Try very hard to clean this up. How?

~~Begin by constructing the Morita context~~

Construct a Morita context. ~~Start with~~

$Y = jB$ Can you adjoin $i$, $j$

Consider the Morita context $B$

$X = Bi$ $B$ ($= M_2$ graded alg) generated by the alg $B$ in degree 22, an element $i$ degree 21, an element $j$ degree 12, satisfying relations

$ij = h$ $\sum_t h_t i = i$ $\sum_s j h_s = j$

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Start again Let's begin with $A$, construct $(X, Y, \langle x, y\rangle)$ as dual pair over $A$.

~~This is the question about dual~~ ~~This is~~

$$V \longmapsto F(V) = p(\mathbb{C}\Gamma \otimes V)$$

$$\begin{tikzcd}
X = F(\tilde{A}) \arrow[r, "\alpha"] \arrow[dr, "j"'] & \mathbb{C}\Gamma \otimes \tilde{A} \arrow[r, "\beta"] \arrow[d, shift right, "\eta_1"'] & F(\tilde{A}) = X \\
 & \tilde{A} \arrow[u, shift right, "\varepsilon_1"'] \arrow[ur, "i"'] &
\end{tikzcd}$$

$\boxed{p(s) j = jsh}$

right module version Let $N$ be a right red $B$-mod.

factor
$$\begin{tikzcd}
N \arrow[rr, bend left, "\cdot h"] \arrow[r, "\cdot i"'] & M & N \arrow[l, "\cdot j"]
\end{tikzcd}$$

$p(s) = jsi$

$n i\, p(s) = nhsi$

$\boxed{i p(s) = hsi}$

$\boxed{p(s) j = jsh}$

$s \notin \mathcal{I}$, $0 = hsh = \underbrace{i}_{\text{surj}}(jsi)\underbrace{j}_{\text{inj}}$

Right module picture. $N$ $B^{op}$-module such that $NB = N$, equiv. $N$ is a $\Gamma^{op}$ module equipped with an operator $h: n \mapsto nh$ sat $hsh = 0$ for $s \notin \Phi$, $\sum_s nshs^{-1} = n$ $\forall n \in N$.

Put $M = \mathrm{Im}\{N \xrightarrow{\cdot h} N\}$, whence canon maps $N \xrightarrow{\cdot i} M \xrightarrow{\cdot j} N$ $h = ij$

(Observe: When you form $(M \otimes \mathbb{C}\Gamma)p$, the image of the projection $p$, ~~let~~ you can say let

$$F(M) = \mathrm{Im}\{M \otimes \mathbb{C}\Gamma \xrightarrow{\cdot p} M \otimes \mathbb{C}\Gamma\}$$

whence there are canonical maps of $B^{op}$-modules

$$F(M) \xrightarrow{\cdot \alpha} M \otimes \mathbb{C}\Gamma \xrightarrow{\cdot \beta} F(M)$$

such that $\alpha\beta = 1_{F(M)}$, $\beta\alpha = \cdot p$)

Next: the diagram

$$\begin{tikzcd}
F(M) \arrow[r, "\cdot\alpha"] \arrow[dr, "\cdot i"'] & M \otimes \mathbb{C}\Gamma \arrow[r, "\cdot\beta"] \arrow[d, shift right, "\eta_1"'] & F(M) \\
& M \arrow[u, shift right, "\varepsilon_1"'] \arrow[ur, "j"'] &
\end{tikzcd}$$

$$M \otimes \mathbb{C}\Gamma = \left\{ \sum_s m(s) \otimes s \;\middle|\; m: \Gamma \to M \text{ fin. supp} \right\}$$

$$\left( \sum_t m(t) \otimes t \right) u = \sum_s m(tu^{-1}) \otimes t$$

$$mj = m\varepsilon_1\beta = (m \otimes 1)\beta$$

$$\begin{tikzcd}
F(M) \arrow[r, "\alpha"] \arrow[dr, "\iota = \alpha\eta_1"'] & M \otimes \mathbb{C}\Gamma \arrow[r, "\beta"] \arrow[d, shift right, "\eta_1"'] & F(M) \\
 & M \arrow[u, shift right, "\varepsilon_1"'] \arrow[ur, "j = \varepsilon_1\beta"'] &
\end{tikzcd}$$

$$\left(\sum_t m(t)\otimes t\right)\beta = \left(\sum_t m(t)\varepsilon_1 t\right)\beta = \sum_t m(t) j t$$

Let $n\alpha = \sum_t m(t)\otimes t \implies n\alpha t^{-1}\eta_1 = m(t) = nt^{-1}\iota$

$$\therefore \boxed{n\alpha = \sum_t nt^{-1}\iota \otimes t} \quad \boxed{\left(\sum_t m(t)\otimes t\right)\beta = \sum_t m(t) j t}$$

$$n\alpha\beta = \sum_t nt^{-1}\iota j t = n$$

$$\left(\sum_s m(s)\otimes s\right)\beta\alpha = \left(\sum_s m(s) j s\right)\alpha = \sum_{s,t} m(s)\overbrace{j s t^{-1}\iota}^{p(st^{-1})} \otimes t$$

So there is the formula for $p$.

$$\sum_s m(s)\otimes s \xrightarrow{p} \sum_{s,t} m(s)p(st^{-1})\otimes t$$

action of $(mu)(s) = m(su^{-1})$

Recap. Given $A$-module $V$ ~~????~~ you get $\overbrace{p}^{\Gamma\text{-inv idemp}}$ on $\mathbb{C}\Gamma\otimes V$ and $F(V) = \mathrm{Im}\{p : \mathbb{C}\Gamma\otimes V \longrightarrow \mathbb{C}\Gamma\otimes V\}$, whence canonical maps

$$\begin{tikzcd}
F(V) \arrow[r, "\alpha."] \arrow[dr, "i"'] & \mathbb{C}\Gamma\otimes V \arrow[r, "\beta."] \arrow[d, shift right, "\eta_1"'] & F(V) \\
 & V \arrow[u, shift right, "\varepsilon_1"'] \arrow[ur, "j"'] &
\end{tikzcd}
\quad \left\{\begin{array}{l} \beta\alpha = 1_{F(V)} \\ \alpha\beta = p \end{array}\right.$$

such that $F(V)$ becomes a $\overset{\text{red}}{B}$-mod. with $h = \iota j$

$hsh = \iota p(s) j$

You are beginning with $A$, more precisely with left $A$-modules $V$ and right $A$-modules $M$, then you have a
$$\left\{\begin{array}{lll} p\cdot & \Gamma\text{-compatible idemp on} & \mathbb{C}\Gamma\otimes V \\ \cdot p & \Gamma^{op} \quad " \quad " & M\otimes\mathbb{C}\Gamma \end{array}\right.$$

$$p\Big(\sum_t t\otimes v(t)\Big) = \sum_s s\otimes\sum_t p(s^{-1}t)v(t)$$

$$\Big(\sum_s m(s)\otimes s\Big)p = \sum_t\Big(\sum_s m(s)p(st^{-1})\Big)\otimes t$$

So you have left $B$-module $p(\mathbb{C}\Gamma\otimes V)$
right —— $(M\otimes\mathbb{C}\Gamma)p$

Important cases.

$$\boxed{\begin{array}{lll} B, A^{op} \text{ bimodule} & p(\mathbb{C}\Gamma\otimes A) & = X \\ A, B^{op} \quad \text{——} & (A\otimes\mathbb{C}\Gamma)p & = Y \end{array}}$$

Now you want to construct pairings $Y\times X\longrightarrow A$ and $X\times Y\longrightarrow B$

Repeat:

$$\begin{tikzcd} X \arrow[r, "\alpha"] \arrow[rr, bend left, "h\cdot"] \arrow[dr, "j\cdot"'] & \mathbb{C}\Gamma\otimes\tilde{A} \arrow[r, "\beta"] \arrow[d, shift right, "\eta_1"'] & X \\ & \tilde{A} \arrow[u, shift right, "\varepsilon_1"'] \arrow[ur, "\iota\cdot"'] & \end{tikzcd}$$

$$\begin{tikzcd} Y \arrow[r, "\alpha"] \arrow[rr, bend left, "\cdot h"] \arrow[dr, "\cdot\iota"'] & \tilde{A}\otimes\mathbb{C}\Gamma \arrow[r, "\beta"] \arrow[d, shift right, "\eta_1"'] & Y \\ & \tilde{A} \arrow[u, shift right, "\varepsilon_1"'] \arrow[ur, "j"'] & \end{tikzcd}$$

$$\alpha\xi = \sum s\otimes js^{-1}\xi$$

$$\beta\Big(\sum_t t\otimes a(t)\Big) = \sum_t t\iota a(t)$$

$$\beta\alpha\xi = \sum_s s\iota js^{-1}\xi = \xi$$

$$\alpha\beta\Big(\sum_t t\otimes a(t)\Big) = \alpha\Big(\sum_t t\iota a(t)\Big) = \sum_s s\otimes\sum_t \underbrace{js^{-1}t\iota}_{p(s^{-1}t)} a(t)$$

$$\boxed{(pa)(s) = \sum_t p(s^{-1}t)a(t)}$$

$$\eta\alpha = \sum_t \eta t^{-1}\iota\otimes t$$

$$\Big(\sum_s a(s)\otimes s\Big)\beta = \sum_s a(s)js$$

$$\eta\alpha\beta = \sum_s \eta t^{-1}\iota jt = \eta$$

$$\Big(\sum_s a(s)\otimes s\Big)\underbrace{\beta\alpha}_{p} = \sum_t\sum_s a(s)jst^{-1}\iota\otimes t$$

$$\boxed{(ap)(t) = \sum_s a(s)p(st^{-1})}$$

$$\begin{pmatrix} A & Y \\ X & B \end{pmatrix}$$

You know that any Morita equivalence corresponds to a firm Morita context. Go back to

$$\mathcal{M}(A) \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{M}(B)$$

$$F(V) = p(\mathbb{C}\Gamma \otimes V) = \underbrace{p(\mathbb{C}\Gamma \otimes \tilde{A})}_{X} \otimes_A V$$

$$G(W) = hW$$

There seems to be a viewpoint where it is unimportant to worry about the type of module, whether firm or reduced etc.

$F(V) = p(\mathbb{C}\Gamma \otimes V)$ equipped with $B$-module structure obtained from $\Gamma$ action and operator

$$h = \underbrace{\beta \varepsilon_1}_{i} \underbrace{\eta_1 \alpha}_{j} = ij$$

$$\beta\alpha = 1_{F(V)}$$
$$\alpha\beta = p$$

$$p\Big(\sum_t t \otimes v(t)\Big) = \sum_s s \otimes \sum_t p(s^{-1}t)\, v(t)$$

$$p(1 \otimes v) = \sum_s s \otimes p(s^{-1})\, v$$

$$hth = ijtij = \beta\varepsilon \boxed{\eta_1 \alpha t \beta \varepsilon_1} \eta_1 \alpha$$

$$\eta_1 t \overset{p}{\alpha\beta} \varepsilon_1 v = \eta_1 t p \sum 1 \otimes v$$
$$= \eta_1 t \sum_s s \otimes p(s^{-1}) v$$
$$= p(t) v$$

Given a left $B$-module $W$ put $V = hW$ with $p(s)\,hw = hshw$. Then $p(s) = 0$ for $s \notin \Phi$ as $hsh = 0$, and $\sum_t p(st^{-1})p(t)\,hw = \sum_t hs\underbrace{t^{-1}ht}hw$ $= hshw = p(s)\,hw$.

Similarly given a right $B$ module $N$ put $M = Nh$ with $nh\,p(s) = nhsh$. Again $p(s)=0$ if $s \notin \Phi$ and $nh\sum_t p(st^{-1})p(t) = \sum_t nhs\underbrace{t^{-1}ht}h$ $= nhsh = (nh)\,p(s)$

Back to left modules. $W = \sum_s shW \Rightarrow$ $hW = \sum_s hshW = \sum_s p(s)hW = AhW$. And if $p(s)hw = hshw = 0 \ (\forall s)$, then $hw = \sum_s s^{-1}hs\,hw = 0$. So ${}_A hW = 0$, and $hW$ is reduced.

Right picture is the same, namely

$$N = \sum_s Nhs \qquad \therefore\ Nh = Nhsh$$

$$N = \sum_s Ns^{-1}hs = \sum_s Nhs$$

$$Nh = \sum_s Nhsh = \sum_s Nh\,p(s)$$

if $nh\,p(s) = nhsh = 0$ $\forall s$, then

$$0 = \sum_s nh\underbrace{shs^{-1}} = nh$$

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$X \otimes_A M \qquad (V \otimes_A Y) \otimes_B (X \otimes_A M) = V \otimes_A M$

$$\begin{pmatrix} A & Y \\ X & B \end{pmatrix} \qquad M \otimes_A Y \otimes_B \overset{X}{W}$$

$$\cap$$

$$M \quad A^{op}\text{-mod}$$

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Yesterday, made a simplification, namely

$W \longmapsto hW$ with $p(s)hw = hshw$

~~[illegible]~~ $\cdot\ W = \sum shW \implies$ ~~[illegible]~~ $hW = \sum_s hshW = \sum_s p(s)hW$

also $hw = \sum_s s^{-1}hshw = \sum_s s^{-1}p(s)hw$ ~~shows~~ shows

$p(s)\,hw = 0 \implies hw = 0.$ $\therefore$ $hW$ is red $A$-module

~~[illegible] [illegible] X [illegible]~~ This looks good as far as constructing ~~the~~ Morita context $\begin{pmatrix} hBh & hB \\ Bh & B \end{pmatrix}$

~~Also you have to do it to You have $X = Bh$ is an $B, A$ bimodule $X = Bh$~~

$Bh$    $B, A$ bimod

$hB$    $A, B$ bimod

$hBh$    $A$ bimodule, there's an $A$-bimodule surj

$$hB \otimes_B Bh \longrightarrow hBh$$

You want to identify $hBh$ with $A_{red}$

Idea:
$$\begin{pmatrix} hBh & hB \\ Bh & B \end{pmatrix} = \begin{pmatrix} jBi & jB \\ Bi & B \end{pmatrix}$$

what's important is that $Bh$ is both a ~~submodule~~ subspace of $B$ and a quotient space.

Put into words: $Bh$ is a left $B$-module generated ~~by~~ the element $h$, i.e. $B/B_h$ ??

$Bh = \{bh \in B \mid b \in B\} \simeq B/B_h$
where $B_h = \{b' \in B \mid b'h = 0\}$. $Bh$ is a principal $B$-module. Now when you factor $\cdot h = \cdot ij : B \xrightarrow{i} B_h \overset{j}{\hookrightarrow} B$ what does $Bi$ mean?

$$Bh = Bij$$

$$B \xrightarrow{\cdot h = i} Bh \xhookrightarrow{inc = j} B$$

$Bh = Bi$ means $\{bh \mid b \in B\} \longrightarrow \underbrace{(B)i}_{\text{image of } B \text{ under } i}$

Similarly

$$h\cdot = ij : B \xrightarrow{h\cdot = j} hB \xhookrightarrow{inc = i} B$$

$\{hb \mid b \in B\} = hB = j(B)$ IDEA.

introduce the obvious module for the Morita context.

$$\begin{pmatrix} jBi & jB \\ Bi & B \end{pmatrix} \begin{pmatrix} hB \\ B \end{pmatrix}$$

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Your problem is this: You have the equivalence $m(A)$ $m(B)$ for left modules, ~~you translate~~ want the corresp. M.C. $\begin{pmatrix} A & Y \\ X & B \end{pmatrix}$, you know what good modules for this MC are. If you're ~~starting~~ with $B$, then good modules have the form $\begin{pmatrix} hW \\ W \end{pmatrix}$

So what to do? You have $M(A) \simeq M(B)$ and want the corresp. M.C. $D = \begin{pmatrix} A & Y \\ X & B \end{pmatrix}$. What are the good D-modules? Answer $\begin{pmatrix} hW \\ W \end{pmatrix}$, with W a good B-module. You've seen that such a pair $\begin{pmatrix} hW \\ W \end{pmatrix}$ comes equipped with canon maps $i : hW \hookrightarrow W$ and $j : W \twoheadrightarrow hW$ such that $h = ij$. You expect D to be

$$D = \begin{pmatrix} A & jB \\ Bi & B \end{pmatrix}$$

$jB = hB$

By symmetry $\cdot h : B \xrightarrow{\cdot i} Bh \xhookrightarrow{\cdot j} B$ so that $Bi = Bh$

$h\cdot : B \xrightarrow{j\cdot} hB \xhookrightarrow{i\cdot} B$ so that $jB = hB$

~~Ultimately I think you want to form~~

Repeat. You have $M(A)$ equiv. to $M(B)$, to find the assoc. M. cont. $D = \begin{pmatrix} A & Y \\ X & B \end{pmatrix}$. What are good D-modules: Answer $\begin{pmatrix} hW \\ W \end{pmatrix}$, W a good B module, where this pair is equipped with the factorization $h\cdot = (i\cdot)(j\cdot) : W \xrightarrow{j\cdot} hW \xhookrightarrow{i\cdot} W$. $jw = hw$, $jW = hW$

Our idea for D is $\begin{pmatrix} jBi & jB \\ Bi & B \end{pmatrix} = \begin{pmatrix} hBh & hB \\ Bh & B \end{pmatrix}$

What's going on? Mainly you have the functor $W \longmapsto hW$ with $p(s)hw = hshw$. Another notation $W \longmapsto hW$ with $p(s)jw = i(jsi)jw$

meaning is not yet clear. Write

$$\underbrace{D = \begin{pmatrix} jBi & jB \\ Bi & B \end{pmatrix}}_{\text{think in this } \uparrow \text{ way}} \qquad \underbrace{D = \begin{pmatrix} hBh & hB \\ Bh & B \end{pmatrix}}_{\text{work with}}$$

$$(b_1 i)(j b_2) = b_1 h b_2 \qquad (b_1 h)(h b_2) \overset{\text{def}}{=} b_1 h b_2$$

~~In~~ other version you have a dual pair over $B$ given by $hB$, $Bh$, and $\langle b_1 h, h b_2 \rangle = b_1 h b_2$

Then get Morita context $D = \begin{pmatrix} hB \otimes_B Bh & hB \\ Bh & B \end{pmatrix}$

(digress. Let $(X, Y, \langle y, x \rangle)$ be a dual pair over $A$. Use $M_2$ grading to construct a M context.

$$D \xrightarrow{\Delta} M_2 \otimes D$$

$$\begin{aligned} a &\longmapsto e_{11} \otimes a \\ x &\longmapsto e_{21} \otimes x \\ y &\longmapsto e_{12} \otimes y \end{aligned}$$

When you give the dual pair you use 4 of the products $a_1 a_2$, $ay$, $xa$, $yx$

Consider alg $D$ gen. by $A \to D$, $X \to D$, $Y \to D$ subject to ~~the relations~~ requiring that the products in $D$ are the specified ones.) and you want $XX = 0$, $YY = 0$.

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Back to $D = \begin{pmatrix} hBh & hB \\ Bh & B \end{pmatrix}$. Check this is a Morita context.

You have dual pair over $B$ given by $hB, Bh$, and the pairing $\langle b_1h, hb_2\rangle = b_1hb_2$. Associated to this dual pair is an algebra $hB\otimes_B Bh$ with product given by $(hb_1\otimes b_2h)(hb_3\otimes b_4h)$

$$= hb_1\otimes (b_2hb_3)b_4h = hb_1b_2hb_3\otimes b_4h$$

Consider $hB\otimes_B Bh \longrightarrow hBh \qquad hb_1\otimes b_2h \longmapsto hb_1b_2h$

Define product in $hBh$ by

$$(hb_1h)(hb_2h) = hb_1hb_2h$$

$$\begin{array}{ccc}
(hb_1\otimes b_2h\ ,\ hb_3\otimes b_4h) & \longmapsto & hb_1\otimes b_2hb_3b_4h \\
\downarrow & & \downarrow \\
(hb_1b_2h\ ,\ hb_3b_4h) & \longmapsto & h(b_1b_2)h(b_3b_4)h
\end{array}$$

~~Special~~ What should be true in this ~~stupid~~ situation?

What been accomplished. dual pair $hB, Bh$, $\langle b_1h, hb_2\rangle = b_1hb_2$ yield M.cont.

$$\begin{pmatrix} hB\otimes_B Bh & hB \\ Bh & B\end{pmatrix} \twoheadrightarrow \begin{pmatrix} hBh & hB \\ Bh & B\end{pmatrix}$$

$$\boxed{(hb_1\otimes b_2h)hb_3 = hb_1b_2hb_3}$$

$$\| \qquad\qquad$$

$$hb_1b_2hb_3$$

you should know that $hB\otimes_B Bh \to hBh$ is surjective kernel killed by the ring

$$(hb_1\otimes b_2h)b_3h$$

$$hb_1b_2hb_3h$$

~~Last step~~. Repeat again. Given a ring $B$ and an element $h \in B$, you ~~can define~~ get a Morita context

$$\begin{pmatrix} hBh & hB \\ Bh & B \end{pmatrix}$$

~~Last step is~~ How: Have dual pair over $B$ given by $hB$, $Bh$, $\langle b_1h, hb_2\rangle = b_1hb_2$, whence a Morita context

$$\begin{pmatrix} hB\otimes_B Bh & hB \\ Bh & B \end{pmatrix}$$

~~and then you notice~~, a surjective (assuming $B^2 = B$ which is true when $B = BhB$) alg map $hB\underset{B}{\otimes}Bh \twoheadrightarrow hBh$

$$\begin{tikzcd}
(hb_1\otimes b_2h) \arrow[d, mapsto] & \cdot & (hb_3\otimes b_4h) \arrow[d, mapsto] & = & (hb_1\otimes b_2hb_3b_4h) \arrow[d, mapsto] \\
(hb_1b_2h) & \cdot & (hb_3b_4h) & = & hb_1b_2hb_3b_4h
\end{tikzcd}$$

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Last step is to consider $X\otimes_A Y \longrightarrow B$ i.e.

$$Bh\otimes_{hBh} hB \longrightarrow B.$$

$$b_1h\otimes_A hb_2 \longmapsto b_1hb_2$$

$$\begin{tikzcd}
(b_1h\otimes hb_2) \arrow[d, mapsto] & \cdot & (b_3h\otimes hb_4) \arrow[d, mapsto] & = & b_1h\otimes hb_2b_3hb_4 \arrow[d, mapsto] \\
(b_1hb_2) & \cdot & (b_3hb_4) & & b_1hb_2b_3hb_4
\end{tikzcd}$$

~~[illegible]~~ In the $\Gamma$-situation, the map $Bh \otimes_{hBh} hB \longrightarrow B$ should be an isomorphism because of $\sum_s shs^{-1}b = b$. Given $\sum b_i h \otimes hb_i'$ in $Bh \otimes_{hBh} hB$ ~~[illegible]~~ with image $0 = \sum_i b_i h b_i'$

Then
$$\sum_s \underbrace{\sum_i shs^{-1}b_i h \otimes hb_i'}_{s \otimes \sum_i hs^{-1}b_i h b_i' = 0} = \sum_i b_i h \otimes hb_i'$$

What else happens?
$$\begin{pmatrix} hBh & hB \\ Bh & B \end{pmatrix}$$

$\langle bh, hb' \rangle = bhb'$

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next maybe ~~[illegible]~~ you want to understand role of $h = \iota j$. You think that in order to control $hB$ you need "the" canonical fact of $h$.

$$\begin{tikzcd} B \arrow[rr, bend left, "h\cdot"] \arrow[dr, two heads, "j"'] & & B \\ & hB \arrow[ur, hook, "\iota"'] & \end{tikzcd} \Rightarrow j(B) = hB$$

Similarly to control $Bh$ you need

$$\begin{tikzcd} B \arrow[rr, bend left, "\cdot h"] \arrow[dr, two heads, "\iota"'] & & B \\ & Bh \arrow[ur, hook, "j"'] & \end{tikzcd}$$

$B\iota = Bh$

$\langle bh, hb' \rangle$ $\quad Bh \times hB$

$B\iota j \times \iota j B$

---

logic $\quad hB = \iota j B \simeq jB$ because $\iota$ is an inclusion

$Bh = B\iota j \simeq B\iota$ because $j$ is an inclusion?

So exactly what remains to understand. One thing would be an explicit identification of $Bh$ with $p(\mathbb{C}\Gamma \otimes A)$. ~~At the moment you assume~~ How to proceed? ~~Where~~ The idea: The basic Morita equivalence for left modules is

$$W \longmapsto hW \leftarrow hB \otimes_B W$$
$$\mathcal{M}(B) \longrightarrow \mathcal{M}(A)$$

$p(\mathbb{C}\Gamma \otimes \tilde{A}) \otimes_A V \to p(\mathbb{C}\Gamma \otimes V) \longleftarrow\!\!| \; V$. So certainly you have a construction $V \longmapsto F(V) = p(\mathbb{C}\Gamma \otimes V)$, as this is right cont & exact in $V$ you get canon. isom

$$F(\tilde{A}) \otimes_A V \xrightarrow{\sim} F(V)$$

where $F(\tilde{A})$ ~~is flat~~ firm over $A^{op}$. Diagram

$$\begin{tikzcd} F(V) \arrow[r, "\alpha"] \arrow[dr, "j"'] & \mathbb{C}\Gamma \otimes V \arrow[r, "\beta"] \arrow[d, shift right, "\eta_1"'] & F(V) \\ & V \arrow[u, shift right, "\varepsilon_1"'] \arrow[ur, "i"'] & \end{tikzcd}$$

$\alpha, \beta$ $\Gamma$-module maps. $\quad \beta\alpha = 1_{F(V)}$

$\alpha\beta = p$ where

$$\alpha w = \sum_{s \in \Gamma} s \otimes j s^{-1} w \quad , \quad \beta\Big(\sum_t t \otimes v(t)\Big) = \sum_t t i \, v(t)$$

$$\beta\alpha w = \sum_s s i j s^{-1} w = w \quad , \quad \alpha\beta\Big(\sum_t t \otimes v(t)\Big) = \sum s \otimes \sum_t \underbrace{p(s^{-1}t)}_{j s^{-1} t i} v(t)$$

Go over the Mor. equiv.

$\mathcal{A}$ = red. A-modules $V$ with $p(s)$ operators | supp cond, idemp.

$\mathcal{B}$ = — B-modules = $\Gamma$-mods with $h$ | supp cond $hsh=0$, part. of 1

$\mathcal{D}$ = cat of $(V, W, i, j)$, $V$ v.s., $W$ $\Gamma$-module

$i: V \hookrightarrow W$, $j: W \twoheadrightarrow V$ | supp: $jsi = 0$ $s \notin \Phi$ | part $\sum_s s i j s^{-1} w = w$

$\mathcal{D} \to \mathcal{B}$ sends $(V, W, i, j)$ into the $\Gamma$-mod $W$ equipped with $h = ij$ on $W$. But $ij = h$, $i$ inj, $j$ surj $\Rightarrow$ canon isom $V = hW$, $i = \text{inc}$, $j = h$

$\therefore$ $\mathcal{D} \to \mathcal{B}$ equivalence of categories.

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Next define $\mathcal{D} \to \mathcal{A}$ by $(V, W, i, j) \longmapsto V$ eq. w. $p(s) = jsi$

supp cond. $jsi = 0 \to hsh = 0$. ~~$\sum p(s) V = \sum jsiV$~~

idemp. $\sum_t j s t^{-1} i j t i = jsi$

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$$\sum p(s) V = \sum jsijW = j \sum \underbrace{shW}_{shs^{-1}W} = jW = V$$

~~$0 = ip(s)hw = ijshw = hshw$ $\forall s$.~~

Assume $p(s)hw = 0$ $\forall s$. then $ip(s)hw = ijshw$

$\Rightarrow hw = \sum_s s^{-1} h s h w = 0$. $\therefore$ $V$ reduced.

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Next. show $W$ determined uniquely by $V$ / $i$ extends uniquely to a $\Gamma$-map

$$\begin{tikzcd} W \arrow[r, "\alpha"] \arrow[dr, "j"'] & \mathbb{C}\Gamma \otimes V \arrow[r, "\beta"] \arrow[d, shift right, "\eta_1"'] & W \\ & V \arrow[u, shift right, "\varepsilon_1"'] \arrow[ur, "i"'] & \end{tikzcd}$$

$\beta \sum t \otimes v(t) = \sum_t t i v(t)$

$\varepsilon_1(v) = 1 \otimes v$

$\eta_1(\sum t \otimes v(t)) = v(1)$

$\alpha w = \sum s \otimes j s^{-1} w$

$$\beta\alpha w = \sum s i j s^{-1} w = w$$

$$\alpha\beta \sum_t t \otimes v(t) = \alpha \sum_t t i v(t) = \sum_s s \otimes \sum_t \underbrace{j s^{-1} t i}_{p(s^{-1}t)} v(t)$$

Are the details of the Mor. eq. clear? You showed that given $(V, W, i, j)$ in $\mathcal{D}$ there is a diagram.

$$\begin{tikzcd} W \arrow[r, "\alpha"] \arrow[dr, "j"'] & \mathbb{C}\Gamma\otimes V \arrow[r, "\beta"] \arrow[d, shift right, "\eta_1"'] & W \\ & V \arrow[u, shift right, "\varepsilon_1"'] \arrow[ur, hook, "i"'] & \end{tikzcd}$$

where $\alpha, \beta$ $\Gamma$-maps, $\beta\alpha = 1_W$, $\alpha\beta = p$ on $\mathbb{C}\Gamma\otimes V$ $j = \eta_1\alpha$ and $i = \beta\varepsilon_1$. Thus $\alpha : W \underset{\text{canonical}}{\simeq} \operatorname{Im}(p)$,

What should you be trying to say?

Given $(V, W, i, j)$ in $\mathcal{D}$, then there are canonical $\Gamma$-maps $W \xrightarrow{\alpha} \mathbb{C}\Gamma\otimes V \xrightarrow{\beta} W$ such that $\beta\alpha = id_W$, $\alpha\beta = p$ on $\mathbb{C}\Gamma\otimes V$, $i = \beta\varepsilon_1$, $j = \eta_1\alpha$

---

Given $(V, W, i, j)$ in $\mathcal{D}$

Define $W \xrightarrow{\alpha} \mathbb{C}\Gamma\otimes V \xrightarrow{\beta} W$ by

$$\alpha w = \sum_s s \otimes j s^{-1} w \qquad \beta\Big(\sum_t t\otimes v(t)\Big) = \sum_t t\, i\, v(t).$$

Then $\beta\alpha w = \sum s\, i\, j\, s^{-1} w = w$

$$\underbrace{\alpha\beta}_{p}\Big(\sum t\otimes v(t)\Big) = \sum_s s\otimes \sum_t \underbrace{j s^{-1} t\, i}_{p(s^{-1}t)}\, v(t)$$

Conclude $\alpha, \beta$ identify $W$ with the retract of $\mathbb{C}\Gamma\otimes V$ corresp to the proj op $p$

$p(s) = j s i$ ; $p(s) = 0 \Rightarrow hsh = 0$ in general and want $i j s i j = 0 \Rightarrow j s i = 0$ you need $i$ inj $j$ surj.

but if you want to go from $B$-mod $W$ to $V$

You want to start with A module V
~~The problem~~ Having reviewed the Mor. equiv.
you now want to understand why $Bh = p(\mathbb{C}\Gamma\otimes\tilde{A})$
better, why $Bh = p(\mathbb{C}\Gamma\otimes A_r)$ where $A_r = A/\{a \mid Aa = aA = 0\}$.
Notice there is a difficulty. $p(\mathbb{C}\Gamma\otimes V)$ ?
~~Start again. Given $D = \begin{pmatrix} A & Y \\ X & B \end{pmatrix}$ you know~~

Start with left modules. An A-module V determines a B-module $X(V) = p(\mathbb{C}\Gamma\otimes V)$ which has the form $X\otimes_A V$ where $X = p(\mathbb{C}\Gamma\otimes\tilde{A})$. The problem is that you can't recover V from $X(V) = p(\mathbb{C}\Gamma\otimes\tilde{A})$, unless you specify that V is reduced (or firm). Now you can recover the reduced version of V by applying h. So there should be a canon isom. $hX(V)$ where V is reduced: $AV = V$, ${}_A V = 0$. So take V = reduced left **A-module** version of $\tilde{A} \supset A \twoheadrightarrow A/{}_A A$. It seems that if $X(A) = Bh$, then $hX(A)$ should be both $hBh$ and $A/{}_A A$

$X(V) = Bh\otimes_A V$ $\qquad\qquad$ $X(\tilde{A}) = Bh$

$hB$

$Bh \quad B$

Review yesterday's idea about the algebra $A = hBh$ being reduced as both left and right $A$-module.
(Put down this mornings idea about a (strictly) reduced Morita context)

Begin with $B$ which has left + right $\Gamma$ action giving rise to $\sum s h_s s^{-1}$ on either side. Form $hB$ with $p(s) = hs$ acting on the left

$$\Big( B = \sum shB \implies hB = \sum hshB = \sum p(s)hB$$

$$p(s)hb = hshb = 0 \quad \forall s,$$

$$hb = \sum s^{-1}hshb \implies hb = 0. \Big).$$

Similarly $A = hBh$ should be reduced left $A$-module. $Bh = \sum shs^{-1}Bh = \sum shBh$

$$\implies hBh = \sum_s hshBh = \sum_s p(s)\,hBh$$

Also if $p(s)hbh = hshbh = 0 \quad \forall s$, then

$$hbh = \sum_s s^{-1}hshbh = 0.$$

~~So you learn that~~

Different notation

Let $p(s) = jsi$ meaning

$$jsi(hbh) = hshbh \qquad bhjsi = bhsh$$

$$jsi\, hb = hshb$$

Check again you have

$$D = \begin{pmatrix} hBh & hB \\ Bh & B \end{pmatrix}$$

Let $A = hBh$ with product $(hb_1h)^*(hb_2h) = hb_1hb_2h$

assume $BhB = B$. Then

$$AY = (hBh)^*(hB) = hBhB = hB = Y$$

$$XY = (Bh)^*(hB) = BhB = B$$

$$YX = hBBh \subset hBh$$

$$hBBh \supset hBhBh = hBh$$

$$YBX = hBhBh$$

$$BX = BBh$$

$$\cancel{YXYX = YBX}$$

Start again with a ring $B$ an elt $h \in B$ assume $BhB = B$. Then get

$$D = \begin{pmatrix} hBh & hB \\ Bh & B \end{pmatrix}$$

a Morita context (strictly idempotent)

Can do most of this via dual pair $hB$, $Bh$,

$(b_1h) \times (hb_2) = b_1hb_2$. $\quad B = BhB \subset B^3$

Next what?

Keep on going. Is $D$ reduced?

$BX = BBh \supset B^3h \supset BhBh = Bh = X$

$YB = hBB \supset hBhB = hB = Y.$

Look at $A = hBh$ acting via $*$ on $\overset{hB}{\phantom{x}} = Y$

Let $y = hb'$ be such that $hbhb'h = 0$ for all $b$. ~~So you find the~~

You would like to take $a' = hb'h$ such that $\forall a = hbh$ $\quad aa' = hbhb'h = 0$, and conclude that $a' = hb'h = 0$

So why does this work in the $\Gamma$ context?

Something involving $hb'h = \sum s^{-1}hshb'h$

At some point you use $hb' = \sum s^{-1}hshb'$

You seem to be using the partition of unity. Any individual element $hb' \in hB$ can be reconstructed from the $p(s)hb'$, in fact from $\sum_s s^{-1}p(s) = \sum_s s^{-1}ysi$ ???

---

Start again with $\begin{pmatrix} A & Y \\ X & B \end{pmatrix} = \begin{pmatrix} hBh & hB \\ Bh & B \end{pmatrix}$

the Morita context is strictly idemp when $BhB = B$

$XY = BhB = B \qquad YB = hBB \supset hBhB = hB = Y$

$BX = BBh \supset BhBh = Bh = X$

$B = BhB \subset B^3 \implies B = B^2$

$YX = hBBh = hBh = A$

$XA = (Bh)*(hBh) = BhBh = Bh = X$

$AY = (hBh)*(hB) = hBhB = hB = Y$

---

~~Obs~~ Observation: Can $h$ be a multiplier of $B$? ~~to stuck~~

Continue with the example: ~~Start with~~ $h$,

$A = hBh$ $\qquad A^2 = hBhBh = hBh = A$

$\boxed{\text{Thus } BhB = B \implies A^2 = A}$ Now what about

${}_A A = \{ hb'h \mid \forall b \quad hbhb'h = 0 \}$? What happens in our example? ~~$W = Bh$~~

You first do $W = B$, $A$-module $hB$ with $p(s)hb' = hshb'$. To see $hB$ is reduced you use $hb' = \sum s^{-1}hshb'$. Similarly you take case $W = Bh$, $A$-module $hBh$ which $p(s)hb'h = hshb'h$.
Then $p(s)hb'h = 0 \quad \forall s \implies hshb'h = 0 \quad \forall s \implies hb'h = \sum s^{-1}hshb'h = 0$

So what? Look at $V = hB \quad p(s)hb' = hshb'$, seems to involve $\sum_s s^{-1}p(s)hb' = \sum s^{-1}hshb' = hb'$.

Something new here with $\sum s^{-1}p(s)$. Reminds me of ~~the grading~~ graded algebras.

$$A \xrightarrow{\Delta} \mathbb{C}\Gamma \otimes A$$

$$A_s \ni a_s \longmapsto s \otimes a_s$$

$$V \xrightarrow{\Delta} \mathbb{C}\Gamma \otimes V$$ ? Maybe this will become clearer with repetition.

$$p(s)v \longmapsto s \otimes p(s)v$$

Try again to show $hB$ as left $A = hBh$ is reduced.

Take $hb' \in hB$ ∋ $p(s)hb' = hshb' = 0 \quad \forall s$

Then $0 = \sum_s s^{-1}p(s)hb' = \sum_s s^{-1}hs\,hb' = hb'$

---

$BhB = B$, $\{b' \mid hbhb' = 0 \quad \forall b\}$. You want this to be zero, sort of a nondegenerate pairing $hB \times hB \longrightarrow hB$ better $hBh \times hBh \longrightarrow hBh$

$$A \times A \longrightarrow A$$

~~It works because of partition of~~ Leave alone.

---

Back to the example

$$\begin{pmatrix} A & Y \\ X & B \end{pmatrix} = \begin{pmatrix} hBh & hB \\ Bh & B \end{pmatrix}$$

$Y = hB$ and $A = hBh$ are left $A$-reduced.

$X = Bh$ and $A = hBh$ are right $A$-reduced

Why. $(hbh) * (hb') = hbhb' = 0$ for all $b$

take $b = s$ get $hshb' = 0 \quad \forall s$

$$0 = \sum_s s^{-1}hshb' = hb'$$

---

let's check this:

Remaining problem: To see clearly why $Bh \simeq p(\mathbb{C}\Gamma \otimes A_h)$.

$$\begin{matrix} A & Y \\ X & B \end{matrix}$$

You feel that this should follow easily from the Morita equivalence

$$\mathcal{A} \leftarrow \mathcal{D} \rightarrow \mathcal{B}$$

of left modules. ~~Ok the~~

$$V \longmapsto p(\mathbb{C}\Gamma \otimes V)$$

$$hW \longleftarrow\!\shortmid\; W$$

$$V \longleftarrow\!\shortmid\; (V, W, \iota, j) \longmapsto W$$

$p(s) = js\iota$  $h = \iota j$

This is all very simple, so start with $W = Bh$ whence the diagram

$$\begin{tikzcd}
Bh \arrow[r, "\alpha"] \arrow[dr, "j=h"'] & (\mathbb{C}\Gamma \otimes hBh) \arrow[r, "\beta"] \arrow[d, shift right, "\eta_1"'] & Bh \\
& hBh \arrow[u, shift right, "\varepsilon_1"'] \arrow[ur, "\iota = \mathrm{incl.}"'] &
\end{tikzcd}$$

$$hBh = A_h$$

---

You need to believe in this diagram.

$$\begin{tikzcd}
W \arrow[r, "\alpha"] \arrow[dr, "j=h"'] & \mathbb{C}\Gamma \otimes V \arrow[r, "\beta"] \arrow[d, shift right, "\eta_1"'] & W \\
& V \arrow[u, shift right, "\varepsilon_1"'] \arrow[ur, "\iota = \mathrm{incl}"'] &
\end{tikzcd}$$

maybe the notation $\mathbb{C}\Gamma \overset{p}{\otimes} V$

$\boxed{\text{Question}}$ projection in a $M$ context.
leads to an alg generated by $p_{ij}$ $1\le i,j \le 2$ subject to ~~satisfying~~ the relations
$$p_{ik} = \sum_j p_{ij}p_{jk}$$

---

Problem: canon. isom. $Bh \simeq \mathbb{C}\Gamma \otimes (hBh)$ should follow from the diagram

$$\begin{tikzcd}
Bh \arrow[r, "\alpha"] \arrow[dr, "j=h"'] & \mathbb{C}\Gamma\otimes hBh \arrow[r, "\beta"] \arrow[d, shift right] & Bh \\
 & hBh \arrow[u, shift right] \arrow[ur, hook, "i=inc"'] &
\end{tikzcd}$$

---

Big Hope is to somehow develop the techniques found recently for dealing with the image of an operator $h: W \to W$ as if it were idempotent. Hope to crack Volodin

---

In the above replace $Bh$ by $B$.

$$\begin{tikzcd}
B \arrow[r, "\alpha"] \arrow[dr, two heads, "j=h"'] & \mathbb{C}\Gamma\otimes hB \arrow[r, "\beta"] \arrow[d, shift right, "\eta_1"'] & B \\
 & hB \arrow[u, shift right, "\varepsilon_1"'] \arrow[ur, hook, "i=inc"'] &
\end{tikzcd}$$

$\beta\alpha = id_B$

$\alpha\beta = p$

$$\begin{tikzcd}
W = B\otimes_B W \arrow[r] \arrow[dr, "j\otimes 1_W = h"'] & \mathbb{C}\Gamma\otimes hB\otimes_B W \arrow[r] \arrow[d, shift right] & B\otimes_B W = W \\
 & hB\otimes_B W \arrow[u, shift right] \arrow[ur, "i\otimes 1_W"'] &
\end{tikzcd}$$

press

Review the argument that $A = hBh$ satisfies ${}_{A}A = 0$ and $A_{A} = 0$.

$A'$ is the alg. gen. by the elements $p(s) = hsh$ for $s \in \Gamma$ subject to the relns. ~~which satisfy~~ $p(s) = 0$ for $s \notin \Phi$, OK and

$$\sum_s p(s) * p(s^{-1}t) = \sum_s hshs^{-1}th = hth = p(t). \quad ?$$

---

Start with $B, \Gamma, h$ as usual, then form M. context

$$\begin{pmatrix} A & Y \\ X & B \end{pmatrix} = \begin{pmatrix} hBh & hB \\ Bh & B \end{pmatrix}$$

with the product defined as if $h$ were idempotent

| |
|---|
| IDEA Can this be linked to the algebra $A\langle D\rangle$ you once tried to understand, especially the role of $D^2$? |

So $A$ has element $hbh$ with product $(hb_1h)(hb_2h) = hb_1hb_2h$. So in $A$ you have elements $p(s) = hsh$ for $s \in \Gamma$ satisfying the relations. Do they generate $hBh = A$? You know $B = \Gamma \ltimes \mathcal{E}$ so every element of $B$ is a linear comb. of elts. $t h_{s_1} \cdots h_{s_n} =$

$$t s_1 h s_1^{-1} (s_2 h s_2^{-1}) \cdots (s_{n-1} h s_{n-1}^{-1}) (s_n h s_n^{-1})$$

$$p(ts_1)\, p(s_1^{-1}s_2) \cdots\cdots$$

$$p(s_1)\cdots p(s_n)$$
$$= hs_1hs_2\cdots hs_nh \quad \in hBh$$
~~But~~
$$= h(s_1hs_1^{-1})(s_1s_2hs_2^{-1}s_1^{-1})(s_1s_2s_3h\,s_3^{-1}s_2^{-1}s_1^{-1})s_1s_2s_3h \quad ?$$

$$h\,h_{t_1}h_{t_2}\cdots h_{t_3}$$

---

$$p(s_1)\cdots p(s_n) = (hs_1h)*(hs_2h)*\cdots*(hs_nh)$$
$$= hs_1hs_2\cdots hs_nh$$

$$hs_1hs_2h = h(s_1hs_1^{-1})s_1s_2h$$

You want to show that any elt of $hBh = A$ is a linear ~~elt~~ comb. of products $p(s_1)\cdots p(s_n)$ i.e. $hs_1hs_2h\cdots$

$$h(s_1hs_1^{-1})s_1s_2h \qquad \text{take } s_2 = s_1^{-1} \qquad h\,h_{s_1}h$$

$$\underline{h(s_1hs_1^{-1})(s_2hs_2^{-1})s_3h}, \quad \underbrace{\text{span } B}$$

seems obvious: take $h\,h_{s_1}\cdots h_{s_n}\,t\,h$

$$hs_1h(s_1^{-1}s_2)hs_2^{-1}s_3h\cdots hs_n^{-1}th$$

$$h\,t\,h_{s_1}h_{s_2}h = hts_1hs_1^{-1}s_2hs_2^{-1}h$$

---

So conclude that the $p(s) = hsh$ generate $hBh$ for $*$ product. & ~~satisfy~~ the relns.

~~[illegible]~~ You should write up something especially
$$\begin{pmatrix} hBh & hB \\ Bh & BhB \end{pmatrix}$$
with products defined as if $h$ were idempotent

discuss case $B = \Gamma \ltimes \mathcal{E}$

$\mathcal{E}$ generators $h_s$ $s \in \Gamma$
$$\left( \begin{array}{l} \sum_s h_s h_t = h_t \\ \sum_t h_s h_t = h_s \end{array} \right.$$

---

$$hb_1 \otimes b_2 h \qquad \begin{pmatrix} hB \otimes_B Bh & hB \\ Bh & B \end{pmatrix}$$

$$(hb_1 \otimes b_2 h) * hb_3 = hb_1 \otimes b_2 h b_3$$

$$hb_1 b_2 h * hb_3 = hb_1 b_2 h b_3$$

---

~~[illegible]~~ You want to put the construction in perspective, a good framework. How to proceed? There is something related to factoring $h = y$ ~~You want to factor~~ Start with $B$, then construct $hB$, $Bh$ and the pairing $Bh \times hB \to B$

~~[illegible]~~ What is ~~[illegible]~~ significant is the left $B$ module $X = Bh$, right module $Y = hB$ and the pairing $Y \otimes_B X \longrightarrow B$ being ~~[illegible]~~ surjective. It's perhaps not important that there are inclusions ~~[illegible]~~ $X = Bh \hookrightarrow B$ or $Y = hB \hookrightarrow B$

In fact you have seen that you don't want to think of both $X = Bh$ and $Y = hB$ being contained in $B$, rather you would like one to be a $B$ submodule and the other a $B$ quotient module.

Given $(V, W, \iota, j)$ with $h = \iota j : W \xrightarrow{j} V \xhookrightarrow{\iota} W$

If you $W = B$, then you get

$$B \xrightarrow{j = h} hB \xhookrightarrow{\iota = in} B$$

$$hB = Y$$

$$B \xrightarrow{\iota} Bh \xhookrightarrow{\iota} B$$

$$\begin{pmatrix} & jB = hB \\ B\iota = Bh & B \end{pmatrix}$$

Examine the pairing $Bh \times hB \longrightarrow BhB$

to find $\langle \overset{x}{b_1 h}, \overset{y}{hb_2} \rangle = b_1 h b_2$, you do

either, lift $x$ to $b_1$ and apply to $y$ getting $b_1 y$

or lift $y$ to $b_2$ and apply to $x$ getting

$x b_2 = b_1 h b_2$

---

Again: To define $x * y$ where $x = b_1 h$ and $y = h b_2$ you lift $x$ to $b_1$ and right mult by $y$ to get $x * y = b_1 y = b_1 h b_2$, or you lift $y$ to $b_2$ and left mult by $x$ to get $x * y = x b_2 = b_1 h b_2$

Is it possible to understand better modules over the Morita context $\begin{pmatrix} hBh & hB \\ Bh & B \end{pmatrix}$ ?

$$\begin{pmatrix} hBh & hB \\ Bh & B \end{pmatrix}\begin{pmatrix} V \\ W \end{pmatrix}.$$

I think that you want to focus on factorization. So replace $\begin{pmatrix} hBh & hB \\ Bh & B \end{pmatrix}$ by $\begin{pmatrix} jBi & jB \\ Bi & B \end{pmatrix}$, which you understand ~~as a dual pair~~ as associated to the dual pair $jB$, $Bi$, $Bi \times jB \longrightarrow B$.

---

Let us start with $B$, $h \in B$ such that $BhB = B$. Then define a dual pair by $Y = B$ as $B^{op}$ module, $X = B$ as $B$-module, and $\langle x, y \rangle = xhy$. Then you have a surjective map of dual pairs over $B$ given by

$$\begin{pmatrix} & B \\ B & B \end{pmatrix} \longrightarrow \begin{pmatrix} & hB \\ Bh & B \end{pmatrix}$$

$$(b_1 \in Y,\ b_2 \in X) \longmapsto (hb_1 \in hB,\ b_2 h \in Bh)$$

$$\langle b_2, b_1 \rangle = b_2 h b_1 \qquad b_2 h * h b_1 = b_2 h b_1$$

Repeat what you learned namely that your dual pair $(hB, Bh, b_2h * hb_1 = b_2hb_1)$ is a quotient of the dual pair $(B, B, \langle b_2, b_1\rangle = b_2hb_1)$. Thus ~~one~~ by factoring

$$\begin{array}{ccccc} B \times B & \longrightarrow & Bh \times hB & \longrightarrow & B \\ (b_2, b_1) & \longmapsto & b_2h * hb_1 & = & b_2hb_1 \end{array}$$

you make the pairing less degenerate.

Is this discussion relevant to the ~~[illegible]~~ $\Gamma$ situation? no

---

Yesterday you demystified the Morita context $\begin{pmatrix} hBh & hB \\ Bh & B \end{pmatrix}$, which you hoped might serve to treat any element of a ring as an idempotent. Idea is ~~[illegible]~~ that the dual pair $(hB, Bh, \langle b_2h, hb_1\rangle = b_2hb_1)$ is a quotient of $(B, B, \langle b_2, b_1\rangle = b_2hb_1)$. ~~This~~

---

~~[illegible]~~ So you get the following picture

$$\begin{pmatrix} hB\otimes_B Bh & hB \\ Bh & B \end{pmatrix} = \begin{pmatrix} \cancel{B\otimes_B B} & B/{}_hB \\ B/B_h & B \end{pmatrix}$$

But notice that ~~[illegible]~~ this looks different from $\begin{pmatrix} & jB \\ Bi & B \end{pmatrix}$

You think of $i \big| j$ as injective, $j$ — surjective $h = ij$

[redacted] Given $(V, W, W \overset{j}{\twoheadrightarrow} V \overset{i}{\hookrightarrow} W)$ you?

What is $Bi$? It is $B/\{b \mid bi = 0\}$.

$bi = 0 \iff bij = 0$ when $j$ surj.

So $Bi = B/B_h$. $jB = B/\{b \mid jb = 0\}$.

$jb = 0 \iff ijb = 0$ when $i$ injective

$\therefore\ jB = B/{}_hB$.

---

To find what to do next? Let's go over what happens when $B = \Gamma \ltimes \mathcal{E}$ $\mathcal{E}$ has generators $h_s$ $s \in \Gamma$ subject to relations $h_s h_t = 0$ $s^{-1}t \in \Phi$ and $\sum_s h_s h_t = h_t$, $\sum_t h_s h_t = h_s$. At this point today you want to complete your understanding of the Morita context & Morita equivalence. How?

First recall the categorical equivalence

$$\mathcal{A} \longleftarrow \mathcal{D} \xrightarrow{\sim} \mathcal{B}$$

$\mathcal{A}$ consists

$\mathcal{B}$ cons of $W$ ($\Gamma$), $h : W \to W \mid \sum s h s^{-1} = 1_W$, $hsh = 0$

$\mathcal{D}$ Consists of $(V, W, i : V \hookrightarrow W, j : W \twoheadrightarrow V)$ ($V$ vs, $W$ $\Gamma$ mod) such that

$$jsi = 0 \quad s \notin \Phi$$

$$\sum_s sijs^{-1} = 1_W$$

Notice that the $\mathcal{D}, \mathcal{B}$ equivalence is easy, just like constructing $D$ [redacted] from $B, h$. You may be able now to deal with adjoining $i, j$ so that $ij = h$

What to do: Essentially look again at the generators + relations construction of something like $D$.

Recall the def: $D$ is the Morita context ($M_2$-graded alg) with gens $x_t$ of degree $21$, $y_s$ ——— $12$, $t, s \in \Gamma$

subject to the relations $y_s x_t = y_{us} x_{ut}$ ($= 0$ if $s^{-1}t \notin \Phi$)

$$\sum_s x_s y_s x_t = x_t, \quad \sum_t{}' y_s x_t y_t = y_s$$

You want to fit this into the $\begin{pmatrix} & hB \\ Bh & B \end{pmatrix}$ pattern. Is it possible to get $X = Bh$, $Y = hB$ using $\Gamma i = \{x_t \mid t \in \Gamma\}$ and $j\Gamma = \{y_s \mid s \in \Gamma\}$?

You were hoping for some version of $X = B/B_h$. What happens? $B = \mathbb{C}\Gamma \otimes \mathcal{E} \Rightarrow Bh = \mathbb{C}\Gamma \otimes \mathcal{E}h$. What about $\mathcal{E}h$? Recall $\mathcal{E}$ has generators $h_s = shs^{-1}$ and $h_s h = shs^{-1}h = 0$ for $s^{-1} \notin \Phi$.

You have $B$ defined nicely with $\Gamma$ acting as multipliers. Same for $D$

---

Idea: You have this M-cont $\begin{pmatrix} hBh & hB \\ Bh & BhB \end{pmatrix}$

assume $BhB = B$ so that $B \subset BhB \subset B^3 \subset B^2$.

Then there's a Morita equivalence around which might be easy to describe as $(V, W \dots)$ etc.

$$\begin{pmatrix} & B/B_h \\ B/B_h & B \end{pmatrix} = \begin{pmatrix} & hB \\ Bh & B \end{pmatrix}$$

So $\begin{pmatrix} & B/_hB \\ B/B_h & B \end{pmatrix} = \begin{pmatrix} & hB \\ Bh & B \end{pmatrix}$

Put $A' = Y \otimes_B X = hB \otimes_B Bh$

$$= \left(B/_hB\right) \otimes_B \left(B/B_h\right) = \frac{B\otimes_B B}{{}_hB\otimes_B B + B \otimes_B B_h}$$

You have ring surjection $A' \longrightarrow hBh = A$
kernel killed by $A$ on left + right

Can you interpret this Morita context via modules? First idea is to use $\begin{pmatrix} hBh & hB \\ Bh & B \end{pmatrix}$

~~Modules should be~~ $\cancel{({}_A X, {}_B\ldots)}$ to obtain the M.eq.

$$W \longmapsto hB \otimes_B W \longrightarrow hW$$

$$W = Bh \otimes_{hBh} V \longleftarrow\!\!\!| \; V$$

Question: Is $W \longmapsto hW$ from $B$-modules to $hBh = A$-modules a Morita equivalence?

---

You have this Morita ~~equa~~ context $\begin{pmatrix} hBh & hB \\ Bh & B \end{pmatrix}$

~~now ... ...~~ what is the corresponding equivalence of cats? Point is that for a left $B$-mod $W$ such that $BW = W$ you have

$$hB \otimes_B W \twoheadrightarrow hW$$

the kernel should be killed by $A = hBh$

$$hbh * \sum hb_i \otimes w_i = \sum hbhb_i \otimes w_i$$

What's ~~what~~ new is that instead of the usual possibilities for the functors

$$W \longmapsto \left\{ \text{either of } Y \otimes_B W \longrightarrow \mathrm{Hom}_B(X, W) \right\}$$
$$y \otimes w \longmapsto (x \mapsto (xy)w)$$

or the image of this map. Is it possible that

$$\mathrm{Im}\left\{ Y \otimes_B W \longrightarrow \mathrm{Hom}_B(X, W) \right\} \qquad \text{where } \begin{array}{l} Y = hB \\ X = Bh \end{array}$$

actually gives $hW$?

$$hb \otimes_B w \longmapsto \left( b'h \mapsto \underbrace{(b'h * hb)w}_{b'hbw} \right)$$

$$Y \otimes_B W \longrightarrow \mathrm{Hom}_B(X, W)$$

$$hB \otimes_B W \longleftrightarrow \mathrm{Hom}_B(Bh, W)$$

---

OKAY. you have the map

$$hB \otimes_B W \longrightarrow \mathrm{Hom}_B(Bh, W) \qquad \text{does what?}$$
$$hb \otimes w \longmapsto (b'h \mapsto (b'hb)w)$$
$$hbw \mapsto (b'h \mapsto b'h * hbw)$$

So basically ~~you~~ corresp to $BW = W$, you want ${}_BW = 0$, and then ~~[illegible]~~

$$hW \longrightarrow \mathrm{Hom}_B(Bh, W)$$
$$hw \longmapsto (b'h \longmapsto b'hw)$$

is injective.

Repeat this. Given a $B$ module $W$ ~~the~~ which is reduced: $BW = W$, ${}_BW = 0$, then $\mathrm{Im}\{ Y \otimes_B W \longrightarrow \mathrm{Hom}_B(X, W)\}$ should be the reduced $A = hBh$ module corresponding to $W$:

$$\mathrm{Im}\{ hB \otimes_B W \longrightarrow \mathrm{Hom}_B(Bh, W)\}$$

the map is $hb \otimes w \longmapsto (b'h \longmapsto b'hbw)$

and it factors

$$hb \otimes w \qquad\qquad\qquad (b'h \mapsto b'hbw)$$

$$\begin{tikzcd}
hB \otimes_B W \arrow[rr] \arrow[dr, twoheadrightarrow, "\text{if } BW=W"'] & & \mathrm{Hom}_B(Bh, W) \\
& hW \arrow[ur, hook, "\text{if } {}_BW = 0"'] &
\end{tikzcd}$$

$$hbw, \quad hw' \longmapsto (b'h \mapsto b'hw')$$

---

Your program is to describe nicely the Morita equivalence assoc. to $\begin{pmatrix} hBh & hB \\ Bh & B \end{pmatrix}$ when $B = BhB$

~~And~~ It seems that

To understand the Morita equivalence to the Morita context $\begin{pmatrix} hBh & hB \\ Bh & B \end{pmatrix}$ when $BhB = 1$ with $*$ product

It seems that it is best to use the reduced module picture.

Claim: $\begin{pmatrix} A & Y \\ X & B \end{pmatrix}$ as above

satisfies $A^2 = A = YX \qquad Y = AY = YB$

$XA = BX = X \qquad B = B^2 = XY$

$$A^2 = (hBh)*(hBh) = hBhBh = hBh = A$$
$$YX = hBBh = hBh = A.$$

(need $B = BhB \Rightarrow B \subset B^3 \subset B^2$ $\therefore B = B^2$.)

Better $B = BhB \subset B \cdot B$

$$XY = Bh * hB = BhB = B$$

$$AY = hBh * hB = hBhB = hB = Y$$

$$YB = hBB = hB = Y$$

$$XA = Bh * hBh = BhBh = Bh = X$$

$$BX = BBh = Bh = X.$$

What I learned is that the functors

$W \mapsto hB \otimes_B W$ ,

You know I think that the equivalence

$$\mathcal{M}(B) \xrightarrow{\sim} \mathcal{M}(A) \qquad \text{is given by}$$

$$W \longmapsto hW$$

the inverse ~~functor~~ being $V \mapsto Bh \otimes_A V$ roughly. Did show that the image of

$$\underset{\underset{hB}{\parallel}}{Y} \otimes_B W \longrightarrow \mathrm{Hom}_B(\underset{\underset{Bh}{\parallel}}{X}, W)$$

$$hb \otimes w \longmapsto (b'h \mapsto b'hbw)$$

$$hbw, \; hw \mapsto ($$

is $hW$.

$$hB \otimes_B W \xrightarrow{\text{since } BW=W} hW \xhookrightarrow{\text{if } {}_BW=0} \mathrm{Hom}_B(Bh, W)$$

$$hb \otimes w \mapsto hbw \;, \quad hw \mapsto (b'h \mapsto b'hw)$$

So where are we?? In control of the functors

$$\mathcal{M}_r(B) \longrightarrow \mathcal{M}_r(A) \qquad \text{Check directly} \quad \cancel{hW \subset W}$$
$$W \longmapsto hW. \qquad \cancel{\text{so that if } Bhw = 0, \text{ then } hw = 0}$$

$$(hBh)*(hW) = hBhW = hBhBW = hBW = hW$$

If $(hBh)*(hw) = hBhw = 0$, then $BhBhw = 0$

$$\underset{Bhw}{\parallel} \;= 0$$

so $hw = 0$. So $hW$ is $A$-reduced. In particular $hB$ and $hBh$ should be $A$-reduced

So let's check this carefully.

$$\begin{pmatrix} A & y \\ X & B \end{pmatrix} = \begin{pmatrix} hBh & hB \\ Bh & B \end{pmatrix}$$

$BhB = B$ with $*$ product $(bh)*(hb') = bhb'$ etc.

Now what you should be able to prove is the equivalence $\mathcal{M}_r(B) \longrightarrow \mathcal{M}_r(A)$

$$W \longmapsto hW$$

Check: $B = BhB \subseteq BB = B^2$. Assume $BW = W$ and ${}_BW = 0$

Then $A(hW) = (hBh)*(hW) = hBhW = hBhBW = hW$

$A(hw) = 0 \Rightarrow hBhw = 0 \Rightarrow 0 = BhBhw = Bhw$

$\Rightarrow hw = 0$. Next given $V$ $A$-reduced module: $AV = V$ and ${}_AV = 0$ let

$$W = \mathrm{Im}\{Bh \otimes_A V \longrightarrow \mathrm{Hom}_A(hB, V)\}$$

$$\overbrace{Bh \otimes_A V}^{B\text{ conilfree}} \twoheadrightarrow W \hookrightarrow \overbrace{\mathrm{Hom}_A(hB, V)}^{B\text{ nilfree}}$$

$$bh \otimes v \longmapsto (hb' \longmapsto hb'bhv)$$

$$\mathrm{Hom}_B(B, \mathrm{Hom}_A(hB, V)) = \mathrm{Hom}_A(hB \otimes_B B, V)$$

$$\begin{array}{c} \mathrm{Hom}_A(hB \otimes_B B, V) \\ \uparrow s \\ \mathrm{Hom}_A(hB, V) \end{array}$$

$$0 \longrightarrow K \longrightarrow hB \otimes_B B \longrightarrow hB \longrightarrow 0$$

$$0 \longleftarrow \mathrm{Hom}_A(hB \otimes_B B, V) \xleftarrow{\sim} \mathrm{Hom}_A(hB, V) \longleftarrow 0$$

So where are you $\begin{pmatrix} A & Y \\ X & B \end{pmatrix} = \begin{pmatrix} hBh & hB \\ Bh & B \end{pmatrix}$

Assuming $B^2=0$, ${}_B B = B_B = 0$ then you know that $hB$, $Bh$ are A-reduced. You would like $Bh$ to be B reduced. Part of this $B(Bh) = B^2h = Bh$ is OKAY. But there might be problems with ${}_B Bh = 0$, NO

$${}_B Bh = \{ bh \in Bh \mid Bbh = 0 \}$$

---

$$\begin{pmatrix} A & Y \\ X & B \end{pmatrix} = \begin{pmatrix} hBh & hB \\ Bh & B \end{pmatrix}$$

Assume
$BhB = B$
${}_B B = 0 = B_B$

Can you weaken the latter to ${}_B B_B = 0$

---

Consider $\begin{pmatrix} A & Y \\ X & B \end{pmatrix} = \begin{pmatrix} hBh & hB \\ Bh & B \end{pmatrix}$ with $*$ prod. as if $h^2=h$

$BhB = B$.

Claim: ~~strictly idempotent.~~ Claim $\mathcal{M}_r(B) \rightleftarrows \mathcal{M}_r(A)$

$W \mapsto hW$, inverse $V \mapsto Bh \otimes_A V$

$$V \mapsto \mathrm{Im}\{ Bh \otimes_A V \longrightarrow \mathrm{Hom}_A(hB, V) \}$$

$$\mathrm{Hom}_B(B, \mathrm{Hom}_A(hB, V)) = \mathrm{Hom}_A(hB \otimes_B B, V)$$
$$= \mathrm{Hom}_A(hB, V)$$

$\cancel{W \mapsto \mathrm{Im}\{ hB \otimes_B W \to \mathrm{Hom}_B(Bh, W) \}}$

$$\operatorname{Im}\{hB\otimes_B W \xrightarrow{\ hW\ } \operatorname{Hom}_B(Bh, W)\}$$

$$hb\otimes w \mapsto hbw, \quad hw \mapsto (b'h \mapsto b'hw)$$

$\cancel{\operatorname{Hom}_B(B, \operatorname{Hom}_B(Bh, W)) \simeq \operatorname{Hom}_B}$

It seems that all you can say is that $W \mapsto hW = V$ is A-red : when ${}_B B \cancel{h} = 0$.

$0 = hBh * hw = hBhw \implies Bhw = BhBhw = 0 \quad \therefore hw = 0$

You want to write up something

---

~~As~~ $U(n,1)$ acts on $V = \mathbb{C}^n \oplus \mathbb{C}$ preserving the hermitian form ~~xxxx~~ $H(\xi) = \|\xi_+\|^2 - \|\xi_-\|^2$ then $U(n,1)$ acts on $V \otimes H$, a Krein space, observation: Lagrangian ~~sub~~ subspace of $V\otimes H = H^{\oplus n} \oplus H$ is same as the graph of a unitary isom $H^n \simeq H$, that is, an orth family of isometries $s_i : H \to H$ $s_i^* s_j = \delta_{ij}$ such that $\sum s_i s_i^* = 1$, same as a unital $*$hom. $\mathcal{O}_n \longrightarrow \mathcal{L}(H)$. The action of $U(n,1)$ on Lagrangian subspaces

First understand $n=1$, where $H$ can be finite diml.

---

The idea is for each $g \in U(n,1)$ to produce an alg autom $\mathcal{O}_n \longrightarrow \mathcal{O}_n$. You propose to make $\mathcal{O}_n$ act on the $\mathcal{O}_n^{op}$-module $\mathcal{O}_n$. What does this mean?

The idea should be ~~x~~ to mimic the Hilbert

You have $K = V \otimes H$ a Krein space

$$K = V_+ \otimes H \oplus V_- \otimes H \qquad \text{polarization}$$

$L \subset K$ $\quad$ $L$ Lagrangian. You know that $L = \begin{pmatrix} g \\ 1 \end{pmatrix} K$

---

need to understand $n = 1$.

$$K = \begin{pmatrix} V_+ \\ V_- \end{pmatrix} \otimes H = \begin{pmatrix} V_+ \otimes H \\ V_- \otimes H \end{pmatrix}$$

$$L = \begin{pmatrix} 1 \\ T \end{pmatrix} (V_+ \otimes H) \subset \begin{pmatrix} V_+ \otimes H \\ V_- \otimes H \end{pmatrix}$$

$L$ is the graph of a unitary isom from $V_+ \otimes H$ to $V_- \otimes H$. Now given $g \in U(V)$ you get a different Lagrangian subspace $g(L)$

$$L = \begin{pmatrix} z \\ 1 \end{pmatrix} \mathbb{C} \qquad g = \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix}$$

$$g(L) = \begin{pmatrix} az + b \\ \bar{b} z + \bar{a} \end{pmatrix} \mathbb{C} = \begin{pmatrix} \frac{az+b}{\bar{b}z + a} \\ 1 \end{pmatrix} \mathbb{C}$$

---

Start again. $A$ $*$ homom. (unital) $\mathcal{O}_n \to \mathcal{L}(H)$ same as a unitary isom $\mathbb{C}^n \otimes H \xleftarrow{\sim} H$, same as a Lag. $L \subset \begin{pmatrix} \mathbb{C}^n \otimes H \\ H \end{pmatrix}$ $\qquad L = \begin{pmatrix} u \\ 1 \end{pmatrix} H$

You take $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in U(n,1)$. Then $gL$

space action. Form $V \otimes H$ Krein space and consider all polarizations. If you have chosen one $V \otimes H = V_+ \otimes H \oplus V_- \otimes H$ then any other is given by the graph of an isometry $V_+ \otimes H \xrightarrow{\sim} V_- \otimes H$.

---

First do Hilbert space reps of $O_n$. $V = \begin{pmatrix} \mathbb{C}^n \\ \mathbb{C} \end{pmatrix}$ equipped with $v^* \varepsilon v$ herm. form $\varepsilon = \begin{pmatrix} 1_n & 0 \\ 0 & -1 \end{pmatrix}$.

$H$ Hilbert space form $V \otimes H$ equip with tens. prod herm form $(v \otimes \xi)^* (\varepsilon \otimes 1)(v \otimes \xi) = (v^* \varepsilon v)|\xi|^2$. Has polarization $(V_+ \otimes H) \oplus (V_- \otimes H)$ Krein space.

Result describes ~~polarizations~~ Lagrangian subspace as a unitary equiv. of $V_+ \otimes H \xrightarrow{\sim} V_- \otimes H$

~~You know then that given~~

$V$ standard rep of $U(n,1)$, has natural herm. form $v^* \varepsilon v$ preserved by $U(n,1)$, Tensor with $H$ to get a Krein space $V \otimes H$.

The point you have missed maybe is the fact that there's ~~a polar~~ both a polarization and a Lagrangian subspace involved. Thus you have $V \otimes H$ acted on by $G = U(n,1) \otimes 1$

~~[illegible]~~ So $G$ acts on the Lag. subspaces

---

You have Lagrangian subspaces $W \subset V \otimes H$ so if you are given the polarization $V_+ \otimes H \oplus V_- \otimes H$

$\mathcal{O}_n \longrightarrow \mathcal{L}(H)$ same as $u : H \xrightarrow{\sim} \mathbb{C}^n \otimes H$

same as $\Gamma_u = \begin{pmatrix} u \\ 1 \end{pmatrix} H \subset \begin{pmatrix} \mathbb{C}^n \otimes H \\ H \end{pmatrix}$

Now take $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in U(n,1)$. It should be ~~clear~~ true that $g\Gamma_u = \begin{pmatrix} au+b \\ cu+d \end{pmatrix} H$ is a Lagrangian subspace of $\begin{pmatrix} \mathbb{C}^n \otimes H \\ H \end{pmatrix}$, hence should be $\Gamma_{u'} \subset \begin{pmatrix} \mathbb{C}^n \otimes H \\ H \end{pmatrix}$

where $u' = (au+b)(cu+d)^{-1}$

$$u'' = (\alpha u' + \beta)(\gamma u' + \delta)^{-1}$$

$$= \left[\alpha(au+b)(cu+d)^{-1} + \beta\right]\left[\gamma(au+b)(cu+d)^{-1} + \delta\right]^{-1}$$

$$= \left[\alpha(au+b) + \beta(cu+d)\right]\cancel{(cu+d)^{-1}(cu+d)}\left[\gamma(au+b) + \delta(cu+d)\right]^{-1}$$

$$= \left[(\alpha a + \beta c)u + (\alpha b + \beta d)\right]\left[(\gamma a + \delta c)u + (\gamma b + \delta d)\right]^{-1}$$

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \alpha a + \beta c & \alpha b + \beta d \\ \gamma a + \delta c & \gamma b + \delta d \end{pmatrix}$$

---

Let's see if it is possible to ~~settle the establ~~ understand the action. First you want to show how $U(n,1)$ acts on the unital $*$ repns of $\mathcal{O}_n$ on a fixed $H$. $\underbrace{\mathrm{Hom}^*(\mathcal{O}_n, \mathcal{L}(H))}$

What is an element of $\wedge$, answer an isom $u : H \xrightarrow{\sim} \mathbb{C}^n \otimes H$

equiv. $s_i \in \mathcal{L}(H)$ $s_i^* s_j = \delta_{ij}$, $\sum s_i s_i^* = 1$.

equiv. a Lag subspace $L \subset \begin{pmatrix} \mathbb{C}^n \otimes H \\ H \end{pmatrix}$ $\qquad L = \begin{pmatrix} u \\ 1 \end{pmatrix} H$

$\mathrm{Hom}^{\mathrm{un}*\mathrm{alg}}(\mathcal{O}_n, \mathcal{L}(H)) = \{u : H \xrightarrow{\sim} \mathbb{C}^n \otimes H \text{ unitary}\}$

Such a $u$ same as the Lag subsp $\Gamma_u = \begin{pmatrix} u \\ 1 \end{pmatrix} H \subset \begin{pmatrix} \mathbb{C}^n \otimes H \\ H \end{pmatrix}$

Action of $U(n,1)$ on $\begin{pmatrix} \mathbb{C}^n \\ \mathbb{C} \end{pmatrix}$ preserving $\xi^* \varepsilon \xi = \|\xi_+\|^2 - \|\xi_-\|^2$

Thus the group $U(n,1)$ acts on the set of unital $*$ alg homoms. $\mathcal{O}_n \longrightarrow \mathcal{L}(H)$ for any Hilbert space $H$. The problem is to understand why this action $g \mapsto g(u)$ is given by an action of $U(n,1)$ on the $C^*$-algebra $\mathcal{O}_n$.

~~[illegible]~~

~~Given $g \in U(n,1)$ and $\tilde{u} : \mathcal{O}_n \to \mathcal{L}(H)$~~ $\widetilde{g(u)} : \mathcal{O}_n \to$

Given

$$\begin{tikzcd} \mathcal{O}_n \arrow[r, "\tilde{u}"] & \mathcal{L}(H) \\ \mathcal{O}_n \arrow[u, dashed, "g_*"] \arrow[ur, "\widetilde{g(u)}"'] & \end{tikzcd}$$

why does $\exists$ a $g_* : \mathcal{O}_n \longrightarrow \mathcal{O}_n$ $\ni$ $\tilde{u} g_* = \widetilde{g(u)}$

Maybe this follows from a formula.

Let $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in U(n,1)$, $u : H \xrightarrow{\sim} \mathbb{C}^n \otimes H$

$$g \Gamma_u = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} u \\ 1 \end{pmatrix} H = \begin{pmatrix} au+b \\ cu+d \end{pmatrix} H$$

$$= \begin{pmatrix} (au+b)(cu+d)^{-1} \\ 1 \end{pmatrix} H = \Gamma_{g(u)}$$

where $g(u) = (au+b)(cu+d)^{-1}$.