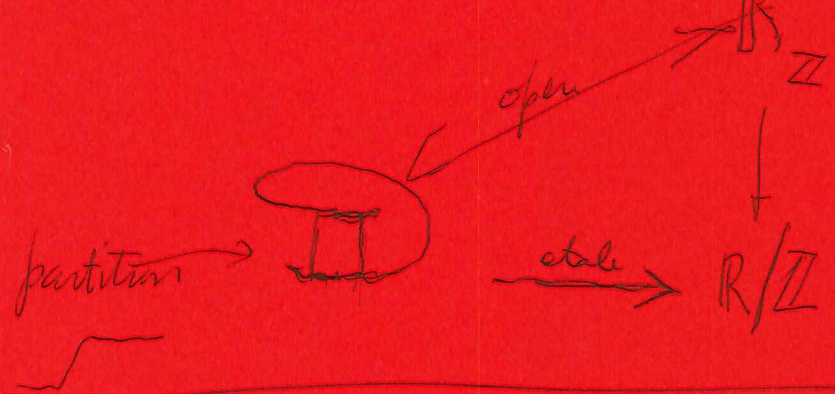
Let us try to construct an equivalence of the groupoid $\mathbb{R}_{\mathbb{Z}}$ with $\mathbb{R}/\mathbb{Z}$. You have a functor $\mathbb{R}_{\mathbb{Z}} \longrightarrow \mathbb{R}/\mathbb{Z}$ which is fully faithful.

$$\mathrm{Hom}_{\mathbb{R}_{\mathbb{Z}}}(x, x') = \begin{cases} \phi & \text{if } x - x' \notin \mathbb{Z} \\ x - x' & \text{if } x - x' \in \mathbb{Z} \end{cases} \quad ??$$

The only idea available at the moment is to use an (open) covering of $\mathbb{R}/\mathbb{Z}$. This introduces?



Repeat:

Back to rings. $C(\mathbb{R}) \rtimes \mathbb{Z}$ or the algebraic version $C_c(\mathbb{R}) \otimes \mathbb{C}[\mathbb{Z}]$. You still have to understand the Morita equivalence with $C(\mathbb{R}/\mathbb{Z})$.

Go back to the old viewpoint, which is fairly alg. The basic bi-module is $P = C_c(\mathbb{R})$ acted on the left by $C_c(\mathbb{R}) \otimes \mathbb{C}[\mathbb{Z}] = B$ and on the right by $C(\mathbb{R}/\mathbb{Z}) = A$.

$\mathrm{Hom}_A(P, A) = ?$ Do you have any feeling about P as an A-module? Look at the submodule $\{f \in P = C_c(\mathbb{R}) \mid f = 0 \text{ on } \mathbb{Z}\} = P_0$. We have continuous functions on $\mathbb{R}$ with compact support and vanishing on $\mathbb{Z}$. So

$$P_0 = \bigoplus_{n\in\mathbb{Z}} C((n,n+1)) \simeq \mathbb{C}[\mathbb{Z}]\otimes C((0,1))$$

what are you learning?

$P = C_c(\mathbb{R})$ left $C_c(\mathbb{R})\otimes\mathbb{C}[\mathbb{Z}]$, right $\overbrace{C(\mathbb{R}/\mathbb{Z})}^{A}$

B should be $P\otimes_A Q$, B is $C_c(\mathbb{R})\otimes_{\mathbb{C}}\mathbb{C}[\mathbb{Z}]$

What is Q? There's a problem here with A being unital, NO. A firm dual pair will consist of P unitary A^{op}-mod, Q unitary A-mod., surj. A-bimod pairing $Q\otimes_{\mathbb{C}} P \twoheadrightarrow A$. Then $Q\otimes_B P \simeq A$ which implies $\sum_1^n \langle q_i, p_i\rangle = 1$

$$p \longmapsto (pq_i)_1^n \longmapsto \sum_1^n (pq_i)p_i = p$$

$$P \xrightarrow{(\cdot q_i)} B^n \subset \tilde{B}^n \xrightarrow{(\cdot p_i)} P$$

$B = P\otimes_A Q = C_c(\mathbb{R}) \otimes_{A=C(\mathbb{R}/\mathbb{Z})} Q$ What is Q? Something like $C(\mathbb{R}/\mathbb{Z})\otimes\mathbb{C}[\mathbb{Z}]$

It seems that Q must be $A\otimes_{\mathbb{C}}[\mathbb{Z}] = A[u,u^{-1}]$.

Functions on a torus? So what can I do? Specify P, Q. $P = C_c(\mathbb{R})$. $A = C(\mathbb{R}/\mathbb{Z})$ acts (right) on it by mult. $Q = A[\mathbb{Z}]$ where A acts via mult. Pairing $Q\otimes P = A[\mathbb{Z}]\otimes C_c(\mathbb{R}) \overset{?}{\to} A$

The obvious thing coming to mind should

involve $f \longmapsto \sum_n f(x+n)$

Today you hope to finally understand the Morita equivalence between $B = C_c(\mathbb{R}) \tilde{\otimes} \mathbb{C}[\mathbb{Z}]$ and $A = C(\mathbb{R}/\mathbb{Z})$.

A is commutative, geometrically you have a situation over the circle $\mathbb{R}/\mathbb{Z}$, should examine what happens over a point $\pi(y) = y + \mathbb{Z} \in \mathbb{R}/\mathbb{Z}$. Then A, B become $A_{\pi y} = \mathbb{C}$, $B_{\pi y} = C_c(y+\mathbb{Z}) \tilde{\otimes} \mathbb{C}[\mathbb{Z}]$

$P_{\pi y} = C_c(y+\mathbb{Z}) = \bigoplus_{n \in \mathbb{Z}} \mathbb{C} e_{y+n}$. You seem to be involved with ~~[illegible]~~ a situation holding for any group Γ.

$$B = C_c(\Gamma) \otimes \mathbb{C}[\Gamma]$$

Basic equivalence: Γ-modules with Γ-gradings

$$M = \bigoplus_{s \in \Gamma} M_s \qquad tM_s = M_{ts} \qquad \underset{\parallel}{\bigoplus_s} sN$$

functor is $\quad M \longmapsto M_1$, $\quad N \longmapsto \overline{\mathbb{C}[\Gamma] \otimes N}$

So P here appears to be $\mathbb{C}[\Gamma]$

$$M \longmapsto M_1 = e_1 M. \qquad C_c(\Gamma) = \bigoplus_{s \in \Gamma} \mathbb{C} e_s$$
$$= e_1 B \otimes_B M$$

$$Q = e_1 B = \bigoplus_{s \in \Gamma} \mathbb{C} e_1 s \simeq \mathbb{C}[\Gamma]$$ considered as a right B-module

---

$$\sum_{s \in \Gamma} [s] n_{t^{-1}s}$$
$$\parallel$$

$$N \qquad \mathbb{C}[\Gamma] \otimes N = \bigoplus_{s \in \Gamma} [s] N$$

$$e_t \left(\sum_{s \in \Gamma} [s] n_s \right) = [t] n_t \qquad t \sum_s [s] n_s = \sum_s [ts] n_s$$

B has basis $e_s[t]$ $\qquad P = \mathbb{C}[\Gamma]$ has basis $[t]$

$$(e_{s_1}[s])[t] = e_{s_1}[st] = \begin{cases} 0 & s_1 \neq st \\ [s_1] & s_1 = st \end{cases}$$

$$(p\,q)\,p' = p\langle q, p'\rangle$$

confused again.

$B = C_c(\Gamma) \otimes \mathbb{C}[\Gamma]$ $\qquad$ basis $e_s \otimes t$

$$(e_s \otimes t)(e_{s'} \otimes t') = e_s e_{ts'} \otimes tt' = \begin{cases} e_{ts'} \otimes tt' & \text{if } s = ts' \\ 0 & \text{otherwise} \end{cases}$$

$$(p \otimes q)\,p' = q\langle q, p'\rangle$$

$$P = \bigoplus_{s \in \Gamma} \mathbb{C}[s]$$

$$e_t[s] = \delta_{ts}[s]$$
$$t[s] = [ts].$$

$$e_s[t] = \delta_{st}[t]$$
$$s[t] = [st]$$

$$s e_{s'} = e_{ss'} s \quad ?$$

$$s e_{s'}[t] = \delta_{s',t}[st]$$

$$e_{ss'} s[t] = e_{ss'}[st] = \underbrace{\delta_{ss', st}}_{\delta_{s',t}}[st]$$

$$s e_{s'}[t] = \delta_{s',t}[st]$$

$$e_{s''} s[t] = e_{s''}[st] = \delta_{s'', st}[st]$$

$$\delta_{s^{-1}s'', t}$$

$$e_{s''} s = s e_{s^{-1}s''}$$

So where am I? You now have P described by the basis $[t]$, $t \in \Gamma$, where the B action is

$$\begin{cases} e_s[t] = \delta_{s,t}[t] \\ s[t] = [st] \end{cases} \qquad \boxed{s e_{s'} = e_{ss'} s}$$

Is it clear that $Be_1 \xrightarrow{\sim} P$? Make B operate on $[1]$.

$$e_s t[1] = e_s[t] = \delta_{s,t}[st] \quad ?$$

Use crossproduct property

$$\cancel{s}e_1 \longmapsto se_1[1] = s[1].$$

$$B \xleftarrow{\sim} \mathbb{C}[\Gamma] \otimes C_c(\Gamma) \Longrightarrow Be_1 = \mathbb{C}[\Gamma]e_1 \xrightarrow{\sim} P$$

$$te_s \qquad t \otimes e_s$$

$$\| $$

$$e_{ts} t$$

$$\begin{tikzcd} B \arrow[rr, "b \mapsto b[1]"] \arrow[d] & & P \\ Be_1 \arrow[urr] & & \\ te_1 \arrow[r, mapsto] & te_1[1] = t[1] = [t] & \end{tikzcd}$$

So what next? ~~[illegible]~~

$$e_1 B = \cancel{\ldots}\ e_1\Big(\bigoplus_{s,t} \mathbb{C}e_s t\Big) = \bigoplus_{t \in \Gamma} \mathbb{C}e_1 t$$

Maybe it is better to use B as a ring of operators on $P = \mathbb{C}[\Gamma][1]$. So go ~~back~~ to

$$(t \otimes e_s)[s'] = t\,\delta_{s,s'}[s'] = [ts']\delta_{s,s'} \qquad \text{This doesn't work}$$

What's confusing is that $B = P \otimes_{\mathbb{C}} Q$ where $P = Be_1$, $Q = e_1 B$, + you want $P = C[\Gamma]e_1$ $\quad Q = e_1 C_c(\Gamma)$

missing twist.

B = ~~space~~ ring of operators with basis te_s

$B e_1$ = space of ops. with basis te_1

$e_1 B$ = ———————————— $e_1 s$

It seems the pairing is obvious, except you want to write it in terms of compactly supported fns instead of basis elts. You hunt for a suitable notation.

P should be $C_c(\mathbb{R})$ or $C_c(y+\mathbb{Z})$

How do you ~~treat~~ interpret e_1? e_y?

$B = C_c(\mathbb{R}) \rtimes \mathbb{Z} = C_c(\mathbb{R}) \otimes \mathbb{C}[\mathbb{Z}]$ cross prod.

~~where do you put~~ It should not matter whether you write $B = C_c(\mathbb{R}) \otimes \mathbb{C}[\mathbb{Z}]$ or $= \mathbb{C}[\mathbb{Z}] \otimes C_c(\mathbb{R})$. In any case B is some alg of operators on $C_c(\mathbb{R}) = P$. It is a B-module. Come on. B acts on P. You seek a Morita equivalence of B with $A = C(\mathbb{R}/\mathbb{Z})$, which is unital. Everything takes place over the circle. A unital should force $P, Q \in \mathcal{P}(B)$, $\mathcal{P}(B^{op})$ resp.

The idea is to restrict supports to lie over a closed arc of $\mathbb{R}/\mathbb{Z}$. This will define your e_1. You expect P, Q to be dual over B.

Let $e^2 = e$ in B, are Be and eB dual over B?

$$\mathrm{Hom}_B(Be, \tilde{B}) = ?$$

$$\tilde{B} \xrightarrow{\cdot e} Be$$

$$\begin{tikzcd} Be \arrow[r, hook] \arrow[rr, bend right, "1"'] & \tilde{B} \arrow[r, "e"] & Be \end{tikzcd}$$

$\therefore$ $Be \hookrightarrow \tilde{B} \xrightarrow{\cdot e} Be$ is the identity

$$\tilde{B} \xrightarrow{\cdot e} Be \xhookrightarrow{\iota} \tilde{B}$$

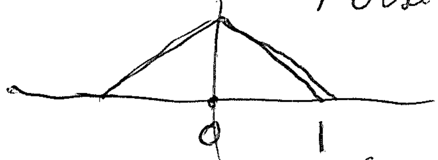
$$\mathrm{Hom}_B(Be, \tilde{B}) = \mathrm{Hom}_{\tilde{B}}(\tilde{B}e, \tilde{B}) = e\tilde{B} = eB$$

Take $B = C_c(\mathbb{R}) \times \mathbb{Z} = C_c(\mathbb{R}) \tilde{\otimes} \mathbb{C}[\mathbb{Z}]$ cr. prod.

$P = C_c(\mathbb{R})$. Question whether $P = Be$ for some e in B.

$P = C_c(\mathbb{R})$ $\qquad B = C_c(\mathbb{R}) \tilde{\otimes} \mathbb{C}[\mathbb{Z}]$. You first show that $P \in \mathcal{P}(\tilde{B})$ and $P = BP$. Idea: use covering of $\mathbb{R}/\mathbb{Z}$ by two sets. First you have

$$B \xrightarrow{\cdot h_0} P$$



because $u^{\pm n} h_0 = h_n$ and $\left(\sum h_n - 1\right) f = 0$ for $f \in P$. Thus Bh_0 contains approx. to the identity, etc.

Since $B \xrightarrow{\cdot h_0} P$ is surjective, assuming P proj B-module, there is a lift $P \longrightarrow B$ wrt $\cdot h_0$. Can you construct interesting B-linear map from P to B. Actually can you compute $\mathrm{Hom}_B(P, B)$ which should be the module Q for the Morita equivalence. What else

Replace P by $P_{\pi(y)} = \{ f \in C_c(\mathbb{R}) \mid f(y+\mathbb{Z}) = 0 \}$

$$= \bigoplus_{n \in \mathbb{Z}} \underbrace{C(y+n+I)}_{(y+n,\, y+n+1)} = \bigoplus_n u^n C(y+I)$$

$$= \mathbb{C}[\mathbb{Z}] \otimes C(y+I)$$

$$\mathrm{Hom}_B(P_{\pi y}, B) = \mathrm{Hom}_{C_c(\mathbb{R})}(C(y+I), B)$$

$$\overset{?}{=} \mathrm{Hom}_{C_c(\mathbb{R})}(C(y+I), C_c(\mathbb{R})) \otimes \mathbb{C}[\mathbb{Z}]$$

$$\mathrm{Hom}_{C_c(\mathbb{R})}\big(C(I), C_c(\mathbb{R})\big)$$

$$\begin{tikzcd}
C(I) \arrow[r, hook] \arrow[d, equal] & C_c(\mathbb{R}) \arrow[d] \\
C(I) \arrow[r, hook] & C(I_+)
\end{tikzcd}$$

Let I be an open arc on the circle S.
Can you determine

$$\mathrm{Hom}_{C(S)}\big(C(I), C(S)\big)$$

I open

$$\begin{tikzcd}
 & C(S-\bar{I}) \arrow[d, hook] & \\
C(I) \arrow[r, hook] & C(S) \arrow[r, two heads] \arrow[d, two heads] & C(S-I) \arrow[d] \\
 & C(\bar{I}) \arrow[r] & C(pt \sqcup pt)
\end{tikzcd}$$

the wrong approach I think.

Start again. $P = C_c(\mathbb{R})$ B, A bimodule

$B \xrightarrow{\cdot h_0} P$ but now you need B-mod maps

$P \longrightarrow B$.   Localize.   Fix a proper arc
of $\mathbb{R}/\mathbb{Z}$. You want to replace P by
$P_{\pi y} = \{ f \in P = C_c(\mathbb{R}) \mid f(y+\mathbb{Z}) = 0 \}$. This is a B-submod

~~[illegible]~~ of P with following structure

$$P_{\pi y} = \bigoplus_{n \in \mathbb{Z}} C(y+n+(0,1))$$

$$= \bigoplus_n u^n C(y+(0,1)) = \mathbb{C}[\mathbb{Z}] \otimes C(y+(0,1))$$

more generally given (a,b) a small int. of $\mathbb{R}$, consider

$$P_{\pi(a,b)} = \bigoplus_n C(n+(a,b)) = \mathbb{C}[\mathbb{Z}] \otimes C((a,b)).$$

Can you produce B-linear map from $P_{\pi(a,b)}$ to B. You think it should be enough to produce a ~~[illegible]~~ $C_c(\mathbb{R})$-module map from $C(a,b)$ to $B = \mathbb{C}[\mathbb{Z}] \otimes C_c(\mathbb{R})$ i.e.

$$C((a,b)) \longrightarrow \mathbb{C}[\mathbb{Z}] \otimes C_c(\mathbb{R})$$

~~[illegible] Consider what goes wo~~

Start again. $P = C_c(\mathbb{R})$ is a left B-module, where $B =$ ~~[illegible]~~ $C_c(\mathbb{R}) \rtimes \mathbb{Z}$ $= \mathbb{Z} \ltimes C_c(\mathbb{R})$. You want to show ~~[illegible]~~ ~~[illegible]~~ that P is a f.g. proj. firm B-module, in fact you want to produce $e \in B$ s.t. $P = Be$.

~~[illegible]~~ Let's review the situation. Aim to construct Mor. eq. between $A = C(\mathbb{R}/\mathbb{Z})$ and $B = \mathbb{Z} \ltimes C_c(\mathbb{R}) = \mathbb{C}[\mathbb{Z}] \otimes C_c(\mathbb{R})$ with twisted or crossed mult, cross prod. mult. You want to produce a firm dual pair $(P, Q, Q \otimes_{\mathbb{C}} P \to A)$ such that, together with $P \otimes_A Q \simeq B$. Since A is unital P, Q must be unitary over A, and it should be the case that P, resp Q is f.g. proj over $\tilde{B}$ (resp $\tilde{B}^{op}$) and firm $P = BP$, $Q = QB$. Since everything is A-linear, you are working over the circle, so you should be able to ~~reduce~~ specialize to any pt $\pi(y) \in \mathbb{R}/\mathbb{Z}$, whence $C_c(\mathbb{R}) = P$ becomes $P_y \simeq C(y+\mathbb{Z}) = \mathbb{C}[y+\mathbb{Z}]$, ~~Thus you have~~ and B becomes $B_y = \mathbb{Z} \ltimes C(y+\mathbb{Z})$. (Still confused with left + right – you want $B = P \otimes_A Q = Be \otimes_A eB$.)

Need a way to sort things out. The important idea seems to involve $\mathbb{Z}$-grading. Focus on ~~getting~~ a geometric picture for B-modules. A B-module is like the space of sections of a $\mathbb{Z}$-equivariant vector bundle, or sheaf over $\mathbb{R}$. It is ~~[illegible]~~ continuously graded with respect to $\mathbb{R}$ in some sense.

Look locally near a point $\pi y \in \mathbb{R}/\mathbb{Z}$. Locally here probably should means you take an small enough open interval πJ around πy and replace $C_c(\mathbb{R})$ with $C_c(J+\mathbb{Z})$. ~~You want to keep sections~~

Using $\pi_!$

Take $P = C_c(\mathbb{R}) \Rightarrow \Gamma(\mathbb{R}/\mathbb{Z}, \pi_! \mathcal{O}_{\mathbb{R}})$ as the basic object. Aim: Čech covering argument to explain why P is fin. gen. proj over B.

$X = \mathcal{U} \cup V$, get Mayer-Vietoris

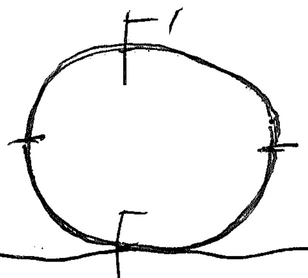
$$0 \longrightarrow C_c(\pi^{-1}(U \cap V)) \to C_c(\pi^{-1}U) \oplus C_c(\pi^{-1}V) \longrightarrow C_c(\mathbb{R}) \to 0$$

splitting given by partition of unity on X, subordinate to the covering.

So what happens is: $C_c(\pi^{-1}U) = C_c(U \times \mathbb{Z})$ $= C_c(U) \otimes \mathbb{C}[\mathbb{Z}]$. Problem: What about the B-module $C_c(\pi^{-1}U) = \mathbb{C}[\mathbb{Z}] \otimes C_c(U')$?

You probably want to avoid asking what sort of $C_c(X)$-module is $C_c(U)$

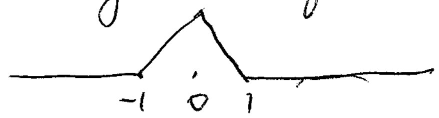
$$0 \to C_c(U \cap V) \longrightarrow C_c(U) \oplus C_c(V) \longrightarrow C_c(\mathbb{R}/\mathbb{Z}) \to 0$$



$$C(X) \xrightarrow{\sim} C(F) \times_{C(F \cap F')} C(F')$$

Important point: To show the identity map of P is nuclear. So you need to construct finitely many elements of P and B-linear maps $P \longrightarrow \tilde{B}$ with certain properties.

You seem to be heading toward a link between nuclearity and functions with compact support. In the locally compact space setting bounded functions pair with functions vanishing at ∞.

Back to the module $P = C_c(\mathbb{R})$ over the crossproduct ring $B = \mathbb{C}[\mathbb{Z}] \overset{\triangle}{\otimes} C_c(\mathbb{R})$. You have a surjection of B-modules $B \xrightarrow{\cdot h_0} P$ where h_0 :  , and you would like enough B-module maps $P \longrightarrow B$ to construct a section of $\cdot h_0$.

Idea ~~is to~~ use $\qquad 1 = |\cos \pi x|^2 + |\sin \pi x|^2$

So $|\sin \pi x|$ has period 1, vanishes on $\mathbb{Z}$
$\quad\ |\cos \pi x|$ ——————————— $\frac{1}{2} + \mathbb{Z}$

Take $f \in P$ mult. by $|\sin \pi x|$

$|\sin \pi x| f \in P_{(0+\mathbb{Z})}$ van. on $0+\mathbb{Z}$

$I = (0,1)$

$$P_{(0+\mathbb{Z})} = \bigoplus_{n \in \mathbb{Z}} \underbrace{C(n+I)}_{\substack{\text{cont fns. on } [n,n+1] \\ \text{vanishing on end points}}}$$

$$= \bigoplus_n u^n C(I) \simeq \mathbb{C}[\mathbb{Z}] \otimes_{\mathbb{C}} C(I)$$

and this nicely embeds into $B = \mathbb{C}[\mathbb{Z}] \otimes C_c(\mathbb{R})$

Why? The point is that you have a $C_c(\mathbb{R})$-linear map $C(I) \xrightarrow{\alpha_0} C_c(\mathbb{R})$

Let $\varphi \in P_{(0+\mathbb{Z})}$ cont. comp supp van. on $\mathbb{Z}$.

$$\varphi = \sum \underbrace{u^n \chi_{[0,1]} u^{-n}\varphi} \quad \left(= \sum_n \chi_{[n,n+1]} \varphi\right)$$

$$\alpha(\varphi) = \sum_n u^n \alpha_0\left(\chi_{[0,1]} u^{-n}\varphi\right)$$

---

so if $\varphi = \sum u^n \varphi_n$ $\qquad \varphi_n = \chi_{[0,1]} u^{-n}\varphi$

then $\alpha(\varphi) = \sum u^n \alpha_0(\varphi_n)$.

Is α a B-module map?

YES

$$u^m \varphi = \sum_n u^{m+n} \varphi_n$$

$$\alpha(u^m\varphi) = \sum_n{}' u^{m+n} \alpha_0(\varphi_n) = u^m \overbrace{\sum_n u^n \alpha_0(\varphi_n)}^{\alpha(\varphi)}$$

$f \in C_c(\mathbb{R})$.

$$f\varphi = \sum_n u^n \underbrace{\chi_{[0,1]}(u^{-n}(f\varphi))}_{f_n \varphi_n}$$

f_n bdd cont on $[0,1]$

$$= \sum_n (u^n f_n)(u^n \varphi_n)$$

$$f\alpha(\varphi) = f \sum_n u^n \alpha_0(\varphi_n) = \sum_n f u^n (\alpha_0(\varphi_n))$$

can be rep. by $u^n f_n$

Keep trying to clarify the picture.

Let's try other ideas. The importance of a suitable $\mathbb{Z}$-grading. Maybe go back to Poisson summation. You still need to understand the dual pair P, Q over $A = C(\mathbb{R}/\mathbb{Z})$. Other ideas involve being over the circles. You know that $C_c(\mathbb{R})$ is ~~[illegible]~~ a unitary A-module. If you take a Hilbert space viewpoint, i.e. interpret P as ~~[illegible]~~ inside $L^2(\mathbb{R})$, ~~[illegible]~~ and B as inside a C*-alg of operators on $L^2(\mathbb{R})$ containing $f(x)$ $f \in C(\mathbb{R})$ (mult. ops.) and the translation operators $\{u^n\}_{n\in\mathbb{Z}}$, also you have mult. by periodic fns. e.g. $e^{2\pi i n x}$,

Recall ~~[illegible]~~ points of view. Originally instead of $C_c(\mathbb{R})$ you considered the Schwartz space $S(\mathbb{R})$. ~~[illegible]~~ The aim then was to ~~[illegible]~~ construct the Hopf line bundle on the 2-torus, a line bundle of degree 1. Somewhere along the line you brought in $\mathbb{Z} \ltimes S(\mathbb{R})$. Maybe this stuff works for any Γ.

For general Γ ~~you have $E\Gamma$~~ say torsion-free you have $E\Gamma \xrightarrow{\pi} B\Gamma$, you get a flat? line bundle $\mathcal{L}$ for the group ring $\mathbb{C}[\Gamma]$ over $B\Gamma$, whose fibre at a point $x \in B\Gamma$ is $\mathbb{C}[\pi^{-1}(x)]$ which is a free rank 1 $\mathbb{C}[\Gamma]$-mod. Can complete

$\mathbb{R} \xrightarrow{\pi} \mathbb{R}/\mathbb{Z}$ I think it's true that if $\mathcal{O}_{\mathbb{R}}$ = sheaf of cont. fns. on $\mathbb{R}$, then $\pi_! \mathcal{O}_{\mathbb{R}}$ is the sheaf of continuous sections of the "canonical line bundle" for the group ring.

Something interesting here is the way you produce the family of compact operators over the circle.

Questions: Exact Nature, details, of the Morita equivalence between $\mathbb{C}[\mathbb{Z}] \ltimes C_c(\mathbb{R})$ and $C(\mathbb{R}/\mathbb{Z})$, explicit description of P, Q and the pairing. What does the multiplier algebra look like?

$P = C_c(\mathbb{R})$ is a right $\underbrace{C(\mathbb{R}/\mathbb{Z})}_{A}$-module

~~You can't do anything~~

Replace $C_c(\mathbb{R})$ by $\mathcal{S}(\mathbb{R})$. Then what is $\mathrm{Hom}_A(P, A)$. What is the smooth version? Is there a smooth version?

$P = \mathcal{S}(\mathbb{R})$ $\qquad B = \mathbb{Z} \ltimes \mathcal{S}(\mathbb{R})$

Can you find $\mathrm{Hom}_B(P, B)$. Note that there is a B-mod surjection $B \twoheadrightarrow P$ Take  translate by $\mathbb{Z}$, divide to get partition of 1.

$$\mathrm{Hom}_B(P, B) \longrightarrow \mathrm{Hom}_B(B, B)$$

What is

$\mathrm{Hom}_B(P, P)$ $\qquad P = C_c(\mathbb{R})$

$T : P \longrightarrow P$ $\qquad T$ commute with B

guess that if T commutes with P-mult then T is multiplication by a cont. function; then T commutes with $\mathbb{Z}$ translations if T is periodic i.e. $T \in C(\mathbb{R}/\mathbb{Z})$. This seems OK. Next you want $\mathrm{Hom}_B(P, B)$. Can you construct some map from P to $B = (\oplus u^n) \otimes P$

$$B = \bigoplus_{n \in \mathbb{Z}} P u^n \qquad \text{as a } P\text{-module.}$$

So a hom $P \longrightarrow \oplus P u^n$, let's assume known that $\mathrm{Hom}_P(P, P)^n =$ all cont. functions on $\mathbb{R}$

Let $T \in \mathrm{Hom}_B(P, \oplus P u^n)$, so it seems clear that

$$T = \sum_{n \in \mathbb{Z}} T_n u^n \qquad \text{with } T_n \text{ cont on } \mathbb{R}$$

$$Tu = uT \qquad u T_n u^{-1} u^{n+1} = T_n u^{n+1}$$

so each $T_n \in \mathbb{C}(\mathbb{R}/\mathbb{Z})$, so it seems that

$$Q = \mathrm{Hom}_B(P, B) = A \otimes \mathbb{C}[\mathbb{Z}]$$

So how is $Q = \bigoplus A u^n$ a right B module

$$q = \sum a_n u^n \in \mathrm{Hom}_B\Big(P, \underbrace{\bigoplus_{n\in\mathbb{Z}} P u^n}_{B}\Big) = Q$$

~~If $p \in P$, then~~

Begin again. B is a ring of ops. on $P = C_c(\mathbb{R})$ commuting with $A = C(\mathbb{R}/\mathbb{Z})$ mult, not all

$$Q \longrightarrow \mathrm{Hom}_B(P, B)$$

$B = P \otimes_A Q$ should be true.

need $Q \times P \longrightarrow A$

$Q \otimes_B P \to A$

Start with B

$\Rightarrow Q \longrightarrow \mathrm{Hom}_{A^{op}}(P, A)$

$P \otimes_A Q \longrightarrow B$

$Q \longrightarrow \mathrm{Hom}_B(P, B)$

$B = P \overset{\Delta}{\otimes} \mathbb{C}[\mathbb{Z}]$

$\mathrm{Hom}_B(P, B)$
ψ

$$T = \sum T_n(x)\, u^n$$

$$Tu = \sum T_n u^{n+1}$$

$$uT = \sum (u T_n u^{-1})\, u^{n+1}$$

$$f \longmapsto af$$

At the end yesterday you found Q.
$P = C_c(\mathbb{R})$ $\quad B = C_c(\mathbb{R}) \overset{\triangleleft}{\otimes} \mathbb{C}[\mathbb{Z}]$ ring of operators on P.

Let $T \in \mathrm{Hom}_B(P,P)$. T commutes with the subring of multiplication operators given by elements in P.
So $T \in \mathrm{Hom}_P(P,P)$ which should mean that T is multiplication by a continuous function $T(x)$ on $\mathbb{R}$

$(Tf)(x) = T(x)f(x)$. But T also commutes with σ: $(\sigma f)(x) = f(x-1)$. So

$$(T\sigma f)(x) = T(x)f(x-1)$$
$$\parallel$$
$$(\sigma T f)(x) = T(x-1)f(x-1)$$

$\therefore$ $T(x) = T(x-1)$
T is periodic.

So $\mathrm{Hom}_B(P,P) = \mathbb{C}(\mathbb{R}/\mathbb{Z}) = A$ as expected.

Next Let $T \in \mathrm{Hom}_B(P,B)$, given $f \in P$

$$Tf = \sum (T_n f) u^n \quad \text{imagine}$$

so $T_n \in \mathrm{Hom}_P(P,P)$ $\therefore T_n$ is mult. by $T_n(x)$ cont on $\mathbb{R}$

$$T(\sigma f) = \sum (T_n \sigma f)\, u^n$$
$$u(Tf) = \sum u(T_n f)\, u^n = \sum \sigma(T_n f)\, u^{n+1}$$

$$\therefore \quad T_{n+1}\sigma f = \sigma(T_n f)$$

$$T_{n+1}(x) f(x-1) = T_n(x-1) f(x-1)$$

So you find that $T_{n+1}(x) = T_n(x-1)$

$$T_{n+1}(x+1) = T_n(x) \quad \cancel{\varphi(x)}$$

$$\therefore \quad T_n(x) = \varphi(x-n) \qquad \text{for all } n, x.$$

$\varphi = T_0$

$$T_n = \sigma^n \varphi$$

$\therefore$ ~~$Tf = \sum \varphi(x-n)u^n \in B$~~

$$Tf = \sum_n \sigma^n(\varphi) f u^n = \sum_n u^n \varphi u^{-n} f u^n$$

$$= \sum_n u^n \underbrace{(\varphi \, \sigma^{-n} f)}_{\varphi(x) f(x+n)} = \sum_n \varphi(x+n) f(x) u^n$$

so you have a finite sum iff $\varphi \in C_c(\mathbb{R})$.

$$Tf = \sum_n T_0(x-n) f(x) u^n$$

Repeat $T : P \longrightarrow B = \bigoplus_{n \in \mathbb{Z}} P u^n$

is B-linear, i.e. P linear and comp. with u.

P linear $\Rightarrow$ $Tf = \sum (T_n f) u^n$ where $T_n = T_n(x)$. is a cont. fn on $\mathbb{R}$.

$$T\sigma f = \sum (T_n \sigma(f)) u^n \qquad \therefore T_{n+1} \sigma\cancel{(f)} = \sigma(T_n)\cancel{\sigma(f)}$$

$$uTf = \sum \sigma(T_n)\sigma(f) u^{n+1} \qquad T_{n+1}(x) = T_n(x-1)$$

$$T_n(x) = T_{n-1}(x-1) = T_{n-2}(x-2) = \cdots T_0(x-n)$$

Violin 2

$$\therefore \quad Tf = \sum_n T_0(x-n) f(x) u^n$$

this sum should be finite for any f.

$\Rightarrow T_0 \in C_c(\mathbb{R})$.

$$\mathrm{Hom}_B(P, B) \simeq \underbrace{C_c(\mathbb{R})}$$

$$\Big(f \mapsto \underbrace{\sum_n \varphi(x-n) f(x) u^n}_{\sum_n u^n \varphi(x) f(x+n)} \Big) \longleftarrow\!\shortmid \ \varphi$$

This looks like a pairing from $C_c(\mathbb{R}) \times C_c(\mathbb{R})$ to $A = C(\mathbb{R}/\mathbb{Z})$.

So let's go over this for a better understand. ~~So back to~~ the motivation arising from the assembly map for the group $\mathbb{Z}$. In general given Γ descrete with universal bundle $E\Gamma \xrightarrow{\pi} B\Gamma$, ~~you have a~~ ~~...~~ which is a locally trivial fibre bundle with $\pi^{-1}(x) \simeq \Gamma$ as right Γ-set, hence taking chains ~~you~~ get a locally trivial fibre bundle $x \mapsto \mathbb{C}[\pi^{-1}(x)]$ with fibre a free rank 1 right $\mathbb{C}[\Gamma]$-module. Get a tautological [redacted] line bundle for the group ring $\mathbb{C}[\Gamma]$ over $B\Gamma$. You then can ~~...~~ get ~~the~~ line bundle bundles for other unital rings A via unital homom. $\mathbb{C}[\Gamma] \longrightarrow A$, the most important being $C_r(\Gamma)$, the reduced C^*-algebra.

For $\Gamma = \mathbb{Z}$, $\quad C_r(\mathbb{Z}) \simeq C(S^1)$ character group of $\mathbb{Z}$

So you get a family of line bundles over the circle $\mathbb{Z}^\vee$ parametrized by the circle $\mathbb{R}/\mathbb{Z}$, i.e. a ~~~~ line bundle over the 2 torus $\mathbb{R}/\mathbb{Z} \times \mathbb{Z}^\vee$

Go back to $T \in \mathrm{Hom}_B(P, B)$. Since T is P linear you should have

$$Tf = \sum (T_n f) u^n$$

with T_n a mult. op. ~~~~ from $T(\sigma f) = u(Tf)$ you get

$$\sum_n (T_n \sigma f) u^n = \sum_n ({}^\sigma T_n {}^\sigma f)\, u^{n+1}$$

whence ${}^\sigma T_n = T_{n+1}$ or $T_n(x-1) = T_{n+1}(x)$

or $T_n(x) = T_{n-1}(x-1) = \cdots = T_0(x-n)$. So

$$Tf = \sum_n T_0(x-n) f(x)\, u^n \in B$$

But you ~~want to~~ should write elements of $B = P \otimes_A Q$ as sums of "rank 1 operators" $p \otimes q$ $p = f$, $q = T_0$ above.

$$Tf = \sum_n f u^n T_0 = f\left(\sum u^n\right) T_0 .$$

~~~~ What seems to be true is that $P = C_c(\mathbb{R}) = Q$, and the pairing is

$$\langle \overset{\varphi}{q}, \overset{f}{p} \rangle = \left(\sum \sigma^n\right) \overset{\varphi}{q} \overset{f}{p} = \sum_n (\overset{\varphi f}{qp})(x-n) \in A = C(\mathbb{R}/\mathbb{Z})$$

~~~~ and

$$(f \otimes \varphi) f_1 = f \left(\sum \sigma^n\right)(\varphi f_1)$$

Yesterday you found the dual pair over $A = C(\mathbb{R}/\mathbb{Z})$, namely $P = Q = C_c(\mathbb{R})$ and the pairing $Q \times P \longrightarrow A$ is $\langle q, p \rangle = \sum_{n \in \mathbb{Z}} \sigma^n(qp) = \sum_{n\in\mathbb{Z}} (qp)(x-n)$. One then has a canonical ring homom. $P \otimes_A Q \longrightarrow B$, which should turn out to be an isom. Recall $B = \underbrace{P \otimes \mathbb{C}[\mathbb{Z}]}_{P \otimes_A A[\mathbb{Z}]}$

$A[\mathbb{Z}]$ and Q are close, but not the same. Is ~~there~~ it possible to specify "close". Both are $A[\mathbb{Z}]$-modules ?

Review yesterday calculation.

$$B = C_c(\mathbb{R}) \otimes \mathbb{C}[\mathbb{Z}] = \bigoplus_{n \in \mathbb{Z}} C_c(\mathbb{R}) \otimes u^n$$

$$u f u^{-1} = \tilde{f} \qquad (\tilde{f})(x) = f(x-1).$$

$P = C_c(\mathbb{R})$ considered as B module with $C_c(\mathbb{R})$ acting via mult and $u \cdot f = \tilde{f}$

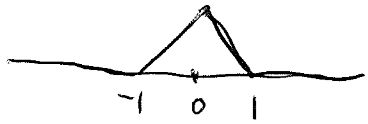
$\mathrm{Hom}_B(P, P)$

What do you need to make a proof that $P \otimes_A Q \xrightarrow{\sim} B$?
What do you need to ~~make~~ prove $P \otimes_A Q \xrightarrow{\sim} B$. One idea is to localize on the circle. Fix $\pi(y) \in \mathbb{R}/\mathbb{Z}$, evaluate $f \in C_c(\mathbb{R})$ on the coset $y + \mathbb{Z}$

so let see what happens.

$$P = \mathbb{C}[y+\mathbb{Z}] \cdot \qquad Q = \mathbb{C}[y+\mathbb{Z}]$$

pairing should be induced by $\langle q p \rangle = \pi_*(qp)$
simply the diagonal pairing.

So what is your aim. Repeat. $A = C(\mathbb{R}/\mathbb{Z})$
unital ring, $P = C_c(\mathbb{R})$, $\pi_* : C_c(\mathbb{R}) \longrightarrow C(\mathbb{R}/\mathbb{Z})$
Thus P is an A-module with a trace,
π_* is surjective $\quad \pi_*(h_0) = 1$.  .

Now P is a ring, so you get a form
dual pair P, Q over A with $P = Q$ $\quad \langle q, p \rangle = \pi_*(qp)$
So you get an algebra $P \otimes_A Q = B$,
operates on the left of P, right of Q

$$(p \otimes q)\, p' = (p\, \pi_* q)\, p'$$

Ask how good the duality between P, Q is?
Ask about $\mathrm{Hom}_A(C_c(\mathbb{R}), A)$

Consider $X = [0,1] \qquad Y = [\tfrac{1}{2}, 1] \qquad X - Y = [0,1)$

Find $\mathrm{Hom}_{C(X)}\big(C(X-Y), C(X)\big)$ $\quad \left(\begin{array}{l} C(X-Y) = \text{ideal of } f \in C(X) \\ \ni f(Y) = 0. \end{array} \right.$

Let $\Phi : \underbrace{C(X-Y)}_{J} \longrightarrow \underbrace{C(X)}_{A}$ be $\underbrace{C(X)}_{A}$ linear

$\Phi \in \mathrm{Hom}_A(J, A)$ but $J = J^2$ so $\Phi(J) = J\Phi(J) \subset J$

therefore $\Phi \in \mathrm{Hom}_A(J,J)$ which should be the same as $\mathrm{Hom}_J(J,J)$ (since J/J^2 and ${}_J J = 0$)

Now $\mathrm{Hom}_J(J,J) =$ bdd cont functions on $X-Y$

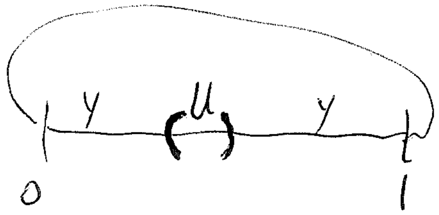
So we get $\mathrm{Hom}_{C(X)}\big(C(X-Y), C(X)\big) = \mathrm{Hom}_{C(X-Y)}\big(C(X-Y), C(X-Y)\big)$

$= C\big($Stone-Čech compact of $X-Y\big)$

Now return to $\mathrm{Hom}_{C(\underbrace{\mathbb{R}/\mathbb{Z}}_{X})}\big(C_c(\mathbb{R}), C(\underbrace{\mathbb{R}/\mathbb{Z}}_{X})\big)$

Take a small $\neq \phi$ closed interval or arc in the circle X Y and let $U = X - Y$. Y is closed arc $\neq \phi$ also $\neq X$, $U = X - Y$. Simplest case is for $Y =$ pt.

Then $\qquad$ Then $C_c(\mathbb{R}) = \bigoplus C(n+$



Try to say this better. $X = \mathbb{R}/\mathbb{Z}$ $\quad Y = \{y + \mathbb{Z}\}$

i.e. you replace $P = C_c(\mathbb{R})$ by the subring $P_{y+\mathbb{Z}}$ of functions vanishing on the coset $y+\mathbb{Z}$, getting

$$P_{y+\mathbb{Z}} = \bigoplus_{n\in\mathbb{Z}} C(y+n, y+n+1)$$

But

$$\mathrm{Hom}_{C(\mathbb{R}/\mathbb{Z})}\big(C(y+n, y+n+1), C(\mathbb{R}/\mathbb{Z})\big)$$

should be the ring of bdd cont. functions on $(y+n, y+n+1)$

So $\mathrm{Hom}_{C(\mathbb{R}/\mathbb{Z})}\left(P_y, C(\mathbb{R}/\mathbb{Z})\right) = \prod_n$ bdd cont fns on $(y+n, y+n+1)$

Now can you use the presentation

$$0 \longrightarrow P_{y,y_1} \longrightarrow P_y \oplus P_{y_1} \longrightarrow P \longrightarrow 0$$

It seems that $\mathrm{Hom}_{C(\mathbb{R}/\mathbb{Z})}\left(P, C(\mathbb{R}/\mathbb{Z})\right) =$ all cont fns. on $\mathbb{R}$

What is the assembly map [redacted] in the case $\Gamma = \mathbb{Z}$.
In general it goes from [redacted] ? to ? There is a canonical family, [redacted] parametrized by points of the classifying space $B\Gamma$, of free rank 1 modules over the group ring $\mathbb{C}[\Gamma]$. In particular, if you have a homom. $\mathbb{C}[\Gamma] \longrightarrow C([redacted]\Omega)$ any comm. C^* alg then you have a family param. by $B\Gamma$ of line bundles over Ω, this should mean you have a canonical line over $B\Gamma \times \Omega$. When Γ abelian, Ω = Pontryagin dual $\Gamma^\vee$, so you have a canon. line bundle over $B\Gamma \times \Gamma^\vee$

Now ~~what does~~ how is this picture related to Morita equivalence ?.

But **first** [redacted] spend some time on the K-theory. The family over $B\Gamma$ of line bundles for $C_R(\Gamma)$ [redacted] should represent an elt of $KK^0(C(B\Gamma), C_r(\Gamma))$?

$$K^*(A) = KK^*(\mathbb{C}, A)$$

$$KK(C(X), C(Y)) \quad ??$$

Go over the rudiments of KK:

$KK^0(\mathbb{C}, C(X))$ is contravariant in X $\therefore = K^0(X)$.

A class is represented by a bimodule roughly. Actually you use a $\mathbb{Z}/2$-graded Hilbert C^*-module over $C(X)$ e.g. $C(X, H)$ where H is $\mathbb{Z}/2$ graded. and comes with an $F = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Then left $\mathbb{C}$-action is a degree 0 family of projectors on H over X. Ultimately you get two projectors on H congruent modulo compacts.

~~one picture emerging. Let~~

You need to understand Kasparov theory

There are puzzles to sort out. First try to reconstruct what you learned from Cuntz's talk. Given Γ and a finite subset F of Γ you get a simplicial complex Σ_F whose simplices are nonempty subsets M of Γ such that $MM^{-1} \subset F$. You need to assume $1 \in F$ for Σ_F to be nonempty. Since $(MM^{-1})^{-1} = MM^{-1}$ you can replace F by $F \cap F^{-1}$ with is closed under -1. Question: Do you want $\underbrace{MM^{-1} \subset F}_{\Gamma \text{ right acts on } \Sigma_F}$ or $\underbrace{M^{-1}M \subset F}_{\Gamma \text{ left acts on } \Sigma_F}$?

~~[illegible]~~ You need to understand the assembly map for a group Γ. ~~[illegible]~~ This links K-homology of $B\Gamma$ to K-theory of $C_r(\Gamma)$. There is a tautological K^0-class ~~[illegible]~~ in $K_0(C(B\Gamma) \otimes \mathbb{C}[\Gamma])$.

~~[illegible]~~ You have $E\Gamma \xrightarrow{\pi} B\Gamma$ universal bundle gives rise to a fibre bundle $E\Gamma \times^{\Gamma} \mathbb{C}[\Gamma]$ over $B\Gamma$ where the fibre is a free rank 1 right $\mathbb{C}[\Gamma]$-modules.

Anyway you have this tautological line bundle for the group ring $\mathbb{C}[\Gamma]$ over $B\Gamma$. Question

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More assembly map. You reviewed a bit about the Novikov Conjecture. M closed orient. manifold with fund. gp π, get map $M \longrightarrow B\pi$, so get $H^*(B\pi) \longrightarrow H^*(M)$ subring of coh. obtained from π. NC conj asserts pairing such a coh. class with $\mathcal{L}$ class is a homotopy invariant of M.

Index thm. reformulation, ~~says that~~ K-theory form., says that the signature operator tensored with a vector bundle coming from $B\pi$ is a homot. invariant

Still missing something. K_{homology} of $M \longrightarrow K_{\text{hom}}$ of $B\pi$ missing the idea that a representation of π gives a local system on M, whose cohomology is a homotopy invariant

More on the assembly map. You start with $\pi: E\Gamma \longrightarrow B\Gamma$. Instead take a principal Γ-bundle $E \xrightarrow{\pi} X$, and form the associated bundle with fibre $\mathbb{C}[\Gamma]$; this is a a [redacted] flat bundle with fibre $\mathbb{C}[\pi^{-1}(x)]$ over $x \in X$.

Program to replace geometry by rings + modules. You are viewing

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Still no progress. Start with the geometry – a group Γ and its universal bundle $E\Gamma \longrightarrow B\Gamma$. The universal bundle is well-defined up to homotopy equivalence. There is a specific theorem stating this, whose proof you might study and abstract. You have geometric constructions related to homotopy Homotopy extension, Covering Homotopy Thms, partitions of unity,

There is a sense in which spaces can be replaced by C^* algs. Gelfand thm. that locally compacts spaces are equiv. to comm. C^* algs.

Too much hot air. Let's go over again the simplest interesting example. $\mathbb{R} \xrightarrow{\pi} \mathbb{R}/\mathbb{Z}$ where $\Gamma = \mathbb{Z}$, $E\mathbb{Z} = \mathbb{R}$, $B\mathbb{Z} = \mathbb{R}/\mathbb{Z}$. This is a geometric situation. Assembly in this case replaces $E\mathbb{Z}$ by the ~~space~~ algebra $C_c(\mathbb{R})$ with $\mathbb{Z}$ acting via translation [redacted]

[redacted]

$$\sigma^n(f(x)) = f(x-n)$$

You always avoid ~~saying~~ statement what's important and new on the C^* alg side. This is the non-comm. C^* alg $C(\mathbb{R}) \rtimes \mathbb{Z}$. This is something new, not a "geometric object" i.e. space, manifold which is familiar.

The question to ~~answer~~ consider is how to get insight here. You might look the cross product for a finite group acting on a locally compact space, or maybe even a compact (Lie) gp acting on such a space.

In the $\mathbb{Z}$ example the ~~space~~ $C(\mathbb{R})$ is a fin. proj module over the crossproduct

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Look at $\mathbb{Z}$-example slowly. Take the "algebraic" model where the C^* alg $C(\mathbb{R}) = \{$all cont fns. on $\mathbb{R}$ vanishing at $\infty\}$ is replaced by the alg $C_c(\mathbb{R})$ of cont fns with comp. supp., and the cross product uses the algebraic group ring $\mathbb{C}[\mathbb{Z}] = \bigoplus_{n\in\mathbb{Z}} \mathbb{C}u^n$

Let's review Cuntz's exposition. Γ discrete group F finite subset of Γ containing 1, Σ_F = simp. complex of finite $\neq \emptyset$ $M \subset \Gamma$ such that $M^{-1}M \subset F$ (Can replace F by $F \cap F^{-1}$ etc.)

$$\mathcal{E}_{\Sigma_F} = \text{noncomm } C^* \text{ alg corresp to } \Sigma_F$$

$$= C^*\left(h_s \geq 0,\ s \in \Gamma;\ h_s h_t = 0 \quad \overbrace{s^{-1}t \notin F}^{s \in tF} \right\}$$

$$\sum_{s \in \Gamma} h_s h_t = h_t \Big)$$

get action straight. $\boxed{s h_t = h_{st} s}$ defines the crossproduct. $\mathcal{E}_{\Sigma_F} \rtimes \Gamma$ You want Γ to acts on Σ_F by left mult.

$(sM)^{-1}(sM) = M^{-1}M$ OK

So as to reconstruct the canonical projection in $\mathcal{E}_{\Sigma_F} \rtimes \Gamma$. $\mathcal{E}_{\Sigma_F}$ is a Γ-algebra ~~so~~ you can form the cross product, and then the crossproduct is Γ-graded. Let A be a Γ-alg, ~~and~~ form $A \rtimes \Gamma$ and ask for $p \in A \rtimes \Gamma$

$p^* = p = p^2$. $\quad p = \sum_{s\in\Gamma} p_s s \quad$ assume finite and

~~what do you get~~

$$p^* = \sum_{s\in\Gamma} s^{-1} p_s^* = \sum_{s\in\Gamma} s\, p_{s^{-1}}^*$$

$$p = \sum_{s\in\Gamma} p_s s = \sum_{s\in\Gamma} s\, p_1 \quad ?$$

$$p^2 = \sum_{\substack{s\in\Gamma\\ t\in\Gamma}} p_s s\, p_t t = \sum_{s,t\in\Gamma} p_s p_{st}\, st = \sum_{t\in\Gamma}\Big(\sum_s p_s p_{st}\Big) st$$

$st = u$

$$= \sum_{u\in\Gamma}\Big(\sum_{\substack{s,t\in\Gamma\\ st=u}} p_s p_t\Big) u$$

$$= \sum_{u\in\Gamma}\Big(\sum_{t\in\Gamma} p_{ut^{-1}} p_t\Big) u$$

$$\boxed{\therefore\ p_u = \sum_t p_{ut^{-1}} p_t}$$

$$\cancel{p = \sum_{s\in\Gamma} p_s s \qquad p^* = \sum s^{-1} p_s^*}$$

$$\cancel{= \sum_{s\in\Gamma} (s p_1) \qquad p^* = \sum p_1^* s^{-1}}$$

What you want is a projector in a Γ-graded alg. $\quad B = \bigoplus_{s\in\Gamma} B_s \qquad B_s B_t = B_{st}$

$$(B_s)^* = B_{s^{-1}}$$

Then $\quad p = \sum_{s\in\Gamma} p_s \qquad p_s \in B_s$

$$\boxed{\begin{aligned} p = p^2 &\iff p_s = \sum_t p_{st^{-1}} p_t \\ p = p^* &\iff p_s^* = p_{s^{-1}} \end{aligned}}$$

Finally you take $\quad B = \mathcal{E}^{A}_{\Sigma_F} \rtimes \Gamma = \bigoplus_{s\in\Gamma} \overset{B_s}{A s}$

You want the insight to produce the formula.

Problem: Find Cuntz's formula for the ~~projector~~ projector

$B = A \rtimes \Gamma$ $\qquad$ A C^*-alg with left Γ action.

Example $\quad B = C(\mathbb{R}) \rtimes \mathbb{Z}$. naturally acts on $A = C(\mathbb{R})$. The construction you seek should mimic the embedding of a vector bundle as a summand of a trivial vector bundle. Review this carefully

X compact space, E vector bundle over X, ~~thus~~ whence $\exists$ $X = \bigcup U_j$ and isos. $\theta_j : E|_{U_j} \xrightarrow{\sim} U_j \times W$,

choose $\sum \chi_j^2 = 1$ partition of 1. χ_j pos. supp in U_j

Wait you need.

$$\Gamma(X, E) \longrightarrow C(X) \otimes \bigoplus_{j=1}^{N} W \longrightarrow \Gamma(X, E)$$

$$\xi \longmapsto \left(\chi_j^{\text{pt}} \theta_j (\xi|_{U_j}) \right)_{1 \le j \le N}$$

$$\left(f_j \in C(X, W) \right) \longmapsto \sum_j \chi_j \quad \theta_j^{-1} f_j$$

notation unclear.

E is a vector bundle over X hence have

$$\Gamma_c(U_j, E) \simeq C_c(U_j) \otimes W \qquad \text{isom of } C(X)\text{-modules.}$$

$$s \longmapsto \theta_j s$$

$$f\, \theta_j^{-1}(1 \otimes w) \longleftarrow f \otimes w$$

$X = \bigcup_{j=1}^{N} U_j \qquad \theta_j : E|_{U_j} \xrightarrow{\sim} U_j \otimes W$

$$\Gamma_c(U_j, E) \xrightarrow{\sim} C_c(U_j) \otimes W$$

$$s \longmapsto \theta_j s$$

$$\theta_j^{-1}(1 \otimes w) \qquad 1 \otimes w$$

So now construct the maps.

$$\begin{tikzcd}
\Gamma(X,E) \arrow[r] & \prod \Gamma_c(U_j, E) \arrow[r] & \Gamma(X,E) \\
\xi \arrow[r, mapsto] & (\chi_j \cdot \xi|_{U_j}) \arrow[r, mapsto] \arrow[d, "\wr"] & \sum_j \chi_j^2 \xi = \xi \arrow[d, dash] \\
 & \bigoplus_j C_c(U_j) \otimes W & \bigoplus_j C(X) \otimes W \arrow[l, hook'] \\
 & \bigcap & \\
 & \bigoplus_j C(X) \otimes W &
\end{tikzcd}$$

$$\xi \longmapsto \chi_j(\xi|_{U_j}) \sim \chi_j \theta_j(\xi|_{U_j}) \longmapsto \chi_j \theta_j(\xi|_{U_j})$$

$$\Gamma_c(U_j, E) \simeq C_c(U_j) \otimes W \xhookrightarrow{\chi_j} C(X) \otimes W$$

E v.b. over X compact. $X = \bigcup U_i$ $\quad E_{U_i} \simeq U_i \times W$

aim to show $\Gamma(X,E)$ is a f. prg. $C(X)$-module. Partition of 1

$\sum x_i^2 = 1$. $\quad x_i \in C_c(U_i)$ $\quad E_{U_i}$

$$\Gamma(X,E) \longrightarrow \bigoplus \Gamma_c(U_i, E|U_i) \longrightarrow \Gamma(X,E)$$

$E_{U_i} \simeq$ ~~[illegible]~~ $U_i \times W$

$\Gamma_c(U_i, E_{U_i}) \simeq C_0(U_i) \otimes W$

Still confused

~~you a map~~

you need maps $\Gamma(X,E) \longrightarrow C(X) \otimes W \longrightarrow \Gamma(X,E)$
~~of~~ $C(X)$-modules whose composition is the identity
Pick a basis for W. Better you seem to want

$W \xrightarrow{\alpha} \Gamma(X,E) \qquad W^* \xrightarrow{\beta} \Gamma(X,E^*)$

such that $W \otimes W^* \longrightarrow \Gamma(X,E) \otimes \Gamma(X,E^*) \longrightarrow C(X)$

Consider the dual pair over $C(X)$ given by $\Gamma(X,E)$, $\Gamma(X,E^*)$ with the obvious pairing. Then $\Gamma(X,E) \underset{C(X)}{\otimes} \Gamma(X,E^*) =$
~~[illegible]~~ $\Gamma(X, End(E))$. $\qquad P \otimes_A Q = B$ but B is unital say $\sum_i p_i \otimes q_i = 1$

~~[illegible]~~ obvious point. Over $X \times X$ consider $\sum_{i=1}^{n} f_i(x) g_i(y)$

Things become clearer. You have E/X and E^*/X v.b.'s actually just a v.b E over X, and you need to prove the identity map of E is nuclear

$$\Gamma(X,E) \underset{C(X)}{\otimes} \Gamma(X,E^*) \longrightarrow \Gamma(X, \overset{\overset{id}{\cap}}{E \otimes E^*})$$

suppose ~~[illegible]~~ you write $id = \sum_{i=1}^{N} \overset{\overset{\Gamma(X,E)}{\cup}}{p_i} \otimes \overset{\overset{\Gamma(X,E^*)}{\cup}}{q_i}$, then

you have $\quad E \xrightarrow{(q_i)} \mathbb{C}^N_X \xrightarrow{(p_i)} E$

so factoring the identity map thru ~~the~~ a trivial bundle is the same as writing $\Gamma(X,E)$ as direct sum.

Start again. Yesterday you looked at the proof that a v.b. E over X compact is a summand of a trivial v.b. This means $\exists$ W v.s. and v.b. maps

$$E \xrightarrow{q} W_X \xrightarrow{p} E$$

with composition the identity. ~~You can suppose W = $\mathbb{C}^N$~~

~~whence~~

$$p \in \mathrm{Hom}(W, \Gamma(X,E))$$
$$q \in \mathrm{Hom}(W^*, \Gamma(X,E^*))$$

$$p \otimes q \in \mathrm{Hom}\,(W \otimes W^*, \ \Gamma(X,E) \otimes \Gamma(X,E^*))$$
$$\downarrow$$
$$\Gamma(X \times X, E \boxtimes E^*)$$
$$\downarrow$$
$$\Gamma(X, E \otimes E^*)$$

$W = \bigoplus \mathbb{C} e_i \qquad W^* = \bigoplus \mathbb{C} e_i^*$

$p = (p_i) \qquad p_i = p(e_i) \in \Gamma(X,E)$

$q = (q_i) \qquad q_i = \underbrace{q^t(e_i^*)}_{e_i^* q} \in \Gamma(X,E^*)$

$$p \circ q = \sum_i p e_i \otimes e_i^* q \quad \in \Gamma(X, E \otimes E^*)$$

Thus you want $\sum p_i \otimes q_i = \mathrm{id} \quad \in \Gamma(X, E \otimes E^*)$

How to summarize? You have $A = C(X)$

$P = \Gamma(X,E)$, $\quad Q = \Gamma(X,E^*)$, $\quad B = P \otimes_A Q = \Gamma(X, \mathrm{End}(E))$

Your point is that factoring the id_E through a trivial v.b. is equivalent to showing id_E is nuclear, ~~Look carefully!!~~ i.e. writing $id_E = \sum p_i \otimes q_i$

~~How do you ... about~~

E vector bundle over X, E^* dual v. bundle

$$\Gamma(X,E) \otimes \Gamma(X,E^*) \longrightarrow \Gamma(X, \overset{E\otimes E^*}{\mathrm{End}(E)})$$
$$\qquad\qquad\qquad\qquad\qquad\qquad \mathrm{Id} \in$$

Assume $\mathrm{Id} \in$ Image i.e. $\exists\ p_i \in \Gamma(X,E)$ $q_i \in \Gamma(X,E^*)$ $\ni$ $\sum p_i \otimes q_i \longmapsto \mathrm{Id}$. i.e. $\forall\, x \in X$, $\xi \in E_x$ $\sum_i p_i^{(x)} \langle q_i^{(x)}, \xi \rangle = \xi$. Another way to say this is that the composition

$$E \xrightarrow{(q_i)} \mathbb{C}^N_X \xrightarrow{(p_i)} E$$

is the identity, i.e. you have embedded E as a summand of a trivial v.b.

How do you construct such a factorization. The idea is that such factorizations exist locally. suppose then given $X = \bigcup U_j$ + $\forall j$ over.

$$E_{U_j} \xrightarrow{f_j} (W_j)_{U_j} \xrightarrow{g_j} E_{U_j} \qquad g_j f_j = 1 \text{ on } E_{U_j}$$

Let $1 = \sum x_j^2$ be a partition of 1 with $\mathrm{Supp}\, x_j \subset U_j$

form

$$E \xrightarrow{f=(f_j x_j)} \Big(\bigoplus_j W_j\Big)_X \xrightarrow{g=(x_j g_j)} E$$

This seems sufficiently clear.

Now return to a principal Γ bundle over X ~~[illegible]~~ denote it $\pi: Y \to X$. Then get $\pi_!(\mathbb{C}_Y)$ locally coeff. system on X with fibre at x equal to $\bigoplus_{y \in \pi^{-1}(x)} \mathbb{C}_y = \mathbb{C}[Y_x]$ which is a free $\mathbb{C}[\Gamma]$-module of rank 1, I guess you want to assume Γ left acts on Y. Cuntz's model Σ_F = simplicial complex of nonempty finite subsets $M \subset \Gamma$ such that $M^{-1}M \subset F$ (F a finite subset of Γ cont. 1 and closed under -1).

~~[illegible]~~ $\pi_!(\mathbb{C}_Y)$ is the sheaf on X whose sections over $U \subset X$ are the continuous functions on $\pi^{-1}(U)$ with support proper over U.

~~This sheaf is ...~~

Wait: Over the base X you have a kind of vector bundle, where the fibre is the Γ-module $\mathbb{C}[\Gamma]$. Natural question is whether this ~~vector~~ $\mathbb{C}[\Gamma]$ line bundle can be embedded as a summand of a trivial $\mathbb{C}[\Gamma]$-bundle. You expect the ~~small~~ previous argument to work, assuming X compact.

What about Cuntz's model? Any simplex M can be moved by an element of Γ to one $\ni 1 \in M$. Then $M^{-1}M \subset F \Rightarrow M \subset F$ so there are only finitely many simplices modulo Γ.

Aim: ~~[illegible]~~ show 1 is nuclear

Puzzle: Take $Y = \mathbb{R} \xrightarrow{\pi} \mathbb{R}/\mathbb{Z} = X$. $C_c(\mathbb{R})$ is a module over both $\underset{\text{unital}}{C(\mathbb{R}/\mathbb{Z}) \otimes \mathbb{C}[\mathbb{Z}]}$ and $\underset{\text{nonunital}}{C(\mathbb{R}) \rtimes \mathbb{Z}}$

$Y = \mathbb{R} \xrightarrow{\pi} \mathbb{R}/\mathbb{Z} = X$. $\Gamma(X, \pi_! \mathbb{C}_Y) = C_c(\mathbb{R})$.

$C_c(\mathbb{R})$ is the main object of interest, it is the space of global sections of the line bundle for $\mathbb{C}[\mathbb{Z}]$ over $B\mathbb{Z} = X$.

Take the bundle viewpoint. $\mathcal{L}$ for $\mathbb{C}[\mathbb{Z}]$ is easily described, namely, it the bundle over the circle $\mathbb{R}/\mathbb{Z} = [0,1]/\{0,1\}$ assoc. to the clutching autom. a Moebius bundle. You want to focus upon the identity being nuclear. What does this mean? Anyway you cover $\mathbb{R}/\mathbb{Z}$ by the complements of the points $0+\mathbb{Z}$, $\frac{1}{2}+\mathbb{Z}$. ~~OK everything~~

Look at $\{f \in C_c(\mathbb{R}) \mid f(\mathbb{Z}) = 0\}$.

$$= \bigoplus_{n \in \mathbb{Z}} C(n+(0,1)) = \mathbb{C}[\mathbb{Z}] \otimes C((0,1))$$

~~Take the last.~~ Take the sections of $\mathcal{L}$ vanishing at $x = (0+\mathbb{Z})$

The point is that $\mathcal{L}$ becomes trivial over the complement of a point, i.e. $U \times \mathbb{C}[\mathbb{Z}]$ so sections with ~~proper supp~~ proper support become ? (this is slightly messy). Instead the proper thing to take is maybe $C_c(U) \otimes \mathbb{C}[\mathbb{Z}]$. There's an obvious map.