A coalgebra map $\mathbb{C}\Gamma \otimes \mathbb{C}\Gamma \xrightarrow{\mu} \mathbb{C}\Gamma$ is equivalent to a map $\Gamma_+ \wedge \Gamma_+ \longrightarrow \Gamma_+$ of pointed sets. which is the same as a map $\Gamma_+ \times \Gamma_+ \longrightarrow \Gamma_+$, denote it as a product $(x, x') \longmapsto xx'$ satisfying $0x' = 0$ and $x'0 = 0$. Check associativity. The square

$$
\begin{tikzcd}
\mathbb{C}\Gamma \otimes \mathbb{C}\Gamma \otimes \mathbb{C}\Gamma \arrow[r, "1\otimes\mu"] \arrow[d, "\mu\otimes 1"'] & \mathbb{C}\Gamma \times \mathbb{C}\Gamma \arrow[d, "\mu"] \\
\mathbb{C}\Gamma \otimes \mathbb{C}\Gamma \arrow[r, "\mu"] & \mathbb{C}\Gamma
\end{tikzcd}
$$

$$
\begin{tikzcd}
\mathbb{C}\Gamma \otimes \mathbb{C}\Gamma \otimes \mathbb{C}\Gamma \arrow[r, shift left, "\mu\otimes 1"] \arrow[r, shift right, "1\otimes\mu"'] & \mathbb{C}\Gamma \times \mathbb{C}\Gamma \arrow[r, "\mu"] & \mathbb{C}\Gamma
\end{tikzcd}
$$

$$\mu(\mu\otimes 1) = \mu(1\otimes\mu)$$

$$
\Leftrightarrow \quad
\begin{tikzcd}
\Gamma_+ \wedge \Gamma_+ \wedge \Gamma_+ \arrow[r, shift left, "\mu\wedge 1"] \arrow[r, shift right, "1\wedge\mu"'] & \Gamma_+ \wedge \Gamma_+ \arrow[r, "\mu"] & \Gamma_+ \\
\Gamma_+ \times \Gamma_+ \times \Gamma_+ \arrow[u, two heads] \arrow[r, shift left, "\tilde\mu\times 1"] \arrow[r, shift right, "1\times\tilde\mu"'] & \Gamma_+ \times \Gamma_+ \arrow[u] \arrow[r, "\tilde\mu"] & \Gamma_+ \arrow[u, equal]
\end{tikzcd}
$$

?

---

Let $X$ be a set with basepoint $0$

Let $\mu : X \times X \longrightarrow X$ be a binary operation such that $\mu(0,x) = \mu(x,0) = 0$. Consider

$$X \times X \times X$$

$$(X \wedge X) \times X$$

Way to handle the problem. Let $X$ be a pointed set with basepoint denoted $0$, let $\bar{\mathbb{C}}X = \mathbb{C}X/\mathbb{C}0$ be the corresp coalg with $X = \text{Points}(\bar{\mathbb{C}}X)$. Then $\bar{\mathbb{C}}X \otimes \bar{\mathbb{C}}X = \bar{\mathbb{C}}[X \wedge X]$. I should have pointed out that $X \hookrightarrow \bar{\mathbb{C}}X$, $X$ can be identified with the subset of points. Now use $\bar{\mathbb{C}}X \otimes \bar{\mathbb{C}}X = \bar{\mathbb{C}}[X \wedge X]$.
Then a product $\bar{\mathbb{C}}X \otimes \bar{\mathbb{C}}X \longrightarrow \bar{\mathbb{C}}X$ respecting the comodule structure is equivalent to a map $\mu : X \wedge X \longrightarrow X$, equivalently a product $X \times X \longrightarrow X$ say $(x,y) \longmapsto xy$ such that $x0 = 0$, $0x = 0$, $\forall x$.
Note that

$$\begin{tikzcd}
X \wedge X \arrow[r, "\mu"] \arrow[d] & X \arrow[d] \\
\bar{\mathbb{C}}X \otimes \bar{\mathbb{C}}X \arrow[r] & \bar{\mathbb{C}}X
\end{tikzcd}$$

Now the associativity of the product on $\bar{\mathbb{C}}X$ is equivalent to

$$\begin{tikzcd}
X \wedge X \wedge X \arrow[r, "1 \wedge \mu"] \arrow[d, "\mu \wedge 1"'] & X \wedge X \arrow[d, "\mu"] \\
X \wedge X \arrow[r, "\mu"] & X
\end{tikzcd}$$

comm.

Important that the product on $X$ which is

$$\begin{array}{ccccc}
X \times X & \longrightarrow & X \wedge X & \xrightarrow{\mu} & X \\
(x, y) & & (x \wedge y) & & \mu(x \wedge y)
\end{array}$$

is given by embedding $x, y$ into $\bar{\mathbb{C}}X$ and using the product in $\bar{\mathbb{C}}X$. Thus assoc. for $\mu$

Back to the Morita context. ~~[illegible]~~

$\Gamma$ = "units" $e_{ij}$ in $M_2\mathbb{C}$. Then $\Gamma_+ = \Gamma \cup \{0\}$ is closed under multiplication in $M_2\mathbb{C}$, so can define a Morita context as a $\Gamma$ graded algebra

Some lemma to formulate. Notion of $\Gamma$-graded algebra makes sense for a set $\Gamma$ ~~[illegible]~~ such that $\Gamma_+$ is equipped with an associative multiplication such that the basepoint $0$ is absorbing: $0s = s0 = 00 = 0$

$\Gamma_+ = \Gamma \cup \{0\}$ has semigroup structure such that $0$ is absorbing. ~~[illegible]~~

Form $\mathbb{C}\Gamma$ coalgebra with points $\Gamma_+ \hookrightarrow \mathbb{C}\Gamma$

Then
$$\begin{tikzcd} \Gamma_+ \wedge \Gamma_+ \arrow[r, hook] \arrow[d, "\text{prod}"'] & \mathbb{C}\Gamma \otimes \mathbb{C}\Gamma \arrow[d] \\ \Gamma_+ \arrow[r, hook] & \mathbb{C}\Gamma \end{tikzcd}$$

So $\mathbb{C}\Gamma$ becomes a Hopf algebra

Let $A$ be a ring, let $\Gamma$ be a subset of $A$ not containing $0$ such that $\Gamma_+ = \Gamma \sqcup \{0\}$ is closed under multiplication in $A$. There is a universal such alg namely $\mathbb{C}\Gamma$.

---

Turn to comodules $V \xrightarrow{\Delta_V} \mathbb{C}\Gamma \otimes V$

$$\Delta_V(v) = \sum_s s \otimes e_s v \quad \begin{array}{l} \xrightarrow{\Delta\otimes 1} \sum_s s\otimes s \otimes e_s v \\ \qquad\qquad \| \\ \xrightarrow{1\otimes\Delta_V} \sum_{s,t} s \otimes t \otimes e_t e_s v \end{array}$$

$\therefore$ $e_s = e_s^2$
$e_s e_t = 0 \quad s \neq t$

$$V = \bigoplus_{s\in\Gamma} e_s V \oplus V_0 \qquad V_0 = (1-e)V \quad e = \sum_{s\in\Gamma} e_s$$

$$V \xrightarrow{\Delta} \mathbb{C}\Gamma \otimes V \quad , \quad W \xrightarrow{\Delta} \mathbb{C}\Gamma \otimes W$$

$$\begin{array}{ccc} V \otimes W & \longrightarrow & \mathbb{C}[\Gamma \times \Gamma] \otimes (V \otimes W) \\ & & \downarrow \mu \\ & & \mathbb{C}\Gamma \otimes (V \otimes W) \end{array}$$

$$V = \bigoplus_{s \in \Gamma_+} V_s \qquad W = \bigoplus_{t \in \Gamma_+} W_t$$

$$V \otimes W = \bigoplus_{s,t} V_s \otimes W_t \qquad s,t \in \Gamma_+$$

$$(V \otimes W)_u = \bigoplus_{u = st} V_s \otimes W_t$$

Easier should be to define $\Gamma$-graded algebra.

---

Review $\Gamma$ set, $\mathbb{C}\Gamma$ is coalg $\Delta s = s \otimes s \quad s \in \Gamma$

point of a coalg $C$ is $\xi \in C \quad \Delta \xi = \xi \otimes \xi$.

Points of $\mathbb{C}\Gamma = \Gamma \amalg 0 = \Gamma_+$

get equivalence between the category of pointed sets (~~sets~~ with basepoint) and the category of setlike coalgebras.

$$\Gamma_+ \longmapsto \mathbb{C}\Gamma_+ / \mathbb{C}[0] \simeq \mathbb{C}\Gamma$$

$$\text{Points of } \mathbb{C}\Gamma \longleftarrow\!\!\shortmid \mathbb{C}\Gamma$$

Points$(\mathbb{C}\Gamma \otimes \mathbb{C}\Gamma') = (\Gamma \times \Gamma')_+ = \Gamma_+ \wedge \Gamma'_+$

---

~~Now can define~~ Next consider products on $\mathbb{C}\Gamma$ $\ni$ the map $\mathbb{C}\Gamma \otimes \mathbb{C}\Gamma \longrightarrow \mathbb{C}\Gamma$ is a coalg morph

$$\mathbb{C}\Gamma \otimes \mathbb{C}\Gamma = \mathbb{C}[\Gamma \times \Gamma]$$

Such a prod. same as a map $\mu : \Gamma_+ \wedge \Gamma_+ \longrightarrow \Gamma_+$ in the cat of pointed sets. The prod on $\mathbb{C}\Gamma$ is assoc. if

$$\mu(\mu \wedge 1) = \mu(1 \wedge \mu) : \Gamma_+ \wedge \Gamma_+ \wedge \Gamma_+ \underset{1 \wedge \mu}{\overset{\mu \wedge 1}{\rightrightarrows}} \Gamma_+ \wedge \Gamma_+ \xrightarrow{\mu} \Gamma_+$$

Start with

$$\begin{array}{ccc} \mathbb{C}\Gamma \otimes \mathbb{C}\Gamma & \longrightarrow & \mathbb{C}\Gamma \\ \cup & & \cup \\ \Gamma_+ \wedge \Gamma_+ & \longrightarrow & \Gamma_+ \end{array}$$ coalg map

---

Compare three things

a bilinear product $\mathbb{C}\Gamma \otimes \mathbb{C}\Gamma \longrightarrow \mathbb{C}\Gamma$ which is a coalgebra map.

a map $\mu : \Gamma_+ \wedge \Gamma_+ \longrightarrow \Gamma_+$

a map $\tilde{\mu} : \Gamma_+ \times \Gamma_+ \longrightarrow \Gamma_+$ $\ni$ $\tilde{\mu}(s,0) = 0$, $\tilde{\mu}(0,s) = 0$.

These amount to the same thing.

Look at assoc. $\mathbb{C}\Gamma \otimes \mathbb{C}\Gamma \otimes \mathbb{C}\Gamma \rightrightarrows \mathbb{C}\Gamma$

The product in $\mathbb{C}\Gamma$ is assoc. $\Longleftrightarrow$ $\tilde{\mu}$ on $\Gamma_+$ is assoc.

$\Longleftrightarrow$ $\mu(\mu \wedge 1) = \mu(1 \wedge \mu) : \Gamma_+ \wedge \Gamma_+ \wedge \Gamma_+ \longrightarrow \Gamma_+$. ($\mu$ is assoc.)

$$\Gamma_+ \wedge \Gamma_+ \wedge \Gamma_+ = (\Gamma \times \Gamma \times \Gamma)_+$$

---

If product $\mu$ on $\mathbb{C}\Gamma$ is assoc., then $\tilde{\mu}$ on $\Gamma_+$ is assoc. Conversely if $\tilde{\mu}$ on $\Gamma_+$ is assoc. then the identity $(xy)z = x(yz)$ holds in $\mathbb{C}\Gamma$ for $x,y,z \in \Gamma_+$.

To check assoc. it should be enough to take $x,y,z \in \Gamma$

So it seems there might be a partial semi group notion. A bilinear product $\mathbb{C}\Gamma \times \mathbb{C}\Gamma \longrightarrow \mathbb{C}\Gamma$ pres. coalg st. is determined on the set $\Gamma \times \Gamma$, is the same as a map $\Gamma \times \Gamma \longrightarrow \Gamma_+$. Assoc. is determined on the set $\Gamma \times \Gamma \times \Gamma$

$$\begin{array}{ccc} \overset{x,y,z}{\Gamma \times \Gamma \times \Gamma} & \longrightarrow & \Gamma \times \Gamma_+ \\ \downarrow & & \downarrow \\ \Gamma_+ \times \Gamma & \longrightarrow & \Gamma_+ \\ (xy, z) & & \end{array}$$

$(\Gamma \times \Gamma)_+$

A bilinear product $\mathbb{C}\Gamma \otimes \mathbb{C}\Gamma \longrightarrow \mathbb{C}\Gamma$ resp. the coalg structure is equivalent to the corresp map of ptd sets: $(\Gamma \times \Gamma)_+ \longrightarrow \Gamma_+$, or to the induced binary operation $\Gamma_+ \times \Gamma_+ \longrightarrow \Gamma_+$ on $\Gamma_+ \subset \mathbb{C}\Gamma$, equivalent also to the product $\Gamma \times \Gamma \longrightarrow \Gamma_+$

Associativity of the product on $\mathbb{C}\Gamma$ is equivalent to $(xy)z = x(yz)$ for $x, y, z \in \Gamma$

Point to focus upon, ~~A coalg map~~ ~~$\mathbb{C}\Gamma \otimes \mathbb{C}\Gamma \to \mathbb{C}\Gamma$~~ A bilinear product $\mathbb{C}\Gamma \otimes \mathbb{C}\Gamma \longrightarrow \mathbb{C}\Gamma$ which ~~respects comult~~ restricts (under $\Gamma_+ \subset \mathbb{C}\Gamma$) to a binary operation on $\Gamma_+$ such that $0$ is absorbing: $s0 = 0s = 0$. The product on $\mathbb{C}\Gamma$ is associative $\Longleftrightarrow$ the binary operation on $\Gamma_+$ is associative.

---

Next do $\Gamma$-graded vector spaces an algebras.

Comodule $V$ for $\mathbb{C}\Gamma$, $\Delta_V : V \longrightarrow \mathbb{C}\Gamma \otimes V$

$$(\Delta_{\mathbb{C}\Gamma} \otimes 1)\Delta_V = (1 \otimes \Delta_V)\Delta_V$$

$$\Delta_V(v) = \sum_s s \otimes e_s v \xrightarrow{\Delta_{\mathbb{C}\Gamma} \otimes 1} \sum_s s \otimes s \otimes e_s v$$

$$\xrightarrow{1 \otimes \Delta_V} \sum_s s \otimes \sum_t t \otimes e_t e_s v$$

$\therefore$ $e_s e_t = 0 \quad s \neq t$

$e_s^2 = e_s$ $\qquad$ also $s \mapsto e_s v$ finite support $\forall v$.

$$\therefore \quad V = \bigoplus_{s \in \Gamma} e_s V \oplus e_0 V \qquad e_0 = 1 - \sum_{s \in \Gamma} e_s$$

Conclusion: A comodule for $\mathbb{C}\Gamma$ is $\overbrace{\text{a}}^{\text{the same as}}$ $\Gamma_+$-graded vector space

$V_0 = 0 \Longleftrightarrow V$ is counital $\quad (\eta \otimes 1)\Delta_V = 1_V$

Now you want to use your $\mathbb{C}\Gamma$ theory to construct a Morita context via gens + rels.

a $\mathbb{C}\Gamma$ comodule $=$ $\hat{\Gamma}_+$-module $= \Gamma_+$ graded v.s.

Given $V \to \mathbb{C}\Gamma \otimes V$, $W \to \mathbb{C}\Gamma \otimes W$ get

$V \otimes W \longrightarrow \mathbb{C}(\Gamma \times \Gamma) \otimes V \otimes W$ whence $V \otimes W$ is $\Gamma_+ \times \Gamma_+$ graded

via $(V \otimes W)_u = \bigoplus_{u=st} V_s \otimes W_t$ $\quad s,t,u \in \Gamma_+$ ? ?

a $\mathbb{C}\Gamma$-comodule $=$ a unital $\mathbb{C}[\Gamma_+]$-comodule.

Given $V, W$ such get $V \otimes W \longrightarrow \mathbb{C}[\Gamma_+ \times \Gamma_+] \otimes V \otimes W$

whence $V \otimes W$ is $\Gamma_+ \times \Gamma_+$ graded: $(V \otimes W)_{(s,t)} = V_s \otimes W_t$.

Then push forward via $\Gamma_+ \times \Gamma_+ \longrightarrow \Gamma_+$ to get

$(V \otimes W)_u = \bigoplus_{u=st} V_s \otimes W_t$.

a ~~[illegible]~~ . ? ~ .

Another thing to do is to treat $\mathbb{C}\Gamma$ non-unitally, but not just as a non unital ring, I mean you want to restrict comodules to avoid nil comodules. Then it's clear what to do.

$$\begin{array}{c} V \otimes W \longrightarrow \mathbb{C}\Gamma \otimes \mathbb{C}\Gamma \otimes V \otimes W \\ \downarrow{\scriptstyle \mu \otimes 1_{V \otimes W}} \\ \mathbb{C}\Gamma \otimes V \otimes W \end{array}$$

this gives a $(\Gamma \times \Gamma)_+ = \Gamma_+ \wedge \Gamma_+$ grading where $(V \otimes W)_0 =$

---

Review the idea. $\mathbb{C}\Gamma$ is coalgebra with counit, ~~but you~~ counital comodules are the same as $\Gamma$-graded vector spaces. The interesting point in that you can define a tensor product for these comodules?

associated to an associative product on $\Gamma_+ = \Gamma \sqcup \{0\}$ such that $0$ is absorbing. Check why/how:

$$V \xrightarrow{\Delta_V} \mathbb{C}\Gamma \otimes V \quad , \quad W \xrightarrow{\Delta_W} \mathbb{C}\Gamma \otimes W$$

$$\begin{tikzcd} V\otimes W \arrow[r, "\Delta_V\otimes\Delta_W"] \arrow[dr] & \mathbb{C}\Gamma\otimes\mathbb{C}\Gamma\otimes V\otimes W \arrow[d, "s\otimes t \mapsto st"] \\ & \mathbb{C}\Gamma\otimes V\otimes W \end{tikzcd}$$

This ~~commute~~ gives a $\mathbb{C}\Gamma$-comodule structure on $V\otimes W$

$$(V\otimes W)_u = \bigoplus_{\substack{s,t\in\Gamma \\ u=st}} V_s \otimes W_t$$

~~[illegible]~~ $V\otimes W$, is not necessarily $\Gamma$ graded

$V\otimes W$ is $\Gamma_+$ graded but not necessarily $\Gamma$ graded because there can be pairs $s,t\in\Gamma$ such that $st=0$.

But then we can ~~make~~ kill these components to get a counital comodule

---

Idea: Let $A$ be a $\Gamma_+$-graded algebra

$$A = \bigoplus_{s\in\Gamma_+} A_s \qquad A_s A_{s'} \subset A_{ss'}$$

note that $A_0$ is an ideal $\quad A_s A_0 \subset A_0$

$\therefore$ $A/A_0$ is an algebra

$$A/A_0 = \bigoplus_{s\in\Gamma} A_s \qquad \begin{aligned} A_s A_{s'} &\subset A_{ss'} && \text{if } ss'\in\Gamma \\ &= 0 && \text{if } ss' = 0 \end{aligned}$$

$A/ \qquad \mathbb{C}\Gamma\otimes A/A_0$

---

$$A \longrightarrow \mathbb{C}[\Gamma_+]\otimes A$$

$$a_s \longmapsto s\otimes a_s$$

what am I trying to understand?

You want to define $\Gamma$ graded algebra where $\Gamma_+ = \Gamma \amalg \{0\}$ is a semigroup with absorbing elt $0$. Equiv. $\mathbb{C}\Gamma$ with ~~set~~ like $\Delta$ has a prod. $\mathbb{C}\Gamma \otimes \mathbb{C}\Gamma \to \mathbb{C}\Gamma$ respecting ~~[illegible]~~ coalg structure.

$$A \longrightarrow \mathbb{C}\Gamma \otimes A$$

$$a_s \longmapsto s \otimes a_s \qquad s \in \Gamma$$

A $\Gamma$-graded alg is an ~~[illegible]~~ alg $A$ equipped with a $\Gamma$-grading $A = \bigoplus_{s\in\Gamma} A_s$ such that

$A_s A_t \subset A_{st}$ where the product ~~$A$~~ $st$ (which is defined in $\Gamma_+$) is in $\Gamma$, otherwise $A_s A_t = 0$.

Alternative. A $\Gamma$-graded alg is a $\Gamma_+$-graded alg such that $A_0 = 0$.

~~[illegible]~~

---

two structures grading $\bigoplus_{s\in\Gamma} A_s = A$

alg str

$$A \xrightarrow{\Delta} \mathbb{C}\Gamma \otimes A$$

$$A_s \longrightarrow s \otimes A_s$$

$$A_s A_t \xrightarrow{\Delta} (s \otimes A_s)(t \otimes A_t) = st \otimes A_{st}$$

$$\begin{pmatrix} & y_s = g s^{-1} \\ x_t = t i & \end{pmatrix}$$

$x_t x_{t'} = 0$

$y_s y_{s'} = 0$ $\quad x_t \in C_{21} \qquad y_s \in C_{12}$

$y_s x_t$ depends only on $s^{-1}t$

$y_s x_t = y_{us} x_{ut} \qquad \sum_s x_s y_s x_t = x_t$

Summarize.

$\Gamma$ set, $\mathbb{C}\Gamma$ has coproduct ~~coalg~~ $\Delta s = s \otimes s$ ~~making~~ which it a coassoc., and cocomm ~~coalg~~

Outline

$\Gamma$ set $\mathbb{C}\Gamma$ has coproduct $\Delta : \mathbb{C}\Gamma \to \mathbb{C}\Gamma \otimes \mathbb{C}\Gamma$, $\Delta s = s \otimes s$ for $s \in \Gamma$.

$\Gamma$ set, $\mathbb{C}\Gamma$ the vector space gen. by $\Gamma$ equipped with $\Delta : \mathbb{C}\Gamma \to \mathbb{C}\Gamma \otimes \mathbb{C}\Gamma$, $\Delta s = s \otimes s$ for $s \in \Gamma$. (the coproduct) The $\mathbb{C}\Gamma$ is a coass, cocomm. coalg with the counit $\eta : \mathbb{C}\Gamma \to \mathbb{C}$, $\eta(s) = 1$. TOO HARD

---

$\Gamma$ set coalg $\mathbb{C}\Gamma$ $\Delta(s) = s \otimes s$.

point ~~[illegible]~~ $\xi \in C$ $\Delta(\xi) = \xi \otimes \xi$, same as $\mathbb{C} \longrightarrow \mathbb{C}\Gamma$ coalg morphism

Points of $\mathbb{C}\Gamma$ = $\Gamma_+ \subset \mathbb{C}\Gamma$

recover $\Gamma$ as points of $\mathbb{C}\Gamma$, $\eta(\xi) = 1$

category of set like coalgebras and (resp) morphisms respecting counit $\equiv$ cat. of sets. (pointed sets).

Examine prod. $\mathbb{C}\Gamma \otimes \mathbb{C}\Gamma \longrightarrow \mathbb{C}\Gamma$

$$\mathbb{C}\Gamma \otimes \mathbb{C}\Gamma = \mathbb{C}[\Gamma \times \Gamma]$$

$$\Gamma_+ \wedge \Gamma_+ = (\Gamma \times \Gamma)_+ \longrightarrow \Gamma_+$$

Back to Morita context.

$$C = \begin{pmatrix} A & Y \\ X & B \end{pmatrix}$$ operate on ($W$ $\Gamma$-mod, $V$ v.s., $W \overset{j}{\twoheadrightarrow} V \overset{i}{\hookrightarrow} W$) satisfying

$$\left\{ \begin{array}{ll} jsi = 0 & s \notin \Phi \\ \sum s i j s^{-1} = 1_W & \\ j \text{ surj}, \ i \text{ inj.} & \end{array} \right.$$

Given $(V, W, i, j)$ as above, then $V$ is a red. $A$-mod with op. $p(s) = jsi$, and $W$ is a reduced $B$-mod with $h = ij$

$$\begin{tikzcd} W \arrow[r, "\alpha"] \arrow[dr, "j"'] & \mathcal{O}\Gamma \otimes V \arrow[r, "\beta"] \arrow[d, shift right, "\eta_1"'] & W \\ & V \arrow[u, shift right, "\varepsilon_1"'] \arrow[ur, "i"'] & \end{tikzcd}$$

$\alpha, \beta$ $\Gamma$-mod maps $\ni$
$\beta\alpha = 1_W$
$\alpha\beta = p$

$$p\Big(\sum_t{}' t \otimes v(t)\Big) = \sum_s s \otimes \Big(\sum_t p(s^{-1}t)\, v(t)\Big)$$

$\alpha, \beta$ the ! $\Gamma$-mod maps $\ni$

$$\eta_1 \alpha = j, \quad \beta \varepsilon_1 = i$$

$$W = \sum_t tiV$$

$$V = jW = \sum_t p(t) V$$

$$\alpha i v = \alpha\beta(1 \otimes v) = p(1 \otimes v) = \sum_s s \otimes p(s^{-1}) v$$

$$hsh = i\underbrace{(jsi)}_{p(s)}j$$

$W$ $\Gamma$-mod, $h$
$hsh = 0$, $\sum s h s^{-1} = 0$

$$W \overset{j}{\twoheadrightarrow} V \overset{i}{\hookrightarrow} W \qquad V = \operatorname{Im}(h \text{ on } W)$$

$$\alpha w = \sum_s s \otimes j s^{-1} w$$

$$w \in \sum t i V \qquad j s^{-1} t i v = 0 \quad \forall s$$

The equivalences

$\{V\}$ $\{(V, W, i, j)\}$ $\{W\}$

seem clear.

What do you need to do?

$$C = \begin{pmatrix} A' & Y \\ X & B' \end{pmatrix} \qquad x_t = ti \qquad y_s = js^{-1}$$

In $A'$ ~~$p(s)$~~ you have $p(s^{-1}t) = y_s x_t$

In $B'$ you have $x_t y_s = ths^{-1} = h_t ts^{-1} = ts^{-1}h_s$

Construct $B \longrightarrow B'$ $\qquad \mathcal{E}$ gen. $h_s$ $s \in \Gamma$

rels $\sum_s h_s h_t = h_t$, $\sum_t h_s h_t = h_s$

In $B'$ you have $\sum_s x_s y_s x_t y_t = x_t y_t$. So you get a unique alg homom. $\mathcal{E} \longrightarrow B'$

$$h_s \longmapsto x_s y_s$$

Maybe you [redacted] want to use that $B$ is $\Gamma$ graded.

$$B = \mathbb{C}\Gamma \otimes \mathcal{E}$$

So what you have ~~there~~ is $\mathcal{E} \twoheadrightarrow \mathcal{E}' \subset B'$

$$h_s \qquad x_s y_s$$

[redacted] You are missing something

You have specific generators $\overbrace{\text{and relations}}$ ~~and~~ for $B'$, which should be enough to define a map $B \to B'$. The problem ~~??~~ seems to be that $\Gamma$ is not contained in $B'$. It seems that one ought to be able to form a semidirect product $\mathbb{C}\Gamma$ ~~$\otimes$~~ $\oplus$ $B'$. Looks interesting.

Maybe you want to look at $\Gamma$ acting on $C$. $x_t = ti$, $y_s = js^{-1}$

$u*(x_t) = x_{tu}$ $\quad u*(y_s) = y_{us}$

$u*(\underbrace{y_s x_t}_{p(s^{-1}t)}) = y_{us} x_{ut} = y_s x_t$ $\qquad (us)^{-1}(ut) = s^{-1}t.$

supp.

$\sum x_s y_s x_t = x_t$ $\quad$ ok.

There apparently $\exists$ a $\Gamma$-grading on the Morita context.

$$C \longrightarrow \mathbb{C}\Gamma \otimes C$$

$x_t \qquad t \otimes x_t$

$y_s \qquad s^{-1} \otimes y_s$

~~Life goes on.~~

$(t\otimes x_t)(t'\otimes x_{t'})$
$= tt' \otimes x_{tt'} = 0$

$(s^{-1}\otimes y_s)(t\otimes x_t) = s^{-1}t \otimes y_s x_t$ depends only on $s^{-1}t$.

$\sum_s (s\otimes x_s)(s^{-1}\otimes y_s)(t\otimes x_t)$

$= \sum_s t \otimes (x_s y_s x_t) = t\otimes x_t$

So it seems you get grading wrt the ~~groupoid~~ groupoid which is the product of $M_2$ and $\Gamma$. Assume true that $C$ has such a grading, then look at your $\Gamma$ action.

---

$C$ is $\Gamma$-graded $\qquad x_t = ti$ degree $t$
$\qquad y_s = js^{-1}$ — $s^{-1}$

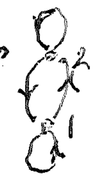
$y_s x_t \longrightarrow s^{-1}t \otimes y_s x_t$

$$C \longrightarrow \mathbb{C}\Gamma \otimes C$$

$x_t \qquad t \otimes x_t$

$y_s \qquad s^{-1} \otimes y_s$

$C$ should be graded wrt

$\tilde{\Gamma} =$  $\times \Gamma$

$M_2(\mathbb{C}\Gamma)$ $\qquad e_{ab} \otimes s \in M_2(\mathbb{C}\Gamma)$

$$x_t = e_{21} \otimes t$$
$$y_s = e_{12} \otimes s^{-1}$$

$$\begin{pmatrix} & s^{-1} \\ t & \end{pmatrix} \begin{pmatrix} V \\ W \end{pmatrix}$$

$$C \longrightarrow M_2 \otimes \mathbb{C}\Gamma \otimes C$$

$$x_t \qquad e_{21} \otimes t \otimes x_t$$

$$y_s \qquad e_{12} \otimes s^{-1} \otimes y_s$$

So what comes next? ~~Next!~~

You have a $\Gamma$ grading and a $\Gamma$ action.

Perhaps this is enough to specify $B$.

Maybe it's true that a $\Gamma$-graded $\Gamma$ algebra is cross product.

$$B = \bigoplus_{s \in \Gamma} B_s \qquad B_s B_t \subset B_{st} \qquad \text{NO}$$

More like multipliers.

There ~~seems to~~ might be a general result ~~over~~ about a $\Gamma$ graded algebra having a natural extension by $\mathbb{C}\Gamma$.

~~[illegible]~~ You are missing something

$\Gamma$ action on $C$ $\qquad u*(x_t) = x_{ut}$

$u*(y_s) = y_{ut}$

$$u*(y_s x_t) = y_{us}\, x_{ut} = y_s x_t$$

$$\cancel{u*(x_s y_s)} =$$

$$u*(y_s x_t) = j s^{-1} u^{-1}\, u t i = j s^{-1} t i$$

$$u*(x_t y_s) = \cancel{[illegible]}\; x_{ut}\, y_{us} = u t i j s^{-1} u^{-1}$$

So you get conjugation action of $\Gamma$ on $B'$.

Question to ask is whether ~~they~~ there is a canon homom. from $\mathbb{C}\Gamma$ to the multiplier algebra of $B'$.

Try to define left mult by $\Gamma$ on $B'$

$$x_t y_s = t i j s^{-1} = t h s^{-1} = t s^{-1} h_s$$

Look maybe ~~[illegible]~~ $\sum x_t y_{ut}$

$$\sum_{ts^{-1}=u} x_t y_s \qquad\qquad \sum_{u=ts^{-1}} x_t y_s = \sum_t x_{us} y_s$$

$t = us$

$$t \sum_s x_s y_s = \sum_s x_{ts} y_s = \sum_s x_{tt^{-1}s}\, \overset{js^{-1}t}{y_{t^{-1}s}} = \sum_s x_s y_{t^{-1}s}$$

$$\sum_s x_s y_s t = \sum_s x_s y_{t^{-1}s} \qquad js^{-1}t \qquad \sum h_s t$$

$$s h s^{-1} t \qquad \sum t h_s \qquad \|$$

You would like $\quad t \sum_s x_s y_s = \left(\sum_s x_s y_s\right) t$

So you know that

$$t \sum x_s y_s t^{-1} = \sum_s x_{ts} y_{ts} = \sum_s x_s y_s$$

$$t \sum x_s y_s = \sum x_s y_s t$$

$u = ts$

$t^{-1}u = s$

$$\sum_{us^{-1}=t} x_u y_s = \sum x_u y_{t^{-1}u} = \left(\sum x_u y_u\right) t$$

So you would like to ~~prove that $\mathbb{C}\Gamma$~~ construct $\mathbb{C}\Gamma \longrightarrow \mathrm{Mult}(\mathcal{B})$. So what to do

What to do? $\mathcal{B} = X \otimes_A Y$

$$\mathrm{Mult}(\mathcal{B}) = \underbrace{\{(\lambda, \rho) \in \mathrm{Hom}_{A^{op}}(X, X) \times \mathrm{Hom}_A(Y, Y)^{op} \mid \langle y\rho, x\rangle = \langle y, \lambda x\rangle\}}$$

definition of Mult for dual pair $(X, Y, \langle\,,\rangle)$

$$\mathrm{Mult}(B) = \{(\lambda, \rho) \in \mathrm{Hom}_{B^{op}}(B, B) \times \mathrm{Hom}_B(B, B)^{op} \mid \langle b, \lambda b'\rangle = \langle b\mu, b'\rangle\}$$

$$(\lambda, \rho)(\lambda', \rho') = (\lambda\lambda', \rho\rho')$$

B ideal in R

$$\langle b, \lambda\lambda' b'\rangle = \langle b\rho, \rho' b'\rangle$$
$$= \langle b\rho\rho', b'\rangle$$

---

The Mult(C) should arise from the dual pair X, Y. Obvious left $\Gamma$ action on X

$$C = \begin{pmatrix} A' & Y \\ X & B' \end{pmatrix}$$ has been constructed as a graded alg. wrt a Groupoid.

Is it possible to associate

---

C is const from gen. + relns. It should be clear that $\Gamma$ ~~operates~~ as multipliers on the dual pair X, Y over $A'$

Think of $C$ as $\begin{pmatrix} A' \\ X \end{pmatrix} \otimes_{A'} (A' \;\; Y)$

and ~~also~~ see how $\Gamma$ acts as multipliers.

$\Gamma$ acts trivially ~~on~~ the ring $A'$.

$\Gamma$ acts by left mult on the $x$'s $\qquad x_t = ti$

$ux_t = x_{ut}$, $\qquad X = \sum_t x_t A'$,

$\Gamma$ right acts on the $y$s $\qquad y_s u = j s^{-1}u = y_{u^{-1}s}$

$$\cancel{\langle (y_s a)u, a'x_t \rangle = \langle (y_s u)a, a'x_t \rangle}$$

$$\langle (ay_s)u,\ x_t a' \rangle = \langle a y_{u^{-1}s},\ x_t a' \rangle$$

$a j s^{-1}u = a y_{u^{-1}s}$ $\qquad = a\,p((u^{-1}s)t)\,a'$

$$\langle a y_s,\ u(x_t a') \rangle = \langle a y_s,\ x_{ut} a' \rangle = a\,p(s^{-1}ut)\,a'$$

You have defined a multiplier on

Puzzle: Why multipliers seem to be linked to objects in a groupoid. A Morita context $\Gamma$ idempotents?

You need to ~~[illegible]~~ find good assertions. $\Gamma$ graded algebra

When is an algebra a crossproduct?

If $\Gamma$ group, you want $B$ to be $\Gamma$-graded:

$$B = \bigoplus_{s \in \Gamma} B_s \qquad B_s B_t \subset B_{st}$$

and to have a unital homom. $\mathbb{C}\Gamma \longrightarrow Mult(B)$

this means $\Gamma$ acts on $B$ as $B^{op}$ module and $\Gamma$ right acts on $B$ as $B$-module assoc. $(bs)b' = b(sb')$

$\forall b \in B,\ s \in \Gamma$ ~~the~~ products $sb$ and $bs$ are given so that

$$s(bb') = (sb)b' \qquad (st)b = s(tb)$$
$$b(sb') = (bs)b' \qquad b(st) = (bs)t$$
$$(bb')s = b(b's)$$

~~assuming all this holds~~

comp. with grading $B_s B_t \subset B_{st}$

$$sB_t \subset B_{st}, \qquad B_t s \subset B_{ts}$$

Now take $A = B_1$. ~~See what you have~~

$\Gamma$ left acts on $\bigoplus_{s \in \Gamma} B_s$ $\quad tB_s \xrightarrow{\sim} B_{ts}$

right $\quad B_t s \xrightarrow{\sim} B_{ts}$

$B_1$ is an algebra $\quad sB_1 s^{-1} \subset B_s s^{-1} = B_1$

$$A = \bigoplus_{s\in\Gamma} A_s \qquad \Delta : A \longrightarrow \mathbb{C}\Gamma \otimes A$$
$$\Delta(a) = \sum s \otimes e_s a$$

Go over $\otimes$ for $\mathbb{C}\Gamma$ comodule.

$$V \longrightarrow \mathbb{C}\Gamma \otimes V, \quad W \longrightarrow \mathbb{C}\Gamma \otimes W$$
$$V\otimes W \longrightarrow \mathbb{C}\Gamma \otimes \mathbb{C}\Gamma \otimes V \otimes W \longrightarrow \mathbb{C}\Gamma \otimes V \otimes W$$
$$V \longrightarrow \mathbb{C}\Gamma_+ \otimes V \qquad W \longrightarrow \mathbb{C}\Gamma_+ \otimes W$$
$$V\otimes W \longrightarrow \underbrace{\mathbb{C}\Gamma_+ \otimes \mathbb{C}\Gamma_+}_{\mathbb{C}[\Gamma_+ \times \Gamma_+]} \otimes V \otimes W \longrightarrow$$
$$\mathbb{C}[\Gamma_+ \times \Gamma_+] \longrightarrow \mathbb{C}[\Gamma_+]$$
$$V\otimes W = \bigoplus_{s,t\in\Gamma_+} V_s \otimes W_t \longmapsto \bigoplus_{u\in\Gamma_+} \underbrace{(V\otimes W)_u}_{\bigoplus_{\substack{u=st\\ s,t\in\Gamma_+}} V_s\otimes W_t}$$

If you stick to $\mathbb{C}\Gamma$, get

$$V\otimes W \longrightarrow \mathbb{C}\Gamma\otimes\mathbb{C}\Gamma \otimes V\otimes W \xrightarrow{\text{is } \Gamma_+ \wedge \Gamma_+ \to \Gamma_+} \mathbb{C}\Gamma \otimes V \otimes W$$

$$V\otimes W = \bigoplus_{s,t\in\Gamma_+} V_s\otimes W_t$$
$$= \bigoplus_{\substack{u=st\\ \text{in } \Gamma}} V_s\otimes W_t \oplus \bigoplus_{\substack{0=st\\ \text{in } \Gamma_+}}$$

$V, W$ are graded wrt $\Gamma_+$

$V\otimes W$ graded wrt $\Gamma_+\times\Gamma_+$

pushing forward $\Gamma_+\times\Gamma_+ \longrightarrow \Gamma_+$ gives a $\Gamma_+$ grading on $V\otimes W$, then you kill the $st=0$ components. Something

Construction of grading on $D = \begin{pmatrix} A' & Y \\ X & B' \end{pmatrix}$

D has gens $x_t$ $(= ti)$, $y_s$ $(= ys^{-1})$ subject to relations

$$x_t x_{t'} = 0, \quad y_s y_{s'} = 0, \quad y_s x_t = y_{us} x_{ut} = 0 \quad \text{for } s^{-1}t \notin \Phi$$

$$\sum_s x_s y_s x_t = x_t \qquad \sum_t y_s x_t y_t = y_s$$

grading bialgebra $\mathbb{C}[M_2 \times \Gamma] = M_2\mathbb{C} \otimes \mathbb{C}\Gamma$

Take first the case of $\mathbb{C}[M_2] = M_2\mathbb{C}$

Form the alg $M_2\mathbb{C} \otimes D$, and introduce the generators with degree point. Better to say that we construct ~~the~~ alg map

$$D \xrightarrow{\Delta} M_2\mathbb{C} \otimes \mathbb{C}\Gamma \otimes D$$

sending gen

$$x_t \quad e_{21}\, t \otimes x_t$$

$$y_s \quad e_{12}\, s^{-1} \otimes y_s$$

$$y_s x_t \quad e_{11}\, s^{-1}t \otimes y_s x_t$$

$$x_s y_s \quad e_{22}\, 1 \otimes x_s y_s$$

$$\sum_s x_s y_s x_t \quad e_{21} \otimes \sum_s x_s y_s x_t$$

---

So you check that $D \xrightarrow{\Delta} \mathbb{C}[M_2 \times \Gamma] \otimes D$

$\Delta$ ? Take simpler example

$$\exists!\, \Delta : A \xrightarrow{\Delta} \mathbb{C}\Gamma \otimes A \underset{1_{\mathbb{C}\Gamma} \otimes \Delta_A}{\overset{\Delta \otimes 1_A}{\rightrightarrows}} \mathbb{C}\Gamma \otimes \mathbb{C}\Gamma \otimes A$$

$$\ni p(s) \longmapsto s \otimes p(s) \;\underset{\forall s}{\rightrightarrows}\; \begin{matrix} s \otimes s \otimes p(s) \\ s \otimes s \otimes p(s) \end{matrix}$$

---

Review the situation. You want to show that an alg defined by gens + rels is $\Gamma$ graded, when the generators + relations are homogeneous.

The $\overset{\text{desired}}{\Gamma}$-grading $A = \bigoplus_{s \in \Gamma} A_s$ is equiv. to ~~an alg map~~ a coproduct 

$$\Delta : A \longrightarrow \mathbb{C}\Gamma \otimes A$$

which is counital and respects the algebra structures on $A$, $\mathbb{C}\Gamma$. To say $a \in A$ is homog of degree $s$ means that $\Delta(a) = s \otimes a$. Can

Better to say that a $\Gamma$-grading on $A = \bigoplus_{s \in \Gamma} A_s$ equivalent to a ~~counital~~ coproduct

$$\Delta : A \longrightarrow \mathbb{C}\Gamma \otimes A$$

$$\Delta(a) = \sum_s s \otimes e_s a$$

making $A$ a counital comodule for $\mathbb{C}\Gamma$. Moreover $\Delta$ respect algebra structure iff $A_s A_t \begin{cases} \subset A_{st} & st \neq 0 \\ 0 & st = 0. \end{cases}$

Then If $A$ is defined by homogeneous relations, you get a unique hom. $\Delta : A \longrightarrow \mathbb{C}\Gamma \otimes A$ sending a gen. $p(s)$ of degree $s$ to $\Delta p(s) = s \otimes p(s)$

---

So now what you need to do is ~~go over the~~ work out the details of the Morita context.

$$D = \begin{pmatrix} A' & Y \\ X & B' \end{pmatrix} \qquad \text{gen } x_t = \begin{pmatrix} 0 & 0 \\ \underset{ti}{\phantom{x}} & 0 \end{pmatrix} \longleftarrow \text{degree } (21, t)$$

$$\text{gen } y_s = \begin{pmatrix} 0 & j s^{-1} \\ 0 & 0 \end{pmatrix} \longleftarrow \text{degree } (12, s^{-1})$$

gives rise to a grading of the alg $D$ ~~into~~ indexed by $M_2 \times \Gamma$.

Now make $\Gamma$ act on $X$, really as multipliers on $D$, i.e $\mathbb{C}\Gamma \longrightarrow \mathrm{Mult}(D)$.

Idea $\Gamma$ should left act on $(X, B')$ and right act on $\begin{pmatrix} Y \\ A' \end{pmatrix}$. Why? because you should be able to form the semi-direct product of $B'$ and $\mathbb{C}\Gamma$.

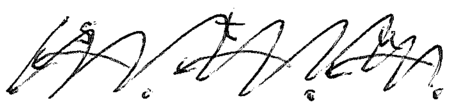
Puzzle. What is the multiplier algebra $\Theta$ for $\mathbb{C}\Pi$ where $\Pi_+$ is a semi group with absorbing elt $0$?

In the case of Morita contexts the diagonal ring $\begin{pmatrix} \mathbb{C} & 0 \\ 0 & \mathbb{C} \end{pmatrix}$ acts as multipliers. Why? Because Left mult. by $D = \begin{pmatrix} A & Y \\ X & B \end{pmatrix}$ on itself ~~preserves~~ carries each column into ~~itself~~, consequently $\underbrace{\begin{pmatrix} \mathbb{C} & 0 \\ 0 & \mathbb{C} \end{pmatrix}}_{\text{right mult}}$ lies in $\mathrm{Hom}_D(D,D)^{op}$. Similarly $\begin{pmatrix} \mathbb{C} & 0 \\ 0 & \mathbb{C} \end{pmatrix} \in \mathrm{Hom}_{D^{op}}(D,D)$

~~One can ask what it means for~~

~~Example. Let ... be a groupoid~~

Example. Consider the set of all arrows in a category. This is a set $\Gamma$ with partially defined multiplication, ~~should~~ namely the domain is the subset of $\Gamma \times \Gamma$ consisting of $(s,t)$ such that $\mathrm{dom}(s) = \mathrm{range}(t)$. $\mathbb{C}\Gamma$ should be the path algebra of $\Gamma$, which I assume is known to be associative. But this should be easy




4 cases

What about modules over the path algebra?

First case. Single object $X$, identity $1_X =$

$$F(X) = e_X F(X) \oplus (1-e_X)F(X)$$

get rid of $(1-e_X)F(X)$.

$$\begin{tikzcd} X \arrow[loop left, "e_X"] & Y \arrow[l, "u"'] \arrow[loop right, "e_Y"] \end{tikzcd}$$

$$F(e_X)F(u) = F(u) = F(u)F(e_Y)$$

$\therefore$

$$F(Y) = \overbrace{F(1_Y)}^{e_Y} F(Y) \oplus e_Y^{\perp} F(Y)$$

$F(u) = F(u)\, e_Y \implies F(u)$ kills $e_Y^{\perp} F(Y)$

$F(u) = e_X F(u) \implies e_X = 1$ on $\mathrm{Im}\, F(u)$

Organize. the category $\mathcal{C}$ gives rise to a path algebra $\mathbb{C}\Gamma$ where $\Gamma$ = set of arrows in $\mathcal{C}$. The path algebra is idempotent, it should have left and right units, in fact, $\sum_{X \in \text{Objects}} e_X$ should be an approximate identity.

The path alg has the set of arrows as basis for each arrow $f$ in $\Gamma$ there is exactly one object $X$ such that $1_Y f$ is defined namely $Y$ = target$(f)$, and exactly one object $X$ such that $f 1_X$ is defined namely source$(f)$, so one is going to have $\sum_{X \in Ob} 1_X$ is a local identity.

Let's finish the calculation of the Morita context.

$D = \begin{pmatrix} A' & Y \\ X & B' \end{pmatrix}$ Define a left action of $\Gamma$ on $(X \;\; B')$ commuting with right mult by $\begin{pmatrix} A' & 0 \\ 0 & B' \end{pmatrix}$ using partition of 1. $X = \sum x_t A'$, $B' = \sum x_t Y$

left mult by $u$ should be def'd using the local unit $\sum x_s y_s$ as follows:

better $u\phi \overset{\text{def}}{=} \sum_s x_{us} y_s \phi \qquad \phi \in X \oplus B'$.

this is well-defined since $\phi$ is a fin. sum of terms $x_t \psi$ with $\psi \in A' \oplus Y$ and we have $y_s x_t \neq 0$ for finitely many $s$.

Note

$$\sum_s x_{us} y_s x_t \psi = \sum_s x_{us} y_{us} x_{ut} \psi = x_{ut} \psi$$

So multiplication by $u$ on $X, B'$ is specified by $u(x_t \psi) = x_{ut} \psi$.

---

Question: Is there an obstruction to an idempotent ring being Morita equivalent to a ring with local units?

---

Go over what we know.

Today you want progress on the Morita context – identify it with $\begin{pmatrix} hBh & hB \\ Bh & B \end{pmatrix}$. Idea: To show $hBh = A$, can you treat $B$ as $B \otimes B^{op}$ module so that $hBh$ is the ind. $A \otimes A^{op}$ module corresp to $B$?

Start with the Morita context you construct by generators and relations.

The point is that $B$ has local units so that $Bh$ and $hB$ are firm over $B$, also $B$ is firm, so that by the M equiv. you know $hB$ and $Bh$ are firm over $A$. Then ~~you know that~~ ~~$Bh \otimes_A hB \xrightarrow{\sim} B$~~ $(hB, Bh)$ is a firm dual pair over $A$, and $Bh \otimes_A hB \longrightarrow B$ must be an isom. as $B$ is firm (this also results from the M eq. $Bh \otimes_A hB \otimes_B B \xrightarrow{\sim} B$. So the only thing you don't know is whether $hB \otimes_B Bh \longrightarrow A$ is an isom.), equivalently whether $A$ is firm.

Let's start from the $A$ end — you want to understand the dual pair $X, Y$.

IDEA: Because $D$ generated by $X, Y$, $D$ should be as free as possible

---

$\Gamma$ ring consists of two ~~sets~~ v.s. $X, Y$ and two linear maps.

$$X \otimes Y \otimes X \longrightarrow X \qquad [x_1, y_1, x_2]$$
$$Y \otimes X \otimes Y \longrightarrow Y \qquad [y_1, x_1, y_2]$$

relns.
$$[[x_1, y_1, x_2], y_2, x_3] = [x_1, [y_1, x_2, y_2], x_3]$$
$$= [x_1, y_1, [x_2, y_2, x_3]]$$

$$x_1 \otimes y_1 \otimes x_2 \longmapsto x_1 y_1 x_2$$

relation $(x_1 y_1 x_2) y_2 x_3 = x_1 (y_1 x_2 y_2) x_3 = x_1 y_1 (x_2 y_2 x_3)$

Put $A = Y \otimes X \Big/ [y_1, x_1, y_2] \otimes x_2 = y_1 \otimes [x_1, y_2, x_2]$

$B = X \otimes Y \Big/ x_1 y_1 x_2 \otimes y_2 = x_1 \otimes y_1 x_2 y_2$

Define $D = \begin{pmatrix} A & Y \\ X & B \end{pmatrix}$ by gens + relns.

$$X \oplus Y \longrightarrow D$$
$$x \longmapsto \hat{x}$$
$$y \longmapsto \hat{y}$$

$$\left( \begin{array}{l} \hat{x}^2 = 0 \\ \hat{y}^2 = 0. \end{array} \right.$$

$$\widehat{x_1 y_1 x_2}\, \hat{y}_2 = \hat{x}_1\, \widehat{y_1 x_2 y_2} \qquad ?$$
$$\widehat{y_1 x_1 y_2}\, \hat{x}_2 = y_1\, x_1\, y_2\, \hat{x}_2$$

relations $x \mapsto \hat{x}$, $y \mapsto \hat{y}$ linear, $\hat{x}^2 = 0$, $\hat{y}^2 = 0$

$\hat{x}_1 \hat{y} \hat{x}_2 = \widehat{x_1 y x_2}$, $\hat{y}_1 \hat{x} \hat{y}_2 = \widehat{y_1 x y_2}$

Let be generated by $\hat{x} : X \longrightarrow D$, $\hat{y} : Y \longrightarrow D$ satisfying these relations.

$$D \xrightarrow{\Delta} M_2 \otimes D$$
$$\hat{x} \longmapsto e_{21} \otimes \hat{x}$$
$$\hat{y} \longmapsto e_{12} \otimes \hat{y}$$

$$\underbrace{\hat{x}_1 \hat{y} \hat{x}_2}_{\widehat{x_1 y x_2}} \qquad \Delta(\hat{x}_1)\Delta(\hat{y})\Delta(\hat{x}_2) = \underbrace{e_{21} e_{12} e_{21}}_{e_{21}} \otimes \underbrace{\hat{x}_1 \hat{y} \hat{x}_2}_{\widehat{x_1 y x_2}}$$

$D$ is a Morita context. $D_{21} = X \qquad D$

By gens and relations you construct $D$

$$x_t x_{t'} = 0 \qquad y_t y_{t'} = 0$$

$$y_s x_t = y_{us} x_{ut} = 0 \quad \text{for } s^{-1}t \notin \Phi$$

$$\sum_s x_s y_s x_t = x_t \qquad \sum_t y_s x_t y_t = y_s$$

~~what do you~~

You get a Morita context $D = \begin{pmatrix} A' & Y \\ X & B' \end{pmatrix}$

$A'$ spanned by products of $y_s x_t$

by words in the generators ~~with start~~ beginning with a $y$, ending with $x$

| | | | | |
|---|---|---|---|---|
| $A'$ | ——— | $y$ | ——— | $x$ |
| $X$ | ——— | $x$ | ——— | $x$ |
| $Y$ | ——— | $y$ | ——— | $y$ |
| $B'$ | ——— | $x$ | ——— | $y$ |

So it is obvious that

$$YX = A, \quad XY = B$$

$$\left.\begin{matrix} XYX = X \\ YXY = Y \end{matrix}\right| \text{ from sum relns.}$$

$$AY = YXY = Y$$

$$AA = AYX = YX = A$$

Because of the local identity in $B$ you know that $BX = X \iff B \otimes_B X \xrightarrow{\sim} X$

$$YB = B \iff Y \otimes_B B \xrightarrow{\sim} Y$$

$$BB = B \iff B \otimes_B B \xrightarrow{\sim} B$$

It should then follow that $X, Y$ are firm for $A$

$$\begin{array}{ccccccc} K & \longrightarrow & X\otimes_A A & \longrightarrow & X & \longrightarrow & 0 \\ & & \uparrow s & & \uparrow s & & \\ B\otimes_B K & \longrightarrow & B\otimes_B X\otimes_A A & \longrightarrow & B\otimes_B X & \longrightarrow & 0 \end{array}$$

$$\sum x_i \otimes a_i \in X\otimes_A A \quad \ni \quad \sum x_i a_i = 0$$

$$xy \sum_i x_i \otimes a_i = \sum_i x \otimes y x_i a_i = 0$$

This arguments shows that since $X, Y$ firm over $B$ that $X, Y$ firm over $A$. Also follows from Mor. eq.

$X, B$ firm $B$-mod $\Longrightarrow$ $Y\otimes_B X$, $Y\otimes_B B$

~~[illegible]~~ So you have $B, X, Y$ firm Bimodules

But $A$ might not be firm. $A$ firm $\Leftrightarrow$ $Y\otimes_B X \xrightarrow{\sim} A$.

$$\begin{pmatrix} Y\otimes_B X & Y \\ X & B \end{pmatrix} \longrightarrow \begin{pmatrix} A & Y \\ X & B \end{pmatrix}$$

---

Exercise: ~~[illegible]~~ Consider a dual pair $(X, Y, \langle\,,\,\rangle)$ over $A$. Use $M_2$-grading idea to construct from the dual pairs a Morita context. Given a Morita context $C$

$$C \xrightarrow{\Delta} M_2 \otimes C \qquad \text{So what}$$

$$C_{ij} \longrightarrow e_{ij} \otimes C_{ij} \qquad ?$$

---

So what to do. Go back to $D = \begin{pmatrix} A' & Y \\ X & B' \end{pmatrix}$

$x_t x_{t'} = 0$ $\qquad y_s x_t = y_{us} x_{ut} = 0 \qquad s^{-1}t \notin \Phi$

$y_s y_{s'} = 0$ $\qquad \sum_s x_s y_s x_t = x_t, \quad \sum_t y_s x_t y_t = y_t$

Define $\Gamma$ action on $X$, $Y$ by the formulas

$$ux = \sum_s x_{us} y_s x \qquad , \qquad yu = \sum_t y x_t y_{u^{-1}t}$$

well defined because any $x$ is sum of $x_t \xi$ $\quad \xi \in B'$

and $y_s x_t \xi = 0$ a$\forall$ s. Moreover

$$u(x_t\xi) = \sum_s x_{us} y_s x_t \xi = \sum_s x_{us} y_{us} x_{ut} \xi = x_{ut}\xi$$

so $\boxed{u(x_t \xi) = x_{ut}\xi \quad , \quad (\xi y_s)u = \xi y_{u^{-1}s}}$

$$\sum_t \xi y_s x_t y_{u^{-1}t} = \sum_t \xi y_{u^{-1}s} x_{u^{-1}t} y_{u^{-1}t} = \xi y_{u^{-1}s}.$$

Defines a left $\Gamma$ action on $X, B$
and a right $\Gamma$ action on $Y, B$

~~What else to try. You have the multiplier algebra~~

Let's straighten out the pairing between $X, Y$ over $A$

Simple situation $p = p^2 \in M_n A$. $\boxed{\text{IDEA.}}$ Generalize $P_\Gamma$ which is universal wrt projections in a $\Gamma$ graded algebra, $\Gamma$ a group, to $\Gamma \to \Gamma_+$ is a semigroup with absorbing element $0$.

Go back to simplest case.

$$\begin{matrix} A & Y \\ X & \end{matrix} \qquad \begin{matrix} A & A & A \\ A & & \\ A & & \end{matrix}$$

what you want is to have a fg free module and its dual

<u>Example</u> $\quad B \quad Bh$ $\quad$ pairing $Bh \times hB \longrightarrow BhB < B$
$\quad hB \quad hBh$ $\quad$ is defined by
$$\langle b_1 h, h b_2\rangle = (b_1 h) b_2 = b_1 (h b_2)$$

~~[illegible]~~ Can you use this in the case $\begin{pmatrix} A & A \\ A & A \end{pmatrix}$ ?

---

go to simple case namely $p = p^2 \in M_n\tilde{A} \simeq \begin{pmatrix} \tilde{A} \\ \vdots \\ \tilde{A} \end{pmatrix} \otimes_A (\tilde{A} \cdots \tilde{A})$

$E = \begin{pmatrix} \tilde{A} \\ \vdots \\ \tilde{A} \end{pmatrix}$, $F = (\tilde{A}, \cdots \tilde{A})$ $\qquad F \otimes E \longrightarrow \tilde{A}$ dot product

$\begin{pmatrix} \tilde{A} & F \\ E & M_n\tilde{A} \end{pmatrix}$ $\qquad \tilde{A} \xrightarrow{\varepsilon_i} \tilde{A}^n \xrightarrow{\eta_j} \tilde{A}$

$$\eta_j \varepsilon_i = \delta_{ji}, \quad \sum \varepsilon_i \eta_i = 1$$

$$E = \bigoplus_1^n \varepsilon_i \tilde{A} \qquad F = \bigoplus_1^n \eta_j \tilde{A}$$

confusion: row column, left right. You want $M_n\tilde{A}$ to left act on $E$ a right $\tilde{A}$-module

Basic stuff is $\tilde{A}$, $E$ an $A^{op}$-mod, $F$ an $A$-mod.

Duality pairing $F \otimes E \longrightarrow A$, nuclear op.

$$E \otimes_A F \longrightarrow E \otimes_A \mathrm{Hom}_{A^{op}}(E, A) \longrightarrow \mathrm{Hom}_{A^{op}}(E, E)$$

$$x \otimes y \longmapsto x \otimes \langle y, -\rangle \longmapsto (x \mapsto x\langle y, x\rangle)$$

and the key point (choice?) is the element

$\sum x_i \otimes y_i \in E \otimes_A F$ yielding the identity op.

<u>IDEA</u> Ultimately you seek a calculus of "kernels" which yield the K-groups. You may be dealing with vector spaces, but the choice of partition of unity should

should lead to the geometric object you want.

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~~So what ... forget about~~

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You want to understand, review duality. finite case

$\begin{pmatrix} A & Y \\ X & B \end{pmatrix}$ where: $\sum_1^n x_i y_i = 1 \in B$ (B unital)

You have $X \xrightarrow{(y_i)} A^n \subset \tilde{A}^n \xrightarrow{(x_i)} X$

so $X$ is a retract of $\tilde{A}^n$, hence $X$ is a fg proj $A^{op}$ module, also $X = XA$ because $x = \sum x_i(y_i x)$. Claim

$Y \xrightarrow{\sim} \mathrm{Hom}_{A^{op}}(X, \tilde{A})$ $\quad y \longmapsto (x \longmapsto yx$

$$X \xrightarrow{(y_i \cdot)} \tilde{A}^n \xrightarrow{(x_i \cdot)} X$$

$$(\lambda(x_i)) \longleftarrow\!\!| \ \lambda \qquad\qquad y \mapsto (x \mapsto yx)$$

$$\begin{tikzcd}
X^\vee & \tilde{A}^n \arrow[l, "(\cdot y_i)"'] & X^\vee \arrow[l] \\
Y \arrow[u] & \tilde{A}^n \arrow[l, "\cdot(y_i)"'] \arrow[u] & Y \arrow[l, "\cdot(x_i)"'] \arrow[u]
\end{tikzcd} \qquad y \qquad ?$$

~~$Y \otimes X$~~

$$\begin{tikzcd}
M \otimes_A \mathrm{Hom}_{A^{op}}(X, A) \arrow[r] & \mathrm{Hom}_{A^{op}}(X, M) \\
M \otimes_A Y \arrow[u] &
\end{tikzcd}$$

$$m \otimes y \longmapsto (x \mapsto myx)$$

$$\sum_i m(yx_i) \otimes_A y_i$$

You want a direct proof that

$$M \otimes_A Y \longrightarrow \mathrm{Hom}_{A^{op}}(X, M)$$
$$m \otimes y \longmapsto (x \longmapsto myx)$$

is an isom. The inverse should send $\phi : X \longrightarrow M$ to

$$\sum_i \phi(x_i) \otimes y_i \longmapsto \left(x \mapsto \sum \phi(x_i) y_i x\right)$$
$$= \sum \phi(x_i y_i x) = \phi(x)$$

Summary. Given $\begin{pmatrix} A & Y \\ X & B \end{pmatrix}$ strictly idempotent with $B$ unital

Choose $\sum_1^n x_i y_i = 1$, then construct isom.

$$M \otimes_A Y \xrightarrow{\sim} \mathrm{Hom}_{A^{op}}(X, M)$$

$$m \otimes y \longmapsto (x \mapsto m(yx))$$

$$\sum_1^n \underbrace{\phi(x_i)}_{m(yx_i)\otimes y_i} \otimes y_i \longleftarrow \phi$$

$$\longmapsto (x \mapsto \sum \phi(x_i)(y_i x))$$

What are you trying to accomplish?
Given $\begin{pmatrix} A & Y \\ X & B \end{pmatrix}$ s. idemp. $B$ unital you find

$X, Y$ dual fg proj $A$-modules

$$B = \mathrm{Hom}_{A^{op}}(X, X) = \mathrm{Hom}_A(Y, Y)$$

Things to understand. Choice of $\sum x_i \otimes y_i = 1$, partition of unity in $B$. Meaning

$$X \xrightarrow{(y_i)\cdot} A^n \subset \tilde{A}^n \xrightarrow{(x_i)\cdot} X$$

Thus the partition of unity is equivalent to embedding $X$ as a retract of a free $\tilde{A}$ module.

---

Cuntz's $C^*$ algebra $\mathcal{O}_n$ is acted upon by $U(n,1)$.
What is a $*$ repn of $\mathcal{O}_n$ on $H$? It consists of $s_i : H \longrightarrow H$ satisfying $s_i^* s_j = \delta_{ij}$, $\sum s_i s_i^* = 1$.
Trick is to form $V \otimes H$ where $V = \mathbb{C}^{n+1}$ with the hermitian form of ~~type~~ signature $(n,1)$. $V \otimes H$

$V \otimes H$ is a Krein space, this means effectively that admits a ?

Take a polarization of $V$: $V = V_+ \oplus V_-$ where $V_+, V_-$ are $\perp$ for the herm. form which is

$$z = e^{i\theta} \qquad dz = e^{i\theta}\, i\, d\theta$$

$$\oint z^n dz = \int_0^{2\pi} (e^{i\theta})^n e^{i\theta} i\, d\theta$$

$$= i \int_0^{2\pi} e^{i(n+1)\theta} d\theta$$

$$= i \left[ \frac{e^{i(n+1)\theta}}{i(n+1)} \right]_0^{2\pi} \qquad n+1 \neq 0$$

$$= \begin{cases} 0 & n \neq -1 \\ 2\pi i & n = -1 \end{cases}$$

---

The idea you want is that you should work with $H_+ \oplus H_-$ equipped with hermitian form $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. This is the standard form for a Krein space. Isotropic subspaces same as partial unitaries from $H_+$ to $H_-$. So a full (Lagrangian) subspace is a unitary isom between $H_+$ and $H_-$. In our case where $V_+ = \mathbb{C}^n$ and $V_- = \mathbb{C}$, $(V \otimes H)_+ = \mathbb{C}^n \otimes H$, $(V \otimes H)_- = \mathbb{C} \otimes H$ you get a representation of $\mathcal{O}_n$ on $H$.

Idea: Let $V = \mathbb{C}^{n+1}$ with $U(n,1)$ symmetry. $V = \underset{+1}{\mathbb{C}^n} \oplus \underset{-1}{\mathbb{C}}$. For any Hilb. space $H$ you get $V \otimes H = (V\otimes H)_+ \oplus (V \otimes H_-)$. This is a Hilbert space with tensor product hermitian form. ~~preserved by $U(n,1)$~~ with $U(n,1)$ symmetry.

A Lagrangian subspace in a polarized Hilbert space is ~~equal~~ the graph of a unitary. Get equivalence between unitaries $\mathbb{C}^n \otimes H \xrightarrow{\sim} H$ and Lag. subspaces, i.e. ~~between~~ $\mathcal{O}_n$ representations on $H$

~~and~~ But $U(n,1)$ acts. So you have a family of $*$ homos. $\mathcal{O}_n \longrightarrow \mathcal{L}(H)$.

You're missing something. You want the group $U(n,1)$ to act on the alg $\mathcal{O}_n$, then can form cross product I guess? Does this crossproduct act on $V \otimes H$.

---

So look at the relations $y_j x_i = \delta_{ij}$ and $\sum x_i y_i = 1$

Pimsner constructs ~~a limit~~ an inductive limit of $T(V) \otimes T(V^*)$

---

Get back to lecture. $D = \begin{pmatrix} A' & Y \\ X & B' \end{pmatrix}$

gens $x_t, y_s$ $\quad t,s \in \Gamma$

rels $\quad x_t x_{t'} = 0, \quad y_s y_{s'} = 0$

$y_s x_t = y_{us} x_{ut}$ $\quad$ i.e. $y_s x_t$ dep on $s^{-1}t$.

$y_s x_t = 0 \qquad s^{-1}t \notin \Phi$

$\sum_s x_s y_s x_t = x_t, \quad \sum_t y_s x_t y_t = y_s$

$M_2(\mathbb{C}\Gamma) = \mathbb{C}[M_2 \times \Gamma]$ basis $e_{ij} \otimes s$

$$D \xrightarrow{\Delta} \underbrace{M_2(\mathbb{C}\Gamma)}_{M_2\mathbb{C} \otimes \mathbb{C}\Gamma} \otimes D$$

$$\begin{pmatrix} & y_s \\ x_t & \end{pmatrix}$$

$x_t \qquad \Delta x_t = e_{21} \otimes t \otimes x_t$

$y_s \qquad \Delta y_s = e_{12} \otimes s^{-1} \otimes y_s$

$y_s x_t \longmapsto e_{11} \otimes s^{-1}t \otimes y_s x_t$ dep ~~only~~ on $s^{-1}t$. supp. cond.

$\sum_s x_s y_s x_t \longmapsto e_{21} \otimes \cancel{s}t \otimes \sum_s x_s y_s x_t$

---

What can you say about $B'$, $A'$,

$A'$ gen. by $\underbrace{y_s x_t}_{p(s^{-1}t)}$, by Morita context properties.
$\quad \| \quad j s^{-1} t i$

$$\sum_t p(s^{-1}t)\, p(t^{-1}u) = \sum_t y_s x_t y_t x_u = y_s x_u = p(s^{-1}u)$$

get surjection $A \twoheadrightarrow A'$

$B'$ gen by $x_t y_s = t s^{-1} x_s y_s$

---

$$\begin{pmatrix} A' & Y \\ X & B' \end{pmatrix}$$

So you define left multiplier $u$ by $(\sum_s x_{us} y_s)\circ$ on $X, B'$. Then

$$\sum_s x_{us} y_s x_t \eta = \sum_s x_{us} y_{us} x_{ut} \eta = x_{ut} \eta$$

right mult. $\circ(\sum_t x_t y_{u^{-1}t})$ on $Y, B'$

$$\eta y_s \sum_t x_t y_{u^{-1}t} = \sum_t \eta y_{u^{-1}s} x_{u^{-1}t} y_{u^{-1}t} = \eta y_{u^{-1}s}$$

$\therefore \boxed{u(x_t \eta) = x_{ut} \eta} \qquad \boxed{(\eta y_s) u = \eta y_{u^{-1}s}}$

This ought to define $u$ as a multiplier on $B'$. Simple enough. You want to check the three associativity conditions.

To show $\quad u(b_1 b_2) = (u b_1) b_2$

Can assume $\quad b_1 = x_t y$, then

$$u(b_1 b_2) = u(x_t y b_2) = x_{ut} y b_2 = (u b_1) b_2$$

$$\cancel{(ub)u' = (u(x_t y))u' = x_{ut} y u'}$$

$$\cancel{(ub)u' = u(x_t y_s) u'}$$

$$\big(u(x_t a y_s)\big) u' = (x_{ut} a y_s) u' = x_{ut} a y_{u'^{-1}s}$$

$$u\big((x_t a y_s) u'\big) = u(x_t a y_{u'^{-1}s}) = x_{ut} a y_{u'^{-1}s}$$

---

Look at $B'$ which is gen. by $x_t y_s$ $\left(= t\, \iota\, y\, s^{-1} = t h_1 s^{-1} = h_t (ts^{-1}) = (ts^{-1}) h_s\right)$. You should write it as ~~$ts^{-1}$~~ $\boxed{t h_s = t x_s y_s}$

---

Look at $B'$ with $\mathbb{C}\Gamma \hookrightarrow \mathrm{Mult}(B')$ and also the $\Gamma$ grading. $\quad B' = \bigoplus_{s \in \Gamma} B'_s$. But $B'$ has the generators $x_t y_s = ts^{-1} h_s = h_t\, ts^{-1}$ of degree $ts^{-1}$!

So it's clear that $B'_1$ is generated by $x_s y_s = h_s$ which satisfy the relations $\sum_s h_s h_t = h_t$, $\sum_t h_s h_t = h_s$

You certainly have the relations defining $\mathcal{C}$

$$h_s h_t = 0 \qquad s^{-1}t \notin \Phi$$

$$h_1 s^{-1} t h_1 = 0$$

You need somehow to show that $B'$ involves the tensor algebra, not the commutative one. ~~Unless~~ It seems that you want to start with $B$, then construct the Morita context.

Let's try to understand $\sum x_s y_s = 1$ better, $\sum x_s y_s$. With luck there will be a link between this $\mathcal{E}$ and the tensor alg. Take finite $\Gamma$ case?

$$\mathbb{C}\langle h_s\rangle \Big/ \textstyle\sum h_s = 1$$

You maybe need to understand more about $\mathcal{E}h_1$ or $h_1\mathcal{E}$?

Look at $B = \Gamma \ltimes \mathcal{E}$. Think about the grading on $X$. Conj. $X \simeq Bh_1$ $B$ as $B^{op}$-module should corresp to the red $A^{op}$-module $Bh_1$. Wait: ~~The Morita equivalence~~ Have basis M. eq.

$$\begin{tikzcd} W \arrow[r, "\alpha"] \arrow[dr] & \mathbb{C}\Gamma \otimes V \arrow[r, "\beta"] \arrow[d, shift right, "\eta_1"'] & W \\ & V \arrow[u, shift right, "\varepsilon_1"'] \arrow[ur, hook, "i"'] & \end{tikzcd}$$

$\alpha w = \sum_s s \otimes j s^{-1} w$

$\beta \sum_t t \otimes v(t) = \sum_t t\, i\, v(t)$

$\beta\alpha = 1_W$

$(\alpha\beta)\left(\sum t \otimes v(t)\right) = \sum_s s \otimes \sum_t p(s^{-1}t) v(t)$

this diagram summarizes the relations between reduced $B$ and reduced $A$-modules. $V \simeq hW$.

$Y = Y \otimes_B B \simeq hB$

$$\begin{pmatrix} hB \otimes_B Bh & hB \\ Bh & B \end{pmatrix}$$

firms version

better might be

$$\begin{pmatrix} & jB \\ Bi & B \end{pmatrix}$$

$W = B$

$$\begin{tikzcd} B \arrow[r] & \overset{hB = jB}{V} \arrow[r, "i"] & B \end{tikzcd}$$

So let's start with trying to identify

$$\begin{pmatrix} & hB \\ Bh & B \end{pmatrix} \quad \text{with} \quad \begin{pmatrix} & jB \\ B\iota & B \end{pmatrix}$$

these are two dual pairs over $B$ which should be isomorphic in an obvious way. Certainly

$$\begin{tikzcd} B \arrow[rr, "h\cdot"] \arrow[dr, twoheadrightarrow, "j=h\cdot"'] & & B \\ & hB \arrow[ur, hook, "\iota=\mathrm{inc.}"'] & \end{tikzcd} \qquad \begin{tikzcd} B \arrow[rr, "\cdot h"] \arrow[dr, twoheadrightarrow] & & B \\ & Bh \arrow[ur, hook] & \end{tikzcd}$$

~~[illegible]~~ Focus upon what to do. First work out right module notation.

$$\begin{tikzcd} B \arrow[rr, "\cdot h"] \arrow[dr, "\cdot x_s"'] & & B \\ & \phantom{X} \arrow[ur, "\cdot y_s"'] & \end{tikzcd} \qquad b\sum x_s y_s = 1.$$

---

Idea: you have found ~~[illegible]~~ $\Gamma$ in the ring of multipliers, so it should ~~be possible to~~ follow that reduced $D$ modules are the same as ~~[illegible]~~ $V, W, \iota, j$   $W$ a $\Gamma$-module, $V$ a vector space, $W \overset{j}{\twoheadrightarrow} V \overset{i}{\hookrightarrow} W$ linear maps $\ni$ $jsi = 0$   $s \notin \Phi$.   $\sum_{s\in\Gamma} s\iota j s^{-1} = 1_W$

---

$$\mathcal{M}(A) \underset{Y\otimes_B -}{\overset{X\otimes_A -}{\rightleftarrows}} \mathcal{M}(B)$$

$Y = Y\otimes_B B \simeq hB$ firm $A$, ${}^{\text{red}}B^{op}$

$X = B\otimes_B X \simeq Bh$ firm $A^{op}$, ${}^{\text{red}}B$

$Y\otimes_B X = hB\otimes_B Bh$ firm version of $hBh$

$\Longrightarrow hBh \simeq A_{\text{red}}$

What do you propose?

What to do? Form $\begin{pmatrix} hBh & hB \\ Bh & B \end{pmatrix}$

$h = ij$. Is $\begin{pmatrix} ijBij & ijB \\ Bij & B \end{pmatrix} \simeq \begin{pmatrix} jBi & jB \\ Bi & B \end{pmatrix}$ ?

$$\| $$

$$\begin{pmatrix} ij & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} B & B \\ B & B \end{pmatrix}\begin{pmatrix} ij & 0 \\ 0 & 1 \end{pmatrix}$$

$$\|$$

$$\begin{pmatrix} i & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} j & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} B & B \\ B & B \end{pmatrix}\begin{pmatrix} i & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} j & 0 \\ 0 & 1 \end{pmatrix}$$

---

How to make progress? Let us start with the simplest module picture, namely $V$ vector space, $W$ $\Gamma$-module, $i : V \to W$, $j : W \to V$ sat $jsi = 0$ for $s \notin \Phi$ and $\sum_s sijs^{-1} = 1_W$. Now you want an idemp. ring to yield these modules. $\begin{pmatrix} \mathbb{C} & j \\ i & \mathbb{C}\Gamma \end{pmatrix}$ will give ~~rise a~~ a unital ring. Finite $\Gamma$, $\Phi = \Gamma$. GNS ?

$$\begin{array}{ccc} A & & B \\ V & \underset{i}{\overset{j}{\rightleftarrows}} & W \end{array}$$

$\Gamma \longrightarrow \mathcal{L}(W)$ a unitary rep. $\quad V \underset{i}{\overset{i^*}{\rightleftarrows}} W$

$\rho(s) = i^* s i \in \mathcal{L}(V)$ $\qquad \mathbb{C}\Gamma$ acts on $W$

get $\rho(s) = i^* s i$ the compression of $s$ to $V$

$\cancel{W \oplus \mathbb{C}\Gamma \otimes \mathcal{L}(V) \otimes \mathbb{C}\Gamma}$ You get certain ops.

on

Go back to your GNS notation.

$\rho : A \longrightarrow B$ linear. Consider $\left(\underset{A}{M}, \underset{B}{N}, \begin{matrix} i: N \to M \\ j: M \to N \end{matrix}\right)$

$$j a i n = \rho(a) n$$

module type

~~[illegible]~~

$$\Gamma = A \oplus A \otimes B \otimes A$$

$$a \qquad a_1 \otimes b \otimes a_2$$

$$a \qquad a_1 i b j a_2$$

---

Proceed as follows. Begin with the unital ring

$$\left(\begin{array}{c|c} \mathbb{C} & 0 \\ \hline 0 & \mathbb{C}\Gamma \end{array}\right)$$

whose good modules are pairs $\begin{pmatrix} V \\ W \end{pmatrix}$ $V$ vector space, $W$ $\Gamma$-module.

adjoin elements $\begin{pmatrix} 0 & 0 \\ i & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & j \\ 0 & 0 \end{pmatrix}$ to this ring satisfying the relations: $jsi = 0 \quad s \notin \Phi$.

$$\underbrace{\sum_s s i j s^{-1} i = i}_{\Downarrow}, \qquad \sum_t j t i j t^{-1} = j$$

$$\sum_s t s i j s^{-1} i = t i$$

$$\|$$

$$\sum_s s i j s^{-1} t i = t i$$

$$\sum_s \cancel{\quad} x_s y_s x_t = x_t$$

---

Today you want to ~~[illegible]~~ understand two constructions of a Morita context.

Two constructions of $D$ ~~are possible~~ should yield the same result. First do it for $B$. First ~~...~~ definition. generators $h_s$ $s \in \Gamma$

---

Today work out equivalence of different definitions

$$D = \begin{pmatrix} A' & Y \\ X & B' \end{pmatrix}$$

Morita context ($= M_2$-graded alg)
(in fact graded alg wrt $M_2 \times \Gamma$)

gens $x_t, y_s$ $t, s \in \Gamma$ $x_t x_{t'} = y_s y_{s'} = 0$

$y_s x_t = y_{us} x_{ut} = 0$ for $s^{-1}t \in \Phi$.

$$\sum_s x_s y_s x_t = x_t \quad , \quad \sum_t y_s x_t y_t = y_s$$

~~So~~ $D$ is ~~clearly~~ (strictly) idempotent: ~~Since every element~~ ~~of~~ Since $D$ spanned by words in the gens. with $x, y$ alt. ~~$D$ is strictly idemp.~~ you have

~~$x_t \in XA'$ ... $x_t \in B'X$~~

$y_s x_t$ gen $A'$, $x_t y_s$ gen $B'$ | $YX = A'$, $XY = B'$

$$\sum_s x_s y_s x_t = x_t \implies X = B'X = XA'$$

$\implies B' = XY = B'XY = B'B'$

$$\sum_t y_s x_t y_t = y_s \implies Y = A'Y = YB'$$

~~$B' = XY = XY$~~

Relations: $p(s^{-1}t) = y_s x_t$

$$\sum_t p(s^{-1}t)\, p(t^{-1}u) = \sum_t y_s x_t y_t x_u = y_s x_u = p(s^{-1}u)$$

So you get $A \twoheadrightarrow A'$, $p(s) \mapsto y_s x_t$

In $B'$ have $x_s y_s$

---

~~path alg of a category~~ Start again

Define $D$ by gens & relations. ~~D is idempotent~~
~~so Mult(D) is defined. Look at Mult(D)~~

Consider $Mult(D)$, this is ~~only~~ a unital algebra with a homom.

$$D \xrightarrow{\phi} Mult(D)$$

$$d \longmapsto \phi_d = (\phi_d ; \phi_d) \in Hom_{D^{op}}(D,D) \times Hom_D(D,D)^{op}$$

$$(\mu\phi_d)d_1 = \mu(\phi_d d_1) = \mu(dd_1) = (\mu d)d_1 = \phi_{\mu d} d_1$$

$$d_1(\mu\phi_d) = (d_1\mu)\phi_d = (d_1\mu)d = d_1(\mu d) = d_1\phi_{\mu d}$$

$$\therefore \quad \mu\phi_d = \phi_{\mu d} \quad \text{in } Mult(D).$$

---

$$\mu = (d \mapsto \mu d,\ d \mapsto d\mu)$$

$$\in \{Hom_{D^{op}}(D,D),\ Hom_D(D,D)^{op}\}.$$

$$\mu(dd_1) = (\mu d)d_1 \qquad (d_1 d)\mu = d_1(d\mu)$$

$$d(\mu d_1) = (d\mu)d_1$$

$$(\mu\nu)d = \mu(\nu d) \qquad d(\mu\nu) = (d\mu)\nu$$

$$\cancel{d((\mu\nu)d_1) = d(\mu(\nu d_1))} \qquad (d_1(\mu\nu))d$$

$$d_1((\mu\nu)d) = d_1(\mu(\nu d)) = (d_1\mu)(\nu d) = ((d_1\mu)\nu)d$$

Apparently $D = \begin{pmatrix} A' & Y \\ X & B' \end{pmatrix}$ any Morita ~~context~~ admits multipliers $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\left(\begin{pmatrix} a & y \\ x & b \end{pmatrix}\begin{pmatrix} a_1 & y_1 \\ x_1 & b_1 \end{pmatrix}\right) = \begin{pmatrix} a & y \\ 0 & 0 \end{pmatrix}\begin{pmatrix} a_1 & y_1 \\ x_1 & b_1 \end{pmatrix}$$

$$= \begin{pmatrix} aa_1 + yx_1 & ay_1 + yb_1 \\ 0 & 0 \end{pmatrix}$$

Do it carefully first

$A = \bigoplus A$ Anyway

To understand multipliers. Examples.

$A = \bigoplus_{(i,j)} A_{ij}$. The thing to check is that certain multipliers occur - involved with identity maps in the category.

Deal with matrices

Start with a small category $\mathcal{C}$ and the category of functors $\mathcal{C} \longrightarrow \mathrm{Mod}(\mathbb{C})$. $x \longmapsto F_x$, $(u : x \to y) \longmapsto (F_u : F_x \to F_y)$. Let $\Gamma$ be the set of arrows in $\mathcal{C}$, $\mathbb{C}\Gamma$ has basis $\{s \in \Gamma\}$ with multiplication defined by

$$st = \text{composition of } s \text{ and } t \text{ if defined}$$
$$= 0 \text{ otherwise.}$$

You need to understand assoc.

$$\Gamma_2 \equiv\!\!\!\rightarrow \Gamma_1 \rightrightarrows \Gamma_0$$

given three arrows

$$X \xrightarrow{u} Y, \quad X' \xrightarrow{v} Y', \quad X'' \xrightarrow{w} Y''$$

$uv$ defined iff $Y = X'$

$vw$ defined iff $Y' = X''$

$(uv)w = u(vw)$

$$X \xrightarrow{u} Y, \quad X' \xrightarrow{v} Y', \quad X'' \xrightarrow{w} Y''$$

the left side defined iff $Y = X'$ and $Y' = X''$
the right side defined iff $Y' = X''$ and $Y = X'$.
So the path algebra for a category is definitely associative.
What structure does the path algebra have?
~~local~~ approx unit, namely $\sum_X' e_X$

So you take $\sum_i c_i f_i$ finite linear comb. of arrows. Let $f_i : X_i \to Y_i$ $\quad f_i \in Hom(X_i, Y_i)$

$$X_i \xrightarrow{id_{X_i}} X_i \xrightarrow{f_i} Y_i \qquad id_{X_j} f_i = \begin{cases} f_i & j = i \\ 0 & j \neq 0 \end{cases}$$

So it seems that $\sum_j' id_{X_j}$ is an ~~local~~ approx unit in the path alg. You know $(id_X)^2 = id_X$
$id_X \, id_Y = 0 \qquad X \neq Y$. It should follow that firm $\Leftrightarrow$ red.

---

Start again $D$ generated by $x_t \; y_s \quad s \in \Gamma$

$$x_t x_{t'} = y_s y_{s'} = 0 \qquad y_s x_t = y_{us} x_{ut} = 0 \qquad s^{-1}t \notin \Phi$$

$$\sum_s x_s y_s x_t = x_t, \quad \sum_t y_s x_t y_t = y_s$$

Define $\mathbb{C}\Gamma \to Mult(D)$

$$u \longmapsto \sum_s x_{us} y_s = \sum_t x_t y_{u^{-1}t}$$

Review Mult(A) especially semi-direct prod.

A ring ~~[illegible]~~ $\mu \in \mathrm{Mult}(A)$ is two operators ~~maps~~ on A $(a \mapsto \mu a,\ a \mapsto a\mu)$ s.t.

$$\mu(a_1 a_2) = (\mu a_1) a_2$$
$$a_1(\mu a_2) = (a_1 \mu) a_2$$
$$(a_1 a_2)\mu = a_1(a_2 \mu)$$

product $\mu\nu$ defined by $(\mu\nu)a = \mu(\nu a)$
$a(\mu\nu) = (a\mu)\nu$

$$a_1((\mu\nu)a_2) = a_1(\mu(\nu a_2)) = (a_1\mu)(\nu a_2)$$
$$(a_1(\mu\nu))a_2 = ((a_1\mu)\nu)a_2 = (a_1\mu)(\nu a_2)$$

Mult(A) subring of $\mathrm{Hom}_{A^{op}}(A,A) \times \mathrm{Hom}_A(A,A)^{op}$ satisfying $\langle a_1, \mu a_2\rangle = \langle a_1 \mu, a_2\rangle$

$$\phi : A \longrightarrow \mathrm{Mult}(A) \qquad a \longmapsto \phi_a$$

$\phi_a a_2 = a a_2$ $\quad a_1(\phi_a a_2) = (a_1\phi_a)a_2$ ✓

$a_1 \phi_a = a_1 a$

~~Semidirect product. Before~~

$$(\mu\phi_a)a_2 = \mu(a a_2) = (\mu a)a_2 = \phi_{\mu a} a_2$$
$$a_1(\mu\phi_a) = (a_1\mu)a = a_1(\mu a) = a_1 \phi_{\mu a}$$

$$(\phi_a \mu)a_2 = a(\mu a_2) = (a\mu)a_2 = \phi_{a\mu} a_2$$

~~$a_1(\phi_a\mu) = (a_1 a)\mu = a_1(a\mu) = a_1\phi_{a\mu}$~~

$$\therefore\ a_1(\phi_a \mu) = (a_1 a)\mu = a_1(a\mu) = a_1 \phi_{a\mu} \qquad \therefore\ \phi_a \mu = \phi_{a\mu}$$

~~$\therefore\ \phi_a\mu = \phi_{a\mu}$ [illegible]~~

$$(\mu\phi_a)a_2 = \mu(a a_2) = (\mu a)a_2 = \phi_{\mu a} a_2 \qquad \mu\phi_a = \phi_{\mu a}$$
$$a_1(\mu\phi_a) = (a_1\mu)a = a_1(\mu a) = a_1\phi_{\mu a}$$

~~[illegible]~~ Next define semi-direct product ring

$\mathrm{Mult}(A) \oplus A$ $\quad (\mu_1 + a_1)(\mu_2 + a_2) = \mu_1\mu_2 + (\mu_1 a_2 + a_1\mu_2 + a_1 a_2)$

8 assoc. cond.

2 $\mu_1, \cancel{\mu_2}, \mu_3$ $\quad a_1, \cancel{a_2}, a_3$ $\qquad \mu, \nu$ 3 $\qquad \mu, a_1, a_2$ 3

int. one is $\mu(a\nu) = (\mu a)\nu$

needs $A^2 = A$.

$$\mu((a_1 a_2)\nu) = \mu(a_1(a_2\nu)) = (\mu a_1)(a_2\nu)$$
$$(\mu(a_1 a_2))\nu = ((\mu a_1)a_2)\nu = (\mu a_1)(a_2\nu)$$

~~[illegible]~~

Question: Given a bialg with setlike coalg structure $\mathbb{C}\Gamma$ when ~~[illegible]~~ can projection operators associated with $\Gamma$ grading yield multipliers? You have the examples $M_n$.

~~[illegible]~~ $\Gamma$ ~~category~~ the arrows in a category. In the path alg you have the elements $1_X$ $\forall$ object $X$. These are ann. idempotents, and $\sum_X 1_X$ is an approx unit ~~[illegible]~~ both left + right.

What are the corresp good modules? $\mathbb{C}\Gamma$

Idea: You saw that counital comodules for $\mathbb{C}\Gamma$ same as $\Gamma$-graded vector spaces, and the product $\mathbb{C}\Gamma \otimes \mathbb{C}\Gamma \longrightarrow \mathbb{C}\Gamma$ leads to a tensor product on the category of $\Gamma$-graded V.S. ~~You are interested in~~

~~[illegible]~~ Start again: $\Gamma$ = set of arrows in a small category. $\mathbb{C}\Gamma$ is the path algebra, has basis the arrows $s$, product is composition ~~[illegible]~~ when defined $0$ otherwise. $\mathbb{C}\Gamma$ has approx id. $\sum_X 1_X$

Obvious modules for the ~~path~~ arrow alg should be given by functors. Use normal composition

$[f][g] = [fg]$ $\qquad \xleftarrow{f}\xleftarrow{g}$

So a covariant functor should yield a left $\mathbb{C}\Gamma$-module.

~~How to proceed?~~

Given $F(X)$ $\quad \forall X \in \mathcal{O}$

$\mathbb{C}\Gamma \otimes M \longrightarrow M$ $\qquad$ Assume $\bigoplus 1_X M = M$

~~$\mathbb{C}\Gamma$ = arr~~ $\Gamma$ = arrows in $\mathcal{C}$ with partial product $fg$ defined $\Leftrightarrow$ $dom(f) = range(g)$.

$\xleftarrow{}\xleftarrow{}$

Let $F : \mathcal{C} \longrightarrow Mod(\mathbb{C})$ be covariant functor

Put $M = \bigoplus_x F(x)$ $\qquad \mathbb{C}\Gamma \otimes M \longrightarrow M$

$\otimes(\xi \in F(x)) \mapsto f_*\xi \in F(y)$

$(y \xleftarrow{f} x)$ $\qquad f_*\xi = 0$ if $d(f) \neq r(\xi)$

Start again.

$\Gamma$ is the set of arrows in a category $\mathcal{C}$

$\mathbb{C}\Gamma$ arrow ring $\qquad [f][g] = \begin{cases} [fg] & d(f) = r(g) \\ 0 & \neq \end{cases}$

a functor $x \longmapsto F_x$ $\qquad (y \xleftarrow{f} x) \longmapsto (F_y \xleftarrow{f_*} F_x)$

In this way you get nice $\mathbb{C}\Gamma$ modules.

In $\mathbb{C}\Gamma$ you have orth. idempotents $1_x$

$(y \xleftarrow{f} x)$ $\qquad f 1_x = f \qquad 1_y f = f$

$f 1_x = f$ if $d(f) = x$, $0$ otherwise $\qquad 1_y f = f$ if $y = r(f)$, $0$ if not.

So $\exists$ approximate identity in $\mathbb{C}\Gamma$ namely $\sum_{x \in Ob.} 1_x$

Prop. Assume $A$ has local left units $\forall a_1, \dots, a_n \ \exists a \quad (1-a)a_j = 0 \quad j = 1, \dots, n$

For an $A$-module $M$ TFAE.

(i) $M = AM$ $\qquad k = \sum a_i \otimes m_i$

(ii) $M$ is firm: $A \otimes_A M \xrightarrow{\sim} M$

(iii) $M$ is red. $AM = M$, ${}_A M = 0$.

(iv) $\forall m_i$ finite $\exists a \quad (1-a)m_i = 0$.

$m_i = \sum_j a_{ij} m_j \qquad (1-a)a_{ij} = 0 \implies (1-a)m_i = 0.$

$\forall m \ \exists a \qquad (1-a)m = 0 \implies M = AM.$

~~OKAY, so what~~ So for $A = \mathbb{C}\Gamma$, you have

$M = AM \iff M \xleftarrow{\sim} \bigoplus_{Ob} 1_x M \qquad e_\Phi$

$\Downarrow$

$m = \sum a_i m_i = \sum f_j m_i \qquad \sum_{x \in \Phi} e_x m = m$

Idea: Properties of $e_x$ as far as left modules, (covariant functors) are concerned.

$e_x F = F_x$

$x \xleftarrow{g} y \qquad 1_x g = g \implies e_x g_* = g_* : F_x \leftarrow F_y$

You want to show that $e_x$ for any object $x$ is a multiplier

$\Gamma$ set of arrows in cat $\mathcal{C}$

$\mathbb{C}\Gamma$ has approx identity $\sum_{x \in Ob} id_x$

so $\mathbb{C}\Gamma$ is unital when only finitely many objects.

---

~~[illegible]~~

Return to original program, that is, to ~~[illegible]~~ describe firm or reduced $D$-modules.

$D$ is defined by generator $x_t, y_s$ subject to relations. You construct a $M_2 \times \Gamma$ grading on $D$, so it's a Morita context $\begin{pmatrix} A' & Y \\ X & B' \end{pmatrix}$ with $\Gamma$ grading.

Next you can define $\mathbb{C} \times \mathbb{C}\Gamma \longrightarrow \mathrm{Mult}(D)$

via $u \longmapsto \sum_s x_{us} y_s = \sum_t x_t y_{u^{-1}t}$.

$e_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ $\quad e_{22} \otimes u = \begin{pmatrix} 0 & 0 \\ 0 & \sum_s x_{us} y_s \end{pmatrix}$

You going to

---

The problem to discuss. You have the alg $D$ graded wrt $M_2$, so the first thing to show is that there is a ~~[illegible]~~ homom. $\begin{pmatrix} \mathbb{C} & 0 \\ 0 & \mathbb{C} \end{pmatrix} \longrightarrow \mathrm{Mult}(D)$.

In fact ~~[illegible]~~ you may be missing the idea that $\mathrm{Mult}(D)$ is naturally a Morita context. So is it possible for a $\Gamma$ grading on $D$ to extend? ~~So [illegible] So [illegible]~~

How to proceed?

$$A \xrightarrow{\Delta} \mathbb{C}\Gamma \otimes A$$
$$\Delta(a_s) = s \otimes a_s$$

$\mathrm{Hom}_{A^{op}}(A, A)$ left multipliers

perhaps there's a $\Gamma$-graded version of this.

$$\mu : \bigoplus_s A_s \longrightarrow \bigoplus A.$$

The obvious conjecture is that we can take Hom in the graded context.

Let's see how this works at least for $M_2$

$$\underbrace{\begin{pmatrix} \mu_{11} & \mu_{12} \\ \mu_{21} & \mu_{22} \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}} \begin{pmatrix} a'_{11} & a'_{12} \\ a'_{21} & a'_{22} \end{pmatrix}$$

$$\sum_{j,k} (\mu_{ij} a_{jk}) a'_{kl} = \sum_{j,k} \mu_{ij} (a_{jk} a'_{kl})$$

$$\mu_{ij} : A_{jk} \times A_{kl} \longrightarrow$$

---

$\Gamma$ group. : ~~Restrict~~ attention to $\mu : \bigoplus_{s \in \Gamma} A_s \longrightarrow \bigoplus_{s \in \Gamma} A_s$

where $\mu$ ~~has~~ is homogeneous, i.e. $\exists\, t \ni \mu A_s \subset A_{ts}$

---

You need to understand $D = \begin{pmatrix} A' & Y \\ X & B' \end{pmatrix}$. ~~Yes~~ The basic idea: D defined by generators + relations. Your problem is to ~~describe~~ find all reduced D-modules.

$$D \xrightarrow{\Delta} \mathbb{C}M_2 \otimes D$$

$$\Delta(x_t) = e_{21} \otimes x_t$$

$$\Delta(y_s) = e_{12} \otimes y_s$$

$x_t x_{t'} = y_s y_{s'} = 0$

$y_s x_t \mapsto e_{11} \otimes y_s x_t$

$x_s y_s x_t \mapsto e_{21} \otimes x_s y_s x_t$

You want to construct a semi-direct product

$$\begin{pmatrix} \mathbb{C} & 0 \\ 0 & \mathbb{C}\Gamma \end{pmatrix} \oplus \begin{pmatrix} A' & Y \\ X & B' \end{pmatrix}$$

Check what you need to form the semi-direct product Mult(A) $\oplus$ A. You ~~need to~~ define $(\mu + a)(\mu' + a') = \mu\mu' + (\mu a' + a\mu' + aa')$ and need to check assoc. 3 $\mu$'s ✓, 2 $\mu$'s

$$(\mu\mu')a = \mu(\mu' a) \quad \text{prod for left mult}$$

$$(\mu a)\mu' = \mu(a\mu') \quad \text{uses } A^2 = A.$$

$$a(\mu\mu') = (a\mu)\mu' . \quad \text{prod for rt. mult.}$$

1 $\mu$ def of mult.

So all you have to do is to construct a homom. $\begin{pmatrix} \mathbb{C} & 0 \\ 0 & \mathbb{C}\Gamma \end{pmatrix} \longrightarrow \text{Mult}(D)$

There is a homom. (nonunital) $\mathbb{C}\Gamma \to \text{Mult}(D)$ sending $u \in \Gamma$ to $\sum x_{us} y_s = \sum x_s y_{u^{-1}s}$. ~~How does this work~~. Formula

$$\sum_s x_{us} y_s \, x_t \Big|_a^y = \sum_s x_{us} y_{us} x_{ut} \Big|_a^y = x_{ut} \Big|_a^y$$

$$u \, x_t \Big|_a^y = x_{ut} \Big|_a^y$$

need to get straight $\begin{pmatrix} A' & Y \\ X & B' \end{pmatrix}$

You want to associate to each $u \in \Gamma$ an element of Mult(D), call this $\mu_u$; it consists of two operators on D.

$$\mu_u \begin{pmatrix} a & y \\ x & b \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ ux & ub \end{pmatrix}$$

$$\begin{pmatrix} a & y \\ x & b \end{pmatrix} \mu_u = \begin{pmatrix} 0 & yu \\ 0 & bu \end{pmatrix}$$

It might help to write

$$\mu_u \begin{pmatrix} a & y \\ x & b \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & u \end{pmatrix}\begin{pmatrix} a & y \\ x & b \end{pmatrix}$$

---

The point is that $\sum_s \begin{pmatrix} 0 & 0 \\ 0 & x_{us} y_s \end{pmatrix}$.

is a well defined left mult.

Put another way, you have $\phi \begin{pmatrix} 0 & 0 \\ 0 & x_{us} y_s \end{pmatrix} \in$ Mult(D)

~~What~~ The infinite sum make sense

$$\underbrace{\sum_s \begin{pmatrix} 0 & 0 \\ 0 & x_{us} y_s \end{pmatrix}\begin{pmatrix} a & y \\ x & b \end{pmatrix}}_{\begin{pmatrix} 0 & 0 \\ \sum x_{us} y_s x & \sum x_{us} y_s b \end{pmatrix}}$$

~~So what is going on!~~

Maybe you to get away from matrix mult.

D gen $x_t$ $y_s$ ...

$\sum x_{us} y_s$ The point is that this sum infinite makes sense in Mult(D). For every $\xi \in D$ $\sum_s x_{us} y_s \xi$ is a finite sum.

Try ~~further~~ more Assume $\left(\begin{array}{c|c} \mathbb{C} & 0 \\ \hline 0 & \mathbb{C}\Gamma \end{array}\right) \subset \mathrm{Mult}(D)$.

$$R = \left(\begin{array}{c|c} \mathbb{C} & 0 \\ \hline 0 & \mathbb{C}\Gamma \end{array}\right) \oplus \underbrace{\left(\begin{array}{c|c} A' & Y \\ \hline X & B' \end{array}\right)}_{D}$$ defined

reduced firm D-modules $\simeq$ ~~B~~unital R-mods D-reduced firm.

$$V = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} Z$$

$Z$

$$W = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} Z.$$

so wh

---

Let's get your act together. ~~This time look at~~

First go over the construction of $B$. Let $\mathcal{E}$ = alg gens $h_s$ $s \in \Gamma$, rels. $h_s h_t = 0$ if $s^{-1}t \in \Phi$ and $\sum_s h_s h_t = h_t$, $\sum_t h_s h_t = h_s$. Observe $\Gamma$ action $u(h_s) = h_{us}$, ~~permutes~~ then can form cross product $B = \Gamma \ltimes \mathcal{E} = \mathbb{C}\Gamma \otimes \mathcal{E}$ with product $(s \otimes \xi)(t \otimes \xi') = st \otimes t^{-1}(\xi)\xi'$. $B$ has approx identity $\sum h_s$. $B = B^2 = \sum h_s B$ $\qquad \mathcal{E} = \sum_s h_s \mathcal{E}$

$B = \mathcal{E} \otimes \mathbb{C}\Gamma = \sum h_s \mathcal{E} \otimes \mathbb{C}\Gamma = \sum h_s B$. Thus ~~B~~ if $M$ is a $B$ mod, ~~then~~ $BM = M \Leftrightarrow \sum h_s M = M$ in which case $\sum h_s$ is an approx id.

~~$\sum h_s h_t$~~ Define $\Gamma$ action on $M$.

$$\sum_t h_t \sum_s h_s m_s$$

$$B = \mathcal{E} \otimes \mathbb{C}\Gamma = \sum ?$$

$$\frac{1}{(s-a)(s-b)} = \frac{A}{s-a} + \frac{B}{s-b}$$

---

$$B = \mathbb{C}\Gamma \otimes \mathcal{E} \simeq \mathcal{E} \otimes \mathbb{C}\Gamma \underset{\text{ideal}}{\subset} \tilde{\mathcal{E}} \otimes \mathbb{C}\Gamma$$

$$B = \mathcal{E} \otimes \mathbb{C}\Gamma \hookrightarrow \tilde{\mathcal{E}} \otimes \mathbb{C}\Gamma \longrightarrow \mathbb{C}\Gamma$$

so that $\mathbb{C}\Gamma$ maps to $\mathrm{Mult}(B)$.

You want to check that there is an alg morphism $\mathbb{C}\Gamma \longrightarrow \mathrm{Mult}(B)$. ~~for all clear~~

In fact you would like an approximate identity to be multiplied on the left and right by $\Gamma$. Now $B = \mathcal{E} \otimes \mathbb{C}\Gamma = \bigoplus_{s \in \Gamma} \mathcal{E} \otimes s$

$\mathcal{E}$ has approx id $\sum h_s$. Question: Clearly $\sum h_s \otimes 1$ times to $\xi \otimes s$ with $\xi \in \mathcal{E}$ is $\xi \otimes s$. What about on the other side?

$$\left(\sum h_s \otimes 1\right)(\xi \otimes t) = \sum_s h_s \xi \otimes t = \xi \otimes t$$

$$(\xi \otimes t) \sum_s h_s \otimes 1 = \sum_s \xi h_{ts} \otimes t = \xi \otimes t$$

So the ~~group~~ left action of $u \in \Gamma$ on $B$ should be $\sum_{s \in \Gamma} h_s \otimes u$. Check

$$\left(\sum_{s \in \Gamma} h_s \otimes u\right)(\xi \otimes t) = \sum_s h_s u(\xi) \otimes ut = \underbrace{u(\xi)}_{u \xi u^{-1}} \otimes ut$$

$$u(\xi \otimes t) = u \xi u^{-1} ut = u(\xi) \otimes ut$$

So what is the next step? Check other side $(\xi \otimes t)\left(\sum_s h_s \otimes u\right) = \sum_S \xi(t * h_s) \otimes tu$

$$= \sum_s \xi h_{ts} \otimes tu = \xi \otimes tu$$

Befor. $\left(\sum_s h_s \otimes u\right)(\xi \otimes t) = \sum_s h_s(u*\xi) \otimes ut = (u*\xi) \otimes ut$

which is correct.

Can you summarize what you have learned?

You have ~~seen~~ examined $B = \mathcal{E} \rtimes \Gamma = \mathcal{E} \otimes \mathbb{C}\Gamma$

$(\xi \otimes s)(\xi' \otimes t) = \xi(s * \xi') \otimes st$. Now you define a multiplier ~~~~ on $B$ by

left mult.

$$\xi \otimes t \longmapsto \sum_s (h_s \otimes u)(\xi \otimes t) = \sum_s h_s(u*\xi) \otimes ut \quad \overset{(u*\xi)\otimes ut}{\shortparallel}$$

right mult.

$$\xi \otimes t \longmapsto \sum_s (\xi \otimes t)(h_s \otimes u) = \sum_s \xi h_{ts} \otimes tu = \xi \otimes tu$$

So what does this **all** mean????? ~~Answer!~~

It should tell you what good modules are.

Let $W$ be a $B$-module such that $BW = W$. Since $B = \sum_s h_s B$ you find

$W = \sum_s h_s BW \subset \sum h_s W \subset W$. $\therefore W = \sum h_s W$

and so $W$ has a $\Gamma$-action given by

$$u : h_t w \longmapsto \sum_s (h_s \otimes u)(h_t w)$$

$$= \sum_s \quad ?, ?$$

You need $\quad W = \sum_{s,t} (h_s \otimes t) W$

What did you learn yesterday. Looked at $B$. $\mathcal{E}$ defd by gens $h_s$, $s \in \Gamma$ rels $h_s h_t = 0$ for $s^{-1}t \in \Phi$ and $\sum_s h_s h_t = h_t$, $\sum_t h_s h_t = h_s$. Action of $\Gamma$ on $\mathcal{E}$ given by $u * h_s = h_{us}$. Then $B = \Gamma \ltimes \mathcal{E} = \mathbb{C}\Gamma \otimes \mathcal{E}$ $\quad (s \otimes \xi)(t \otimes \xi') = st \otimes (t^{-1} * \xi)\xi'$. $\quad B = \mathcal{E} \rtimes \Gamma = \mathcal{E} \otimes \mathbb{C}\Gamma = \bigoplus_{s \in \Gamma} \mathcal{E} s$ abbrev. $\xi \otimes s$ to $\xi s$. Then an element of $B$ is a finite linear comb. $\sum_s \xi_s s$ and product is $(\xi s)(\xi' s') = \xi(s * \xi') s s'$. ~~Now using~~ In other words $s\xi' = (s * \xi')s$ $\qquad s * \xi' = s \xi' s^{-1}$

$$\begin{aligned} \mathbb{C}\Gamma \otimes \mathcal{E} &\xrightarrow{\sim} \mathcal{E} \otimes \mathbb{C}\Gamma \\ s \otimes \xi &\longmapsto s * \xi \otimes s \\ t \otimes (t^{-1} * \eta) &\longleftarrow\!\shortmid\ \eta \otimes t \end{aligned} \qquad s\xi \longmapsto (s * \xi)s$$

?

$(s\xi)(s_1 \xi_1)$

Again
$$\begin{aligned} \mathbb{C}\Gamma \otimes \mathcal{E} &\longrightarrow \mathcal{E} \otimes \mathbb{C}\Gamma \\ s\xi &\longmapsto (s * \xi)s \\ t(t^{-1} * \eta) &\longleftarrow\!\shortmid\ \eta t \end{aligned} \qquad \begin{aligned} &\eta(t * \eta')tt' \\ &\quad \| \\ &(\eta t)(\eta' t') \end{aligned}$$

$$s\xi s'\xi' \longmapsto (s * \xi)ss'\xi' = \underbrace{(s * \xi)(ss' * \xi')}_{s * (\xi(s' * \xi'))} ss' \quad ?$$

$$s\xi = \underbrace{s\xi s^{-1}}_{(s * \xi)} s \qquad s\xi s'\xi' = ss'\underbrace{(s'^{-1}\xi s')}_{s'^{-1} * \xi}\xi'$$

~~[illegible]~~

$$\begin{array}{cc} \mathbb{C}\Gamma \otimes \mathcal{E} & \mathcal{E} \otimes \mathbb{C}\Gamma \\ \sum_s s \otimes \xi_s & \sum_s (s * \xi_s) \otimes s \end{array}$$

$$\sum_s s\xi_s = \sum_s (s\xi_s s^{-1})s = \sum_s (s * \xi_s)s$$

$B = \mathbb{C}\Gamma \otimes \mathcal{E} \qquad (s\otimes\xi)(t\otimes\eta)$

$D$ defined by gen. $x_t, y_s$ $s\in\Gamma$ relns.

$x_t x_{t'} = y_s y_{s'} = 0 \qquad y_s x_t = y_{us} x_{ut} = 0 \quad s t \notin \Phi$

$$\sum_s x_s y_s x_t = x_t \qquad \sum_t y_s x_t y_t = y_s$$

---

$$\sum_s x_s y_s \cdot \qquad D = \begin{pmatrix} A' & Y \\ X & B' \end{pmatrix}$$

Point is that $\sum_s x_s y_s$ is well defined in $Mult(D)$

$$\sum_s x_s y_s \begin{vmatrix} y \\ x_t \end{vmatrix} = \begin{vmatrix} 0 \\ x_t \end{vmatrix}$$

So how do you make this clear? To get the complete picture

$$0 \to \cancel{K} \longrightarrow D \longrightarrow \overbrace{Mult(D)}^{R} \qquad K = \{d \in D \mid Dd = dD = 0\}$$

Point is that If

---

multipliers for $B = \mathcal{E} \otimes \mathbb{C}\Gamma = \bigoplus_{s\in\Gamma} \mathcal{E}s$

$(\xi\otimes s)(\xi'\otimes s') = \xi\underbrace{(s*\xi')}_{''}\otimes ss'$

$\xi s \xi' s' = \xi(s\xi' s^{-1}) ss'$

$$\sum_s h_s h_t = h_t \text{ in } \mathcal{E}$$

$$\sum_t h_s h_t = h_s$$

$$\sum_s (h_s \otimes u)(\xi \otimes t)$$

$$= \sum_s h_s(u*\xi)\otimes ut = (u*\xi)\otimes ut$$

Simpler $(1\otimes u)(\xi\otimes t) = (u*\xi)\otimes ut \qquad (\xi\otimes t)(1\otimes u) = \xi\otimes tu$

So $B = \mathcal{E} \rtimes \Gamma$ is an ideal in $\tilde{\mathcal{E}} \rtimes \Gamma$

$$0 \longrightarrow \mathcal{E} \longrightarrow \tilde{\mathcal{E}} \longrightarrow \mathbb{C} \longrightarrow 0$$

$$0 \to B \longrightarrow \tilde{\mathcal{E}} \rtimes \Gamma \longrightarrow \mathbb{C}\Gamma \longrightarrow 0$$

<u>Point:</u> Given $0 \longrightarrow A \longrightarrow R \longrightarrow R/A \longrightarrow 0$

get $R \longrightarrow \mathrm{Mult}(A)$

$r \longmapsto (a \mapsto ra,\ a \mapsto ar)$

---

~~If $A \hookrightarrow R$~~

$$\begin{array}{ccccccc} 0 \longrightarrow & A & \longrightarrow & B & \xrightarrow{\text{Ring } \cancel{\text{map}}} & A/B & \longrightarrow 0 \\ & \| & & \downarrow & & \downarrow & \\ & A & \xrightarrow{\phi} & \mathrm{Mult}(A) & \longrightarrow & \mathrm{Out}(A) & \longrightarrow 0 \end{array}$$

$$b(a_1a_2) = (ba_1)a_2$$
$$a_1(ba_2) = (a_1b)a_2$$
$$(a_1a_2)b = a_1(a_2b).$$

$\phi_a$

$$(\phi_a\mu)a' = \phi_a(\mu a')$$
$$= a(\mu a') = (a\mu)a'$$
$$= \phi_{a\mu}a'$$

$$\phi_a a_2 = a a_2$$
$$a_1 \phi_a = a_1 a$$

$$a'(\phi_a\mu) = (a'\phi_a)\mu = (a'a)\mu$$
$$= a'(a\mu)$$
$$= a'\phi_{a\mu}$$

$$\mathrm{Mult}(A) = \{ \mathrm{Hom}_{A^{op}}(A,A) \times \mathrm{Hom}_A(A,A)^{op}$$

$$(a \longmapsto \mu a,\quad a \longmapsto a\mu) \in$$

such that $a'(\mu a) = (a'\mu)a$

$$a'((\mu\nu)a) = a'(\mu(\nu a)) = (a'\mu)(\nu a)$$
$$(a'(\mu\nu))a = ((a'\mu)\nu)a = (a'\mu)(\nu a)$$

$(\mu a)\nu$

So consider forming $\mathrm{Mult}(A) \oplus A$

$$(\mu a)\nu \overset{?}{=} \mu(a\nu) \qquad \text{true if } a = a_1 a_2$$

and true after applying $\phi$.

$$\phi_{(\mu a)\nu} = \phi_{\mu a}\nu = (\mu\phi_a)\nu$$
$$\phi_{\mu(a\nu)} = \mu\phi_{a\nu} = \mu(\phi_a \nu)$$

---

So what's up. $D$ defined by generators + relations

Try different approach. Start with $A$ for a change:

gen $p(s)$ $s \in \Gamma$, supp reln. $p(s) = 0$ $s \notin \Phi$, idemp rel $p(s) = \sum_t p(st^{-1})p(t)$. ~~You are going to defin~~ Go over case $\Gamma = \Phi$ finite. Then $\sum s \otimes p(s) \in \mathbb{C}\Gamma \otimes A$ idemp.

Better – go back to $\begin{pmatrix} A & Y \\ X & B \end{pmatrix}$ $\qquad \sum x_s y_s = 1 \in B$

Situation: $A$ ring $X, Y$ dual pair over $A$.

canon. map
$$W \otimes_A Y \longrightarrow \mathrm{Hom}_{A^{op}}(X, W)$$
$$w \otimes y \longmapsto (x \longmapsto w\langle y, x\rangle)$$

~~assume~~ take $W = X$, assume $\sum x_s \otimes y_s \longmapsto \mathrm{id}_{\cancel{X}}$

$$\begin{tikzcd} X \arrow[r, "(y_s)"] \arrow[rr, bend right, "1_X"'] & \tilde{A}^n \arrow[r, "(x_s)"] & X \end{tikzcd}$$

$\phi : X \longrightarrow W$ $\quad A^{op}$ linear

send $\phi$ to $\sum \phi(x_s) \otimes y_s$

$$w \otimes y \longmapsto (x \longmapsto w\langle y, x\rangle) \longmapsto \sum_s w\langle y, x_s\rangle \otimes y_s$$
$$\parallel$$
$$\sum_s w \otimes \langle y, x_s\rangle y_s = w \otimes y$$

You are assuming a M. cont. $\begin{pmatrix} A & Y \\ X & B \end{pmatrix}$ ~~ ~~ with $\sum x_s y_s = 1 \in B$. Then find

$$W \otimes_A Y \xrightarrow{\sim} \mathrm{Hom}_{A^{op}}(X, W)$$

$$w \otimes y \longmapsto (x \longmapsto w(yx))$$

$$\sum_s w(yx_s) \otimes y_s \qquad \sum_s \phi(x_s) \otimes y_s \longleftarrow\!\shortmid \ \phi$$

$$\| $$

$$\sum_s w \otimes y x_s y_s \qquad \left(x \longmapsto \underbrace{\sum_s \phi(x_s)(y_s x)}_{\phi\left(\sum_s x_s (y_s x)\right)}\right)$$

here use $\sum_s y x_s y_s = y$.

here use $\sum x_s y_s x = x$

Special cases $W = \tilde{A} \Rightarrow Y = \mathrm{Hom}_{A^{op}}(X, \tilde{A})$ ~~ ~~

$$W = X \Rightarrow X \otimes_A Y = \mathrm{Hom}_{A^{op}}(X, X).$$

So what seems important is to use the M. cont. structure $\begin{pmatrix} A & Y \\ X & B \end{pmatrix}$ and the identities $\left|\begin{array}{l} \sum x_s y_s x = x \\ \sum y x_s y_s = y. \end{array}\right.$

---

multiplier ring for $A$ ??

~~ ~~ Let's try to start with $A$. Go over from $A$ viewpoint.

$$X \longrightarrow \mathrm{Hom}_A(Y, A)$$

---

You need precise statements and proofs. Let's try a simple ~~ ~~ "finite" case

Start with $\Gamma = \Phi$ finite. $A$ gen $p(s)$

$s \in \Gamma$ $\quad p(s) = \sum_{s=tu} p(t)\,p(u)$. $\quad A$ is idempotent. Try to construct $X$ and $Y$, and the pairing $yx$,

$$Y = p(\mathbb{C}\Gamma \otimes A)$$

$$\mathbb{C}\Gamma \otimes A = \Big\{ \sum_{t\in\Gamma} t \otimes a_t \;\Big|\; t \mapsto a_t,\ \text{map } \Gamma \to A \Big\}$$

$$\begin{tikzcd}
Y \arrow[r, "\alpha"] \arrow[dr, "j"'] & \mathbb{C}\Gamma \otimes A \arrow[r, "\beta"] \arrow[d, shift right, "\eta_1"'] & Y \\
& A \arrow[u, shift right, "\varepsilon_1"'] \arrow[ur, "i"'] &
\end{tikzcd}$$

$$\alpha y = \sum_s s \otimes j s^{-1} y, \qquad \beta\Big(\sum_t t \otimes a_t\Big) = \sum_t t i a_t$$

$$\beta\alpha y = \beta \sum_s s \otimes j s^{-1} y = \sum_s s i j s^{-1} y = y \qquad \text{Why?}$$

because
$$\sum_s s i j s^{-1} = \sum_s s \beta \varepsilon_1 \eta_1 \alpha s^{-1} = \beta \sum_s s \varepsilon_1 \eta_1 s^{-1} \alpha = \beta\alpha$$

$$\alpha\beta \sum_t t \otimes a_t = \sum_s s \otimes j s^{-1} \sum_t t i a_t = \sum_s s \otimes \sum_t j s^{-1} t i a_t$$

Still not transparent. From this viewpoint you have to define
$$p \sum t \otimes a_t = \sum_s s \otimes \sum_t p(s^{-1}t)\, a_t$$

Still not transparent. How to improve the situation? Basically you need

Idea. Mult() =

Let's discuss philosophy: Look at good modules for $A$, $B$, $D$ resp. For $A$ you want reduced ~~A~~ modules i.e. $V$ such that $AV = V$, ${}_AV = 0$. For $B$ you want $B$-modules $W \ni BW = W$, whence $W$ is the same as a $\Gamma$-module with $h_1 \ni \sum h_s = 1$ on $W$. For $D$ you want $(V, W, i, j)$ with ~~...~~ $i$ injective, $j$ surjective. You know these module cats are equivalent, ~~But if A is~~. BUT if $A$ is not reduced, then $A$, $D$ are not in the appropriate module categories.

One thing to try is to make

~~...~~ Take $B$

---

I think you might try the following.

~~... Start with ... adjoin an h~~

Start with ~~unital ring~~ $\mathbb{C}\Gamma$ ~~adjoin~~ form unital alg ~~gen~~ by $h$ sat. $hsh = 0$ $s \notin \Phi$

$$\sum_s h_s h_t = h_t, \quad \sum_t h_s h_t = h_s \qquad h_s = shs^{-1}.$$

get unital ring $R$ ~~idempotent~~ ideal $B$ gen. by the $h_s$. Clear that $R = \mathbb{C}\Gamma \oplus B$. Look at unital modules over $R$

---

Problem. ~~[illegible]~~ Given a ring $A$, say $A = A^2$, how does $\mathrm{Mult}(A)$ compare to $\mathrm{Mult}(A/K)$, $K = \{a \mid Aa = aA = 0\}$.

$$\begin{array}{ccc} A/K & \hookrightarrow & \mathrm{Mult}(A) \\ \| & ? & \downarrow \\ A/K & \hookrightarrow & \mathrm{Mult}(A/K) \end{array}$$

$\mathrm{Hom}_{A^{op}}(A, A)$

---

good $\mathrm{Hom}_A(Y, Z) = \mathrm{Hom}(A \otimes_A Y, \mathrm{Hom}_A(A, Z))$

$= \mathrm{Hom}_A(AY, Z/_A Z)$

$\mathrm{Hom}_A(Y, Z) \longrightarrow \mathrm{Hom}_A(AY, Z/_A Z)$ inj? NO

But if $AY = Y$, then

$0 \to \cancel{\mathrm{Hom}_A(AY, {}_A Z)} \to \mathrm{Hom}_A(AY, Z) \longrightarrow \mathrm{Hom}_A(AY, Z/_A Z)$

OKAY. $\mathrm{Hom}_{A^{op}}(A, A) \hookrightarrow \mathrm{Hom}_{A^{op}}(A, A/K)$

$\| $

$\mathrm{Hom}_{A^{op}}(A/K, A/K)$.

---

$$\begin{array}{ccc} A/K & \hookrightarrow & \mathrm{Mult}(A) \\ \| & & \downarrow \\ A/K & \hookrightarrow & \mathrm{Mult}(A/K) \end{array}$$

How do you propose to go?

~~Assume $A = A^2$. Then~~

~~$\mathrm{Hom}_A($~~

You know that if $AY = Y$ and ${}_AZ = 0$, then $\mathrm{Hom}_A(Y, Z) \xrightarrow{\sim} \mathrm{Hom}_{m(A)}(Y, Z)$

$$\begin{gathered} \| \text{ in general} \\ \mathrm{Hom}_A(A \otimes_A Y, \mathrm{Hom}_A(A, Z)) \\ \| \\ \mathrm{Hom}_A(AY, Z/{}_AZ) \end{gathered}$$

~~So in general~~ So take $Y = Z = A$.

$$\mathrm{Hom}_A(A, A/{}_AA)$$

---

There is a point I don't understand.

$$0 \longrightarrow K \longrightarrow A \longrightarrow \mathrm{Mult}(A) \longrightarrow \mathrm{Outmult}(A) \longrightarrow 0$$

Problem you vaguely remember, whether $\mathrm{Mult}(A)$ is intrinsic. First look an $A$-module $Y$ and compare endos of $Y$ with endos of $Y$ in $\mathcal{M}(A)$.

$$\mathrm{Hom}_A(Y, Y) \longrightarrow \mathrm{Hom}_A(A \otimes_A Y, \mathrm{Hom}_A(A, Y))$$

$$0 \longrightarrow {}_AY \longrightarrow Y \longrightarrow \mathrm{Hom}_A(A, Y)$$

---

$$\mathrm{Hom}_A(Y, Z) \longrightarrow \mathrm{Hom}_A(A \otimes_A Y, \mathrm{Hom}_A(A, Z))$$

$$0 \longrightarrow {}_AZ \longrightarrow Z \longrightarrow \mathrm{Hom}_A(A, Z)$$

$$0 \longrightarrow Z/{}_AZ \to \mathrm{Hom}_A(A, Z) \longrightarrow \underset{\text{nil}}{Q} \longrightarrow 0$$

$$\mathrm{Hom}_A(Y,Z) \longrightarrow \mathrm{Hom}_A(A\otimes_A Y, \mathrm{Hom}_A(A,Z))$$
$$\qquad\qquad\qquad\qquad\qquad\qquad \parallel$$
$$\qquad\qquad\qquad\qquad \mathrm{Hom}_A(AY, Z/_A Z)$$

Think of picking an object of $\mathfrak{M}(A)$ and taking its endo ring.

$$\mathrm{Hom}_A(Y,Y) \longrightarrow \mathrm{Hom}_{\mathfrak{m}(A)}(Y,Y) = \mathrm{Hom}_A(AY, Y/_A Y)$$

$$\begin{tikzcd} & Y \arrow[dr, two heads] & \\ AY \arrow[ur, hook] \arrow[d, two heads] & & Y/_A Y \\ \mathrm{Im}\{AY \to Y/AY\} \arrow[urr, hook] & & \end{tikzcd}$$

Anyway - summarize the situation. You start with $A = A^2$. You get

$$\begin{array}{ccccccc} 0 \to & A/K & \longrightarrow & \mathrm{Mult}(A) & \longrightarrow & \mathrm{Outmult}(A) & \to 0 \\ & \parallel & & \downarrow & & & \\ 0 \to & A/K & \longrightarrow & \mathrm{Mult}(A/K) & & & \end{array}$$

$$0 = (A+K)(a+K) = Aa + K$$

$$AK = KA = 0$$

$$Aa,\ aA \subset K \implies A^2 a,\ aA^2 \subset AK,\ KA = 0.$$

So if $A = A^2$, then it looks like $\mathrm{Mult}(A/K)$ is "bigger" than $\mathrm{Mult}(A)$.

Work on Morita equivalence details.

$$W \longmapsto V = hW \qquad \text{with} \qquad p(s) = jsi$$

here $h = ij$ : $W \xrightarrow{j} hW \xhookrightarrow{i} W$

In $B$ have $0 = hsh = ysij$ for $s \notin \Phi$.

$i$ inj, $j$ surj $\implies p(s) = jsi = 0$ for $s \notin \Phi$.

$$\sum_t p(st^{-1})p(t) = \sum jst^{-1}hti = jsi = p(s)$$

Reduced $\quad W = \sum_t tijW \implies jW = \sum p(t)\, jW$

$$w = \sum sijs^{-1}w$$

Assume $p(t)\sigma = jti\sigma = 0$ for all $t$. Then

$\iota\sigma = \sum s\iota j s^{-1} i\sigma = 0$ so $\sigma = 0$ as $\iota$ inj.

---

$V \longmapsto W = p(\mathbb{C}\Gamma\otimes V)$ $\qquad p\left(\sum_t t\otimes v(t)\right) = \sum_s s\otimes\sum_t p(s^{-1}t)v(t)$.

$p$ comm with $\Gamma$

$p^2 = p$.

$$\begin{tikzcd} W \arrow[r, "\alpha"] \arrow[dr, "j"'] & \mathbb{C}\Gamma\otimes V \arrow[r, "\beta"] \arrow[d, shift right, "\eta_1"'] & W \\ & V \arrow[u, shift right, "\varepsilon_1"'] \arrow[ur, "i"'] & \end{tikzcd}$$

$t\varepsilon_1 v(t)$

$\beta\left(\sum_t t\otimes v(t)\right) = \sum_t t\iota v(t)$

$\alpha w = \sum_s s\otimes v(s) \Rightarrow \underbrace{\eta_1 s^{-1}\alpha w}_{j s^{-1} w} = v(s)$

$W$ is a $\Gamma$-module with operator $h = h_1 = ij = \beta\varepsilon_1\eta_1\alpha$

[redacted] $hsh = i(jsi)j = \iota p(s) j = 0$ for $s\notin\Phi$.

$\sum shs^{-1}w = \sum s\beta\varepsilon_1\eta_1\alpha s^{-1}w$

$= \sum\beta\cancel{\varepsilon_s\eta_s}\alpha w = \cancel{\beta}w$

---

$\Gamma$-mod $W \longmapsto V = hW$ [redacted] with $h: W \xrightarrow{j} \twoheadrightarrow$ [redacted] $\xhookrightarrow{i} W$

$p(s) = jsi$. Check that $V$ is a reduced $A$-module

red. $A$-mod $V \longmapsto W = p(\mathbb{C}\Gamma\otimes V)$ $\qquad p\left(\sum t\otimes v(t)\right) = \sum_s s\otimes\underbrace{\sum_t p(s^{-1}t)v(t)}_{p\sigma}$

Canonical isos. $V \xmapsto{F} W = p(\mathbb{C}\Gamma\otimes V) \xmapsto{G} hW = hp(\mathbb{C}\Gamma\otimes V)$

$W \longmapsto V = hW \longmapsto p(\mathbb{C}\Gamma\otimes hW)$

---

~~Also~~ Claim: If $V$ red. $A$-module, then $\exists$ canon. isom. of

[redacted] $GFV = hp(\mathbb{C}\Gamma\otimes V)$ with $V$

Why, because

It seems you want to introduce symbols $F, G$ for the functors

$$F(V) = p(\mathbb{C}\Gamma\otimes V) = \Big\{\sum_t t\otimes\phi(t) \;\Big|\; p\phi=\phi\Big\}.$$

31
31
30
92

$$\begin{tikzcd}
F(V) \arrow[r, "\alpha"] \arrow[dr, "j"'] & \mathbb{C}\Gamma\otimes V \arrow[r, "\beta"] \arrow[d, shift right, "\eta_1"'] & F(V) \\
& V \arrow[u, shift right, "\varepsilon_1"'] \arrow[ur, "i"'] &
\end{tikzcd}$$

---

So start again.

Given $V$ an $A$-module, define $p$ on $\mathbb{C}\Gamma\otimes V$ by $p\left(\sum_t t\otimes v(t)\right) = \sum_s s\otimes\sum_t p(s^{-1}t)\,v(t)$ $\quad p$ well defined

$p$ comm. with $\Gamma$-action: $u\sum_t t\otimes v(t) = \sum_t t\otimes v(u^{-1}t)$

$p^2 = p$.

$F(V) = p(\mathbb{C}\Gamma\otimes V)$. There are canonical $\Gamma$-maps

$$\begin{tikzcd}
F(V) \arrow[r, "\alpha"] \arrow[dr, "j"'] & \mathbb{C}\Gamma\otimes V \arrow[r, "\beta"] \arrow[d, shift right, "\eta_1"'] & F(V) \\
& V \arrow[u, shift right, "\varepsilon_1"'] \arrow[ur, "i"'] &
\end{tikzcd}$$

$\beta\alpha = 1$

$\alpha\beta = p$

$$\beta\Big(\sum_t \overset{t\,\varepsilon_1 v(t)}{\cancel{\phantom{xx}}}\, t\otimes v(t)\Big) = \sum_t t\,\iota\, v(t)$$

$s^{-1}\alpha w = \sum_s s\otimes v(s)$

$v(s) = \eta_1 s^{-1}\alpha w = j s^{-1} w$

$\alpha w = \sum_s s\otimes j s^{-1} w$

wait:

---

Start again. First go over the Mor. eq.

$W \longmapsto GW = hW$, let $h = ij : W \overset{j}{\twoheadrightarrow} hW \overset{i}{\hookrightarrow} W$

Given $W$ factor $h = ij : W \twoheadrightarrow hW \xhookrightarrow{i} W$ 441

$GW = hW$ with $p(s) = jsi$. Check relations

$0 = hsh = i(jsi)j = i\,p(s)\,j$ for $s \notin \Phi \Rightarrow$

$p(s) = 0$ since $i$ inj $j$ surj. $\sum_t p(st^{-1})p(t) = p(s)$.
$jst^{-1}ij\,t\,i$

Given $V$ define

$p$ on $\mathbb{C}\Gamma \otimes V$ by $p\left(\sum_t t \otimes v(t)\right) = \sum_s s \otimes \sum_t p(s^{-1}t)\,v(t)$

$p$ commutes with left $\Gamma$-mult.

Given $V$, form $\mathbb{C}\Gamma \otimes V$ space of $\sum_{t\in\Gamma} t \otimes v(t)$ where $v(t)$ fn on $\Gamma$ with fin supp values in $V$. $\Gamma$-mod. with $u \sum_t t \otimes v(t) = \sum_t ut \otimes v(t) = \sum_t t \otimes v(u^{-1}t)$. Define

$p$ comm. with $\Gamma$ action. $p^2 = p$.

~~$F(V) = p(\mathbb{C}\Gamma \otimes V)$~~. Form diag.

$$\begin{tikzcd}
F(V) \arrow[r, "\alpha"] \arrow[dr, "j"'] & \mathbb{C}\Gamma \otimes V \arrow[r, "\beta"] \arrow[d, shift right, "\eta_1"'] & F(V) \\
& V \arrow[u, shift right, "\varepsilon_1"'] \arrow[ur, "i"'] &
\end{tikzcd}$$

$F(V) = p(\mathbb{C}\Gamma \otimes V)$

$\alpha$ inclusion

$\beta$

---

Construction of the Morita context.

functor $\mathcal{M}(B) \longrightarrow \mathcal{M}(A)$ takes $W$ to $hW$ equipped with $p(s)$ operators as follows. Factor $h$ on $W$ canon. $h = ij$ where $j : W \twoheadrightarrow hW$, $i : hW \hookrightarrow W$

[redacted] $p(s) = jsi$

Seems good to factor

$$\begin{tikzcd} W \arrow[r, two heads, "j"] \arrow[rr, bend left, "h"] & hW \arrow[r, hook, "\iota"] & W \end{tikzcd} \qquad h = \iota j \qquad \begin{array}{l}\iota \text{ inj} \\ j \text{ surj.}\end{array}$$

define $\quad p(s) = js\iota \quad$ on $jW$ $(= hW)$. Thus

$$p(s) jw = js\iota jw = jshw$$

you can write this simply as $\quad p(s)hw = hshw$

define: $\quad p(s)(hw) = hshw$

Then $\quad s \notin \Phi \implies hsh = 0 \implies p(s) = 0.$

$$\sum_s p(s)\, p(s^{-1}t) hw = \sum_s hshs^{-1}thw = hthw$$
$$= p(t)hw$$

$\therefore \quad G(W) = hW \quad$ equipped with $\quad p(s)(hw) = hshw$

$\cancel{X} = G(B) = hB$,

$$G(B) \otimes_B W = hB \otimes_B W \twoheadrightarrow hW$$

$$\sum_i hb_i \otimes w_i \longmapsto \sum hb_i w_i = 0.$$

$$p(s)\Big(\sum_i hb_i \otimes w_i\Big) = \sum_i hs\,hb_i \otimes w_i = 0$$

---

Other side. $\quad W' \quad B^{op}$-module

Define functor $\quad W' \longmapsto W'h \quad$ with $\quad whp(s) = whsh$

$$\sum_s whp(s)\,p(s^{-1}t) = \sum_s whshs^{-1}th = whth = whp(t)$$

$X = Bh. \qquad$ Then $\quad W' \otimes_B Bh \twoheadrightarrow W'h$

$A^{op}$ nil isom. $\quad$ Now what?