Review ~~what happens~~ the assembly situation for $Y = \mathbb{R} \xrightarrow{\pi} \mathbb{R}/\mathbb{Z} = X$. ~~The basic obj~~ The main object is $\pi_! \mathbb{C}_Y$, which is a "line bundle" for the ring $\mathbb{C}[\mathbb{Z}]$ over X. The aim is to embed this line bundle, denote it L, as a summand of a trivial line bundle for $\mathbb{C}[\mathbb{Z}]$ over X. The idea is to use a covering of X over which the line bundle is trivial, and a subordinate partition of 1.

Notice what you are doing. Locally over X you have trivial bundles: $\mathbb{C}[\mathbb{Z}]^n_{\mathcal{U}}$: base $\mathcal{U}$, total sp $\mathcal{U} \times \mathbb{C}[\mathbb{Z}]^n$, fibre $\mathbb{C}[\mathbb{Z}]^n$. think of these as objects in a category, a fibred category over $\mathrm{Open}(X)$.

You feel the urge to replace this geometry of bundles by modules.

Look at the example closely. $X = \mathbb{R}/\mathbb{Z}$

$$\mathbb{C}[\mathbb{Z}]_X$$

So what happens? Graeme's nerve.

You have all these ~~effects~~ viewpoints, angles, Begin with yesterday's idea that embedding a v.b. E as summand of trivial bundle is equivalent to the identity map being nuclear, which you can show locally using locally trivial property, then combining via a partition of the form $\sum x_i^2 = 1$. Need details. $X = \bigcup U_i$, $E_{|U_i} = U_i \times W_i$, whence

~~If~~ map $U_i \times W_i \xrightarrow{\sim} E_{|U_i} \longrightarrow U_i \times W_i$

(module level $\mathcal{O}(U_i) \otimes W_i \xrightarrow{\sim} \mathcal{O}(U_i, E)$

$\mathcal{O}(U_i) \otimes W_i^* \xrightarrow{\sim} \mathcal{O}(U_i, E^*)$

Combine $\mathcal{O}(U_i) \otimes W_i \otimes W_i^* \xrightarrow{\sim} \mathcal{O}(U_i, E \otimes E^*)$

Canonical elt of $W_i \otimes W_i^*$ gives identity map on $E|U_i$ whence you have bundle maps

$$\mathcal{O}(U_i, E) \xrightarrow{q} \mathcal{O}(U_i) \otimes W_i \xrightarrow{P} \mathcal{O}(U_i, E)$$

but its still not crystal clear.

So how to proceed? The geometric picture should be transparent. A proper notation should make this clear. ~~Translate into equations~~ You need to translate from geometry to **modules**.

$\Gamma \longrightarrow Y \xrightarrow{\pi} X$ principal Γ-bundle, X comp.

Introduce assoc fibre bundle $L = Y \times^{\Gamma} \mathbb{C}[\Gamma]$; as a set$/X$ this is locally trivial i.e. $\forall x \ \exists U \ni x$ and $L_{|U} \simeq U \times \mathbb{C}[\Gamma]$. So if you have a top. on $\mathbb{C}[\Gamma]$ preserved by left mult, then get induced top on L. Can define conts section.

Review $\Gamma \to Y \xrightarrow{\pi} X$ princ. bundle, Γ disc, X compact.

$C_c(Y)$ unitary module (alg) over $C(X) \otimes \mathbb{C}[\Gamma]$

We know $X = U_1 \cup \cdots \cup U_N$ $\quad U_i$ open in X

such that $\exists$

$$\begin{tikzcd} & Y \arrow[d, "\pi"] \\ U_i \arrow[ur, dashed, "s"] \arrow[r, hook] & X \end{tikzcd}$$

whence $\pi^{-1}(U_i) = U_i \times \Gamma$

Let φ be continuous fn on X $\quad$ supp $\varphi \in$ some U_i.

$$\pi^{-1}(U) = \coprod_{g} g V_0$$

You want something transparent. Your basic module in $C_c(Y)$. ~~Let $K \subset X$ $\quad K$ compact~~

Let $K \subset U \subset X$
$\quad$ K comp, U open

$\exists\ \tilde{U} \subset \pi^{-1}(U) \ni \quad \pi : \tilde{U} \xrightarrow[\text{homeo.}]{} U$

Inside $C_c(Y)$ is $C_c(\pi^{-1}U) = C_c(\tilde{U} \times \Gamma)$

$= C_c(\tilde{U}) \otimes \mathbb{C}[\Gamma]$

---

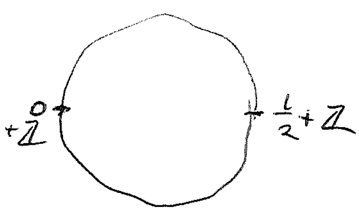
do things carefully for $\mathbb{Z}$.

$$\begin{matrix} Y & & X \\ \shortparallel & & \shortparallel \\ \mathbb{R} & \xrightarrow{\pi} & \mathbb{R}/\mathbb{Z} \end{matrix}$$

$P = C_c(\mathbb{R})$ $\quad$ this is the space which will turn out to be a fin. proj. module, i.e. identity is nuclear

Specific open covering of X $\quad U_0, U_1$ $\quad$ complements of $0+\mathbb{Z}$, $\frac{1}{2}+\mathbb{Z}$.

$$\begin{aligned} \pi^{-1}(U_0) &= \tilde{U}_0 \times \mathbb{Z} \\ \pi^{-1}(U_1) &= \tilde{U}_1 \times \mathbb{Z}. \end{aligned}$$



Let χ_0 have support contained in U_0, Consider $\chi_0 \cdot$ on $C_c(\mathbb{R})$ $\quad$ image contained in functions vanishing on $0+\mathbb{Z}$. But this subspace is induced a

$\Gamma \longrightarrow Y \xrightarrow{\pi} X$ principal Γ-bundle, X compact

$P = C_c(Y)$ is naturally a module over $C_c(Y) \rtimes \Gamma$,

by which I mean the cross product algebra $B = C_c(Y) \overset{\triangleleft}{\otimes} \mathbb{C}[\Gamma]$.

You want to prove that P is a firm finite proj. B-module.

Suppose the bundle trivial: $Y = X \times \Gamma = \coprod_{\Gamma} X$

$$C_c(Y) = C(X) \otimes \mathbb{C}[\Gamma] \quad — \quad C_c(\Gamma) = \bigoplus_{s\in\Gamma} \mathbb{C} e_s$$

$$B = C(X) \otimes \underset{C_c(\Gamma)}{\cancel{\mathbb{C}[\Gamma]}} \overset{\triangleleft}{\otimes} \mathbb{C}[\Gamma]$$ finite matrix ops. on $C_c(\Gamma)$.

So you have a Morita equivalence between B and $A = C(X)$.

At this point you ought to be able to write things out precisely starting from $\Gamma \longrightarrow Y \xrightarrow{\pi} X$ princ. bundle X compact. Get a precise Morita equivalence of the crossproduct $\mathbb{C}[\Gamma] \overset{\triangleright}{\otimes} C_c(Y)$ with $C(X)$. ~~You have to~~ At the moment the basic idea is because the bundle is locally trivial you can reduce to $Y = \Gamma \times X$

$$C_c(Y) = C_c(\Gamma) \otimes \underbrace{C(X)}_{A} \qquad B = \mathbb{C}[\Gamma] \overset{\triangleright}{\otimes} C_c(\Gamma) \otimes C(X)$$

There's an idea here that might be useful, which starts with the way local nuclearity is pieced together (using a partition of unity) to get global nuclearity. For example if E is a vb over X compact and $X = \bigcup u_i$, $E|_{u_i}$ trivial then from $E|_{u_i} \xrightarrow{\hat{q}} (W_i)_{u_i} \xrightarrow{\hat{p}} E|_{u_i}$, $\hat{p}\hat{q} = 1$

your can assemble $E \longrightarrow \left(\bigoplus W_i\right)_X \longrightarrow E$

using a partition of 1 of the form $\sum x_i^2 = 1$.

~~[illegible]~~ There should be a simple way to see, describe, how local Morita equivalences can be pieced together via a partition of 1 to get a global Morita equivalence. ~~[illegible]~~

$\Gamma \to Y \xrightarrow{\pi} X$, X compact. Result is that the crossproduct alg $P \overset{\Delta}{\otimes} \mathbb{C}[\Gamma]$, where $P = C_c(Y)$, is Morita equivalent to $C(X)$. Can you prove this? The proof might proceed via Mayer-Victoris

Ex: $\mathbb{Z} \to \mathbb{R} \to \mathbb{R}/\mathbb{Z}$ $\qquad P = C_c(\mathbb{R})$, $Q = C_c(\mathbb{R})$

$$\langle q, p \rangle(\cancel{x}) = \sum_{n \in \mathbb{Z}} q(y+n)\,p(y+n)$$

$$P \otimes_A Q = C_c(\mathbb{R}) \otimes_{C(\mathbb{R}/\mathbb{Z})} C_c(\mathbb{R})$$

$$\overset{?}{=} C_c(\mathbb{R} \times_{\mathbb{R}/\mathbb{Z}} \mathbb{R}) = C_c(\mathbb{R}) \otimes C_c(\mathbb{Z})$$

so it seems OKAY.

In general given $\Gamma \to Y \xrightarrow{\pi} X$ princ. Γ bundle.

$$P = C_c(Y) \qquad B = C_c(Y \times_X Y) = C_c(Y \times \Gamma) = C_c(Y) \otimes C_c(\Gamma)$$

This seems to work easily. Return now to Cuntz's model, the s. complex $\Sigma_F = \{\phi \neq M \subset \Gamma \mid M^{-1}M \subset F\}$. ~~[illegible]~~
$\qquad\qquad$ finite

~~[illegible]~~ What can you do?

In the case of $\mathbb{Z}$ you know $B \xrightarrow{\cdot h_0} P$ is surjective, ~~Is this true in general?~~ this is a map of firm left B-modules, so corresp to $Q \xrightarrow{\langle\,,h_0\rangle} A$, $h_0 \in P$, $Q = C_c(Y)$, so is pairing with $h_0 \in P$. ~~Then you have something non trivial~~ So all you have to do to obtain a projector is to lift 1 i.e. produce k such that ~~[illegible]~~ $\langle k, h_0 \rangle = 1$.

C Review a little. So for

$\Gamma \to Y \to X$, say $\mathbb{Z} \to \mathbb{R} \to \mathbb{R}/\mathbb{Z}$ you get a dual pair over $A = C(X)$ given by $P = C_c(Y) = Q$ as A-modules. $\langle g, p \rangle^{(x)} = \sum_{\Gamma} (gp)(y\gamma)$. Then have

$$B = C_c(Y \times_X Y) = C_c(Y) \otimes C_c(\Gamma)$$

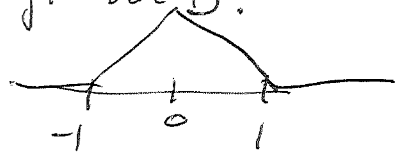
$$\uparrow S \qquad \longleftarrow \text{ this you might prove by MV on the 2nd component. ??}$$

$$C_c(Y) \otimes_A C_c(Y)$$

needs work!!

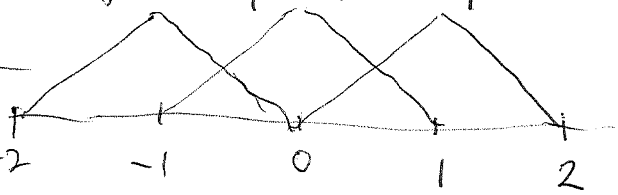
But now use A unital to show P proj. over B.

Take $\mathbb{Z}$-case h_0:



Then $f \mapsto \langle f, h_0 \rangle = \sum_{n \in \mathbb{Z}} (fh_0)(y+n)$ will give $1 \in A$ when $f = 1$ on Supp h_0. e.g.

$f = h_{-1} + h_0 + h_1$



It should be clear that a similar thing works for the simplicial $B\Gamma$. $\left(\sum_{s \in \Gamma} h_s - 1\right) h_t = 0$

$$Q \xrightarrow{\langle -, h_0 \rangle} A \qquad \text{Is this onto}$$

answer is yes. $\sum_s \langle h_s, h_1 \rangle = \sum_s \theta(h_s h_1) = \theta(h_1) = 1$

$\theta(f) = \sum_s sf$

Question: Where do you get the $h^{1/2}$?

d Let's review what you learned yesterday. $\Gamma \to Y \xrightarrow{\pi} X$ principal bundle (X compact). The claim is that there's a Morita equivalence between ~~[illegible]~~ $A = C(X)$ and the cross product alg. ~~[illegible]~~ $C_c(Y) \hat{\otimes} \mathbb{C}[\Gamma]$. ~~What is the~~

Consider
$$\begin{tikzcd} Y \times_X Y \arrow[r] \arrow[d] & Y \arrow[d] \\ Y \arrow[r] \arrow[u, bend left, dashed] & X \end{tikzcd}$$
You get two lifts of Y into the fibreproduct; so

$$Y \times_X Y \simeq \Gamma \times Y$$ should lead to an isom of $C_c(Y \times_X Y)$ with $C_c(\Gamma) \otimes C_c(Y)$ and also $C_c(Y) \otimes C_c(\Gamma)$

In addition you expect the cartesian square above to yield an isom:
$$\begin{array}{ccc} P \otimes_A Q & \longrightarrow & B \\ \| & & \| \\ C(Y) \otimes_{C(X)} C(Y) & \xrightarrow{\sim} & C_c(Y \times_X Y) \end{array}$$

In fact this is probably the way to start the proofs. You should maybe use $\pi^! \ \pi_!$ etc. There are lots of details to work out.

~~But what [illegible] the~~ $\langle g, p \rangle = \sum_\gamma \gamma(gp)$

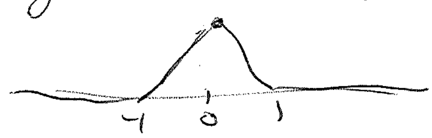
Now for the pairing $Q \times P \longrightarrow A$

Go to your example $\mathbb{Z} \to \mathbb{R} \longrightarrow \mathbb{R}/\mathbb{Z}$. Since A ~~that~~ is unital and $QP = A$, P, Q are dual finite proj [redacted] B-modules, firm besides, of course.

~~In the [illegible]~~ look at $\sum_\Gamma \gamma f$. ~~[illegible]~~

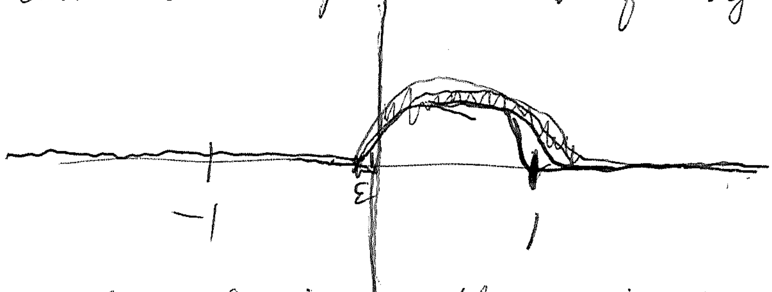
~~h~~ $\in P$ $\quad B \xrightarrow{\cdot h} P$ $\quad$ In your basic example

$Q \xrightarrow{\langle\ ,h\rangle} A$ $\quad h_0$:



$$\sum \gamma h_0 = 1.$$

e But you want P to lift into B as left module, so you want q, p so that $\langle q,p\rangle = \sum_{\Gamma} \gamma(qp) = 1 \in A$. How nice can you make things? Basic requirement is that you start with a pos >0 function on an open interval of length >1, ~~and~~



The obvious thing is to take $p = q = f(x)$ where f is >0 on (a,b) with $b-a > 1$.

Then $\sum_{\gamma \in \mathbb{Z}} \gamma(f^2)$ is periodic pos.

The point is that it is easy to find p, q such that $\langle q,p\rangle = 1 \in A$, and then $p \otimes q$ is a projector in B with image

$$P \xleftarrow{p\,\cdot} A \qquad\qquad B \xrightarrow{\cdot p_0} P$$

$$P \qquad Q \qquad\qquad Q \xrightarrow{\langle -, p_0\rangle} A$$

$$B \underset{-\otimes q_0}{\overset{\cdot p_0}{\rightleftarrows}} P$$

$$(p\otimes q) \longmapsto p\langle q, p_0\rangle$$

$$p\otimes q_0 \longleftarrow\!\shortmid\; p$$

$$p\otimes q_0 \longmapsto p\langle q_0, p_0\rangle = p$$

proj op. on the left B-module $B = P\otimes Q$ is

$$p\otimes q \longmapsto p\langle q, p_0\rangle \longmapsto \underbrace{p\langle q,p_0\rangle \otimes q_0}_{(p\otimes q)(p_0\otimes q_0)}$$

so the idempotent in the crossproduct B is $p_0 \otimes q_0$.

f So how is this related to Cuntz's constructions? Apparently his $p_0, q_0 = h_1^{1/2}$. Then the condition becomes

$$\sum_{s\in\Gamma} {}^s(q_0 p_0) = \sum_{s\in\Gamma} {}^s h_1 = \sum_{s\in\Gamma} h_s = 1$$

and the projector is $h_1^{1/2} \otimes h_1^{1/2} \in B_\cdot = C_c(Y \times_X Y)$

$= C_c(Y) \otimes C_c(\Gamma)$. Cuntz uses the Γ grading on B, and writes his projector $p = \sum p_s$ $\quad p_s \in B_s = P_s$

It should be easy to get from $h_1^{1/2} \otimes h_1^{1/2} \in C_c(Y \times_X Y)$ to $\sum_s h_1^{1/2} s\, h_1^{1/2} = \sum_s (h_1^{1/2} h_s^{1/2}) s$ $\quad \therefore\ p_s = h_1^{1/2} h_s^{1/2}$.

$$p_{st^{-1}} p_t = h_1^{1/2} h_{st^{-1}}^{1/2} h_1^{1/2} h_t^{1/2} \qquad p_s^* = h_s^{1/2} h_1^{1/2} = s h_1^{1/2} s^{-1} h_1^{1/2}$$

$\cancel{= h_1^{1/2}\, st^{-1} h_1^{1/2} (st^{-1})^{-1} h_1^{1/2}\, t h_1^{1/2} t^{-1}}$

$$= h_1^{1/2} st^{-1} h_1^{1/2} t s^{-1} \qquad p_s = h_1^{1/2} h_s^{1/2} = h_1^{1/2} s\, h_1^{1/2} s^{-1}$$

$$p_s^* = s h_1^{1/2} s^{-1} h_1^{1/2}$$

try. $p_s = h_s^{1/2} h_1^{1/2}$

$p_{st^{-1}} = st^{-1} h_1^{1/2} t s^{-1}$ $\quad \cancel{(p_s s)^* = s^{-1} p_s ?}$

$$p_s = h_1^{1/2} h_s^{1/2} \qquad p_{s^{-1}} = h_1^{1/2} s^{-1} h_1^{1/2} s$$

$$p_s^* = h_s^{1/2} h_1^{1/2}$$

$$(p_s s)^* = s^{-1} p_s = \underbrace{(s^{-1} p_s s)}_{p_{s^{-1}}} s^{-1}$$

need $s^{-1} p_s s = p_{s^{-1}}$

$p_{s^{-1}} = h_1^{1/2} s^{-1} h_1^{1/2} s$

$s^{-1} p_s s = s^{-1}(h_1^{1/2} h_s^{1/2}) s$

p_s not self adj.

g

$$p_s = h_1^{1/2} h_s^{1/2}$$

$$p_s^* = h_s^{1/2} h_1^{1/2} = s\left(h_1^{1/2} h_{s^{-1}}^{1/2}\right) s^{-1} = s\, p_{s^{-1}} s^{-1}$$

$$\left(\sum p_s s\right)^* = \sum_s s^* p_s^* = \sum_s p_{s^{-1}} s^{-1} = \sum p_s s .$$

$$\left(\sum_s p_s s\right)\left(\sum_u p_u u\right) = \sum_{s,u} p_s \left(s p_u s^{-1}\right) su$$

$$= \sum_t \left(\sum_{su=t} p_s\left(s p_u s^{-1}\right)\right) t$$

$$p_t = \sum_{\substack{s,u \ni \\ su=t}} p_s\, s\, p_u\, s^{-1} = \sum_s p_s\, s\, p_{s^{-1}t}\, s^{-1} \qquad ?$$

$$t(p_s s)t^{-1}$$

---

go through it properly. $\Sigma_F = \{ M \mid \phi \neq M \subset \Gamma,\ M^{-1}M \subset F\}$

Γ left acts on Σ_F.

$$\mathcal{E}_{\Sigma_F} = C^*\{ h_s \mid h_s \geq 0,\quad h_s h_t = 0 \text{ if } s^{-1}t \notin F\}$$

$$\sum_t h_s h_t = h_s$$

$\mathcal{E}^{ab}_{\Sigma_F}$ is essentially $C_c(Y)$

Y = the geometric real. of Σ_F $\qquad t(h_s) = h_{ts}$

in the cross product alg you have $t h_s t^{-1} = h_{ts}$

You ought to be able to carry over earlier ideas

$P = Q = C_c(Y)$ $\qquad \langle q, p \rangle = \sum_s {}^s(qp)$

h But then $p_0 = h_1^{1/2} = q_0$ $\langle q_0, p_0\rangle = \sum_s {}^s h_1 = \sum h_s = 1$

$$B = P\otimes_A Q \underset{-\otimes q_0}{\overset{\cdot p_0}{\rightleftarrows}} P$$

$$Q \underset{\cdot q_0}{\overset{\langle -, p_0\rangle}{\rightleftarrows}} A \qquad q \longmapsto \langle q, p_0\rangle \mapsto \langle q, p_0\rangle q_0$$

So the projection you get in B is $p_0 \otimes q_0$

Idea: apply $C_c(-)$ systematically to infinite coverings.

First you finish with the h_s.

$\Gamma \to Y \to X$ $Y = |\Sigma_F|$ geom. real.

$C_c(Y)$ contains functions h_s $s\in\Gamma$

$h_s \geq 0$ $h_s h_t = 0$ if $s^{-1}t \notin F$

$$\sum_{s\in\Gamma} h_s h_t = h_t$$

Point is you have a Morita equiv.

$A = C(X)$, $P = Q = C_c(Y)$, $B = C_c(Y \times_X Y)$

$B = P\otimes_A Q$ acts on P $\langle q, p\rangle = \sum_s s*(qp)$

given $p_0 \in P$ get $B = P\otimes_A Q \underset{-\otimes q_0}{\overset{\cdot p_0}{\rightleftarrows}} P$ ⟳ mult by $\langle q_0, p_0\rangle$

⟲ $\cdot\, p_0\otimes q_0$. In the case of Σ_F you take $p_0 = q_0 = h_1^{1/2}$, then $\langle q_0, p_0\rangle = \sum_s s* h_1 = \sum h_s$ which is the identity function

So Y = specific simplicial complex ~~with~~ having as vertices elts of Γ and as simplices finite non-empty subsets M such that $\forall s,t\in M$ ~~this~~ $s^{-1}t \in F$ holds. The action of Γ on Y will not be true unless Γ ~~has~~ is torsion-free.

~~What is the~~ $C_c(Y)$ contains $h_s = (s*h_1)$ satisfying

$$\Big(\sum_s h_s - 1\Big)h_t = 0 \qquad \forall t.$$

Now produce the projector

Recall $B = P\otimes_A Q \ni p_0\otimes q_0$ s.t. $\underbrace{\sum_{s\in\Gamma} s*(q_0p_0)}_{\langle q_0,p_0\rangle} = 1$

$$\therefore\ (p_0\otimes q_0)(p_0\otimes q_0) = p_0\otimes \overbrace{\langle q_0,p_0\rangle}^{1} q_0$$

so $p_0\otimes q_0$ is ~~a projector~~ idempotent. Take $p_0=q_0=h_1^{1/2}$

$$\sum_s s*(q_0p_0) = \sum_s s*h_1 = \sum h_s = 1$$

~~If you have to~~ Point: $B = C_c(Y\times_X Y)$ which should be $C_c(Y)\otimes_{C(X)} C_c(Y)$, because it's locally true over X.

However to get Cuntz's formula for the canonical idemp. you need the ~~graded~~ components of $h_1^{1/2}\otimes h_1^{1/2}$ ~~with related~~ wrt the Γ grading. I think what you want to do is $\sum_{s\in\Gamma} h_1^{1/2}\, s\, h_1^{1/2} = \sum_{s\in\Gamma} h_1^{1/2}h_s^{1/2}\otimes s$, because formally

$$\Big(\sum_{s\in\Gamma} h_1^{1/2} s h_1^{1/2}\Big)\Big(\sum_{t\in\Gamma} h_1^{1/2} t h_1^{1/2}\Big) = \sum_{s,t\in\Gamma} h_1^{1/2} s h_1 t h_1^{1/2}$$

$$= \sum_s h_1^{1/2} s h_1 \sum_t t h_1^{1/2} = \sum_s h_1^{1/2} h_s \sum_t st\, h_1^{1/2} = \sum_s h_1^{1/2} h_s \sum_u u\, h_1^{1/2}$$

$$= h_1^{1/2}\sum$$

A clearer way would be

$$\sum_s h_1^{1/2} s h_1^{1/2} \sum_t h_1^{1/2} t h_1^{1/2}$$

$$= \sum_s \sum_t h_1^{1/2} h_s s t h_1^{1/2} \qquad t = s^{-1}u$$

$$= \sum_s \sum_u h_1^{1/2} h_s u h_1^{1/2}$$

$$= \sum_u h_1^{1/2} \underbrace{\sum_s h_s}_{1} u h_1^{1/2} = \sum_u h_1^{1/2} u h_1^{1/2}.$$ I think I understand this now.

Next project is to coordinate the two ~~rings~~ algebras over which $P = C_c(Y)$ is finite projective. One is $B = C_c(Y \times_X Y)$ and the other is $A \otimes \mathbb{C}[\Gamma]$.

Focus on the case $\Gamma = \mathbb{Z}$ $Y = \mathbb{R}$. You have the multiplier algebra for $B = P \otimes_A Q$, which is the multiplier alg $\subset \mathrm{Hom}_A(Q,Q) \times \mathrm{Hom}_{A^{op}}(P,P)$ satisfying

$$\mathrm{Hom}_B(B,B) = \mathrm{Hom}_A(Q,Q)$$

$$\mathrm{Hom}_{B^{op}}(B,B) = \mathrm{Hom}_{A^{op}}(P,P)$$

Compat. cond. is $\langle q\mu, p \rangle = \langle q, \mu p \rangle$.

So what is $\mathrm{Hom}_A(P,P)$. Arbitrary cont. functions on $\mathbb{R}$. It seems that $\mathrm{Hom}_A(P,P)$ might be cross product of $C(\mathbb{R})$ and $\mathbb{C}[\mathbb{Z}]$.

The next project is to explain how $P = C_c(Y)$ happens to be a finite proj module over both $A^{[\Gamma]} = C(X) \otimes \mathbb{C}[\Gamma]$ and $B = C_c(Y \times_X Y)$. Both A, B act on P

$P = C_c(\mathbb{R})$ has operators of mult. by continuous functions on $\mathbb{R}$ and also translations by $\mathbb{Z}$. The natural thing to do might be to look at the C^*-picture. This means introducing a basic Hilbert space on which our algs + modules become operators. $\underline{L^2(\mathbb{R})}$? $C_c(\mathbb{R}) \underset{\text{completion}}{\rightsquigarrow} C(\mathbb{R})$

Another way to proceed is to tentatively assume

$$\mathrm{End}_{C(X)}(C_c(Y)) = \mathrm{Cont}(Y) \overset{\triangleleft}{\otimes} \mathbb{C}[\Gamma]$$

OKAY

I really think I can prove this

In any case ~~we sh~~ you should have a good part of the multiplier alg of B. ~~So the~~ The situation is commutative except for the group Γ. Yes

So you assume that the (left) multiplier algebra of $P = C_c(\mathbb{R})$ as an $A = C(\mathbb{R}/\mathbb{Z})$-module is the cross product $\mathrm{Cont}(\mathbb{R}) \overset{\triangleleft}{\otimes} \mathbb{Z}$, and that this is the multiplier alg for $B = P \otimes_A Q = C_c(Y \times_X Y)$.

$$B = C_c(Y) \overset{\triangleleft}{\otimes} \mathbb{C}[\mathbb{Z}]$$

$$\mathrm{Mult}(B) = \mathrm{Cont}(Y) \overset{\triangleleft}{\otimes} \mathbb{C}[\mathbb{Z}].$$

B should be an ideal in $\mathrm{Mult}(B)$

What about $A[\mathbb{Z}] = C(\mathbb{R}/\mathbb{Z}) \otimes \mathbb{C}[\mathbb{Z}]$ obviously sits in $\mathrm{Mult}(B)$ as a subring, probably injects into the "Calkin" alg $\mathrm{Mult}(B)/B$

What do you know about K-theory?

B is Morita equiv. to $A = C(\mathbb{R}/\mathbb{Z})$. ?

Obvious to consider $\underbrace{(C_c(Y) \oplus A)}$ $\overset{\triangleleft}{\otimes} \mathbb{C}[\mathbb{Z}]$

add periodic fns. $A = C(\mathbb{R}/\mathbb{Z})$ to $C_c(Y)$.

Baum-Connes for $\mathbb{Z}$.

There seems to be a general result that

$C(X)$ Morita equiv. to $C_c(Y) \overset{\triangleleft}{\otimes} \mathbb{C}[\Gamma]$

for a principal Γ-bundle with X compact.

Baum-Connes class $\in$

$$K_0\big(C(X) \otimes C_r(\Gamma)\big) \qquad \underbrace{KK\big(C(X), \mathbb{C}\big)}$$

Use this class to cap-map K-homology of X to K-theory of $C_r(\Gamma)$.

$$K\big(C_r(\Gamma)\big) = KK\big(\mathbb{C}, C_r(\Gamma)\big)$$

This discussion seems OK, but what is mysterious is the role of the cross product algebra $C_c(Y) \overset{\triangleleft}{\otimes} \mathbb{C}[\Gamma]$.

The point might be that the cross product occurs in an extension:

$$\underbrace{C_c(Y) \overset{\triangleleft}{\otimes} \mathbb{C}[\Gamma]}_{B} \hookrightarrow (\overset{\overset{C(X)}{\downarrow}}{A} \oplus C_c(Y)) \overset{\triangleleft}{\otimes} \mathbb{C}[\Gamma] \twoheadrightarrow C_{[redacted]}(X) \otimes \mathbb{C}[\Gamma]$$

B

$\int$ Morita eq

$C(X)$

This ~~extension~~ splits so it can't be ~~...~~ right.
So ask if there is an interesting extension of $C(X)$ by ~~...~~ $C_c(Y)$?

You now need to understand the part of C's lecture concerning BC conjecture. Where to start? There should be a BC map relating K-homology of $B\Gamma$ to the K-theory of $C^*_r(\Gamma)$ i.e.

$$KK_*(C(B\Gamma), \mathbb{C}) \dashrightarrow KK_*(\mathbb{C}, C^*_r(\Gamma))$$

geometrically you have $\Gamma \longrightarrow Y \longrightarrow X$ and finite projective module $C_c(Y)$ over $C(B\Gamma) \otimes \mathbb{C}[\Gamma]$ i.e. an element of $KK_0(\mathbb{C}, C(B\Gamma)\otimes\mathbb{C}[\Gamma])$ except that KK is probably ~~...~~ only defined for C^*algs. so the element should be in

$$KK_0(\mathbb{C}, C(B\Gamma)\otimes C^*_r(\Gamma))$$

Joachim uses $KK^\Gamma_*(\ ,\ )$ for Γ, C^* algebras, and you need to understand the rules for this equivariant K-theory. You expect

$$KK^\Gamma_*(C(Y), \mathbb{C}) = KK_*(C(X), \mathbb{C})$$

$C(Y)$ is the C^* version of $C_c(Y)$. Maybe

$$KK^\Gamma_*(C(Y), \mathbb{C}) = KK_*(C(Y)\rtimes\Gamma, \mathbb{C}) = KK_*(C(X), \mathbb{C})$$

is true? because $C(Y)\rtimes\Gamma$ is M.eq to $C(X)$

Why don't you develop the idea that Γ equivariant K-theory of a Γ-alg A ~~should~~ might be essentially the K-theory of

$A \rtimes \Gamma$. ~~What might~~ topological construction

Z a Γ-space $\longmapsto$ $E\Gamma \times^{\Gamma} Z$ homot. ~~quotient~~

~~extend $E\Gamma \times Z$~~

You have princ. bdl

$$\Gamma \longrightarrow E\Gamma \times Z \longrightarrow E\Gamma \times^{\Gamma} Z$$

so you expect a Morita equivalence between

$C_c(E\Gamma \times^{\Gamma} Z)$ and $C_c(E\Gamma \times Z) \overset{\triangleleft}{\otimes} \mathbb{C}[\Gamma]$?

$C_c(E\Gamma) \otimes C(Z)$

This get tricky.

Continue to focus on the idea that Γ-equivariant K-theory for a Γ-alg A should be close to K-theory of $A \rtimes \Gamma$. $A \overset{\triangleleft}{\otimes} \mathbb{C}[\Gamma]$

Recall that the algebra $B = A \rtimes \Gamma$ is naturally Γ-graded, i.e. $B = \bigoplus_{s \in \Gamma} B_s$, $B_s B_t \subset B_{s+t}$, $B_s^* = B_{s^{-1}}$. Alt. terminology is that B has a $\hat{\Gamma}$-action, or is a $\check{\Gamma}$-algebra. There should be a functor $B \longmapsto B \rtimes \check{\Gamma}$, $B \rtimes \check{\Gamma} = B \otimes C_c(\Gamma)$

If A is Γ-alg, then it is natural to look at A-modules M ~~which and are~~ which are Γ equivariant. These are unitary $\tilde{A} \rtimes \Gamma$-modules, in particular they are $A \rtimes \Gamma$ modules. Similarly if $B = \bigoplus_{s \in \Gamma} B_s$ is a Γ-graded algebra, it is natural to consider B-modules M with Γ-grading compatible with that of B.

this means $B_s M_t \subset M_{s+t}$, such modules are modules over $B \rtimes \check{\Gamma}$ - means you adjoin projectors e_s, $s \in \Gamma$ to B. You need to formulate the appropriate firmness conditions to identify Γ-graded B-modules with $B \rtimes \check{\Gamma}$-modules.

~~[illegible] you need to formulate the appropriate~~

Continue with reviewing C's talk. ~~[illegible]~~

At some point he introduces $A \rtimes \Gamma = B$ which is Γ-graded, and he introduces the Γ-graded algebra P_F which is universal for projectors $p = \sum_{s \in F} p_s$ in a Γ-graded algebra. ~~[illegible]~~ (The sum here probably should be finite.) $p = p^2 = p^* \iff (p_s)^* = p_s^{-1}$ and $p_s = \sum_t p_{st^{-1}} p_t$

There seem to be interesting relations between P_F and the non comm. simplicial complex $\mathcal{E}_{\Sigma_F}$ which amounts to a map $P_F \longrightarrow \mathcal{E}_{\Sigma_F} \rtimes \Gamma$ of Γ-graded algebras.

Review again the formulas.

$$p = \sum_s h_1^{1/2} s h_1^{1/2} = \sum_s \underbrace{h_1^{1/2} h_s^{1/2}}_{p_s} s$$

~~[illegible]~~ Ultimately you have to understand the role of F.

There's a lot to understand, but ~~you need to begin~~ it seems that the key to assembly is to be found among partitions of unity. You need to ~~develop~~ control gluing. Try to formulate a successful program.

Problem: In the case of $\Gamma \to Y \to X$ you still need to relate $P = C_c(Y)$ as $B = C_c(Y) \rtimes \Gamma$-module to P as a unitary module over $C(X) \otimes \mathbb{C}[\Gamma]$.

So what? You are still puzzled by the crossproduct $C_c(Y) \hat{\otimes} \mathbb{C}[\Gamma]$ versus $C(X) \otimes \mathbb{C}[\Gamma]$. Let's go over the situation. You start with a principal bundle $\Gamma \longrightarrow Y \longrightarrow X$ with X compact. This is the basic object; what can you do with it. X compact leads naturally to $C(X)$, then Y leads to $C_c(Y)$. When the $\overset{C_c(\Gamma)}{}$ bundle is trivial: $Y = X \times \Gamma$ then $C_c(Y) = C(X) \otimes \mathbb{C}[\Gamma]$, a ~~[illegible]~~ free rank 1 module over the unital alg $C(X) \otimes \mathbb{C}[\Gamma]$. In general you find $C_c(Y)$ is a f.g. proj. unitary module over $C(X) \times \mathbb{C}[\Gamma]$. $C_c(Y)$ is a nuclear module over this ring, meaning that the identity map is nuclear.

Maybe you should analyze ~~[illegible]~~ ~~what you need~~ in order to ~~[illegible]~~ show that the identity map (of a vector bundle) is nuclear

E Q: What viewpoint for X? Open cover + ~~partition~~ of 1
$\downarrow$ Simplest case is ~~[illegible]~~ MV: [illegible] $X = U \cup V$.
X Maybe ~~[illegible]~~ $X = \bar{U}$, i.e. compactification like attaching a cell, might be relevant.

First discuss $X = U \cup V$. At this point I recall some nice features of the C^*-theory. You ~~restrict~~ attention to hermitian vector bundles, and then it makes sense to consider bounded continuous sections over an open set U, and it makes sense to multiply such sections by a continuous function ~~[illegible]~~ which is zero ~~[illegible]~~ outside U.

~~[illegible]~~ Return to Serre's thm.

Vector bundle $\begin{array}{c} E \\ \downarrow \\ X \end{array} = \bigcup U_i$ hermitian v.b. + partition of 1

~~[illegible]~~ What is it that you want to do? Given a principal Γ-bundle $Y \to X$ with X compact, you ~~[illegible]~~ choose a partition of unity on X over which the bundle is trivial. partition of unity means a finite family of continuous functions $\{h_i \geq 0 \mid i \in I\}$ such that $\sum h_i = 1$, the bundle $Y \to X$ is assume to be trivial over each open set $U_i = \{x \mid h_i(x) > 0\}$. partition is same as a cont. map $X \longrightarrow \Delta =$ the simplex of ~~[illegible]~~ prob. measures on the index set I. ~~[illegible]~~ You choose a trivalization $U_i \times_X Y \simeq U_i \times \Gamma$ ~~[illegible]~~ in addition to the choice of partitions.

~~[illegible]~~

So you end up with an open covering + partition and a cocycle. There's (maybe) a problem with intersections $U_{i_1} \cap U_{i_2}$ not being connected.

Say $Y = X \times \Gamma$, whence $C_c(Y) \simeq C(X) \otimes C_c(\Gamma)$

so ~~you have~~ $C_c(Y)$ is a free module over the ring $C(X) \otimes \mathbb{C}[\Gamma]$. So you need ~~[illegible]~~ module map

$$C_c(Y) \longrightarrow C(X) \otimes \mathbb{C}[\Gamma] \longrightarrow C_c(Y)$$

$$C_c(Y) = C(X) \otimes C_c(\Gamma) \xrightarrow{\sim} C(X) \otimes \mathbb{C}[\Gamma] \longrightarrow C_c(Y)$$

there probably is still stuff ~~to understand~~

Maybe Γ graded mods important

So I am still confused, but it gets clearer.

~~...~~ You know that when $Y = X \times \Gamma$ that $C_c(Y)$ is Γ-graded.

Here you know that $C_c(X \times \Gamma) = \bigoplus_{s \in \Gamma} C_c(X \times s)$

So the idea in general: $\Gamma \to Y \to X$ is to choose $U_i \subset X$ such that $\bigcup U_i = X$. $\sum h_i = 1$, $h_i \geq 0$, $h_i(X - U_i) = 0$

$Y|_{U_i} \simeq U_i \times \Gamma$ and you use $h_i^{1/2}$

$U_i \times \Gamma \simeq U_i \times_X Y$ $\qquad C_c(U_i \times_X Y) = C_c(U_i) \otimes \mathbb{C}[\Gamma]$

~~...~~ What is your goal? Same sort of model for $C(B\Gamma)$, e.g. like $C(E\Gamma) \rtimes \Gamma$, except $B\Gamma$ ~~...~~ should be replaced by an ind. limit of compact spaces. $C(E\Gamma)$ becomes $\varinjlim_F \mathcal{E}^{ab}_{\Sigma_F}$ and then $C(B\Gamma)$ becomes

$\varinjlim_F \mathcal{E}^{ab}_{\Sigma_F} \rtimes \Gamma$. But you still miss the link between $C_c(Y)$ as $C(X) \otimes \mathbb{C}[\Gamma]$-module on one hand and as $C_c(Y) \rtimes \Gamma$-module on the other hand.

Go back to $\Gamma \to Y \to X$, try ~~something~~ looking for ~~...~~ a model of $B\Gamma$. $X = U \cup V$

Continue – this time focus a bit on the crossproduct $C_c(Y) \overset{\triangleleft}{\otimes} \mathbb{C}[\Gamma]$. What you believe to be true is that this crossproduct algebra is Morita equivalent to $C(X)$, maybe $C_c(X)$ in general. Stick to X compact

One point is the action of $C(X)$ which should allow localization of some sort

Go back to $\mathbb{Z} \longrightarrow \mathbb{R} \longrightarrow \mathbb{R}/\mathbb{Z}$.

Go back to $A = C(\mathbb{R}/\mathbb{Z})$ $\qquad B = C_c\left(\mathbb{R} \underset{\mathbb{R}/\mathbb{Z}}{\times} \mathbb{R}\right) = P \otimes_A Q$

$P = C_c(\mathbb{R}) = Q$

what's interesting here is the pairing $\quad Q \otimes P \longrightarrow A$

$$q \otimes p \longmapsto \sum_{s\in\Gamma} s^*(qp)$$

This should be a trace on B with values in A.
You must work out proofs at some point.

What do you learn about $C_c(\mathbb{R})$ as a B module? By Morita theory since A unital and the pairing is surjective P must be a nuclear B-module
Q ———————— B^{op}-module

What's the role of $\underline{C(\mathbb{R}/\mathbb{Z}) \otimes \mathbb{C}[\mathbb{Z}]}$? ~~What for~~

From $C(\mathbb{R}/\mathbb{Z})$ you get the partition of unity allowing you to localize. First of all everything is a module or alg. over $C(\mathbb{R}/\mathbb{Z})$. Look at the bundle picture. Over a arb. point $x \in \mathbb{R}/\mathbb{Z}$ $\qquad x = y + \mathbb{Z}$

At this point you need to look again at C's talk.

Γ, $\quad \Sigma_F$, $\quad \mathcal{E}_{\Sigma_F}$ noncomm. alg assoc. to this simp. cx., $\quad \mathcal{E}_{\Sigma_F} \rtimes \Gamma$ some sort of noncomm. quotient space

simp. cx. Γ acts by left mult

it seems that

$$\mathcal{E}^{ab}_{\Sigma_F} \rtimes \Gamma = C_c(|\Sigma_F|) \rtimes \Gamma = C\big(\underbrace{|\Sigma_F|/\Gamma}_{\text{base}}\big)$$

In $\mathcal{E}_{\Sigma_F} \rtimes \Gamma$ you should have a projector

$$\sum_s p_s s = \sum_s h_1^{1/2} h_s^{1/2} s \qquad h_1 h_s = 0 \implies h_1^{1/2} h_s^{1/2} = 0$$

so you have the canonical projector $\sum_{s \in F} h_1^{1/2} s h_1^{1/2}$

Thus there is a canonical homom. of Γ-graded algebras.

$$P_F \longrightarrow \mathcal{E}_{\Sigma_F} \rtimes \Gamma$$

Confusing: $\mathcal{E}_{\Sigma_F}$ is like $C_c(|\Sigma_F|)$ Morita eq.

so $\mathcal{E}_{\Sigma_F} \rtimes \Gamma$ is like $C_c(|\Sigma_F|) \rtimes \Gamma \sim C(|\Sigma_F|/\Gamma)$

Try some more. A Γ-alg

$$\varinjlim_F KK^{\check{\Gamma}}(P_F, A \rtimes \Gamma) \longrightarrow K_*(A \rtimes \Gamma)$$

Baaj-Skandalis: $\wr\wr$

$$\varinjlim_F KK(P_F \rtimes \check{\Gamma}, A) \qquad A \rtimes \Gamma \rtimes \check{\Gamma} \sim \mathcal{K} \otimes A$$

But $\boxed{\underline{\text{Thm.}}\ P_F \rtimes \check{\Gamma} \text{ "stably" isom to } \mathcal{E}_{\Sigma_F}}$

related to the map $P_F \longrightarrow \mathcal{E}_{\Sigma_F} \rtimes \Gamma$

Look at $\mathbb{Z}$. $F = \{-1, 0, 1\}$.

Today you try to understand the Baaj-Skandalis part of C's talk.

Γ group $\quad F$ finite subset containing 1 closed under inverse.

$$\Sigma_F = \text{poset } \{M \subset \Gamma \mid M \neq \emptyset,\ M^{-1}M \subset F\} \quad \Rightarrow M \text{ finite} \neq \emptyset.$$

$$\mathcal{E}_{\Sigma_F} = C^*\{h_s, s\in\Gamma \mid h_s \geq 0,\ h_s h_t = 0 \quad t^{-1}s \notin F\}$$

$$\sum_{s\in tF} h_s h_t = h_t$$

If the $h_s \in \mathbb{R}$ ~~are numbers~~, then $h_t > 0 \Rightarrow \sum_{s\in tF} h_s = 1.$

Γ acts on $\mathcal{E}_{\Sigma_F}$ $\qquad s * h_t = h_{st}$

can form $\qquad \mathcal{E}_{\Sigma_F} \rtimes \Gamma = \bigoplus_{s\in\Gamma} \mathcal{E}_{\Sigma_F}\, s$

In general look at a Γ-alg A and form $A \rtimes \Gamma$ which is Γ-graded. $\cancel{p = \sum p_s s \in A\rtimes\Gamma}$

$$\cancel{p^2 = \sum_{s,t} p_s s p_t t = \sum_{s,t} p_s\, st^{-1} p_t t = \sum_{s,t} p_{st^{-1}} p_s s}$$

$$\cancel{= \sum_t \Big(\sum_s p_s s\Big) p_t t = \sum_t \Big(\sum_s p_{st^{-1}} st^{-1}\Big) p_t t}$$

$$\cancel{= \sum_t \sum_s p_{st^{-1}} p_s s}$$

$$p = \sum p_s \in \bigoplus_{s\in\Gamma} B_s$$

$$p = p^* = p^2 \Longrightarrow \quad p_s^* = p_{s^{-1}} \text{ and } \sum_t p_{st^{-1}} p_t = p_s$$

Candidate is $\quad p_s = h_1^{1/2} s h_1^{1/2} = h_1^{1/2} h_s^{1/2} s$

$$p^2 = \sum_{s,t} h_1^{1/2} s h_1 t h_1^{1/2} = \sum_{s,t} h_1^{1/2} h_s\, st\, h_1^{1/2} = \sum_u h_1^{1/2}\, u\, h_1^{1/2}$$

There should be an intelligent picture behind these formulas. You notice that $p_s = 0$ for $s \notin F$.

So let's try to put some order into all of this stuff.

$$p = \sum_{s\in\Gamma} h_1^{1/2} s\, h_1^{1/2} = \sum_{s\in\Gamma} h_1^{1/2} h_s^{1/2} s .$$ ~~And this~~

This p is canonical in $\mathcal{E}_{\Sigma_F} \rtimes \Gamma$, so you get a Γ-graded map.

$$\boxed{P_F \longrightarrow \mathcal{E}_{\Sigma_F} \rtimes \Gamma}$$

universal for projections in a Γ-graded alg $B = \bigoplus_{s\in\Gamma} B_s$ with support in the finite subset F.

~~Is~~ this map an isomorphism on the C^* level?

$p_s = h_1^{1/2} s\, h_1^{1/2}$. It looks like this is a positive definite function on Γ. ~~So what do we do here.~~

$$p_1 = h_1^{1/2} 1\, h_1^{1/2} = h_1$$

$P_F \rtimes \check{\Gamma}$ stably isom to $\mathcal{E}_{\Sigma_F}$. It looks like

$P_F \longrightarrow \mathcal{E}_{\Sigma_F} \rtimes \Gamma$ is an isom of Γ-graded algs.

Can you see this is true?

Problem is following: $\Gamma = \mathbb{Z}$ $\quad F = \{-1, 0, 1\}$

P_F is universal Γ-graded algebra generated ~~the~~ by components of a projector $p = p_{-1} + p_0 + p_1$ supp in F.

~~[illegible]~~ Consider Hilbert space representations of ~~[illegible]~~

P_F. Can look at $\mathbb{Z}$-graded or ungraded reps. ~~S~~ince P_F is $\mathbb{Z}$-graded it should be easier to look at $\mathbb{Z}$-graded representations. So you consider a $\mathbb{Z}$-graded Hilbert space $H = \bigoplus_{n\in\mathbb{Z}} H_n$, equiv. a Hilb. space representation of $\check{\mathbb{Z}} = S^1$, ~~[illegible]~~ and then you have a projector $p = p_{-1} + p_0 + p_1$. to say $p_k = 0$ for $|k| > 1$ means what?

$$\iota_n : H_n \hookrightarrow H \qquad j : W \hookrightarrow H$$

$$p = jj^* \qquad p_k : H_n \longrightarrow H_{n+k} \quad \forall n$$

$$p_k = \sum_n \iota_{n+k}\, p\, \iota_n^*$$

to say that $p_k = 0$ for $k \geq 2$ seems to mean ~~that~~

Review. A a Γ-alg given, get $A \rtimes \Gamma$ a $\check{\Gamma}$-alg;

given $B = \bigoplus_{s\in\Gamma} B_s$ a $\Gamma^\vee$ alg, get $B \rtimes \Gamma^\vee$ a Γ alg

how to view: naturally assoc. to a Γ-alg A is the cat of Γ-equiv. A-modules, and these are ~~[illegible]~~ B-modules
naturally assoc. to a $\Gamma^\vee$-alg B is the cat. of Γ-graded B-modules and these are modules over $B \rtimes \Gamma^\vee$.

Take $\Gamma = \mathbb{Z}$. A a Γ- C^*-alg. ~~[illegible]~~

a

What do you need to identify P_F with the cross product $\mathcal{E}_{\Sigma_F} \rtimes \Gamma$? First interpret the map $P_F \to \mathcal{E}_{\Sigma_F} \rtimes \Gamma$; this gives a way to go from Γ-equivariant $\mathcal{E}_{\Sigma_F}$-modules to modules over P_F.

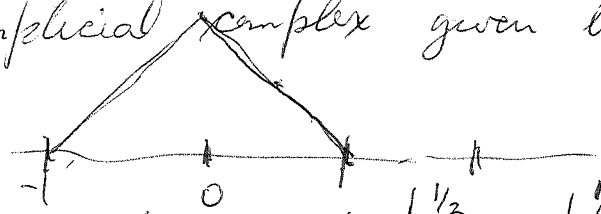
Roughly, given an equivariant module over the functions on the total space of the principal bundle, you get a P_F module, so it seems like P_F is the functions on the base. So C's statement about $P_F \longrightarrow \mathcal{E}_{\Sigma_F} \rtimes \Gamma$ being an isom (equivalence in the appropriate sense, "stable isomorphism").

What to do: It should be true that $\mathcal{E}_{\Sigma_F} \rtimes \Gamma$ is Morita equivalent to P_F (otherwise what C says about Baaj - Skandalis is beyond your abilities to reconstruct). The bimodule you need for the Morita equivalence should be the image of the canonical projector $p = \sum_{s \in \Gamma} h_1^{1/2} s h_1^{1/2}$

There ~~should~~ might be some link between your grid spaces and the $\mathbb{Z}$-case of the preceding. Actually what you should do is understand John Roe's picture of finite propagation speed kernels. $\exists$ Some resemblance between a projector of the form $p_{-1} + p_0 + p_1$ and the way you propagate in grid space. Also

$p = \sum_{s\in\mathbb{Z}} h_1^{1/2} s h_1^{1/2}$ reminds one of positive definite functions

~~Start~~ Form the $*$ alg. $\mathcal{E}_{\Sigma_F}$ noncomm. version of simplicial complex given by $\mathbb{R}$ with $\mathbb{Z}$-triangulation



basic generators $u^n * h_0^{1/2} = h_s^{1/2}$ $\quad s \in \mathbb{Z}$

relations are $\left(\sum h_s - 1\right) h_t = 0 \qquad h_n h_m = 0 \quad |n-m| \geq 2$

Anyway you get this algebra $\mathcal{E}_{\Sigma_F}$ acted on by Γ. $\mathcal{E}_{\Sigma_F}$ is a Γ-algebra, so you can form $\mathcal{E}_{\Sigma_F} \rtimes \Gamma$. First show that $\mathcal{E}_{\Sigma_F}$ is a ~~fin. pro~~ nuclear module. Same basic idea.

$$e = \sum_{n\in\mathbb{Z}} h_0^{1/2} u^n h_0^{1/2} = \sum_{n\in\mathbb{Z}} h_0^{1/2} h_n^{1/2} u^n$$

$$= h_0^{1/2} h_{-1}^{1/2} u^{-1} + h_0 + \underbrace{h_0^{1/2} h_1^{1/2} u}_{\left(\sqrt{1-t}\,\sqrt{t}\right) u}$$

Think of $\mathcal{E}_{\Sigma_F} \rtimes \Gamma$ as left acting on $\mathcal{E}_{\Sigma_F}$. What is the effect of e??

What seems to be happening is that this simplicial formalism differs from what you looked at before. You now have a more complicated e. Before with $C_c(Y) \rtimes \Gamma$ you took ?

What happens? You should be able to prove for any $\Gamma \longrightarrow Y \longrightarrow \underset{\text{compact}}{X}$ that $C_c(Y) \rtimes \Gamma \sim C(X)$. Proof via partition of unity on X. Morita theory should tell you that $C_c(Y)$ is fin. proj. over $C_c(Y) \rtimes \Gamma$. Now your proof uses

$$B = C_c(Y \times_X Y) = C_c(Y) \rtimes \Gamma$$
$$\parallel$$
$$C_c(Y) \otimes_{C(X)} C_c(Y)$$

How do things work

C claims a "stable" equivalence between P_F and $\mathcal{E}_{\Sigma_F} \rtimes \Gamma$ induced by a homom. $P_F \longrightarrow \mathcal{E}_{\Sigma_F} \rtimes \Gamma$. Study this. You begin with an explicit projector

$$e = \sum_{s \in \Gamma}' h_1^{1/2} s h_1^{1/2} \quad \text{in} \quad \mathcal{E}_{\Sigma_F} \rtimes \Gamma.$$

You have a homom. $P_F \longrightarrow \mathcal{E}_{\Sigma_F} \rtimes \Gamma$ hence ~~an~~ a hom. $P_F \rtimes \check{\Gamma} \longrightarrow \mathcal{E}_{\Sigma_F} \rtimes \Gamma \rtimes \check{\Gamma}$ Mor. eq. to $\mathcal{E}_{\Sigma_F}$

Let's try to make these Meq's explicit. How? You need the appropriate dual pairs. ~~What does this mean?~~ When does a hom $A \longrightarrow B$ induce a M. eq. conditions that kernel K killed by A: $AK = KA = 0$, other is $BAB = B$ $\qquad ABA = A$

$$\begin{pmatrix} A & AB \\ BA & B \end{pmatrix}$$

One natural module for $\mathcal{E}_{\Sigma_F} \rtimes \Gamma$ is $\mathcal{E}_{\Sigma_F}$

Anytime you find a $\mathcal{E}_{\Sigma_F}$ module M with compatible Γ action you get a projection e on M.

Try a different approach: ~~Classify~~

Look a rep. of $\mathcal{E}_{\Sigma_F} \rtimes \Gamma$ on $\mathcal{H}$

Better what is a rep. of $\mathcal{E}_{\Sigma_F}$ on $\mathcal{H}$.

~~Look~~ You seem be stuck on showing an equivalence between P_F and $\mathcal{E}_{\Sigma_F} \rtimes \Gamma$.

$\mathcal{E}^{ab}_{\Sigma_F} = C(\underline{E}_F\Gamma)$ $\quad$ $\underline{E}\Gamma$ is the geometric simplicial complex whose simplices are all non empty subsets M of Γ s.t. $M^{-1}M \subset F$.

$\mathcal{E}_{\Sigma_F}$ generators h_s $s \in \Gamma$ $\quad$ $h_s \geq 0$

$h_s h_t = 0$ if $t^{-1}s \notin F$ $\quad$ $\sum_s h_s h_t = h_t$

Γ acts via $s * h_t = h_{st}$

Fix $\Gamma = \mathbb{Z}$, what is a rep. of $\mathcal{E}_{\Sigma_F} \rtimes \Gamma$ on a Hilbert space $\mathcal{H}$? ~~[illegible]~~ Come on Dan! unitary operator

In $\mathcal{H}$ you have a unitary operator u and a non negative operator $h_0 \geq 0$ such that $h_0 u^n h_0 = 0$ for $|n| > 1$. Put $h_n = u^n * h_0 = u^n h_0 u^{-n}$. Can you describe better the picture. ~~[illegible]~~

Assume the h_n commute.

Wait, look at the subspaces $\overline{h_n \mathcal{H}}$ ~~[illegible]~~ $= \overline{u^n h_0 u^{-n} \mathcal{H}}$ $= u^n \overline{h_0 \mathcal{H}}$. You assume these generate $\mathcal{H}$

Look closer. Suppose given operators $h_n \geq 0$ on $\mathcal{H}$ for $n \in \mathbb{Z}$ such that $h_k h_\ell = 0$ for $|k-\ell| \geq 2$.

A self adjoint operator A yields a decomp. $\mathcal{H} = \mathrm{Ker}(A) \oplus \overline{A\mathcal{H}}$. Better to take $A \geq 0$.

$$h_k h_\ell = h_\ell h_k = 0 \qquad h_k \overline{h_\ell \mathcal{H}} = 0$$

$$\Rightarrow \quad \overline{h_\ell \mathcal{H}} \subset \mathrm{Ker}(h_k) = \overline{h_k \mathcal{H}}^{\perp}$$

$\therefore$ $h_\ell \mathcal{H}$ and $h_k \mathcal{H}^{\perp}$ are $\perp$

So you find scattering ~~situation~~

Let's begin again. Consider $\mathcal{E}_{\Sigma_F} \rtimes \Gamma$, where $\Gamma = \mathbb{Z}$ $F = \{-1, 0, 1\}$. This is the C^*-alg generated by $h_0 \geq 0$ and a unitary u, ~~with~~ subject to the relations $h_0 u^n h_0 = 0$ for $|n| \geq 2$ and

$$\underbrace{h_0 \sum_{n \in \mathbb{Z}} u^n h_0 u^{-n}}_{h_0(h_{-1} + h_0 + h_1)} = h_0$$

? You lack an understanding of ~~this~~ simplex condition

But consider a Hilbert space $\mathcal{H}$ with non-neg operator $h_0 \geq 0$ and unitary op u $\ni$ $h_0 u^n h_0 = 0$ for $|n| \geq 2$. Get subspace $Y = \overline{h_0 \mathcal{H}}$ such that $(Y | u^n Y) = 0$ for $|n| \geq 2$. Can assume $\sum u^n h_0 \mathcal{H} = \sum h_n \mathcal{H}$ dense in $\mathcal{H}$. Now what happens? You have a partial unitary situation $X = Y \cap u^{-1}Y$

$X \underset{u}{\overset{e}{\rightleftarrows}} Y$. Note that you have replaced h_0 by the projector onto Y which is limit $\lim_{n\to\infty} h_0^{1/n}$. What can you say?

$$\boxed{h_0 u^{-1} h_0 + h_0^2 + h_0 u h_0 = e} \quad ?$$

$$= h_0 h_{-1} u^{-1} + h_0^2 + h_0 h_1 u$$

Let's go in the other direction! Start with $\mathcal{H}, u, Y$ closed in $\mathcal{H}$, $Y \perp u^n Y$ for $|n| \geq 2$. Then you ~~[illegible]~~ have this ~~perfect~~?

$$u^{-1}Y \quad Y \quad uY \quad u^2Y$$

$$\oplus u^{-2}V_- \oplus u^{-1}V_- \oplus \begin{pmatrix} \overbrace{X + V_+}^{Y} \\ V_- + uX \end{pmatrix} \begin{matrix} \oplus uV_+ \oplus \cdots \\ + \end{matrix}$$

begin with a contraction operator on X.

H Hilbert space with operator $h_0 \geq 0$ and unitary operator u, such that $h_0 u^n h_0 = 0$ for $|n| \geq 2$, i.e.

$$0 = \langle H | h_0 u^n h_0 | H \rangle = \langle h_0 H | u^n | h_0 H \rangle. \quad \text{i.e.}$$

$\overline{h_0 H} \perp u^n \overline{h_0 H}$ for $|n| \geq 2$. Let $Y = \overline{h_0 H}$

Then Y is closed in H and $u^m Y \perp u^n Y$ for $|m-n| \geq 2$

You are interested in the condition

$$\left(\sum h_n\right) h_0 = h_0 \qquad h_n = u^n h_0 u^{-n}$$

so

$$(h_{-1} + h_0 + h_1) h_0 = h_0$$

i.e. you want $h_{-1} + h_0 + h_1$ to be equal 1 on Y. Suppose $Y = \overline{h_0 H}$ is 1-dimensional, so that h_0 is hermitian of rank 1. Pick a unit vector $\xi_0 \in Y$ so that $h_0 = c\,\xi_0 \xi_0^*$ with $c > 0$. ~~Thats~~ Assume H closure of $\{u^n \xi_0\}$, get a spectral measure.

$$\int z^n \, d\mu = (\xi_0, u^n \xi_0) = \begin{cases} \bar\alpha & n=-1 \\ 1 & n=0 \\ \alpha & n=1 \\ 0 & \text{otherwise} \end{cases}$$

$$d\mu = \rho \frac{d\theta}{2\pi} \qquad \rho = \alpha z^{-1} + 1 + \bar\alpha z$$

$$\int z^n (\alpha z^{-1} + 1 + \bar\alpha z) \frac{d\theta}{2\pi} = \begin{cases} \alpha & n=1 \\ 1 & n=0 \\ \bar\alpha & n=-1 \\ 0 & \text{otherwise} \end{cases}$$

$Y = \overline{h_0 H}$ assume 1 diml., ξ_0 unit vector

Three lines Y, uY, $u^{-1}Y$. These must be independent otherwise say $u^{-1}\xi_0 = c_1 \xi_0 + c_2 u \xi_0$

and then $\mathbb{C}\xi_0 + \mathbb{C}u\xi_0$ is stable under u^{-1}.

~~So now you have~~ Is it possible for $(h_{-1}+h_0+h_1)h_0 = h_0$

$$h_0 = c\,\xi_0 \xi_0^* \qquad h_{-1} h_0 = c^2 u^{-1}\xi_0 \xi_0^* u \xi_0 \xi_0^*$$

$$h_1 = c\,u\xi_0 \xi_0^* u^{-1} \qquad h_0^2 = c^2 \xi_0 \xi_0^*$$

$$h_{-1} = c\,u^{-1}\xi_0 \xi_0^* u \qquad h_1 h_0 = c^2 u \xi_0 \xi_0^* u^{-1} \xi_0 \xi_0^*$$

$$h_{-1}h_0 = c^2 u^{-1}\xi_0 \alpha \xi_0^*$$
$$h_0^2 = c^2 \xi_0 \xi_0^*$$
$$h_1 h_0 = c^2 u\xi_0 \bar{\alpha} \xi_0^*$$

this is not ~~+~~ prop. to ξ_0 when $\alpha \neq 0$.

$$\therefore (h_{-1}+h_0+h_1)h_0 = c^2\Big(\underbrace{\alpha u^{-1}\xi_0 + \xi_0 + \bar\alpha u\xi_0}\Big)\xi_0^*$$

$$1+\bar\alpha z + \alpha z^{-1} = 1 + 2\mathrm{Re}(\bar\alpha z) \qquad |z|=1$$
$$= 1 + 2|\alpha| \cos(-\arg\alpha + \arg z) \quad ?$$

Start again. $\mathcal{H} = \underbrace{\mathcal{E}_{\Sigma_F}} \rtimes \mathbb{Z}$ $\qquad F = \{-1, 0, 1\}$.

generators h_0, u $\qquad h_n = u^n h_0 u^{-n}$

Hilb. space H with unit of $\mathcal{Z}$ and $h_0 \geq 0$

Put $h_n = u^n h_0 u^{-n}$ $\qquad$ Put $Y = \overline{h_0 H}$

$$(Y, u^nY) = (\overline{h_0H}, \underbrace{u^n\overline{h_0H}}_{\overline{h_nH}}) = 0 \quad \text{for } |n| \geq 2.$$

Assume Y 1-dim.

Then H contains the lines u^nY permuted by $\mathbb{Z}$.

so why is ~~that~~ $|\alpha| \leq \frac{1}{2}$. ~~Let's try~~

$\oplus\, \mathbb{C}^n$

Still trying to understand $\mathcal{O}$'s ~~a~~ $\mathcal{E}_{\Sigma_F} \rtimes \Gamma$ in the simplest case. ~~This alg is gen~~ A *rep of this alg on a Hilb. space H is given by a unitary u and a pos. herm. h_0 satisfying $h_0 u^n h_0 = 0$ for $|n| \geq 2$ and $(h_{-1}+h_0+h_1)h_0 = h_0$.

~~[illegible]~~ Let $V = \overline{h_0 H}$

Then $(V, u^n V) = (\overline{h_0 H}, \overline{u^n h_0 V}) = 0$ for $|n| \geq 2$

$(h_0 H, u^n h_0 H) = (H, \underbrace{h_0 u^n h_0}_{0} H)$. ~~So you have~~

can assume $\sum u^n V$ dense in H, the point is that the

You understand the abelianization: $\mathcal{E}^{ab}_{\Sigma_F}$ it is a commutative C^*-alg ~~whose spectrum is~~ generated by non-negative fns. h_n sat $h_0 h_n = 0$ $|n| \geq 2$.

and $\underbrace{\left(\sum_n h_n\right) h_m = h_m}$. What the solutions.

$$(h_{m-1} + h_m + h_{m+1}) = 1 \qquad \text{where } h_m \neq 0.$$

~~[illegible]~~ $U_m = \{x \mid h_m(x) > 0\}$ maps natural to

$$\underbrace{h_{m-1}\left(\sum h_n\right) h_m} = h_{m-1} h_m$$

$$h_{m-1}(h_{m-1} + h_m) h_m = h_{m-1} h_m$$

h_0 commutes with $h_{-1} + h_0 + h_1$, hence with $h_{-1} + h_1$

check this ~~$\sum_{n \in \mathbb{Z}} h_0 h_n h_1 = h_0 h_0 h_1 + h_0 h_1 h_1$~~

basic relation is $\sum_n h_m h_n = h_m$

$$h_0(h_{-1} + h_0 + h_1) = h_0$$

$$h_0(h_0 h_1) + (h_0 h_1) h_1 = h_0 h_1$$

$$h_0(h_{-1} + h_0 + h_1) h_1 = h_0 h_1$$

$$h_0(h_0 + h_1) h_1 = h_0 h_1$$

alg defined by generators and relations.

new viewpt. Let H be a Hilb. rep of $\mathcal{E}_{\Sigma_F} \rtimes \mathbb{Z}$, so you have $h_0 \geq 0$ on H, u unitary on H and relations. $h_0 u^n h_0 = 0$ $|n| \geq 2$. Also the relation $\left(\sum_{|n|\leq 1} h_n\right) h_0 = h_0$ which you don't understand.

Ignore last relation and let $W = \overline{h_0 H} \subset H$. Then $W \perp u^n W$ for $|n| \geq 2$, because

$$(h_0 \xi, u^n h_0 \xi') = (\xi, \underbrace{h_0 u^n h_0}_{0} \xi')$$

Can suppose $H = \overline{\sum_{n\in\mathbb{Z}} u^n W}$, so is a gen. subspace for $\mathbb{Z}$-module H. GNS theorem ~~tells~~ us that H arises from a pos. def fn on $\mathbb{Z}$ w. values in $\mathcal{L}(W)$ which is equivalent to a pos. measure $d\mu$ on the circle. You only need $(w, u^n w')$ to reconstruct H.

$\|$ Moments in $\mathcal{L}(W)$

Here things are simple as all moments with $|n| > 1$ are zero. $\varepsilon : W \hookrightarrow H$ $\varepsilon^* u^n \varepsilon$.

Your aim now: You start with the repn of $\mathcal{E}_{\Sigma_F} \rtimes \Gamma$ and get $W \subset H \circlearrowleft u$ $\ni W \perp u^n W$ for $|n| > 1$. H, u ~~can be fo~~ are det. by

$$\mathbb{Z} \ni n \longmapsto \varepsilon^* u^n \varepsilon \in \mathcal{L}(W)$$

zero ~~unless~~ for $|n| > 1$.

get $\rho(z) = \sum_{|n|\leq 1} z^{-n} \varepsilon^* u^n \varepsilon$ $\qquad \varepsilon^* u^n \varepsilon = \int z^n \rho(z) \frac{d\theta}{2\pi}$

$$= z\, \varepsilon^* u^{-1} \varepsilon + \underset{\overset{\|}{\varepsilon^*\varepsilon}}{1_W} + z^{-1} \varepsilon^* u \varepsilon$$

When is $\rho(z) = z(\alpha)^* + 1 + z^{-1}\alpha$ a positive operator valued fn. on S^1? Here $\alpha \in \mathcal{L}(W)$ is a contraction.

You want to understand, study $H, u, W \subset H$ such that $\overline{\sum u^n W} = H$ and $W \perp u^n W$ for $|n| \geq 2$. simplest case: $\dim(W) = 1$. ~~Let $W = \xi\mathbb{C}$ $\|\xi\| = 1$. Then there's a prob measure $d\mu$ such that Then there's a~~ So you have a unitary representation of $\mathbb{Z}$ with a cyclic v. ξ, know $\exists$ $d\mu$ on S^1 and an isom

$$L^2(S^1, d\mu) = H \qquad \text{i.e. } 1 \longmapsto \xi$$

$$z^n \longmapsto u^n\xi \qquad z\cdot \Longleftrightarrow u.$$

Start again. You consider $H, u, W \subset H$ such that $W \perp u^n W$ for $|n| \geq 2$. Assume $H = \overline{\sum_n u^n W}$, W generates H as .... Question is how to construct H from W data. Answer Let $\varepsilon : W \hookrightarrow H$ be inclusion, $\varepsilon^*\varepsilon = 1$. Then H is specified, determined by the function $n \longmapsto \varepsilon^* u^n \varepsilon$ from $\mathbb{Z} \longrightarrow \mathcal{L}(W)$. an arbitrary pos. def. function

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Where are you? Consider partial unitary $X \underset{b}{\overset{a}{\rightrightarrows}} Y$ and dilate

$$\cdots \oplus u^{-1}V_- \oplus \boxed{\begin{matrix} \overbrace{aX \oplus V_+}^{Y} \\ \| \\ V_- \oplus bX \end{matrix}} \oplus uV_+ \oplus \cdots$$

$uY \perp (V_- + u^{-1}V_- + \cdots$

$uY \subset bX \oplus uV_+ + \cdots$

$u^{-1}Y \perp V_+ + uV_+ + \cdots$ $\quad \underbrace{aX \oplus u^{-1}V_-}_{u^{-1}Y} \oplus \cdots$

so $u^{-1}Y \perp uY$

$\Longleftrightarrow aX \perp bX$

$\Longleftrightarrow b^*a = 0.$

Something ~~seems to be~~ is wrong. Begin again with $H, u, W \xrightarrow{\varepsilon} H$ such that $\varepsilon^* u^n \varepsilon = 0$ for $|n| \geq 2$, and ε injective. Assume H generated by εW under u: $\sum u^n \varepsilon W$ dense in H.

H is determined by the function $n \mapsto \varepsilon^* u^n \varepsilon$ from $\mathbb{Z}$ to $\mathcal{L}(W)$ because given $f(z) \in \mathbb{C}[u, u^{-1}] \otimes W$ so $f(z) = \sum z^n f_n$, then

$$\left\| \sum u^n \varepsilon f_n \right\|^2 = \sum_{n,n'} f_n^* \varepsilon^* u^{-n+n'} \varepsilon f_{n'}$$

put $\rho(z) = \sum z^{-n} (\varepsilon^* u^n \varepsilon)$

i.e. $\boxed{\varepsilon^* u^n \varepsilon = \int z^n \rho(z) \frac{d\theta}{2\pi}}$

$$\left\| \sum u^n \varepsilon f_n \right\|^2 = \cancel{\int \sum_{n,n'} f_n^* \varepsilon^* z^{-n+n'} \varepsilon f_{n'} \frac{d\theta}{2\pi}}$$

$$= \sum_{n,n'} f_n^* \cdot \int z^{-n+n'} \rho(z) \frac{d\theta}{2\pi} f_{n'}$$

$$= \int \frac{d\theta}{2\pi} \sum_n f_n^* z^{-n} \rho(z) \sum_{n'} z^{n'} f_{n'}$$

$$= \int \frac{d\theta}{2\pi} f(z)^* \rho(z) f(z)$$

should be true that $\rho(z) \geq 0 \quad \forall z \in S^1 \iff$ this integral is ≥ 0 for any $f \in \mathbb{C}[z, z^{-1}] \otimes W$

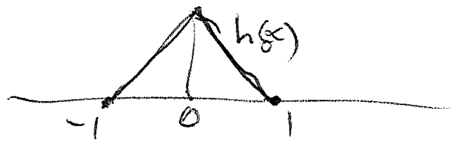
In our case $\varepsilon^* u^n \varepsilon = 0$ except for $-1, 0, 1$ so

$\rho(z) =$ [redacted] $z^{-1} \varepsilon^* u \varepsilon + \varepsilon^* \varepsilon + z \varepsilon^* u^{-1} \varepsilon$

Natural question is when such a family over S^1 of hermitian ops is ≥ 0.

~~Eg~~ Example. $H = L^2(\mathbb{R})$ $W = L^2(-1,1)$ u = shift by 1

$\varepsilon = h_0^{1/2}$



$\rho(z) \in L^2(-1,1)$ is what. $\varepsilon^*\varepsilon = h_0$

Something simpler is $\varepsilon = 1$, really $\chi_{[-1,1)}$

~~$\rho(z) = z^{-1}\chi_{(0,1)} + \chi_{(-1,1)} + z\chi_{(-1,0)}$~~

$$\varepsilon^* u \varepsilon = \chi_{(-1,1)}\chi_{(0,2)} = \chi_{(0,1)}$$

$$\varepsilon^* u^{-1}\varepsilon = \chi_{(-1,1)}\chi_{(-2,0)} = \chi_{(-1,0)}$$

$$\rho(z) = z^{-1}\chi_{(0,1)} + \chi_{(-1,1)} + z\chi_{(-1,0)} \qquad \text{NO}$$

~~Let~~ Just take ~~that~~ example simplest

$\dim(W) = 1$. $a = \varepsilon^*\varepsilon > 0$ $\varepsilon^* u \varepsilon = b \in \mathbb{C}$.

$|b| < a$

~~$\rho(z) = z^{-1}b + b$~~

$$\rho(z) = z^{-1}b + a + z\bar{b}$$

$\arg b - \arg z$

$$z^{-1}b + z\bar{b} = 2\,\mathrm{Re}(z^{-1}b) = 2|b|\cos\theta$$

~~so~~ for $\rho(z)$ to be positive you must have $a \geqslant 2|b|$

~~Rather than~~

Question: Given $b \in \mathcal{L}(W)$ when is

$1 + z^{-1}b + zb^* \geqslant 0$ for $|z|=1$. If $[b, b^*] = 0$ this is true iff $\|b\| \leqslant \frac{1}{2}$. ~~It is probably true that~~

$\|b\| \leqslant \frac{1}{2} \implies$ true. because $z^{-1}b + zb^*$ is hermitian and $\|z^{-1}b + zb^*\| \leqslant \|b\| + \|b^*\| = 2\|b\| \leqslant 1$, so the spectrum of $z^{-1}b + zb^*$ ~~is~~ is $\subset [-1,1]$, so spec. of $1 + z^{-1}b + zb^*$ is $\subset [0,2]$

So where are we?

You consider $H, u, W \overset{\varepsilon}{\subset} H$ such that $W \perp u^n W$ for $|n| \geq 2$. $\varepsilon : W \longrightarrow H$ be the inclusion: $\varepsilon^* \varepsilon = 1$. Know $\varepsilon^* u \varepsilon = b$ is such that $1 + z^{-1} b + z b^* \geq 0$ $\forall z \in S$!

$b = x + iy$ $\qquad x = x^*, y = y^*$

$$1 + (z^{-1} + z)x + i(z^{-1} - z)y$$

want $1 + 2(x\cos\theta + y\sin\theta)$ not easily understood

Best to proceed with condition $\|b\| \leq \frac{1}{2}$

So let's return to Cuntz's claim that $\mathcal{E}_{\Sigma_F} \rtimes \mathbb{Z} = B$ is equivalent to P_F. You have analyzed a representation of B and found something. You have $h_0 \geq 0$, which together with the unitary u generates B. Your approach was to form the subspace $\overline{h_0 H} = W$, then use relations $h_0 u^n h_0 = 0$ for $n \geq 2$. By working with W you effectively replace h_0 by the corresp. projector. ~~Everything is simple.~~ So in $\mathcal{L}(W)$ you have this positive family $1 + z^{-1} b + b^* z$ on the circle. In general you want 1 to become h_0^2. So what happens is you introduce $h_0^{1/2}$ $\quad h_n^{1/2} = u^n h_0^{1/2} u^{-n}$ and how does this behave? What's important is

$$h_0^{1/2} = \varepsilon = \varepsilon^*$$

$$z \varepsilon^* u^{-1} \varepsilon + \varepsilon^* \varepsilon + z^{-1} \varepsilon^* u \varepsilon$$

Let's go over things carefully. Consider $H, u, \varepsilon \geq 0$ such that $\varepsilon u^n \varepsilon = 0$ $|n| \geq 2$. Let

$$h_n = u^n \varepsilon^2 u^{-n}$$
$$= (u^n \varepsilon u^{-n})^2 \qquad \text{Let } \varepsilon_n = u^n \varepsilon u^{-n}$$

so that $\varepsilon_m \varepsilon_n = u^m \varepsilon u^{-m+n} \varepsilon u^{-n} = 0$ if $|m-n| \geq 2$

Go over this again. Given $H, u, \varepsilon = \varepsilon^* \geq 0$ such that $\varepsilon u^n \varepsilon = 0$ for $|n| \geq 2$. Put $\varepsilon_n = u^n \varepsilon u^{-n}$ and then $\varepsilon_m \varepsilon_n = 0$ for $|m-n| \geq 2$. Assume that $\sum_n u^n \varepsilon H$ is dense in H, i.e. the subspace εH generates H under u.

$$u^n \varepsilon H = u^n \varepsilon u^{-n} H = \varepsilon_n H$$

Assume that $\forall \xi \in H$ that $\sum \varepsilon_n^2 \xi = \xi$.
Suffices to take $\xi \in \varepsilon_0 H$.

It seems that a partition of id in the Hilbert space context is a family of operators $k_j \geq 0$ $\ni$ $\sum_j k_j^2 = 1$. Is there an analog (non-comm.

$$a_1, \ldots, a_n \geq 0. \qquad a_i (\sum a_i^2)^{-1/2} \quad \text{not hermitian.}$$

Start again. Begin with $\varepsilon : W \to H \circlearrowleft u$ get positive definite function $n \mapsto \varepsilon^* u^n \varepsilon$, $\mathbb{Z} \mapsto \mathcal{L}(W)$, which means I think that the "function" on S^1 $\sum z^{-n} \varepsilon^* u^n \varepsilon$ on S^1 with hermitian operator values is ≥ 0. (In general function is to be replaced by measure)

Now assume $\varepsilon^* u^n \varepsilon = 0$ for $|n| \geq 2$. Your function amounts to 3 operators $\underbrace{\varepsilon^* u^{-1} \varepsilon}_{P_{-1}}$, $\underbrace{\varepsilon^* \varepsilon}_{P_0}$, $\underbrace{\varepsilon^* u \varepsilon}_{P_1}$