~~[illegible]~~

Let's start at C's end: $\mathcal{E}_{\Sigma_F} \rtimes \mathbb{Z}$ with generators $h_0 \geq 0$, u unitary, form elements $h_n = u^n h_0 u^{-n}$ relations $h_0 u^n h_0 = 0$ $|n| \geq 2$ and the elusive relation, obscure $\left(\sum_n h_n\right) h_m = h_m$, which amounts to $(h_{-1} + h_0 + h_1) h_0 = h_0$. Then have $p = \sum_n \varepsilon u^n \varepsilon$ where $\varepsilon = h_0^{1/2}$. $p = \varepsilon u^{-1} \varepsilon + \varepsilon\varepsilon + \varepsilon u \varepsilon$ is a projector in $\mathcal{E}_{\Sigma_F} \rtimes \mathbb{Z}$. ~~You are amazed~~ You have now related the pos. def function on the group to this proj. p

Put $p = \underbrace{\varepsilon^* u^{-1} \varepsilon}_{p_{-1}} + \underbrace{\varepsilon\varepsilon}_{p_0} + \underbrace{\varepsilon u \varepsilon}_{p_1} \in B_{-1} + B_0 + B_1$ and ask for $p^2 = p$. One way to get positivity is to ~~ask~~. If $a = a^*$ then $a^2 = a \implies a \geq 0$.

$$p_{-1}^2 = 0 \;, \quad p_1^2 = 0$$

$$p_{-1}p_0 + p_0 p_{-1} = 0, \quad p_1 p_0 + p_0 p_1 = 0.$$

$$p_{-1}p_1 + p_1 p_{-1} + p_0^2 = p_0$$

Let's sort certain things out

Begin with Hilb space H, unitary u, herm. $\varepsilon \geq 0$.

~~assume~~ assume $\varepsilon u^n \varepsilon = 0$ for $|n| \geq 2$.

New idea is that an operator $h \geq 0$ is an ingredient in a noncomm. partition of unity. Given $a : X \to H$ from the viewpoint of H you have $(aa^*)^{1/2}$ on H which should have the same image. Now go back to GNS. You want H, u unitary, $\varepsilon \geq 0$, $\varepsilon u^n \varepsilon = 0$ $|n| \geq 2$. then you get pos. def. $n \longmapsto \varepsilon u^n \varepsilon$ on $\mathbb{Z}$.

$p_n = \varepsilon u^n \varepsilon$ Claim that $p = p_{-1} + p_0 + p_1$ is a projector $\iff$ $\sum (u^n \varepsilon u^{-n})^2 = 1$ on H.

~~$p_n = u^n \varepsilon u^{-n}$~~

Explicit formula $\quad p_n = \varepsilon u^n \varepsilon, \quad p = \sum_n \varepsilon u^n \varepsilon$

$$p^2 = \sum_{n,n'} \varepsilon u^n \varepsilon \, \varepsilon u^{n'} \varepsilon$$

Easy direction

$$p^2 = \sum_{n,n'} \varepsilon_0 \varepsilon_n^2 u^{n+n'} \varepsilon_0$$

$$= \sum_{n,n''} \varepsilon_0 \varepsilon_n^2 u^{n''} \varepsilon_0$$

$$= \sum_{n''} \left(\sum_n \varepsilon_0 \varepsilon_n^2 \right) u^{n''} \varepsilon_0$$

$$p = \sum_{n''} \varepsilon_0 u^{n''} \varepsilon_0$$

so $p^2 = p$ together with the grading yields $\sum_n \varepsilon_0 \varepsilon_n^2 = \varepsilon_0$,

no, not quite $\quad \left(\sum_n \varepsilon_0 \varepsilon_n^2 \right) u^m \varepsilon_0 = \varepsilon_0 u^m \varepsilon_0 \quad \forall m$

but then you win because $\overline{\sum u^m \varepsilon_0 H} = H$.

H, u unitary, $\varepsilon \geq 0$, $\quad \overline{\sum_n u^n \varepsilon H} = H$

$p_n = \varepsilon u^n \varepsilon \quad = 0$ for $|n|$ large.

$p(z) = \sum z^{-n} p_n \qquad p(z)^2 = \sum_m z^{-m} p_m \sum_n z^{-n} p_n =$

$$= \sum_k z^{-k} \sum_{m+n=k} p_m p_n \qquad \sum_m \varepsilon u^m \varepsilon^2 u^{k-m} \varepsilon =,$$

$$\sum_m \varepsilon u^m \varepsilon^2 u^{-m} u^k \varepsilon \qquad p(z)^2 = p(z) \iff$$

$$p(z)^2 = p(z) \iff \forall k \quad \sum_m \varepsilon\,(u^m \varepsilon u^{-m})^2\, u^k \varepsilon = \varepsilon u^k \varepsilon$$

$$\iff \sum_m \varepsilon (u^m \varepsilon u^{-m})^2 = \varepsilon$$

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again: H, u, ε | $\sum u^n \varepsilon H$ dense

$p_n = \varepsilon u^n \varepsilon = 0$ $|n|$ large

let $\varepsilon_n = u^n \varepsilon u^{-n}$ $\qquad \varepsilon_n \varepsilon_m = 0$ $|n-m|$ large

$$p(z) = \sum z^{-n} p_n = \sum z^{-n} \varepsilon u^n \varepsilon \qquad \text{herm. on } S^1.$$

$$p(z)^2 = \sum_k z^{-k} \sum_{m+n=k} p_m p_n$$

$$= \sum_k z^{-k} \underbrace{\sum_{m} \varepsilon_0 u^m \varepsilon_0^2 u^{k-m} \varepsilon_0}_{\displaystyle\sum_m \varepsilon_0 \varepsilon_m^2 u^k \varepsilon_0 \overset{?}{=} \varepsilon_0 u^k \varepsilon = p_k}$$

$p(z)^2 = p(z)$? You get exactly $\sum_m \varepsilon_0 \varepsilon_m^2 = \varepsilon_0$ since $\sum u^k \varepsilon_0 H$ dense in H.

~~So~~ what do you understand? You have this good case.

What do you learn? Given a Laurent poly loop in the Grassmannian of W, you get an interesting unitary operator

$H, u, \quad Y \hookrightarrow H, \quad X = u^{-1}Y \cap Y$

$X \underset{b}{\overset{a}{\rightrightarrows}} Y$ glue to get $\overline{Y + uY} = aY \oplus V_+$

$$\overline{u^{-1}Y + Y} = V_- \oplus bY$$

should be true that

$aX \oplus V_+ \simeq V_- \oplus bX$

Start with $X \subset H, \quad Y = X + uX$

Begin with W closed in $H \circlearrowright u$ $\qquad u^n W \perp W \quad |n| \geq 2.$

form $\overline{W + uW} = W \oplus V_+$

$$\overline{W+uW} = V_- \oplus uW$$

You think it's true that $\overline{u^{-1}W + W + uW} = u^{-1}V_- \oplus W \oplus V_+$
where in general $u^{-1}V_-$, V_+ need not be $\perp$.

$$u^{-1}V_- \oplus \begin{pmatrix} aX \overset{\perp}{\oplus} V_+ \\ V_- \overset{\perp}{\oplus} bX \end{pmatrix} \oplus uV_+ \qquad ?$$

Need review X. Need to think clearly.

When you dilate a partial unitary you do something special for the higher moments.

Dilating a contraction c on X means looking for j:
$j : X \hookrightarrow H \circlearrowright u$ such that $j^* u^n j = \begin{cases} c^n & n \geq 0 \\ (c^*)^{-n} & n \leq 0 \end{cases}$

so if you ~~ask for~~ take $c^2 = 0$, you get examples

This is exactly what one needs, what you missed. Let's check this. Suppose given $W \underset{j}{\subseteq} H \circlearrowleft u$ such that $j^* u^n j = 0$ for $|n| \geq 2$. First case to understand is where $j^* u j$, this is a contraction, satisfies $(j^* u j)^2 = 0$.

$$p(z) = z^{-1}c + 1 + zc^* \quad ?$$
$$p(z)^2 = 2z^{-1}c + 1 + cc^* + 2zc^*$$

Go back to ε_0, u $\qquad \varepsilon_0 u^n \varepsilon_0 = 0 \quad |n|$ large

$$p(z) = \sum z^{-n} p_n \qquad p_n = \varepsilon_0 u^n \varepsilon_0$$

$$p(z)^2 = \sum_n z^{-n} \sum_{j+k=n} p_j p_k = \sum_n z^{-n} \sum_j p_j p_{n-k}$$

$$\sum_j p_j p_{n-j} = \sum_j \varepsilon_0 \underbrace{u^j \varepsilon_0^2 u^{-j}}_{\varepsilon_j^2} u^n \varepsilon_0$$

so you see that $p(z)^2 = p(z)$ exactly when $\sum \varepsilon_0 \varepsilon_j^2 = \varepsilon_0$ on the Hilbert space. What's critical.

$$P_F \longrightarrow \mathcal{E}_{\Sigma_F} \rtimes \Gamma \qquad \text{is this an isom.}$$

$$\sum_u p_u = \sum_s p_s \sum_t p_t \qquad p_u = \sum_{u=st} p_s p_t$$

The situation becomes clearer now. If you take a hermitian operator valued function on the circle which is ≥ 0, then you get a Hilbert space, namely, first you have a ~~[illegible]~~ hermitian scalar product pointwise which can be integrated over the circle. Next requiring the hermitian operator to be ~~projector~~ idempotent guarantees positivity. In this case you have a hermitian vector bundle over the circle given by the image of the projector. ~~[illegible]~~

Next question: Is $P_F \to \mathcal{E}_{\Sigma_F} \rtimes \Gamma$ an isomorphism at least for $\mathbb{Z}$ and $F = \{-1, 0, 1\}$. ~~[illegible]~~ The idea will be to show there's an equivalence of Hilb. space representations. So suppose given a Hilbert space H' ~~with [illegible]~~ with a * action of P_F, this means 3 operators p_{-1}, p_0, p_1 satisfying $p_k^* = p_{-k}$ and the componentwise version of $p^2 = p$ for $p = p_{-1} + p_0 + p_1$ (It only may be a Morita equivalence in this situation). Then you have a family $p(z) = zp_{-1} + p_0 + z^{-1}p_1$ of projections on H' over the circle, so it seems you can take H to be $p\,L^2(S^1, H')$. It looks like

Easy direction: Given $H \circlearrowleft u$, $\varepsilon_0 \geq 0$ such that $\overline{\sum_n u^n \varepsilon_0 H} = H$ $\varepsilon_0 u^n \varepsilon_0 = 0$ $|n| \geq 2$ and $\sum u^n \varepsilon_0^2 u^{-n} = 1$ on H, you form $p(z) = z\,\varepsilon_0 u^{-1} \varepsilon_0 + \varepsilon_0^2 + z^{-1}\varepsilon_0 u \varepsilon_0$ on $H' = \overline{\varepsilon_0 H}$

Problem whether, why $P_F \to \mathcal{E}_{\Sigma_F} \rtimes \Gamma$ is a Morita equivalence, at least for Hilbert space representations.

~~[illegible]~~ Easy direction: Given $\overline{H, u, \varepsilon_0 \geq 0}$ $\varepsilon_0 u^n \varepsilon_0 = 0$ $n \notin F$. $\sum \varepsilon_n^2 = 1$ on $H = \sum u^n \varepsilon H$. Then get a

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aim to setup a Morita equivalence between P_F and $\mathcal{E}_{\Sigma_F} \rtimes \mathbb{Z}$

You want a M. eq. Begin with $H, u, \varepsilon_0 \geq 0$

$\overline{\sum_n u^n \varepsilon_0 H} = H$ You know H is a completion of Laurent polys with values in H, ~~[illegible]~~ where the scalar product is given by a ~~[illegible]~~ measure on the circle with positive ~~hermitian~~ operator values, this measure $d\rho$ has the moments $\int (f^*, d\rho\, f) \sim \varepsilon_0 u^n \varepsilon_0$?

Begin again with $H, u, \varepsilon_0 \geq 0 \ni H = \overline{\sum u^n \varepsilon_0 H}$.

Assume $\varepsilon_0 u^n \varepsilon_0 = 0$ for $|n| \geq N$. To simplify suppose ~~[illegible]~~ ε_0 has finite rank

---

Maybe you are too devoted ~~to~~ Hilbert space. Instead try to construct a Morita equivalence. Given (H, u, ε_0) a module over $\mathcal{E}_{\Sigma_F} \rtimes \mathbb{Z}$, the homom.

$P_F \longrightarrow \mathcal{E}_{\Sigma_F} \rtimes \mathbb{Z}$ gives you a Laurent polynomial projector $p(z) = \sum_{n \in F} z^{-n} \varepsilon_0 u^n \varepsilon_0$ on $\varepsilon_0 H$. If $F = \{-1, 0, 1\}$

~~[illegible]~~ you have $\varepsilon_0 u^n \varepsilon_0 \quad n = -1, 0, 1$

$$p_n = \sum_j \underbrace{p_j p_{n-j}}$$

$$\varepsilon_0 u^j \varepsilon_0 \, \varepsilon_0 u^{-j+n} \varepsilon_0 = \varepsilon_0 \sum_j^2 u^n \varepsilon_0$$

~~[illegible]~~ P_F is a $\mathbb{Z}$-graded nonunital algebra with generators p_n, $n \in \mathbb{Z}$ subject to the relations

$$\begin{cases} p_n^* = p_{-n}, & p_n = \sum_j p_j p_{n-j} \\ p_n = 0, \ n \notin F \end{cases}$$

observe that $p_n = p_0 p_n +$ other quad. terms. where $p_0 = \varepsilon_0^2$ on H.

so $P_F = P_F^2 \qquad P_F = p_0 P_F = P_F p_0$

Is $\mathcal{E}_{\Sigma_F}$ idempotent? Yes.

$\varepsilon_m = \sum_n \varepsilon_m \varepsilon_n^2$ shows $\varepsilon_m \in \mathcal{E}_{\Sigma_F}^3$

so $\mathcal{E}_{\Sigma_F} \subset \mathcal{E}_{\Sigma_F}^3$ etc. What's next? Do you actually get a Morita equivalence?

$\mathcal{E}_{\Sigma_F}$ gen. by ε_m $m \in \mathbb{Z}$

$\varepsilon_m \varepsilon_n = 0$ $m-n \notin F$

$\varepsilon_m \sum_n \varepsilon_n^2 = \varepsilon_m$

action of Γ: $u^n \varepsilon_m u^{-n} = \varepsilon_{n+m}$. Have canonical

$p \in \mathcal{E}_{\Sigma_F} \rtimes \Gamma$ $\quad p = \sum_{n \in \mathbb{Z}} \varepsilon_0 u^n \varepsilon_0 = \sum_{n \in F} \varepsilon_0 \varepsilon_n u^n$

You want to construct a Morita equivalence

Given an $\mathcal{E}_{\Sigma_F} \rtimes \Gamma$ module H you get these operators $p_n = \varepsilon_0 u^n \varepsilon_0$ on H. What is $\sum_n p_n H$? $\quad P_F H = P_F(P_F H)$

Clearly $p_n H = \varepsilon_0 u^n \varepsilon_0 H = \varepsilon_0 \varepsilon_n H$

You assume algebraically that $H = \sum_n u^n \varepsilon_0 H$

whence $\varepsilon_0 H = \sum_n \varepsilon_0 u^n \varepsilon_0 H = \sum_n p_n H$

Try $F = \{0\}$. whence $\varepsilon_0 u^n \varepsilon_0 = 0$ $n \neq 0$ so

$\varepsilon_n \sum_m \varepsilon_m^2 = \varepsilon_n \implies \varepsilon_n^2 = \varepsilon_n$

Let's review this. $\mathcal{E}_{\Sigma_F}$ gen. ε_n relations

$\varepsilon_m \varepsilon_n = 0$ for $m-n \notin F$, $\sum_n \varepsilon_m \varepsilon_n^2 = \varepsilon_m = \sum_n \varepsilon_n^2 \varepsilon_m$

$\Gamma = \mathbb{Z}$ acts via $u^n \varepsilon_m u^{-n} = \varepsilon_{n+m}$. Let $B = \mathcal{E}_{\Sigma_F} \rtimes \mathbb{Z}$

A = subring generated by $p_n = \varepsilon_0 u^n \varepsilon_0 = \varepsilon_0 \varepsilon_n u^n$

$p_n = 0$ for $n \notin F$. $p(z) = \sum_{n\in F} z^{-n} p_n$. Then

$$p(z)^2 = \sum_{n} z^{-n} \sum_{i+j=n} p_i p_j \qquad \underline{p_n = \sum_i p_i p_{n-i} \ ?}$$

$$\sum_i p_i p_{n-i} = \sum_i \varepsilon_0 u^i \varepsilon_0^2 u^{-i+n} \varepsilon_0$$

$$= \sum_i \varepsilon_0 \varepsilon_i^2 u^n \varepsilon_0 = \varepsilon_0 u^n \varepsilon_0 = p_n$$

P_F = ring gen. by p_n, $n \in \mathbb{Z}$ relns. $p_n = 0$ $n \notin F$

Note $p_n \in P_F^2$ so $P_F = P_F^2$ $\qquad p_n = \sum_i p_i p_{n-i}$

If $F = \{0\}$, then $P_F = \mathbb{C} p_0$, $p_0 = p_0^2$

Take an $B = \mathcal{E}_{\Sigma_F} \rtimes \mathbb{Z}$ module H such that $H = BH$

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problem of the implementation of the commutation relations namely, you have Γ acting on $\mathcal{E}_{\Sigma_F}$ and a rep of the latter. Can you describe $\mathcal{E}_{\Sigma_F}$.

Let's keep to your example $\Gamma = \mathbb{Z}$ $F = \{-1, 0, 1\}$.

$\mathcal{E}_{\Sigma_F}$ generators ε_n $n \in \mathbb{Z}$, relations $\varepsilon_m \varepsilon_n = 0$ $m-n \notin F$

and $\sum_{n\in\mathbb{Z}} \varepsilon_n^2 \varepsilon_m = \varepsilon_m = \sum_{n\in\mathbb{Z}} \varepsilon_m \varepsilon_n^2$

Recap. Take a Hilbert Space repn. H of $\mathcal{E}_{\Delta_F} \rtimes \Gamma$ such that $\sum u^n \varepsilon_0 H = H = \sum \varepsilon_n H$. Then you get $p(z) = \sum_{n\in\mathbb{Z}} z^{-n} \underbrace{\varepsilon_0 u^n \varepsilon_0}_{p_n}$

$$\sum_j p_j p_{n-j} = \sum \varepsilon_0 u^j \varepsilon_0^2 u^{-j+n} \varepsilon_0 = \left(\sum \varepsilon_0 \varepsilon_j^2\right) u^n \varepsilon_0 = \varepsilon_0 u^n \varepsilon_0 = p_n$$

$p(z) = p(z)^* = p(z)^2$ on S^1. Now what about $p_n = \varepsilon_0 u^n \varepsilon_0$ this lies in $e_0 \mathcal{L}(H) e_0$

Let's try to understand the simplest situations

$H \circlearrowleft u$, $\varepsilon_0 \geq 0$, $\sum u^n \varepsilon_0 H = H$.

$\varepsilon_0 u^n \varepsilon_0 = 0 \quad |n| \geq N.$, $\quad \varepsilon_n = u^n \varepsilon_0 u^{-n}$, $\quad \varepsilon_m \varepsilon_n = u^n \varepsilon_0 u^{-n+m} \varepsilon_0 u^{-m}$
$= 0$ for $|-n+m| \geq N$.

Basic point is that H can be recovered from the function $n \longmapsto \varepsilon_0 u^n \varepsilon_0 = p_n$, or equivalently the function $z \mapsto \sum z^{-n} \varepsilon_0 u^n \varepsilon_0$ which is Laurent poly with ≥ 0 values on S^1! What is the idea? Assume ε_0 finite rank, so that $\varepsilon_0 H$ is fin. diml. Then $z \longmapsto \sum z^{-n} \varepsilon_0 u^n \varepsilon_0$ gives a hermitian metric on the trivial bundle over S^1 with fibre $\varepsilon_0 H$ and you integrate this to get the norm. The good case is when $p(z)^2 = p(z)$ is a projector. Then its rank shouldn't change. What you get is $p = p^* = p^2$ on $L^2(S^1, \varepsilon_0 H)$ and $H = pL^2(S^1, \varepsilon_0 H)$. This is incredibly confusing!

Do things in the other direction if possible. Suppose you have a Laurent polynomial loop in the Grassmannian $Gr(V)$.

$p(z) \in \mathcal{L}(V)$, $z \in S^1$, even simpler suppose you have $S^1 \xrightarrow{f} S^2 = \mathbb{CP}_1$ rational map

$f = \sum$. Are there any interesting cases?

Back to $H \circlearrowleft u$, $\varepsilon_0 \geq 0$ $\quad \varepsilon_n = u^n \varepsilon_0 u^{-n}$

get $p(z) = \sum z^{-n} \varepsilon_0 u^n \varepsilon_0 : S^1 \longrightarrow \mathcal{L}(\varepsilon_0 H)_+$

Go the other way, start with $p(z) = p(z)^* = p(z)^2$ $z \in S^1$ ~~xx~~ with values in $Gr(V)$. ~~loop space~~
Then you get an H ~~namely~~ $pL^2(S^1, V)$

Next attempt. Begin with H, u, ε_0 such that
$\varepsilon_0 u^n \varepsilon_0 = 0$ for $|n| \geq N$, and $\sum_n \varepsilon_m \varepsilon_n^2 = \varepsilon_m = \sum_n \varepsilon_n^2 \varepsilon_m$

you get $p(z) = \sum_{|n| \leq N} z^{-n} \underbrace{\varepsilon_0 u^n \varepsilon_0}_{p_n}$ sat $p(z)^* = p(z)^2 = p(z)^*$

for $|z| = 1$. $p_n \in \mathcal{L}(\varepsilon_0 H)$!

The Hilbert space picture is confusing
Maybe it's better to start with P_F which is the algebra gen. by elts $p_n, n \in \mathbb{Z}$ subject to the relations $p_n = 0$ $n \notin F$, $p_n = \sum_{j \in \mathbb{Z}} p_j p_{n-j}$.
P_F is universal wrt idempotent Laurent polynomials with monomial degrees in F.
Special case $F = \{0\}$
Worry about $F = \{-1, 0, 1\}$ case. Let's see if an algebraic statement is possible. You consider a P_F-module V, ~~and you produce~~ this means a vector space together with family of operators p_n satisfying the above relations. The idea is then to take $p(V \otimes \mathbb{C}[u, u^{-1}])$. Call this H. it is a $\mathbb{C}[u,u^{-1}]$ module with ~~a canonical~~ what extra structure?

$$pV \subset u^{-1}V + V + uV$$

$A = P_F$ generators p_n $n \in \mathbb{Z}$

relations $p_n = 0$ $n \notin F = \{-1, 0, 1\}$

$$p_n = \sum_j p_j p_{n-j} \qquad \therefore P_F = P_F^2$$

~~[illegible]~~

Idea: Given an $A = P_F$-module V we form $p(z)\left(\mathbb{C}[z, z^{-1}] \otimes V\right)$, where $p(z) = \sum_{n \in F} z^{-n} p_n$ is idempotent,

Note that A is commutative. $p(z)$ is family of idempotent operators on V, and $\mathbb{C}[z, z^{-1}] \otimes V$ is the space of Laurent polynomials with values in V, so (say V fin. diml) $p(z)\left(\mathbb{C}[z, z^{-1}] \otimes V\right)$ is the space of sections of a vector bundle.

Things are very surprising. ~~Alternative description~~

Basically you form two A. modules $p(z)\left(A[z, z^{-1}]\right)$ and a pairing with values in A, this is essentially clear from the commutative picture, bundle, dual bundle, pairing to functions. Corresp. fin. rank is $P \otimes_A Q$. sections of endo bundle

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How it should work. You begin with an $A = P_F$ module V, i.e. a Laurent poly family $p(z) = \sum_{n \in F} z^{-n} p_n$ of ~~idempotents~~ operators on V. ~~[illegible]~~ $p(z) \in \mathrm{End}_{\mathbb{C}[z, z^{-1}]}(\mathbb{C}[z, z^{-1}] \otimes V)$

V gives rise to (the section of) the trivial bundle $\mathbb{C}[z, z^{-1}] \otimes V$, and $p(z)$ gives splitting $\mathbb{C}[z, z^{-1}] \otimes V = E \oplus E^1$ as $\mathbb{C}[z, z^{-1}]$-modules.

The point is now to understand the ~~~~ good ring ~~~~ of operators on E. You have factorization of 1_E

$$E \hookrightarrow \mathbb{C}[z,z^{-1}]\otimes V \longrightarrow E$$

~~~~ the second map is determined by the linear map $V \hookrightarrow \mathbb{C}[z,z^{-1}]\otimes V \xrightarrow{p} E$, which should be p_0? No

$$p(z) = \sum z^{-n} p_n$$

$$\begin{tikzcd} \mathbb{C}[z,z^{-1}]\otimes V \arrow[r, two heads] & E \arrow[r, hook] & \mathbb{C}[z,z^{-1}]\otimes V \end{tikzcd}$$

~~One of the maps is cle~~

To clarify. Consider

$$f\otimes v \longmapsto f\,p(z)v$$

$$p(z) : \mathbb{C}[z,z^{-1}]\otimes V \longrightarrow \mathbb{C}[z,z^{-1}]\otimes V$$

the linear extension of $p(z) : V \longrightarrow \mathbb{C}[z,z^{-1}]\otimes V$. ~~So you~~

$$\begin{tikzcd} \mathbb{C}[z,z^{-1}]\otimes V \arrow[r, two heads] \arrow[rr, bend left, "p(z)"] & E \arrow[r, hook] & \mathbb{C}[z,z^{-1}]\otimes V \end{tikzcd}$$

you have $V \xrightarrow{p(z) = \sum z^{-n} p_n} \mathbb{C}[z,z^{-1}]\otimes V = \bigoplus z^n V$

so you have $V \xrightarrow{p(z)} E \subset \mathbb{C}[z,z^{-1}]\otimes V$

$$v \qquad\qquad \sum z^{-n} p_n v$$

~~~~ You want a nice ring of operators ~~~~ on E, ~~~~ You have mult. by z, i.e. the op u.

~~~~ Also you have $E \subset \mathbb{C}[z,z^{-1}]\otimes V \xrightarrow[\text{comp.}]{\text{degree } 0} V$

which can be combined with $p(z) : V \longrightarrow E$?

$$v \longmapsto \sum z^{-n} p_n v$$

$$\begin{array}{ccc} E \subset \mathbb{C}[z,z^{-1}]\otimes V & & \\ \downarrow \int \frac{d\theta}{2\pi} & & \\ V & \xrightarrow{p(z)} & E \end{array}$$

nearly there.

~~You are combining~~ You have ~~~~ obvious maps

$E \xrightarrow{\pi_n} V$ giving the coeff of z^{-n} $\qquad \pi_n u = \pi_{n+1}$

and a map $V \longrightarrow E$, namely $v \longmapsto p(z)v$
~~~~

can compose to get $p\pi_n : E \to V \to E$

$$\sum_n z^{-n} p_n \pi_n$$

$\therefore$ $V \xrightarrow{\varepsilon_0} \mathbb{C}[z,z^{-1}]\otimes V \xrightarrow{p} E$

$E \hookrightarrow \mathbb{C}[z,z^{-1}]\otimes V \xrightarrow{\pi_0} V$

Let V be a P_F-module i.e. a vector space equipped with a Laurent poly family of idempotent operators: $p(z) = \sum_n' z^{-n} p_n = p(z)^2$

& $p_n = 0 \quad n \notin F.$ $\qquad \sum_j p_j p_{n-j} = p_n$

Now interpret $p(z)$ as a proj on $\mathbb{C}[z,z^{-1}]\otimes V$

$E = p(\mathbb{C}[z,z^{-1}]\otimes V)$. have maps.

$$\begin{tikzcd} \mathbb{C}[z,z^{-1}]\otimes V \arrow[r, two heads] \arrow[rr, bend left, "p"] & E \arrow[r, hook] \arrow[rr, bend left, "id"] & \mathbb{C}[z,z^{-1}]\otimes V \arrow[r, two heads] & E. \end{tikzcd}$$

important here is

$P_F V = V$ assume

How do you organize this? On $\mathbb{C}[z,z^{-1}]\otimes V$ you have $p^2 = p$ so you get $E \xrightarrow{\alpha} \mathbb{C}[\ ]\otimes V \xrightarrow{\beta} E$

$\beta\alpha = 1$, $\alpha\beta = p$ usually Morita stuff, but somehow you have to make it precise. Begin with zilch

OK you have $E \xrightarrow{\alpha_n} V \xrightarrow{\beta_n} E$

such that $\sum \beta_n \alpha_n = 1$

$$E \xrightarrow{(z^n\alpha_n)} \bigoplus_n z^n V \xrightarrow{(\beta_n z^{-n})} E$$

$\alpha_n z^k = \alpha_{n-k}$

$\alpha_0 z^k = \alpha_{-k}$

$\boxed{\alpha_0 z^{-n} = \alpha_n}$

$$\xi \longmapsto \sum z^n \alpha_n(\xi) \xrightarrow{\cdot z^k} \sum z^{k+n} \alpha_n(\xi)$$

$$z^k\xi \longmapsto \sum z^n \alpha_n(z^k\xi) \qquad \alpha_n(z^k\xi) = \alpha_{n-k}(\xi)$$

$$E \xrightarrow{(z^n\alpha_n)} \bigoplus z^n V \xrightarrow{(\beta_n z^{-n})} E$$

$$\downarrow z\cdot$$

$$z^n \alpha_n z^k = z^n \alpha_{n-k}$$

$$\boxed{\alpha_n z^k = \alpha_{n-k}} \qquad \alpha_{-k} = \alpha_0 z^k$$

$$\xi \longmapsto \sum z^n \alpha_n(\xi)$$

$$z^k\xi \qquad \downarrow z^k\cdot$$

$$\sum_n z^n \alpha_n(z^k\xi) = \sum_n z^{k+n}\alpha_n(\xi)$$

$$z^n \eta \longmapsto \beta_n \eta$$

$$z^{k+n}\eta \longmapsto z^k \beta_n \eta$$

$$\boxed{z^k \beta_n = \beta_{k+n}}$$

$$\sum_n \beta_{n+k}\alpha_{n+k} = \sum z^k \beta_n \alpha_n z^{-k} = z^k z^{-k} = 1.$$

$$\boxed{z^k \beta_0 = \beta_k} \qquad \boxed{\alpha_k = \alpha_0 z^{-k}}$$

$$1 = \sum_k \beta_k \alpha_k = \sum_k z^k (\beta_0\alpha_0) z^{-k}$$

this is your p_0

What should I say?

What is α_0? Let $\xi = \sum z^m v_m \in E$

Start again. $p(z) = \sum z^{-n} p_n \in \mathcal{L}(V)$ $\quad p(z)^2 = p(z)$

$$\begin{tikzcd} \mathbb{C}[z]\otimes V \arrow[r, "\beta"] \arrow[rr, bend left, "p"] & E \arrow[r, "\alpha"] \arrow[rr, bend left, "1_E"] & \mathbb{C}[z]\otimes V \arrow[r, "\beta"] & E \end{tikzcd}$$

$\beta\alpha = 1$

$\alpha\beta = p$

$$\beta(f(z)v) = f(z)\,\beta(v) = f(z)\,p(z)\,v$$

Start again: Given V a vector space equipped with ~~[illegible]~~ a Laurent poly family of operators

$$p(z) = \sum_{n \in F} z^n p_n \quad \text{which is idempotent: } p(z)^2 = p(z)$$

equiv. $p_n = \sum_j p_j p_{n-j}$

Can interpret p as an idempotent operator on the module $\mathbb{C}[\mathbb{Z}] \otimes V = \bigoplus u^n V$ over the ring of Laurent polys. $E = p(u)(\mathbb{C}[\mathbb{Z}] \otimes V)$.

$$\begin{tikzcd} \mathbb{C}[\mathbb{Z}]\otimes V \arrow[r, "\beta"] \arrow[rr, bend left, "p = p(u)"] & E \arrow[r, "\alpha"] \arrow[rr, bend left, "1_E"] & \mathbb{C}[\mathbb{Z}]\otimes V \arrow[r, "\beta"] & E \end{tikzcd}$$

$p = \alpha\beta$, $1_E = \beta\alpha$. Aim: On V you have the operators p_n, V is a module over P_F. You want E to be a module over some natural ring B including the group ring $\mathbb{C}[\mathbb{Z}]$, ~~[illegible]~~ On $\mathbb{C}[\mathbb{Z}]\otimes V$ you have projection operators for each degree.

$$V \xrightarrow{\iota_n} \mathbb{C}[\mathbb{Z}]\otimes V \xrightarrow{j_n} V \qquad j_m \iota_n = \delta_{mn}$$

$$\sum u^k v_k \longmapsto v_n \qquad \sum \iota_n j_n = 1$$

$$v \longmapsto u^n \otimes v$$

to combine with α, β to get $\beta \iota_n j_n \alpha$ ~~[illegible]~~ $\in \mathcal{L}(V)$.

~~[illegible]~~

You have basic maps.

$$j_n\alpha : E \longrightarrow \mathbb{C}[\mathbb{Z}]\otimes V \longrightarrow V$$

$$\beta i_n : V \longrightarrow \mathbb{C}[\mathbb{Z}]\otimes V \longrightarrow E$$

~~[illegible]~~ satisfying $\sum_n (\beta i_n)(j_n \alpha) = \beta\alpha = 1_E$

and $(j_m\alpha)(\beta i_n) = j_m p \iota_n \overset{?}{=} p_{m-n}$

Let's check this

$$V \xrightarrow{\iota_n} \mathbb{C}[z,z^{-1}]\otimes V \xrightarrow{j_n} V$$
$$v \longmapsto z^n v, \quad \sum z^n v_n \longmapsto v_n$$

$$j_n \iota_m = \delta_{mn} \qquad \sum_n \iota_n j_n = 1 \text{ on } \mathbb{C}[z,z^{-1}]\otimes V$$

now combine with

$$\begin{tikzcd} \mathbb{C}[z,z^{-1}]\otimes V \arrow[r, "\beta"] \arrow[rr, bend left, "p"] & E \arrow[r, "\alpha"] \arrow[rr, bend left, "1_E"] & \mathbb{C}[z,z^{-1}]\otimes V \arrow[r, "\beta"] & E \end{tikzcd}$$

$$\alpha\beta = p, \quad \beta\alpha = 1.$$

$$j_n\alpha : E \xrightarrow{\alpha} \mathbb{C}[z,z^{-1}]\otimes V \xrightarrow{j_n} V$$
$$\beta\iota_m : V \xrightarrow{\iota_m} \mathbb{C}[z,z^{-1}]\otimes V \xrightarrow{\beta} E$$

$$1_E = \beta\alpha = \beta \sum_n \iota_n j_n \alpha = \sum_n (\beta\iota_n)(j_n\alpha)$$

$$(j_n\alpha)(\beta\iota_m) = \underline{j_n p \iota_m} : V \longrightarrow V.$$

$$V \xrightarrow{\iota_m} \mathbb{C}[z,z^{-1}]\otimes V \xrightarrow{p} \mathbb{C}[z,z^{-1}]\otimes V \xrightarrow{j_n} V$$

Perhaps you want to use $p = p^2$.

Question: Is $j_n\alpha = j_n p$, $p\iota_m = \beta\iota_m$. NO because $\underline{j_n\alpha}$ is defined on E, and $j_n p$ is defined on $\mathbb{C}[z,z^{-1}]\otimes V$.

What is $j_n p \iota_m$? $\quad v \in V \quad \iota_m v = z^m v$

$$p\iota_m v = \sum_k z^{k} p_k z^m v = \sum_k z^{m+k} p_k v$$

$$j_n p \iota_m v = p_k v \quad \text{where } k \ni m+k = n, \; k = n-m$$

$$\boxed{j_n p \iota_m = p_{n-m}}$$

$$p_s = h_0^{1/2} s h_0^{-1/2} = h_0^{1/2} h_s^{-1/2} s$$

~~[illegible]~~ You should be able to produce a Morita equivalence at this point. You have operators linking V and E

On E you have operators u = mult by z and

$$\underbrace{\beta\iota_0 \jmath_0 \alpha}: E \xrightarrow{\alpha} \mathbb{C}[z,z^{-1}]\otimes V \xrightarrow{\jmath_0} V \xrightarrow{\iota_0} \mathbb{C}[z,z^{-1}]\otimes V \xrightarrow{\beta} E$$

this should be the analog of h_0 $\parallel p_0$?, so the relation to check are $h_0 u^n h_0 = 0$ for n large and

~~$\sum_0 u^n$~~

Repeat. V a vector space equipped with a Laurent poly loop of idempotent operators

$$p(z) = \sum_{n\in F} z^n p_n \qquad p(z)^2 = p(z)$$

$$p_n = \sum_j p_j p_{n-j}$$

Given V define E = Image of $p(z)$ on $\mathbb{C}[z,z^{-1}]\otimes V$

$$\begin{tikzcd} \mathbb{C}[z,z^{-1}]\otimes V \arrow[rr, "p"] \arrow[dr, two heads, "\beta"] & & \mathbb{C}[z,z^{-1}]\otimes V \arrow[dr, two heads, "\beta"] & \\ & E \arrow[ur, hook, "\alpha"] \arrow[rr, "1_E"] & & E \end{tikzcd}$$

$$p = \alpha\beta \qquad 1_E = \beta\alpha$$

Next have
$$V \xrightarrow{\iota_n} \mathbb{C}[z,z^{-1}]\otimes V \xrightarrow{\jmath_n} V \qquad \jmath_n = \jmath_0 u^{-n}$$
$$\iota_n = u^n\iota_0 \qquad v \longmapsto z^n v, \quad \sum z^k v_k \longmapsto v_n$$

$$\underline{\jmath_m \iota_n = \delta_{mn}}, \qquad \sum_n \iota_n \jmath_n = 1 \quad \text{(agrees)} \sum_n u^n \iota_0 \jmath_0 u^{-n} = 1$$

$$\jmath_0 u^{-m+n}\iota_0 = \delta_{-m+n,0}$$

So what are you doing? Need operators on E.

u^n,
$$\jmath_n\alpha = (\jmath_0\alpha)u^{-n}: E \longrightarrow \mathbb{C}[z,z^{-1}]\otimes V \xrightarrow{\jmath_n} V$$

$$\beta\iota_m = \beta u^m\iota_0 = u^m(\beta\iota_0): V \xrightarrow{\iota_m} \mathbb{C}[z,z^{-1}]\otimes V \xrightarrow{\beta} E$$

Clearly ~~one some~~ get ops $E \xrightarrow{j_n \alpha} V \xrightarrow{\beta \iota_m} E$

$$\beta \iota_m j_n \alpha = u^m (\beta \iota_0 j_0 \alpha) u^{-n}.$$ $\beta \iota_0 j_0 \alpha$ is the key operator on E, basic ~~[illegible]~~ small support operator you sum the translates of to get a partition of unity!! Check: $\sum_n u^n \beta \iota_0 j_0 \alpha u^{-n} =$

$$\sum_n \beta u^n \iota_0 j_0 u^{-n} \alpha = \sum_n \beta \iota_n j_n \alpha = \beta \alpha = 1_E.$$

~~[illegible]~~ $(\beta \iota_0 j_0 \alpha) u^n (\beta \iota_0 j_0 \alpha) \overset{?}{=} 0$ for $n \notin F$?

$$j_0 \alpha u^n \beta \iota_0 = j_0 u^n \alpha \beta \iota_0 = j_0 u^n p \iota_0$$

$$p = \sum u^k \otimes p_k \quad \text{on} \quad \mathbb{C}[z,z^{-1}] \otimes V$$

$$j_0 = j_0 \otimes 1$$

$$(j_0 \otimes 1) \sum (u^{n+k} \otimes p_k)(\iota_0 \otimes 1) = \sum_k \underbrace{j_0 u^{n+k} \iota_0}_{\delta_{n+k,0}} \otimes p_k = p_{-n}$$

In any case you need to understand $\beta \iota_0 j_0 \alpha$

~~Now all you have to do is to decide on yields.~~

Recap. You have gone from V a module over $A = P_F$ to E a module over the ring B generated by an invertible (? be careful, u is **not** actually **in the ring** B)

But B is a cross product. $C \rtimes \mathbb{Z}$.

So what happens is that you have a typical Morita equivalence situation, namely, you will have bimodules P, Q s.t. $PQ = B$ $QP = A$

& ~~Even~~ B ~~not~~ comm.

Come on. $E = P \otimes_A V$ so P should be $p(\mathbb{C}[z,z^{-1}]\otimes A)$, Given E with its operators u and $(\beta\iota_0)(j_0\alpha) = h_0$.

Check things again. Given $p(z) = \sum_n z^n p_n = p(z)^2$ a L.poly family of idemp. ops of V, you form

$$E = p(\mathbb{C}[z,z^{-1}]\otimes V)$$

$$\begin{tikzcd} E \arrow[r, "\alpha"] \arrow[rr, bend right, "1_E"'] & \mathbb{C}[z,z^{-1}]\otimes V \arrow[r, "\beta"] & E \end{tikzcd}$$

$$V \xrightarrow{\iota_n} \mathbb{C}[z,z^{-1}]\otimes V \xrightarrow{j_n} V$$

$$\iota_n = u^n \iota_0 \qquad j_n = j_0 u^{-n}$$

$$j_m \iota_n = j_0 u^{-m+n} j_0 = \delta_{mn}.$$

$$1 = \sum_n \iota_n j_n = \sum_n u^n(\iota_0 j_0)u^{-n}$$

$$1_E = \sum_n \beta u^n(\iota_0 j_0)u^{-n}\alpha = \sum_n u^n \underbrace{(\beta\iota_0)(j_0\alpha)}_{h_0} u^{-n}$$

Next you want something about

$h_0 u^k h_0 = \beta\iota_0 \underbrace{j_0 \alpha u^k \beta \iota_0} j_0 \alpha$ being zero.

$$j_0 \alpha u^k \beta i_0 = j_0 u^k \alpha\beta \iota_0 = j_0 u^k \underline{p(u)}\, i_0$$

$$\sum u^n p_n = p_{-k}$$

$$p(u) \in \mathbb{C}[z,z^{-1}]\otimes \mathcal{L}(V)$$

It seems that

$h_0 u^k h_0 = 0$ for $k \in F$

Maybe what's important is the existence of a factorization. Suppose then that $h_0 = x_0 y_0$.

Now consider
$$\sum_{n\in\mathbb{Z}} x_0 u^n y_0 \quad ?$$

$$p = \sum h_0^{1/2} u^n h_0^{1/2} \qquad p^2 = \sum_{m,n} h_0^{1/2} u^m h_0 u^n h_0^{1/2}$$

$$\cancel{p^2 = \sum_n h_0^{1/2} \sum_m \left[u^m h_0 u^{-m} u^{m+n} h_0 \right]}$$

$$\cancel{p^2 = \sum_{m,n} h_0^{1/2} u^m h_0 u^{-m} u^{m+n} h_0^{1/2} = h_0^{1/2} \sum_m h_m \, u^m h_0^{1/2}}$$

$$p^2 = \sum_m h_0^{1/2} u^m h_0 u^{-m} \underbrace{\sum_n u^m u^n h_0^{1/2}}_{\sum u^n h_0^{1/2}} \quad ?$$

$$p^2 = \sum_{m,n} h_0^{1/2} u^m h_0 u^n h_0^{1/2}$$
$$= \sum_{m,n} h_0^{1/2} h_m u^{m+n} h_0^{1/2}$$
$$= \sum_m h_0^{1/2} h_m \underbrace{\sum_n u^{m+n} h_0^{1/2}}_{\text{looks indep of } m}$$
$$= h_0^{1/2} \cancel{\sum_m} h_m \sum_n u^n h_0^{1/2} = \sum_n h_0^{1/2} u^n h_0^{1/2}$$

$$p = \sum_n y_0 u^n x_0 \qquad p^2 = \sum_{m,n} y_0 u^m \overbrace{\cancel{x_0} \, y_0}^{h_0} u^n \cancel{x_0}$$
$$= \sum_{m,n} y_0 h_m u^{m+n} x_0$$
$$= \sum_{m,k} y_0 h_m u^k x_0 = \sum_k y_0 u^k x_0$$

You now have a chance to construct a M. eq.
~~[illegible]~~ Is it clear that P_F is flat? Better look at $B\text{-}\mathcal{E}_{\Sigma_F}$, does this have approx or local identities

$B = \mathcal{E}_{\Sigma_F}$ generators h_n, $n \in \mathbb{Z}$ relations $h_m h_n = 0$ $m-n \notin F$

and $\sum_n h_n h_m = h_m = \sum_n h_m h_n$

~~[illegible]~~ Recall a ring A has local left identities when
$\forall x \in A \quad \exists a \in A \quad ax = x$

mult. system $\{1-a \in A \mid a \in A\} = 1+A$

Claim this $\Longrightarrow$ $\forall x_1, \dots, x_n \quad \exists a \ni (1-a)x_i = 0. \quad i=1,\dots,n$

$\exists a_1 \ni (1-a_1)x_1 = 0$ $\qquad s_1 x_1 = 0$

$\exists a_2 \ni (1-a_2)(1-a_1)x_2 = 0$ $\qquad s_2(s_1 x_2) = 0$

$\exists a_3 \ni (1-a_3)[(1-a_2)(1-a_1)x_3] = 0$ $\qquad s_3(s_2 s_1 x_3) = 0$

$\Rightarrow s_3 s_2 s_1$ kills x_1, x_2, x_3

same arguments works when the x's belong to an A-module ~~M~~.

$\forall x \in M \quad \exists a \in A \quad xa = a$

$\Longrightarrow \forall x_1, \dots, x_n \in M \quad \exists a \ni ax_i = x_i$

Does this imply M flat?

$$\begin{tikzcd}
\tilde{A} \arrow[r, "\cdot\alpha"] \arrow[dr, "0"'] & \tilde{A}^k \arrow[r, dotted, "\cdot\alpha'"] \arrow[d, "\cdot m"] & \tilde{A}^k \arrow[dl, ".m'"] \\
& M &
\end{tikzcd}$$

$(1-a)m = 0$

$m = am$

$$\begin{tikzcd}
\tilde{A} \arrow[r, "\cdot(a_{ij})"] \arrow[dr, "0"'] & \tilde{A}^k \arrow[r, "a"] \arrow[d, "\cdot(m_j)"] & \tilde{A}^k \arrow[dl, "\cdot(m_j)"] \\
& M &
\end{tikzcd}$$

$\exists a \ni \quad (1-a)m_j = 0 \quad \forall j$

Assume $\forall x \in A \quad \exists a \quad (1-a)x = 0$.

Given $\sum_j a_{ij} x_j = 0$ [redacted] so choose a so that

$a a_j = a_j$, then / Your missing the idea that some category is filtering. You want to prove that $1+A$ is a filtering category?

$$\cdot \rightrightarrows \cdot \qquad \cdot \underset{1-a_2}{\overset{1-a_1}{\rightrightarrows}} \cdot \xrightarrow{1-a} \cdot \qquad (1-a)(a_1 - a_2) = 0.$$

: You are not making sense.

A module M is flat when the cat of f.g. free modules over M : $\tilde{A}^n \longrightarrow M$ is filtering.

mult sys. S ~~is filter~~ what is $S^{-1}M$? Pairs (s, m). So S is a monoid and M is a functor from S to Ab. You want the colimit

$$\bigoplus_{s \xleftarrow{\phi} s_0} M \rightrightarrows \bigoplus_{s_0} M \longrightarrow S^{-1}M \longrightarrow 0$$

$s_1 = s s_0$ $\qquad m \longmapsto (s_0, m)$, $\longmapsto (s s_0, sm)$

$S^{-1}M = S \times M$. How do you get into this?

~~Given (s, m)~~ $\qquad \dfrac{m}{s} = \dfrac{m'}{s'} \iff s'm = sm'$

$$\begin{matrix} \dfrac{m}{s} & & \dfrac{m'}{s'} \\ \parallel & & \parallel \\ \dfrac{s'm}{s's} & & \dfrac{sm'}{ss'} \end{matrix}$$

So you consider the ~~quotient~~ $S \times M$ / equivalence relation generated by $(s, m) \sim (ts, tm)$. So consider the cat

$Ob = S \quad Ar = S \times S \qquad Hom(s_1, s_2) = \{t \in S \mid ts_1 = s_2\}$

Good case when this cat is filtering $\quad s_1 \xrightarrow{t_1} s_3 \xleftarrow{t_2} s_2 \qquad s_1 \rightrightarrows s_2 \dashrightarrow s_3$

One point then is that ~~we~~ you want the category with $Ob = S$ and $Ar = S \times S$ to be filtering.

~~Given $1 \to a_1, 1 \to a_2$~~ But our $S = 1 + A$ ~~has~~ only one object. S filtering means given $s_1, s_2 \in S$ $\exists\, t \in S$ such that $ts_1 = ts_2$

$\cdot \rightrightarrows \cdot \to \cdot$

Let S act on X, $X/S = X/$ equiv. reln gen by $x \sim sx$ $\forall s$.

What should the equiv. reln. be? ~~Ans.~~ $x \sim y \iff \exists s,\ sx = sy$.

Check this is an equiv. rel. Ref ✓ Symm ✓, so Trans?

$x \sim y \sim z$ $\quad sx = sy \quad ty = tz$, but $\exists u \ni us = ut$

$\Rightarrow usx = usy = uty = utz \Rightarrow x \sim z$

~~[illegible]~~

Let S act on $\tilde{A}$ by mult. Is $\tilde{A}/S = A$?

Obvious map $A \longrightarrow \tilde{A}/S$ $\quad a \longmapsto$ equiv cl of a.

~~inj. $a_1 \sim a_2$ means $\exists s$ $sa_1 = sa_2$~~ Onto.

Let $n + a \in \tilde{A}$, then $n + a \sim (1+$

$A = P_F$ alg (noncommutal) generated by elts p_n, $n \in \mathbb{Z}$
relation $p_n = \sum_j p_j p_{n-j}$ $\qquad p_n = 0 \quad n \notin F$.

If V is an A-module, let $E(V) = p(z)(\mathbb{C}[z,z^{-1}] \otimes V)$.
this is an exact functor from A-modules to modules over B ~~[illegible]~~ the cross product algebra $\mathbb{C}[\mathbb{Z}] \ltimes \mathcal{E}_{\Sigma_F}$

where $\mathcal{E}_{\Sigma_F}$ generators h_n
relations $h_m h_n = 0 \qquad m - n \notin F$

$$\sum_n h_n h_m = h_m = \sum_n h_m h_n$$

this is the reason for your digression about

The relations show that $\mathcal{E}_{\Sigma_F}$ has local left (resp. right) identities
$\mathcal{E}_{\Sigma_F}$ should be left + right flat.

$$\begin{tikzcd} \tilde{A} \arrow[r, "(x_i)"] & \tilde{A}^n \arrow[r, "(y_i)"] \arrow[d, "(y_i)"] & \tilde{A} \arrow[dl, "a"] \\ & A & \end{tikzcd} \qquad \sum_{i=1}^n y_i x_i = 0$$

choose a so that $a y_i = y_i \quad \forall i$.

$$0 = \sum y_i x_i = \sum (a y_i) x_i$$

$$\begin{tikzcd} \tilde{A} \arrow[r, "\cdot(x_i)"] & \tilde{A}^n \arrow[d, "\cdot(y\quad)"] \\ & \end{tikzcd} \qquad \begin{tikzcd} \tilde{A} \arrow[r, "\cdot\vec{x}"] & \tilde{A}^n \arrow[r, "\cdot\vec{y}"] \arrow[d, "\cdot\vec{y}"] & \tilde{A} \arrow[dl, "\cdot a"] \\ & A & \end{tikzcd}$$

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nonunital algs

$A = P_F$ gens. p_n relns. $p_n = 0 \quad n \notin F$
$p_n = \sum_j p_j p_{n-j}$

$B = \mathcal{E}_{\Sigma_F}$ gens. h_n relns $h_m h_n = 0 \quad m-n \notin F$

$$\sum_n h_n h_m = h_m = \sum_n h_m h_n$$

Given V an A-mod form $E(V) = \underbrace{p(u)}_{\sum u^n \otimes p_n}(\mathbb{C}[u,u^{-1}] \otimes V)$

$V \mapsto E(V)$ clearly exact.

also right cont. ~~...~~

$$E(V) = E(\tilde{A}) \otimes_A V$$

$E(\tilde{A}/A) =$ suppose $AV = 0$ i.e. the $p_n = 0$ on V then $E(V) = \mathbb{C}[u,u^{-1}] \otimes V$. Review the operators naturally occurring on $E(V)$ in general.

$\beta\alpha = 1_E$

$$E \xrightarrow{\alpha} \mathbb{C}[z,z^{-1}] \otimes V \xrightarrow{\beta} E$$

$$j_0 z^{-n} = j_n \uparrow \iota_n = z^n \iota_0$$

$$V$$

$$1_E = \beta\alpha = \beta \sum \iota_n j_n \alpha = \sum_n u^n \overbrace{(\beta \iota_0 j_0 \alpha)}^{h_0} u^{-n}$$

On $E(V)$ then you have $h_n = u^n \underbrace{(\beta \iota_0 j_0 \alpha)}_{h_0} u^{-n}$ such that $\sum h_n = 1$. Thus $B\, E(V) = E(V)$.

The question now is how to start with a B module, e.g. B (which you know is firm flat because of the local identities. How do you recover A?

There a notation problem $p_n = h_0^{1/2} h_n^{1/2} u^n$, you must distinguish between the p_n which are generators for A and the h_n which are generators for $\mathcal{E}_{\Sigma_F}$. Clarify.

$$\beta\alpha = 1_E$$

$$E \xrightarrow{\alpha} \mathbb{C}[z,z^{-1}]\otimes V \xrightarrow{\beta} E$$

$$1_E = \beta \sum_n u^n \iota_0 j_0 u^{-n} \alpha = \sum_n u^n \underbrace{(\beta \iota_0)(j_0 \alpha)}_{h_0} u^{-n}$$

$$1_E = \sum_n (\beta \iota_n)(j_n \alpha) \qquad \sum u^k p_k \qquad -m+k+n=0$$

But $$(j_m \alpha)(\beta \iota_n) = j_m\, p(u)\, \iota_n = j_0 u^{-m} p(u) u^n \iota_0$$
$$= p_{m-n}$$

Now you need to develop the idea of interchange and Morita equivalence — this should generalize the results

$$tr(pq) = tr(qp), \qquad \underset{K_1(A)}{[1-qp]} = \underset{K_1(B)}{[1-pq]}$$

Part of this picture is making choices (i.e. representing a nuclear map as an element of a tensor product. So how do you start? You have E which is a firm module for B.

The bimodule ${}_B P_A$ should be $E(A)$

$$E(M) = p(u)\left(\mathbb{C}[z,z^{-1}]\otimes M\right) \qquad \text{exact in } M \text{ right cont.}$$
$$= E(\tilde{A})\otimes_A M$$

$$0 \longrightarrow E(A) \longrightarrow E(\tilde{A}) \longrightarrow \overset{0}{\overset{\|}{E(\tilde{A}/A)}} \longrightarrow 0$$

If $AV=0$, then $p(u)=0$. and $E(V)=0$.

So $\quad E(M) = E(A)\otimes_A M.$

So you understand the P-bimodules. It should have natural generators.

$$\begin{array}{ccccc} E(A) & \xrightarrow{\ \alpha\ } & \mathbb{C}[z,z^{-1}]\otimes A & \xrightarrow{\ \beta\ } & E(A) \\ & & j_0 \big\downarrow \big\uparrow i_0 & & \\ & & A & & \end{array}$$

namely βi_0. $\quad \left(\begin{array}{l} \beta i_0 : A \longrightarrow E(A) \\ j_0\alpha : E(A) \longrightarrow A \end{array} \right.$

You expect $P=E(A)$ to be generated by βi_0 and u^n and this is clear. Dually Q should be generated by $j_0\alpha \in \mathrm{Hom}_{A^{op}}(E(A),A)$ and u^n.

At this point it seems clear that the expected Morita equivalence holds, but it remains to get the details straight.

Given M a vector space, an A-module, i.e. M is a v.s. equipped with operator $p(z)$ on $\mathbb{C}[Z]\otimes M$
$p(u) = \sum u^n p_n$, $p_n \in V$, then form

$$E(M) = p(u)\left(\mathbb{C}[Z]\otimes M\right).$$

$E(M)$ is a $\mathbb{C}[Z]$-module

$$\begin{tikzcd}
E(M) \arrow[r, hook, "\alpha"] \arrow[rr, bend left] & \mathbb{C}[Z]\otimes M \arrow[r, "\beta"] \arrow[d, shift right, "j_0"'] & E(M) \arrow[r, "\alpha"] & {} \\
 & M \arrow[u, shift right, "\iota_0"'] & &
\end{tikzcd}$$

Of course this is related to the GNS business.

$$u^n \iota_0 = \iota_n \qquad j_0 u^{-n} = j_n.$$

$$\sum_{n\in\mathbb{Z}} \iota_n j_n = id \quad \text{on } \mathbb{C}[Z]\otimes M$$

$$1_E = \beta\left(\sum \iota_n j_n\right)\alpha = \beta\left(\sum u^n \iota_0 j_0 u^{-n}\right)\alpha$$

$$= \sum_n u^n(\beta\iota_0 j_0 \alpha)u^{-n}$$

$$p = \alpha\beta, \quad j_m p \iota_n = (j_0\alpha)u^{-m+n}(\beta\iota_0)$$

To make this clearer. So far you have distinguished maps $\beta\iota_0 : M \longrightarrow E$ and $j_0\alpha : E \longrightarrow M$, which can be moved by the group $\{u^n\}$ to obtain ~~an embedding~~ a surj. $\mathbb{C}[u,u^{-1}]\otimes M \xrightarrow{\beta} E$, $u^n \otimes m \longmapsto u^n \underbrace{\beta\iota_0}_{\beta i_n}(m)$

and inj. $E \xrightarrow{\alpha} \mathbb{C}[u,u^{-1}]\otimes M$

$\xi \longmapsto$

$$\alpha = \sum_n \iota_n j_n \alpha = \sum_n u^n \iota_0 j_0 \alpha u^{-n}$$

$$\beta = \beta \sum \iota_n j_n = \sum_n u^n \beta\iota_0 j_0 u^{-n}$$

$$1_E = \sum \beta u^n \iota_0 j_0 u^{-n} \alpha = \sum_n u^n (\beta\iota_0 j_0 \alpha) u^{-n}$$

$$p = \alpha\beta \qquad j_m p \iota_n = j_0 u^{-m} \alpha\beta u^n \iota_0 = (j_0\alpha) u^{-m+n} (\beta\iota_0)$$

You really need a nice way to handle partitions of unity, embedding as a summand of a trivial bundle, expressing ~~map~~ the identity map as a sum of "rank 1" maps, writing an element of A as a sum of products $a = \sum q_i p_i$

Essence of Morita equivalence

$pq = 1_E$

$qp = e$ idemp

Important is $E \xrightarrow{q} A^n \xrightarrow{p} E$

$$E^* \xleftarrow{{}^t q} A^n \xleftarrow{{}^t p} E^*$$

Start again, focus upon GNS, look for a variant involving idempotents

Recall GNS idea. A, B unital algs, $\rho : A \to B$ linear $\rho(1) = 1$ given

universal ρ with A fixed is RA. In the nonunital setting RA is $TA = A \oplus A^{\otimes 2} \oplus \cdots\cdots$.

Basic idea to look at data given by an A-module M, B-module N, $i : N \to M$, $j : M \to N$

$j a i = \rho(a)$ So what happens in the nonunital situation is that you add $ji = \mathrm{id}_N$.

Look at your ideas this past week. You looked at $H, u,\ j : V \to H$, adjoint j^*; ~~and you~~ you get pos. def. function on $\mathbb{Z}$: $\rho : u^n \longmapsto j^* u^n j \in \mathcal{L}(V)$. The unusual aspect here is that ρ generally lies in the positive cone, but you require it to be idempotent. Γ graded setting

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Start again Given H Hilbert space, u unitary op, V closed subspace $j : V \to H$ the inclusion, consider $u^n \longmapsto j^* u^n j$, $\mathbb{Z} \longrightarrow \mathcal{L}(V)$. This is a pos. definite function on the group $\mathbb{Z}$. Make assumption that $j^* u^n j = 0$ for $|n|$ large, take F.T.

Different idea H Hilbert space acted on by $C(S^1)$ V closed subspace. You are confused by the measure aspects. But take the case of finite support to reduce to Laurent poly functions + Lebesgue measure.

Given H Hilbert space, u unitary on H, $V \overset{j}{\hookrightarrow} H$ closed subspace, ~~form the~~ form the moments:

$$j^* u^n j \in \mathcal{L}(V)$$

Assume $j^*u^nj = 0$ for $|n| > N$, form

$$\rho(z) = \sum_n z^{-n} j^* u^n j \in \mathbb{C}[z, z^{-1}] \otimes \mathcal{L}(V)$$

$\rho(z)$ is a Laurent poly function on the unit circle $|z| = 1$ with values in the subspace of herm. ops, in fact ~~non-negative~~ positive herm. ops. Why? Assume $\exists z_0 \in S^1$ such that $\rho(z_0)$ is not ≥ 0, i.e. $\exists v \in V$ such that $0 > v^*\rho(z_0)v = \sum_{|n| \leq N} z_0^{-n} v^* j^* u^n j v$.

~~If you have~~ You now have reduced to the case where $V = \mathbb{C}v$ and j incl. You have

$$\rho : z \longmapsto \sum_{|n| \leq N} z^{-n} v^* u^n v \qquad \text{real valued function on } S^1$$

which is < 0 at some point z_0. Let $f(z) \in C(S^1)$ have support in < 0 region of ρ.

$$\int \overline{f(z)}\, \rho(z) f(z) \frac{d\theta}{2\pi} = \langle f(u)v \,|\, f(u)v \rangle$$
$$< 0,$$

What you want to understand is? Take an example. $\Gamma = \mathbb{Z}$ repn. H, u subsp $V \subset H$, then get

$$u^n \longmapsto j^* u^n j \qquad \text{pos. def fn. on } \mathbb{Z} \text{ values in } \mathcal{L}(V)$$

GNSS thm. gives converse. Pos. means ~~take~~ matrix

$\rho_{k\ell} = j^* u^{-k+\ell} j$ is positive so that

if f_n is a function on the group to V

then $\sum_{k,\ell} f_k^* \rho_{k\ell} f_\ell$ is ≥ 0.

$$\left(\sum_k u^k j f_k\right)^* \left(\sum_\ell u^\ell j f_\ell\right) = \sum_{k,\ell} f_k^* j^* u^{-k+\ell} j f_\ell$$

~~GNSS gives converse. i.e. given~~

completely pos. map. $\rho : A \longrightarrow \mathcal{L}(V)$

$\forall\, a_1, \dots, a_n$ $\quad \rho(a_k^* a_\ell)$ should be positive

On $A \otimes V$ you want a ≥ 0 scalar product

$$\left(\sum a_k \otimes v_k\right)^* \left(\sum a_\ell \otimes v_\ell\right) = \sum_{k,\ell} v_k^* \rho(a_k^* a_\ell) v_\ell$$

When $A = \mathbb{C}[\Gamma]$, then $A \otimes V = \bigoplus_{s \in \Gamma} s \otimes V$

$$\left(\sum_s s \otimes v_s\right)^* \left(\sum_t t \otimes v_t\right) = \sum_{s,t} v_s^* \rho(s^{-1} t) v_t$$

This seems OKAY to me. So now you can ask when idempotent + hermitian $\Rightarrow$ completely pos?

You want to start with $\rho : \mathbb{C}[\Gamma] \longrightarrow \mathcal{L}(V)$
$\rho(s)^* = \rho(s^{-1})$ and you want ρ comp. pos.

What happens with $\Gamma = \mathbb{Z}$.

You have a block matrix $\rho(u^{-k+\ell})$ entries in $\mathcal{L}(V)$. Complete Positivity means $\rho(u^{-\ell+k}) = \rho(u^{-k+\ell})^*$ and

$\sum_{k,\ell} f_k^* \rho(u^{-k+\ell}) f_\ell \geq 0$ all $\sum u^k \otimes f_k \in \mathbb{C}[\Gamma] \otimes V$

[redacted] You want $\rho(u^{-k+l})$ a block matrix

$\left(\begin{array}{c|c} & \\ \hline & \end{array}\right)$ with entries in $\mathcal{L}(V)$ to be ≥ 0.

This ~~means~~ $\sum f_k^* \rho(u^{-k+l}) f_l \geq 0$ all $\sum_k u^k \otimes f_k \in \mathbb{C}[\mathbb{Z}] \otimes V$,

~~so it's enough to look at one f at a time which means looking~~ $f = \sum u^k \otimes f_k \in \mathbb{C}[\mathbb{Z}] \otimes V$. ~~Look~~

Associate to $f(u)$ the function $z \mapsto f(z) = \sum z^{+k} f_k$ from S^1 to V. ~~Then~~ Put $\rho(z) = \sum_n \bar{z}^n \rho(u^n)$

$$\int f(z)^* \rho(z) f(z) \frac{d\theta}{2\pi} = \int \sum_k f_k^* z^{-k} \sum_n \bar{z}^n \rho(u^n) \sum_l z^l f_l \frac{d\theta}{2\pi}$$

$$-k - n + l = 0 \qquad n = -k+l$$

$$= \sum_{k,l} f_k^* \rho(u^{-k+l}) f_l .$$

But now you can analyze positivity of $\rho(z)$ pointwise on S^1, certainly in the case where $\rho(z)$ is a continuous function.

Now comes the interesting ~~amusing~~ point, where $\rho(z)$ is **idempotent** $\forall z$.

[redacted] $\rho(z) = \rho(z)^*$ + $\rho(z) = \rho(z)^2 \implies \rho(z) \geq 0$

So what next?

non abelian case. Question: Consider a Γ-graded projection

[redacted] $p = (p_s)_{s \in \Gamma}$ $\qquad p_s = 0 \quad s \notin$ some finite F

$p_s^* = p_{s^{-1}}$

$$\sum_s p_s \sum_t p_t = \sum_{s,t} p_s p_t \qquad p_s = \sum_t p_{st^{-1}} p_t \overset{?}{=} \sum_t p_t p_{t^{-1}s}$$

$$= \sum_u \Big(\sum_{u=st} p_s p_t \Big)$$

~~Now it makes sense to~~

Consider a Γ-graded projection $p = (p_s)_{s \in \Gamma}$ with values in $\mathcal{L}(V)$. Question: is $s \mapsto p_s$ from Γ to $\mathcal{L}(V)$ completely positive?

~~Note~~ Suppose given $p_s \in \mathcal{L}(V)$ $\left(\text{for } s\in\Gamma\right.$ sat $p_s = 0$ $\forall s \notin F$ some

$$p_s^* = p_{s^{-1}}, \quad p_s = \sum_{\cancel{tu}} p_{s\cancel{u}^{-1}} p_{\cancel{u}} = \sum_{\substack{t,u \\ \ni s = tu}} p_t p_u$$

~~Look at ...~~

~~...~~ Look at $\mathbb{C}[\Gamma]\otimes V$; a $\mathbb{C}[\Gamma]\otimes\mathcal{L}(V)$-module

$$\mathrm{Hom}_{\mathbb{C}[\Gamma]}\left(\mathbb{C}[\Gamma]\otimes V, \mathbb{C}[\Gamma]\otimes V\right)$$

$$= \mathrm{Hom}\left(V, \mathbb{C}[\Gamma]\otimes V\right) \supset \mathbb{C}[\Gamma]\otimes \mathrm{End}(V)$$

Does $p = (p_s)$ yield a projector on this free Γ-module $\mathbb{C}[\Gamma]\otimes V$?

$M = \mathbb{C}[\Gamma]\otimes V$ is naturally Γ-graded

$$\| $$

$$\bigoplus_{s\in\Gamma} M_s = \bigoplus_{s\in\Gamma} s\otimes V \qquad tM_s = ts\otimes V = M_{ts}$$

so $\mathrm{Hom}_{\mathbb{C}[\Gamma]}\left(\mathbb{C}[\Gamma]\otimes V, \mathbb{C}[\Gamma]\otimes V\right)$ should have elements of ~~a given ally~~ deg $= s$ for $s\in\Gamma$.

$$\mathrm{Hom}\left(V, \mathbb{C}[\Gamma]\otimes V\right) \longleftrightarrow \mathbb{C}[\Gamma]\otimes \mathrm{End}(V)$$

so $\mathbb{C}[\Gamma]\otimes \mathrm{End}(V)$ acts on $\mathbb{C}[\Gamma]\otimes V$

$$(s\otimes a)(t\otimes v) = st\otimes av$$

So now write down $p = \sum_{s\in\Gamma} s\otimes p_s \in \mathbb{C}[\Gamma]\otimes \mathrm{End}(V)$

$$p^2 = \sum_{s,t} (s\otimes p_s)(t\otimes p_t) = \sum_{u\in\Gamma} u\otimes \sum_{\substack{s,t \\ st=u}} p_s p_t = \sum_{u\in\Gamma} u\otimes p_u = p.$$

clear that $*$ should work.

details. Γ group, V vector space, then $\mathbb{C}[\Gamma]\otimes V = \bigoplus_{s\in\Gamma} s\otimes V$ is Γ-graded vector space and a left Γ-module. Compatible condition: $t(s\otimes V) \subset ts\otimes V$, i.e. left mult by $t\in\Gamma$ is an operator "homogeneous of degree t" wrt the Γ-grading

$$\operatorname{Hom}_{\mathbb{C}[\Gamma]}(\mathbb{C}[\Gamma]\otimes V, \mathbb{C}[\Gamma]\otimes V) = \operatorname{Hom}(V, \mathbb{C}[\Gamma]\otimes V)$$

contains $\mathbb{C}[\Gamma]\otimes \operatorname{Hom}(V,V)$. You want to view $\mathbb{C}[\Gamma]\otimes V$ as the trivial "vector bundle" with fibre V and $\mathbb{C}[\Gamma]\otimes \operatorname{Hom}(V,V)$ as its "finite support" endomorphisms. Consider an idempotent

$$p = \sum_{s\in\Gamma} s\otimes p_s \quad \in \mathbb{C}[\Gamma]\otimes \operatorname{Hom}(V,V)$$

$$p^2 = \sum_{s,t\in\Gamma} st\otimes p_s p_t, \quad \therefore \quad p_u = \sum_{\substack{s,t\\ st=u}} p_s p_t$$

p gives rise to a splitting of the trivial bundle $\mathbb{C}[\Gamma]\otimes V$; let E be the image, so that there are Γ-module maps

$$\begin{tikzcd} & V \arrow[d, "\iota_t"] & \\ E \arrow[r, "\alpha"] & \mathbb{C}[\Gamma]\otimes V \arrow[r, "\beta"] \arrow[d, "j_s"] & E \\ & V & \end{tikzcd}$$

such that $\beta\alpha = 1_E$, and $\alpha\beta = p$.
(Actually you should probably focus on positivity. You want the link with GNSS)

$$1_{\mathbb{C}[\Gamma]\otimes V} = \sum_s \iota_s j_s = \sum_s s\,\iota_1 j_1 s^{-1}$$

$$1_E = \beta\, 1_{\mathbb{C}[\Gamma]\otimes V}\,\alpha = \sum_{s\in\Gamma} s(\beta\iota_1)(j_1\alpha)s^{-1}$$

$$(j_s\alpha)(\beta\iota_t) = j_1 s^{-1} p\, t\, \iota_1$$
$$= \sum_u j_1 s^{-1} t u\, p_u\, \iota_1 = p_{t^{-1}s}$$

So what you want ultimately is to take the operator s on E and compress it using the maps $V \xrightarrow{\beta i_1} E \xrightarrow{j_1 \alpha} V$, i.e. $j_1 \alpha s \beta i_1$

$$= j_1 s \alpha \beta i_1 = j_1 s p i_1 = j_1 s \sum_u u p_u i_1 = p_{s^{-1}}$$

~~absolutely ...~~ Look at this stuff from Hilbert space viewpoint. Take a $*$ repn of P_F and dilate.

You have to find the obstruction, but you are almost there. For a general Γ you can begin with a

$p \in \mathbb{C}[\Gamma] \otimes \mathrm{End}(V)$ $\qquad p = \sum s \otimes p_s$ finite sum

$p^* = \sum s^{-1} \otimes p_s^*$

$p^2 = p = p^*$ $\qquad p_s = \sum_{\substack{t,u \\ \ni tu = s}} p_t p_u \qquad = p_{s^{-1}}^*$

$$E = p(\mathbb{C}[\Gamma] \otimes V) \xrightarrow{\alpha} \mathbb{C}[\Gamma] \otimes V \xrightarrow{\beta} E$$

$$1_E = \beta \alpha = \beta \Big(\sum_s i_s j_s \Big) \alpha = \sum_s s (\beta i_0 j_0 \alpha) s^{-1}$$

Try to incorporate the Hilbert space situation. This means for any module there's a dual module around.

So V, which so far is a vector space, has a corresponding V^* and an anti-linear $V \xrightarrow{\sim} V^*$ get $*$ on $\mathcal{L}(V)$. What about $\mathbb{C}[\Gamma] \otimes V$? Hilbert space structure is clear $\mathbb{C}[\Gamma] \otimes V = \bigoplus s \otimes V$, ~~[illegible]~~

You want ~~the module~~ $i_s : v \longmapsto s \otimes v$ to make $\mathbb{C}[\Gamma] \otimes V$ the orth direct sum of $s \otimes V$.

Mystery. H Hilbert space with Γ-action, $V \xrightarrow{j} H$ map, get $s \mapsto j^* s j \in \mathrm{End}(V)$ which should determine the scalar prod. on $\mathbb{C}[\Gamma] \otimes V \longrightarrow H$

$$s \otimes v \longmapsto sjv$$

$$\langle sjv, tjv' \rangle = \langle v, j^* s^{-1} t j v' \rangle.$$

$\rho(s) = j^* s j$ is an operator on V, $\rho(s)^* = \rho(s^{-1})$

? $\rho : \mathbb{C}[\Gamma] \longrightarrow \mathrm{End}(V)$ completely pos.?

$\forall$ $s_1, \dots, s_n$ $\qquad \rho(s_k^{-1} s_\ell) \in M_n(\mathrm{End}(V)) = \mathrm{End}(V^n)$

given $\sum c_k s_k \in \mathbb{C}[\Gamma]$

you want $\qquad \rho\left(\sum_k (c_k s_k)^* \sum_\ell (c_\ell s_\ell) \right)$

$$= \sum_{k,\ell} \bar{c}_k \underbrace{\rho(s_k^{-1} s_\ell)}_{j^* s_k^{-1} s_\ell j} c_\ell \quad \text{to be } \geq 0 \text{ in } \mathcal{L}(V)$$

$$= j^* \left(\sum_k \bar{c}_k s_k^{-1} \right) \left(\sum_\ell c_\ell s_\ell \right) j = j^* a^* a j$$

what are you trying to get? You know ~~that~~ for any H with Γ-action and $j : V \longrightarrow H$ that $\rho(s) = j^* s j \in \mathrm{End}(V)$ is completely positive. On the other hand, given a map $s \longmapsto \rho(s)$, $\Gamma \longrightarrow \mathrm{End}(V)$ you ~~have~~ have the notion that $\rho(s)$ is a Γ-graded proj.

The question is whether Γ-graded proj $\Longrightarrow$ completely pos.?
This seems false in general, but true for Γ abelian.

Return to Morita equivalence. Given a Γ-graded projection on a vector space V, equivalently $p = \sum_s s \otimes p_s \in \mathbb{C}[\Gamma] \otimes \mathrm{End}(V)$ such that $p^2 = p$, you get

$$E = p(\mathbb{C}[\Gamma] \otimes V) \xrightarrow{\alpha} \mathbb{C}[\Gamma] \otimes V \xrightarrow{\beta} E$$

$$1_E = \beta\alpha = \beta\Big(\sum_{s\in\Gamma} s\, \iota_1 j_1 s^{-1}\Big)\alpha = \sum_{s\in\Gamma} s(\beta\iota_1)(j_1\alpha)s^{-1}$$

$$p = \alpha\beta \qquad j_s p \iota_t = j_1 s^{-1} p t \iota_1 = p_{st^{-1}}$$

so for $st^{-1} \notin F$ you have $j_s p \iota_t = 0$.

Put $h_1 = \beta\iota_1 j_1 \alpha \qquad h_s = \beta\iota_s j_s \alpha$

$$h_s h_t = \beta\iota_s \underbrace{j_s \alpha\beta\iota_t}_{\substack{0 \\ \text{for } st^{-1}\notin F}} j_t \alpha = 0 \qquad st^{-1} \notin F$$

What is the basic idea? $A = \varinjlim_F P_F$ No it's an inverse system

Fix an $F \subset \Gamma$, let $A = P_F$

form corresp. $E = p(\mathbb{C}[\Gamma] \otimes A)$. You have natural generators $\beta\iota_s : A \longrightarrow E(A)$, and natural cogenerators $j_s\alpha : E(A) \longrightarrow A$. generate means

$$\mathbb{C}[\Gamma] \otimes A \xrightarrow{\sum \beta\iota_s} E(A)$$

$$E(A) \xrightarrow[\sum (j_s\alpha)]{} \mathbb{C}[\Gamma] \otimes A$$

$$j_1 \alpha s^{-1}$$

and cogen means

$$F(A) \subset \mathrm{Hom}_{A^{op}}(E(A), A)$$

So what are you attempting to do? You have $E(A) = p(\mathbb{C}[\Gamma]\otimes A) \hookrightarrow \mathbb{C}[\Gamma]\otimes A$

$$\mathbb{C}[\Gamma]\otimes A \xrightarrow{\beta} E \xhookrightarrow{\alpha} \mathbb{C}[\Gamma]\otimes A$$

$$(\mathbb{C}[\Gamma]\otimes A)^{*\vee} \dashrightarrow \check{E} \qquad (\mathbb{C}[\Gamma]\otimes A)^{\vee}$$

Is P_F commutative? ~~$p_s p_t = p_{st}$~~ p_t NO.

$$p_s = \sum_{\substack{t,u \\ tu=s}} p_t p_u \quad \text{in } P_F$$

$$p_s = \sum_{\substack{t,u \\ tu=s}} p_u p_t \quad \text{in } P_F^{op}$$

$$= \sum_{ut=s} p_t p_u$$

You want to consider a noncomm. unital alg Δ and its "Grassmannians" projectors p in $\Delta\otimes V$. To understand this variety.

$\Delta = \mathbb{C}[\Gamma]$, V vector space say f.d. Form

$\mathbb{C}[\Gamma]\otimes V \qquad \mathrm{Hom}_{\mathbb{C}[\Gamma]}(\mathbb{C}[\Gamma]\otimes V, \mathbb{C}[\Gamma]\otimes W) = \mathrm{Hom}(V, \mathbb{C}[\Gamma]\otimes W)$

$$= \mathbb{C}[\Gamma]\otimes \mathrm{Hom}(V,W)$$

Let $p = \sum s\otimes p_s \in \mathbb{C}[\Gamma]\otimes \mathrm{Hom}(V,V)$ be a projection: $p^2 = p$ $\quad \sum_{s,t} st\otimes p_s p_t = \sum u\otimes p_u$

$$p_u = \sum_{s,t \; st=u} p_s p_t$$

nuclear = finite sum of elementary maps.

what do you propose? Let's start with a simple type of Morita equivalence. Let A be an algebra, unital to begin with, V a vector space say f.d. Put

$$E = V \otimes A$$
$$F = A \otimes V^* \qquad B = E \otimes_A F = V \otimes \underbrace{(A \otimes_A A)}_{=A} \otimes V^*$$

$$\left(\begin{array}{c|c} A & A \otimes V^* \\ \hline V \otimes A & V \otimes A \otimes V^* \end{array}\right)$$ Morita context.

$$E \otimes_A F$$

now let $p \in \mathrm{Hom}_{A^{op}}(V \otimes A, V \otimes A) = \overbrace{V \otimes A \otimes V^*}$ at least for $\dim V < \infty$.

Then you get another Morita context.

$$\begin{pmatrix} A & (A \otimes V^*)p \\ p(V \otimes A) & p(V \otimes A \otimes V^*)p \end{pmatrix}$$

for this to yield a Morita equivalence you need $FE = A$

$$(A \otimes V^*)\overset{p}{\overbrace{p \cdot p}}(V \otimes A) = A \,.$$ So you need

$V^* p V$ to generate A as ideal. Now return to your group case. $p \in V \otimes A \otimes V^*$

$$\begin{tikzcd} E \arrow[r, "\alpha"] \arrow[rr, bend left, "id"] & V \otimes A \arrow[r, "\beta"] \arrow[rr, bend left, "p"] & E \arrow[r, "\alpha"] & V \otimes A \end{tikzcd}$$

$V \otimes$

$p \in V \otimes A \otimes V^*$ $\qquad E = p(V \otimes A) \qquad F = (A \otimes V^*)p$

$FE = (A \otimes V^*)p(V \otimes A) = A(\underbrace{V^*_p V}_{\text{components of } p})A$

condition $A(V^*_p V)A = A$, for example if $\overset{p}{A}$ is ?

Start again. You've confused things by using A twice. ~~In the example $A = P_F$ E~~

~~[illegible]~~ You need to distinguished the group ring $\mathbb{C}[\Gamma]$ from the ring P_F generated by the coeffs of p.

Start with the Morita equivalence

$R \qquad \cancel{R} \otimes \cancel{V}^*$

$\cancel{V}^* \otimes R^* \qquad \cancel{R} \otimes 1$

???

partition of 1

~~[illegible]~~ The basic

$R = \mathbb{C}[\mathbb{Z}] = \mathbb{C}[u, u^{-1}]$ Laurent polys.

~~[illegible]~~ f.g. free module $R \otimes V$ $\qquad \dim V < \infty$

$p \in \mathrm{Hom}_R(R \otimes V, R \otimes V) = \mathrm{Hom}(V, R \otimes V) = R \otimes V \otimes V^*$

what's tricky is that $\mathrm{Hom}_R(R, R) = R^{op}$

$R \xrightarrow{\cdot x} R \xrightarrow{\cdot y} R \qquad (\cdot y)(\cdot x) = (\cdot xy)$

So therefore you want to begin with $V \otimes R$ as R^{op} module

$\mathrm{Hom}_{R^{op}}(V \otimes R, V \otimes R) = \mathrm{Hom}(V, V \otimes R) = V \otimes R \otimes V^*$

So you go back to the M. eq.

$$\begin{pmatrix} R & R\otimes V^* \\ V\otimes R & V\otimes R\otimes V^* \end{pmatrix}$$

and let $p \in V\otimes R\otimes V^*$ $\quad p^2 = p$.

$$\begin{pmatrix} R & \overset{F}{(R\otimes V^*)p} \\ \underset{E}{p(V\otimes R)} & \underset{E\otimes_R F}{p(V\otimes R\otimes V^*)p} \end{pmatrix}$$

Your simplest example might be $R = \mathbb{C}[\mathbb{Z}]$.

How does this compare to your ~~original~~ original approach? Confusion reigns. Go back to P_F in the case $\Gamma = \mathbb{Z}$, $F = \{-1, 0, 1\}$, P_F is a ring defined by gens. + relations. gen p_s, $s \in \Gamma$ relations $p_s = 0$ $s \notin F$, $p_s = \sum_t p_{st^{-1}} p_t$. Let V be a left P_F-module

$R = \mathbb{C}[\mathbb{Z}]$. Identify $V\otimes R\otimes V^*$ with the ring $\mathrm{End}(V)\otimes R$ as follows.

$$(v\otimes r\otimes \lambda)(v_1\otimes r_1)$$
$$= v\otimes r\lambda(v_1)r_1$$
$$= v\lambda(v_1)\otimes r r_1$$

$$v\otimes r\otimes\lambda \longmapsto v\lambda\otimes r$$

$$(v_1\otimes r_1\otimes\lambda_1),(v_2\otimes r_2\otimes\lambda_2) \qquad (v_1\lambda_1\otimes r_1),\ (v_2\lambda_2\otimes r_2)$$

$$\downarrow \qquad\qquad \downarrow$$

$$v_1\otimes r_1(\lambda_1 v_2) r_2\otimes\lambda_2 \qquad v_1\lambda_1 v_2\lambda_2\otimes r_1 r_2 \qquad ??$$

$$\longleftrightarrow v_1(\lambda_1 v_2)\lambda_2\otimes r_1 r_2$$

$\mathbb{C}[\mathbb{Z}] = \mathbb{C}[z, z^{-1}]$.

P_F generators p_n $n \in \mathbb{Z}$

rel. $p_n = 0$ $|n| > N$ $\qquad F = \{n \mid |n| \le N\}$.

$$p_n = \sum_j p_j p_{n-j}$$

$p(z) = \sum_j z^j p_j$ is a Laurent poly w coeffs in P_N idempotent. $\qquad p(z) \in \mathbb{C}[z, z^{-1}] \otimes P_F$

Check that the ring $(V \otimes R) \otimes_R (R \otimes V^*) = V \otimes R \otimes V^*$ with product $(v_1 \otimes r_1 \otimes \lambda_1)(v_2 \otimes r_2 \otimes \lambda_2) = v_1 \otimes r_1 \lambda_1(v_2) r_2 \otimes \lambda_2$ is isom. to the ring $(V \otimes V^*) \otimes R$ via

$$v \otimes r \otimes \lambda \xmapsto{\theta} v \otimes \lambda \otimes r$$

$$\begin{array}{ccc}
v_1 \otimes r_1 \otimes \lambda_1 \quad v_2 \otimes r_2 \otimes \lambda_2 & \xmapsto{\text{prod}} & v_1 \otimes r_1 \lambda_1(v_2) r_2 \otimes \lambda_2 \\
\downarrow \theta \qquad\qquad \downarrow \theta & & \downarrow \theta \\
v_1 \otimes \lambda_1 \otimes r_1 \quad v_2 \otimes \lambda_2 \otimes r_2 & & v_1 \otimes \lambda_2 \otimes r_1 \lambda_1(v_2) r_2 \\
\downarrow \text{prod} & & \parallel \\
\underbrace{v_1 \otimes \lambda_1(v_2) \lambda_2 \otimes r_1 r_2} & & v_1 \otimes \lambda_1(v_2) \lambda_2 \otimes r_1 r_2
\end{array}$$

$$p \in V \otimes R \otimes V^* = (V \otimes V^*) \otimes R$$

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So you learn that for $R = \mathbb{C}[\Gamma]$ that

$$p = \sum_s p_s \qquad p_s \in (V \otimes V^*) \otimes s \qquad ?$$

Confusion still reigns. Let's check things carefully. Basic idea is to look at operators on free R^{op}-modules: $V \otimes R$ say $R = \mathbb{C}[\Gamma]$.

$$\mathrm{Hom}_{R^{op}}(V \otimes R, W \otimes R) = \mathrm{Hom}(V, W \otimes R)$$
$$= W \otimes R \otimes V^* \qquad \text{if } \dim(V) < \infty.$$

But there is something else involved when $R = \mathbb{C}[\Gamma]$, namely, the Γ grading.

You want to start with a standard free Morita context.

$$\begin{pmatrix} R & R\otimes V' \\ V\otimes R & V\otimes R\otimes V' \end{pmatrix}$$

V' a suitable dual for V.

then take $p = p^2$ in $V\otimes R\otimes V'$ to get

$$\begin{pmatrix} R & (R\otimes V')p \\ p(V\otimes R) & p(V\otimes R\otimes V')p \end{pmatrix}$$

Check that the ring $V\otimes R\otimes V'$ with mult.

$$(v_1\otimes r_1\otimes\lambda_1)(v_2\otimes r_2\otimes\lambda_2) = v_1\otimes r_1\langle\lambda_1, v_2\rangle r_2\otimes\lambda_2$$

is isomorphic to $(V\otimes V')\otimes R$ with mult.

$$(v_1\otimes\lambda_1\otimes r_1)(v_2\otimes\lambda_2\otimes r_2) = v_1\otimes\langle\lambda_1, v_2\rangle\lambda_2\otimes r_1 r_2$$

via θ $\quad v\otimes r\otimes\lambda \xrightarrow{\theta} v\otimes\lambda\otimes r$

$$\cancel{(v_1\otimes\lambda_1\otimes r_1)(v_2\otimes\lambda_2\otimes r_2)} \qquad v_1\otimes\lambda_2\otimes r_1\langle\lambda_1, v_2\rangle r_2$$

This seems OKAY. $\quad \bigoplus_{s} (V\otimes V')s$

$$p \in \overline{V\otimes V'\otimes R} \qquad R = \mathbb{C}[\Gamma]$$

so $\quad p = \sum p_s\otimes s$

$$p^2 = \sum_{s,t} (p_s\otimes s)(p_t\otimes t) = \sum_u \Big(\sum_{st=u} p_s p_t\Big)\otimes u$$

You should almost be there. Wait look at $A\otimes R = A\otimes \mathbb{C}[\Gamma] = \bigoplus_{s\in\Gamma} As \simeq \bigoplus_{s\in S} A$

where an element is a ~~function~~ family $(a_s)_{s\in S}$ of finite support such that $(ab)_u = \sum_{st=u} a_s b_t$. Put another way a Γ-graded alg ~~B~~ B is ~~an~~ alg. equ. w. ~~the~~ splitting $B = \bigoplus_s B_s$ such that $B_s B_t \subset B_{st}$

What's interesting suppose ~~[illegible]~~ V is a P_F-module, ~~&~~ equiv. $V\otimes R$ ~~[illegible]~~ is given a projection p, then you get $E(V) = p(V\otimes R)$ exact in V, hence $E(V) \leftarrow E(\tilde{A})\otimes_A V = E(A)\otimes_A V$? $A = P_F$

If W is a right P_F^A module

$F(W) = p(R\otimes W) \xleftarrow{\sim} F(A$?,

V a P_F-module, $P_F = A$, $E(V) = p(V\otimes R)$ exact in V, so $E(V) = E(\tilde{A})\otimes_A V = E(A)\otimes_A V$.

Similarly for a A^{op}-module W, put $F(W) = (R\otimes W)p$ then $F(W) = W\otimes_A F(\hat{A}) = W\otimes_A F(A)$.

$$\begin{pmatrix} R & \overset{F}{\overset{\shortparallel}{(R\otimes A)p}} \\ \underset{\overset{\shortparallel}{E}}{p(A\otimes R)} & p(A\otimes R\otimes A)p \end{pmatrix}$$

$FE = (R\otimes A)p(A\otimes R)$ work this out in your example.

Fix Γ, F get P_F ~~nonunital alg~~ nonunital alg, gen p_s, $s \in \Gamma$ + relations.
If V a P_F module get

$$E(V) = p(V \otimes R) \quad \text{exact rt cont. in } V$$

$$\therefore \quad E(V) = E(\tilde{P}_F) \otimes_{P_F} V = E(P_F) \otimes_{P_F} V$$

dually, also for P_F^{op}-modules W get

$$F(W) = (R \otimes W)p = W \otimes_{P_F} F(P_F)$$

Put $A = P_F$

$$E(V) = E(A) \otimes_A V = p(V \otimes R)$$

$$F(W) = W \otimes_A F(A) = (R \otimes W)p$$

$$E(V) = E(A) \otimes_A V = p(A \otimes R) \otimes_A V$$

$$F(W) = W \otimes_A F(A) = W \otimes_A (R \otimes A)p$$

$$\begin{pmatrix} A & F = (R \otimes A)p \\ E = p(A \otimes R) & p(A \otimes R \otimes A)p \end{pmatrix} \qquad FE = (R \otimes A)p(A \otimes R)$$

V an $A = P_F$ module, get p on $V \otimes R$

so get

$$E \xrightarrow{\alpha} V \otimes R \xrightarrow{\beta} E \xrightarrow{\alpha} V \otimes R$$

$$V \otimes R = \bigoplus_s V \otimes s \qquad \iota_s j_s \text{ proj. onto } V \otimes s.$$

$$1_E = \sum_s \beta \iota_s j_s \alpha = \sum_{s \in \Gamma} s \, \underbrace{(\beta \iota_1 j_1 \alpha)}_{h_1} s^{-1}$$

$$\beta\iota_s \in \mathrm{Hom}(V, E), \quad j_s\alpha \in \mathrm{Hom}(E, V), \quad \beta\iota_s j_s \alpha = h_s$$

$\cancel{p}$ $\qquad j_s p \iota_t = j_s \alpha\beta \iota_t = j_1 s^{-1}\alpha\beta t \iota_1$

Organize. $R = \mathbb{C}[\Gamma]$, V vector space, $V\otimes R$ = free R^{op}-module gen by V.

isom if $\dim(V) < \infty$.

$$\operatorname{Hom}_{R^{op}}(V\otimes R, V\otimes R) = \operatorname{Hom}(V, V\otimes R) \hookleftarrow End(V)\otimes R$$

Note that $R \xrightarrow{\sim} \operatorname{Hom}_{R^{op}}(R,R)$

$r \longmapsto r\cdot$

$$(rs)\cdot x = (r\cdot)((s\cdot)x)$$

$$R \xrightarrow{s\cdot} R \xrightarrow{r\cdot} R$$

$$x \longmapsto sx \longmapsto rsx$$

$$(r\cdot)(s\cdot) = (rs)\cdot$$

Yes, very clear!

$$R = \operatorname{Hom}_{R^{op}}(R,R) \longrightarrow \operatorname{Hom}_{R^{op}}(V\otimes R, V\otimes R)$$

$\therefore$

$$\boxed{\operatorname{Hom}(V,V)\otimes R \longrightarrow \operatorname{Hom}_{R^{op}}(V\otimes R, V\otimes R) \quad \text{basic homo}}$$

So what game do you want to play.

$$p = \sum p_s \otimes s \in$$

$p \in End(V)\otimes R \qquad p^2 = p$

$\downarrow$

$End_{R^{op}}(V\otimes R)$ so you get $\overline{p(V\otimes R)}$ Yes!!!

So what gives? You do not think V f.d. is important. Important the maps arising from

$$p(V\otimes R) \xhookleftarrow{\alpha} \underset{\oplus V's}{V\otimes R} \xrightarrow{\beta} p(V\otimes R)$$

interesting goals? Fix Γ, F so as to have a definite ring $A = P_F$. Then for any A-module V you get $E(V) = p(V \otimes R)$ ~~a right~~ a proj R^{op}-module. You get an exact functor $\mathrm{Mod}(\tilde{A}) \longrightarrow \mathrm{Mod}(R^{op})$

$$V \longmapsto E(A) \otimes_A V.$$

$2 \times 310 \,|\,$. 210, 20, 300, 530   620

V a module for $A = P_F$. This means there is a canon. $p \in A \otimes R$ s.t. $p^2 = p$, ~~[illegible]~~ so get p on $V \otimes R$.

$$\mathrm{Hom}(V,V) \otimes R \longrightarrow \mathrm{Hom}_{R^{op}}(V \otimes R, V \otimes R)$$

$$\varphi \otimes r \longmapsto (\varphi \otimes r)(v \otimes r_1) = \varphi v \otimes r r_1$$

What to do? Ultimately you want to setup a Mor. equiv. between $A = P_F$ and some alg of operators on $E = p(A \otimes R)$. At the moment we have on E the operator $h_1 = \beta \iota_1 \delta_1 \alpha$ and conjugates $h_s = s h_1 s^{-1} = \beta \iota_s \delta_s \alpha$ which add to id_E. Lets try the following, a version of $h_1^{1/2}$. Formally factor h_1. Better to assume $h_1 : E \longrightarrow E$ is nuclear, then choose a V and fact.

$$\begin{tikzcd} E \arrow[dr, "\alpha_1"'] & & E \\ & V \arrow[ur, "\beta_1"'] & \end{tikzcd}$$

, then extend

$$E \longrightarrow V \otimes R \longrightarrow E$$

Start again. ~~from that view point~~ To describe finite projective right $R = \mathbb{C}[\Gamma]$-modules. Same as projections on $V \otimes R$ for V fin. dim.

Let E be an R^{op}-module.

$$E \otimes_R \mathrm{Hom}_{R^{op}}(E, R) \longrightarrow \mathrm{Hom}_{R^{op}}(E, E)$$

$$\xi \otimes \varphi \longmapsto \left(\xi' \longmapsto \xi\varphi(\xi')\right)$$

Then E is finite projective R^{op}-module $\iff$ $\exists\, \xi_1, \dots, \xi_n \in E$ $\varphi_1, \dots, \varphi_n \in E^*$ such that $\sum_j \xi_j \otimes \varphi_j \longmapsto id_E$

$$\sum_j \xi_j \varphi_j(\xi) = \xi \qquad \forall\, \xi \in E$$

OKAY

$$E \xrightarrow{\begin{pmatrix}\varphi_1 \\ \vdots \\ \varphi_n\end{pmatrix}} R^n \xrightarrow{(\xi_1, \dots, \xi_n)} E$$

So what's going on? Go back to

$$\begin{tikzcd} E \arrow[r, "\alpha"] & V \otimes R \arrow[r, "\beta"] \arrow[d, shift right, "j_1"'] & E \\ & V \arrow[u, shift right, "i_1"'] & \end{tikzcd}$$

The point is that if V is f.d. then

$$E \xrightarrow{j_1 \alpha} V \xrightarrow{\beta i_1} E \qquad \text{is nuclear.}$$

So now start with an R^{op}-module E equipped with a nuclear map h such that

Let's do more. Let E be a fin. proj $R = \mathbb{C}[\Gamma]$-module

Then

$$\mathrm{Hom}_R(E, R) \otimes_R M \xrightarrow{\sim} \mathrm{Hom}_R(E, M)$$

$$\varphi \otimes m \longmapsto (\xi' \longmapsto \varphi(\xi') m)$$

taking $M = E$, choosing $\sum_{j=1}^{n} \varphi_j \otimes \xi_j \mapsto \sum_1^n \varphi_j(\xi) \xi_j = \xi$

you get

$$E \xrightarrow{(\varphi_j)} R^n \xrightarrow{(\xi_j)} E$$

factorization of the identity. Now $R^n = R \otimes V \qquad V = \mathbb{C}^n$.

$$R \otimes V = \bigoplus_{s \in \Gamma} sV$$

specifically you have $\iota_s : V \xrightarrow{\sim} sV \qquad \iota_s = s\iota_1$

$j_s : sV \xrightarrow{\sim} V \qquad j_s = j_1 s^{-1}$

$$E \xrightarrow{\alpha} R \otimes V \xrightarrow{\beta} E \qquad \begin{aligned} \beta\alpha &= id_E \\ \alpha\beta &= p \end{aligned}$$

where

p is ~~a proj~~ an idempotent endo of the R-module $R \otimes V$.

$$id_E = \beta \sum_{s \in \Gamma} \iota_s j_s \alpha = \sum_{s \in \Gamma} s (\beta \iota_1 j_1 \alpha) s^{-1}$$

thus $\beta \iota_1 j_1 \alpha : E \longrightarrow V \longrightarrow E$ is a nuclear map of E as a vector space such that $\sum s(\beta\iota_1 j_1 \alpha) s^{-1} = id_E$

$$p \in \mathrm{Hom}_R(R \otimes V, R \otimes V) = \mathrm{Hom}(V, R \otimes V) = R^{op} \otimes \mathrm{Hom}(V, V)$$

If $r \in R$ and $\theta \in \mathrm{Hom}(V,V)$, ~~then~~ let $\rho_r(x) = xr \quad x \in R$

$$(r \otimes \theta)(x \otimes v) = (\rho_r \otimes \theta)(x \otimes v) = xr \otimes \theta(v).$$

$$(\rho_r \rho_{r'})(x) = xr'r = \rho_{r'r}(x).$$

$$\therefore \quad p = \sum_{s\in\Gamma} s \otimes p_s \qquad p_s \in \mathrm{End}(V)$$

Look carefully at $\underbrace{R^{op} \otimes \overbrace{\mathrm{End}(V)}^{C}}_{B} = \bigoplus_{s\in\Gamma} s \otimes C$

$B = \bigoplus_s B_s$ where $B_s = \cdot s \otimes C$ and

$$(s_1 \otimes c_1)(s_2 \otimes c_2) = s_2 s_1 \otimes c_1 c_2$$

$\therefore \quad B_s B_t$ in this case is B_{ts}

$$p = \sum_{s\in\Gamma} s \otimes p_s \in R^{op} \otimes \mathrm{End}(V)$$

then $p(t\otimes v) = \sum_{s\in\Gamma} (\rho_s \otimes p_s)(t\otimes v) = \sum_{s\in\Gamma} ts \otimes p_s v$

So it might be natural to change the notation.

Do the preceding with right R-modules.

$$E \otimes_R \mathrm{Hom}_{R^{op}}(E,R) \longrightarrow \mathrm{Hom}_R(E,E)$$
$$\xi \otimes \varphi \longmapsto (\xi_1 \longmapsto \xi\,\varphi(\xi_1))$$

$$E \xrightarrow{\alpha} V \otimes R \xrightarrow{\beta} E$$
$$V\otimes R = \bigoplus_s V \otimes s$$

?

$$\operatorname{Hom}_R(R\otimes V, R\otimes V) \overset{\text{alg isom}}{=} R^{op}\otimes \operatorname{End}(V)$$

$$\boxed{p = \sum_{s\in\Gamma} s^{-1}\otimes p_s} \qquad \text{defines } p_s$$

$$p^2 = \sum_s \sum_t (s^{-1}\otimes p_s)(t^{-1}\otimes p_t)$$

$$= \sum_{s,t} \underbrace{\cancel{(ts)^{-1}}}_{\substack{t^{-1}s^{-1} \\ \| \\ (st)^{-1}}} \otimes p_s p_t = \sum_{s,t} (st)^{-1}\otimes p_s p_t$$

$$= \sum_u u^{-1} \sum_{\substack{s,t \\ st=u}} p_s p_t \qquad \therefore \quad p_u = \sum_{\substack{s,t \\ st=u}} p_s p_t$$

$$E \xrightarrow{\alpha} R\otimes V \xrightarrow{\beta} E$$

$$j_1 s^{-1} = j_s \downarrow \ \uparrow \iota_s = s\iota_1 \qquad V$$

$$id_{R\otimes V} = \sum_s \iota_s j_s = \sum_s s\iota_1 j_1 s^{-1}$$

$$id_E = \sum_s s(\beta\iota_1 j_1 \alpha)s^{-1}$$

$$h_1 = \beta\iota_1 j_1\alpha \qquad h_s = [redacted]\ s h_1 s^{-1}$$

$$h_1 h_s = \beta\iota_1 \boxed{j_1\alpha s\beta\iota_1} j_1 \alpha s^{-1}$$

$$\iota_1 = 1\otimes id_V$$

$$j_1 = \underbrace{\text{proj on } 1}_{\pi_1} \otimes id_V$$

$$j_1\alpha \cancel{t}\beta\iota_1 = j_1 \cancel{t} p\,\iota_1 = j_1 t\left(\sum_s s^{-1}\otimes p_s\right)\iota_1$$

$$= \sum_s \pi_1 t s^{-1}\iota_1 \otimes p_s = p_t$$

$$u = ts^{-1}$$

$$s^{-1}u^{-1}t = 1$$

$$j_s\alpha\beta\iota_t = j_1 s^{-1}\left(\sum_u u^{-1}\otimes p_u\right)t\,\iota_1 = p_{ts^{-1}}$$

Right R modules. $\mathrm{Hom}_{R^{op}}(V\otimes R, V\otimes R) = \mathrm{End}(V)\otimes R$

$$p = \sum p_s \otimes s \in \mathrm{End}(V)\otimes R$$

$$p^2 = \sum_{s,t} p_s p_t \otimes st = \sum_u p_u \otimes u \implies p_u = \sum_{\substack{s,t\\ st=u}} p_s p_t$$

$$\begin{tikzcd} E \arrow[r, hook, "\alpha"] & V\otimes R \arrow[r, "\beta"] \arrow[d, shift right, "j_1"'] & E \\ & V \arrow[u, shift right, "\iota_1"'] & \end{tikzcd}$$

You certainly have

$$id_{V\otimes R} = \sum_s \boxed{\iota_s j_s}$$

inclusions + proj in degree s

$$\therefore 1_E = \sum_{s\in\Gamma} \beta\, \iota_s j_s \alpha$$

$$\boxed{\begin{aligned} \iota_s &= \rho_s \iota_1 \\ j_s &= j_1 \rho_{s^{-1}} \end{aligned}}$$

$$\iota_s v = v\otimes s = \rho_s(v\otimes 1) = \rho_s \iota_1 v$$

$$\cancel{j_s \rho_t(v\otimes 1) = j_s(v\otimes t) = \delta_{st^{-1}} v = j_{st}(v\otimes 1)}$$

$$\therefore \cancel{j_s \rho_t = j_s \rho_t = j_{st}}$$

$$j_s \sum v_t \otimes t = v_s$$

$$j_s \rho_u \sum v_t \otimes t = j_s \sum_t v_t \otimes \overset{s}{\overset{\shortparallel}{tu}} = v_{su^{-1}}$$

$$= j_s \sum v_{tu^{-1}} \otimes t = v_{su^{-1}} = j_{su^{-1}} \sum v_t \otimes t$$

$$j_s \rho_u = j_{su^{-1}}$$

$$\boxed{\iota_s = \rho_s \iota_1}$$

$$j_s = \cancel{}\ j_{su^{-1}} \rho_{u^{-1}}$$

$$\boxed{j_s = j_1 \rho_{s^{-1}}}$$

$$j_1 \rho_u = j_{u^{-1}}$$

$$\boxed{j_u = j_1 \rho_{u^{-1}}}$$

$$1_E = \sum_s \rho_s \left(\beta \iota_1 j_1 \alpha\right) \rho_s^{-1}$$

Review. $R = \mathbb{C}[\Gamma]$ looking at ~~the~~ free fin. free R-module $R \otimes V$, $\dim(V) < \infty$.

$$\operatorname{End}_R(R \otimes V) = R^{op} \otimes \operatorname{End}(V) \qquad \Gamma^{op} \text{ graded alg.}$$

$$\qquad\qquad\qquad\qquad s \otimes \theta$$

$$(s \otimes \theta)(r \otimes v) = \overbrace{\rho_s r}^{rs} \otimes \theta v \qquad \rho_s \rho_t (r) = rts = \rho_{ts}(r)$$

$$p = p^2 \in R^{op} \otimes \operatorname{End}(V) \qquad p = \sum_{s \in \Gamma} \rho_{s^{-1}} \otimes p_s$$

$$\rho_{t^{-1}} \rho_{u^{-1}} = \rho_{u^{-1}t^{-1}} = \rho_{(tu)^{-1}}$$

$$p^2 = \sum_{t,u \in \Gamma} \underbrace{\rho_{t^{-1}} \rho_{u^{-1}}}_{\rho_{(tu)^{-1}}} \otimes p_t p_u = \sum_s \rho_{s^{-1}} \otimes \sum_{\substack{t,u \\ \ni\, tu = s}} p_t p_u$$

$$\therefore \quad p_s = \sum_t p_t p_{t^{-1}s}$$

So now you get $\quad E = p(R \otimes V)$

$$\begin{tikzcd} E \arrow[r, "\alpha"] & R \otimes V \arrow[r, "\beta"] \arrow[d, shift right, "j_1"'] & E \\ & V \arrow[u, shift right, "\iota_1 = 1 \otimes -"'] & \end{tikzcd}$$

Γ acts on R via ρ_s = right mult by s^{-1} ?

Let's try right R-module approach.

$$\operatorname{End}_{R^{op}}(R \otimes V) = R \otimes \operatorname{End}(V)$$

$$\lambda_r \otimes \theta \longleftarrow r \otimes \theta$$

λ_r = left mult by r.

$$\begin{tikzcd} E \arrow[r, "\alpha"] & V\otimes R \arrow[r, "\beta"] \arrow[d, shift right, "j_s"'] & E \\ & V \arrow[u, shift right, "\iota_s"'] & \end{tikzcd}$$

$$\mathrm{End}_{R^{op}}(V\otimes R) = \mathrm{End}(V)\otimes R$$
$$p = \sum p_s \otimes \lambda_s$$

$$\iota_s v = v\otimes s \qquad \iota_s v = \rho_s \iota_1 v$$

$$j_s\left(\sum v_s \otimes s\right) = v_s \qquad j_s v = j_1 \rho_{s^{-1}}(\xi)$$

$$1_E = \beta\left(\sum \iota_s j_s\right)\alpha = \sum \rho_s \underbrace{(\beta \iota_1 j_1 \alpha)}_{h_0} \rho_{s^{-1}}$$

$$1_E = \sum_{s\in\Gamma} \rho_s h_0 \rho_{s^{-1}}$$ [redacted]

need overlap to be finite

$$h_0 h_s = \beta\iota_1 \underbrace{\left[j_1 \alpha \rho_s \beta \iota_1\right]} j_1 \alpha \rho_{s^{-1}}$$

$$j_1 p \rho_s \iota_1 v = j_1\left(\sum_t p_t \otimes \rho_t\right)\rho_s (v\otimes 1)$$
$$= j_1 \sum_t (p_t\otimes\rho_t)(v\otimes s)$$
$$= j_1 \sum_t p_t v \otimes st = p_{s^{-1}}(v).$$

Better

$$h_s h_t = \beta \iota_s j_s \alpha \beta \iota_t j_t \alpha$$

$$j_s \alpha\beta \iota_t = j_s p \iota_t = \sum_u j_s(p_u\otimes\lambda_u)\iota_t$$

$\sum_u j_s(p_u\otimes\lambda_u)\iota_t$ sends v to $v\otimes t$ to $\sum_u p_u v \otimes \overline{ut}^{=s}$

to $p_{st^{-1}} v$

$$\boxed{j_s \alpha \beta \iota_t = j_s p \iota_t = p_{st^{-1}}}$$

~~so you conclude~~

so $h_s h_t = 0$ for $p_{st^{-1}} = 0$ $\therefore$ if $st^{-1} \notin F = \{s \mid p_s \neq 0\}$

Idea for the rest. Following Cuntz define an alg $\mathcal{E}_{\Sigma_F}$ to be generated by elements h_s $s \in \Gamma$ such that $h_s h_t = 0$ for $st^{-1} \notin F$

Program. First discuss finite projective modules over $\mathbb{C}[\Gamma]$. This discussion should amount to the standard Morita theory over a unital ring, but modified by the fact the left and right module categories are equivalent in a canonical way. Need to define a pairing between two Γ-modules with values $\mathbb{C}[\Gamma]$, to define dual pair $(E, F, F \times E \to R)$.

Things might look different?

free case

$$\begin{pmatrix} R & R \otimes V^* \\ V \otimes R & V \otimes R \otimes V^* \end{pmatrix}$$

f. proj case

$$\begin{pmatrix} R & (R \otimes V^*)p \\ p(V \otimes R) & p(V \otimes R \otimes V^*)p \end{pmatrix}$$

$$p \in V \otimes R \otimes V^* = \mathrm{End}(V) \otimes R \cong \mathrm{End}_{R^{op}}(V \otimes R) \quad \dim V < \infty$$

What does the general unital case look like?

$V \otimes R = \begin{pmatrix} R \\ \vdots \\ R \end{pmatrix}$ right R module of column vectors

$R \otimes V^* = (R \cdots R)$ left R-mod. of row vectors

~~[illegible]~~ At this point I do not see anything interesting arising in this group ring situation. Wait. Let E be $\mathbb{C}[\Gamma]^{op}$-mod, F be $\mathbb{C}[\Gamma]$-mod. + form $E \otimes_R F$. Do you have different tensor products?

Let E, F be Γ-modules, i.e. vector spaces with a left action of Γ. Then $E \otimes F$ has a natural $\Gamma \times \Gamma$ action. So ??

Review the situation. You are studying the K-theory of a group ring $R = \mathbb{C}[\Gamma]$, that is, fin. projective modules over R, say right modules. The finite free modules ~~have~~ form $V \otimes R$ with $\dim V < \infty$

$$\mathrm{End}_{R^{op}}(V \otimes R) = \mathrm{End}(V) \otimes R \qquad \lambda_r = (r\cdot) : R \to R$$
$$\theta \otimes \lambda_r \longleftarrow \theta \otimes r$$

and a finite proj. R^{op} module has the form $E \simeq p(V \otimes R)$ for such a V, with $p = p^2 \in \mathrm{End}(V) \otimes R$.

Usual Morita picture

$$\begin{pmatrix} R & R \otimes V^* \\ V \otimes R & V \otimes R \otimes V^* \end{pmatrix} \qquad \begin{pmatrix} R & (R \otimes V^*)p \\ p(V \otimes R) & p(V \otimes R \otimes V^*)p \end{pmatrix}$$

Natural question is whether for group rings it's always true that $R = R(V^* p V)R$. What about $\mathbb{Z}$? Here R = Laurent polys on S^1, this is a PID, so the module E is ~~[illegible]~~ free and $\neq 0$. etc.

Now what's ~~[illegible]~~ next? So far you have looked at R-linear operators, but you want to enlarge the ring of operators ~~in~~ as follows: For the free module $V\otimes R$ you have $\mathbb{C}$-linear maps

$$\begin{tikzcd} V\otimes R = \bigoplus_{s\in\Gamma} V\otimes s \arrow[d, shift right, "j_s"'] \\ V \arrow[u, shift right, "\iota_t"'] \end{tikzcd}$$

$$\iota_t(v) = v\otimes t$$

$$j_s\left(\sum v_u \otimes u\right) = v_s$$

$V\otimes R$ is naturally Γ-graded $\quad V\otimes R = \bigoplus_{s\in\Gamma} V\otimes s$

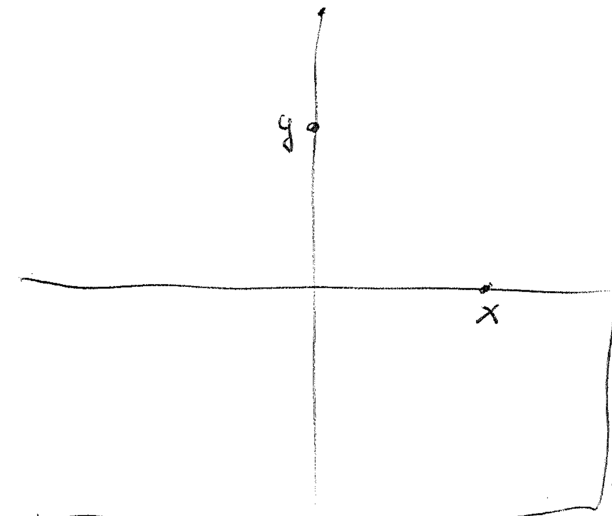
and $t(V\otimes s) = V\otimes ts$. $\quad M = V\otimes R, \; M_s = V\otimes s$

$tM_s \subset M_{ts}$; ~~so~~ $V\otimes R$ is naturally a Γ-graded ~~[illegible]~~ left Γ-module

So what next? For $E = p(V\otimes R)$ you want the operators $\beta\iota_t j_s\alpha \in \mathrm{Hom}_{\mathbb{C}}(E,E)$

$$\begin{tikzcd} E \arrow[r, "\alpha"] & V\otimes R \arrow[r, "\beta"] \arrow[d, shift right, "j_s"'] & E \\ & V \arrow[u, shift right, "\iota_t"'] & \end{tikzcd}$$

These are nuclear operators (finite ~~$\mathbb{C}$~~ rank) but more. In terms of the grading $V\otimes R = \bigoplus_s V\otimes s$ these operators are finite support matrices. ~~Better to say that the operators [illegible] lie in~~. Can describe as diagonal finite supp ~~$\mathrm{End}(V)$~~ $\rtimes \Gamma$.



Get back to your program. The aim is to produce a Morita equivalence between P_F and $\mathcal{E}_{\Sigma_F} \rtimes \Gamma$. Given a P_F module V you get a projection p on the free module $V\otimes R$, and then a proj. R^{op}-module $E(V) = p(V\otimes R)$. This should be the basic operation going from P_F-modules to $\mathcal{E}_{\Sigma_F}\rtimes\Gamma$-modules. Formulas

$$E \xrightarrow{\alpha} V\otimes R \xrightarrow{\beta} E \qquad \beta\alpha = id_E$$

Note $V\otimes R$ is a Γ-module: $t(v\otimes s) = v\otimes ts$ and is Γ-graded $V\otimes R = \bigoplus_s V\otimes s$, compatibly condition $R_t(V\otimes R)_s = \mathbb{C}t(V\otimes s) = V\otimes ts = (V\otimes R)_{ts}$.

You get natural algebra of operators on $V\otimes R$ as vector space, namely

$\hat\Gamma$-operators: $\mathbb{C}[\hat\Gamma] = \bigoplus_{s\in\Gamma}\mathbb{C}e_s$ functions on Γ with fin. support

Γ-operators, form cross product $\mathbb{C}[\hat\Gamma]\rtimes\Gamma$

combine these to finite support operators on $\bigoplus_{s\in\Gamma}\mathbb{C}s = R$.

So on $V\otimes R$ you expect $(V\otimes V^*)\otimes\bigoplus_{s,t}\mathbb{C}e_s\otimes t$

Where are you now? At this point on R you have operator of left mult by R and all finite support matrices relative to the grading $R=\bigoplus_s \mathbb{C}s$

$$1_E = \beta\alpha = \sum \beta e_s\alpha = \sum s(\beta e_1\alpha)s^{-1}$$

now you have an obvious candidate for $h_0^{1/2}$ name $\beta e_1\ e_1\alpha$

So things should be [redacted] a lot clearer. What is your next step? You need ~~to~~ still to specify the ring of operators [redacted] on E. ~~What~~ Ring of Toeplitz operators? Probably the cross product, hopefully ~~with~~ the condition of firmness will then give you the action of Γ.

Review: ~~Ot~~ You have to understand the free module case first. $R = \mathbb{C}[\Gamma]$, $E = V \otimes R$, $\dim(V) < \infty$, E is a right R-module, its ~~$\notin$~~ ring of endomorphisms is $\mathrm{End}(V) \otimes R$, ~~action~~ tensor product algebra, action $(\theta \otimes t)(v \otimes s) = \theta v \otimes ts$. other point is the $\hat{\Gamma}$-action, the Γ-grading, ~~$\mathrm{End}(V)\otimes R = \bigoplus \mathrm{End}$~~

$$A \otimes R = \bigoplus_{s \in \Gamma} (A \otimes R)_s \qquad (A \otimes R)_s = A \otimes s$$

$$R_t \cdot (A \otimes R)_s = \underline{(\mathbb{C}t)(A \otimes s)} = A \otimes ts = (A \otimes R)_{ts} \qquad \emptyset$$

~~So $\mathrm{End}\, V \otimes R$ is a $E = V \otimes R$~~

first R is Γ-graded (has a $\hat{\Gamma}$ action) algebra

$V \otimes R$ is Γ-graded, is a Γ-graded R-~~algebra~~ module

$\underline{\mathrm{End}\, V} \otimes R$ ——— ——————— algebra

You are looking for ~~an~~ a natural algebra of operators on R which should contain ~~the~~ left mult by Γ and action of $\hat{\Gamma}$, i.e. projections e_s on R, (diagonal matrices and shifts)

Review: Consider $\Gamma = \mathbb{Z}$ $\quad R = \mathbb{C}[\mathbb{Z}]$. You are looking for a good ring of operators on $\mathbb{C}[\mathbb{Z}]$.

$\Gamma = \mathbb{Z}$, $F = \{-1, 0, 1\}$,

P_F is nonunital + commutative, gen. p_{-1}, p_0, p_1 relns are $p_{-1}^2 = 0$, $p_{-1}p_0 + p_0 p_{-1} = p_{-1}$, $p_{-1}p_1 + p_1 p_{-1} + p_0^2 = p_0$, $p_1 p_0 + p_0 p_1 = p_1$, $p_1^2 = 0$. This ring has a spectrum, a variety in $\mathbb{C}^3$. No this ring is not ~~commutative~~, only isomorphic to its opposite. Suppose you divide by the ideal gen. by commutators to get the comm. ring with same gen. + relns. Look at spectrum. $(p_{-1} + p_0 + p_1)^2 = p_{-1} + p_0 + p_1$

but $p_{-1}^2 = 0 \Longrightarrow p_{-1} = 0$ $\quad p_1 = 0$ $\quad p_0 = 1$ or 0.

Repeat. Take $\Gamma = \mathbb{Z}$, $\Phi = \{-1, 0, 1\}$, let $A = P_\Phi$, ~~then~~ ~~given an A-module V there is a canonical $p \in$ $V\otimes R$~~

then given an A-module V, i.e. homom. (nonunital) $A \longrightarrow End(V)$, you get from the universal idemp. $p \in A \otimes R$ an idem $p \in End(V) \otimes R \longrightarrow End_{R^{op}}(V \otimes R)$, whence get $E(V) = p(V \otimes R)$ and maps of R^{op}-modules.

$$E(V) \xrightarrow{\alpha} V \otimes R \xrightarrow{\beta} E(V) \qquad \beta\alpha = id$$

α, β commute with ρ_s $\forall s \in \Gamma$. Note that $E(V)$ is a right R-modules, but $V \otimes R$ has ~~both~~ R-bimodule structure. You have ρ_s already on $E(V)$. Next you want the Γ-grading projections e_s on $V \otimes R$ $\quad V \otimes R = \bigoplus_t V \otimes t$ and e_s is the map $e_s(\sum_t v_t \otimes t) = v_s$

~~[illegible]~~ Something is ~~[illegible]~~ wrong ~~[illegible]~~ or confusing.

Start again. The basic construction assigns to idemp $p \in End(V) \otimes R$, the R^{op} module $E = p(V \otimes R)$ which occurs in

$$p(V\otimes R) \xrightarrow{\alpha} V \otimes R \xrightarrow{\beta} p(V\otimes R), \qquad \beta\alpha = id$$

one gets a functor from $A = P_F$-modules V to R^{op}-modules $E(V) = p(V\otimes R)$ which should have the form $E(V) = E(\tilde{A}) \otimes_A V = E(A) \otimes_A V$

What is peculiar about this? For the purpose of Morita equivalence you want an ?, A bimodule

$$E(A) = p(A \otimes R)$$

Pick Γ, $\Phi \subset \Gamma$ $\quad$ Φ finite, contains 1, $\Phi^{-1} = \Phi$.

$$\operatorname{End}_{R^{op}}(V \otimes R) = End(V) \otimes R \qquad p = p^2 \in End(V) \otimes R$$
$$\in End(V) \otimes \mathbb{C}[\Phi]$$

$$p = \sum_{s \in \Phi} p_s \otimes s$$
$$p(v \otimes r) = \sum_s p_s v \otimes rs$$

$$E = p(V\otimes R) \xrightarrow{\alpha} V \otimes R \xrightarrow{\beta} p(V\otimes R) = E$$

$A = P_\Phi$ $\quad$ universal idemp $p = p^2 \in A \otimes R$, $p \in A \otimes \mathbb{C}[\Phi]$.

~~[illegible]~~ $\quad V \longmapsto p(V \otimes R) = E(V)$ $\quad$ V an A module

exact functor rt out, ~~[illegible]~~ $\quad p(V\otimes R) = p(A \otimes R) \otimes_A V$

Point is that $A \xrightarrow{\cdot a} A$ is an A-mod. map, so induces $p(A\otimes R) \longrightarrow p(A \otimes R)$, making $p(A \otimes R)$ a right A, right R - bimodule

$$A \left| \begin{array}{lll} \text{gen} & p_s & \text{for } s \in \Gamma \\ \text{reln} & p_s = 0 & \text{for } s \notin \Phi \\ & p_s = \sum_t p_{st^{-1}} p_t & \end{array} \right.$$

$\sum p_s \otimes s$: $A \otimes R \longrightarrow A \otimes R$ $\qquad p(a \otimes t) = \sum_s p_s a \otimes st$

$$E(A) = \Big(\sum_{s \in \Phi} p_s \otimes s\Big)(A \otimes R)$$ naturally an $(A \otimes R)^{op}$ module

$$\begin{tikzcd} E(A) \arrow[r, "\alpha"] \arrow[rr, bend left, "1"] & A \otimes R \arrow[r, "\beta"] & E(A) \end{tikzcd}$$

So we now see a right $A \otimes R$ - module structure on $E(A)$. You notice also that $E(A) =$ the right ideal $p(A \otimes R)$ inside $A \otimes R$. Next you want to look at $A \otimes R$ noncunital ring because A is noncunital. Confusion is diminishing. ~~Then~~

You want control over the operators on $E(A)$ as right $A \otimes R$ module. Yes. I think you want to ~~examine~~ examine "nuclear" operators on $A \otimes R$

Back to $V \otimes R$ for a general A-module V

$E = E(V)$

$$\begin{tikzcd} E \arrow[r, "\alpha"] & V \otimes R \arrow[r, "\beta"] \arrow[loop below, "e_s"] & E \end{tikzcd}$$

$$\begin{tikzcd} E(A) \arrow[r, "\alpha"] & A \otimes R \arrow[r, "\beta"] \arrow[loop below, "e_s"] & E(A) \\ & A \arrow[u, dotted, shift left] \arrow[u, dotted, leftarrow, shift right] & \end{tikzcd}$$

Next stage. $R = \mathbb{C}[\Gamma]$, A arb. algebra (nonunital)
can form $A \otimes R$ tensor product alg. Let $p \in A \otimes R$
be idempotent, let $p = \sum_{s \in \Gamma} p_s \otimes s$ be the homog. components
of p wrt the Γ-grading. Let $\Phi = \{s \mid p_s \neq 0\}$ be the Supp p.
Note that $p^2 = p$ means $p_s = \sum_t p_{st^{-1}} p_t$.

~~An idempotent p in a Γ-graded algebra is the same as a family~~

Form $E = p(A \otimes R)$, right ideal in
$A \otimes R$ gen. by idemp. p; E is a right $A \otimes R$-module
with module maps

$$E \xrightarrow{\alpha} \underset{\cup}{A \otimes R} \xrightarrow{\beta} E \qquad \tilde{A} \otimes R \longrightarrow E$$

such that $\beta\alpha = 1_E$. Something's happened here! What?

$$p = \sum_{s \in \Phi} p_s \otimes s \in A \otimes R \subset \tilde{A} \otimes R$$

$$\sum_s p_s \tilde{A} \otimes R = \sum_s p_s A \otimes R = p(A \otimes R)$$

$$\overbrace{p(A \otimes R)}^{E} \xrightarrow{\alpha} A \otimes R \subset \tilde{A} \otimes R \xrightarrow{\beta} p(\tilde{A} \otimes R) = E$$

So perhaps you want to write

$$E \overset{\alpha}{\hookrightarrow} A \otimes R \subset \tilde{A} \otimes R \overset{\beta}{\twoheadrightarrow} E \qquad \begin{array}{l} \beta\alpha = 1_E \\ \alpha\beta = p : \tilde{A} \otimes R \to A \otimes R \end{array}$$

Maybe focus on the extension $A \otimes R \subset \tilde{A} \otimes R \longrightarrow R$
and the idempotent p in $A \otimes R$. Is there a
chance this resembles Toeplitz algebra?

So what happens? next, What
should you recall? Go back
to $p = p^2 \in \mathrm{End}(V) \otimes R \longrightarrow \mathrm{End}_{R^{op}}(V \otimes R)$ for V an A-module

$E(V) = p(V \otimes R) = p(A \otimes R) \otimes_A V$. Your hope is
to find a natural ring of operators on $E(V)$ for all A-mods

so you want to produce A^{op}-linear operators on $p(A\otimes R)$. Method: use

$$E(A) \xrightarrow{\alpha} \underbrace{A\otimes R}_{\parallel} \xrightarrow{\beta} E(A)$$

$$\bigoplus_t A\otimes t \qquad e_s\left(\sum a_t\otimes t\right) = a_s\otimes s$$

So you have $\hat{\Gamma}$ action operators e_s

α, β are right $A\otimes R$ module maps.

e_s is (left + right A-linear

$\boxed{\text{left mult by } s \text{ is right } A\otimes R\text{-linear}}$

Go over stuff again. Fix Γ and Φ so that there's a fixed non-unital alg $A = P_{\Phi}$ universal for $p \in A\otimes R$ such that $p^2 = p$ and $p \in A[\Phi]\otimes R$

First claim should be that any A-module V gives rise to $E(V) = p(V\otimes R)$ etc.

so $E(V) = p(\tilde{A}\otimes R)\otimes_A V = \underbrace{p(A\otimes R)}_{E(A)}\otimes_A V$.

So you get an exact functor from A-modules to right R-modules which kills V such that $AV = 0$. It obvious that $\mathrm{End}_{A^{op}}(p(A\otimes R))$ acts on $E(V)$ for all V.

$$E(A) \xleftrightarrow{\alpha} A\otimes R \xrightarrow{\beta} E(A)$$

α, β are $(A\otimes R)^{op}$ linear - note $p(A\otimes R)$ is a right ideal in $A\otimes R$. On $A\otimes R$ you have the $\hat{\Gamma}$ idemp. e_s for $s\in\Gamma$ which project (A⊗R onto $A\otimes R_s = As$

First list what might happen for any A-module V.

$p \in \mathrm{End}(V) \otimes \mathbb{C}[\Phi] \subset \mathrm{End}(V) \otimes R$

$$p = \sum_{s \in \Phi} p_s \otimes s \qquad \theta \otimes s : v \otimes t \longmapsto \theta v \otimes st$$

thus the s here represents left multiplication on R.

$$\begin{tikzcd} p(V \otimes R) = E \arrow[r, hook, "\alpha"] \arrow[rr, bend left, "id_E"] & V \otimes R \arrow[r, "\beta"] & E \end{tikzcd}$$

these maps are right R-linear. But now on $V \otimes R$ we have left mult by R which is R^{op}-linear (this is the Γ action on $V \otimes R$), also the grading projections e_s which are just $\mathbb{C}$-linear. The idea now is to look at $\hat{\Gamma}$-action: $\sum e_s = 1$ on $V \otimes R$. Now we can move the e_s around by both left + right mult by elements of Γ. The difference between left + right doesn't affect the family of e_s, you still have

$$id_E = \sum_{s \in \Gamma} \beta e_s \alpha$$

~~and you want~~ But you want this to be something like $\sum_{s \in \Gamma} s h_0 s^{-1}$. What makes sense? Only right mult. makes sense on E. So introduce the operator ρ_s on $V \otimes R$ $\quad \rho_s(v \otimes t) = v \otimes ts$

$E(A)$ is a right $A \otimes R$ module

you want an appropriate alg B to left operate on $E(A)$ to make it a B, A-bimodule. You put into B the operator ρ_s = right mult by s^{-1}, so that

$$\rho_s(\rho_t r) = \rho_s(rt^{-1}) = rt^{-1}s^{-1} = r(st)^{-1} = \rho_{st}(r)$$

And then put $h_0 = \beta e_1 \alpha$

$R = \mathbb{C}[\Gamma]$, A alg equipped with idempotent $p = p^2 \in A\otimes R$, $p = \sum_s p_s \otimes s$, $E = p(A\otimes R)$ a right ideal in $A\otimes R$, maps of right $\tilde{A}\otimes R$ modules

$$E \xhookrightarrow{\alpha} A\otimes R \xrightarrow{\beta} E \qquad \beta\alpha = 1_E$$

$$A\otimes R = \bigoplus_{s\in\Gamma} A\otimes s$$

is a Γ-grading, compatible with left + right mult by Γ. Let e_s = projection on $A\otimes s$, so that

$$\sum_{s\in\Gamma} e_s = 1_{A\otimes R}.$$

What is your aim? You want to construct an alg B left operating on E, making E a B,A-bimodule. First candidate might be $E\otimes_A \mathrm{Hom}_{A^{op}}(E,A)$. ?

Go back to V vector space equipped with idempotent

$p \in \mathrm{End}(V)\otimes R$ which maps to $\mathrm{End}_{R^{op}}(V\otimes R)$

$E(V) = p(V\otimes R)$ is an R^{op}-module

$p = \sum p_s \otimes s$ left mult by s on R, does not comm. with other left mult. in general.

You are interested in, concerned with, operators naturally occurring on the vector space $E(V)$, for example there is a Γ-action on $V\otimes R$, and on $E(V)$ given by $\rho_s(v\otimes r) = v\otimes rs^{-1}$. Also have projection ops for the Γ-grading e_s

$$e_s(v\otimes t) = \begin{cases} 0 & s\neq t \\ v\otimes s & s = t. \end{cases}$$

~~so~~ what to do? Take $A = \mathbb{C}e$ and

$\rho = e \otimes 1 \in A \otimes R$ whence $E(A) = \rho(A \otimes R) \cong R$

on $E(\mathbb{C}e) = R$ you have Γ-operators $\rho_s = \cdot s^{-1}$

and $\hat{\Gamma}$ operators e_s.

$$(\rho_t e_s \rho_t^{-1})(v \otimes u) = \rho_t e_s (v \otimes ut) = \begin{cases} 0 & s \neq ut \\ \rho_t (v \otimes s) & s = ut \end{cases}$$

$$= \begin{cases} v \otimes st^{-1} & \text{if } s = ut \\ 0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} v \otimes st^{-1} & \text{if } u = st^{-1} \\ 0 & \text{otherwise} \end{cases}$$

$$\boxed{\rho_t e_s \rho_t^{-1} = e_{st^{-1}}}$$

So what algebra do you have? ~~Not close~~

Let V be finite diml, let ρ = identity on $\mathrm{End}(V) \otimes R$ = id operator on $V \otimes R$. Then $E(V) = V \otimes R$ with

Γ operations $\rho_s(v \otimes t) = v \otimes ts^{-1}$ and

$\hat{\Gamma}$ ——— $e_s(v \otimes t) = \begin{cases} v \otimes s & \text{if } t = s \\ 0 & \text{otherwise} \end{cases}$

$$(\rho_t e_s \rho_{t^{-1}})(v \otimes u) = \rho_t e_s (v \otimes ut)$$

$$= \begin{cases} \rho_t (v \otimes s) & \text{if } s = ut \\ 0 & \text{otherwise} \end{cases}$$

$$e_{st^{-1}}(v \otimes u) = \begin{cases} v \otimes st^{-1} & \text{if } \cancel{s = ut}\ st^{-1} = u \\ 0 & \text{otherwise} \end{cases}$$

So on $E(V) = V \otimes R$ you get ~~for~~

$$\begin{cases} \Gamma \text{ operations} & \rho_s(v\otimes t) = v\otimes ts^{-1} \\ \hat{\Gamma} \text{ ———} & e_s(v\otimes t) = \begin{cases} v\otimes s & \text{if } s=t \\ 0 & \text{otherwise} \end{cases} \end{cases}$$

What is the ring generated by these operators? The crossproduct?

Analyze this situation for $\Gamma = \mathbb{Z}$. $R = \mathbb{C}[\mathbb{Z}]$ is the ring $\mathbb{C}[u, u^{-1}]$ of Laurent polys.

Γ-operators are translations = mult. by u^n $n \in \mathbb{Z}$

$\hat{\Gamma}$-operators e_n = projection onto $\mathbb{C}u^n$

Let $D = \mathbb{C}[\hat{\Gamma}] = \bigoplus_{n\in\mathbb{Z}} \mathbb{C}e_n$ functions w. fin. supp. on $\mathbb{Z}$ under mult.

D is stable under conjugation by Γ. ~~So what happens is that~~ What happens? Can form the crossproduct alg. $\mathbb{C}[\hat{\Gamma}] \rtimes \Gamma$ with basis $e_s \otimes t$ and crossproduct mult $(e_s \otimes t)(e_{s_1} \otimes t_1)$ No.

$e_s \otimes \rho_t$ = $e_s \rho_t$ operator on $R = \mathbb{C}[\Gamma]$

~~$(e_s\rho_t)(\xi\otimes u) = e_s(\xi\otimes ut^{-1}) = \begin{cases}\xi\otimes s & \text{if } s = ut^{-1} \\ 0 & \text{otherwise}\end{cases}$~~

$$(\rho_t e_s \rho_{t^{-1}})(\xi\otimes u) = \rho_t e_s(\xi \otimes ut) = \rho_t \begin{cases} \xi\otimes s & \text{if } s = ut \\ 0 & \text{if not} \end{cases}$$

$$= \begin{cases} \xi \otimes st^{-1} & \text{if } s = ut \\ 0 & \text{if not} \end{cases}$$

$$= \begin{cases} \xi \otimes u & \text{if } u = st^{-1} \\ 0 & \text{if not} \end{cases} \qquad \boxed{\rho_t e_s \rho_{t^{-1}} = e_{st^{-1}}}$$

What is the point? You have described $\mathbb{C}[\hat{\Gamma}] \times \Gamma$ as the ring with basis $e_s \rho_t$ $s,t \in \Gamma$ (right translation by t^{-1} followed by proj onto degree s. So on R you [redacted] have $\mathbb{C}[\hat{\Gamma}]$ diagonal operators of finite support, and then you move them to any?

Start again with the trivial case $A = \mathbb{C}e$ with $p = e \otimes 1_R$ on $A \otimes R$ ~~R~~ i.e. $p = 1_R$ on R, so you consider $R = \mathbb{C}[\Gamma]$ with

Γ-operations $\rho_s(t) = ts^{-1}$

$\hat{\Gamma}$-operations $e_s(t) = \begin{cases} s & \text{if } s=t \\ 0 & \text{if not} \end{cases}$

$\mathbb{C}[\hat{\Gamma}]$ = functions with fin supp on Γ under mult.

$$(\rho_s e_t \rho_{s^{-1}})(u) = \rho_s e_t(us) = \begin{cases} \rho_s t & \text{if } t = us,\ u = ts^{-1} \\ 0 & \text{if not} \end{cases}$$

$$= \begin{cases} ts^{-1} & \text{if } u = ts^{-1} \\ 0 & \text{if not} \end{cases}$$

$$\rho_s e_t \rho_{s^{-1}} = e_{ts^{-1}}$$

So what happens is that on $R = \mathbb{C}[\Gamma]$ you get an alg of operators generated by ~~$\mathbb{C}[\hat{\Gamma}]$ and $\mathbb{C}[\Gamma]$~~ the ~~e_s $\mathbb{C}[\hat{\Gamma}]$ and the ρ_s~~ e_s and the ρ_t.

This algebra should be $\mathbb{C}[\hat{\Gamma}]^{\sim} \rtimes \Gamma$. Now how are you supposed to understand this. It is a semi direct product situation. ~~the basis product~~ You have an extension

$$\underset{\substack{\text{finite matrix ops} \\ \text{on } R}}{\mathbb{C}[\hat{\Gamma}] \rtimes \Gamma} \subset \overbrace{\mathbb{C}[\hat{\Gamma}]^{\sim}}^{\text{unital}} \rtimes \Gamma \longrightarrow \mathbb{C}[\Gamma]$$

What seems to be the case is the ring B operating on $E=R$ that you seek is the cross product $\mathbb{C}[\Gamma] \rtimes \Gamma$ viewed as the finite matrix operators on R relative to the basis Γ. The Morita equivalence is clear. But something is

---

start again. ~~Let~~ Let $R = \mathbb{C}[\Gamma]$, A an alg, $p = p^2$ in $A \otimes R$ t.p. alg. Thus $p = \sum_{s \in \Gamma} p_s \otimes s$ where $p_u = \sum_{st=u} p_s p_t$. Let $E = p(A\otimes R)$, a right ideal in $\tilde{A} \otimes R$. One has right $\tilde{A}\otimes R = A \otimes R \oplus R$ module maps.

$$E \xrightarrow{\alpha} A\otimes R \xrightarrow{\beta = p\cdot} E \qquad \beta\alpha = id_E$$

$$E \xrightarrow{\alpha} A\otimes R \subset \tilde{A}\otimes R \xrightarrow{\beta = p\cdot} E$$

Question is p in E. Yes because $p = p^2 \in p(A\otimes R)$.

Have Γ-grading $A \otimes R = \bigoplus_{s\in\Gamma} A \otimes s$ compatible with Γ grading on R and right module structure $(A\otimes s)t \subset A \otimes st$

$e_s =$ proj of $A\otimes R$ onto summand $A\otimes s$. $\sum_{s\in\Gamma} e_s = id$ on $A\otimes R$

Have $\rho_t(a\otimes u) = a \otimes ut^{-1}$ on $A\otimes R$

$$(\rho_t e_s \rho_{t^{-1}})(a\otimes u) = \rho_t e_s (a\otimes ut) = \rho_t \begin{cases} a\otimes s & \text{if } s = ut \quad (u = st^{-1}) \\ 0 & \text{if not} \end{cases}$$

$$= \begin{cases} a \otimes st^{-1} & \text{if } u = st^{-1} \\ 0 & \text{if not} \end{cases} = e_{st^{-1}}(a\otimes u)$$

$\boxed{\rho_t e_s \rho_{t^{-1}} = e_{st^{-1}}}$. $\therefore\ id_E = \beta \, id_{A\otimes R} \, \alpha = \sum_t \beta \rho_{t^{-1}} e_1 \rho_t \alpha$

$\rho_t e_1 \rho_{t^{-1}} = e_{t^{-1}}$ $\qquad = \sum_t \rho_{t^{-1}} \underbrace{\beta e_1 \alpha}_{h_1} \rho_t$

$(\rho_s \rho_t)(a\otimes u) = \rho_s(a\otimes ut^{-1}) = (a\otimes ut^{-1}s^{-1}) = a\otimes u(st)^{-1} = \rho_{st}(a\otimes u)$

So on E you have the group operators ρ_s and the ~~basic operator~~ operators $h_s = \rho_s h_1 \rho_s^{-1} = \rho_s \beta e_1 \alpha \rho_s^{-1}$. So what do you do

$$\begin{tikzcd} E \arrow[r, "\alpha"] & A\otimes R \arrow[r, "e_1"] \arrow[dr, "j_1"'] & A\otimes R \arrow[r, "\beta"] & E \\ & & A \arrow[u, "\iota_1"'] & \end{tikzcd}$$

Now you have to understand Cuntz's $\mathcal{E}_{\Sigma_F}$.

$$h_s h_t = \rho_s \beta \underbrace{e_1 \alpha \rho_s^{-1} \rho_t \beta e_1} \alpha \rho_t^{-1}$$

$$e_1 \alpha \rho_s^{-1} \rho_t \beta e_1 = e_1 \rho_{s^{-1}t}\, p\, e_1$$

$$p = \sum p_u \otimes u \in A\otimes R$$

$$\cancel{\iota_1} j_1 \rho_{s^{-1}t}\left(\sum p_u \otimes u\right)\iota_1 \cancel{j_1}$$

$$\begin{array}{ccccccccc} A & \xrightarrow{\iota_1} & A\otimes R & \xrightarrow{p} & A\otimes R & \xrightarrow{\rho_{s^{-1}t}} & A\otimes R & \xrightarrow{j_1} & A \\ a & \longmapsto & a\otimes 1 & \longmapsto & \sum p_u a\otimes u & \longmapsto & \sum p_u a \otimes u t s^{-1} & \mapsto & p_{st^{-1}}a \end{array}$$

---

again repeat. $R = \mathbb{C}[\Gamma]$ $\quad p = p^2 \in A\otimes R$ tp alg

$p = \sum_s p_s \otimes s$, $\quad E = p(A\otimes R)$ right ideal in $\tilde{A}\otimes R$

$$E \underset{\alpha}{\hookrightarrow} A\otimes R \xrightarrow[\beta]{} E$$

right $\tilde{A}\otimes R$-module maps

$$A\otimes R = \bigoplus_{s\in\Gamma} A\otimes s \qquad e_s(a\otimes u) = \begin{cases} a\otimes s & \text{if } s = u \\ 0 & \text{if not} \end{cases}$$

$\rho_s(a\otimes u) = a\otimes us^{-1}$ left Γ-action of Γ on $A\otimes R$ ~~as right~~

maybe its not important that be an action

$A \otimes R$ as right $\tilde{A} \otimes R$ - module.

$$e_s(a\otimes u) = \begin{cases} a\otimes s & \text{if } s = u \\ 0 & \text{otherwise} \end{cases}$$

$$\rho_s(a\otimes u) = a\otimes us$$

$$(\rho_s e_t \rho_{s^{-1}})(a\otimes u) = \rho_s e_t(a\otimes us^{-1}) = \rho_s \begin{cases} a\otimes t & \text{if } t = us^{-1} \\ 0 & \text{if not} \end{cases}$$

$$= \begin{cases} a\otimes ts & \text{if } t = us^{-1} \iff s = ts \\ 0 & \text{if not} \end{cases}$$

$$\therefore\ \rho_s e_t \rho_s^{-1} = e_{ts}$$

check $\boxed{\rho_s \rho_t = \rho_{ts}}$

$$\boxed{\rho_s e_1 \rho_s^{-1} = e_s}$$

$$\rho_{s_1 s_2} e_t \rho_{s_1 s_2}^{-1} = \overbrace{\rho_{s_2}\underbrace{\rho_{s_1} e_t \rho_{s_1}^{-1}}_{e_{ts_1}}\rho_{s_2}^{-1}}^{e_{ts_1s_2}}$$

$$id_{A\otimes R} = \sum_{s\in\Gamma} e_s = \sum_{s\in\Gamma} \rho_s e_1 \rho_s^{-1}$$

$\beta\alpha = id_E$
$\alpha\beta = p$

$$\begin{tikzcd} A\otimes R \arrow[r, "\beta"] \arrow[rr, bend right, "p"'] & E \arrow[r, "\alpha"] & A\otimes R \arrow[r, "\beta"] & E \end{tikzcd}$$

$$id_E = \beta\alpha = \beta\, id_{A\otimes R}\,\alpha = \sum_s \rho_s \underbrace{\beta e_1 \alpha}_{h_1} \rho_s^{-1} = \sum h_s$$

$$h_s h_t = \rho_s \beta e_1 \alpha \rho_s^{-1}\rho_t \beta e_1 \alpha \rho_t^{-1}$$

$$A\otimes R \xrightarrow{\epsilon_1} A \xrightarrow{u} A\otimes R$$
$$a \longmapsto a\otimes 1$$

$$e_1 \, p\, \rho_s^{-1}\rho_t e_1 = e_1 \sum_u (p_u\otimes u)\rho_s^{-1}\rho_t e_1$$

$$e_1 \sum (p_u\otimes u)\rho_s^{-1}\rho_t (a\otimes 1) = e_1 \sum (p_u\otimes u)(a\otimes ts^{-1}) = e_1 \sum_u p_u a \otimes uts^{-1}$$

$$= \sum_u \begin{cases} p_u a & \text{if } uts^{-1} = 1 \\ 0 & \text{if not.} \end{cases}$$

and with $p_{st^{-1}}$

$h_s h_t \neq 0$ when $p_{st^{-1}} \neq 0$

Again. $R = \mathbb{C}[\Gamma]$, A alg, $A \otimes R$ t.p. alg

$p = p^2 \in A \otimes R$, $\quad p = \sum_s p_s \otimes s \qquad p_u = \sum_{\substack{s,t \\ st=u}} p_s p_t$

$E = p(A\otimes R)$ right ideal in $A \otimes R$ also $\tilde{A} \otimes R$

$$\begin{tikzcd} E \arrow[r, hook, "\alpha"] \arrow[rr, bend left, "id_E"] & A \otimes R \arrow[r, two heads, "\beta"] & E \end{tikzcd}$$

$\beta\alpha = id_E$

α, β right $\tilde{A}\otimes R$-modules maps.

$A \otimes R = \bigoplus_{s \in \Gamma} A \otimes s$

$$e_s(a \otimes u) = \begin{cases} a \otimes s & \text{if } s = u \\ 0 & \text{if not} \end{cases}$$

$$\sum e_s = id_{A\otimes R}$$

$id_E = \sum_s \beta e_s \alpha \qquad \rho_t(a \otimes u) = a \otimes ut^{-1}$

$$(\hat{\rho}_t e_s \hat{\rho}_{t^{-1}})(a \otimes u) = \rho_t e_s (a \otimes ut) = \begin{cases} a \otimes st^{-1} & \text{if } s = ut \quad u = st^{-1} \\ 0 & \text{if not} \end{cases}$$

$$\hat{\rho}_t e_s \hat{\rho}_{t^{-1}} = e_{st^{-1}}$$

~~Question~~ Let $B = \bigoplus B_s$ be Γ-graded algebra:

$B_s B_t \subset B_{st}$, $\quad p = p^2$ in B means

$$p = \sum_{p_s \in B_s} p_s \qquad p_u = \sum_{\substack{s,t \\ st=u}} p_s p_t$$

Start from the $\mathcal{E}_{\Sigma_F} = B$ end, generators h_s, relations $h_s h_t = 0$ if $s^{-1}t \notin F$, $\sum_s h_s h_t = h_t = \sum_s h_t h_s$

Let E be a left B-module, ~~firm~~ assume $E = BE$ i.e. $E = \sum h_s E$. Action of Γ on B $\; th_s = h_{ts}$

$B \rtimes \Gamma$ cross product. If E firm ~~then~~

$E = B \otimes_B E$. ~~If~~ Because