Buy train ticket to Bonn from Ostende ~~&~~ return
Return sailing times (next Saturday + Sunday).

	15.00	18.15
arr	16.30	19.45

Somehow you have to start thinking [illegible] about mathematics again. Start writing mathematical terms. What to discuss? Your talk on Morita invariance of higher algebraic K theory for h-unital rings. Cuntz will again be here, speaking W 16.30. You need to make an abstract of your talk. Present idea is to begin with [illegible] $K_*(A)$.

[illegible] Recall the formula used by Cuntz, but probably due to (Baaj) Skandalis? Given Γ discrete, have non comm. model for $B\Gamma$ which abelianizes to a big simplex. There's a simplicial complex vertices are elts of Γ, a finite $F \subset \Gamma$ is given and simplexes are finite $M \subset \Gamma$ such that $M^{-1}M \subset F$, so you may assume $1 \in F$, $F = F^{-1}$. Get [illegible] s.complex Σ_F left acted on by Γ. Then [illegible] have non comm. C^*-alg. $\mathcal{E}_{\Sigma_F}$ generated by h_s $s \in \Gamma$, subject to $h_s^* = h_s \geq 0$ $h_s h_t = 0$ if $s^{-1}t \notin F$, also want $h_s \sum_{t \in \Gamma} h_t = h_s$. $\mathcal{E}^{ab}_{\Sigma_F}$ is an alg of functions where the (image of) h_s is the barycentric coord.

Γ-graded algebras enter

Connes Renormalization Group, Riemann-Hilbert problem, and the missing Galois Theory at Archimedean places.

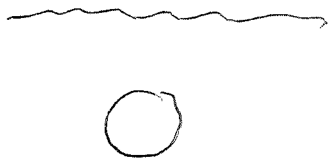
Renormalization (jt with D Kreimer)

Physics Idea: Pb. complete initial a

$F = ma$ $F = -Mg$ $\overset{\text{initial}}{a \simeq 100g}$

m mass of ping pong ball

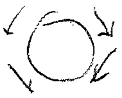
M mass of water



actually $a \leq 2g$

GREEN (1830's)

new inertial mas $m \longmapsto m + \frac{1}{2}M$ (spherical case)



$\int E^2$

you can only measure the sum $m + \frac{1}{2}M$ for electron

QFT (Computational Aspect)

$$S(A) = \int \mathcal{L}(A)\, d^4x$$

$$\mathcal{L}(A) = \frac{(\partial A)^2}{2} - \frac{m^2}{2}A^2 + \mathcal{L}_{int}(A)$$

$$G_N(x_1, \dots, x_N) = \langle 0 | T\phi(x_1) \cdots \phi(x_N) | 0 \rangle$$

$$= N \int e^{i\frac{S(A)}{\hbar}} A(x_1) \cdots A(x_N)\, [dA]$$

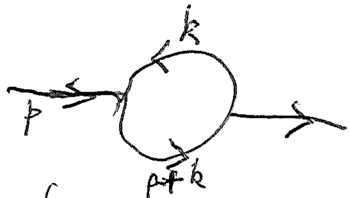
Expansion $S(A) = S_0(A) + S_{int}(A)$

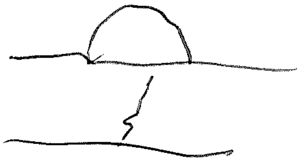
$$G_N(x_1, \dots x_N) = \sum_{n=0}^{\infty} \frac{i^n}{n!} \int \phi(x_1) \cdots \phi(x_N)\, \underline{d\mu}$$

$$\Big(\sum_{n=0}^{\infty} \quad \underline{d\mu}\Big)^{-1}$$

Gaussian meas.

Exame





$$U(\Gamma) = \int \frac{1}{k^2+m^2} \frac{1}{(p+k)^2+m^2} d^Dk$$

DIVERGENT

PHYSICS RESOLUTION $\int_{|k|<\Lambda}$ Λ = cutoff

CUTOFF CHANGING THE RULES AS YOU GO ALONG

B.P. Preparation. $\bar{R}(\Gamma) = U(\Gamma) + \sum_{\gamma<\Gamma} C(\gamma) U(\Gamma/\gamma)$

$C(\Gamma) = -\underset{\text{pole part}}{T}(\bar{R}(\Gamma))$ is local

Renormalized Graph $R(\Gamma) = \bar{R}(\Gamma) + C(\Gamma)$ is finite

D. Kreimer

$$\Delta\Gamma = \Gamma \otimes 1 + 1 \otimes \Gamma + \sum_{\gamma<\Gamma} \gamma \otimes \Gamma/\gamma$$

is a ~~coproduct~~ Hopf algebra $\Delta : \mathcal{H} \longrightarrow \mathcal{H} \otimes \mathcal{H}$

$\mathcal{H}$ = poly alg gen by 1PI graphs

graded Hopf algebr = $\mathcal{U}$(graded Lie algebra of Feynmann graphs.

Problem - what is the mathematical meaning of renormalization

Hilbert 21st Problem

$y'(z) = A(z)\, y(z)$ $A(z) = \sum_{\alpha \in S} \frac{A_\alpha}{z-\alpha}$

$$\gamma(z) = \gamma_-(z)^{-1}\gamma_+(z) \qquad z \in \mathbb{C}$$

$$\gamma_-(\infty) = 1.$$

$$\Downarrow$$

General Principle of extraction of finite value

Given some loop γ, want value at a point such as 0 where it is not defined, answer take $\gamma_+(0)$.

Alg.	Geom.
$\mathcal{H} \xrightarrow{\phi} \mathcal{A}$ alg of Laurent series	$\varepsilon \longmapsto \gamma(\varepsilon)$ loop of etc
$\mathcal{H} \xrightarrow{\phi_+} \mathcal{A}_+$ reg at $\varepsilon=0$	loop reg at $\varepsilon=0$
$\mathcal{H} \xrightarrow{\phi_-} \mathcal{A}_-$ gen. by $\frac{1}{\varepsilon}$	γ reg. for $\varepsilon \neq 0$ $\gamma_-(\infty) = 1$

Thm. (acdk) $\gamma(\varepsilon) = \gamma_-(\varepsilon)^{-1}\gamma_+(\varepsilon)$

BIRK. DEC. of ϕ is given by (inductively)

$$\Delta X = X \otimes 1 + 1 \otimes X + \sum X' \otimes X''$$

$$\phi_-(X) = -T\left(\phi(X) + \sum \phi_-(X')\phi(X'')\right)$$

$$\phi_+(X) = \phi_-(X) + \phi(X) + \sum \phi_-(X')\phi(X'')$$

for any graded conn. Hopf alg.

duality between standard simplices

polytope X $\quad p(X,t)=\sum_{i\geq 0} a_i t^i$

generating series $(X_n)_{n\geq 0}$

$$f_i^X(x)=\sum_{n\geq 1}(-1)^n\, p(X_{n-1},t)\, x^n$$

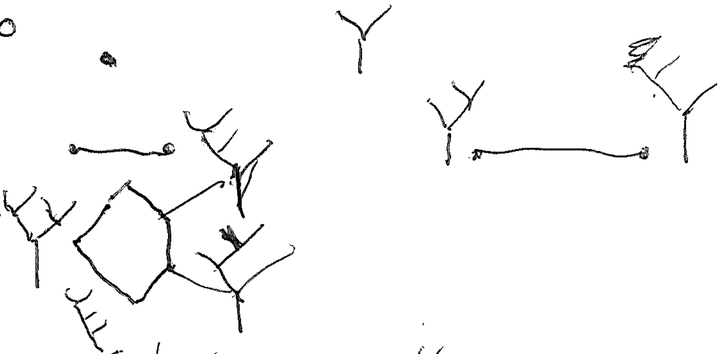
for Δ^n $\quad f_i^\Delta(x)=\frac{-x}{(1+x)(1+(1+t)x)}$

Stasheff polytopes

K^n $\quad K^0$

K^1

K^2



K^3 6 pentagons + 4 squares

in general

$$f_i^K(x)=\frac{-1-(2+tx)+\sqrt{1+2(2+t)x+t^2x^2}}{2(1+t)x}$$

$$f_i^\Delta(f_i^K(x))=x$$ inverse sense composition

Associate dialgebras $\quad \dashv, \vdash : A\otimes A\longrightarrow A$

5 axioms.

Leibniz algebras — noncomm. version of Lie Algs. $[x,y]$.

$$[[x,y],z]=[[x,z],y]+[x,[y,z]]$$

$$\begin{tikzcd}
\mathrm{Ass} \arrow[r, hook] \arrow[d] & \boxed{\mathrm{Dialgs}} \arrow[d] \\
\mathrm{Lie} \arrow[r, hook] & \mathrm{Leibniz}
\end{tikzcd}$$

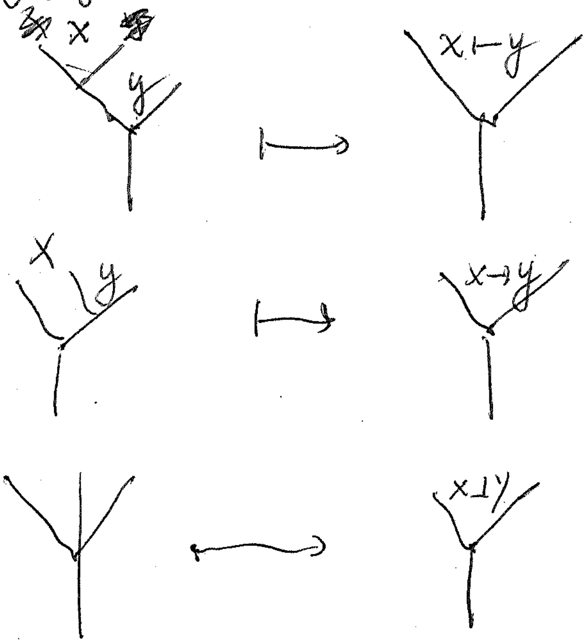
$[x,y]\approx x\dashv y - y\vdash x$

Thm. free assoc. dialg on one generator

$P_1 = \{x\}$, $\{\hat{x}x, x\hat{x}\}$

$\check{u} \dashv \check{v} = \check{u}v$, $\check{u} \vdash \check{v} = u\check{v}$

trialgebras



Thm. free assoc. trialg P_m = subsets of $\{0, \cdots, m-1\}$

$\check{u} \dashv \check{v} = \check{u}V$, $\check{u} \vdash \check{v} = u\check{v}$, $\check{u} \perp \check{v} = \check{u}\check{v}$

Alg operads + Koszul duality

$\mathcal{P}$ $\quad \mathcal{P}(V)$ = free alg type $\mathcal{P}$ on V

$$\begin{tikzcd} W \arrow[r] \arrow[dr] & \mathcal{P}(W) \arrow[d, dashed, "\exists!\ \mathcal{P}\text{alg extn.}"] \\ & A \end{tikzcd} \qquad \mathrm{Id} \to \mathcal{P} \qquad \begin{tikzcd} \mathcal{P}(V) \arrow[r] \arrow[dr, "\mathrm{id}"'] & \mathcal{P}(\mathcal{P}(V)) \arrow[d, "\mathcal{P}\mathrm{alg}"] \\ & \mathcal{P}(V) \end{tikzcd}$$

$\mathcal{P}: \mathrm{Vect} \longrightarrow \mathrm{Vect}$

$\gamma: \mathcal{P} \circ \mathcal{P} \longrightarrow \mathcal{P}$ associative unital

Get assoc. alg in cat of Poly Functors: $V \to V$ equipped with composition

Ginsburg-Kapranov (quadratic) $\mathcal{P}$ $\quad \mathcal{P}^!$ dual of chain complex $\underbrace{\left(\mathcal{P}^{!}(\mathcal{P}(V)), d\right)}_{\text{the Koszul cx,}}$ acyclic?

$Ass^{!} = Ass$ Koszul $\Leftrightarrow$ HH of $T(V)$ is trivial

$Lie^{!} = Comm$ } Koszul $\Leftrightarrow$ $H_*(Lie\,V)$ trivial

$Comm^{!} = Lie$ }

Assoc. trialgs. $\longrightarrow$ $\overset{\text{Trias}}{\text{operad}}$

1) Dual operad = cat of alg. over Dias

2) Koszul

Dendriform trialgs $\prec$ $\succ$ $\cdot$

$Y \prec Y =$ 

$Y \succ Y =$ 

$Y \cdot Y =$ 

Renormalyzation gp is what?

Dimensional Analysis

$d^D p \to \mu^{-D} d^D p$ $\quad\mu$: unit of m.

Unren. th. $\quad \gamma_N(\varepsilon) \in G.$ $\quad \gamma_{e^t\mu}(\varepsilon) = \theta_{\varepsilon t}(\gamma_\mu(\varepsilon))$

FACT $\quad \gamma_\mu^-$ is ind. of μ.

THM $\quad \gamma^-(\varepsilon)^{-1}\theta_{\varepsilon t}(\gamma^-(\varepsilon))$ is FINITE at $\varepsilon = 0$

and $F_t = \exp t\beta$ is a 1 Par-group $F_t \in G$.

$\beta = \underset{\substack{\uparrow \\ \text{grading}}}{Y} \operatorname{Res} \gamma$

Thm. $\gamma^-(\varepsilon) = \lim_{t\to\infty} e^{-t(\frac{\beta}{\varepsilon}+Z_0)} e^{tZ_0}$

Z_0 adjoined to Lie G $\quad$ (Z_0 = grading?)

Galois Th at Arch Pl.

Artin L fns. —— Hecks L fn

$\mathrm{Gal}(k_{ab}/k) \leftarrow - - - GL_1(A)/GL_1(k)$

Weil Galois interp of D_k? $\quad$ conn. component

Brauer theory

$M \longrightarrow \mathrm{Mod}(M) \subset \mathbb{R}_+^*$

Classification of AFD factors

$\mathbb{C} \subset \mathbb{C}_{un}$ $\quad \mathrm{Fix}(\theta_\lambda) = \mathbb{C}$

Max Unram. extension

α constant

in fact depends on μ $\quad \alpha(e^t\mu) = F_t(\mu)$

High Energy Physics works over $\mathbb{C}_{un}$

Euler-Zagier Multiple Zeta values

$$\sum_{n_1 < n_2} \frac{1}{n_1^{a_1}} \frac{1}{n_2^{a_2}} - \frac{1}{n_k^{a_k}}$$

Stafford (jt with Dennis Keeler), closely related to Artin Tate Van den Bergh. Twisted Homog. Coord ring. everything over $k = \bar{k}$ alg. cl. field, X proj scheme/k, $\mathcal{L}$ ample line bundle, form $B(X, \mathcal{L}) = \bigoplus_n H^0(X, \mathcal{L}^{\otimes n})$. Next take σ autom. of X

form $B(X, \mathcal{L}, \sigma) = \bigoplus H^0(X, \overset{\mathcal{L}_n}{\overbrace{\mathcal{L} \otimes \mathcal{L}^\sigma \otimes \ldots \otimes \mathcal{L}^{\sigma^{n-1}}}})$, with

mult. $B_n \otimes B_m \xrightarrow{\sim} H^0(\mathcal{L}_n) \otimes H^0(\mathcal{L}_m^{\sigma^n}) \longrightarrow H_0(\mathcal{L}_n \otimes \mathcal{L}_m^{\sigma^n}) = B_{n+m}$

<u>Ex,</u> $X = \mathbb{P}^1$, $\mathcal{L} = \mathcal{O}(1)$, $k(X) = k(u)$, $\mathcal{L}$ = sheaf gen. by global sections 1, u. $B(X, \mathcal{L}, Id) = k[x, y]$. Next take σ autom. given by $u \longmapsto u+1$, $B_1 \times B_1 \longrightarrow B_2$

$y \otimes x = u \otimes 1 \longrightarrow u \otimes 1^\sigma = u$

$x \otimes y = 1 \otimes u \longrightarrow 1 \otimes (u+1) \longrightarrow u+1$.

$x \otimes x \longrightarrow \underline{1}$

so $xy = yx + x^2$

$B(\mathbb{P}^1, \mathcal{O}(1), \sigma) = \dfrac{k\{x, y\}}{(xy - yx = x^2)}$

For $B(X, \mathcal{L}, \sigma)$ to be nice we need to change $\mathcal{L}$ ample to $\mathcal{L}$ σ-ample.

Def. $\mathcal{L}$ ample when $\forall$ coh. F $H^i(X, F \otimes \mathcal{L}^{\otimes n}) = 0$ $\forall n >> 0$ $\forall i \geq 1$

replaced by $H^i(X, F \otimes \mathcal{L}_n) = 0$ $\forall i \geq 1$, $n >> 0$.

Thm ① (Artin vd Bergh 1990) $\mathcal{L}$ σ-ample, Then the map $F \longmapsto \bigoplus_n H^0(X, F \otimes L_n)$ (this is a right B module) gives an equiv. of cats. $coh(X) \longrightarrow q\,gr(B)$

coherent sheaves on X

cat of noeth graded B modules modulo those of finite length

② B is right noeth.

Defn: ① A ring R is connected graded if $R = k \oplus R_1 \oplus \cdots$

② A graded ring R has Gelfand-Kirillov dim $= \alpha$
if $\cancel{\alpha} = \underline{\lim} \{ \beta : \dim_k (\bigoplus_{i=0}^{n} R_i) \leq n^\beta \quad n >> 0 \}$

Thm 2 B conn, grad, fin gen, domain GK dim 2, + generated by R_1 as a k algebra. Then $B \cong B(X, \mathcal{L}, \sigma)$ up to to a fin. dim. factor), for some curve X, aut σ and ample $\mathcal{L}$. Moreover $\mathcal{L}$ is σ ample so B is left + right noeth.

($\mathcal{L}$ is σ-ample $\Leftarrow$ $\mathcal{L}$ ample
$\Rightarrow$ if X irred.

Reason can ass. F invertible, by RR need $F \otimes \mathcal{L}_n$ have degree $>> 0$. $\deg(F \otimes \mathcal{L}_n) = \deg F + n \deg(L)$

Artin-Tate-van den Bergh + Stephenson Thm. Let R be a ring (something like weighted poly ring in 3 vbles.) such that

① gl dim $R = 3$
② GK dim $R < \infty$
③ Gorenstein cond: $\mathrm{Ext}^i(k, R) = \begin{cases} 0 & i \neq 3 \\ k & i = 3 \end{cases}$

Then R is noeth domain, with the same Hilbert series as $k[x_0, x_1, x_2]$

Bondal-Polischuk: "any possible" noncomm $\mathbb{P}^2$ is q-gr R for such an R (with $\dim R_1 = 3$) $ch(k) \neq 3$

know (R gen by R_1, $\dim R_1 = 3$) either $R \simeq B(\mathbb{P}^2, \mathcal{O}(1), \sigma)$
or $R \longrightarrow B(E, \mathcal{L}, \sigma)$ E cubic curve in $\mathbb{P}^2$

---

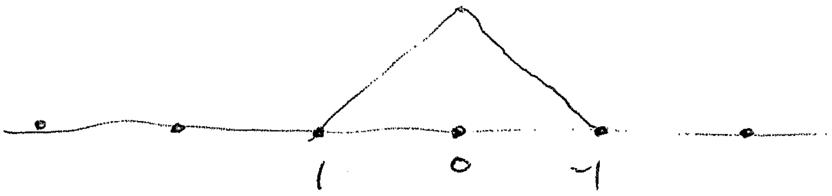
left σ-ample same as right σ^{-1} ample

Def. $A_{Num}(X)$ = group of Cart. div. / num. equiv $\simeq \mathbb{Z}^m$

Let $M = \{t \in \Gamma \mid h_t > 0\}$. Pick $s \in M$. Then $h_s h_t > 0 \Rightarrow s^{-1}t \in F$ so $t \in sF$. $\therefore$ M finite $\therefore$

$$h_s = \sum_{t \in M} h_s h_t \Rightarrow \sum_{t \in M} h_t = 1.$$

example $\Gamma = \mathbb{Z}$, $F = \{-1, 0, 1\}$



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cross product alg $A \rtimes \Gamma$, alg $= \bigoplus_{s \in \Gamma} As$, is naturally Γ graded. Let $B = \bigoplus_{s \in \Gamma} B_s$ be Γ-graded ie $B_s B_t \subset B_{s+t}$, e.g. $A \rtimes \Gamma$. P_Γ universal Γ-graded algebra gen. by a projector $p = p^* = p^2$. $p = \sum_{s \in \Gamma} p_s$ $p_s \in B_s$

$p = p^* \Longleftrightarrow p_s^* = p_{s^{-1}}$, $p = p^2 \Longleftrightarrow p_t = \sum_s p_s p_{s^{-1}t}$.

Consider $\mathcal{E}_{\Sigma_F} \rtimes \Gamma$ ~~essentially~~

Canonical hom. $P_\Gamma \longrightarrow \mathcal{E}_{\Sigma_F} \rtimes \Gamma$ $p_s \longmapsto h_1^0 h_s^0$

In $\mathcal{E}_{\Sigma_F}$ you have h_s, $s \in \Gamma$ $t h_s t^{-1} = h_{ts}$

$p_s \in \mathcal{E}_{\Sigma_F} s$, p_s should become $h_1^{1/2} h_s^{1/2} s$

From Claire O'Kane

Dear Mrs Quillen,
I would like to do Violin over the Summer Holidays. Is it possible
to do this on a Saturday Morning?

$$p_s^* = (h_1^{1/2} h_s^{1/2} s)^* = s^{-1} h_s^{1/2} h_1^{1/2} = h^{1/2} h_{s^{-1}}^{1/2} s^{-1} = p_{s^{-1}}$$

$$\sum_s h_1^{1/2} h_s^{1/2} s \cdot h_1^{1/2} h_{s^{-1}t}^{1/2} s^{-1} t = \sum_s h_1^{1/2} h_s^{1/2} h_s^{1/2} h_t^{1/2} t = h_1^{1/2} h_t^{1/2} t = p_t$$

$P_\sigma \in GL_m(\mathbb{Z})$, say P_σ quasi-unip if all e-values $\overset{?}{=} 1$.

Thm (AvdB) If X smooth surface, $\exists$ σ-ample $\mathcal{L}$ $\Leftrightarrow$ P_σ quasi-unipotent

Thm. Keeler. X any proj scheme/k with auto σ. Then X has a σ ample inv. sh. $\Leftrightarrow$ P_σ is quasi-unipotent

---

You need to get started on your talk, or review Joachim's talk.

Γ discrete group, $F \subset \Gamma$ finite, get simp complex $\{M \text{ finite} \neq \emptyset \text{ in } \Gamma \mid M^{-1}M \subset F\}$, call this Σ_F, can ass. $1 \in F$, $F^{-1} = F$. ~~Def suppose~~

A finite subset $M^{\neq\emptyset}$ of Γ is a simplex when $x^{-1}y \in F$ for all $x, y \in M$.

[redacted] $\mathcal{E}_{\Sigma_F}$ = C^* alg gen. by h_s, $s \in \Gamma$ $\ni$ $h_s = h_s^* \geq 0$, [redacted] $h_s h_t = 0$ if $s^{-1}t \notin F$. Then

[redacted] $h_s =$ [redacted] $\sum_t h_s h_t$. ~~So what~~

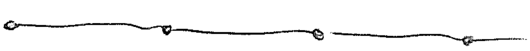
$\mathcal{E}^{ab}_{\Sigma_F}$ impose comm. C^* alg [redacted] $^{\text{cont}}$ functions on spectrum, a point is a ~~[illegible]~~ family $\{h_s\}_{s \in \Gamma}$ of $\mathbb{C}$-numbers. $h_s \geq 0$ all s, $h_s h_t = 0$, $s^{-1}t \notin \Gamma$.

$h_s = \sum_t h_s h_t$ suppose $s_0 \ni h_{s_0} > 0$, $M = \{t \mid h_t > 0\}$. $t \in M \Longrightarrow t^{-1}s_0 \in F$

$\Longleftarrow$

So there is this mysterious formula for a canonical projector inside the cross product $\mathcal{E}_{\Sigma_F} \rtimes \Gamma$. mysterious because you don't understand

Yesterday Joachim mentioned (something about) why the Bott class needs rapidly decreasing matrices. Something involving smooth (differentiable?) functions on the circle. Point perhaps is isom of smooth functions on $[0,1]$ vanishing to infinite order at endpoints and smooth functions on the circle vanishing to inf order at 0. Somehow this isomorphism is needed for periodicity. Non alg part reflected in Voevodski's two circles.

Lets now look at $\Gamma = \mathbb{Z}$ and $E_\Gamma = \mathbb{R}$ with simplicial structure as usual 
maybe you ~~first~~ start with the principal bundle $\mathbb{R} \to \mathbb{R}/\mathbb{Z}$ and you have associated fibre bundle with fiber the alg group ring $\mathbb{C}[\mathbb{Z}]$. This gives a line bundle for the ring $\mathbb{C}[\mathbb{Z}]$ of ~~Laurent series~~, over the circle, better to say you ~~have~~ a line bundle over the circle $\mathbb{R}/\mathbb{Z}$ where the usual field $\mathbb{R}$ or $\mathbb{C}$ is replaced by the ring $\mathbb{C}[\mathbb{Z}]$ of Laurent polys. Anyway you get a line bundle over the torus

What can you do? You have a very specific situation. Essentially you have a space of functions in $\mathbb{R}$

You have a ~~line~~ fibre bundle over the circle with fibre $\mathbb{C}[\mathbb{Z}]$, you know the space of continuous global sections can be identified with $C_c(\mathbb{R})$

Manin Quantized Θ fns and Abelian Varieties

linear spaces, keep these, but deform morphisms

- Affine alg groups

$G \longrightarrow F(G)$

$$\begin{tikzcd} & \cdot \arrow[d, "e"] & \\ G\times G \arrow[r, "m"] & G \arrow[r, "\Delta"] \arrow[d] \arrow[loop left, "i"] & G\times G \\ & \cdot & \end{tikzcd}$$

axioms → diagrams defining Hopf alg

$$\begin{tikzcd} & k & \\ F(G)\otimes F(G) & F(G) \arrow[l, "m^*"'] \arrow[u] & F(G)\otimes F(G) \arrow[l, "\Delta^*"'] \\ & k \arrow[u] & \end{tikzcd}$$

- Alg varieties

$A \longrightarrow \Gamma(A, \overset{\text{v. ample}}{L})$

$$\begin{tikzcd} A\times A \arrow[r, "M"] & A\times A \\ (x, y) & (x+y, x-y) \end{tikzcd}$$

$$\begin{tikzcd} \cdot \arrow[d, "e"] & \\ A \arrow[r, "\Delta"] \arrow[d] \arrow[loop left, "i"] & A\times A \\ \cdot & \end{tikzcd}$$

$M^*(L \boxtimes L) = L^2 \boxtimes L^2$

$i^*(L) = L$

①
$$\begin{tikzcd} k & \\ \Gamma(A,L^2) \arrow[u] & \Gamma(A,L)\otimes\Gamma(A,L) \arrow[l, "\Delta^*"'] \\ k \arrow[u] & \end{tikzcd}$$

② $\Gamma(A,L)\oplus\Gamma(A,L) \xrightarrow{M^*} \Gamma(A,L^2)\oplus\Gamma(A,L^2)$

Upon $\Gamma(A,L)$ acts Heisenberg group Mumfords 1960s

③ Canonical bases in $\Gamma(A,L)$

Plan A ~~[illegible]~~ $\mathbb{C}^n/\mathbb{Z}^{2n} \simeq \underset{\text{alg torus}}{\mathbb{C}^{*n}}/\underset{\text{mult. lattice}}{B}$

$A \rightsquigarrow \underset{\text{alg torus}}{T}/B$ $B \subset T(K)$

$T \rightsquigarrow \underset{\substack{\text{quant}\\\text{not group}}}{T_{\text{quant}}}/B$ $\Gamma(A,L) \rightsquigarrow$ space of quantized Θ fns $\subset Af(T_{\text{quant}})$

satisfying autom. conditions

Category of non comm torus (H free abelian gp, α)

$\alpha : H \times H \longrightarrow K^*$ $\qquad$ $\alpha(h,g) = \alpha(g,h)^{-1}$

$\alpha(h_1+h_2, g) = \alpha(h_1,g)\,\alpha(h_2,g)$

$(H_1, \alpha_1) \xrightarrow{\varphi} (H_2, \alpha_2)$: $\quad H_1 \to H_2$: $\quad \alpha_2^2(f(h), f(g)) = \alpha_1^2(f(h), f(g))$

Characteristic φ $\qquad$ $\varepsilon_f(h,g) = \alpha_1(h,g)\,\alpha_2^{-1}(f(h), f(g)) \in \{\pm 1\}$

q torus: $T(H,\alpha)$: rep by $Al(H,\alpha) = \bigoplus_{h \in H} K e(h)$ $\qquad$ $e(h)e(h') = \alpha(h,h')\,e(h+h')$

$Af(H,\alpha) = \prod K e(h)$

$An(H,\alpha) = \left\{ \sum a_h e(h) \,\middle|\, |a_h| = O\left((\|h\|+1)^{-N} \right) \right\}$

morphisms of tori $\Longleftrightarrow$ homoms. of alg fns.

$T(H,\alpha) \rightsquigarrow T(H,1)$ classical commutative

$B \subset T(H,1)(K)$ acting on $T(H,\alpha)$

(i) standard maps

(ii) $[n]$ changes quant. param

(iii), (iv) M^*

Quantum $n \times n$ matrices $\qquad$ $\mathcal{O}_q(M_n(k))$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad \begin{matrix} ab = qba & cd = qdc \\ ac = qca & bd = qdb \end{matrix}$$

$$bc = cb \qquad ad - da = (q - q^{-1})cb$$

quantum det $\qquad$ $D_q = \sum_{\sigma \in S_n} (-q)^{l(\sigma)} X_{\sigma 1, 1} \cdots X_{\sigma n, n}$

$n=2$ $\quad D_q = ad - qcb$

$[I \mid J]$

comult. $\Delta(X_{ij}) = \sum_k X_{ik} \otimes X_{kj}$

$$\Delta[\underset{\substack{\uparrow\\ \text{rows}}}{I}|\underset{\substack{\uparrow\\ \text{cols.}}}{J}] = \sum_{|K|=|I|} [I|K] \otimes [K|J]$$

$\underline{\text{NYM}}$
Jap J M 9(93)
PW Mem AMS 89

q not a root of 1.

Group $2n$ torus $\underline{\alpha} = (\alpha \ldots ; \beta \ldots)$

$\underline{\alpha} \cdot X_{ij} = \alpha_i \beta_j X_{ij}$ invariant prime ideals

---

$A^2 = A$ Vaserstein $A \to R \to R/A$

$GL(R) \to GL(R/A)$

$$\begin{tikzcd} \tilde{A} \arrow[r] \arrow[d] & R \arrow[d] \\ \mathbb{Z} \arrow[r] & R/A \end{tikzcd}$$

$$\begin{array}{cc} K_1(\tilde{A}) & K_1(R) \\ K_1(\mathbb{Z}) & K_1(R/A) \end{array}$$

Vaserstein

$K_1(\tilde{A}) \to K_1(\mathbb{Z}) \oplus K_1(R) \to K_1(R/A) \to K_0(\tilde{A}) \to K_0(\mathbb{Z}) \oplus K_0(R) \to K_0(R/A)$

Vaserstein identity relates $1-xy$ to $(1-yx)^{-1}$.

$$\begin{pmatrix} A & Q \\ P & B \end{pmatrix} \quad \underbrace{\begin{pmatrix} 1 & -q \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & q \\ p & 1 \end{pmatrix}}$$

$$\cancel{\begin{pmatrix} (1-qp)^{-1} & 0 \\ 0 & 1 \end{pmatrix}} \quad \underbrace{\begin{pmatrix} 1-qp & 0 \\ p & 1 \end{pmatrix}\begin{pmatrix} (1-qp)^{-1} & 0 \\ 0 & 1 \end{pmatrix}}$$

$$\begin{pmatrix} 1 & 0 \\ -p & 1 \end{pmatrix}\begin{pmatrix} 1 & q \\ p & 1 \end{pmatrix}\begin{pmatrix} 1 & -q \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ p\underline{(1-qp)^{-1} \quad 1} \end{pmatrix}\begin{pmatrix} 1 & q \\ 0 & 1-pq \end{pmatrix}\begin{pmatrix} 1 & -q \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1-pq \end{pmatrix}$$

$$\begin{pmatrix} 1 & -q \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & q \\ p & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -p & 1 \end{pmatrix} = \begin{pmatrix} 1-qp & 0 \\ p & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -p & 1 \end{pmatrix} = \begin{pmatrix} 1-qp & 0 \\ 0 & 1 \end{pmatrix}$$

$p_s = h_1^{1/2} h_s^{1/2} s$ $\qquad p_s^* = s^{-1} h_s^{1/2} h_1^{1/2} = h_1^{1/2} h_{s^{-1}}^{1/2} s^{-1} = p_{s^{-1}}$

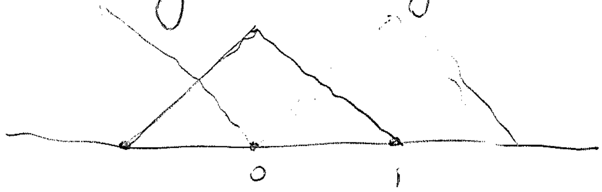
$$\sum_t p_{st^{-1}} p_t = \sum_t h_1^{1/2} h_{st^{-1}}^{1/2} st^{-1} h_1^{1/2} h_t^{1/2} t$$

$$= \sum_t h_1^{1/2} h_{st^{-1}}^{1/2} h_{st^{-1}}^{1/2} h_s^{1/2} s = h_1^{1/2} h_s^{1/2} s = p_s$$

you want to get back to understanding

$C(\mathbb{R}) \rtimes \mathbb{Z} = C(\mathbb{R}/\mathbb{Z}) \otimes \mathbb{K}$? Why

In any case you have a projection in $C(\mathbb{R}) \rtimes \mathbb{Z}$

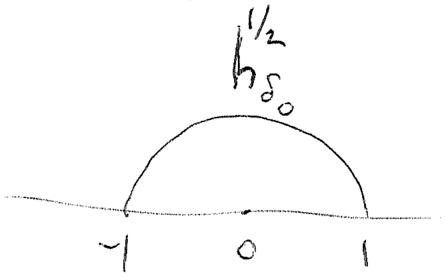


$\sum h_1^{1/2}$

Wait. $C(\mathbb{R})$ is a module over $C(\mathbb{R}) \rtimes \mathbb{Z}$

You have an explicit $p = \sum_{s\in\Gamma} \underbrace{h_1^{1/2} h_s^{1/2}}_{} s$ zero for most s.

$$p = \sum_{n\in\mathbb{Z}} h_{\delta_0}^{1/2} \delta_n h_{\delta_0}^{1/2}$$



$h_{\delta_0} h_{\delta_{\substack{-1\\+1}}} \neq 0$

This p can be written $au + b + cu^{-1}$

$C(\mathbb{R}) \rtimes \mathbb{Z}$ is a twisted Laurent poly ring

$$f u^n g u^m = f \cdot (u^n g u^{-n}) \cdot u^{n+m}$$

$$(p_{-1} + p_0 + p_1)^2 = p_{-1}^2 \quad p_{-1}p_0 + p_0p_{-1} \quad p_0^2$$

$$p_{-1}p_1 + p_1p_{-1} + p_0p_1 + p_1p_0 + p_t^2$$

$C(\mathbb{R}) \rtimes \mathbb{Z}$ Morita eq to ~~$C(S^1)$~~ $C(\mathbb{R}/\mathbb{Z})$

to show this you need ~~the~~ appropriate bimodules

Obv. Candidate is $C(\mathbb{R}) = \{$cont $f(x)$ van. at $\infty.\}$ naturally acted on by $C(\mathbb{R}) \rtimes \mathbb{Z}$. But wait, this "duality" type of duality stems from $C(\mathbb{Z}) \rtimes \mathbb{Z} \simeq \mathbb{K}$

$B = P \otimes_A Q = \varinjlim F_\alpha \otimes_A Q$ $\qquad$ $\overset{\tilde{A}^{n_\alpha}}{\overset{\shortparallel}{F_\alpha}}$ free fg A-mod

$F \longrightarrow P \longrightarrow \operatorname{Hom}_A(Q, A)$

$Q \longrightarrow \operatorname{Hom}_{A^{op}}(P, A) \rightarrow \operatorname{Hom}_{A^{op}}(F, A) \simeq A\check{F} \subset \check{F}$

$$\begin{tikzcd}
F \otimes_A Q \arrow[r] \arrow[d] & P \otimes_A Q \\
F \otimes_A A\check{F} = M_n(A) \arrow[d] & \\
F \otimes_A \check{F} = M_n(\tilde{A}) &
\end{tikzcd}$$

point.
$$\begin{tikzcd}
(F, Q) \arrow[r] \arrow[d] & (P, Q) \\
(F, A\check{F}) &
\end{tikzcd}$$

critical point independence of $\mathcal{F}_P$ arbitrariness of choice. $\qquad$ Choose $F \longrightarrow P$ No.

You have a filtering category: objects are maps $F \longrightarrow P$ with F ~~fg~~ fg free, morphs are

$$\begin{tikzcd}
F' \arrow[rr] \arrow[dr] & & F \arrow[dl] \\
& P &
\end{tikzcd}$$

Formally

$$\begin{tikzcd}
\varinjlim_{F\ \mathcal{F}_P} K_*(F \otimes_A Q) = K_*(P \otimes_A Q) \arrow[d] & \\
K_*(F \otimes_A A\check{F}) \arrow[r] & K_*(A)
\end{tikzcd}$$

But ~~now~~ you must show commutativity of

$$\begin{tikzcd}
K_*(F' \otimes_A Q) \arrow[r] \arrow[d] & K_*(F \otimes_A Q) \arrow[d] \\
K_*(F' \otimes_A A\check{F}') \arrow[dr] & K_*(F \otimes_A A\check{F}) \arrow[d] \\
& K_*(A)
\end{tikzcd}$$

$F' \longrightarrow F$

$$\begin{tikzcd}
Q \arrow[dr] & & \\
& A\check{F} \arrow[r] & A\check{F}'
\end{tikzcd}$$

$$\begin{tikzcd}
F' \otimes_A A\check{F} \arrow[r] \arrow[d] & F \otimes_A A\check{F} \\
F \otimes_A A\check{F} &
\end{tikzcd}$$

~~[illegible]~~ Start with
$$\begin{tikzcd} F' \arrow[r, "\phi"] \arrow[dr] & F \arrow[d] \\ & P \arrow[r] & \mathrm{Hom}_\Lambda(Q, A) \end{tikzcd}$$
and

factor it $F' \xrightarrow{\Gamma_\phi} F' \oplus F \xrightarrow{pr_2} F$

What you must prove: Given
$$\begin{tikzcd} F' \arrow[r, "\phi"] \arrow[dr] & F \arrow[d] \\ & P \end{tikzcd}$$
one has
$$\begin{tikzcd} Q \arrow[r] & \mathrm{Hom}_{A^{op}}(P, A) \arrow[r] & \mathrm{Hom}_{A^{op}}(F, A) \arrow[r] \arrow[d, equal] & \mathrm{Hom}_{A^{op}}(F', A) \\ Q \arrow[rr] & & A\check{F} \arrow[r] & A\check{F}' \end{tikzcd}$$

$$\begin{tikzcd} F' \otimes_A Q \arrow[r, Rightarrow] \arrow[d] \arrow[dr, phantom, "\checkmark"] & F \otimes_A Q \arrow[d] \\ F' \otimes_A A\check{F} \arrow[r] \arrow[d] & F \otimes_A A\check{F} \arrow[d, equal] \\ F' \otimes_A A\check{F}' \arrow[d, equal] & M_n(A) \arrow[d, dashed, "(K_*)"] \\ M_{n'}(A) \arrow[r, dashed, "(K_*)"] & K_*(A) \end{tikzcd}$$

idea $\det(1 + K)$
K finite rank.

Procesi MacDonald positivity Conj + "n!" Conj.

n! conj. n integer ~~[illegible]~~ partition $\lambda \vdash n$

02
01 11
00 10

$D_\lambda(x_1, \dots, x_n, y_1, \dots, y_n) = \det$ of

$$\begin{array}{c|ccccc|} & 00 & 10 & 01 & 11 & 02 \\ 1 & x_1^0 y_1^0 & x_1^1 y_1^0 & & & x_1^0 y_1^2 \\ 2 & x_2^0 y_2^0 & & & & \\ 3 & & & & & \\ 4 & & & & & \\ 5 & & & & & \end{array}$$
← index of exponents

this is a mixed vanderMonde det.

V_λ = span of all derivatives of D_λ

n! conjecture says $\dim V_\lambda = n!$ due to GARSIA - HAIMAN approach to prove MacD + conj

V_λ carries the regular rep of S_n in a bigraded way.

MacD conj. Symm functions useful to compute characters of symm. gps (FROBENIUS) change of basis

$S_\lambda(x)$ basis of Schur fns.

88 MacD found $H_\lambda(x; q, t) = \sum P_{\lambda\mu}(q,t) S_\mu(x)$

MacD + conj says $P_{\lambda\mu}(q,t)$ are polys in q, t with $\geq$ int coeff.

$\mathbb{C}^{2n} = (\mathbb{C}^2)^n$ S_n acts

$$\begin{tikzcd} & \mathbb{C}^{2n}/S_n \\ H_n \arrow[ur] & \end{tikzcd}$$

$\mathbb{C}^{2n}/S_n$ bad var. map not flat (in contrast to $\mathbb{C}^n$)

H_n = Hilbert scheme $= \{ I \subset \mathbb{C}[x,y] \mid \dim \mathbb{C}[x,y]/I = n \}$

turns out to be smooth.

Form

$$\begin{tikzcd} X_n \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{fiber prod as varieties}"] & \mathbb{C}^{2n} \arrow[d] \\ H_n \arrow[r] & \mathbb{C}^{2n}/S_n \end{tikzcd}$$

Thm: FLAT → (left vertical arrow)

H_n^T fixpts indexed by partitions

$\pi^{-1}(p_\lambda)$ should be $= V_\lambda$

---

so what to do? V inf dim'l. v.s. $1+T$ $T \in V \otimes V^* \subset \mathrm{End}\, V$

You want to define $\det(1+T)$. T finite rank

subspaces $\mathrm{Im}(T) \xleftarrow{\sim} V/\mathrm{Ker}(T)$. First reduction

is $\det(1+T, V) = \det(1+T, TV)$

$$\begin{tikzcd} TV \arrow[r, hook] \arrow[d, "1+T|_{TV}"] & V \arrow[r] \arrow[d, "1+T"] & V/TV \arrow[d, "1"] \\ TV \arrow[r, hook] & V \arrow[r] & V/TV \end{tikzcd}$$

$V \to X \to V$ $GL(V \otimes V^*) = \bigcup GL(F \otimes V^*)$

$V \otimes V^* = \bigcup F_i \otimes V^*$ $V = \bigcup F_i$

$$\begin{tikzcd} V \otimes V^* \\ F_i \otimes V^* \arrow[u, hook] \arrow[d] \\ F_i \otimes F_i^* \end{tikzcd}$$

$V \otimes V^* = \bigcup F_i \otimes V^*$

$$\begin{tikzcd} F_i \otimes V^* \arrow[r, hook] \arrow[d] & F_j \otimes V^* \arrow[d] \\ F_i \otimes F_j^* \arrow[r, hook] \arrow[d] & F_j \otimes F_j^* \\ F_i \otimes F_i^* & \end{tikzcd}$$

V infdiml v.s., $B = V\otimes V^*$ ring of fin rk ops.

$V = \bigcup F_i$ $\qquad GL(B) = \bigcup_i GL(F_i\otimes V^*)$

$F_i \subset F_j$.

$$\begin{tikzcd}
F_i\otimes V^* \arrow[r, hook] \arrow[d] & F_j\otimes V^* \arrow[d] \\
F_i\otimes F_j^* \arrow[r, hook] \arrow[d] & F_j\otimes F_j^* \\
F_i\otimes F_i^* &
\end{tikzcd}
\qquad
\begin{tikzcd}
GL(F_i\otimes V^*) \arrow[r, hook] \arrow[d] & GL(F_j\otimes V^*) \arrow[r, hook] \arrow[d] & GL({}_n V\otimes V^*) \\
GL(F_i\otimes F_j^*) \arrow[r, hook] \arrow[d] & GL(F_j\otimes F_j^*) & \\
GL(F_j\otimes F_j^*) & &
\end{tikzcd}$$

$C(\mathbb{R})\rtimes\mathbb{Z}$ is M equiv to $C(\mathbb{R}/\mathbb{Z})$

supposedly it is obvious how $C(\mathbb{Z})\rtimes\mathbb{Z}$ is the compact operators, $C(\mathbb{Z})$ [illegible] $= \bigoplus_{n\in\mathbb{Z}} \mathbb{C}e_n$ so modules over $C(\mathbb{Z})$ are $\mathbb{Z}$-graded vector spaces $\bigoplus_{n\in\mathbb{Z}} V_n$ then when you take $\rtimes\mathbb{Z}$ you ask ask for an invertible operator $\bigoplus_{n\in\mathbb{Z}} V_n$ of degree 1, whence $V_n \simeq V_{n+1} \simeq V_{n+2} \simeq \ldots\ldots$

$C(\mathbb{R})\rtimes\mathbb{Z}$ acts on $C(\mathbb{R})$ also $C(\mathbb{R}/\mathbb{Z})$

I_n = ideal gen by products containing at least $n+1$ pairwise different h_s.

$L: K_*(I_n/I_{n+1}) \xrightarrow{\sim} K_*(I^{ab}_n/I^{ab}_{n+1})$ $\quad \forall n$

Baum's question about Bott class having finite support

Inland Revenue

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PL25 5YN

49.8g

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13 Oakthorpe Road

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Dear Mr QUILLEN

Thank you for contacting the Self Assessment Orderline.

I am pleased to enclose the following items of stationery.

Description	Ref.	Version	Quantity
Employment supplementary pages	SA101	9900	2
Non-residence supplementary pages	SA109	9900	1
Claim for Personal pension contributions	PP120	1999	1
Personal Pension Schemes	PP43	1999	1

This gets clearer. You have $\phi : F_i \longrightarrow F_j$ and $\phi^t : AF_j^\vee \longrightarrow AF_i^\vee$ you want

$$\begin{tikzcd}
F_i \otimes AF_j^\vee \arrow[r, "\phi\otimes 1"] \arrow[d, "1\otimes\phi^t"'] & F_j \otimes AF_j^\vee \arrow[d, phantom, "\cong"] \\
F_i \otimes AF_i^\vee \arrow[d, phantom, "\cong"] & M_{n_j} A \\
M_{n_i} A &
\end{tikzcd}$$

Yours sincerely

C. C. Villa
ORDERLINE MANAGER

$$K_*(B) = K_*(P \otimes_A Q) = \varinjlim K_*(F_j \otimes_A Q) \qquad F_j \simeq \tilde{A}^{n_j}$$

$$Q \otimes_{\mathbb{Z}} P \xrightarrow{\langle\, ,\, \rangle} A$$

$$Q \longrightarrow \mathrm{Hom}_{A^{op}}(P, A) \longrightarrow \mathrm{Hom}_{A^{op}}(F_j, A) = AF_j^\vee$$

$$\begin{tikzcd}
K_*(F_j \otimes_A Q) \arrow[d] & & \\
K_*(F_j \otimes_A A\check{F}_j) \arrow[r, phantom, "\cong"] & K_*(M_{n_j}(A)) \arrow[r] & K_*(A)
\end{tikzcd}$$

$$F_i \xrightarrow{\phi} F_j \xrightarrow{\psi} P$$

Lazard thm.

a flat module is a filtered colimit of f.g. free modules.

$$\begin{tikzcd}
F_i \otimes_A Q \arrow[r, "\phi\otimes 1"] \arrow[d, "1\otimes\psi^t"] & F_j \otimes_A Q \arrow[d, "1\otimes\psi^t"] \\
F_i \otimes A\check{F}_j \arrow[r, "\phi\otimes 1"] \arrow[d, "1\otimes\phi^t"] & F_j \otimes_A A\check{F}_j \arrow[d, phantom, "="] \\
F_i \otimes A\check{F}_i \arrow[d, phantom, "="] & M_{n_j}(A) \\
M_{n_i}(A) &
\end{tikzcd}$$

$$F \longrightarrow P$$

$$F_i \xrightarrow{\phi} F_j$$

$$F_i \xrightarrow{\Gamma_\phi} F_i \oplus F_j \xrightarrow{pr_2} F_j$$

$$F_i \xrightarrow{\begin{pmatrix} 1 \\ 0 \end{pmatrix}} \begin{matrix} F_i \\ \oplus \\ F_j \end{matrix} \xrightarrow{\begin{pmatrix} 1 & 0 \\ \phi & 1 \end{pmatrix}} \begin{matrix} F_i \\ \oplus \\ F_j \end{matrix} \xrightarrow{(0\ 1)} F_j$$

$$\begin{pmatrix} 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \phi & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ \phi \end{pmatrix}$$

$$= \begin{pmatrix} \phi & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \phi$$

Information required if you claim split-year treatment

- Date of your arrival in the UK — Day / Month / Year **9.25** ___ / ___ / ___
- Date of your departure from the UK — Day / Month / Year **9.26** ___ / ___ / ___

Information required if you claim to be not domiciled in the UK

- Have you submitted full facts to the Inland Revenue (for example, on forms DOM1 or P86) regarding your domicile in the six years ended 5 April 2000? Yes **9.27** ☐ No **9.28** ☑
- If you came to the UK before 6 April 1999, has there been a relevant change in your circumstances or intentions during the year ended 5 April 2000? Yes **9.29** ☐ No **9.30** ☑ Not applicable **9.31** ☐

Information required if you are resident in the UK and you also claim to be resident in another country for the purposes of a Double Taxation Agreement

- In which country as well as the UK were you regarded as resident for 1999-2000? **9.32** ______
- Were you also regarded as resident in the country in box 9.32 for 1998-99? Yes **9.33** ☐ No **9.34** ☐

Information required if you are not resident or are resident in another country for the purpose of a Double Taxation Agreement and are claiming relief under a Double Taxation Agreement

- Amount of any relief you are claiming from UK tax if you are not resident in the UK or are dual resident **9.35** £ ______

You must fill in and send me the claim form in ***Help Sheet IR302: Dual residents*** **or** ***Help Sheet IR304: Non residents - relief under Double Taxation Agreements*** **as applicable. These are available from the Orderline.**

Additional information

$\mathrm{Firm}(A) = M$ $\qquad A \otimes_A M \xrightarrow[\mathrm{Mod}(O)]{\sim} M$

Thm 1 $\quad A = A^2 \qquad \mathcal{M}(A)$

$$\begin{tikzcd} \mathrm{Firm}(A) \arrow[r, phantom, "<"] \arrow[dr, squiggly] & \mathrm{Mod}(A) \arrow[d] \\ & \mathcal{M}(A) \end{tikzcd}$$

~~$\mathcal{M}(A)$~~

If $A = A^2$, then $\mathcal{M}(A) \simeq \mathrm{Firm}(A)$

A unital $\quad ea =$ [illegible] $ae = a$

$$M = eM + (1-e)M$$

$$\mathrm{Mod}(A) = \begin{pmatrix}\text{unitary} \\ A\text{-modules}\end{pmatrix} \times \mathrm{Mod}(O)$$

Now fill in any other supplementary Pages that apply to you. Otherwise, go back to page 2 in your Tax Return and finish filling it in.

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$B = P\otimes_A Q$ first part treats $\begin{pmatrix} A & Q \\ P & B \end{pmatrix}$

where A, B are right flat (also P, Q)

$$A \longmapsto A\otimes_A Q \qquad K_*(B) \longrightarrow K_*(A)$$

right A-flat right B-flat

$$P \longmapsto P\otimes_A Q = B$$

$$Q\otimes_{\mathbb{Z}} P \longrightarrow A \qquad \langle aq, pa'\rangle = a\langle q,p\rangle a'$$

$$q\otimes p \longmapsto \langle q,p\rangle \qquad B = P\otimes_A Q$$ ring of fin rk ops.

$$(p\otimes q)p' = p\langle q,p'\rangle$$

$$q'(p\otimes q) = \langle q',p\rangle q$$

$$(p\otimes q)(p'\otimes q') = p\langle q,p'\rangle\otimes q' = p\otimes\langle q,p'\rangle q'$$

$\mathrm{Mod}(A)$ = abel. an cat of all left A-modules

$$\mathrm{Mod}(A) \underset{Q\otimes_B -}{\overset{P\otimes_A -}{\rightleftarrows}} \mathrm{Mod}(B)$$

$$\mathcal{M}(A) = \mathrm{Mod}(A)\big/\{M\in\mathrm{Mod}(A) \mid \exists N \ A^N M = 0\}$$

A idempotent $A = A^2$

$$\mathcal{M}(A) = \mathrm{Mod}(A)\big/\ \mathrm{Mod}(0)$$

$$\begin{tikzcd} & M/AM & \\ \langle q,p\rangle m \in & M \arrow[u] \arrow[r, mapsto] & P\otimes_A M \arrow[dl] \\ q\otimes p\otimes m \in \arrow[u, mapsto] & Q\otimes_B P\otimes_A M \arrow[u] & \end{tikzcd}$$

Assume $A = \langle Q, P\rangle$

~~$Q\otimes_{\mathbb{Z}}$~~

$$\sum q_i\otimes p_i\otimes m_i$$

h unital rings, h-unitary modules.

$$\xrightarrow{b'} A\otimes A\otimes M \xrightarrow{b'} A\otimes M \xrightarrow{b'} M \longrightarrow 0$$

$0 = \mathrm{Tor}_n^{\tilde{A}}(\mathbb{Z}, M)$ bar homology

$\mathrm{Tor}_0^{\tilde{A}}(\mathbb{Z}, M) = M/AM$

Tor_1

M h-unitary ~~means~~ defined by $\mathrm{Tor}_n^{\tilde{A}}(\mathbb{Z}, M) = 0$ $n \geq 0$.

equiv to $\exists$

$$\to F_2 \to F_1 \to F_0 \to M \to 0$$

where F_i flat $\ni$ $AF_i = F_i$.

2nd part of proof. $B = P\otimes_A Q$ A is ~~rt~~ A^{op} flat $\Rightarrow Q = A\otimes_A Q$ is B^{op} flat

$$\begin{pmatrix} A & Q \\ P & B \end{pmatrix}$$

rt flat $P_0 \twoheadrightarrow P$

$B_0 = P_0 \otimes_A Q \twoheadrightarrow P\otimes_A Q = B$

right flat

The first point is to note that B h-unitary over B^{op} $\Leftrightarrow$ P is h-unitary over A^{op}

B h-unitary over B^{op} $\Rightarrow \exists$ form flat res. $\cdots \to E_1 \to E_0 \to B \to 0$

$\Rightarrow \to Q\otimes_B E_1 \to Q\otimes_B E_0 \longrightarrow Q \longrightarrow 0$

---

m.eq for right flat idem rings.

B

O.K. to get the final steps.

M an A-module (A unital) is unitary when $1m = m$

h-unitary

$$\mathrm{Tor}_n^{\mathbb{Z}\times A}(\mathbb{Z}, M) = 0 \qquad \forall n$$

$$\xrightarrow{b'} A\otimes A\otimes M \xrightarrow{b'} A\otimes M \xrightarrow{b'} M$$

$$A \overset{\mathrm{II}}{\underset{A}{\otimes}} M \xrightarrow{\sim} M$$

$$\mathrm{Tor}_0^{\mathbb{Z}\times A}(\mathbb{Z}, M) = M/AM$$

$$\mathrm{Tor}_1^{\mathbb{Z}\times A}(\mathbb{Z}, M) = \mathrm{Ker}\{A\otimes_A M \to M\}$$

Prop. M h unitary over A $\Longleftrightarrow$ $\exists$ f.f. res.

$$\longrightarrow F_2 \longrightarrow F_1 \longrightarrow F_0 \longrightarrow M \longrightarrow 0$$

F_n flat firm ($AF_n = F_n$)

A h unital resp iff h unitary as left (resp. rt) mod.

$$\boxed{F_0} \longrightarrow A$$

flat

$$F_0 \xrightarrow{p} A \qquad \xi f' = p(\xi) f'$$

$$(\overset{Q}{A}, \overset{Q\times P}{F_0}, F_0\times A \to A) \quad (A, A, \mu)$$

$$(\xi, a) \mapsto p(\xi)a$$

$$B_0 = A\otimes_A F_0 = F_0$$

OP semi simp $\quad$ P = semi-simp A-mod corres to F_0

$Q_2 \qquad Q_1 \qquad Q_0$

Q

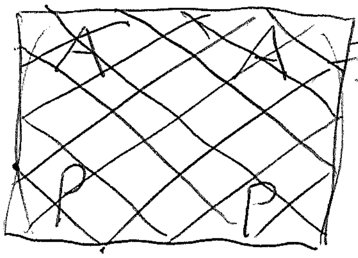
Idea to get across.

A given ~~[illegible]~~ idempotent, choose surj

$P \xrightarrow{f} A$ of A^{op}-modules with P A^{op}-flat firm

dual pair $\left(P, A, \begin{array}{ccc} A \times P & \longrightarrow & A \\ (a, p) & \longmapsto & af(p) \end{array}\right)$

$P \otimes_A A = P$ is a ring: $p_1 p_2 = p_1 f(p_2)$



~~[illegible]~~

$$\underbrace{K_* P}_{\text{ind of choice of } P} \longrightarrow K_* A$$

$Z = \mathbb{Z}[Z_1, Z_2, \cdots]$ $\quad \mu(Z_n) = \sum_{i+j=n} Z_i \times Z_j \quad Z_0 = 1$

called Leibniz-Hopf algebra

$\mathfrak{M} = \mathfrak{z}^*$ (graded dual) $\quad$ conj 1972 $\mathfrak{M}$ free poly (over $\mathbb{Z}$)

$$\begin{tikzcd} & \mathbb{Z}[a_1, a_2, \cdots] \arrow[d, hook] & \text{Symm} \\ \mathfrak{z} = \mathbb{Z}\langle Z_1, Z_2, \cdots\rangle \arrow[r, dotted, dash, "\text{duality}"] \arrow[d] & \mathfrak{M} = \text{Quasi-Symm} & \\ \mathbb{Z}[Z_1, Z_2, \cdots] & & \end{tikzcd}$$

Witt vector alg

$Z_n \nearrow z_n = [1 \ldots 1]$

$\mu [a_1, \cdots, a_n]$

$= \sum_i [a_1, \cdots, a_i] \otimes [a_{i+1}, \cdots, a_n]$

Topics + Objects.

ring A, dual pair $(P, Q, \langle\, ,\rangle : Q \otimes_{\mathbb{Z}} P \to A)$

ring finite rank of $P \otimes_A Q = B$

Morita context $\begin{pmatrix} A & Q \\ P & B \end{pmatrix}$, data needed for an equiv. of module categories.

$\mathrm{Mod}(A)$ = abelian cat of left A-modules, A^{op}

$$\begin{tikzcd} \mathrm{Mod}(A) \arrow[r, shift left, "P\otimes_A -"] & \mathrm{Mod}(B) \arrow[l, shift left, "Q\otimes_B -"] \end{tikzcd}$$

usual Morita thm. unitary modules $1m = m \quad \forall m$

to restrict idempotent rings $A = A^2$

$\mathcal{M}(A) = \mathrm{Mod}(A) / \mathrm{Mod}(0)$ — abelian groups with $am = 0 \quad \forall a, \forall m$

$0 \to M' \to M \to M'' \to 0$

ex. A unital $ea = ae = a$ $M \in \mathrm{Mod}(A)$ $M = eM + (1-e)M$

$\mathrm{Mod}(A) = \boxed{\text{unitary modules}} \times \mathrm{Mod}(0)$

: $\mathrm{Firm}(A) = \{ A\text{-modules } M \mid A \otimes_A M \xrightarrow{\sim} M \}$.

Prop. $\mathrm{Firm}(A) \simeq \mathcal{M}(A)$ $M \longmapsto A \otimes_A A \otimes_A M$

firm dual pair describe all firm rings m.eq. to A.

h-unitary^{A} module M : $\mathrm{Tor}_n^{\mathbb{Z}\times A}(\mathbb{Z}, M) = 0 \quad \forall n.$

$$0 \to A \to \mathbb{Z}\times A \to \mathbb{Z} \to 0$$

$$0 \to \mathrm{Tor}_1 \to A\otimes_A M \to M \to \overset{\mathrm{Tor}_0}{M/AM} \to 0$$

L. $M = AM \implies \exists^{\text{firm}}$ flat $F \twoheadrightarrow M \qquad AF=F$

Prop M h-unital $\iff \exists \quad \cdots \to F_1 \to F_0 \to M \to 0$

$C(\mathbb{R}) \ltimes \mathbb{Z}$, first you need $C(\mathbb{Z}) \ltimes \mathbb{Z}$

$= \left(\bigoplus_m \mathbb{C}e_m\right) \otimes \mathbb{C}[u,u^{-1}] = \sum_{m,n} \mathbb{C}\, e_m u^n$ OK

Ring of operators on $\mathbb{C}[u,u^{-1}] = \bigoplus_{n\in\mathbb{Z}} \mathbb{C}u^n$

where $e_m u^n = u^n e_{m-n}$

$$e_m u^n \sum_k a_k z^k = e_m \sum a_k z^{n+k} = a_{m-n} z^m$$

$$u^n e_{m-n} \sum_k a_k z^k = u^n \left(a_{m-n} z^{m-n}\right) = a_{m-n} z^m$$

$$u^n e_m = e_{m+n} u^n$$

Viewpoint: You consider the ring $\bigoplus_{n\in\mathbb{Z}} \mathbb{C}e_n \rtimes \mathbb{Z}$ where the $e_n e_{n'} = \delta_{nn'} e_n$ and u acts as $u e_n u^{-1} = e_{n+1}$. Is this ring isomorphic to double infinite matrices with finite support? Certainly it looks like this ring arises from a dual pair namely $Q = \mathbb{C}[\mathbb{Z}] = \mathbb{C}[u,u^{-1}]$ and $P = \bigoplus \mathbb{C}e_n$ with pairing $\langle u^k, e_\ell \rangle = \delta_{k\ell}$

Multi Trip Travel Protector Endorsement

Conditions applying to NatWest Travel Protector

1. Any dispute over the interpretation of this Endorsement, or its rights or obligations, shall be referred to an Arbitrator as allowed under current legislation.
2. Where any costs or expenses we pay to you, on your behalf, are not insured under this Endorsement, you must reimburse us within 28 days of our request to you to do so.
3. We will make every effort to apply the full range of services in all circumstances dictated by the terms and conditions. Remote geographical locations may preclude the normal standard of services being provided, but in all such cases, the full monetary benefits under your insurance cover will continue to apply.
4. Anything mentioned in the Conditions shown in your Travel Protector Policy Wording, on Page 16.

So you've described your ring $A \rtimes \mathbb{Z}$, but it's not clear yet why this ring is finite supp. matrices. Something suggestive is the resemblance of $A \rtimes \mathbb{Z}$ to a $B = P \otimes Q$ where $P = A$ is dual to $Q = \mathbb{C}[\mathbb{Z}$

Is this true?

$$\text{Take then } P = \overset{\text{compsupp function on } \mathbb{Z}}{\bigoplus_m \mathbb{C} e_m} \otimes \underbrace{\bigoplus_n \mathbb{C} u^n}_{\text{vector space gen. by } \mathbb{Z}}$$

use pairing $\langle u^n, e_m \rangle = \delta_{mn}$ $\quad \langle u^n, f \rangle = f(n)$.

$$(f u^n)(g u^m) = f \langle u^n, g \rangle u^m \qquad \text{NO}$$

Go next to Weyl picture – functions on group is the basic repn. – this is A and you have the cross product acting $A \rtimes \mathbb{Z}$, A acts by mult, $\mathbb{Z}$ acts by translation

~~g acts on A~~ $\big((f u^m) g\big)(x) = f(x) g(x-m)$

$P = \bigoplus \mathbb{C}e_n$ $\qquad Q = ?$ Look at modules to get things straight. Basic picture is $\overbrace{\bigoplus_m \mathbb{C}e_m}^{P} \otimes \overset{Q}{\bigoplus_{n\in} \mathbb{C}u^n}$ acting on $\overbrace{\bigoplus \mathbb{C}e_m}^{P}$

$Q =$ group ring $\mathbb{C}[\mathbb{Z}]$ Hopf algebra

$P =$ "finite support" dual $=$ functions on the group fin. support

What's the basic representation? You're being stupid again. You should think of the ~~[illegible]~~ regular repn. with translation ops and mult by characters.

$P =$ space of functions comp. supp on $\mathbb{Z}$

$Q =$ group ring

$$A = \text{functions on } \mathbb{Z} = C(\mathbb{Z})$$

$$A \rtimes \Gamma = C(\mathbb{Z}) \tilde{\otimes} \mathbb{C}[\mathbb{Z}]$$

$A =$ functions $\overset{\text{comp supp}}{\text{on}}$ the coset $x + \mathbb{Z}$

$\mathbb{Z}$ acts on this so you can form the cross. product

$A \rtimes \mathbb{Z} = C(x+\mathbb{Z}) \otimes \bigoplus_{n\in\mathbb{Z}} \mathbb{C}u^n.$ Take basis

$e_{x+m}u^n$, product $e_{x+m}u^n e_{x+m'}u^{m'} =$

$$e_{x+m}\delta^{x+m}_{x+m'}$$

let $x=0$.

$$\boxed{e_m u^n e_{m'} u^{m'} = e_m e_{m'+n} u^{n+n'}}$$

$$(f u^n)(g u^{n'}) = f g^{\sigma^n} u^{n+n'}$$

←

Try again to make clear the Morita equivalence of $B = A \ltimes \mathbb{Z}$ $\quad A = C(\mathbb{Z})$ with $\mathbb{C}$. $A = \{\text{functions fin. supp on } \mathbb{Z}\}$ $= \bigoplus_{m\in\mathbb{Z}} \mathbb{C}e_m$. A firm A-module M is a graded module wrt $\mathbb{Z}$: $\quad M = \bigoplus_{m\in\mathbb{Z}} M_m, \; M_m = e_m M$.

Perhaps you should look at B acting on A.

Idea behind the M.eq. is: a firm B-module M is the same as a $\mathbb{Z}$ graded B-module: $M = \bigoplus_{m\in\mathbb{Z}} M_m$. $M_m = e_m M$ and u^n gives $M_0 \xrightarrow{\sim} M_n$. So

$M \mapsto e_0 M = e_0 B \otimes_B M$

$$\begin{pmatrix} \mathbb{C} & e_0 B = Q \\ Be_0 = P & B \end{pmatrix}$$

$P = Be_0$ contains $u^m e_0 = e_m u^m$ should form basis

$Q = e_0 B = \bigoplus_n \mathbb{C} e_0 u^n \qquad \langle e_0 u^n, e_m u^m \rangle =$

$= e_0 u^n e_m u^m = e_0 e_{n+m} u^{n+m} = e_0 \delta_{n+m}$.

$u^m e_0 e_0 u^n = e_m u^{m+n}$

Try for clarity $\quad A_x = \mathbb{C}(x+\mathbb{Z}) = \bigoplus_{m\in\mathbb{Z}} \mathbb{C}e_{x+m}$

$A_x \rtimes \mathbb{Z} = \bigoplus_m \mathbb{C}e_{x+m} \otimes \bigoplus_{n\in\mathbb{Z}} \mathbb{C}u^n$. This doesn't help. Try instead to understand why $C(\mathbb{R}) \rtimes \mathbb{Z}$ m.eq. $C(\mathbb{R}/\mathbb{Z})$. partition of unity.

Go back to $\pi : \mathbb{R} \longrightarrow \mathbb{R}/\mathbb{Z}$ fibres $x+\mathbb{Z}$

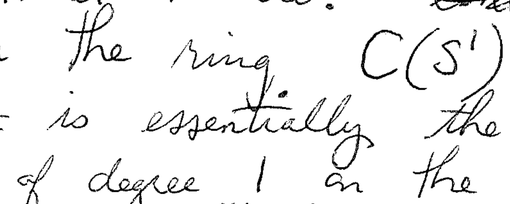
get fibre bundle with fibre

$C_c(x+\mathbb{Z}) = \bigoplus_{m\in\mathbb{Z}} \mathbb{C} c_{x+m}$, considered as a free rank 1 module over $\mathbb{C}[\mathbb{Z}] = \bigoplus_n \mathbb{C}u^n$.

Take constant sheaf $\mathbb{Z}$ on $\mathbb{R}$, apply $\pi_!$ to get ~~what you need~~. a locally constant fibre bundle over the circle - fibre is the group ring $\mathbb{C}[\mathbb{Z}]$ ~~considered as free rank 1 module~~ with $\mathbb{Z}$ acting by multiplication. (You have $\mathbb{Z} \to \mathbb{R} \longrightarrow \mathbb{R}/\mathbb{Z}$. principal $\mathbb{Z}$-bundle, $\mathbb{Z}$ acts on the v.s. $\mathbb{C}[\mathbb{Z}]$, get associated bundle over $\mathbb{R}/\mathbb{Z}$.)

So, simply, you have a line bundle for $\overset{\text{the ring}}{\mathbb{C}[u,u^{-1}]}$ over the circle. What to do? Space of sections is a module over the ring $C(S^1)\otimes\mathbb{C}[u,u^{-1}]$, this space ~~represents the~~ is essentially the space of sections of the line bundle of degree 1 on the torus. At this point you want to get Poisson summation into shape.

What to do? Look at $C_c(\mathbb{R})$ with mult. by $C(\mathbb{R}/\mathbb{Z})$ and translation action by $\mathbb{C}[u,u^{-1}]$ maybe $C_c(\mathbb{Z})$. You recall showing how $E = C_c(\mathbb{R})$ is a finite projective module over $C(\mathbb{R}/\mathbb{Z})[u,u^{-1}]$, using something like $|\sin(\pi x)|$. **Let's try to recall** the situation. One point is that you have $|\sin(\pi x)|^2 + |\cos(\pi x)|^2 = 1$ on the line.



$$\sin(\pi x)^2 = \frac{1-\cos 2\pi x}{2}$$

$$\cos(\pi x)^2 = \frac{1+\cos 2\pi x}{2}$$

Point: $|\sin(\pi x)|$, $|\cos(\pi x)|$ have period 1, unlike $\sin(\pi x)$, $\cos(\pi x)$

Point: multiplying by $\sin(\pi x)$ sends $C_c(\mathbb{R})$ to $C(0,1)[u,u^{-1}]$ allowing you to ~~unwrap~~ map the module $E = C_c(\mathbb{R})$ over $C(\mathbb{R}/\mathbb{Z})[u,u^{-1}] = R$ into R. Get supply of R-linear maps (actual two) $E \longrightarrow R$ which should show E is a fin. gen proj R-module. But instead of R consider $A \rtimes \mathbb{Z} = C_c(\mathbb{R})[u,u^{-1}]$ where this is a twisted Laurent series ring. It seems that $E = A$ considered as $A \rtimes \mathbb{Z}$-module is a direct summand

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$$\frac{\text{locales}}{\text{quantales}} = \frac{\text{topoi}}{?}$$

NC topology

1) Comm. topology (on NC algs)

$$\begin{tikzcd}
 & & & \text{Spec} \\
 & \mathcal{C} \arrow[r, "\text{Rep}"'] & \mathcal{G}\ \left(\text{cat of Groth cats} + \text{exact functors}\right) \arrow[r, "\text{Top}"] & \mathcal{G}\ \text{tors. ths.} \arrow[u, "\text{take irred elts for } \wedge"'] \\
 & a & G_a & G_a\ \text{torsion ths. lattice} \\
\text{Rings} & R \arrow[r] & R\text{-mod} \arrow[r] & R\text{-tors}
\end{tikzcd}$$

basic order top.
Stone top.

Prop. R noeth, then $\wedge$ irred $\Longleftrightarrow$ prime torsion ths. (Popescu)
Every R-tors. is semi prime

Other example

$$\begin{tikzcd}
R\ \text{graded} \arrow[r] & \text{Rigid Q.Th.} \arrow[r] & \text{Spec}^{g} R \\
\text{pos. gr} \\ R_0\ \text{comm} & \{\sigma > K_+\} & \text{Proj}\ R
\end{tikzcd}$$

Abstract. To any ~~idempotent~~ ring A satisfying $A = A^2$ one may associate an abelian cat $\mathcal{M}(A)$ consisting of all firm A-modules M, ~~which~~ where firm means the mult. map $A \otimes_A M \xrightarrow{\sim} M$ is an isom. Suslin result excision holds, corresp to an equivalence between ~~It~~. is natural to ask whether to an corresponding to an equivalence $\mathcal{M}(A) \simeq \mathcal{M}(B)$

Given two idempotent rings A, B with ~~the~~ equivalent firm module categories ~~it~~ is natural to ask whether they have the same higher alg. K-theory. ~~This is false in general,~~ ~~but~~ Using Suslin's excision results on higher alg. K theory for h-unital rings

In general this is false, but it can be shown to hold for h-unital rings using Suslin's excision results on higher alg. K-theory.

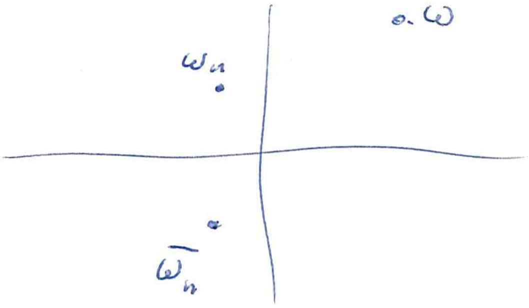
Let's ~~take~~ try for a continuous version

Question: Do there exist de Branges functions with infinitely many zeroes? Example:

$$E(\omega) = \prod_{n=1}^{\infty}\left(1-\frac{\omega}{\omega_n}\right) \qquad \text{where} \quad \operatorname{Im}(\omega_n) > 0.$$

and this converges i.e. $\sum_{n=1}^{\infty} \frac{1}{|\omega_n|} < \infty$

$$\left|\frac{E(\omega)}{E(\bar{\omega})}\right| = \prod_{n=1}^{\infty} \overbrace{\left|\frac{1-\frac{\omega}{\omega_n}}{1-\frac{\omega}{\bar{\omega}_n}}\right|}^{} \qquad \boxed{\frac{|\omega-\omega_n|}{|\omega-\bar{\omega}_n|}} < 1$$



Answer Yes. Next: Does ∃ de Branges function with zeroes $i, 2i, 3i, \ldots$

Another question: $\prod \frac{\omega-\omega_n}{\omega-\bar{\omega}_n}$ how to make sense of such a product Blashke products. ~~[illegible]~~

How to make sense. Note that if we can make sense of this thing, then we can make sense of Re log

$$\frac{d}{dz}\log \overset{E(z)}{\cancel{\prod\left(1-\frac{z}{\alpha_n}\right)}} = \sum_n \frac{1}{1-\frac{z}{\alpha_n}}\left(-\frac{1}{\alpha_n}\right) = \sum \frac{1}{z-\alpha_n}$$

$$-\frac{d^2}{dz^2}\log(E(z)) = \sum_n \frac{1}{(z-\alpha_n)^2} \qquad \text{convergent say.}$$

You Can you integrate once, this amounts to forming $\sum'\left(\frac{1}{z-\alpha_n}+\frac{1}{\alpha_n}\right)$, can certainly integrate again and exponentiate. Does this yield ~~[illegible]~~ Γ.

Back to mathematics. Alg version of Cuntz's C^* Morita equivalences, where to begin? what is the aim, goal? to understand Ranicki's proof that Wall invariant of a compact ANR vanishes, equivalently, any such space is homotopy equiv to a finite CW complex. You have Groth + Verd. duality approach to this.

Immediate problem is to recall assembly map, partitions of unity, etc.

Given Γ a discrete group. The basic idea, called assembly, takes a principal Γ bundle $Y \xrightarrow{\pi} X$ into the associated fibre bundle E over X with fibre $\mathbb{C}[\Gamma]$. Assume there is given a finite partition of unity $\sum h_i = 1$ on X and trivialization of Y over the supports of the partitions. Then one gets from this data an isom. of E with a retract of the "trivial bundle" $X \times \mathbb{C}[\Gamma]^N$ where N is the number of h_i. Leave this idea.

Start with a retract of a free Γ-module

$$\begin{tikzcd}
E \arrow[r, "\alpha"] \arrow[dr, "j_s"'] & \Lambda \otimes V \arrow[r, "\beta"] \arrow[d, shift right, "\eta_s"'] & E \\
& V \arrow[u, shift right, "\varepsilon_s = s\varepsilon_1"'] \arrow[ur, "\iota_s"'] &
\end{tikzcd}$$

$\Lambda = \mathbb{C}\Gamma$

α, β are Γ maps st $\beta\alpha = 1_E$

$\iota_s = \beta(s\varepsilon_1) = s\beta\varepsilon_1 = s\iota_1$

$j_s = \eta_1 s^{-1}\alpha = j_1 s^{-1}$

$$E \xrightarrow{\alpha} \Lambda \otimes V \xrightarrow{\beta} E$$

$$\alpha\xi = \sum_s s \otimes j_1 s^{-1}\xi \qquad \beta\alpha\xi = \sum_s s \overset{h_1}{\boxed{\iota_1 j_1}} s^{-1}\xi$$

$$\xi = \sum_s s h_1 s^{-1}\xi$$

$$\beta \sum_t t \otimes f(t) = \sum_t t \iota_1 f(t)$$

$$\alpha\beta \sum_t t \otimes f(t) = \sum_s s \otimes \underbrace{\boxed{j_1 s^{-1} t \iota_1}}_{p(s^{-1}t)} f(t)$$

$$\sum p(s^{-1}t) f(t)$$

$$Y = \sum z^{-n} y(n)$$

$$\frac{dy}{dt} + ay = bx$$

$$\sum z^{-n} y(n-1) = \sum z^{-n-1} y(n) = z^{-1} Y$$

$$y(t) = \int_{t_0}^{t} y'(t)\,dt + y(t_0)$$

$$y(nT) = y(nT-T) + \frac{T}{2}\left(y'(nT) + y'(nT-T)\right)$$

$$y(n) = y(n-1) + \frac{T}{2}\left(\begin{array}{l} -ay(n) + bx(n) \\ -ay(n-1) + bx(n-1) \end{array}\right)$$

~~$Y = z^{-1}Y + \frac{T}{2}(\quad -a(1+z^{-1}) + b(1+z^{-1}))$~~

~~$Y = z^{-1}Y + \frac{T}{2}(-a)(1+z^{-1})Y + \frac{T}{2}($~~

$$Y = z^{-1}Y + \frac{T}{2}\left(\begin{array}{l} -aY + bX \\ -az^{-1}Y + bz^{-1}X \end{array}\right)$$

$$Y - z^{-1}Y + \frac{T}{2}a(1+z^{-1})Y = \frac{T}{2}b(1+z^{-1})X$$

$$\frac{1-z^{-1}}{1+z^{-1}}Y + \frac{T}{2}aY = \frac{T}{2}bX$$

$$\frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}Y + aY = bX$$

$$Y = \left(a + \frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}\right)^{-1} bX = (a+s)^{-1} bX$$

$$s = \frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}$$

phase space $\mathbb{R}^2$ equipped with $p\,dq$ contact form.
phase space is the affine line $\mathbb{R}$, which is a torsor for $(\mathbb{R}, +)$, where the coordinate is something like δq. What would you like to do?
Define an object with appropriate symmetries. Config space is tangent bundle to affine line, coords $q, \dot{q}$

$$q_n = \bar{h}_n p_n + k_n q_{n-1}$$

$$= \bar{h}_n p_n + k_n \{ \bar{h}_{n-1} p_{n-1} + k_n k_{n-1} q_{n-2} \}$$

$$= \bar{h}_n p_n + k_n \bar{h}_{n-1} p_{n-1} + k_n k_{n-1} \{ \bar{h}_{n-2} p_{n-2} +_n \overset{k_n k_{n-1}}{k_{n-2}} q_{n-3} \}$$

$$q_n = \bar{h}_n p_n + k_n h_{n-1} p_{n-1} + \cdots + k_n \cdots k_{m+1} \bar{h}_{m+1} p_{m+1} + k_n \cdots k_{m+1} q_m.$$

orth set

discrete 1-diml DE $\qquad \psi(n,z) = \begin{pmatrix} p_n(z) \\ q_n(z) \end{pmatrix}$

$$\begin{pmatrix} p_n \\ q_n \end{pmatrix} = \frac{1}{k_n} \begin{pmatrix} 1 & h_n \\ \bar h_n & 1 \end{pmatrix} \begin{pmatrix} z p_{n-1} \\ q_{n-1} \end{pmatrix} \qquad n \in \mathbb{Z}$$

where $|h_n| < 1$ and $k_n = \sqrt{1-|h_n|^2}$

Ex (cont limit) $\qquad x = n\varepsilon \qquad z^\varepsilon = e^{2\lambda\varepsilon}$

$$\psi_x = \begin{pmatrix} p_x \\ q_x \end{pmatrix} = \frac{1}{\sqrt{1-|h_x\varepsilon|^2}} \begin{pmatrix} 1 & h_x\varepsilon \\ \bar h_x\varepsilon & 1 \end{pmatrix} \begin{pmatrix} e^{2\lambda\varepsilon} & 0 \\ 0 & 1 \end{pmatrix} \psi_{x-\varepsilon}$$

$$\partial_x \psi_x = \begin{pmatrix} 2\lambda & h_x \\ \bar h_x & 0 \end{pmatrix} \psi_x \qquad \tilde\psi_x = e^{-\lambda x} \psi_x$$

$$\begin{pmatrix} \partial_x & 0 \\ 0 & -\partial_x \end{pmatrix} \tilde\psi_x = \begin{pmatrix} \lambda & h_x \\ -\bar h_x & +\lambda \end{pmatrix} \tilde\psi_x$$

$$\boxed{\begin{pmatrix} \partial_x & -h_x \\ \bar h_x & -\partial_x \end{pmatrix} \tilde\psi_x = \lambda \tilde\psi_x} \qquad \tilde\psi_x = \begin{pmatrix} u \\ v \end{pmatrix}$$

$h = \bar h$

$$\begin{cases} (\partial_x - \lambda) u = h v \\ (\partial_x + \lambda) v = +h u \end{cases}$$

$$\begin{aligned} \partial u - h v &= \lambda u \\ -\partial v + h u &= \lambda v \end{aligned}$$

$$\begin{aligned} (\partial + h)(u - v) &= \lambda (u + v) \\ (\partial - h)(u + v) &= \lambda (u - v) \end{aligned}$$

$(\partial_x^2 - \lambda^2) u = (\partial_x + \lambda) h v$ ~~$= h'v + h(\partial_x+\lambda)v$~~

~~$= h'v + hv' + \lambda h v$~~

$(\partial_x + \lambda)(\partial_x - \lambda) u = (\partial_x + \lambda) h v$ ~~$= h'v + hv' + \lambda h v$~~

~~$\begin{pmatrix}\partial_x & -h\\ \bar{h} & -\partial_x\end{pmatrix}\begin{pmatrix}u\\ v\end{pmatrix} = \lambda\begin{pmatrix}u\\ v\end{pmatrix}$~~ ~~$\begin{pmatrix}u\\ v\end{pmatrix} = e^{-\lambda x}\begin{pmatrix}p\\ q\end{pmatrix}$~~

---

lst $\quad \partial_x\psi = \begin{pmatrix}2\lambda & h\\ \bar{h} & 0\end{pmatrix}\psi \qquad \begin{pmatrix}u\\ v\end{pmatrix} = e^{-\lambda x}\psi$

$$\begin{pmatrix}\partial_x & -h\\ +\bar{h} & -\partial_x\end{pmatrix}\begin{pmatrix}u\\ v\end{pmatrix} = \begin{pmatrix}\lambda & 0\\ 0 & +\lambda\end{pmatrix}\begin{pmatrix}u\\ v\end{pmatrix}$$

$$\begin{pmatrix}\partial_x & -h\\ \bar{h} & -\partial_x\end{pmatrix}\begin{pmatrix}u\\ v\end{pmatrix} = \lambda\begin{pmatrix}u\\ v\end{pmatrix}$$

eigenvector eqn for a skewadjoint op.

If $\bar{h} = h$, then

$$(\partial + h)(u - v) = \lambda(u + v)$$
$$(\partial - h)(u + v) = \lambda(u - v)$$

$$\underbrace{(\partial + h)(\partial - h)}\,(u + v) = \lambda^2(u + v)$$

~~[illegible]~~

$$\boxed{\left[-\partial^2 + (h' + h^2)\right](u + v) = (-\lambda^2)(u + v)}$$

non commutative Weil algebra (Meinrenken, Alekseev

Duflo isomorphism: G Lie gp $\mathfrak{g}$, e_a basis

$[e_a, e_b] = f_{ab}^{c}\, e_c$

$\mathcal{E}'(\mathfrak{g}) \supset S\mathfrak{g}$ symm generators v_a exp.: $\mathcal{E}'(\mathfrak{g}) \longrightarrow \mathcal{E}'(G)$

$\mathcal{E}(G) \supset U\mathfrak{g}$ UEA " u_a $S(\mathfrak{g})^G \xrightarrow{\sim} U(\mathfrak{g})^G$

map not a ring isom

Duflo ~~abstracts~~ modifies

$\mu \in \mathfrak{g}$ $J(\mu) = \det\left(g(\mathrm{ad}_\mu)\right)$ $g(s) = \frac{1-e^{-s}}{s}$ Jacobian for the exp map

Thm (Duflo) The map $\exp_* J^{1/2}$ restricts to a ring isom for the invariant part.

Weil model for equivariant DR theory

$W_G = S\mathfrak{g}^* \otimes \Lambda \mathfrak{g}^*$, has operators

L_a, ι_a v^a gen for $S(\mathfrak{g}^*)$ $\iota_a y^b = \delta_a^b$

y^a —— $\Lambda(\mathfrak{g}^*)$ $\iota_a v^b = 0$

$dy^a = u^a - \frac{1}{2} f^a_{bc}\, y^b y^c$

Morally $W_G = \Omega(BG)$ $H_G(M) = H(W_G \otimes \Omega(M))_{basic}$

$H_G(pt) = S(\mathfrak{g}^*)^G$

$\hat{W}_G = \mathcal{E}'(\mathfrak{g}^*) \otimes \Lambda(\mathfrak{g}^*)$ $\hat{H}_G(M) = H\left((\hat{W}_G \otimes \Omega(M))_{basic}\right)$

surface waves in shallow

for CH waves both direction + break

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} + \frac{\partial p}{\partial x} = 0$$

$$p = \frac{1}{2}\int e^{-|x-y|}(v^2 + v'^2) \quad = (1-D^3)^{-1}$$

Arnold $SD^0(\mathbb{R}^3)$ Euler $\Leftrightarrow$ geodesic flow

- GEODESIC FLOW in $D'(\mathbb{R}^1)$
- INTEGRABILITY Poisson bracket $[A,B] = \int \frac{\partial A}{\partial m}(mD - Dm)\frac{\partial B}{\partial m}$

$(1-D^2)v = v - v'' = m$

- SOLITONS

$$p e^{-|x-y|-pt}$$

$$\sum p_i e^{-|x-y_i|} \qquad n\text{-soliton}$$

- BREAKDOWN

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like KdV spectral problem assoc to Schroedinger

SPEC. PROBLEM $-f'' + \frac{1}{4}f = \lambda m f$

$\int |m| < \infty \Rightarrow$ purely discrete spectrum.

$$K = \frac{1}{\sqrt{-D^2 + \frac{1}{4}}} \qquad K m K\psi = \frac{1}{\lambda}\psi \qquad |KmK| \leq \int |m|$$

HAMILTONIAN PICTURE $H = \frac{1}{4}\sum \frac{1}{\lambda_k^2}$

$\mathring{m} = -(mD - Dm)v \quad \Longleftrightarrow \text{CH}$