_{hBh} V \right) \longleftarrow V$$

Question: Is  $W \longmapsto hW$  from  $B$ -modules to  $hBh = A$ -modules a Morita equivalence?

You have this Morita ~~equa~~ context  $\begin{pmatrix} hBh & hB \\ Bh & B \end{pmatrix}$

~~more~~ what is the corresponding equivalence of cats? Point is that for a left  $B$ -mod  $W$  such that  $BW = W$  you have  $hB \otimes_B W \rightarrow hW$  the kernel should be killed by  $A = hBh$   
 $hbh * \sum hb_i \otimes w_i = \sum \underbrace{hb hb_i}_{hBh} \otimes w_i$

What's ~~new~~ new is that instead of the usual possibilities for the functors

$$W \mapsto \left\{ \begin{array}{l} \text{rather of } Y \otimes_B W \longrightarrow \text{Hom}_B(X, W) \\ \text{or the image of this map. } y \otimes w \mapsto (x \mapsto (xy)w) \end{array} \right\}$$

or the image of this map. Is it possible that

$$\text{Im} \left\{ Y \otimes_B W \longrightarrow \text{Hom}_B(X, W) \right\} \quad \text{where } Y = hB \\ X = Bh$$

actually gives  $hW$ ?

$$hb \otimes_B w \mapsto (b'h \mapsto \underbrace{(b'h * hb)w}_{b'hbw})$$

$$Y \otimes_B W \longrightarrow \text{Hom}_B(X, W)$$

$$hB \otimes_B W \xrightarrow{\cong} \text{Hom}_B(Bh, W)$$

OKAY. you have the maps

$$hB \otimes_B W \longrightarrow \text{Hom}_B(Bh, W) \quad \text{does what?}$$

$$hb \otimes w \longmapsto (b'h \mapsto (b'hb)w)$$

$$\swarrow \text{rather of } hbw \mapsto (b'h \mapsto b'h * hbw)$$

So basically ~~you~~ ~~corresp~~ to  $BW = W$ , you want  ${}_B W = 0$ , and then ~~rather of~~

$$hW \longrightarrow \text{Hom}_B(Bh, W)$$

$$hw \mapsto (b'h \mapsto b'hw)$$

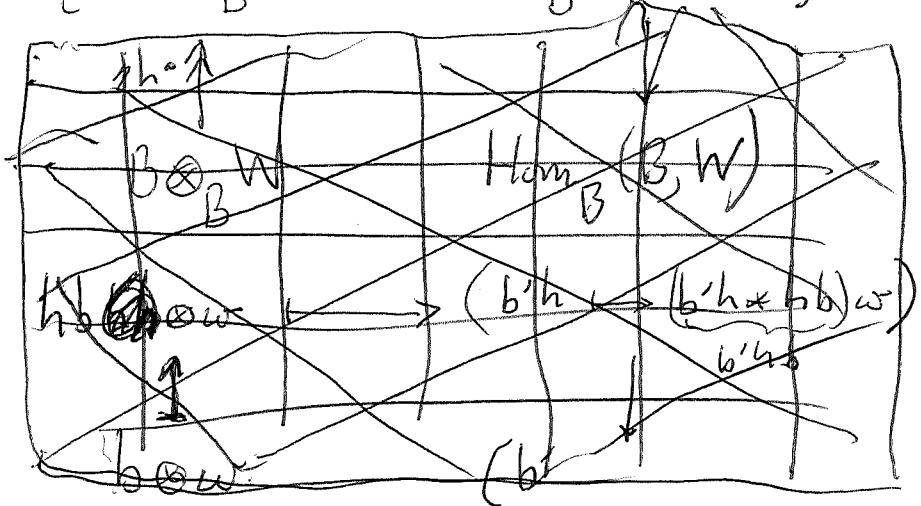
is injective.

Repeat this. Given a  $B$  module  $W$  ~~the~~

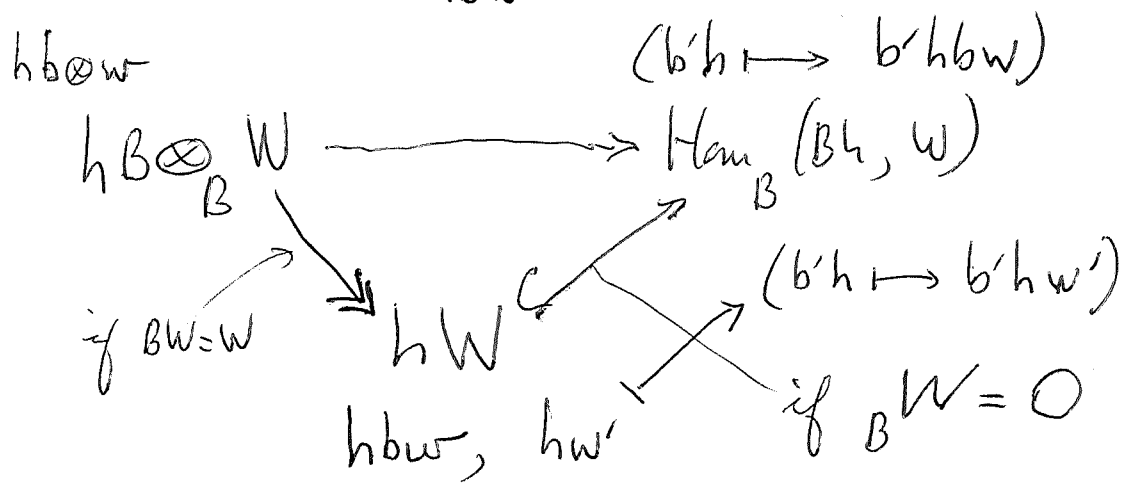
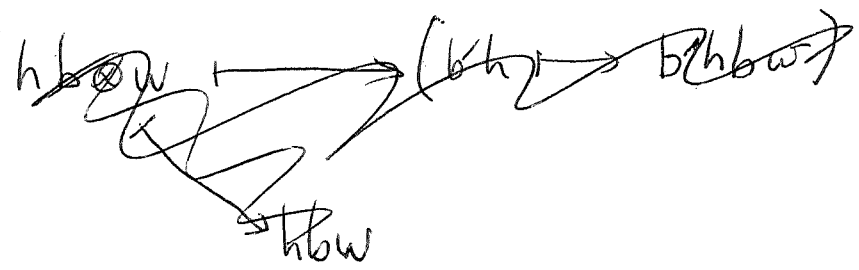
which is reduced:  $BW = W, {}_B W = 0$

then  $\text{Im} \{ Y \otimes_B W \rightarrow \text{Hom}_B(X, W) \}$  <sup>should be</sup> ~~is~~ the reduced  $A = hBh$  module corresponding to  $W$ :

$$\text{Im} \{ hB \otimes_B W \rightarrow \text{Hom}_B(Bh, W) \}$$



the map is  
and it factors



Your program is to describe nicely the Morita equivalence assoc. to  $\begin{pmatrix} hBh & hB \\ Bh & B \end{pmatrix}$  when  $B = BhB$

~~And~~ It seems that

To understand the Morita equivalence to the Morita context  $\begin{pmatrix} hBh & hB \\ Bh & B \end{pmatrix}$  with  $*$  product when  $BhB = I$ .

It seems that it is best to use the reduced module picture.

~~Claim:~~ Claim:  $\begin{pmatrix} A & Y \\ X & B \end{pmatrix}$  as above

satisfies  $A^2 = A = YX$   $Y = AY = YB$

~~XXXXXXXXXX~~

$XA = BX = X$   $B = B^2 = XY$

$A^2 = (hBh) * (hBh) = hBhBh = hBh = A$   
 $YX = hBBh = hBh = A$

(need  $B = BhB$   ~~$BhBhB$~~   $\Rightarrow B \subset B^3 \subset B^2$   
 $\therefore B^2 = B^3$  )

Better  $B = BhB \subset B \cdot B$

$XY = Bh * hB = BhB = B$

$\textcircled{Q}$   
 $MM$

~~XXXXXXXXXX~~  $AY = hBh * hB = hBhB = hB = Y$

$YB = hBB = hB = Y$

$XA = Bh * hBh = BhBh = Bh = X$

$BX = BBh = Bh = X$

What I learned is that the functors  ~~$A \otimes B$~~

~~$A \otimes B$~~   $W \mapsto hB \otimes_B W$  ,

You know I think that the equivalence 479

$$M(B) \xrightarrow{\sim} M(A) \quad \text{is given by}$$

$$W \longmapsto hW$$

the inverse functor being  $V \longmapsto Bh \otimes_A V$   
 roughly. Did show that the image of

~~$$Y \otimes_B W \longrightarrow \text{Hom}_B(X, W)$$~~

$$hB \otimes W \longmapsto (b'h \longmapsto b'hbw)$$

is  $hW$ .

$$hB \otimes_B W \xrightarrow{\text{since } BW=W} hW \xrightarrow{\text{if } B^W=0} \text{Hom}_B(Bh, W)$$

$$hb \otimes w \longmapsto hbw, hw \longmapsto (b'h \longmapsto b'hbw)$$

So where are we?? In context of the functors

$$M_2(B) \longrightarrow M_2(A)$$

$$W \longmapsto hW.$$

Check directly  ~~$hW \in W$~~   
 so that if  $Bhw = 0$ , then  $hw = 0$

$$(hBh)(hW) = hBhW = hBhBW = hBW = hW$$

If  $(hBh)(hw) = hBhw = 0$ , then  $BhBhw = 0$   
 $Bhw = 0$

so  $hw = 0$ . So  $hW$  is  $A$ -reduced. In particular  $hB$  and  $hBh$  should be  $A$ -reduced

So let's check this carefully.  $BhB = B$   
 $\begin{pmatrix} A & Y \\ X & B \end{pmatrix} = \begin{pmatrix} hBh & hB \\ Bh & B \end{pmatrix}$  with  $\times$  product  
 $(bh) \times (hb') = bhb'$   
 etc.

~~Now~~ Now what you should be able to prove is the equivalence  $M_2(B) \rightarrow M_2(A)$   
 $W \mapsto hW$

Check:  $B = BhB \subseteq BB = B^2$ , Assume  $BW = W$  and  $BW = 0$

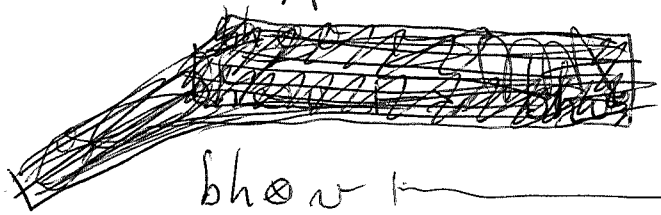
Then  $A(hw) = (hBh) \times (hw) = hBhW = hBhBW = hW$

◆  $A(hw) = 0 \Rightarrow hBhw = 0 \Rightarrow BhBhw = Bhw$

$\Rightarrow hw = 0$ . Next given  $V$   $A$ -reduced module:  $AV = V$  and  $A^2V = 0$  let

$$W = \text{Im} \left\{ \underbrace{Bh \otimes_A V}_{B \text{ conilfree}} \rightarrow \text{Hom}_A(hB, V) \right\}$$

$$\underbrace{Bh \otimes_A V}_{B \text{ conilfree}} \twoheadrightarrow W \hookrightarrow \underbrace{\text{Hom}_A(hB, V)}_{B \text{ nilfree}}$$



$$bh \otimes w \mapsto (hb' \mapsto hb'bhw)$$

$$\text{Hom}_B(B, \text{Hom}_A(hB, V)) = \text{Hom}_A(hB \otimes_B B, V)$$

$$\text{Hom}_A(hB, V)$$

$$0 \rightarrow K \rightarrow hB \otimes_B B \rightarrow hB \rightarrow 0$$

$$0 \leftarrow \text{Hom}_A(hB \otimes_B B, V) \leftarrow \text{Hom}_A(hB, V) \leftarrow 0$$

So where are you  $\begin{pmatrix} A & Y \\ X & B \end{pmatrix} = \begin{pmatrix} hBh & hB \\ Bh & B \end{pmatrix}$  481

Assuming  $B^2=0$ ,  ${}_B B = B_B = 0$  then you know that  $hB, Bh$  are  $A$ -reduced. ~~is~~  
 You would like  $Bh$  to be  $B$  reduced. Part of this  $B(Bh) = B^2h = Bh$  is OKAY. But there might be problems with  ${}_B Bh = 0$ , NO  
 ${}_B Bh = \{bh \in Bh \mid Bbh = 0\}$

$\begin{pmatrix} A & Y \\ X & B \end{pmatrix} = \begin{pmatrix} hBh & hB \\ Bh & B \end{pmatrix}$  Assume  $BhB = B$   
 ${}_B B = 0 = B_B$

Can you weaken the latter to  ${}_B B_B = 0$

Consider  $\begin{pmatrix} A & Y \\ X & B \end{pmatrix} = \begin{pmatrix} hBh & hB \\ Bh & B \end{pmatrix}$  with  $*$  prod. as if  $h^2=h$   
 $BhB = B$ .

Claim: strictly idempotent. Claim  $M_A(B) \iff M_A(A)$   
 $W \mapsto hW$ , inverse  $V \mapsto Bh \otimes_A V$

$V \mapsto \text{Im} \{ Bh \otimes_A V \rightarrow \text{Hom}_A(hB, V) \}$

$\text{Hom}_B(B, \text{Hom}_A(hB, V)) = \text{Hom}_A(hB \otimes_B B, V)$   
 $= \text{Hom}_A(hB, V)$

~~$W \mapsto \text{Im} \{ hB \otimes_B W \rightarrow \text{Hom}_B(Bh, W) \}$~~

$$\text{Im} \{ hB \otimes_B W \xrightarrow{hw} \text{Hom}_B(Bh, W) \}$$

$$hb \otimes w \mapsto hbw, hw \mapsto (b'h \mapsto b'hw)$$

~~$$\text{Hom}_B(B, \text{Hom}_B(Bh, W)) \cong \text{Hom}_B(B, W)$$~~

It seems that all you can say is that  $W \mapsto hW = V$  is  $A$ -red when  $B \neq 0$ .

$0 = hBh * hw = hBhw \Rightarrow Bhw = BhBhw = 0 \therefore hw = 0$

You want to write up something

~~As~~  $U(n, 1)$  acts on  $V = \mathbb{C}^n \oplus \mathbb{C}$  preserving the hermitian form ~~form~~  $H(\xi) = \|\xi_+\|^2 - \|\xi_-\|^2$

then  $U(n, 1)$  acts on  $V \otimes H$ , a Krein space, observation: Lagrangian ~~subspace~~ <sup>the graph of a</sup> subspace of  $V \otimes H = H^{\oplus n} \oplus H$  is same as ~~orth~~ unitary coin

$H^n \cong H$ , that is, an family of isometries  $s_i: H \rightarrow H$  such that  $\sum s_i s_i^* = 1$ , same as a unitary

\* hom.  $O_n \rightarrow \mathcal{L}(H)$ . The action of  $U(n, 1)$  on Lagrangian subspaces

First understand  $n=1$ , where  $H$  can be finite dim.

The idea is for each  $g \in U(n, 1)$  to produce an <sup>alg</sup> autom  $O_n \rightarrow O_n$ . You propose to make  $O_n$  act on the  $O_n^{op}$ -module  $O_n$ . What does this mean?

The idea should be to mimic the Hilbert



You have  $K = V \otimes H$  a Krein space

$$K = V_+ \otimes H \oplus V_- \otimes H \quad \text{polarization}$$

$L \subset K$   $L$  Lagrangian. You know that  $L = \begin{pmatrix} g \\ 1 \end{pmatrix} K$

need to understand  $n=1$ .

$$K = \begin{pmatrix} V_+ \\ V_- \end{pmatrix} \otimes H = \begin{pmatrix} V_+ \otimes H \\ V_- \otimes H \end{pmatrix}$$

$$L = \begin{pmatrix} 1 \\ T \end{pmatrix} (V_+ \otimes H) \subset \begin{pmatrix} V_+ \otimes H \\ V_- \otimes H \end{pmatrix}$$

$L$  is the graph of a unitary ism from  $V_+ \otimes H$  to  $V_- \otimes H$ . Now given  $g \in U(V)$  you get a different Lagrangian subspace  $g(L)$

$$L = \begin{pmatrix} z \\ 1 \end{pmatrix} \mathbb{C} \quad g = \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix}$$

$$g(L) = \begin{pmatrix} az + b \\ \bar{b}z + \bar{a} \end{pmatrix} \mathbb{C} = \begin{pmatrix} \frac{az+b}{\bar{b}z+\bar{a}} \\ 1 \end{pmatrix} \mathbb{C}$$

Start again.  $A$   $\times$  homom. (unital)  $\mathcal{O}_n \rightarrow \mathcal{L}(H)$  same as a unitary ism  $\mathbb{C}^n \otimes H \leftarrow H$ , same as a Lag.

$$L \subset \begin{pmatrix} \mathbb{C}^n \otimes H \\ H \end{pmatrix} \quad L = \begin{pmatrix} u \\ 1 \end{pmatrix} H$$

You take  $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in U(n, 1)$ . Then  $gL$

space action. Form  $V \otimes H$  Krein space <sup>484</sup>  
 and consider <sup>all</sup> polarizations. If you have chosen  
 one  $V \otimes H = \oplus V_+ \otimes H \oplus V_- \otimes H$  then  
 any other is given by the graph of an  
 isometry  $V_+ \otimes H \xrightarrow{\sim} V_- \otimes H$ .

First do Hilbert space reps of  $O_n$ .  $V = \begin{pmatrix} \mathbb{C}^n \\ \mathbb{C} \end{pmatrix}$   
 equipped with  $V^* \varepsilon V$  herm. form  $\varepsilon = \begin{pmatrix} 1_n & 0 \\ 0 & -1 \end{pmatrix}$ .

$H$  Hilbert space form  $V \otimes H$  equip with tensor prod  
 herm form  $(V \otimes \xi)^* (\varepsilon \otimes 1) (V \otimes \xi) = (V^* \varepsilon V) |\xi|^2$ . Has  
 polarization  $(V_+ \otimes H) \oplus (V_- \otimes H)$  Krein space.

Result describes ~~the Lagrangian subspace~~ Lagrangian subspace  
 as a unitary equiv. of  $V_+ \otimes H \xrightarrow{\sim} V_- \otimes H$

~~You know then that given~~

$V$  standard rep of  $U(n,1)$ , has natural  
 herm. form  $V^* \varepsilon V$  preserved by  $U(n,1)$ , Tensor  
 with  $H$  to get a Krein space  $V \otimes H$ .

The point you have missed maybe is the  
 fact that there's ~~a polar~~ both a polarization  
 and a Lagrangian subspace involved. Thus  
 you have  $V \otimes H$  acted on by  $U(n,1) \otimes 1$   
~~So  $G$  acts on the Lag. subspaces~~ So  $G$  acts on the Lag. subspaces

You have Lagrangian subspaces  $W \subset V \otimes H$   
 so if you are given the polarization  $V_+ \otimes H \oplus V_- \otimes H$

$O_n \longrightarrow \mathcal{L}(H)$  same as  $u: H \xrightarrow{\sim} \mathbb{C}^n \otimes H$  485

same as  $\Gamma_u = \begin{pmatrix} u \\ 1 \end{pmatrix} H \subset \begin{pmatrix} \mathbb{C}^n \otimes H \\ H \end{pmatrix}$

Now take  $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in U(n, 1)$ . It should be ~~clear~~ <sup>true</sup> that  $g\Gamma_u = \begin{pmatrix} au+b \\ cu+d \end{pmatrix} H$  is a Lagrangian

subspace of  $\begin{pmatrix} \mathbb{C}^n \otimes H \\ H \end{pmatrix}$ , hence should be  $\Gamma_{u'}$   $\subset \begin{pmatrix} \mathbb{C}^n \otimes H \\ H \end{pmatrix}$

where  $u' = (au+b)(cu+d)^{-1}$

$$u'' = (\alpha u' + \beta)(\gamma u' + \delta)^{-1}$$

$$= [\alpha(au+b)(cu+d)^{-1} + \beta][\gamma(au+b)(cu+d)^{-1} + \delta]^{-1}$$

$$= [\alpha(au+b) + \beta(cu+d)](cu+d)^{-1}(cu+d)[\gamma(au+b) + \delta(cu+d)]^{-1}$$

$$= [(\alpha a + \beta c)u + (\alpha b + \beta d)][(\gamma a + \delta c)u + (\gamma b + \delta d)]^{-1}$$

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \alpha a + \beta c & \alpha b + \beta d \\ \gamma a + \delta c & \gamma b + \delta d \end{pmatrix}$$

Let's see if it is possible to ~~establish~~ understand the action. First you want to show how  $U(n, 1)$  acts on the unitary  $\times$  reps of  $O_n$  ~~on~~ on a fixed  $H$ .  $\text{Hom}^*(O_n, \mathcal{L}(H))$

What is an element of  $\wedge^1$ , answer an isom  $\mathbb{C}^n \otimes H \xrightarrow{\sim} \mathbb{C}^n \otimes H$   
equiv.  $s_i \in \mathcal{L}(H)$   $s_i^* s_j = \delta_{ij}$ ,  $\sum s_i s_i^* = 1$ .

equiv. a Lag subspace  $L \subset \begin{pmatrix} \mathbb{C}^n \otimes H \\ H \end{pmatrix}$   $L = \begin{pmatrix} u \\ 1 \end{pmatrix} H$

$$\text{Hom}^{unital}(\mathcal{O}_n, \mathcal{L}(H)) = \{ u: H \xrightarrow{\sim} \mathbb{C}^n \otimes H \text{ unitary} \} \quad 486$$

Such a  $u$  same as the Lagr subspace  $\Gamma_u = \begin{pmatrix} u \\ 1 \end{pmatrix} H \subset \begin{pmatrix} \mathbb{C}^n \otimes H \\ H \end{pmatrix}$

Action of  $U(n, 1)$  on  $\begin{pmatrix} \mathbb{C}^n \\ \mathbb{C} \end{pmatrix}$  preserving  $\{ \xi \in \mathbb{C}^n \mid \|\xi\|_+^2 = \|\xi\|_-^2 \}$

Then the group  $U(n, 1)$  acts on the set of unital  $\ast$ -alg homoms.  $\mathcal{O}_n \longrightarrow \mathcal{L}(H)$  for any Hilbert space  $H$ . The problem is to understand why this action  $g \mapsto g(u)$  is given by an action of  $U(n, 1)$  on the  $C^\ast$ -algebra  $\mathcal{O}_n$ .

~~For any  $g \in U(n, 1)$  what is the point?~~

Given  $g \in U(n, 1)$  and  $\tilde{u}: \mathcal{O}_n \rightarrow \mathcal{L}(H)$

Given  $\mathcal{O}_n \xrightarrow{\tilde{u}} \mathcal{L}(H)$

why does  $\exists$  a  $g_\circ: \mathcal{O}_n \rightarrow \mathcal{O}_n$  s.t.  $\tilde{u} \circ g_\circ = \tilde{g}(u)$

Maybe this follows from a formula.

Let  $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in U(n, 1)$ ,  $u: H \xrightarrow{\sim} \mathbb{C}^n \otimes H$

$$g \Gamma_u = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} u \\ 1 \end{pmatrix} H = \begin{pmatrix} au + b \\ cu + d \end{pmatrix} H$$

$$= \begin{pmatrix} (au + b)(cu + d)^{-1} \\ 1 \end{pmatrix} H = \Gamma_{g(u)}$$

where  $g(u) = (au + b)(cu + d)^{-1}$ .