So what happening? C^* alg generated by elt p + $p = p^* = p^2$. Spectrum is $\{0,1\}$. In your $\mathbb{Z}$ situation the spectrum is $\mathbb{R}$. What did you find? $h_m \geq 0$, $h_m h_n = 0$ if $|m-n| > 1$

$h_m = h_m(h_{m-1} + h_m + h_{m+1})$. Solution $h_m = 0$ all m

if $\exists\, h_m > 0$, then $h_{m-1} + h_m + h_{m+1} = 1$ so

$(1-h_m)^2 = h_{m-1}^2 + h_{m+1}^2$, Say $h_0 > 0$. Then

$h_{-1} + h_0 + h_1 = 1$ You are stupid if $h_0 > 0$

then $h_n \neq 0 \Rightarrow h_n > 0 \Rightarrow h_0 h_n > 0$

$\Rightarrow n = -1, 0, 1.$ and $h_{-1} + h_0 + h_1 = 1.$ If $h_1 > 0$ then $h_{-1} = 0.$ so you have

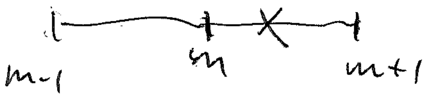
---

Proof. If $h_m > 0$, then $h_n = 0$ except for $n = m-1, m, m+1$ so $h_{m-1} + \underset{>0}{h_m} + h_{m+1} = 1$

Since $h_{m-1} h_{m+1} = 1$ there are two cases:

$h_{m-1} > 0$, $h_{m+1} = 0$ or $h_{m-1} = 0$, $h_{m+1} > 0$.

So you are getting exactly a point in



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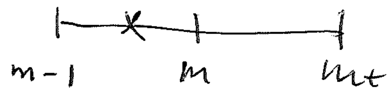


I begin to understand now. Vertex

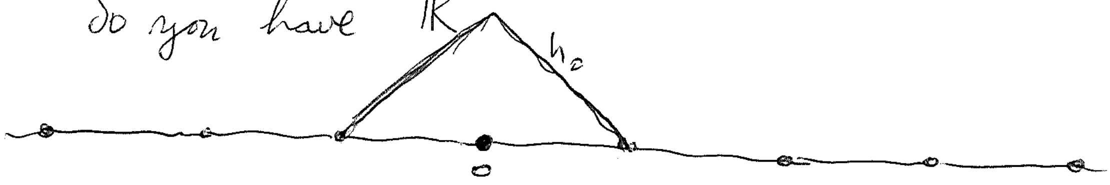
General case: solutions of $h_s \geq 0$ for $s \in \Gamma$, $h_s h_t = 0$ for $s^{-1}t \notin F$

$h_s \sum_{t\in\Gamma} h_t = h_s$ ✓ Ass. solution with $M = \{s \mid h_s > 0\} \Rightarrow \left(\sum_{\xi\in M} h_\xi\right) h_s = h_s$ $\forall t$

To understand the crossproduct $\mathcal{E}^{ab}_{\Sigma_F} \rtimes \Gamma$ next

There seems to be a canonical projection in this cross product $p \longmapsto \sum h_1^{1/2} s\, h_1^{1/2} = \sum_{s\in\Gamma} h_1^{1/2} h_s^{1/2} s$. Where does this come from.

So you have



$s \quad u^n \qquad\qquad u^n h_0 u^{-n} = h_n$

What is the cross product alg $\mathcal{E}^{ab}_{\Sigma_F} \rtimes \mathbb{Z}$?
You produce a canonical projection p in this.
You need to explain what the construction means.

You have this picture from a principal bundle $Y \xrightarrow{p} X$ for Γ, where you produce a "line bundle" E for $\mathbb{C}[\Gamma]$ over X
$E_x = \mathbb{C}[p^{-1}(x)]$ ~~and then you produce a~~
and you show the space of cont sections of E is a f g proj. module using partition of unity.

Something is different ~ [redacted] using $\underline{E}\Gamma$. [redacted] $C(\underline{E}\Gamma) = \text{"}\varinjlim_F\text{"}\, \mathcal{E}^{ab}_{\Sigma_F}$

$\underline{E}\Gamma$ is space of probab. measures on Γ.
a point is [redacted] $(h_s)_{s\in\Gamma}$ $\quad h_s \geq 0$, support (h_s) finite $\sum_s h_s = 1$.

Consider then the set of prob. measures on Γ of finite support, i.e. $(h_s)_{s\in\Gamma}$ s.t. $h_s \geqslant 0$ $\forall s$, $M=\{s \mid h_s > 0\}$ finite, $\sum_s h_s = 1$.

How does this compare with $\varinjlim_{F} \operatorname{Spec} \mathcal{E}^{ab}_{\Sigma_F}$.

Spec $\mathcal{E}^{ab}_{\Sigma_F}$ a point of which is $(h_s)_{s\in\Gamma}$ st. $h_s \geqslant 0$ $\forall s$, $h_s h_t = 0$ when $s^{-1}t \notin F$, $h_s = h_s \sum_{t\in\Gamma} h_t$. Consider a point $\neq 0$, ie $\exists s$ such that $h_s > 0$. Let $M = \{t \mid h_t > 0\}$, then $t \in M \Rightarrow h_s h_t > 0 \Rightarrow s^{-1}t \in F \Rightarrow t \in sF$ in particular M is finite. So $\sum_{t\in\Gamma} h_t = \sum_{t\in M} h_t$ is defined

$$h_s = h_s \sum_{t\in M} h_t \Rightarrow \sum_{t\in M} h_t = 1.$$

So you do find Certainly $\mathcal{E}^{ab}_{\Sigma_F}$ is close to $\underline{E}\Gamma$

Review what you learned yesterday. Given Γ and a finite subset $F \subset \Gamma$, let Σ_F be the simplicial complex ~~with verti~~ whose simplices are non empty subsets $M \subset \Gamma$ s.t. $M^{-1}M \subset F$, vertices = elts $s \in \Gamma$, edge $= \{s,t\}$ $s^{-1}t$ and $t^{-1}s \in F$ $s \neq t$
simplex $= M$ such that every pair is an edge.

~~should~~ $1 \in F$ for $\Sigma_F \neq \phi$, can rep F by $F \cap F^{-1}$

$$\mathcal{E}_{\Sigma_F} = C^*\left\{ h_s, s \in \Gamma \;\middle|\; \begin{array}{l} h_s \geq 0, \; h_s h_t = 0 \text{ for } s^{-1}t \notin F, \\ h_s = \sum_{t \in \Gamma} h_s h_t \quad (\text{finite sum}) \text{ by prec. cond.} \end{array} \right\}$$

$\mathcal{E}^{ab}_{\Sigma_F}$ = comm. C^* alg whose spectrum is the space of ~~solutions of~~ $(h_s)_{s \in \Gamma}$ with $h_s \geq 0$

what does such a thing look like? trivial solution $h_s = 0$ $\forall s$, this indicates we have a locally comp. space

Suppose $\exists s \ni h_s > 0$, say $s = 1$ since Γ operates by left mult on Σ_F. $h_1 > 0$. Then $h_1 h_t = 0$ for $t \notin F$, ~~so there are only finitely many $t \in \Gamma$ s.t.~~

$\Rightarrow M = \{t \mid h_t > 0\}$ finite. Then

$$h_1 = \sum_{t \in \Gamma} h_1 h_t = h_1 \sum_{t \in M} h_t \Rightarrow \sum_{t \in M} h_t = 1.$$

So you have a point in the geom. real. of the s.cx.

~~Conversely given M $h_t > 0$ $t \in M$ a simp~~
~~$\sum h_t = 1$ put $h_t = 0$ for $t \notin M$~~
~~$(h_s)_{s \in \Gamma}$~~

Conversely given $(h_s)_{s \in \Gamma}$ s.t.

$h_s \geq 0$, $M = \{s \in \Gamma \mid h_s > 0\}$ simplex in Σ_F, $\sum h_s = 1$

Given $s, t \in \Gamma$ one has $h_s h_t = 0$ unless

$h_s, h_t > 0 \Rightarrow s, t \in M \Rightarrow s^{-1}t \in F$

<u>Ex.</u> $\Gamma = \mathbb{Z}$ $F = \{-1, 0, 1\}$. Simplex is

$M \subset$ ~~such that~~ $\mathbb{Z}$, $M \neq \emptyset$, finite $|m - n| \leq 1$.

So what's next? You now have to understand $\mathcal{E}_{\Sigma_F} \rtimes \Gamma$, the key point being that there is a canonical projection in this algebra, namely $p = \sum p_s \qquad p_s = h_1^{1/2} s h_1^{1/2} = h_1^{1/2} h_s^{1/2} s$.
Here I am ignoring topology, I mean the kind of group ring.

You need to understand these formulas.

One point: $h_1 h_s = 0$ in $\mathcal{E}_{\Sigma_F}$ for $s \notin F$.

~~so~~ $h_1 h_s = 0 \implies h_1, h_s$ commute $\implies$
$h_1^{1/2} h_s^{1/2} = (h_1 h_s)^{1/2} = 0. \quad \therefore \boxed{p_s = 0 \quad \text{for} \quad s \notin F.}$

This means in the $\mathbb{Z}$ example ~~that~~ $p = p_{-1} + p_0 + p_1$

Your problem is the meaning of all this.

What is going on?

Take $\mathbb{Z}$ example. ~~You have this mod~~

You have a cross product alg $C(\mathbb{R}) \rtimes \mathbb{Z}$.

Here I ignore the type of group ring: there are two candidates $C_r(\mathbb{Z}) \simeq C(S^1)$, $\mathbb{C}[\mathbb{Z}]$ = Laurent polys. There is a canonical idempotent maybe at least when you use the $h_n \in C(\mathbb{R})$, basic partition of unity.

There is a projection $p = \sum_n h_0^{1/2} u^n h_0^{1/2}$ to be understood in $C(\mathbb{R}) \rtimes \mathbb{Z}$.

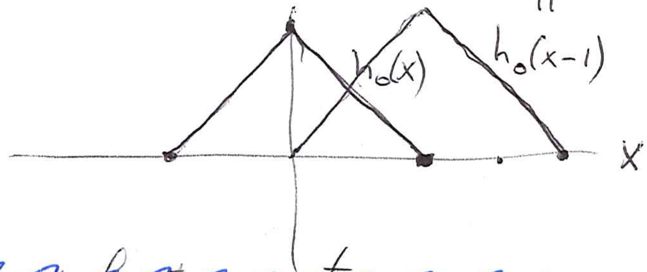
$$p^2 = \sum_{m,n} h_0^{1/2} \underbrace{u^m h_0 u^n}_{h_m u^{m+n}} h_0^{1/2}$$

There is a concrete thing to understand.

$C(\mathbb{R}) \rtimes \mathbb{Z}$ $\qquad \sum_{n\in\mathbb{Z}} f_n(x)u^n$

$$u\,f(x) = f(x-1)\,u$$

$$f(x) = h_0(x)$$



~~Now what next can we~~

Let's try to understand this crossproduct algebra using a representation on Hilbert space. You have $C(\mathbb{R})$ acting in an obvious fashion on $L^2(\mathbb{R})$, also $\mathbb{Z}$ acting via unit translation. Look at the commutator. commuting with $C(\mathbb{R})$ means mult by a bdd meas fn. b on $\mathbb{R}$, then commuting with u means b is periodic. So $C(\mathbb{R}) \rtimes \mathbb{Z}$ is already an alg over $C(\mathbb{R}/\mathbb{Z})$ actually it's a noncumtal alg over C of the torus $\mathbb{R}/\mathbb{Z} \times \mathbb{Z}$

Repeat: ~~You have the cross~~ You want a good understanding of the cross product alg $C(\mathbb{R}) \rtimes \mathbb{Z}$. You have a $*$ repn on $L^2(\mathbb{R})$. Elements of $C(\mathbb{R}) \rtimes \mathbb{Z}$ are roughly twisted Laurent series $\sum_{n\in\mathbb{Z}} f_n(x)u^n$ where $uf(x) = f(x-1)u$, & the $f_n \in C(\mathbb{R})$ i.e. are cont. functions vanishing at ∞. note that the ~~class~~ u^n are not in the cross product so what, you are still very confused

~~You are … identify the correspond~~

Anyway what's going on.

concentrate, Point: $C(\mathbb{R}) \rtimes \mathbb{Z}$ is an alg. over $\mathbb{C}(\mathbb{R}/\mathbb{Z}) \otimes C(S^1) = C(\text{torus})$. You expect an extension

$$0 \longrightarrow C(\mathbb{R}) \longrightarrow \mathbb{C}(\mathbb{R}_+) \longrightarrow \mathbb{C} \longrightarrow 0$$

$$0 \longrightarrow C(\mathbb{R}) \rtimes \mathbb{Z} \longrightarrow \mathbb{C}(\mathbb{R}_+) \rtimes \mathbb{Z} \longrightarrow C_r(\mathbb{Z}) \longrightarrow 0$$

confusing. torus

How to think: Hilb sp rep ~~inside~~ on $L^2(\mathbb{R})$ ~~operators gen. by $C(\mathbb{R})$~~ commutant of $C(\mathbb{R})$ is all bdd meas. functions $L^\infty(\mathbb{R})$ and comm. with $\mathbb{Z}$ action is periodic functions.

It seems important to think of $C(\mathbb{R}) \rtimes \mathbb{Z}$ as an alg. over $C(\text{torus})$. Why? It should be related to the fin gen. proj module over $C(\mathbb{R}/\mathbb{Z}) \otimes C_r(\mathbb{Z})$ arising from the principal bundle $\mathbb{R} \xrightarrow{\text{base}} \mathbb{R}/\mathbb{Z}$. In fact you need to write out details.

But you have

Review the problem: You have still this mysterious crossproduct alg $C(\mathbb{R}) \rtimes \mathbb{Z}$. It's an algebra over $C(\mathbb{R}/\mathbb{Z}) \otimes C_r(\mathbb{Z}) = C(\text{torus})$

Specifically an elt of $C(\mathbb{R}) \rtimes \mathbb{Z}$ is a twisted

Laurent series $\sum_{n\in\mathbb{Z}} f_n(x)\, u^n$ roughly

~~such that~~ where $u f_n(x) = f_n(x-1)\, u$ in the alg.

$C(\mathbb{R}/\mathbb{Z})$ acts by mult: $g \sum f_n u^n = \sum (g f_n) u^n$
and this operator commutes with mult. by u and hence by multiplication. I think you do understand this $C(\mathbb{R}) \rtimes \mathbb{Z}$, somewhat.

Wait: ~~Does~~ form the semi-direct product? No What do you need to embed $C(\mathbb{R}) \rtimes \mathbb{Z}$ into a unital alg! Idea is to embed. Picture:

$C(\mathbb{R}) \rtimes \mathbb{Z}$ doesn't contain $C_r(\mathbb{Z})$ because $C(\mathbb{R})$ not unital, but $C(\mathbb{R}/\mathbb{Z}) \otimes C_r(\mathbb{Z})$ does? Somehow you want to increase $C(\mathbb{R})$ at least by $\mathbb{C}$ but maybe by $\underline{C(\mathbb{R}/\mathbb{Z})}$

$C(\mathbb{R})$
cont functions
van. at ∞

$C(\mathbb{R}/\mathbb{Z})$
cont

$$C(\mathbb{R}) \longrightarrow \underbrace{\begin{pmatrix}\text{continuous}\\ \text{almost}\\ \mathbb{Z}\text{ periodic}\\ \text{functions}\end{pmatrix}}_{} \longrightarrow C(\mathbb{R}/\mathbb{Z})$$

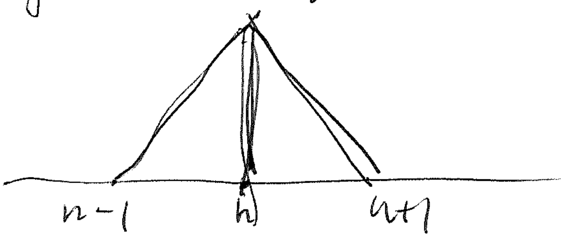
$C(\mathbb{R})$: cont functions van. at ∞

$C(\mathbb{R}/\mathbb{Z})$: continuous periodic functions

$f(x)$ continuous such that $f(x) - f(x-1)$ vanishes at ∞.

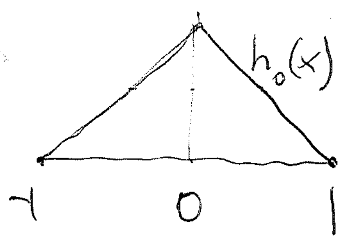
Looks like you can have different periodic functions as $x \to +\infty, -\infty$

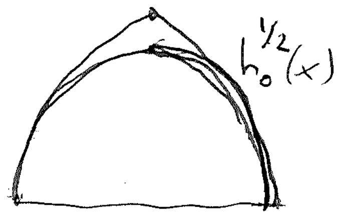
Back to $C(\mathbb{R}) \rtimes \mathbb{Z}$ typical elt is $\sum f_n u^n$ $f_n(x)$. There is a candidate for elements of this C^* algebra, namely continuous functions $f(x, z)$ on $\mathbb{R}\times S^1$ vanishing at ∞. So it's likely that this turns out precise. But ~~it is a~~ there is a p specific projection defined using functions $h_n(x)$ on $\mathbb{R}$



$p = \bigoplus p_n u^n \qquad p_n = h_0^{1/2} h_n^{1/2} u^n$

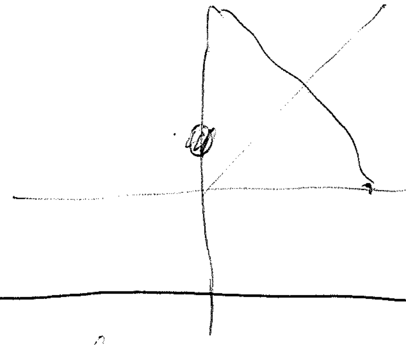
and you know that $h_0 h_n = 0$ for $|n| > 1$.





$h_0(x) = 1 - x$

$h_1(x) = x$



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you are so stupid. Start again You have two constructions you hope to link. One is ~~the ... class~~ for the Bott class on the 2 torus It arises from $\mathbb{R} \longrightarrow \mathbb{R}/\mathbb{Z}$ principal $\mathbb{Z}$-bundle by forming the line bundle E for the group ring $\mathbb{C}[\mathbb{Z}]$ over the circle, the fibre of E over a coset $x + \mathbb{Z}$ is the space $\mathbb{C}[x+\mathbb{Z}]$

~~So what are you aiming~~ Bundle locally constant. In the end you should get a complex line bundle over the torus out of this. You should know this line bundle already, ~~the smooth version is $\mathcal{S}(\mathbb{R})$ with the~~

· $\mathcal{S}(\mathbb{R}) \ni f(x)$ on $\mathcal{S}(\mathbb{R})$ $e^{2\pi i x}$, e^{D}. You have to ~~explain~~ review Poisson summation fmla.

$$f(x) \longmapsto \sum_{n\in\mathbb{Z}} e^{2\pi i y n} f(x+n) = F(x,y)$$

$$F(x,y+1) = F(x,y)$$

$$F(x+1,y) = \sum_{n\in\mathbb{Z}} e^{2\pi i y n} f(x+1+n)$$

$$= \sum_{n\in\mathbb{Z}} e^{2\pi i y(n-1)} f(x+n) = e^{-2\pi i y} F(x,y)$$

A better viewpoint might be to take $f \in \mathcal{S}(\mathbb{R})$ and ~~set~~ make it periodic

$$\sum_n f(x+n) = g(x)$$

$$g(x) = \sum_k e^{2\pi i k x} \underbrace{\int_0^1 e^{-2\pi i k x} \sum_n f(x+n)\,dx}$$

$$\sum_n \underbrace{\int_0^1 e^{-2\pi i k x} f(x+n)\,dx}$$

$$\int_n^{n+1} e^{-2\pi i k x} f(x)\,dx \qquad = \underbrace{\int_{-\infty}^{\infty} e^{-2\pi i k x} f(x)\,dx}_{\hat{f}(k)}$$

$$\sum_{n\in\mathbb{Z}} f(x+n) = \sum_k e^{2\pi i k x}\hat{f}(k)$$

Something magical!

two things to relate. first is the crossproduct algebra $C(X) \rtimes \mathbb{Z}$ containing p.
second is space $\mathcal{S}E$ of sections of the "line bundle" E for the alg $C_r(\mathbb{Z})$ over $\mathbb{R}/\mathbb{Z}$ assoceated to the principal bundle $\mathbb{R} \longrightarrow \mathbb{R}/\mathbb{Z}$. My idea is that the crossproduct should act on $\mathcal{S}E$. In any case something should act on $\mathcal{S}E$. You already have $C(\mathbb{R}/\mathbb{Z}) \otimes C_r(\mathbb{Z})$ acting.

Look in general at $\pi: Y \longrightarrow X$ with group Γ. and X compact. Then E is the fibre bundle over X with $E_x = \mathbb{C}[\pi^{-1}(x)]$, which is a right $\mathbb{C}[\Gamma]$-module. The structural group of this fibre bundle is Γ, i.e. discrete. What is the fibre bundle of automorphism of Y over X?

Mystery: What is $C(Y) \rtimes \Gamma$? This is defined for a discrete group acting on a locally compact space Y. It should be related to the topological ~~category~~ groupoid assoc. to (Y, Γ).

$C(Y) \rtimes \Gamma$ naturally acts on any $C(Y)$ module with compatible Γ action.

Is there a link between $C(Y) \rtimes \Gamma$ which is an alg over $C(X) \times \Gamma$ and

the space of sections of the line [redacted] bundle E
(for ring $\mathbb{C}[\Gamma]$) over X with fibre $E_x = \mathbb{C}[\pi^{-1}(x)]$

Focus on E.
$$\begin{array}{c} Y \leftarrow \Gamma \\ \pi \downarrow \\ X \end{array} \qquad \pi^{-1}(x) \text{ right } \Gamma \text{ torsor}$$

get a local system of right Γ torsors, then a local system $x \longmapsto \mathbb{C}[\pi^{-1}(x)]$ of free rank 1 modules for $\mathbb{C}[\Gamma]$. Now $C(Y)$ acts on $\mathbb{C}[\pi^{-1}(x)]$ by restricting a function $f(y)$ to the orbit $\pi^{-1}(x)$ and multiplying. Thus ~~the sections~~ there should be an action of $C(Y) \rtimes \Gamma$ on $\mathcal{S}E$

Consider $\mathbb{R} \xrightarrow{\pi} \mathbb{R}/\mathbb{Z}$. There's the bundle E over $\mathbb{R}/\mathbb{Z}$ with $E_{y+\mathbb{Z}} = \mathbb{C}[y+\mathbb{Z}]$

$$E_\theta = \mathbb{C}\left[\frac{\theta}{2\pi} + \mathbb{Z}\right]$$

maybe you want to replace this E_θ by the ℓ^2 space having orthogonal unit vectors indexed by the coset $\frac{\theta}{2\pi} + \mathbb{Z}$. Now let f

Can do in general $Y \xrightarrow{\pi} X$ $\quad \ell^2(\pi^{-1}(x)) \xrightarrow{\sim} \ell^2(\Gamma)$

Now take a continuous function $f(y)$ on Y. Then you get a multiplication operator on $\ell^2(\pi^{-1}(x))$ by [redacted] rest. f to the orbit $\pi^{-1}(x)$.

You are interested in the case when f vanishes at ∞.

There's maybe an alg version where f has comp. support

In any case a bounded cont. function ~~is then~~ $f(y)$ determines an operator on the Hilbert space bundle $x \mapsto \ell^2(\pi^{-1}(x))$

Look carefully. Correlate with Poisson summ. You have to keep track of things. ~~First of all~~ Start with $\mathbb{R} \xrightarrow{\pi} \mathbb{R}/\mathbb{Z}$ $\pi(y)$ = the coset $y+\mathbb{Z}$. ~~This also was interested~~

This is a locally trivial fibre bundle where the fibres $y+\mathbb{Z}$ are $\mathbb{Z}$-torsors under $+$. Can form a vector space ~~having~~ $\mathbb{C}[y+\mathbb{Z}]$ having the elements of $y+\mathbb{Z}$ as a basis. Can use instead of $\bigoplus_{y\in\pi^{-1}(x)} \mathbb{C}\delta_y$ the complex functions on the coset $y+\mathbb{Z}$. So you have some kind of vector bundle over the circle $\mathbb{R}/\mathbb{Z}$ where the fibre is a module over $\mathbb{C}[\mathbb{Z}] = \mathbb{C}[u,u^{-1}]$ Laurent polys. The idea is now to explain what sort of vector bundle, ~~better~~, to produce a space of sections. What sort of tools??

What's important is the cross product algebra. (functions on Y) $\rtimes$ $(\mathbb{Z})$. This is the new gadget you want to understand. The simplest version is twisted Laurent polys over functions. Let's see what can be done holomorphically. Let A be a ring of functions on $Y=\mathbb{R}$, closed under unit translation $f(y) \longmapsto f(y-1)$.

Actually you might aim for a small A

Go back to ~~X~~ $\mathbb{R}$ be the line with coord x. $\quad f(x) \mapsto f(x-1)$.

Ask that f be entire. ~~The quotient~~

$\mathbb{C}/\mathbb{Z} \xrightarrow{\sim} \mathbb{C}^\times$ via $e^{2\pi i}$

---

so let us again examine (functions on $\mathbb{R}$) $\rtimes \mathbb{Z}$

$f(x) \longmapsto f(x-1)$ basic action of $\mathbb{Z}$ on functions on the line. Look at $f \in L^2(\mathbb{R})$, you have a ~~and~~ unitary operator, spectrum is S^1, eigenfns: $f(x-1) = \lambda f(x)$, ~~~~ translation invariance of the equation suggests exp. solns. $e^{a(x-1)} = e^{\lambda} e^{ax}$ $\quad e^{-a} = \lambda$, ~~and~~ once a chosen $f(x)e^{-ax} = g(x)$ is periodic

$g(x-1) = g(x)$.

~~Interesting~~

~~Let's~~ Basic objects are $\sum f_n(x) u^n$ twisted Laurent polys, series

with $u f_n(x) = f_n(x-1) u$

of particular interest is $f(x) \in C(\mathbb{R})$ vanishing at ∞.

~~Look at the situation as follows:~~

~~So your task~~

Go back to your program, this concerns ~~the~~ space of sections of the vector bundle E over $\mathbb{R}/\mathbb{Z}$; better, the $\mathbb{C}[\mathbb{Z}] = \mathbb{C}[u,u^{-1}]$ line bundle over the circle. You try to keep this algebra.

The fibre of E over a point $y = x + \mathbb{Z}$ of $\mathbb{R}/\mathbb{Z}$ is $\mathbb{C}[x+\mathbb{Z}]$, denoted E_y. It's a free rank 1 module over $\mathbb{C}[u,u^{-1}]$. You get a generator for each $x \in$ the coset y. ~~that~~

~~[illegible]~~ Sections of E, there are the "flat" sections Is there some hope that this

Start again, you probably have to get straight two pictures of the group ring namely, linear comb. of group elements or functions on the group.

Your aim: You have an explicit infinite diml vector bundle E over $S^1 = \mathbb{R}/\mathbb{Z}$ ~~[illegible]~~ where the fibre is $\mathbb{C}[\mathbb{Z}]$ or something more analytical. ~~[illegible]~~

Important. The space of (suitable) sections of E is a module over $C(\mathbb{R}/\mathbb{Z}) \otimes \mathbb{C}[\mathbb{Z}]$, some kind of functions on torus. This should be a f.g. proj module representing the Bott class.

too bad

What you want to do is to produce a minimal space of sections, as algebraic as possible.

You know what E_y is for each $y \in Y = \mathbb{R}/\mathbb{Z}$, it is either ~~[illegible]~~ the vector space of all functions with finite support on the coset $\pi^{-1}y = x + \mathbb{Z}$, or all finite linear combinations of elements of this coset. (The two descriptions are relevant for algebra structures).

~~[illegible]~~ Any function on R can be restricted to $\pi^{-1}y$ to give an operator, but you want a bdd.

continuous function at least. Let's check this.

$$E_{x+\mathbb{Z}} = \bigoplus_{n\in\mathbb{Z}} \mathbb{C}\,\delta_{x+n} \quad \text{as } x \text{ moves over } 0\le x\le 1$$

there's an obvious trivialization of E pulled back to $[0,1]$ and then an identification $E_{1+\mathbb{Z}} \simeq E_{0+\mathbb{Z}}$.

Can you describe exactly what is a section of E? Assume some sort of finite support in the $\mathbb{Z}$ direction.

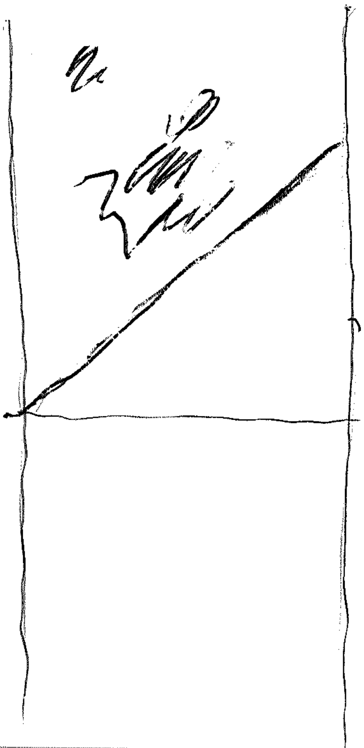
Work with $0 \le x \le 1$. Let $s(x)$ be a cont. section: $s(x) = \big(s(x, x+n)\big)_n$

$s(x)$ is a function on $x+\mathbb{Z}$. Continuity means that if we take x' close to x and identify

$$\begin{array}{ccc} x'+\mathbb{Z} & \simeq & x+\mathbb{Z} \\ \cup & & \cup \\ x'+n' & \longleftrightarrow & x+n \end{array} \quad \text{well} \quad \text{iff} \quad n'=n$$

you have $s(x', x'+n) = s(x, x+n)$. So continuity means that $s(x, x+n)$ is a continuous function of x.

What does this mean?



The point is that a

So, there some hope that this pen will eventually write smoothly

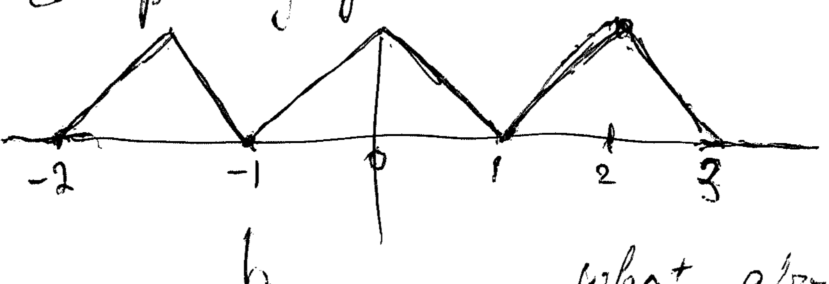
On any cas its clear what a section of this bundle E should be. If we allow

So you learn something namely that ~~the~~ ~~[illegible]~~ if you ignore the compact support condition then a section? You are going to find exactly ~~the~~ ~~[illegible]~~ the Schwartz space. So it seems pretty clear. At this point things became better

I think you know now that the space of sections of E vanishing at ∞ is exactly $C(\mathbb{R})$. Now what else is needed to get things to work? What else ~~[illegible]~~

You seem to have a continuous function version of Poisson summation. Guess: If instead of $S(\mathbb{R})$ you take $C(\mathbb{R})$ continuous functions vanishing at ∞ and treat it as a $\mathbb{Z}\times\mathbb{Z}$ alg., then $C(\mathbb{R})$ = space of continuous sections of the line bundle of degree 1!!! There's a lot to be checked.

So there's a lot to be checked! You have the crossproduct alg $C(\mathbb{R})\rtimes\mathbb{Z}$ acting on the space of sections of E which is $C(\mathbb{R})$. But you know that this module should be a summand of two copies of the crossproduct algebra. You need a partition of unity on the circle. The simplest corresponding functions on $\mathbb{R}$ are



This h_{even} should allow you to do the correct things.

h_{even}

what about $\frac{1+\cos(\pi x)}{2} = \cos^2\left(\frac{\pi x}{2}\right)$?

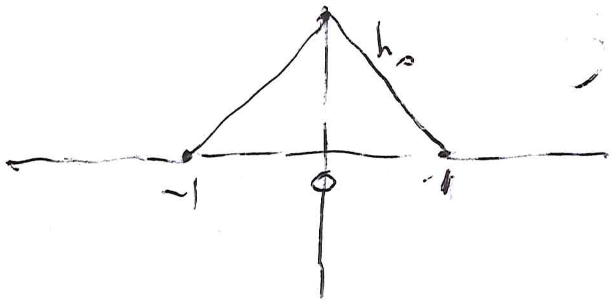
details — what do you want? You need to take $C(\mathbb{R})$ and embed it inside the crossproduct algebra.

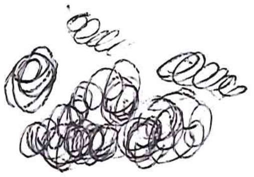
$$f(x) = \sum h_0^{1/2}(x-2n)\, f(x)$$

$=$

Lets straighten this out!!

$\mathcal{S}E = C(\mathbb{R})$, you ~~I~~ think it's true that $C(\mathbb{R})$ is a finitely gen proj mod over $C(\mathbb{R})\rtimes\mathbb{Z}$. ~~Alas~~ You want to understand this point. [redacted] The idea is to use the partition $\sum_{n\in\mathbb{Z}} h_n = 1$ on $\mathbb{R}$





h!

Notice that ~~you have a 2 stage~~ is the inverse of a partition of 1 on the circle $\mathbb{R}/\mathbb{Z}$. Take time

$$h_{ev} \stackrel{!}{=} \sum h_{2n}$$
$$h_{odd} = \sum h_{2n+1}$$

The idea is you take $f \in \mathcal{S}E = C(\mathbb{R})$ and write it as a sum

$$f = h_{ev}^{1/2} f + h_{odd}^{1/2} f \quad ??$$

Go back to the idea that the bundle E becomes trivial over the support of any member of the partition. Boundedness feature of unitary operators. It looks like your partition lives on the barycentric [redacted] subdivision

Start again. Consider the covering $\mathbb{R} \xrightarrow{\pi} \mathbb{R}/\mathbb{Z}$ with group $\mathbb{Z}$. For each coset $y = x + \mathbb{Z} \in \mathbb{R}/\mathbb{Z}$ you form the vector space $\mathbb{C}[x+\mathbb{Z}]$ it generates. This gives a local coefficient system E over the circle $\mathbb{R}/\mathbb{Z}$ with fibre the group ring $\mathbb{C}[\mathbb{Z}]$ considered as a module over itself.

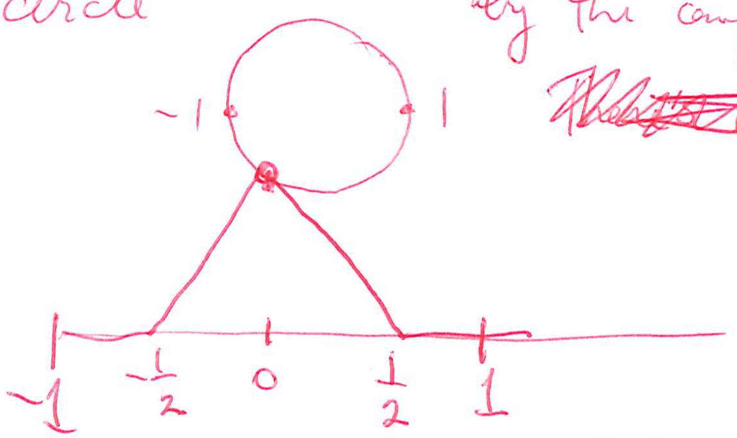
$$\mathbb{R} \times^{\mathbb{Z}} \mathbb{C}[\mathbb{Z}] = E$$

~~As you move around the circle~~

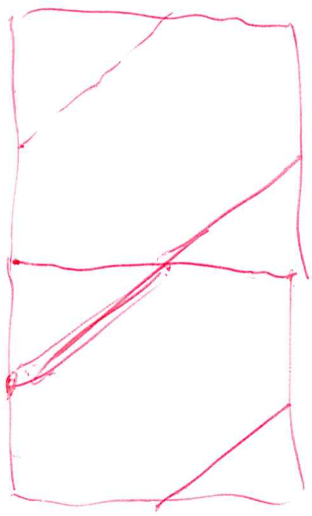
This bundle is flat and the monodromy is mult by the generator of $\mathbb{Z}$.

~~You know how to~~ [illegible] Let $\mathrm{Sect}(E)$ be the space of sections of E.

Consider a section s of E. Cover the circle by the complements of $\mathbb{Z}$, $\frac{1}{2}+\mathbb{Z}$.



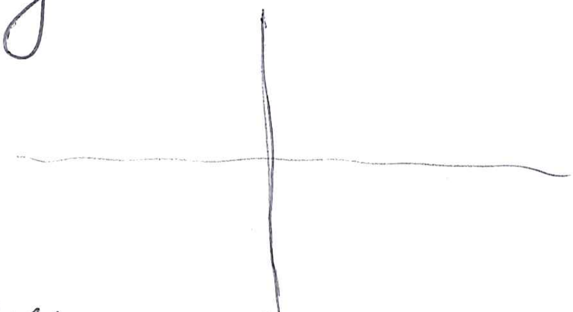
take out $\frac{1}{2}+\mathbb{Z}$



Inside E sits $X = \mathbb{R}$ as a "spiral" and you know that a section amounts to a cont. function on $\mathbb{R}$. So it seems that $C_c(\mathbb{R})$ is the space of sections locally of bounded support vertically.

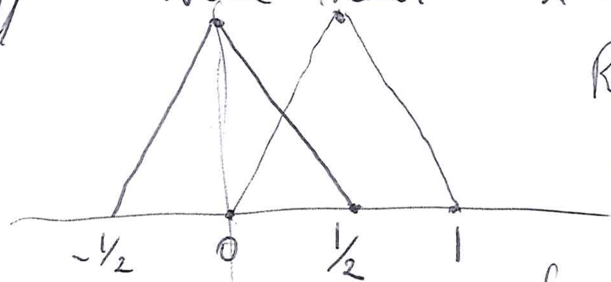
Now you have described $\mathcal{S}E$ and ~~[illegible]~~ you should be able to see that Sect(E) is a fg proj module over $C(\mathbb{R}/\mathbb{Z}) \otimes \mathbb{C}[\mathbb{Z}]$ All you should use is the partition of 1 on $\mathbb{R}/\mathbb{Z}$.

So $\mathcal{S}E$ should $C_c(\mathbb{R})$. Consider then $C_c(\mathbb{R})$ with actions of $C(\mathbb{R}/\mathbb{Z}) \otimes \mathbb{C}[\mathbb{Z}]$. Wait you first want to under why $C_c(\mathbb{R})$ is a fg projective module over $\underbrace{C(\mathbb{R}/\mathbb{Z}) \otimes \mathbb{C}[\mathbb{Z}]}$ actual Laurent polys. Functions on torus algebraic on one side. It should be very simple. Take some partition of unity on the circle $\sin^2(\ ) + \cos^2(\ ) = 1$.



~~Keep then~~ Keep things positive and cont.

Work near $x = 0$. ~~Start~~ with $f(x) \in C_c(\mathbb{R})$



Restrict E to U_0 = complement of $\frac{1}{2} + \mathbb{Z}$ in $\mathbb{R}/\mathbb{Z}$. Then for $|x| < \frac{1}{2}$ f gives a section of E over U_0. $f(x+n)$

$E|U_0 = U_0 \times C_c(\mathbb{Z})$

You need a notation to handle things.

$E_y = C_c(\pi^{-1}y)$. Trivialize E over U_0.

given $x \in (-\frac{1}{2}, \frac{1}{2})$

$$\begin{array}{ccc} C_c(x+\mathbb{Z}) & \simeq & C_c(\mathbb{Z}) \\ \| & & \| \\ E_{x+\mathbb{Z}} & & E_{0+\mathbb{Z}} \end{array}$$

$$(f|_{x+\mathbb{Z}}) \longmapsto (f(x+n))_{n\in\mathbb{Z}}$$

$$\{f(x) \text{ in } C_c(\mathbb{R})\} \longrightarrow \left\{\begin{matrix}\text{section of } E \\ \text{over } U_0\end{matrix}\right\} \longmapsto \left\{\begin{matrix}\text{map from } U_0 \\ \text{to } C_c(\mathbb{Z})\end{matrix}\right\}$$

$\Downarrow$

$(g(x, x+n))_{n\in\mathbb{Z}}$, $x\in(-\frac{1}{2},\frac{1}{2})$

$$f(x) \longmapsto \left(g(x,x+n) = f(x+n)\right) \longmapsto (f(x+n))$$

$(x,n) \in (-\frac{1}{2},\frac{1}{2}) \times \mathbb{Z}$ $\qquad$ $(x,n) \in (-\frac{1}{2},\frac{1}{2}) \times \mathbb{Z}$

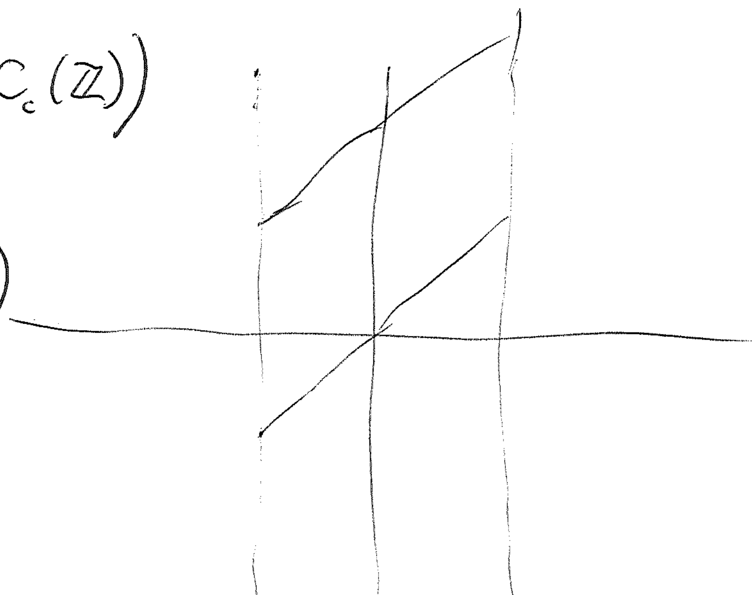
$x \longmapsto (f(x+n))_{n\in\mathbb{Z}}$

~~[illegible]~~ multiply by $h_0(x)$

So suppose given a ~~[illegible]~~ map $U_0 \longrightarrow C_c(\mathbb{Z})$ i.e. a section of E over U_0. It seems that

$$\mathrm{Sect}(U_0, E) = C(U_0, C_c(\mathbb{Z}))$$

$g(x, x+n)$ $\qquad$ $g'(x, n)$



Somehow you are expressing the idea that a function $f(x)$, $x \in \mathbb{R}$ is equiv. to the sequence of functions $\big(\underbrace{f_n(x)}_{f(x+n)},\ x \in [-\frac{1}{2}, \frac{1}{2}),\ n \in \mathbb{Z}\big)$

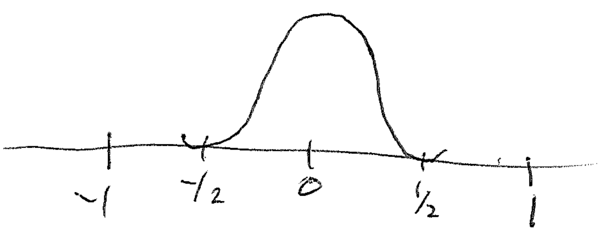
There is of course a problem with continuity which we can circumvent by multiplying by a periodic fn. vanishing on $\frac{1}{2} + \mathbb{Z}$

~~Key idea~~ important idea that $f(x)$ $x \in \mathbb{R}$ is equivalent to a sequence of functions $f_n(x)$ $x \in [-\frac{1}{2}, \frac{1}{2})$, via $f_n(x) = f(x+n)$, or to a sequence $f_n(x)$ $x \in \mathbb{R}$ having supp. $\subset [-\frac{1}{2}, \frac{1}{2})$ (here you extend $f_n(x)$ by zero), then

$$f(x) = \sum_n f_n(x-n) = \sum_{n \in \mathbb{Z}} U^n f_n$$

So now given $f \in C_c(\mathbb{R})$ Pick periodic $\chi_0(x)$, $\chi_1(x) \geq 0$

$$\chi_0 = \cos^2(\pi x)$$
$$\chi_1 = \sin^2(\pi x)$$



Then $\chi_0 f \in C_0(\mathbb{R})$, vanishes at $\frac{1}{2} + \mathbb{Z}$

So $$(\chi_0 f)(x) = \sum_{n \in \mathbb{Z}} U^n g_n \qquad g_n(x) = \begin{cases} (\chi_0 f)(x+n) & \text{for } x \in [-\frac{1}{2}, \frac{1}{2}) \\ 0 & \text{outside} \end{cases}$$

~~Do you understand the map~~

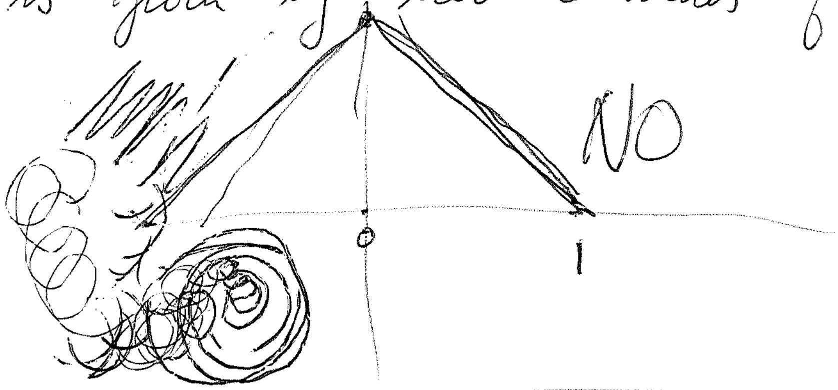
You want to show $C_c(\mathbb{R})$ is a fg proj module over $\boxed{C(\mathbb{R}/\mathbb{Z}) \otimes \mathbb{C}[\mathbb{Z}]} = R$. So you need

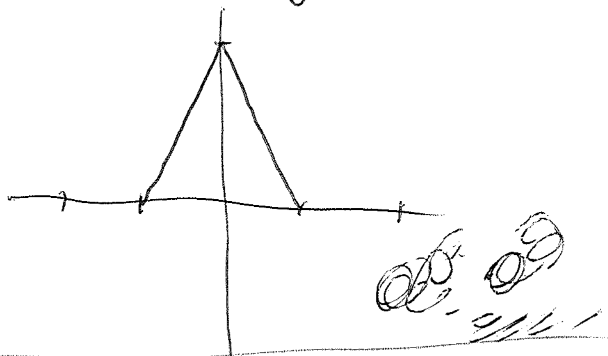
$$C_c(\mathbb{R}) \xrightarrow{\alpha} R^{\oplus 2} \xrightarrow{\beta} C_c(\mathbb{R}).$$

<u>You want to do this as alg. as possible</u>

You have to construct module maps α, β. and you also need to understand how $C_c(\mathbb{R})$ is a fg proj module over $C(\mathbb{R}) \rtimes \mathbb{Z}$.

Continue. Since R is unital the map β is given by two elements of $C_c(\mathbb{R})$ say $h_0^{1/2}$ $h_1^{1/2}$



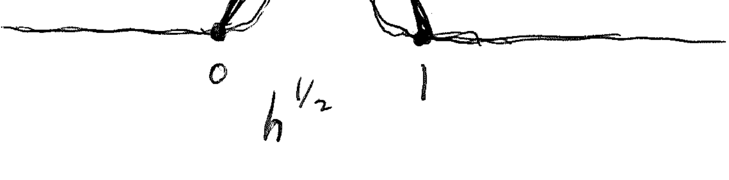


$C_c(\mathbb{R})$ is a unitary R-module, R acts using mult. by cont. periodic functions and unit translation ~~[illegible] Triangulating R~~ So you need to pick a function h in $C_c(\mathbb{R})$ to get a module map $R \to C_c(\mathbb{R})$

You would like to get as much as you ~~can~~ can

$h = \sin^2(\pi x)$ $\qquad$ $h^{1/2} = |\sin(\pi x)|$ $\qquad$ 0 outside.

$0 \le x \le 1$ $\qquad$ $0 \le x \le 1$ $\qquad$ ~~So~~ If you have



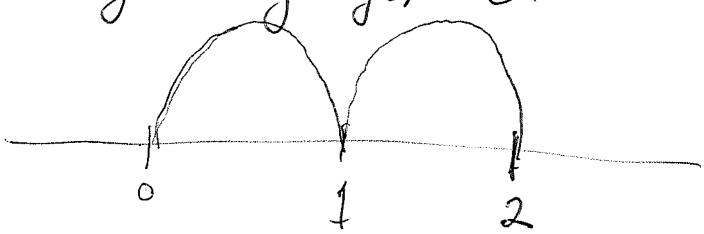
~~So what~~ Describe what you're doing

Given $f \in C_c(\mathbb{R})$, you multiply it by $|\sin(\pi x)|$ and by $|\cos(\pi x)|$

Now $g(x) = |\sin(\pi x)|\, f(x)$ splits into ~~finitely many~~ a sum $\sum_n g_n(x)$, where $g_n(x) \in C((n, n+1))$

$g_n(x)$ cont support $\subseteq [n, n+1]$

finitely many $g_n \neq 0$.



$$|\sin(\pi x)| = \sum_{n \in \mathbb{Z}} \phi_s(x-n)$$

$$|\cos(\pi x)| = \sum_{n \in \mathbb{Z}} \phi_c(x-n)$$

$$\phi(x) = \begin{cases} |\sin(\pi x)| & \text{for } 0 \le x \le 1 \\ 0 & \text{elsewhere} \end{cases}$$

$$|\sin(\pi x)| = \sum_{n \in \mathbb{Z}} \phi_s(x-n)$$

$\phi_s(x) = \sin|\pi x|$ on $0 \le x \le 1$ extended by zero.

$$|\sin(\pi x)| f(x) = \sum_{n \in \mathbb{Z}} \phi_s(x-n) f(x)$$

$$= \sum_{n \in \mathbb{Z}} u^n \phi_s(x)\, u^{-n} f$$

$$= \sum_{n \in \mathbb{Z}} u^n \left(\overbrace{\phi_s u^{-n} f}^{|\sin(\pi x)| \cdot f(x+n)} \right)$$

So given $f \in C_c(\mathbb{R})$, then

$$|\sin(\pi x)|\, f(x) = \sum u^n \underbrace{\phi_s u^{-n} f}$$

this is $|\sin \pi x|\, f(x+n)$ on $[0,1]$ extended by zero to all of $\mathbb{R}$.

Identification between

$C((0,1))$

$C(\mathbb{R}/\mathbb{Z} - \{0+\mathbb{Z}\})$ = cont periodic functions vanishing at 0.

Cont functions on $\mathbb{R}$ supported in $[0,1]$. Yes.

So what you are doing is to take $f \in C_c(\mathbb{R})$, mult by $|\sin(\pi x)|$ so that it vanishes on $0+\mathbb{Z}$, ~~this is the d~~ this can be written uniquely as a sum $\sum_{n\in\mathbb{Z}} f_n(x-n)$

where $f_n \in C_{[0,1]}(\mathbb{R})$, $f_n = 0$ at n,) so you effectively get a twisted Laurent series $\sum_{n\in\mathbb{Z}} u^n f_n = \sum (u^n f_n u^{-n})\, u^n$

Progress is being made, but now you need to look at ρ in $C_c(\mathbb{R}) \rtimes \mathbb{Z}$.

$C_c(\mathbb{R})$ is the space of sections of E, and it is a module over itself as a $*$algebra under mult., also Γ acts. Let $A = \mathbb{C}_c(\mathbb{R}) \rtimes \mathbb{C}[\Gamma]$. A is non unital. You can embed $\mathbb{C}_c(\mathbb{R})$ into A, ~~but there's some~~

There's some variant of the partition of unity argument.

Some variant of the partition of 1 argument.

~~[illegible]~~ $|\sin(\pi x)|^2 + |\cos(\pi x)|^2 = 1$.

Other ideas: multiplier algebra of $C_0(\mathbb{R})$ = C^*-algebra of bdd. contin. functions. This includes the periodic functions $C(\mathbb{R}/\mathbb{Z})$, so it seems that $C_{bdd}(\mathbb{R}) \rtimes \mathbb{Z}$ might be needed.

What is nature of this argument? ~~The module~~

~~[illegible]~~ The partition of unity allows one to take $f \in C_c(\mathbb{R})$ and to split it

$$f = |\sin(\pi x)|^2 f + |\cos(\pi x)|^2 f$$

where $|\sin(\pi x)| f$ and $|\cos(\pi x)| f$ do not see f on $\mathbb{Z}$ resp f on $\frac{1}{2}+\mathbb{Z}$ these belong to a graded subalgebra, there's a grading submodule

Thus if $f \in C_c(\mathbb{R})$ vanishes on a coset $y + \mathbb{Z}$ then the characteristic function of $[y, y+1]$ preserves continuity. If you combine this with the action of Γ ~~you know~~ ?

Consider the subspace of $f \in C_c(\mathbb{R})$, $f(\mathbb{Z}) = 0$ this subspace is stable under $\mathbb{Z}$ translation and is naturally graded wrt $\mathbb{Z}$, so it's a module over $\hat{\mathbb{Z}} \rtimes \mathbb{Z}$ meaning Morita equivalent to a $C((0,1))$-module So you will use $C((0,1)) \otimes \Gamma$ $\mathbb{Z}$-graded.

Things become clearer.

$$C_c(\mathbb{R}) \xrightarrow{\begin{pmatrix}|\sin \pi x| \\ |\cos \pi x|\end{pmatrix}} \begin{matrix} C_c(\mathbb{R}, 0+\mathbb{Z}) \\ \oplus \\ C_c(\mathbb{R}, \frac{1}{2}+\mathbb{Z}) \end{matrix} \xrightarrow{(|\sin \pi x| \ \ |\cos \pi x|)} C_c(\mathbb{R})$$

So look at $C_c(\mathbb{R}, 0+\mathbb{Z})$ as a module over $C_c(\mathbb{R}) \rtimes \mathbb{C}[\mathbb{Z}]$. ~~It is graded over~~ $C_c(\mathbb{R}, 0+\mathbb{Z})$ is naturallly $\mathbb{Z}$ graded. You want to see that $C_c(\mathbb{R}, 0+\mathbb{Z})$ is a fg proj $C_c(\mathbb{R}) \otimes \mathbb{C}[\mathbb{Z}]$ module.

$$C_c(\mathbb{R}, 0+\mathbb{Z}) = \bigoplus_n C([n, n+1))$$
$$= C([0,1)) \times \mathbb{Z}$$

If $C((0,1))$ a fg proj $C_c(\mathbb{R})$ module

$$C_c(\mathbb{R}) \xrightarrow{\begin{pmatrix}\sin \pi x \\ \cos \pi x\end{pmatrix}} \begin{matrix} C_c(\mathbb{R}, 0+\mathbb{Z}) \\ \oplus \\ C_c(\mathbb{R}, \frac{1}{2}+\mathbb{Z}) \end{matrix} \subset \begin{matrix} C_c(\mathbb{R}) \\ \oplus \\ C_c(\mathbb{R}) \end{matrix} \xrightarrow{(\sin \pi x \ \ \cos \pi x)} C_c(\mathbb{R})$$

$$C_c(\mathbb{R}) \xrightarrow{\sin \pi x} \bigoplus_n C([n, n+1))$$
$$\|$$
$$\bigoplus_n u^n C((0,1)) u^{-n} \overset{?}{\subset} C_c(\mathbb{R}) \rtimes \mathbb{C}[\mathbb{Z}]$$

Review. Basic idea is that

$$C_c(\mathbb{R}, y+\mathbb{Z}) = \{ f \in C_c(\mathbb{R}) \mid f(y+\mathbb{Z}) = 0 \}.$$

is ~~natural~~ natural graded wrt $y+\mathbb{Z}$

$$C_c(\mathbb{R}, y+\mathbb{Z}) = \bigoplus_{n \in \mathbb{Z}} C((y+n, y+n+1))$$

So what are we doing.

$$\boxed{C_c(\mathbb{R}, \mathbb{Z}) = \bigoplus_{n \in \mathbb{Z}} C((n, n+1)) \qquad \text{grading}}$$

This is an isom over $\boxed{C_{b}(\mathbb{R}) \tilde{\otimes} \mathbb{C}[\mathbb{Z}]}$

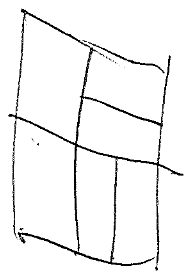
arb cont. functions on $\mathbb{R}$.

$$C_c(\mathbb{R}) \xrightarrow{\begin{pmatrix} |\sin \pi x| \\ |\cos \pi x| \end{pmatrix}} \begin{matrix} C_c(\mathbb{R}, \mathbb{Z}) \\ \oplus \\ C_c(\mathbb{R}, \frac{1}{2}+\mathbb{Z}) \end{matrix}$$

What you need now is an embedding of $C_c(\mathbb{R}, \mathbb{Z})$ into the cross product algebra, or a unital version of it.

Combine

$$\begin{tikzcd} & C_c(\mathbb{R}, \mathbb{Z}) \\ C((0,1)) \arrow[ur, hook] \arrow[r, hook] & C(\mathbb{R}/\mathbb{Z}) \end{tikzcd}$$



Try the following: Take the projection

$$C_c(\mathbb{R}, \mathbb{Z}) \xrightarrow{\text{pro}} C((0,1))$$

$$f(x) \longmapsto (f|(0,1))$$

coextends to a map

$$C_c(\mathbb{R}, \mathbb{Z}) \longrightarrow C((0,1))^{\mathbb{Z}}$$

$$f \longmapsto \left(n \mapsto \underset{x \in (0,1)}{f(n+x)}\right)$$

Problem: Produce a map $C_c(\mathbb{R}) \tilde{\otimes} \mathbb{C}[\Gamma]$ $\longrightarrow C_c(\mathbb{R})$ which is compatible with mult by cont. functions and translation by $\mathbb{Z}$. in other words you want a module map over the cross product algebra. Normally you take an element of $C_c(\mathbb{R})$ and act. Simplest ~~means. What seems simpler~~ Just acts on $|\sin(\pi x)|$.

Review the situation: $C_c(\mathbb{R})$ = cont. fns. on $\mathbb{R}$ comp support module over $C(\mathbb{R}/\mathbb{Z}) \otimes \mathbb{C}[\mathbb{Z}]$ which is fg proj., why $1 = |\sin(\pi x)|^2 + |\cos(\pi x)|^2$

$$C_c(\mathbb{R}) \xrightarrow{\begin{pmatrix}\sin(\pi x)\\ \cos(\pi x)\end{pmatrix}} \begin{matrix} C_c(\mathbb{R}, \mathbb{Z}) \\ \oplus \\ C_c(\mathbb{R}, \frac{1}{2}+\mathbb{Z}) \end{matrix}$$

$$C_c(\mathbb{R},\mathbb{Z}) = \bigoplus_{n\in\mathbb{Z}} C_0((n, n+1))$$ natural grading

$$C_c(\mathbb{R}, \tfrac{1}{2}+\mathbb{Z}) = \bigoplus_{n\in\mathbb{Z}} C_0((n-\tfrac{1}{2}, n+\tfrac{1}{2}))$$

$$C_c(\mathbb{R},\mathbb{Z}) \simeq \begin{matrix} C_0((0,1)) \otimes \mathbb{C}[\mathbb{Z}] \\ \wr| \\ C(\mathbb{R}/\mathbb{Z}, 0+\mathbb{Z}) \end{matrix}$$

so you will get an embedding

$$C_c(\mathbb{R},\mathbb{Z}) \hookrightarrow C(\mathbb{R}/\mathbb{Z}) \otimes \mathbb{C}[\mathbb{Z}].$$

Repeat: $C_c(\mathbb{R})$ = space of sections of the vector bundle E over the circle $\mathbb{R}/\mathbb{Z}$ with fibre $E_y = \mathbb{C}[\pi^{-1}(y)]$ at $y \in \mathbb{R}/\mathbb{Z}$. It's an infinite diml vector bundle with an action of the group ring $\mathbb{C}[\mathbb{Z}]$ such that E_y is a free rank 1 - module over $\mathbb{C}[\mathbb{Z}]$. To show $C_c(\mathbb{R})$ is a f.g. proj module over $C(\mathbb{R}/\mathbb{Z}) \otimes \mathbb{C}[\mathbb{Z}]$.

Basic idea is to use $1 = (\sin \pi x)^2 + (\cos \pi x)^2$ to factor the identity map

replace by $C_c(\mathbb{R},\mathbb{Z})$

$$C_c(\mathbb{R}) \xrightarrow{\begin{pmatrix}\sin \pi x \\ \cos \pi x\end{pmatrix}} \begin{matrix} C_c(\mathbb{R}) \\ \oplus \\ C_c(\mathbb{R}) \end{matrix} \xrightarrow{(\sin \pi x \quad \cos \pi x)} C_c(\mathbb{R})$$

replace by $C_c(\mathbb{R}, \frac{1}{2}+\mathbb{Z})$

~~[illegible]~~ Compatible with mult. by cont. functions on $\mathbb{R}$ and translation, in particular over $C(\mathbb{R}/\mathbb{Z}) \otimes \mathbb{C}[\mathbb{Z}]$.

Now $C_c(\mathbb{R},\mathbb{Z}) = \bigoplus_{n\in\mathbb{Z}} C_0((n,n+1))$ is $\mathbb{Z}$ graded

this is a grading ~~resultn~~ where the nth proj is mult by $\chi_{[n,n+1]}$ which preserves $C_0((n,n+1))$

This grading comp. with $\mathbb{Z}$ action, so that you have an isom.

$$C_c(\mathbb{R},\mathbb{Z}) = \mathbb{C}[\mathbb{Z}] \otimes \boxed{C_0((0,1))}$$

same as cont fns. support in $[0,1]$
periodic cont fns. vanishing at 0

$$\phi(x-n) \longleftarrow u^n \otimes \phi(x)$$

Repeat. $C_c(\mathbb{R})$ = space of sections of E, the v.b. over $\mathbb{R}/\mathbb{Z}$ with fibre $E_y = \mathbb{C}[x+\mathbb{Z}]$ = functions with finite support on the coset $y = x+\mathbb{Z}$. A section of E ~~measurable~~ over U open $\subset \mathbb{R}/\mathbb{Z}$ is a continuous function Nope, you want to avoid the sheaf picture, the open set viewpoint. Instead you want a section of E over a compact subset K ~~to~~ be a continuous function with compact supp. on $\pi^{-1}(K)$. This is an improved def.

$$C_c(\mathbb{R}) = \text{continuous functions with compact support on } \mathbb{R}.$$

$$C_c(\mathbb{R}, \mathbb{Z}) = \{ f \in C_c(\mathbb{R}) \mid f|\mathbb{Z} = 0 \}.$$

Aug 1.

naturally $\mathbb{Z}$ graded.

acted on by $\mathbb{Z}$

$$= \bigoplus_{n\in\mathbb{Z}} C_{[n,n+1]}(\mathbb{R})$$

$$= \bigoplus_{n\in\mathbb{Z}} u^n C_{[0,1]}(\mathbb{R}) \simeq \mathbb{C}[u,u^{-1}] \otimes C_{[0,1]}(\mathbb{R})$$

$$\hookrightarrow \mathbb{C}[\mathbb{Z}] \otimes C(\mathbb{R}/\mathbb{Z}).$$

Your aim is to understand $C_c(\mathbb{R},\mathbb{Z})$ as a $\mathbb{C}[\mathbb{Z}] \otimes C(\mathbb{R}/\mathbb{Z})$ module, and the fact is that it's the ideal $\mathbb{C}[\mathbb{Z}] \otimes C(\mathbb{R}/\mathbb{Z}, 0+\mathbb{Z})$ inside $\mathbb{C}[\mathbb{Z}] \otimes C(\mathbb{R}/\mathbb{Z})$

The point somehow is that when you impose vanishing on the coset $\mathbb{Z}$, then you ~~have an isomorphism between~~ the principal bundle becomes trivial giving ~~an isom~~ the desired isom.

$$\begin{array}{ccccc} \mathbb{R} & \hookleftarrow & \pi^{-1}(K) & \simeq & K\times\mathbb{Z} \\ \downarrow \pi & & \downarrow & & \downarrow \\ \mathbb{R}/\mathbb{Z} & \hookleftarrow & K & = & K \end{array}$$

O

$$\mathrm{Sect}(E,K) \simeq C(K)\otimes\mathbb{C}[\mathbb{Z}] = C_c(K\times\mathbb{Z})$$

---

next Cuntz's framework (Skandalis, etc). Here instead of $C(\mathbb{R}/\mathbb{Z})\otimes\mathbb{C}[\mathbb{Z}]$ you want $C_c(\mathbb{R})\rtimes\mathbb{Z}$

Consider $C_c(\mathbb{R})$ as a module over $A\rtimes\mathbb{Z}$ where A is some ring of continuous functions on $\mathbb{R}$ closed under $\mathbb{Z}$ translation, say $A =$ all continuous bounded functions, or you might like the semi product $C_c(\mathbb{R}) \oplus C(\mathbb{R}/\mathbb{Z})$. So what else to do ???

You want to show that $C_c(\mathbb{R})$ is a fin. proj module over $B = A\rtimes\mathbb{Z}$. Again you use your partition $1 = \sin^2(\pi x) + \cos^2(\pi x)$

$$C_c(\mathbb{R}) \xrightarrow{\binom{s}{c}} \begin{array}{c} C_c(\mathbb{R},\mathbb{Z}) \\ \oplus \\ C_c(\mathbb{R},\tfrac{1}{2}+\mathbb{Z}) \end{array} \xrightarrow{(s \quad c)} C_c(\mathbb{R})$$

So what do you know about the ~~subspace~~ submodule $C_c(\mathbb{R},\mathbb{Z})$ of $C_c(\mathbb{R})$? ~~[illegible]~~

~~[illegible]~~ It's closed under $\mathbb{Z}$ translation and multiplication by bdd functions which are cont except on $\mathbb{Z}$.

There's something you have been sloppy about, namely you need specific generators for $C_c(\mathbb{R})$ over the ring say $C(\mathbb{R}/\mathbb{Z}) \otimes \mathbb{C}[\mathbb{Z}]$

$$
\begin{tikzcd}
C_c(\mathbb{R}) \arrow[r, "s\cdot"] & C_c(\mathbb{R},\mathbb{Z}) \arrow[r, phantom, "\simeq"] & C_0((0,1)) \otimes \mathbb{C}[\mathbb{Z}] \arrow[d, phantom, "\parallel"] & \\
 & & \underbrace{C(\mathbb{R}/\mathbb{Z}, 0+\mathbb{Z})} \arrow[d, phantom, "\cap"] & \\
 & & C(\mathbb{R}/\mathbb{Z}) \otimes \mathbb{C}[\mathbb{Z}] \arrow[r, "S_{[0,1]}"] & C_c(\mathbb{R})
\end{tikzcd}
$$

$S_{[0,1]}$

Take $A \rtimes \mathbb{Z}$ and look at the mult. map

$$
\begin{tikzcd}
 & A \rtimes \mathbb{Z} \arrow[r, "\cdot S_{[0,1]}"] & C_c(\mathbb{R}) \\
 & \cup & \\
C_0(\mathbb{R}) \arrow[r, "\cdot S"] & A &
\end{tikzcd}
$$

This should be easy. $A = C_c(\mathbb{R})$ say
$B = A \rtimes \mathbb{Z}$, you want B-module maps

$$A \xrightarrow{\alpha} B \xrightarrow{\beta} A$$

the map from $B \longrightarrow A$ will be the ?

~~extend~~ α involves $|\sin(\pi x)|$, you mult. by $|\sin(\pi x)|$ on A split $\quad |\sin(\pi x)| = \sum_n u^n S_{[0,1]}$

$$|\sin(\pi x)| = \sum_n u^n \underbrace{s_{[0,1]}(x)}_{\in C_{[0,1]}(\mathbb{R})} u^{-n} = \sum_n s_{[n,n+1]}(x)$$

$$a\,|\sin(\pi x)| = \sum_n a \underbrace{u^n s_{[0,1]}(x) u^{-n}}_{(u^n * s_{[0,1]})}$$

$C_c(\mathbb{R})$ this is a module over $C_c(\mathbb{R}) \rtimes \mathbb{Z}$

$$\sum_{n\in\mathbb{Z}} a_n u^n \qquad a_n \in C_c(\mathbb{R})$$

given $f \in C_c(\mathbb{R})$ mult by $|\sin(\pi x)| = \sum_n u^n * s_{[0,1]}(x)$

get $\sum_n f u^n s_{[0,1]} u^{-n} = \sum_n f s_{[n,n+1]}$

You want a ~~module map~~ map $C_c(\mathbb{R}) \longrightarrow C_c(\mathbb{R}) \rtimes \mathbb{Z}$ which is ~~compl~~ compatible with translation and mult by functions. You are going to use the fact that $|\sin(\pi x)|\, C_c(\mathbb{R})$ admits $\mathbb{Z}$ action and $\mathbb{Z}$ grading.

$$|\sin(\pi x)| = \sum_{n\in\mathbb{Z}} u^n * s_{[0,1]}$$

$$|\sin(\pi x)|\, f = \sum_{n\in\mathbb{Z}} (u^n * s_{[0,1]}) f = \sum_n s_{[n,n+1]} f$$

$$= \sum_{n\in\mathbb{Z}} u^n\, s_{[0,1]}(u^{-n} f)$$

projection

what is the point.

So what can we do? You have this space $|\sin(\pi x)|\, C_c(\mathbb{R})$ of continuous functions; it's contained in $C_c(\mathbb{R}, \mathbb{Z})$ which has grading $C_c(\mathbb{R},\mathbb{Z}) = \bigoplus_{n\in\mathbb{Z}} C_0((n,n+1)) = C_{[n,n+1]}(\mathbb{R})$

$$\mathbb{C}[\mathbb{Z}] \otimes C_0((0,1)) \xrightarrow{\sim} C_c(\mathbb{R},\mathbb{Z})$$
$$u^n \otimes f$$

Therefore if you want a ~~map of~~ map

$$C_c(\mathbb{R},\mathbb{Z}) \longrightarrow \mathbb{C}[\mathbb{Z}] \otimes C_c(\mathbb{R})$$

you need to give

$$C_{[0,1]}(\mathbb{R}) \longrightarrow \mathbb{C}[\mathbb{Z}] \otimes C_c(\mathbb{R})$$

In this way you get a canonical map

$$C_c(\mathbb{R},\mathbb{Z}) \longrightarrow \mathbb{C}[\mathbb{Z}] \ltimes C_c(\mathbb{R})$$
$$\parallel$$
$$\mathbb{C}[\mathbb{Z}] \otimes C_{[0,1]}(\mathbb{R})$$

next you need a map $\mathbb{C}[\mathbb{Z}] \ltimes C_c(\mathbb{R}) \longrightarrow C_c(\mathbb{R})$ acting on a specific element of $C_c(\mathbb{R})$.

Repeat. You are trying to show $C_c(\mathbb{R})$ is a fin. proj. $C_c(\mathbb{R}) \rtimes \mathbb{Z}$-module

$$C_c(\mathbb{R}) \longrightarrow C_c(\mathbb{R},\mathbb{Z}) \simeq \bigoplus_{n\in\mathbb{Z}} C_{[n,n+1]}(\mathbb{R})$$
$$f \longmapsto f\,|\sin(\pi x)| \qquad \simeq \mathbb{C}[\mathbb{Z}] \otimes C_{[0,1]}(\mathbb{R})$$

You are trying to produce enough module maps from $C_c(\mathbb{R})$ to the ring $\underline{C_c(\mathbb{R}) \rtimes \mathbb{Z}} = B$ and module maps $\tilde{B} \longrightarrow C_c(\mathbb{R})$. The latter should clearly be mult. by something like $s_{[0,1]}$. You need an elt of $C_c(\mathbb{R})$, obvious one is $|\sin(\pi x)|\chi_{[0,1]}$.

Next look at module maps from $C_c(\mathbb{R})$ to $B = C_c(\mathbb{R}) \rtimes \mathbb{Z}$. You have

$$C_c(\mathbb{R}) \xrightarrow{\cdot|\sin(\pi x)|} C_c(\mathbb{R}, \mathbb{Z}) = \bigoplus_{n \in \mathbb{Z}} C_{[n,n+1]}(\mathbb{R})$$

$$\simeq \mathbb{C}[\mathbb{Z}] \otimes C_{[0,1]}(\mathbb{R})$$

map of $\mathbb{C}[\mathbb{Z}] \tilde{\otimes} C_c(\mathbb{R})$ modules

Look at $C_c(\mathbb{R}, \mathbb{Z}) = \bigoplus_{n \in \mathbb{Z}} C_{[n,n+1]}(\mathbb{R})$ carefully

$B = C_c(\mathbb{R}) \rtimes \mathbb{Z}$ generated by $C_c(\mathbb{R})$ and u subject to $u f u^{-1} = u * f$ $\quad (u*f)(x) = f(x-1)$

$$C_c(\mathbb{R}, \mathbb{Z}) = \bigoplus_{n \in \mathbb{Z}} \overbrace{u^n C_{[0,1]}(\mathbb{R})}^{C_{[n,n+1]}(\mathbb{R})} = \mathbb{C}[\mathbb{Z}] \otimes C_{[0,1]}(\mathbb{R})$$

To get a B-module map $C_c(\mathbb{R}) \longrightarrow B$, it suffices to give a $C_c(\mathbb{R})$-module map $C_{[0,1]}(\mathbb{R}) \longrightarrow B = C_c(\mathbb{R}) \rtimes \mathbb{Z}$ apparently obvious

Basically obvious

$$C_c(\mathbb{R}) \xrightarrow{|\sin(\pi x)|} C_c(\mathbb{R}, \mathbb{Z}) = \bigoplus_{n\in\mathbb{Z}} C_{[n,n+1]}(\mathbb{R})$$

$$f \qquad \sum s_{[n,n+1]} f$$

$B = C_c(\mathbb{R}) \rtimes \mathbb{Z}$ acts on $C_c(\mathbb{R})$

aim ~~to show~~ enough why $C_c(\mathbb{R})$ is a fin gen proj $\tilde{B}$-module

You need B-module maps $C_c(\mathbb{R}) \longrightarrow B$ and $\tilde{B} \longrightarrow C_c(\mathbb{R})$ to factor the identity map on $C_c(\mathbb{R})$.

$$1 = |\sin(\pi x)|^2 + |\cos(\pi x)|^2$$

mult. by $|\sin(\pi x)|$ gives a B-mod map

$$C_c(\mathbb{R}) \longrightarrow C_c(\mathbb{R}, \mathbb{Z}) = \bigoplus_{n\in\mathbb{Z}} C_{[n,n+1]}(\mathbb{R}) = \mathbb{C}[\mathbb{Z}] \otimes C_{[0,1]}(\mathbb{R})$$

$$f \longmapsto \sum_n s_{[n,n+1]} f =$$

$$\cancel{s_{[n,n+1]} f = u^n s_{[0,1]} u^{-n} f}$$

$$u * f \longmapsto \sum_n (u^n * s_{[0,1]})(u * f)$$

$$= \sum_n u^n s_{[0,1]} u^{-n} u f u^{-1} = u \underbrace{\left(\sum_n u^{n-1} s_{[0,1]} u^{-n+1} f\right)}_{\sum_n (u^{n-1} * s_{[0,1]}) f} u^{-1} \quad ?$$

$C_c(\mathbb{R}) \rtimes \mathbb{Z}$ gen by $f \in C_c(\mathbb{R})$ and u invertible $\ni$ $(ufu^{-1})(x) = f(x-1)$

$$f \longmapsto \sum_{n\in\mathbb{Z}} s_{[n,n+1]} f = \sum_n \left(u^n * s_{[0,1]}\right) f$$

OKAY

$$\begin{array}{ccc} C_c(\mathbb{R}) & \longrightarrow & \bigoplus_{n\in\mathbb{Z}} C_{[n,n+1]}(\mathbb{R}) \\ & \searrow^{|\sin(\pi x)|} & \| \\ & & C_c(\mathbb{R},\mathbb{Z}) \end{array}$$

mult by $|\sin(\pi x)|$

So we have a B-mod. map from $C_c(\mathbb{R})$ to $C_c(\mathbb{R},\mathbb{Z})$. This is step 1. Next ~~you~~ a B-mod map $C_c(\mathbb{R},\mathbb{Z}) \longrightarrow B$.

$$\begin{array}{ccc} C_c(\mathbb{R},\mathbb{Z}) & \longrightarrow & B \\ \| & & \| \\ \mathbb{C}[\mathbb{Z}] \otimes C_{[0,1]}(\mathbb{R}) & \longrightarrow & \mathbb{C}[\mathbb{Z}] \otimes C_c(\mathbb{R}) \end{array}$$

so there's an obvious map: ~~the embedding~~ the inclusion of $C_{[0,1]}(\mathbb{R}) \subset C_c(\mathbb{R})$. This is step 2

Look at $B = \mathbb{C}[\mathbb{Z}] \tilde{\otimes} C_c(\mathbb{R})$ elements are finite sums $\sum u^n f_n$ with $f_n \in C_c(\mathbb{R})$

Point: $C_c(\mathbb{R},\mathbb{Z})$ is the same "size" as $C_c(\mathbb{R})$

$$C_c(\mathbb{R},\mathbb{Z}) = \bigoplus \left(u^n * C_{[0,1]}(\mathbb{R})\right)$$

Repeat. Consider $C_c(\mathbb{R})$ as module over the crossproduct $\mathbb{C}[\mathbb{Z}] \tilde{\otimes} C_c(\mathbb{R})$. In more detail begin with the ring $A = C_c(\mathbb{R}) = \{$cont. functions on $\mathbb{R}$ with compact support. Action of $\mathbb{Z}$ on A:

$$(u*f)(x) = f(x-1). \qquad (u^n*f)(x) = f(x-n).$$

$B =$ crossproduct $\mathbb{Z} \ltimes A = \mathbb{C}[\mathbb{Z}] \otimes A$ in which $ua = (u*a)u$. Can consider A as a left B-module where A acts by mult and u acts as $f \mapsto u*f$.

$$\cancel{(u a f)(x) = u*(af)(x) = (af)(x-1)}$$

$$u(af) = (u*a)(u*f) = (u*a)\cdot uf = \big((u*a)u\big)f$$

$$\begin{matrix} {}^u(af) & = & {}^ua\,{}^uf = {}^ua(u(f)) \\ \| & & \\ u(af) & & \end{matrix} \qquad \therefore \boxed{ua = {}^ua\,u}$$

Next point: Consider the B-modules $C_c(\mathbb{R}, \mathbb{Z})$, $C_c(\mathbb{R}, \tfrac{1}{2}+\mathbb{Z})$.

$$\begin{array}{ccc} C_c(\mathbb{R},\mathbb{Z}) & = & \bigoplus_{n\in\mathbb{Z}} C_{[n,n+1]}(\mathbb{R}) \\ & \nwarrow & \uparrow \wr \\ & & \mathbb{C}[\mathbb{Z}] \otimes C_{[0,1]}(\mathbb{R}) \end{array}$$

This splitting comp. with mult by A and with translation

namely you have $C_{[0,1]}(\mathbb{R}) \subset C_c(\mathbb{R})$

~~Abstract~~ A ring $C_c(\mathbb{R})$, $B = \mathbb{Z} \ltimes A$

Consider A as left B-module. To show ~~A~~ A is ~~a~~ a fin gen. projective B-module, i.e. $\exists$ fact. of identity map on A:

$$A \longrightarrow \tilde{B}^N \longrightarrow A$$

Idea: $$A \xrightarrow{\begin{pmatrix} \sin \\ \cos \end{pmatrix}} \begin{matrix} C_c(\mathbb{R}, \mathbb{Z}) \\ \oplus \\ C_c(\mathbb{R}, \tfrac{1}{2}+\mathbb{Z}) \end{matrix} \xrightarrow{(\sin \ \cos)} A$$

reduces to showing $C_c(\mathbb{R}, \mathbb{Z})$ is a fg proj B module

---

$$A = C_c(\mathbb{R}) \xrightarrow{|\sin|} C_c(\mathbb{R}, \mathbb{Z})$$

Consider the space $C_c(\mathbb{R}, \mathbb{Z}) = \bigoplus_{n \in \mathbb{Z}} C_{[n,n+1]}(\mathbb{R})$

$f \in C_c(\mathbb{R})$, $|\sin(\pi x)| f = \sum_n s_{[n,n+1]} f$

You want to understand $C_c(\mathbb{R}, \mathbb{Z})$ as a module over $\mathbb{C}[\mathbb{Z}] \otimes C(\mathbb{R}/\mathbb{Z})$, and also as a module over $\mathbb{C}[\mathbb{Z}] \ltimes \mathbb{C}_c(\mathbb{R})$. Maybe I should go back over the $\mathbb{C}[\mathbb{Z}] \otimes C(\mathbb{R}/\mathbb{Z})$ situation.

Given $f \in C_c(\mathbb{R})$ consider $|\sin(\pi x)| f \in C_c(\mathbb{R}, \mathbb{Z})$

$$= \underbrace{\bigoplus_{n \in \mathbb{Z}} C_{[n,n+1]}(\mathbb{R})}_{\parallel} \quad \Big| \quad |\sin(\pi x)| f = \sum s_{[n,n+1]} f$$

$$\mathbb{C}[\mathbb{Z}] \otimes C_{[0,1]}(\mathbb{R}) \subset \mathbb{C}[\mathbb{Z}] \otimes C(\mathbb{R}/\mathbb{Z})$$

Take $f \in C_c(\mathbb{R})$ then $|sm| f = \sum_{n\in\mathbb{Z}} s_{[n,n+1]} f$

you want to associate an element of $\mathbb{C}[\mathbb{Z}] \otimes C(\mathbb{R}/\mathbb{Z})$.

$g \in C_c(\mathbb{R}, \mathbb{Z})$ $\quad g = \sum_{n\in\mathbb{Z}} u^n g_n$

You need to start at the beginning.

$\mathbb{R} \xrightarrow{\pi} \mathbb{R}/\mathbb{Z}$ $\quad \pi(x) = x + \mathbb{Z}$

If $y = x+\mathbb{Z} \in \mathbb{R}/\mathbb{Z}$, then $E_y = \mathbb{C}[x+\mathbb{Z}]$
= vector space of finite support functions on the coset. This is a flat vector bundle

---

Repeat but with new idea of using π!

$\mathbb{R} \xrightarrow{\pi} \mathbb{R}/\mathbb{Z}$ $\quad \pi(x) =$ the coset $x+\mathbb{Z}$

If $y = x+\mathbb{Z}$, then $E_y = \mathbb{C}[x+\mathbb{Z}] = \mathbb{C}$-valued functions on the coset with finite support.

~~E is a flat vector bundle over X~~

$\pi : \mathbb{R} \longrightarrow \mathbb{R}/\mathbb{Z}$ is a princ. $\mathbb{Z}$-bundle, so can form

$\mathbb{R} \times^{\mathbb{Z}} \mathbb{C}[\mathbb{Z}]$, this is an assoc. fibre bundle once you make ~~but you haven't made~~ clear the topology on $\mathbb{C}[\mathbb{Z}]$.

Use sections with proper support!

~~What~~ So if you give $\mathbb{C}[\mathbb{Z}]$ the fine topology, then you have a fibre bundle over the circle, locally trivial, so sections over a compact K are essentially cont. maps $K \to \mathbb{C}[\mathbb{Z}]$ and the usual business about fine or ind. limit top yields $C(K) \otimes \mathbb{C}[\mathbb{Z}]$, the alg $\otimes$.

Recall why $C_c(\mathbb{R})$ is a finite projective module over $C(\mathbb{R}/\mathbb{Z}) \otimes \mathbb{C}[\mathbb{Z}]$. Start with $f \in C_c(\mathbb{R})$, that is, a section of E over $\mathbb{R}/\mathbb{Z}$. Partition of unity on $\mathbb{R}/\mathbb{Z}$ $\quad 1 = |\sin(\pi x)|^2 + |\cos(\pi x)|^2$

$$C_c(\mathbb{R}) \xrightarrow{\begin{pmatrix} |\sin| \\ |\cos| \end{pmatrix}} \begin{matrix} C_c(\mathbb{R}) \\ \oplus \\ C_c(\mathbb{R}) \end{matrix} \xrightarrow{\begin{pmatrix} |\sin| & |\cos| \end{pmatrix}} C_c(\mathbb{R})$$

$$C_c(\mathbb{R}) \xrightarrow{|\sin|} C_c(\mathbb{R}, \mathbb{Z}) = \bigoplus_{n \in \mathbb{Z}} C_{[n,n+1]}(\mathbb{R})$$

$$|\sin(\pi x)| f(x) = \sum_{n \in \mathbb{Z}} s_{[n,n+1]} f$$

~~confusing~~ instead, follow geometric picture.

The point is that ~~over~~ if $0+\mathbb{Z}$ is removed from $\mathbb{R}/\mathbb{Z}$ then E becomes trivial.

Finally found what to do.

$$A \rtimes \Gamma = \bigoplus_{s \in \Gamma} As \qquad (a_1 s)(a_2 t) = a_1 (s * a_2) st$$

Let M be an A-module with compatible Γ action

$$s * (am) = (s*a)(s*m)$$

$$s*(a_1a_2) = (s*a_1)(s*a_2)$$

$$\big((a_1s)(a_2t)\big)(a_3u) = \big((a_1(s*a_2))st\big)(a_3u) = a_1(s*a_2)(st*a_3)\,stu$$

$$(a_1s)\big((a_2t)(a_3u)\big) = (a_1s)\big(a_2(t*a_3)tu\big) = a_1\underbrace{s*(a_2(t*a_3))}_{(s*a_2)(st*a_3)}stu$$

Γ group act on an alg A.

$$s*(a_1a_2) = (s*a_1)(s*a_2)$$
$$(st)*a = s*(t*a)$$

Let Γ act on an A-mod. M compatibly

$$s*(am) = (s*a)(s*m)$$
$$(st)*m = s*(t*m)$$

$$A \rtimes \Gamma = \bigoplus_{s\in\Gamma} As \qquad (as)(a't) = a(s*a')\,st$$

If M is an (A,Γ) module then

$(as)m = a(s*m)$ makes M an $A\rtimes\Gamma$-module

$$(as)\big((a't)m\big) = (as)\big(a'(t*m)\big) = a(s*a')\,s*(t*m)$$

$$\big(a(s*a')st\big)m = a(s*a')(st*m)$$

In particular $A\rtimes\Gamma$ acts on A via

$$(as)(a') = a(s*a')$$

Take case $A = C_c(\mathbb{R})$ $\qquad \Gamma = \mathbb{Z}$

$$A\rtimes\mathbb{Z} = \bigoplus_{n\in\mathbb{Z}} Au^n \qquad (u^n*f)(x) = f(x-n)$$

Γ action on A: $\quad s*(aa') = (s*a)(s*a')$

$$(st)*a = s*(t*a)$$

other notation $\quad s*a = {}^{s}a \qquad {}^{s}(aa') = {}^{s}a\,{}^{s}a'$

$${}^{st}a = {}^{s}({}^{t}a)$$

$$A \rtimes \Gamma = A \otimes \mathbb{C}[\Gamma] = A[\Gamma] = \bigoplus_{s\in\Gamma} A_s$$

$$(as)(a's') = a(s*a')\,ss'$$

$$((as)(a_1 s_1))(a_2 s_2) = (a(s*a_1)\,ss_1)(a_2 s_2)$$

$$= a(s*a_1)((ss_1)*a_2)(ss_1)s_2$$

$$\cancel{as(a_1(s_1*a_2)\,s_1s_2) = a(s*a_1)(s*((s_1*a_2)s_1s_2))}$$

$$\cancel{= a(s*a_1)\,s[(s_1*a_1)s_1s_2]}$$

$$\cancel{s*(s_1*a_1)\,ss_1s_2}$$

$$\cancel{= a\ s*a_1(s_1}$$

$$as((a_1s_1)(a_2s_2)) = as(a_1(s_1*a_2)\,s_1s_2)$$

$$= a\ s*(a_1(s_1*a_2))\ ss_1s_2$$

$$= a(s*a_1)(s*(s_1*a_2))\ ss_1s_2$$

$$= a(s*a_1)((ss_1)*a_2)\ ss_1s_2$$

Action of $A \rtimes \Gamma$ on A

$$(as)(a_1) = a(s*a_1)$$

View

$$A = (A \rtimes \Gamma) \otimes_{\Gamma} \overset{\text{trivial }\Gamma\text{-mod}}{\mathbb{C}}$$

idea $A \rtimes \Gamma \subset \tilde{A} \rtimes \Gamma$

so that $(\tilde{A} \rtimes \Gamma) \otimes_{\Gamma} \mathbb{C} \simeq \tilde{A}$

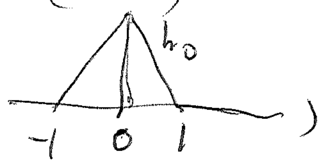
Now onto the example. $A = \mathbb{C}_c(\mathbb{R})$ cont fns. on $\mathbb{R}$ comp. support.

$(u^n * a)(x) = a(x-n)$. So A is automatically an $\underbrace{\boxed{A \rtimes \mathbb{Z}}}_{B}$ module. Q: Is it a fin gen proj module. In fact its a ~~quotient~~ of B

$$\begin{array}{ccc} A \xrightarrow[\text{with } \mathbb{Z}]{\text{not comp}} & A \rtimes \mathbb{Z} & \longrightarrow & A \\ & \parallel & & \parallel \\ & \bigoplus_{n \in \mathbb{Z}} Au^n & \longrightarrow & A \\ & au^n & \longmapsto & a \end{array}$$

To show A is a fin gen proj B-mod you need ~~enough~~ at least enough B-mod. maps $\tilde{B} \longrightarrow A$, ~~i.e. element~~ a finite set of generators for B. The "obvious" generator, i.e. first thing to think of, is 1, since $(au^n) \longmapsto (au^n)\underline{1} = a$. $a(u^n * \underline{1})$:

But $1 \notin A$. However



, this fn. h_0 is such that $\sum_n u^n * h_0 = \sum h_n = \underline{1}$. Thus h_0 is a generator

So far not too clear. You want to ~~take~~ use $h_0^{1/2}$

$\mathcal{B} = C_c(\mathbb{R}) \rtimes \mathbb{Z}$ acts on $\overset{A=}{C_c(\mathbb{R})}$

$B = A[\Gamma] = \bigoplus_{s\in\Gamma} As$ $\qquad sa = (s*a)s$

To show that A is a fin. gen. proj B-module. Note that $A^2 = A$, hence same for B should be true. Let's use what you learned from $C(\mathbb{T})\rtimes\mathbb{Z}$. There is a ~~good theorem~~ question to pose here, namely, why the K class (which is even) can be realized ~~by~~ a projection, i.e. why the class in $K_0(\mathcal{B})$ comes from a ~~firm~~ finitely generated projective module.

Digress ~~of~~ a little to handle Morita invariance of K_0 for idempotent rings. You did prove this I think; it somehow was linked to Vaserstein's identity, I think Vaserstein proved the Whitehead lemma for A idempotent. What do I mean by Whitehead lemma? $[GL(A), GL(A)] =$ $E(A)$ $\quad \left(e_{ij}^{a}, e_{kl}^{a'}\right)$? The fact that

$\oplus$ on matrices induced $+$ on $GL(A)_{ab}$. Begin with basic idea. $(1-pq)$ invertible $\Leftrightarrow$ $(1-qp)$ invertible

$(1-pq)^{-1} = 1 + p(1-qp)^{-1}q$ true by geom. series.

but clear in general $\quad (1+p(1-qp)^{-1}q)(1-pq) = 1-pq$?

$$(1-pq)\left(1+p(1-qp)^{-1}q\right) = 1-pq + \underbrace{(1-pq)p}_{p(1-qp)}(1-qp)^{-1}q = 1.$$

$$p(1-qp)(1-qp)^{-1}q$$

Another version via $\oplus$.

$$X \underset{p}{\overset{q}{\rightleftarrows}} Y$$

$$\begin{tikzcd}
& Y \arrow[d, "{\begin{pmatrix}-q\\1\end{pmatrix}}"] \arrow[dr, bend left, "1-p"] & \\
X \arrow[r, "{\begin{pmatrix}1\\-p\end{pmatrix}}"] \arrow[dr, bend right, "1-qp"'] & X\oplus Y \arrow[r, "(p\ \ 1)"] \arrow[d, "(1\ \ q)"] & Y \\
& X &
\end{tikzcd}$$

so what????

$K_0'A$ length one complexes h.eq. to a finite projective complexes $U_1 \xrightarrow{d} U_0$ over $\tilde{A}$

Basic problem an element of K_0A is rep. by $M(U_1, U_0; \alpha)$ α isom modulo A

$\|$ $U_1 \times_{U_0/AU_0} U_0$ $\alpha: U_1/AU_1 \xrightarrow{\sim} U_0/AU_0$

Milnor shows that this M is a finite proj A-module. How? the point is to construct another such triple V_1, V_0, β such that when you take $\oplus$.

$$\begin{array}{ccc}
U_1\oplus V_1 & U_0\oplus V_0 & \alpha\oplus\beta : (R/A)^N \longrightarrow (R/A)^N \\
\wr\| & \wr\| & \text{lifts to an elt} \\
R^N & R^N & \text{of } GL_n(R).
\end{array}$$

Let's finish off Cuntz stuff and make some notes.

Recall if Γ acts on A:
$$s*(aa') = (s*a)(s*a')$$
$$(st)*a = s*(t*a)$$

then $A \rtimes \Gamma = A \otimes \mathbb{C}[\Gamma] = \bigoplus_s As$ with prod.
$$(as)(a's') = a(s*a')ss'$$
also $A[\Gamma]$ twisted group ring

If M is a A-module equipped with an action of Γ $s*m$ compat. $s*(am) = (s*a)(s*m)$ then M becomes a $A \rtimes \Gamma$ module via $(as)m = a(s*m)$. Check
$$(a_1s_1)((a_2s_2)m) = (a_1s_1)a_2(s_2*m) = a_1(s_1*(a_2(s_2*m)))$$
$$= a_1(s_1*a_2)(s_1*(s_2*m)) = a_1(s_1*a_2)(s_1s_2*m)$$
$$((a_1s_1)(a_2s_2))m = (a_1(s_1*a_2)s_1s_2)m = a_1(s_1*a_2)(s_1s_2*m)$$

So the notation (exponent) ${}^s m$ saves parentheses
$$(as)m \overset{\text{def}}{=} a\,{}^s m$$
$$a_1s_1((a_2s_2)m) = a_1\,{}^{s_1}(a_2\,{}^{s_2}m) = a_1\,{}^{s_1}a_2\,{}^{s_1s_2}m$$
$$(a_1s_1\,a_2s_2)m = (a_1\,{}^{s_1}a_2\,s_1s_2)m = a_1\,{}^{s_1}a_2\,{}^{s_1s_2}m$$

now consider $A = C_c(\mathbb{R})$, $\Gamma = \mathbb{Z} = \{u^n \mid n \in \mathbb{Z}\}$.
$$A[\Gamma] = \bigoplus_{n\in\mathbb{Z}} Au^n$$
$$({}^{u^n}f)(x) = f(x-n)$$

$A[\Gamma]$ acts on A, the claim is that A is a fin. gen. proj. $A[\Gamma]$-module. There's a point here because A is not unital although it is idempotent. A has no idempotents, but enough left (all right) identities.

simplest thing to do is to use the formulas given by Cuntz. You have the simplicial complex $\Sigma_F = \{ M \text{ fin.} \neq \phi \subset \Gamma \mid M^{-1}M \subset F \}$ F given finite subset of Γ, ~~can~~ suppose $1 \in F$, so $\Sigma_F \neq \phi$ and then get same thing for $F \cap F^{-1}$! In our case $\Gamma = \mathbb{Z}$ you take $F = \{-1, 0, 1\}$.

$$\mathcal{E}_{\Sigma_F} = C^*\left\{ h_s, s \in \Gamma \;\middle|\; \begin{array}{l} h_s \geq 0, \quad h_s h_t = 0 \text{ when } s^{-1}t \notin F, \\ \left(\sum h_s - 1\right) h_t = 0 \text{ all } t, \end{array} \right.$$

$$\operatorname{Spec}\left(\mathcal{E}^{ab}_{\Sigma_F}\right) = \left\{ (h_s)_{s \in \Gamma} \;\middle|\; \begin{array}{l} h_s \in [0, \infty), \quad h_s h_t = 0 \quad \text{s,t not a simplex} \\ \sum_s h_s h_t = h_t \end{array} \right\}$$

Let (h_s) be a point in the spectrum which is not zero, let ~~...~~ $M = \{ s \in \Gamma \mid h_s > 0 \}$

~~... then ... assumption any~~

$$s, t \in M \implies h_s h_t > 0 \implies ~~\ldots~~ \; s^{-1}t \in F$$
$$\implies t \in \text{finite set } sF. \quad \therefore M \text{ finite}$$

$$\left(\sum_{s \in M} h_s \right) h_t = h_t \implies \sum_{s \in M} h_s = 1.$$

~~... puzzling ...~~ What's puzzling ~~to~~ me is how a partition of unity gives rise to a projector. — things come to mind like, equivalence between modules and quasi-coh. sheaves: Thm that $\Gamma(\operatorname{Spec} A, \tilde{M}) = M$, also your Serre-Swan proof.

Other ~~abstract~~ ideas. Instead of $A \rtimes \Gamma$ where A = some version of $C_{pr}(E\Gamma)$ you can look at $C(B\Gamma) \otimes \mathbb{C}[\Gamma]$. In the circle situation instead of $C_c(\mathbb{R}) \rtimes \mathbb{Z}$ you looked first at $C(\mathbb{R}/\mathbb{Z}) \otimes \mathbb{C}[\mathbb{Z}]$.

In the $\Gamma = \mathbb{Z}$ case with $F = \{-1, 0, 1\}$ you get a simple kind of projection. There are interesting formulas. Also there is the question of the relation between $C_c(\mathbb{R}) \rtimes \mathbb{Z}$ and $C(\mathbb{R}/\mathbb{Z}) \times \mathbb{Z}$. Both rings act on $C_c(\mathbb{R})$.

~~Viewpoint. There appears to be something new. Cross product behaviour~~

Both $C_c(\mathbb{R}) \rtimes \mathbb{Z}$ and $\overbrace{C(\mathbb{R}/\mathbb{Z}) \otimes \mathbb{C}[\mathbb{Z}]}^{C(\mathbb{R}/\mathbb{Z}) \times \mathbb{Z}}$ act on $C_c(\mathbb{R})$.

$$\begin{pmatrix} 1 & -p \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & p \\ q & 1 \end{pmatrix} \qquad \begin{pmatrix} (1-pq)^{-1} & \\ & 1 \end{pmatrix}\begin{pmatrix} 1-pq & 0 \\ q & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ -q & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ q & 1 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & p \\ 0 & 1 \end{pmatrix}\begin{pmatrix} (1-pq)^{-1} & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ +q & 1 \end{pmatrix}$$

At one time you motivated the argument geometrically. $\underset{\text{closed}}{Y} \subset X$ $\quad (E_1, E_0, \alpha : E_1|Y \xrightarrow{\sim} E_0|Y)$

Add a triple $(F_1, F_1, \mathrm{id} \bmod A)$ to assume E_1 trivial

Special case $E_1 = \mathbb{1}$, So you have E_0 on X and a section $s \in \Gamma(\cancel{Y}, E)$. Suppose geom. you have E a line bundle over X with non van. section s ~~of~~ $E|Y$.

Vaserstein's identity should link $1-pq$, $1-qp$

$$\begin{tikzcd} X \arrow[r, shift left, "p"] \arrow[loop left, "1-qp"] & Y \arrow[l, shift left, "q"] \arrow[loop right, "1-pq"] \end{tikzcd}$$

Somehow there's an identity ~~showing~~ that $1-pq$ and $1-qp$ are inverse in the sense of $\oplus$.

$$\begin{tikzcd} X \arrow[r, "p"] & Y \arrow[dl, "q"] \\ X \arrow[r, "p"] & Y \end{tikzcd}$$

$$\begin{pmatrix} 1-pq & 0 \\ 0 & 1-qp \end{pmatrix}$$

$$\begin{pmatrix} 1-pq & 0 \\ 0 & 1-qp \end{pmatrix}$$

$$\downarrow$$

$$\begin{pmatrix} 1-pq & 0 \\ q(1-pq) & 1-qp \end{pmatrix} \longmapsto \begin{pmatrix} 1-pq & \\ & \end{pmatrix}$$

$$\begin{pmatrix} 1 & p \\ 0 & 1 \end{pmatrix}\begin{pmatrix} (1-pq)^{-1} & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ q & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & p \\ 0 & 1 \end{pmatrix}\begin{pmatrix} (1-pq)^{-1} & 0 \\ q & 1 \end{pmatrix} = \begin{pmatrix} (1-pq)^{-1}+pq & p \\ q & 1 \end{pmatrix}$$

$$\begin{tikzcd}
 & Y \arrow[d, "\binom{q}{1}"] \arrow[dr, "1-pq"] & \\
X \arrow[r, "\binom{1}{p}"] \arrow[dr, "1-qp"'] & X \oplus Y \arrow[r, "(-p\ 1)"] \arrow[d, "(1\ -q)"] & Y \\
 & X &
\end{tikzcd}$$

You are very close to understanding the identity used by Vaserstein (essentially). The claim should be roughly that

$$\begin{pmatrix} (1-qp)^{-1} & 0 \\ 0 & (1-pq)^{-1} \end{pmatrix}$$

is a product of elementary matrices. Now this matrix represents an autos. of $X \oplus Y$

$$\begin{matrix} X \\ \oplus \\ Y \end{matrix}$$

~~$\binom{q}{1}(1-pq)^{-1}(-p\ \ 1)$~~

$$\begin{tikzcd}
X \arrow[r, shift left] & X \oplus Y \arrow[l, shift left, "(1\ \ -q)"] \arrow[r, shift right, "(-p\ \ 1)"'] & Y \arrow[l, shift right, "\binom{q}{1}(1-pq)^{-1}"']
\end{tikzcd}$$

$$X \underset{q}{\overset{p}{\rightleftarrows}} Y \qquad (1-qp)^{-1} \exists \Rightarrow (1-pq)^{-1} =$$

y left to $\begin{pmatrix} 0 \\ y \end{pmatrix}$ apply $(\cancel{1} \ -q)$ to get $-qy$

apply $(1-qp)^{-1}$, then $\begin{pmatrix} 1 \\ p \end{pmatrix}$ to get $-\begin{pmatrix} 1 \\ p \end{pmatrix}(1-qp)^{-1}qy$

which you remove from: $\begin{pmatrix} 0 \\ y \end{pmatrix} + \begin{pmatrix} 1 \\ p \end{pmatrix}(1-qp)^{-1}qy \in \mathrm{Ker}(1\ -q)$

$\begin{pmatrix} q \\ 1 \end{pmatrix}\left(y + p(1-qp)^{-1}qy\right)$ get $(1-pq)^{-1} = 1 + p(1-qp)^{-1}q$

What's somehow

$$\underbrace{\underbrace{\begin{pmatrix} 1 & -q \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & q \\ p & 1 \end{pmatrix}}_{\begin{pmatrix} (1-qp)^{-1} & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1-qp & 0 \\ p & 1 \end{pmatrix}}}_{\begin{pmatrix} 1 & 0 \\ -p & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ p & 1 \end{pmatrix}} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -p & 1 \end{pmatrix}\begin{pmatrix} (1-qp)^{-1} & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & -q \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & q \\ p & 1 \end{pmatrix}$$

$$\boxed{\begin{pmatrix} 1 & q \\ p & 1 \end{pmatrix} = \begin{pmatrix} 1 & q \\ & 1 \end{pmatrix}\begin{pmatrix} 1-qp & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ p & 1 \end{pmatrix}}$$

$$\begin{pmatrix} 1 & q \\ p & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ p & 1 \end{pmatrix}\begin{pmatrix} 1-qp & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & q \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ -p & 1 \end{pmatrix}\begin{pmatrix} 1 & q \\ p & 1 \end{pmatrix}\begin{pmatrix} 1 & -q \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & q \\ 0 & 1-pq \end{pmatrix}\begin{pmatrix} 1 & -q \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1-pq \end{pmatrix}$$

$$\begin{pmatrix} 1 & -q \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & q \\ p & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -p & 1 \end{pmatrix} = \begin{pmatrix} 1-qp & 0 \\ p & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -p & 1 \end{pmatrix} = \begin{pmatrix} 1-qp & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ -p & 1 \end{pmatrix}\begin{pmatrix} 1 & q \\ p & 1 \end{pmatrix}\begin{pmatrix} 1 & -q \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & q \\ 0 & 1-pq \end{pmatrix}\begin{pmatrix} 1 & -q \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1-pq \end{pmatrix}$$

$$\begin{pmatrix} 1 & q \\ p & 1 \end{pmatrix} = \boxed{\begin{pmatrix} 1 & q \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1-qp & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ p & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ p & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1-pq \end{pmatrix}\begin{pmatrix} 1 & q \\ 0 & 1 \end{pmatrix}}$$

$$= \begin{pmatrix} 1 & q \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ p(1-qp)^{-1} & 1 \end{pmatrix}\begin{pmatrix} 1-qp & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ p & 1 \end{pmatrix}\begin{pmatrix} 1 & q(1-pq)^{-1} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1-pq \end{pmatrix}$$

$$\boxed{\begin{pmatrix} 1-qp & 0 \\ 0 & (1-pq)^{-1} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -p(1-qp)^{-1} & 1 \end{pmatrix}\begin{pmatrix} 1 & -q \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ p & 1 \end{pmatrix}\begin{pmatrix} 1 & q(1-pq)^{-1} \\ 0 & 1 \end{pmatrix}}$$

$$\begin{pmatrix} 1-qp & -q \\ p & 1 \end{pmatrix}$$

So after a lot of work you recall the Vaserstein identity. Let's next try to show that K_0 is Morita invariant for idempotent rings. ~~[illegible]~~ It might be easier to show that $A = A^2 \Longrightarrow K_0'A \xrightarrow{\sim} K_0 A$. ~~The point~~ ~~At the following~~ Get your act together. The problem: $K_0^{?}A$ is represented by triples (U_1, U_0, α) $\alpha : U_1/AU_1 \xrightarrow{\sim} U_0/AU_0$ and excision holds. You can lift α to a $\tilde{\alpha} : U_1 \longrightarrow U_0$ to make a complex. But maybe you can prove K_0 is Morita invariant any method at all.

$\begin{pmatrix} A & Q \\ P & B \end{pmatrix}$ suppose given

Vaserstein identity

$$\begin{pmatrix} 1 & -q \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ p & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ -p(1-qp)^{-1} & 1 \end{pmatrix}\begin{pmatrix} 1-qp & -q \\ p & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1-qp & -q \\ 0 & \underset{(1-pq)^{-1}}{1+p(1-qp)^{-1}q} \end{pmatrix}\begin{pmatrix} 1 & (1-qp)^{-1}q \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1-qp & 0 \\ 0 & (1-pq)^{-1} \end{pmatrix}$$

What does this identity do? In a Morita context $\begin{pmatrix} A & Q \\ P & B \end{pmatrix}$ it links $GL_m(A)$ and $GL_n(B)$.

Wait — Try to describe what you want.

Assertion: Given $\begin{pmatrix} A & Q \\ P & B \end{pmatrix}$ there is a unique isom

$K_1A \simeq K_1B$ such that $\forall\, p \in M_{k\ell}(P)$, $q \in M_{\ell k}(Q)$ satisfying $1-qp$ invertible (equiv to $1-pq \in GL(B)$) $\in GL(A)$

one has $\underset{\in K_1A}{[1-qp]} \overset{\sim}{\longleftrightarrow} [1-pq] \in K_1B$

How might this be related to Morita correspondence for complexes? A

u_1 $\quad$ u_0