Year 2000

505q Baum's question: whether for $\Gamma = \mathbb{Z}^2$ the Bott class in $K_*(\mathbb{C}[\Gamma])$ is in the image of $\varinjlim_{\Gamma} KK^{\Gamma}(\mathcal{C}^{ab}_{\Sigma_F}, \mathbb{C})$ under the assembly μ.

INTERVAL 506–1000

Review Cuntz construction.

Γ finite group, two categories: Γ-algebras and $\hat{\Gamma}$-algs = Γ-gr algs. Functors

$$A \quad \Gamma\text{-alg} \longmapsto A \rtimes \Gamma = A \otimes \mathbb{C}\Gamma \qquad sa = s(a)s$$

$$B \quad \hat{\Gamma}\text{-alg} \longmapsto B \rtimes \hat{\Gamma} = ?$$

$$B = \bigoplus_{s \in \Gamma} B_s \longrightarrow B \otimes \mathbb{C}\Gamma$$

$$B_s \longrightarrow B_s \otimes s$$

so what is $\mathbb{C}\Gamma \rtimes \Gamma$ ($=\mathrm{End}(\mathbb{C}\Gamma)$?)

$\mathbb{C}\Gamma \rtimes \Gamma$ has bases $s \otimes t$ with mult

$$(s \otimes t)(s_1 \otimes t_1) = s t s_1 t^{-1} \otimes t t_1$$

first point

a grading $M = \bigoplus_{s \in \Gamma} M_s$ ~~with resp~~ indexed by a finite set Γ is a $\mathbb{C}\Gamma$-comodule, e.g.

given

$$\begin{tikzcd} M \arrow[r, "\mu"] \arrow[d, "\mu"'] & M \otimes \mathbb{C}\Gamma \arrow[d, "\mu \otimes \mathbb{C}\Gamma"] \\ M \otimes \mathbb{C}\Gamma \arrow[r, "1 \otimes \mu"] & M \otimes \mathbb{C}\Gamma \otimes \mathbb{C}\Gamma \end{tikzcd}$$

$$\mu(m) = \sum_{s \in \Gamma} p_s(m) \otimes s \longmapsto \sum_{s \in \Gamma} \sum_{t \in \Gamma} p_t p_s(m) \otimes t \otimes s$$

$$= \sum_{s \in \Gamma} p_s(m) \otimes s \otimes s \qquad \therefore \quad p_t p_s = \begin{cases} p_s & t = s \\ 0 & t \neq s \end{cases}$$

and counit $\quad (1 \otimes \eta)\mu m = \sum_s p_s(m) = m.$

~~So what can we do?~~

A Γ-alg $\longmapsto$ $A \rtimes \Gamma = A \otimes \mathbb{C}\Gamma$ a Γ-graded alg.

$(1 \otimes s)(a \otimes 1) = s(a) \otimes s$

B Γ-graded alg $\longmapsto$ $B \rtimes \hat{\Gamma} = ?$

$B \rtimes \hat{\Gamma}$ should be some sort of $B \otimes \mathbb{C}\Gamma$

$$B \xrightarrow{\mu} B \otimes \mathbb{C}\Gamma \underset{1 \otimes \Delta}{\overset{\mu \otimes 1}{\rightrightarrows}} B \otimes \mathbb{C}\Gamma \otimes \mathbb{C}\Gamma$$

$\otimes$

Try to guess ~~what you can do~~, a Γ action on $B \otimes \mathbb{C}\Gamma$. Probably involves $1 \otimes s$

$$\bigoplus_{s} B \xrightarrow{\mu} \bigoplus_{s \in \Gamma} B_s \otimes \bigoplus_{t \in \Gamma} \mathbb{C}t$$

$\cancel{b_s \otimes s}$

$$\mu(b) = \sum_{s} p_s(b) \otimes s$$

$b_s \otimes t$

~~Try to identify $B \otimes \mathbb{C}\Gamma = B \otimes$~~

$$B = \bigoplus_{s \in \Gamma} B_s \qquad B \otimes \mathbb{C}\Gamma = \bigoplus_{s_2, s_1 \in \Gamma} B_{s_1} \otimes s_2$$

$$(b_{s_1} \otimes s_2)(b_{t_1} \otimes t_2)$$

Γ cyclic of order 2

Review the problem. Γ finite group, two cats Γ algs and Γ-gr-algs. If A is a Γ alg, form $A \rtimes \Gamma = A \otimes \mathbb{C}\Gamma$ $(1\otimes s)(a\otimes 1) = s(a)\otimes s$. Life . B a Γ graded algebra, i.e.

$$B = \bigoplus_{s\in\Gamma} B_s \ni \quad B_s B_t = B_{st}.$$ So if $\Gamma = \mathbb{Z}/m\mathbb{Z}$

$$B = \bigoplus_{s\in\mathbb{Z}/m\mathbb{Z}} B_s$$ such a graded should be equiv. to an action of $(\mathbb{Z}/m\mathbb{Z})^\wedge = \{\zeta \in \mathbb{C}^\times \mid \zeta^m = 1\}$

Why, in one direction: ~~give θ auto of B with $\theta^m = 1$, let $B_s = \{b\in B \mid \theta b = e^{\frac{2\pi i s}{m}} b\}$~~

Given $\mathbb{Z}/m\mathbb{Z}$ grading $B = \bigoplus_{s\in\mathbb{Z}/m\mathbb{Z}} B_s$ $\quad B_s B_t \subseteq B_{s+t}$

define $\theta = e^{\frac{2\pi i s}{m}}$ on B_s.

Let $b' \in B_s$, $b'' \in B_t$

Then $\theta(b') = \zeta^s b'$, $\theta(b'') = \zeta^t b''$, $\theta(b'b'') = \zeta^{s+t} b'b''$

so $\theta(b'b'') = \theta(b')\theta(b'')$. Thus a $\mathbb{Z}/m\mathbb{Z}$ graded alg $B = \bigoplus_{s\in\mathbb{Z}/m\mathbb{Z}} B_s$ has a $(\mathbb{Z}/m\mathbb{Z})^\wedge = \mu_m$ action.

If A is a Γ-algebra, $\Gamma = \mathbb{Z}/m\mathbb{Z}$

let $A_\chi = \{a\in A \mid \gamma(a) = \chi(\gamma)a\}$ here $\chi \in \hat{\Gamma}$

Then $A_\chi A_{\chi'} \subset A_{\chi+\chi'}$ since

~~$a \in A_\chi : \gamma(a) = \chi(\gamma)a$, $a' \in A_{\chi'} : \gamma(a') = \chi'(\gamma)a'$ $\Rightarrow \gamma(aa') = \chi(a)\chi'(a)aa'$~~

$$\left.\begin{aligned} a \in A_\chi &: \quad \gamma(a) = \chi(\gamma)a \\ a' \in A_{\chi'} &: \quad \gamma(a') = \chi'(\gamma)a' \end{aligned}\right\} \Rightarrow \begin{aligned} \gamma(aa') &= \gamma(a)\gamma(a') \\ &= \chi(\gamma)a\,\chi'(\gamma)a' \\ &= (\chi\chi')(\gamma)\,aa'. \end{aligned}$$

$\Rightarrow aa' \in A_{\chi\chi'}$

Γ finite abelian gp, $\hat{\Gamma} = \mathrm{Hom}_{\mathbb{Z}}(\Gamma, \mathbb{C}^\times)$, a $\hat{\Gamma}$ action on M is equivalent to a $\hat{\Gamma}$-grading, M is a complex vector space, to each $\chi \in \hat{\Gamma}$, you assoc. $M_\chi \overset{\text{eigenspace}}{=} \{ m \mid \gamma m = \chi(\gamma) m \}$.

OKAY

Construction $M \rtimes \Gamma$ for M a Γ-module.

Apparently $\hat{\Gamma}$ acts on $M \rtimes \Gamma$, equiv. $M \rtimes \Gamma$ has a natural Γ-grading, obvious

There is a "duality" between Γ and $\hat{\Gamma}$ modules, an equivalence of categories. Namely

If M is a Γ-module ? For example take $\Gamma = \mathbb{Z}/2$, then a $\hat{\Gamma}$

Γ-module $=$ $\hat{\Gamma}$ comodule

If M is a Γ module, then $M \rtimes \Gamma$ is a graded $\hat{\Gamma}$- module.

Change notation? G finite abelian group to get started. $\hat{G}$ = dual group $= \mathrm{Hom}(G, \mathbb{C}^\times)$

Two categories: G-Mod, $\hat{G}$-Mod ~~and then~~.

There's a functor $M \longmapsto M \rtimes \hat{G}$

$$\mathbb{C}G \otimes M = \bigoplus gM = \bigoplus_{g \in G} g \otimes M$$

So what is the construction

$$M \rtimes G = \sum M \otimes g$$

What is $A \rtimes \Gamma$? $\bigoplus_{s \in \Gamma} As$ with ring structure $(as)(a's') = a\,s(a')\,ss'$. This is a ring!!!! What are its modules? representations

A module M over $A \rtimes \Gamma$ should be both an A module and a Γ-module, these being compatible. $s(am) = s(a)\,s(m)$.

$A \longrightarrow \mathrm{End}(M)$. ~~Let G be a finite group~~

$\mathbb{C}\Gamma \nearrow$

You want to take a finite group.

A Γ-algebra e.g. $\mathbb{C}$ with trivial action, then $A \rtimes \Gamma = \mathbb{C}\Gamma$ is a $\hat{\Gamma}$-graded algebra, in fact it's the simplest $\hat{\Gamma}$-graded algebra. Next given a Γ-graded algebra B you want to form $B \rtimes \hat{\Gamma}$ which should be a Γ-alg

$B \rtimes \hat\Gamma = \bigoplus_{x \in \hat\Gamma} Bx$ $\quad B\otimes x$ with product

$$(b_s \otimes x)(b_t \otimes x') = b_s\, x(t)\, b_t \otimes xx'$$

Consider $\mathbb{C}^\Gamma$ the ring of functions on Γ aka $\mathbb{C}\hat\Gamma$. $\mathbb{C}\hat\Gamma \longrightarrow \mathbb{C}^\Gamma$ since $\hat\Gamma$ is a basis for $\mathbb{C}^\Gamma$ Fourier series. Try to see if this makes sense in general.

A Γ-graded module should be the same as a $\mathbb{C}^\Gamma$-module. $M = \bigoplus_{s\in\Gamma} M_s$ $\quad f \in \mathbb{C}^\Gamma$

$fm = \sum_{s\in\Gamma} f(s)m_s$. So now form

$$M \rtimes \mathbb{C}^\Gamma = \{ f : \Gamma \longrightarrow M |$$

$$B \rtimes \mathbb{C}^\Gamma = \{ f : \Gamma \longrightarrow$$

$(m \otimes f)(m' \otimes f')$ Let's use this pen in order to get freely flowing

well $= m(fm') \otimes ff'$

Γ finite group $\mathbb{C}^\Gamma$ = functions $f : \Gamma \longrightarrow \mathbb{C}$ under multiplication. $\mathbb{C}\Gamma$ = group ring.

A $\mathbb{C}^\Gamma$ module structure on M (unitary) is the same as a grading $M = \bigoplus_{s\in\Gamma} M_s$. So $f \in \mathbb{C}^\Gamma$, $m = \sum m_s$ given, you have $fm = \sum_s f(s) m_s$, conversely δ_s fns yield projections on M.

the next thing you need is the analog of $M \rtimes \hat{\Gamma} = M \otimes \mathbb{C}\hat{\Gamma}$ with

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A Γ alg, $B = \bigoplus_{s \in \Gamma} B_s$ Γ-gr alg

$N = \bigoplus_{s \in \Gamma} N_s$ Γ-gr B-module $B_s N_t \subset N_{st}$

$A \rtimes \Gamma = A \otimes \mathbb{C}\Gamma$ $\quad (a \otimes s)(a' \otimes s') = a\, s(a') \otimes ss'$

Keep trying ~~for the~~

$A \rtimes \Gamma = A\Gamma$ twisted group ring

$A\Gamma$ should naturally be a $\hat{\Gamma}$ module

$$\chi(as) = a\chi(s) = a\langle \chi, s\rangle s$$

so you ~~get~~ can form $A\Gamma\hat{\Gamma}$

$= \bigoplus_{s,\chi} As\chi$ $\quad (as\chi)(a's'\chi') = as\,a's'\langle \chi, s'\rangle \chi\chi'$

$$= a\, s(a')\, ss' \langle \chi, s'\rangle \chi\chi'$$

So you want a dual pair ~~A, B~~

What do you understand. A Γ alg, $B = A \rtimes \Gamma$

a $\hat{\Gamma}$-algebra $\quad \chi \cdot as = a\langle \chi, s\rangle s$

$\chi as = a\chi s = as\,\chi(s)$. $\quad \mathrm{Mod}(A) \quad \mathrm{Mod}(B)$

$\mathrm{Mod}(B) =$ A-modules with compatible Γ-action.

Homom. $A \xrightarrow{u} B$ $\quad \mathrm{Mod}(A) \underset{u_*}{\overset{u_!}{\rightleftarrows}} \mathrm{Mod}(B)$ (u^*: $\mathrm{Mod}(B) \to \mathrm{Mod}(A)$)

~~Still~~ Probably $\quad u_! L = B \otimes_A L = L \rtimes \Gamma$

$$u_* L = \mathrm{Hom}_A(B, L) = \mathrm{Hom}_A(A\Gamma, L) = L^\Gamma$$

Γ finite group, A Γ-algebra, $B = A \rtimes \Gamma$,
~~Mod~~ a homom. $B \longrightarrow C$ is the same as pair $A \xrightarrow{i} C$, $\mathbb{C}\Gamma \xrightarrow{j} C$ s.t.

$js\, ia = i(sa)\, js$ $\qquad sas^{-1} = (s * a)$

$\mathrm{Mod}(B) =$ A-mods with compat B action

$$\begin{tikzcd} \mathrm{Mod}(A) \arrow[r, shift left=2, "i_!"] \arrow[r, shift right=2, "i_*"'] & \mathrm{Mod}(B) \arrow[l, "i^*"'] \end{tikzcd}$$

If M an A-module, then $i_! M$ should be something like $B \otimes_A M$, $i_* M$ ——— $\mathrm{Hom}_A(B, M) = \{f : \Gamma \to M\}$.

Likely that the notation $B = A \rtimes G$ for G locally compact is defined via functions on G.

$B = A \otimes \mathbb{C}\Gamma \qquad (a_1 s_1)(a_2 s_2) = a_1 (s_1 * a_2) s_2$

~~$f : \Gamma \to M \qquad f'(as) =$~~

$f \in \mathrm{Hom}_A(B, M) \Rightarrow f(as) = a f(s)$

~~$f(as\, a's') = f(a(s*a')\, ss') = a(s*a') f(ss')$~~

$(as f)(a's') = f(a's' as)$
$= a'(s' * a) f(s's)$

So $\mathrm{Hom}_A(B, M) \simeq M^\Gamma$

$$f \longmapsto (s \mapsto f(s))$$

$$f(as) = af(s)$$

$$((a's')f)(as) = f(asa's') = a(s*a')f(ss')$$

$$(s \mapsto ((a's')f)(s) = (s*a')f(ss')$$

so $M^\Gamma = \{f : \Gamma \to M\}$ $\quad (af)(s) = af(s)$

$(s'f)(s) =$

$\mathrm{Hom}_A(B, M) \simeq M^\Gamma$

$(as \mapsto af(s)) \longleftarrow f$

$((a's')f)(s)$

$= f(sa's') = (s*a')f(ss')$

Check Assume

$a'f = (s \mapsto (s*a')f(s))$

$s'f = (s \mapsto f(ss'))$

~~Then $a's'f = a'(s \mapsto f(ss')) = (s \mapsto a'f(ss'))$~~

$M^\Gamma = \mathrm{Hom}_A(B, M)$

$f \longmapsto (as \longmapsto af(s))$,

$($

$as \longmapsto asa's' = a(s*a')ss' \longmapsto a(s*a')f(ss')$

which comes from $(s \longmapsto (s*a')f(ss'))$

So $(a's')(s \mapsto f(s)) = (a's')(as \longrightarrow af(s))$

$= (as \longmapsto a(s * a') f(ss'))$

$(a's')(s \longmapsto f(s)) = (s \longmapsto (s * a') f(ss'))$

$$\boxed{\begin{aligned} s'(s \mapsto f(s)) &= (s \mapsto f(ss')) \\ a'(s \mapsto f(s)) &= (s \mapsto (s*a') f(s)) \end{aligned}}$$

Repeat $\mathrm{Hom}_A(B,M) =$ $M^\Gamma = \{f : \Gamma \to M\}$ with

OKAY

$(a'f)(s) = (s * a') f(s)$

$(s'f)(s) = f(ss')$

Next what about

$B \otimes_A M$? $\quad B \otimes_A M$

System of imprimitivity – Mackey's ~~connection~~ expression for the grading idea.

$B \otimes_A M = A\Gamma \otimes_A M = \Gamma A \otimes_A M$

$= \mathbb{C}\Gamma \times M.$

Look at $M^\Gamma = \{f : \Gamma \to M\}$ $\quad (s'f)(s) = f(ss')$

$(a'f)(s) = (s*a') f(s)$

splits into direct sum of functions ~~supported~~ with card supp ≤ 1. Look at $B \otimes_A M$

$B \otimes_A M \qquad \bigoplus_{s \in \Gamma} sM$

$asm = s(s^{-1} * a) m$

Review. $B = A \rtimes \Gamma = \bigoplus_{s \in \Gamma} A_s$

$$\mathrm{Hom}_A(B, M) = M^\Gamma$$

$$\big(as \mapsto af(s)\big) \longmapsto \big(as \mapsto as\,a's' = a(s*a')ss' \mapsto a(s*a')f(ss')\big)$$

So B action on M^Γ is $\big((a's')f\big)(s) = (s*a')f(ss')$

i.e.
$$\begin{cases} (a'f)(s) = (s*a')f(s) \\ (s'f)(s) = f(ss'). \end{cases}$$

$\exists$ obvious sys of imp functions of point support.

Next consider $B \otimes_A M = \bigoplus_{s \in \Gamma} sM$

$asm = s(s^{-1}*a)m$. You have the tautological system of imprimitivity. There should be a canonical isom

$$B \otimes_A M \xrightarrow{\sim} \mathrm{Hom}_A(B, M)$$

Here's the idea. Let A be a Γ-algebra. Assume A equipped with a Γ-grading, which means a splitting $A = \bigoplus_{s \in \Gamma} A_s$ into subspaces satisfying $A_s A_t \subset A_{st}$. So far have not used Γ acts on A. Next require $tA_s \subset A_{ts}$ also $A_s t \subset A_{st}$? What do you want? System of imprimitivity.

Try repns. before algebras. Let V be a Γ-module equipped with a Γ-grading:

$V = \bigoplus_{s \in \Gamma} V_s$ $\quad tV_s \subset V_{ts}$. Then $V_1 \xrightarrow[\sim]{s} V_s$

So $V \leftarrow \mathbb{C}\Gamma \otimes V_1$

$W = \bigoplus_{s \in \Gamma} W_s$ Γ-graded.

what is $W \rtimes \hat{\Gamma}$

V Γ-module

~~V Γ-module~~ Basic idea: Let V ~~have~~ be equipped with the following

1) Γ action: $(s, v) \mapsto sv$
2) Γ-grading $V = \bigoplus_{t \in \Gamma} V_t$

assume that $sV_t \subseteq V_{st}$ $\forall s, t \in \Gamma$

Then you have a canonical isom.

$$\mathbb{C}[\Gamma] \otimes V_1 \xrightarrow{\sim} V \qquad \text{ie } s : V_1 \xrightarrow{\sim} V_s$$

for all $s \in \Gamma$. This is essentially obvious, because $V_t \xrightarrow{s} V_{st} \xrightarrow{s^{-1}} V_t$ is the identity, so $V_1 \xrightarrow{s} V_s$ is an isom, with inverse $V_s \xrightarrow{s^{-1}} V_1$. ~~Next you find~~

~~Basic V Γ-module~~

A Γ-algebra $A \rtimes \Gamma = A \tilde{\otimes} \mathbb{C}[\Gamma]$ is the alg describing A-modules with equiv Γ action

$B = \bigoplus_{s \in \Gamma} B_s$ Γ-graded alg $\overset{?}{\Longrightarrow}$ $B \rtimes \hat{\Gamma} = B \hat{\otimes} \mathbb{C}^{\Gamma}$ is the alg describing Γ-graded B modules

assembly. X conn. compact loc. conn. $\tilde{X} \xrightarrow{p} X$
a universal covering, $\Gamma = \mathrm{Aut}(\tilde{X}/X)$.
Suppose Γ acts to the right on $\tilde{X}$, if S a Γ ~~module~~ set, then $\tilde{X} \times^{\Gamma} S$ is a covering of X. A loc coeff system on X i.e. loc const sheaf E should be equiv. to a Γ-module M.
$E = \tilde{X} \times^{\Gamma} M$. Take $\mathbb{C}\Gamma$ the group ring $\tilde{X} \times^{\Gamma} \mathbb{C}\Gamma = p_! \mathbb{C}$. Note that Γ acts on the right, ~~so that~~ OKAY
$(\tilde{X} \times^{\Gamma} \mathbb{C}\Gamma) \times^{\Gamma} M = \tilde{X} \times^{\Gamma} M$. You can think of $\tilde{X}$ as the bundle over X with fibres the orbits $y\Gamma$, $y \in \tilde{X}$.
If M is a f.d. Γ vector space, then $\tilde{X} \times^{\Gamma} M$ is a vector bundle on X. It becomes a $C(X)$ module. Naturally $\tilde{X} \times^{\Gamma} \mathbb{C}\Gamma$ is a $C(X) \otimes \mathbb{C}\Gamma$ module.

$X = \mathbb{R}$ $\quad \Gamma = \mathbb{Z}$, then $X/\Gamma = \mathbb{R}/\mathbb{Z}$

So $\quad \mathbb{R} \times^{\mathbb{Z}} \mathbb{C}\mathbb{Z} = \mathbb{R} \times \mathbb{C}$?

Come on! cosets of $\mathbb{R}$ mod $\mathbb{Z}$. See here the familiar Poisson summ. situation, functions on the circle

$$\begin{array}{ccc} \tilde{X} & & \Gamma \\ \downarrow & & \\ X & & \end{array}$$

get on X a local coeff system $\tilde{X} \times^{\Gamma} \mathbb{C}\Gamma$ of rank 1 free right $\mathbb{C}\Gamma$-modules. Now combine with functions on X.

X compact (say manifold), $\tilde{X} \xrightarrow{p} X$ a universal covering, $\Gamma = \mathrm{Aut}(\tilde{X}/X)$ acting on the right so that $\tilde{X}$ is a principal Γ-bundle. If Γ acts on a vector space V, say finite diml, you get a flat v.b. $\tilde{X} \times^{\Gamma} V$ over X, hence a class in $K^0(X)$.

Take the group ring $\mathbb{C}\Gamma$

Adopt Kasparov (maybe Mischenko) viewpoint. The universal covering $\tilde{X}$ is ~~some~~ a sort of correspondence between X and $B\Gamma$

You need a place to start. Cuntz's picture?

X compact manifold, $\tilde{X} \xrightarrow{p} X$ a universal covering, $\Gamma = Aut(\tilde{X})$, right action. $p^{-1}(x)$ is a Γ orbit, $\tilde{X}$ is a principal bundle with group Γ.

Important thing here is a fin. gen. proj module. $x \mapsto p^{-1}(x)$ is a locally constant sheaf, fibre bundle. 1. Affine version of Γ

OK

Briefly review your look at Cuntz's formalism for Γ finite.

Γ-algs A = algs A with Γ action ($*$ algs)

$\hat{\Gamma}$-algs B = algs B with Γ-grading

natural module category assoc. to the Γ alg A is the cat of A-modules equipped with compatible Γ-action: $s(am) = (s*a)(s*m)$

natural mod cat. assoc to $\hat{\Gamma}$ alg B is the cat of B modules N equipped with compatible Γ-grading $N = \bigoplus_{s \in \Gamma} N_s$, $B_s N_t \subset N_{st}$

Γ, A- modules $= (A \rtimes \Gamma = A \tilde{\otimes} \mathbb{C}\Gamma)$-modules

$\hat{\Gamma}, B$- —— $= (\underbrace{B \tilde{\otimes} \mathbb{C}^{\Gamma}}_{B \rtimes \mathbb{C}^{\Gamma}})$- modules

Now $A \rtimes \Gamma$ has a natural Γ-grading

$= \bigoplus_{s \in \Gamma} As \quad \ni \quad As\, At \subset A\, st$

$as\; a't = a(s*a')\, st$

But look at a $(A \rtimes \Gamma) \rtimes \hat{\Gamma}$ - module D

i.e. a Γ-graded $A \rtimes \Gamma$ - module: $D = \bigoplus_{t \in \Gamma} D_t$

$(As)\, D_t \subset D_{st}$ | $A D_t \subseteq D_t$, $s D_t \subset D_{st}$

Then D_1 is an A-module and

$s : D_1 \xrightarrow{\sim} D_s$ with inverse s^{-1}.

$$\mathbb{C}\Gamma \otimes D_1 \longrightarrow D = \bigoplus D_s$$
$$s \otimes \xi \longmapsto s\xi$$

A $B \rtimes \hat{\Gamma}$ module D is a B-module graded wrt Γ compatibly: $D = \bigoplus_t D_t$, $B = \bigoplus_s B_s$

$B_s D_t \subset D_{st}$. If $B = A \rtimes \Gamma = \bigoplus_s \underbrace{As}_{B_s}$

Then $D = \bigoplus_t D_t$ is an A-module $\ni$ $A D_t \subset D_t$

and $s D_t \subset D_{st}$. So you get

$$\begin{tikzcd} D_t \arrow[r, "s\cdot"] \arrow[rr, bend right, "1"'] & D_{st} \arrow[r, "s^{-1}"] & D_t \end{tikzcd}$$

Also

$$\begin{tikzcd} & D_1 \arrow[dl, "t\cdot"'] \arrow[dr, "st\cdot"] & \\ D_t \arrow[rr, "s\cdot"] & & D_{st} \end{tikzcd}$$

$\therefore$ get $\mathbb{C}\Gamma \otimes D_1 \xrightarrow{\sim} D$

$$s \otimes \xi \longmapsto s\xi$$

This isom. compatible with Γ action and A action

(Thought: Connection (?) between nonunital ring stuff and group completion of monoids.)

$$\begin{array}{ccc} \mathbb{C}\Gamma \otimes D_1 & \longrightarrow & D \\ s \otimes \xi & \longmapsto & s\xi \\ {\scriptstyle a\cdot}\downarrow & & \\ s \otimes (s^{-1}*a)\xi & \longmapsto & s(s^{-1}*a)\xi = as\xi \end{array}$$

$$as = s(s^{-1}as)$$

(Thought: $\hat{\mathbb{Z}} = \varprojlim \mathbb{Z}/n\mathbb{Z}$ inverse limit of cyclic groups. Consider any inverse sequence ~~system~~ of surjections of cyclic groups

$$\longrightarrow C_{n_2} \longrightarrow C_{n_1} \longrightarrow C_{n_0}$$

provided any $m \geq 1$ divides n_k some k you get $\hat{\mathbb{Z}}$ up to isom. Your idea is to explore other correspondences between cyclic groups than homomorphisms. Something to try: You have interesting correspondence between $\mathbb{Z}$ and $\mathbb{Z}$ given by $g \in SU(1,1)$, and between $\mathbb{Z}$ and $\mathbb{R}$.)

So you now appear to understand the duality and perhaps are in a position to understand J's construction. Where to start? For Γ finite consider $\underline{A \rtimes \Gamma}$, which is Γ-graded, ~~and idempotents over this~~ Of interest here are idempotents p in $A \rtimes \Gamma$, which have components in the Γ grading $p = \sum_{s\in\Gamma} p_s$, $p_s^* = p_{s^{-1}}$, $p_t = \sum_s p_s p_{s^{-1}t}$

form universal such alg. P_Γ, autom. Γ-graded. Other things is noncommutative simplex Σ_Γ with Γ as vertices.

$$\Sigma_\Gamma = C^*\{h_s,\ s\in\Gamma \mid h_s = h_s^*,\ \sum_{s\in\Gamma} h_s = 1\}$$

$$P_\Gamma = C^*\{p_s,\ s\in\Gamma \mid p_s^* = p_{s^{-1}},\ p_t = \sum_s p_s p_{s^{-1}t}\}$$

$$p_s = h_1^{\frac12} h_s^{\frac12} s \in \Sigma_\Gamma \rtimes \Gamma$$

$$p_s^* = s^{-1} h_s^{1/2} h_1^{1/2} = h_1^{1/2} s^{-1} h_1^{1/2} = h_1^{1/2} h_{s^{-1}}^{1/2} s^{-1} = p_{s^{-1}}$$

$$\sum_s p_s p_{s^{-1}t} = \sum_s h_1^{1/2} h_s^{1/2} s\, h_1^{1/2} h_{s^{-1}t}^{1/2} s^{-1} t$$

$$= \sum_s h_1^{1/2} \underbrace{h_s^{1/2} h_s^{1/2}}_{\text{sums to } 1} h_t^{1/2} t = h_1^{1/2} h_t^{1/2} t = p_t$$

Thus you have a map of Γ-graded algs.

$$P_\Gamma \longrightarrow \Sigma_\Gamma \rtimes \Gamma, \quad p_s \longmapsto h_1^{1/2} h_s^{1/2} s$$

$$p_1 \longrightarrow h_1 \qquad p_s = h_1^{1/2} h_s^{1/2} s = h_1^{1/2} s\, h_1^{1/2}$$

$$p_s^* p_s = s^{-1} h_s^{1/2} h_1^{1/2} h_1^{1/2} h_s^{1/2} s = h_1^{1/2} h_{s^{-1}} h_1^{1/2}$$

$$p_s^* p_s = h_1^{1/2} s^{-1} \underbrace{h_1^{1/2} h_1^{1/2}}_{h_1} s\, h_1^{1/2} = h_1^{1/2} h_{s^{-1}} h_1^{1/2}$$

$$p_s p_s^* = h_1^{1/2} h_s h_1^{1/2} \qquad \underbrace{h_1^{-1/2} p_s h_1^{-1/2}}_{p_1^{-1/2} p_s p_1^{-1/2} = s\ ?} = s$$

First point is that $p_s^* = p_{s^{-1}}$ and $p_t = \sum_s p_s p_{s^{-1}t}$

$\Rightarrow$ $p_1 = \sum_{s\in\Gamma} p_s p_s^* = p_1^2 + \sum_{s\neq 1} p_s p_s^*$. What is P_Γ^{ab}? This is a comm. C^*-alg, corresp to loc. compact space X, ~~so look at these~~ so on X you have cont $\mathbb{C}$-valued functions vanishing at ∞ $(p_s, s\in\Gamma)$ $\ni$ $\bar{p}_s = p_{s^{-1}}$ $\quad p_t = \sum_s p_s p_{s^{-1}t}$

$$X \longrightarrow \mathbb{C}^\Gamma$$

P_Γ is a C^* alg graded wrt the group Γ universal for proj in C^* alg $B = \bigoplus_s B_s$ graded wrt Γ. So a homom. $P_\Gamma \longrightarrow B$ same as $p_s \in B_s$ $\ni$ $p_s^* = p_{s^{-1}}$, $p_t = \sum_s p_s p_{s^{-1}t}$. Take $B = \mathbb{C}\Gamma$ with $B_s = \mathbb{C}s$, then $p_s = c_s s \qquad c_s \in \mathbb{C}$.

where $c_t t = \sum_s c_s s \, c_{s^{-1}t} s^{-1}t = \sum_s c_s c_{s^{-1}t} t$

i.e. $c_t = \sum_s c_s c_{s^{-1}t}$, $\quad \sum_t c_t t = \sum_{t,s} c_s s \, c_{s^{-1}t} s^{-1}t$

$$= \sum_s c_s s \left(\sum_t c_{s^{-1}t} s^{-1}t \right) — c_t t.$$

$\mathbb{C}\Gamma$ is Γ-graded

and so $\mathrm{Hom}_{\hat{\Gamma}}(P_\Gamma, \mathbb{C}\Gamma) = \{\text{projections } p \text{ in } \mathbb{C}\Gamma\}$

(Thought: Cuntz used S^1 action in his work on $\mathcal{O}_n$). ~~Simplest Z-graded thing~~

~~...~~

~~...~~ Review the universal property of P_Γ: it is a Γ-graded *alg. (equipped with proj $\hat{p}$) such that for any Γ-graded *algebra $B = \bigoplus_{s\in\Gamma} B_s$ and projection $p \in B$ there is a unique homom. of Γ-graded *algs $P_\Gamma \longrightarrow B$ sending $\hat{p} = \sum \hat{p}_s$ to p.

$$P_\Gamma = {}^*\text{alg gen. by } \hat{p}_s,\ s\in\Gamma \text{ subject to}$$
$$\hat{p}_s^* = \hat{p}_{s^{-1}} \qquad \hat{p}_t = \sum_s \hat{p}_s \hat{p}_{s^{-1}t} \qquad \forall s,t.$$

grading such that $\deg(\hat{p}_s) = s$. If we impose commutativity on a Γ-graded algebra $B = \bigoplus_{s\in\Gamma} B_s$, then $B_s B_t = B_{st}$, $B_t B_s = B_{ts}$ imply $B_s B_t = B_t B_s = 0$ if $st \neq ts$.

Special class of Γ-graded *algs are those of the form $B = A \rtimes \Gamma$ where A is a *alg. with Γ action. Then $B = \bigoplus_{s\in\Gamma} sA = \bigoplus_{s\in\Gamma} A_s$ since $sa = (s*a)s$. Can you understand projections p in such a Γ-graded algebra. ~~...~~ Then $p = \sum_{s\in\Gamma} p_s$ with $p_s \in sA = A_s$ and $p_s^* = p_{s^{-1}}$, $p_t = \sum_s p_s p_{s^{-1}t}$. etc.

If $A = \mathbb{C}$ with trivial Γ action, then we have a projection in $\mathbb{C}\Gamma$, a f. diml. C^*-alg, $\prod M_{n_i}\mathbb{C}$, $\underline{\amalg}$ products of Grassmannians. If Γ comm. only finitely many proj.

Next, the simplex C^*alg $\mathcal{E}_{\Sigma_\Gamma} = C^*\left\{h_s, s \in \Gamma \,\middle|\, \begin{matrix} h_s \geq 0 \\ \sum h_s = 1 \end{matrix}\right\}$ acted on by Γ, so can form $\mathcal{E}_{\Sigma_\Gamma} \rtimes \Gamma$, and there is a canonical p in this Γ-graded algebra, namely $p_s = h_1^{1/2} h_s^{1/2} s$: Check:

$$p_s^* = s^{-1} h_s^{1/2} h_1^{1/2} = h_1^{1/2} h_{s^{-1}}^{1/2} s^{-1} = p_{s^{-1}}$$

$$\sum_s p_s p_{s^{-1}t} = \sum_s h_1^{1/2} h_s^{1/2} \underbrace{s\, h_1^{1/2} h_{s^{-1}t}^{1/2} s^{-1}}_{h_s^{1/2} h_t^{1/2}} t = h_1^{1/2} \left(\sum_s h_s\right) h_t^{1/2} t = p_t$$

Thus you have a canonical map of Γ-graded *alg.

$$P_\Gamma \longrightarrow \mathcal{E}_{\Sigma_\Gamma} \rtimes \Gamma$$

$$\hat{p}_s \longmapsto h_1^{1/2} h_s^{1/2} s = h_1^{1/2} s\, h_1^{1/2}$$

You would like an equivalence between $P_\Gamma \rtimes \hat{\Gamma}$ and $\mathcal{E}_{\Sigma_\Gamma}$

$$p_1 = h_1^{1/2} 1\, h_1^{1/2} = h_1$$

$$\sum_s h_1^{1/2} \underbrace{s\, h_1^{1/2}\, h_1^{1/2} s^{-1}}_{h_s} t\, h_1^{1/2} = h_1^{1/2} t\, h_1$$

You seem to be missing something obvious.

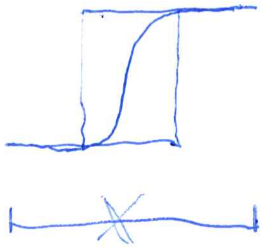
tomorrow: What is the meaning of the simplex algebra $C(\Delta_\Gamma)$ = functions on the simplex with vertices Γ, after taking the cross product. Put another way - significance of the cross product algebra $C(X) \rtimes \Gamma$ when Γ acts on X. BC conjecture says $KK^\Gamma(\underline{E}\Gamma, A) \xrightarrow{\sim} K_*(A \rtimes_r \Gamma)$

There is a key point to be understood that involves constructing a fin gen proj module, analogous to using a partition of unity to embed the module in a free module.

Idea that Σ_F in the case of $\mathbb{Z}$ yields the barycentric subdivision of the $\mathbb{Z}$ tree. ~~So whe~~

There is a finitely generated proj module somewhere. The idea is that there is a bundle over X with fibre $p^{-1}(x)$ a Γ torsor, ~~and hence~~ now bring in $\mathbb{C}$, to form the free $\mathbb{C}\Gamma$-module of rank 1, namely $\bigoplus_{y \in p^{-1}(x)} \mathbb{C}y$, this gives an inf dim vector bundle over X, locally constant, which is finitely generated over $\mathbb{C}\Gamma$. Now you look at sections over X of this bundle. Being of rank 1 things should be simple, YES. you pick an open covering of X over which the bundle becomes trivial, say it has 2 open sets, over each open set you pick a generator then combine by a partition of 1.

How does this work for $\mathbb{R} \xrightarrow{p} \mathbb{R}/\mathbb{Z}$? you need a partition of 1 on $\mathbb{R}/\mathbb{Z}$. There is this other picture. ~~Open covering~~



Go back to general case say $Y \xrightarrow{p} X$ principal Γ-bundle, embed inside $Y \times^{\Gamma} \mathbb{C}\Gamma$ this is a right line bundle over $\mathbb{C}\Gamma$. Choose open covering (say X compact) and trivialization, assume a Weil covering of compact manifold, choose a partition of 1.

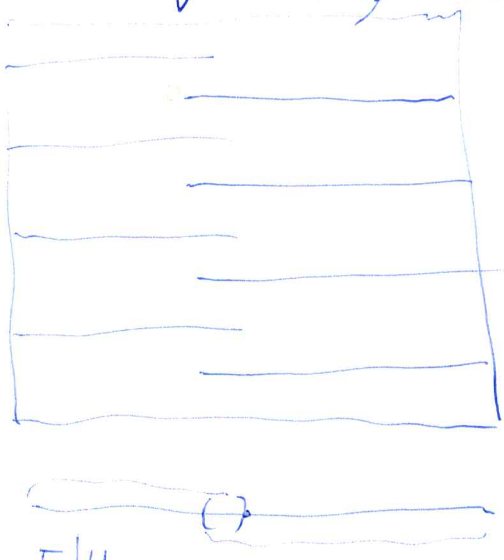
Set it up as simply as possible. You have $p : Y \longrightarrow X$ principal Γ-bundle, say X is a finite simplicial complex. For each vertex v of X you choose point of $p^{-1}(v)$, you then get a 1-cocycle on X with values in Γ.

Analyze carefully what you need in order to construct a fin type proj module over $C(X) \otimes \mathbb{C}\Gamma$. Given a principal Γ-bundle $p : Y \longrightarrow X$ where X is a finite simplicial complex, automatically Y becomes a simplicial complex. For each simplex σ of X, $p^{-1}(\sigma)$ is a Γ-torsor, which you replace by the $\mathbb{C}\Gamma$ right module $\mathbb{C}[p^{-1}(\sigma)]$, effectively start with $x \longmapsto \mathbb{C}[p^{-1}(x)]$ locally bundle of right $\mathbb{C}\Gamma$ modules, of rank 1. Next combine with functions on X, barycentric coordinates

Because $x \mapsto \mathbb{C}[p^{-1}(x)]$ is locally constant we get "flat" sections locally. ~~So we~~

Repeat: $\tilde{X} \xrightarrow{p} X$ princ. Γ-bundle, ~~get~~ a local system on X of Γ torsors, $x \mapsto p^{-1}(x)$, ~~can make flat line vector bundle~~ can form a local system on X of free rank 1 right $\mathbb{C}\Gamma$-modules $x \mapsto \mathbb{C}[p^{-1}(x)]$. View this as a flat inf. dim'l vector bundle E on X, with the extra structure ~~given by~~ of free right $\mathbb{C}\Gamma$-module of rank 1. ~~Basically~~

Locally we have constant sections of this vector bundle which can be multiplied by $C(X)$, ~~continuous~~ to be minimal you want piecewise poly functions on the simplicial complex X, to understand unity you need



$E|U_\alpha$

Basic partition of unity

$\sum_\alpha h_\alpha = 1$ down on X.

You have $p^{-1}(U_\alpha) \xrightarrow[\text{triv.}]{\sim} U_\alpha \times \Gamma$

$E_\alpha \xrightarrow{\sim} U_\alpha \times \mathbb{C}\Gamma$

bundle restricted to U_α

$$\mathrm{Sect}(E_\alpha) \xrightarrow{\sim} C(U_\alpha) \otimes \mathbb{C}\Gamma$$

$$\mathrm{Sect}(E|X) \xrightarrow{\sim} \prod_\alpha \overset{C(U_\alpha)}{\cancel{\mathrm{Sect}(E|U_\alpha)}} \otimes \mathbb{C}\Gamma$$

$$\xi \qquad\qquad h_\alpha \xi$$

flat $\mathbb{C}\Gamma$-line ~~vector~~ bundle E over X
with $E_{(x)} = \mathbb{C}[p^{-1}(x)]$, $E|U_\alpha \xrightarrow{\sim} U_\alpha \times \mathbb{C}\Gamma$

$\mathrm{Sections}(E|U_\alpha) \xrightarrow{\sim} C(U_\alpha)\otimes\mathbb{C}\Gamma$. Idea you missed is that $h_\alpha\xi$ extends by zero to an element of $C(X)\otimes\mathbb{C}\Gamma$. So you should get

$$\begin{tikzcd}
\mathrm{Sect}(E|X) \arrow[r] & \prod_\alpha \underbrace{\mathrm{Sect}_0(E|U_\alpha)}_{\text{vanishing at } \partial U_\alpha} \arrow[r, phantom, "="] & \prod_\alpha C_0(U_\alpha)\otimes\mathbb{C}\Gamma \arrow[d, hook] \\
\xi \arrow[r, mapsto] & (h_\alpha\xi) \arrow[d] & \\
C(X)\otimes\mathbb{C}\Gamma & \sum h_\alpha\xi = \xi \arrow[l, phantom] & \prod_\alpha C(X)\otimes\mathbb{C}\Gamma \arrow[ll, bend left, "\Sigma"']
\end{tikzcd}$$

maybe you want to use $h_\alpha^{1/2}$. This maybe is the other idea. ~~The point is that a partition of unity uses~~

Repeat: $Y \xrightarrow{p} X$ principal Γ-bundle, get ~~make~~ an inf. dim. ~~vector~~ flat vector bundle with structure of right $\mathbb{C}\Gamma$ line bundle with fibre $\mathbb{C}[p^{-1}(x)]$ over x. ~~You want the~~

You get a local system $x \longmapsto \mathbb{C}[p^{-1}(x)]$ on X of line bundles for the ring $\mathbb{C}\Gamma$. View as inf. dim'l flat vector bundle E over X. Look at sections of E over X, continuous (for C^*-alg purposes), to be specific because the top on $\mathbb{C}\Gamma$ hasn't been made precise

When $E|U \xrightarrow{\sim} U \times \mathbb{C}\Gamma$

you want $\mathrm{Sect}(E|U) \xrightarrow{\sim} C(U) \otimes \mathbb{C}\Gamma$

$p: Y \longrightarrow X$ principal Γ bundle, ie a local system $x \longmapsto p^{-1}(x)$ of Γ-torsors, hence a local system $x \longmapsto \mathbb{C}[p^{-1}(x)]$ of right $\mathbb{C}\Gamma$ modules, free of rank 1, hence geometrically you have a ~~vector~~ fibre bundle E over X, locally of the form $U \times \mathbb{C}\Gamma$, where the transitions are left mult by elements of Γ. Now you want to take sections of this vector bundle. You want sections with compact support. To avoid topology problems. To define section over a compact set K. First suppose $K \subset U$ over which E is trivial

A ~~cont~~ bdd section of E over K is a ~~map from~~ section s of E over K such that ~~locally if one picks~~ $\forall y \in p^{-1}(K)$ the support of s in $\mathbb{C}[p^{-1}py]$?

How to say this? the idea is that if one trivializes the principal bundle over a compact K, then the space of bdd sections is defined to be $C(K) \otimes \mathbb{C}\Gamma = \bigoplus_{\gamma \in \Gamma} C(K)\, s_0 \gamma$. family of supports idea? A point of $E = Y \times^{\Gamma} \mathbb{C}\Gamma$ ~~is a~~ over x is an element of $\bigoplus_{\gamma \in \Gamma} \mathbb{C} y \gamma$ where

$y \in p^{-1}(x)$. i.e. a finite linear comb. of elements in $p^{-1}(x)$. ~~This is ...~~

~~If you have a small~~ linearize

let's move on. ~~The point is that if K~~ Let $A \subset X$ such that $\exists$ isom $A \times \Gamma \xrightarrow{\sim} p^{-1}A$

Then $C(A) \otimes \mathbb{C}\Gamma$ you want to be indep of the isom.

So what is an autom of $A \times \Gamma$, answer, it is given by a cont map $A \xrightarrow{g} \Gamma$.

You have problems if the image of g is infinite, why, because? An element of $C(A) \otimes \mathbb{C}\Gamma$ ~~should be the same as~~ is a finite linear comb. $\sum_{\gamma \in \Gamma} f_\gamma \otimes \gamma$ with $f_\gamma \in C(A)$, it should be the same as a cont function $x \longmapsto \sum_{\gamma \in \Gamma} f_\gamma(x) \gamma$ from A to $\mathbb{C}\Gamma$

such that the image is contained in $\mathbb{C}[S]$ for some finite subset S of Γ. ~~So basically you want~~

Where are you? Your problem is to define a space of sections for the flat right line bundle over X for the ring $\mathbb{C}\Gamma$ associated to the ~~locally~~ local system $x \longmapsto \mathbb{C}[p^{-1}(x)]$, Certainly you want sections to be determined locally, better, you want sections to form a sheaf. You know what to do for a small connected subset. Things are basically clear.

~~Suppose for X compact that~~

Suppose now that you have constructed the sheaf of sections of E = loc. system over X, $E_{(x)} = \mathbb{C}[p^{-1}(x)]$. When X is compact you want to show the space of sections over X is a fin. type proj $C(X)\otimes\mathbb{C}[\Gamma]$-module. You know this is true locally. Need partition of 1. argument! See how it works? X compact say smooth manifold - get finite open covering à la Weil!

Let $M = (C(X)\otimes \mathbb{C}\Gamma)$-module of sections over X of this line bundle wrt $\mathbb{C}\Gamma$. ~~Locally we know that~~ For each x the fibre of E is this "line" $\mathbb{C}[p^{-1}(x)]$ wrt $\mathbb{C}\Gamma$. For A small and connected $\mathrm{Sect}(p^{-1}(A), E)$ free rank 1 module over $C(A)\otimes\mathbb{C}\Gamma$. So take Weil covering

You need to construct a fact.

$$\mathrm{Sect}(X,E)\xrightarrow{\Phi}(C(X)\otimes\mathbb{C}\Gamma)^N\xrightarrow{\Psi}\mathrm{Sect}(X,E)$$

of the identity map. N should be the number of elements in a covering. Φ takes a section s restricts it to U_α multiplies by a function h_α so that it extends by zero to $C($.

Do this in steps. Φ takes a section s restricts it U_α: $s|U_\alpha \in \mathrm{Sect}(U_\alpha, E)\simeq C(U_\alpha)\otimes\mathbb{C}\Gamma$, then a partition $\sum h_\alpha = 1$ comes into play.

$$C(X)\otimes\mathbb{C}\Gamma \xrightarrow{\mathrm{res}_\alpha\otimes 1} C(U_\alpha)\otimes\mathbb{C}\Gamma$$

~~Upon~~ You need to multiply $s|U_\alpha$ by an h_α so that $h_\alpha s$ extends continuously ~~by~~ zero to X.

Be careful – this argument is ind of Γ. You want to know why a vector bundle E is a fg proj $C(X)$-module. The argument goes as follows: Choose covering $X = \bigcup U_\alpha$ & $E|U_\alpha \simeq U_\alpha \times \mathbb{C}^N$ and you choose a partition $\sum h_\alpha = 1$ $\quad Supp(h_\alpha) \subset U_\alpha$

Then $s \in \Gamma(X, E) \longmapsto \left(h_\alpha(s|U_\alpha)\right) \in \prod_\alpha \Gamma_c(U_\alpha, E)$

$$\subset \prod_\alpha \Gamma(X,E) \xrightarrow{\sum} \Gamma(X,E).$$

$$\sum_\alpha h_\alpha(s|U_\alpha) = \sum_\alpha h_\alpha s = s.$$

You want to ~~understand why~~ examine carefully the proof that a ~~fg~~ vector bundle E over X is a fin gen proj module over $C(X)$, X compact, say. Locally $\exists$ trivializations $E|_{U_\alpha} \simeq U_\alpha \times \mathbb{C}^N$

$$\text{Sect}(E,X) \longrightarrow \prod_\alpha \text{Sect}(E, U_\alpha) \simeq \prod_\alpha \left(C(U_\alpha) \otimes \mathbb{C}^N\right)$$

$$\xi \longmapsto (\xi|U_\alpha).$$ To simplify suppose $N = 1$, so you get $\xi|U_\alpha = f$ · given trivial. over U_α.

There are two points

What's bothering you is the stuff on nuclearity. Recall what you did before Wasserman's page on Kirchberg's argument. First why $C(X)$, X compact, is nuclear. You need an approximate factorization of the identity map of $C(X)$, must be positive

$$C(X) \xrightarrow{C^* \text{ alg alg hom.}} \mathbb{C}^N \longrightarrow C(X)$$

if you require this then you've got to use N points of X, so you are doing some sort of interpolation. Finite set of points $x_1, \dots, x_N$ in X and a collection of $h_i(x) \geq 0$ $i = 1, \dots, N$ such that $\sum h_i = 1$. Map $f \longmapsto \sum_1^N h_i f(x_i)$. You want

$$f(x) - \sum_1^N h_i(x) f(x_i) = \sum_{i=1}^N h_i(x)\big(f(x) - f(x_i)\big)$$

~~to~~ have small sup norm. The way this works is ~~to get~~

$$\left| \sum_{i=1}^N h_i(x)\big(f(x) - f(x_i)\big) \right| \leq \sum_i h_i(x) \, \|f - f(x_i)\|_{\infty, \operatorname{Supp} h_i}$$

relation between x_i, $h_i(x)$?

Go back to $\mathrm{Sect}(E, X)$, be careful as to what you need! ~~You need~~ You want to map E to the trivial line bundle

$$\begin{tikzcd}
\mathrm{Sect}(X, E) \arrow[rr, "1"] \arrow[dr] & & \mathrm{Sect}(X, E) \\
& C(X)^N \arrow[ur] &
\end{tikzcd}$$

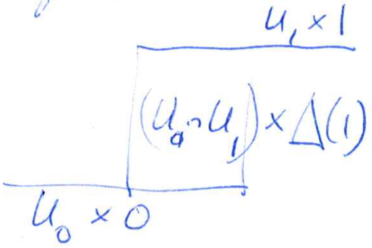
Partition $\sum h_\alpha = 1$ tog with ~~such that~~ $E \xrightarrow{\sim} X \times \mathbb{C}^n$ on some nbd of Support h_α

Let's try to get things clean. Go back to a v.b. E over a compact X. You want to factor

$$\begin{tikzcd} E \arrow[rr, "1"] \arrow[dr] & & E \\ & (\mathbb{C}^r)_X \arrow[ur] & \end{tikzcd}$$

$2500 \times 12 = 30{,}000$

To keep things simple take $r = 1$. You want to combine local trivializations with a partition of unity. Graeme's idea, given an open covering $X = \bigcup_{\alpha \in I} U_\alpha$ you form the $X \times \underset{\text{simplex}}{\Delta}$ with vertices I, and



Repeat - you have no

Repeat. You want to understand why Cuntz's partition of unity algebra works. Something nontrivial is taking place. Take a discrete group Γ and build the simplest nontrivial principal Γ-bundle, this should be the first chunk of Milnor's universal bundle

$$\Gamma * \Gamma \longrightarrow \Sigma\Gamma$$

Problem of adjoining a pt to Γ?

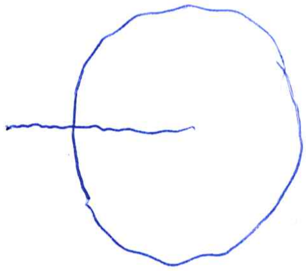
Let us first look at ~~ordinary com~~ real line bundles as concretely as possible. Two choices for structural group: $\{\pm 1\}$, $\mathbb{R}^\times$. Now

$\{\pm 1\}$ is nice because it's both compact and discrete, so you should be able to link two viewpoints.

Describe the key ideas. You start with a line bundle L over X, and assume $X = A \cup B$ where A, B have properties to be determined, an important property is that L is trivial in some sense when restricted to A or to B. You need to construct a factorization

$$\Gamma(X,L) \longrightarrow \Gamma(A,L) \oplus \Gamma(B,L) \longrightarrow \Gamma(X,L)$$

of the identity map. Example to keep in mind $X = \mathrm{Susp}\{\pm 1\} = S^1$ and the Möbius line bundle



. Because the structural group is compact you have hope that the Stone-Čech compactification, or the multiplier algebra, can be used. This seems to be the missing point, also why Cuntz's finite Γ support model should work. So let's see if it works.

So you begin now with $1 = h_a + h_b$, $h_a, h_b \geq 0$ continuous, then get an open covering U_a, U_b where $h_a, h_b > 0$ resp. Basic assumption is that L is trivial when restricted to U_a or U_b, but you want more, L is assumed to have its structural reduced from $\mathbb{R}^{\times}$ to $\{\pm 1\}$. This means you have a line bundle with positive scalar product. ~~of the line bundle~~ So here you ~~can the~~ see a double covering Y of X sitting inside L.

Now you need to construct the map. You have the double covering $Y \xrightarrow{p} X$ and the assoc. line bundle $Y \times^{\{\pm 1\}} \mathbb{R}$. ~~Now lift back~~
You also have the partition $1 = h_a + h_b$ on X. $U_{a|b} = \{x \mid h_{a|b}(x) > 0\}$. By assumption Y is trivial over U_a, so you have a continuous map $s_a : U_a \longrightarrow p^{-1}(U_a) \subset Y$, and similarly for b. ~~So what is the important~~
On $U_a \cap U_b$, s_a and s_b are related by a cont. map $U_a \cap U_b \longrightarrow \{\pm 1\}$. Can you see an embedding of Y into the trivial real vector bundle of rank 2 over X. The first component should ~~be and~~ be the linear map

$$\begin{tikzcd} Y \arrow[rr] \arrow[dr] & & X \times \mathbb{R} \arrow[dl] \\ & X & \end{tikzcd} \qquad s_a(x) \longmapsto (x, 1)$$

i.e. $y \mapsto (s_a | y)$. Multiply by h_a. Does

$x \longmapsto h_a(x)\, s_a(x)$, defined on U_a, extend by zero to a cont. section of L.

This should be clear.

$$\begin{tikzcd} s_a : U_a \arrow[r, hook] \arrow[dr] & Y \subset L \arrow[d] & \\ & X = X & \end{tikzcd}$$

$$\begin{tikzcd} h_a s_a : U_a \arrow[r] \arrow[dr, equal] & p^{-1}(U_a) \subset L \arrow[d] & \\ & U_a \subset X & \end{tikzcd}$$

cont.

define $(h_a s_a)(x) = 0$ for $x \notin U_a$. You need to check continuity. Restrict to U_b.

$$h_a s_a = h_a g_{ab} s_b$$

$(h_a g_{ab})(x) = h_a(x) g_{ab}(x)$ for $x \in U_a \cap U_b$

$= 0$ for $x \in U_b - U_a$

Seems clear because g_{ab} is bounded, and $h_a(x) \to 0$ as $x \to$ pt $\notin U_a$.

So as long as we have this bounded neas it seems that ~~$h_a s_a$ will be a cont.~~

$h_a s_a$ and $h_b s_b$ are cont global sections of L the first non vanish. on U_a, the section non vanish on U_b.

You now have

$$X \times \mathbb{R}^2 \xrightarrow{\begin{pmatrix} h_a s_a \\ h_b s_b \end{pmatrix}} L$$

you can also instead of $s_a(x) \in p^{-1}(x)$

better $\mathbb{R} \longrightarrow L_x$

$c \longmapsto c s_a(x)$

let $g_a : L_x \longrightarrow \mathbb{R}$

$\in \mathbb{R}$

$g_a = (s_a | -)$

$\overset{s_a^*}{g_a}$, s_a inverses over U_a

$$L \xrightarrow{\begin{pmatrix} s_a^* h_a \\ s_b^* h_b \end{pmatrix}} X \times \mathbb{R}^2 \xrightarrow{\begin{pmatrix} h_a s_a & h_b s_b \end{pmatrix}} L$$

$$\begin{pmatrix} h_a s_a & h_b s_b \end{pmatrix} \begin{pmatrix} s_a^* h_a \\ s_b^* h_b \end{pmatrix} = h_a^2 \overbrace{s_a s_a^*}^{1} + h_b^2 \overbrace{s_b s_b^*}^{1}$$

Let's try to make the above work for a finite group Γ. This is the construction of the assembly map. Situation: $Y \xrightarrow{p} X$ is a principal Γ-bundle and you are given a partition of unity. $p^{-1}(x)$ is a Γ-torsor $\forall x$, this is a local system. You probably want to ~~deal~~ think of $\mathbb{C}[p^{-1}(x)]$ as having a natural herm. scalar product where $p^{-1}(x)$ is an orthonormal set. Transitions are then unitary (transition means right mult.

Anyway, consider a partition size 2.

$1 = h_a^2 + h_b^2$, $U_{a/b}$ where $h_{a/b} > 0$, assume

$$\begin{tikzcd} \exists\ s_a : U_a \arrow[r] \arrow[dr, hook] & Y \arrow[d, "p"] \\ & X \end{tikzcd}$$

also for b. Then

$s_a g_{ab} = s_b$ over $U_a \cap U_b$ with $g_{ab} : U_a \cap U_b \longrightarrow \Gamma$

E = vector bundle $Y \times^{\Gamma} \mathbb{C}[\Gamma]$.

to write E as a summand of $X \times \mathbb{C}[\Gamma]^N$ for some N.

~~By Kuiper that~~ s_a is a section of Y over U_a by assumption, what about $h_a s_a$? Yes.
If Γ is finite, then ~~we have a vector~~ E is a f.d. vector bundle (rank $= |\Gamma|$) equipped with a hermitian scalar product. The point again is that $h_a s_a$ ~~makes sense~~ extends continuously (by zero) to ~~all~~ X.

Repeat: Given principal bundle $\pi : Y \longrightarrow X$ for finite group Γ with X compact, form $E = Y \times^{\Gamma} \mathbb{C}\Gamma$ assoc. herm. v.b. over X, note $\mathbb{C}\Gamma$ Hilbert space & elements of Γ form orth basis, hence left + right mult by Γ preserves the inner product, thus E has induced herm. inner product and Γ operates to the right on E in unitary fashion.

Remaining point is to understand ~~the~~ why the space $C(X;E)$ of cont. sections ~~over~~ of E over X is fin gen proj over $C(X) \otimes \mathbb{C}\Gamma$, interpret as embedding as a summand of a fin. gen. free $C(X) \otimes \Gamma$ module. Take partition $1 = \sum h_\alpha$ in $C(X)$ where $h_\alpha \geq 0$ and the prince bundle Y is trivial over $U_\alpha = \{h_\alpha \neq 0\}$, i.e. $\exists$ cont $s_\alpha : U_\alpha \longrightarrow Y$ over X

so s_α induces $s_\alpha : U_\alpha \times \Gamma \xrightarrow{\sim} p^{-1}(U_\alpha)$ and

also $U_\alpha \times \mathbb{C}\Gamma \xrightarrow{\sim} E|_{U_\alpha}$. Also

because s_α is "bounded" $s_\alpha h_\alpha^{1/2}$ is a say this clearly. Fix α and look over U_α where

$$Y|_{U_\alpha} \xleftarrow{\sim} U_\alpha \times \Gamma$$
$$s_\alpha(x)\gamma \longleftarrow\!\shortmid (x,\gamma)$$

~~preserves~~ ~~respects~~ the right γ action

and

$$E|_{U_\alpha} \xleftarrow{\sim} U_\alpha \times \mathbb{C}\Gamma$$
$$\sum s_\alpha(x)\gamma c_\gamma \longleftarrow\!\shortmid (x, \sum c_\gamma \gamma)$$

resp the hermitian v.b. structure, and right $\mathbb{C}\Gamma$ mult.

At each $x \in U$, $E_x \xleftarrow[\tilde{s}_\alpha]{\sim} \mathbb{C}\Gamma$ isom. of ~~Hilbert spaces~~ ~~Γ-modules~~ Hilbert space repns. of Γ, i.e. unitary, i.e. $\tilde{s}_\alpha(x) : \mathbb{C}\Gamma \longrightarrow E_x$ is unitary, so $s_\alpha(x) h_\alpha^{1/2}(x) = t_\alpha(x)$ satisfies $t_\alpha^* t_\alpha = h_\alpha$.

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back to Cuntz's version $\hat{\Gamma}$

Γ given, consider Γ-algs, $\overline{\Gamma\text{-graded algs}}$.

A a Γ-alg, then A-modules with compatible Γ-action $= (A \rtimes \Gamma)$-modules.

B a $\hat{\Gamma}$-alg, then B-modules with compatible $\hat{\Gamma}$-action i.e. Γ-graded B-modules compatible with the Γ-grading of B $= (B \rtimes \hat{\Gamma})$-modules.

$B \rtimes \hat{\Gamma}$

The case to look at now is $\Gamma = \mathbb{Z}$. Your program is to understand Cuntz's formalism. He somehow uses Γ and $\hat{\Gamma}$

in an interesting way. Where to start.
There are two C^*-algs. he associates to Γ and a finite subset F. Assume first Γ is finite and $F = \Gamma$. Then

$$P_\Gamma = C^*\{p_s, s\in\Gamma \mid p_s^* = p_{s^{-1}},\ p_t = \sum_{s\in\Gamma} p_s p_{s^{-1}t}\}$$

universal for projections $p = \sum_{s\in\Gamma} p_s$ in a $\hat{\Gamma}$-alg.

$$\Sigma_\Gamma = C^*\{h_s, s\in\Gamma \mid h_s \geq 0,\ \sum_{s\in\Gamma} h_s = 1\}$$

$$\Sigma_\Gamma^{ab} = \text{continuous functions on the simplex with set of vertices } \Gamma.$$
$$= \text{prob. measures on } \Gamma$$

The "simplex" Σ_Γ is a Γ-algebra, can form $\Sigma_\Gamma \rtimes \Gamma$ which is Γ-graded. P_Γ is Γ graded, can form $P_\Gamma \rtimes \hat{\Gamma}$, here $\hat{\Gamma}$ is the comm. alg. with basis of orthogonal idempotents $\{e_s\}_{s\in\Gamma}$

What might be important here is that you are looking at a projection in a Γ graded algebra like $C(X)\otimes C\Gamma$

$\Sigma_\Gamma \rtimes \Gamma$ has a canonical p.

$$p_s = h_1^{1/2}\, s\, h_1^{1/2} = h_1^{1/2} h_s^{1/2} s$$

$$\sum_s p_s p_{s^{-1}t} = h_1^{1/2} s \underbrace{h_1 s^{-1}}_{\sum h_s = 1} t\, h_1^{1/2} = h_1^{1/2} t\, h_1^{1/2} = p_t$$

So what do you know, what have you learned, the main puzzle is the significance of $\Sigma_\Gamma \rtimes \Gamma$. Σ_Γ^{ab} is the ring of cont fns on the simplex with Γ as vertex set.

$\Sigma_\Gamma \rtimes \Gamma$, hence $\Sigma_\Gamma^{ab} \rtimes \Gamma$ has a canonical p.

I need another example

~~Look at $\Gamma = \mathbb{Z}$ Being a finite gp~~

~~Consider def.~~

$\Gamma = \mathbb{Z}$. Regard def. $\tilde{\Sigma}_F^{a}$ this is a locally finite subcomplex of $\tilde{\Sigma}_\Gamma$

$F \subset \Gamma$ get simp. complex of finite $M \subset \Gamma$ s.t. $M^{-1}M \subset F$. You assume F contains 1 whence all $s \in \Gamma$ are in the simp. complex, can replace F by $F \cap F^{-1}$ to make F symmetric.

$\Gamma = \mathbb{Z}$ $\quad F = \{1, 0, -1\}$ $\quad \tilde{\Sigma}_F^{a}$ is the simplicial complex $\mathbb{R}$ with vertices $\mathbb{Z}$.

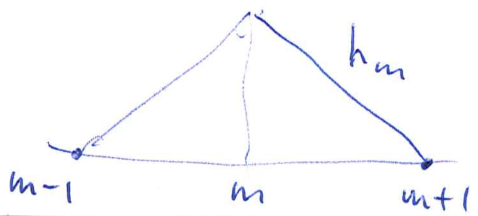
Can you understand ~~K's construction~~

$$\Sigma_F = C^*\left\{ h_n, n \in \mathbb{Z} \;\middle|\; \begin{array}{ll} h_n \geq 0, & \sum h_n = 1 \,? \\ h_m h_n = 0 & |m-n| > 1 \end{array} \right\}$$

'Actual' relations: ~~Point is that~~

$$\underbrace{\left(\sum h_n - 1\right) h_m}_{(h_{m-1} + h_m + h_{m+1} - 1)\, h_m \;=\; 0} = 0 \qquad h_m\left(\sum h_n - 1\right) = 0$$

Σ_F^{ab} ~~relations~~



Back $\Gamma = \mathbb{Z}$, simplicial complex

vertices: $n \in \mathbb{Z}$

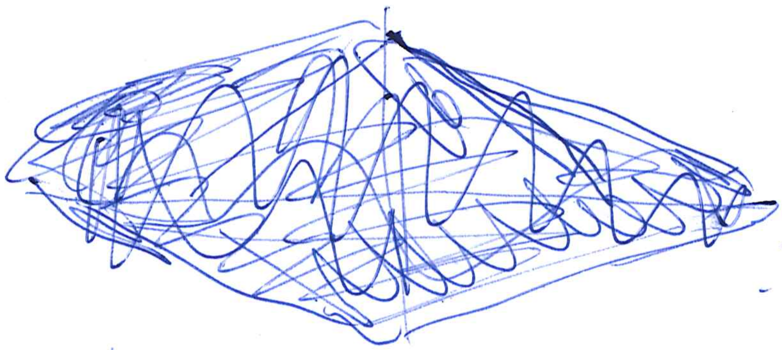
1-simplices: $\{m, n\}$ $\quad |m-n| = 1$.

Then there's Cuntz's algebra

$$C^*\left\{ h_m, n \in \mathbb{Z} \;\middle|\; \begin{array}{l} h_n \geq 0, \quad h_m h_n = 0 \quad |m-n| \geq 2. \\ \underbrace{h_m\left(\sum_n h_n - 1\right) = 0} \end{array} \right\}$$

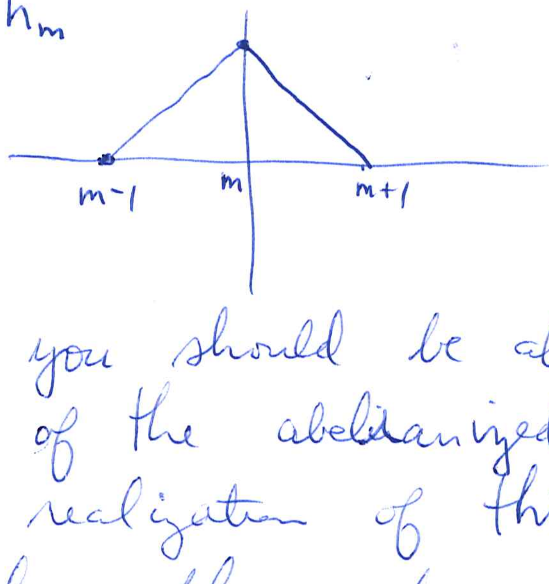
$$h_m = h_m(h_{m-1} + h_m + h_{m+1})$$ another relation by applying $*$

If you abelianize you get



$$h_m h_{m-1} + h_m h_{m+1} = h_m - h_m^2$$
$$h_{m-1} h_m + h_{m+1} h_m = \text{''}$$
$$\therefore [h_m, h_{m-1} + h_{m+1}] = 0$$

graph of h_m



Ideally you should be able to show the spectrum of the abelianized C^* alg is the geometric realization of this simplicial complex. Let's see how this works. Suppose you have a sequence $(h_n)_{n \in \mathbb{Z}}$ of ≥ 0 numbers, satisfying

$h_m h_n = 0 \quad |m-n| \geq 2.$ and $h_m\left(\sum h_n - 1\right) = 0.$

Assume say that $h_0 > 0$, then $h_n \neq 0 \Rightarrow n = -1, 0, 1$

$h_1 > 0 \Rightarrow h_{-1} = 0$, $h_0 + h_1 = 1$.

Now comes the point. Let Σ denote the non comm. alg. Σ^{ab} the abelianization. Assume $\Sigma^{ab} \simeq C_0(\mathbb{R})$. You now need to understand the cross product.

$\Sigma^{ab} \rtimes \mathbb{Z} = C(\mathbb{R}) \rtimes \mathbb{Z}$. You could pose this question in greater generality, namely, consider the universal covering of a manifold M $\tilde{M} \xrightarrow{\Gamma} M$ and form the cross product $C(\tilde{M}) \rtimes \Gamma$. You hope this is Morita equivalent to $C(M)$. How to proceed? Locally over M? ~~More general~~ universal covering is too restrictive. Principal bundle is better. If $\tilde{M} \simeq M \times \Gamma$, then

$C(\tilde{M}) = C(M) \otimes \mathbb{C}\hat{\Gamma}$ some sort of top. group ring is needed here ~~$C(\tilde{M}) \rtimes \Gamma = C(M) \otimes (\mathbb{C}\Gamma \rtimes \Gamma)$~~

probably functions on Γ vanishing at ∞.

So you reach another simple example, namely the $\Gamma, \hat{\Gamma}$ business for $\Gamma = \mathbb{Z}$. So where do we start. Let A be a Γ-algebra, can form $A \rtimes \Gamma$ the ring whose modules are A-modules with compatible Γ action. If B is $\hat{\Gamma}$-alg i.e. equipped with a Γ-grading: $B = \bigoplus_{s \in \Gamma} B_s$ $B_s B_t \subset B_{st}$ $B_s^* = B_{s^{-1}}$, can form $B \rtimes \hat{\Gamma}$ whose modules are ~~Γ-graded~~ B-modules with compatible Γ grading.

Given A with $\mathbb{Z}$-action then $A \rtimes \mathbb{Z}$ is twisted Laurent series ring $\left\{ \sum_{n\in\mathbb{Z}} a_n u^n, \text{ fin. sums} \right\}$.

$u^n a = (u^n * a) u^n$.

Given $B = \bigoplus_{n\in\mathbb{Z}} B_n$ graded ring what is $B \rtimes \hat{\mathbb{Z}}$? $\hat{\mathbb{Z}}$ ~~...~~ should be the ring of functions on $\mathbb{Z}$ under pointwise mult. There's some noncuntal stuff $\hat{\mathbb{Z}} = \sum_{n\in\mathbb{Z}} \mathbb{C} e_n$

By $B \rtimes \hat{\mathbb{Z}}$ you probably mean

$$\bigoplus_{m,n\in\mathbb{Z}} B_m e_n = \left(\bigoplus_{n\in\mathbb{Z}} B_m\right) \otimes \left(\bigoplus_{n\in\mathbb{Z}} \mathbb{C} e_n\right)$$

$$e_n \sum_m b_m = b_n$$

The idea is that if $M = \bigoplus_{n\in\mathbb{Z}} M_n$ is a graded B module, then you are looking at the forge

$$\mathrm{Mod}(R) \xrightarrow{F} \mathrm{Mod}(\mathbb{C})$$

$$M \longmapsto M \qquad F \text{ forgetful fun.}$$

left by r is a map $F \xrightarrow{\cdot} F$

$$\mathrm{Hom}(F, F) \qquad R \xrightarrow{\sim} \mathrm{Hom}_{R^{op}}(R, R)$$

You should now have some control over

$$A \longmapsto A \rtimes \Gamma \quad , \quad B \longmapsto B \rtimes \hat{\Gamma}.$$

$$B \rtimes \hat{\mathbb{Z}} = \bigoplus_{m\in\mathbb{Z}} B_m \otimes \underbrace{\bigoplus_{n\in\mathbb{Z}} \mathbb{C}e_n}_{\text{non unital}}$$

You want to understand how $\mathbb{Z}$ acts on $B \rtimes \hat{\mathbb{Z}}$. First look analytically – $\hat{\mathbb{Z}} = S^1$ equivalence between circle $\mathbb{T}$ action and a $\mathbb{Z}$ grading. Thus $B = \bigoplus_{m\in\mathbb{Z}} B_m$ has a circle action $e^{i\theta} * b = e^{im\theta} b$ when $b \in B_m$. Then $B \rtimes S^1$ should consist of twisted Laurent series

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Start with $A = C(\mathbb{R})$ with $\mathbb{Z}$ acting by translation. Form $A \rtimes \mathbb{Z}$; this is $\mathbb{Z}$-graded hence there's a circle action. $A \rtimes \mathbb{Z} = \bigoplus_{n\in\mathbb{Z}} A u^n$

$ua = (u*a)u$. **circle action is**

$$\cancel{\zeta * (a u^n) = a \zeta^n u^n}$$

$$\cancel{\zeta * (ua) = (\zeta * u) a =}$$

$u a u^{-1} = u * a$ $\qquad$ $\zeta * (a u^n) = \zeta^n a u^n$

$$u a u^{-1} = u * a$$
$$\big\downarrow \zeta *$$
$$\zeta u a \zeta^{-1} u^{-1}$$

So try to understand

$$\bigoplus_{n\in\mathbb{Z}} A u^n \qquad u a u^{-1} = u * a.$$

$C(\mathbb{R})$

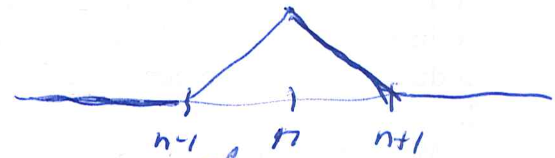
$$u f(x) u^{-1} = f(x+1) = e^D f(x)$$

can you find natural modules for $C(\mathbb{R}) \rtimes \mathbb{Z}$. Example is

$C(a+\mathbb{Z}) \rtimes \mathbb{Z}$. Here you are combining functions on $a+\mathbb{Z}$ vanishing at ∞, tensoring with the group ring $C_r(\mathbb{Z}) = C(S^1)$.

You were looking at the example $\Gamma = \mathbb{Z}$ for Cuntz's theory. $F = \{-1,0,1\}$, Σ_F is the simplicial complex of subsets $M \subset \mathbb{Z} \ni M^{-1}M \subset F$, Vertex $= m \in \mathbb{Z}$, 1-simplex $= \{m,n\}$, $|m-n| = 1$.

$\mathcal{E}_{\Sigma_F}$ is C^* alg generated by h_n



so it should be $C(\mathbb{R})$. $\therefore\ \mathcal{E}^{ab}_{\Sigma_F} = C(\mathbb{R})$

$\Gamma = \mathbb{Z}$ acts on $\mathcal{E}_{\Sigma_F}$ by translation

so you can form $\mathcal{E}_{\Sigma_F} \rtimes \Gamma$, a bigger version than $C(\mathbb{R}) \rtimes \mathbb{Z}$. What do you expect is true about $C(\mathbb{R}) \rtimes \mathbb{Z}$? It should be related to the top category given by $\mathbb{R}$ with $\mathbb{Z}$ acting by translations, but there's the locally compact angle. Obvious question is a link with $C(\mathbb{R}/\mathbb{Z})$. You have around a f.g. projective $C(\mathbb{R}/\mathbb{Z}) \otimes C_r(\mathbb{Z})$ - module.

Review. $\mathbb{R} \xrightarrow{p} \mathbb{R}/\mathbb{Z}$ is a principal bundle group $\mathbb{Z}$.

Notice that $C(\mathbb{R}/\mathbb{Z}) \otimes C_r(\mathbb{Z}) \simeq C(S^1 \times S^1)$ has the Bott class.

All this you should organize

Let's look at what Cuntz does.

$$\mathcal{E}_{\Sigma_F} = C^*\{h_n, n\in\mathbb{Z} \mid h_n \geq 0,\ h_m h_n = 0 \ |m-n|>1,\ h_m\left(\sum_n h_n - 1\right) = 0\}.$$

$$P_F = C^*\{p_s, s\in\Gamma \mid p_s^* = p_{s^{-1}},\ p_t = \sum p_s p_{s^{-1}t},\ p_s = 0,\ s\notin F\}.$$

$P_F \rtimes \hat{\Gamma}$ stably iso to $\mathcal{E}_{\Sigma_F}$

$P_F \longrightarrow \mathcal{E}_{\Sigma_F} \rtimes \Gamma$ specific homom. ~~map~~ which should be a K-equiv. actually should be a Morita equiv.

$p_s \longmapsto h_1^{1/2} h_s^{1/2} s = h_1^{1/2} s h_1^{1/2}$

You want to do this for $F = \{-1, 0, 1\} \subset \mathbb{Z}$

P_F is universal for $p \in \bigoplus_{n\in\mathbb{Z}} B_n$ $\mathbb{Z}$-graded alg. supported in F, i.e. $p = p_{-1} + p_0 + p_1$.

$p = p^* \iff p_1^* = p_{-1},\ p_0^* = p_0$

$p^2 = p$

$$\begin{aligned} &p_{-1}^2 + p_{-1}p_0 + p_{-1}p_1 \\ &+ p_0p_{-1} + p_0^2 + p_0p_1 \\ &+ p_1p_{-1} + p_1p_0 + p_1^2 \end{aligned}$$

$$\begin{aligned} p_{-1}^2 &= 0 \\ p_{-1} &= p_0p_{-1} + p_{-1}p_0 \\ p_0 &= p_1p_{-1} + p_0^2 + p_1p_{-1} \\ p_1 &= p_1p_0 + p_0p_1 \\ p_2^2 &= 0. \end{aligned}$$

~~Let's work it~~ How to proceed? You have very specific C^* algebras. P_F universal for proj $p = p_{-1} + p_0 + p_1$ in a $\mathbb{Z}$-graded alg $\bigoplus_{n\in\mathbb{Z}} B_n$

it seems that $P_F \longrightarrow \mathcal{E}_{\Sigma_F} \rtimes \Gamma$ is a homom. inducing a Morita equivalence. $A \xrightarrow{u} B$ is a meg hom when $BAB = B$, ~~ABA~~ $A(\mathrm{Ker}\, u)A = 0$.

so it seems that the ideal gen by P_F in ~~the~~ $\mathcal{E}_{\Sigma_F} \rtimes \Gamma$ is the ~~[illegible]~~ whole ring.

what do we have specifically.

$$p_n \longmapsto \underbrace{h_0^{1/2}\, u^n\, h_0^{1/2}} \qquad p_0 \longmapsto h_0$$

$$p_n \longmapsto h_0^{1/2} h_n^{1/2} u^n$$

$$\sum_m p_m p_{-m+n} = \sum h_0^{1/2} h_m^{1/2} \underbrace{u^m h_0^{1/2} h_{-m+n}^{1/2} u^{-m+n}}_{h_m^{1/2} h_n^{1/2} u^n}$$

Notice that $h_0 h_n = 0 \iff h_n h_0 = 0$ by s.a.

so $h_0^{1/2} h_n^{1/2} = (h_0 h_n)^{1/2} = 0$ for $|n| > 1$.

Thus you do find. Check support cond. in general: $p_s \longmapsto h_1^{1/2} h_s^{1/2} s$ but $h_1 h_s = 0$ for $s \notin F$. $\therefore h_1^{1/2} h_s^{1/2} = 0$ ~~[illegible]~~

$\mathcal{E}_{\Sigma_F} \rtimes \Gamma$ is Γ graded

$p_s \longmapsto h_1^{1/2} h_s^{1/2} s$ relations $p_s^* = p_{s^{-1}}$, $p_t = \sum_{s \in \Gamma} p_s p_{s^{-1}t}$)
$p_s = 0$ for $s \notin F$.

So you get a $*$ homomorphism

$$P_F \longrightarrow \mathcal{E}_{\Sigma_F} \rtimes \Gamma$$

Is it an isomorphism, onto, or does the image generate the target ring as ideal. ~~So what do we do?~~

$$p_s \longmapsto h_1^{1/2} s h_1^{1/2} \qquad p_1 \longmapsto h_1$$

Look at the example again. Yes.

Possible scaling ~~tricks~~

Take $p_{\pm 1} \longmapsto 2 p_{\pm 1}$

What do you know about $\mathcal{E}_{\Sigma_F} \rtimes \Gamma$ it's roughly generated by h_1 and the $s \in \Gamma$, since $s h_1 s^{-1} = h_s$. The problem here is that $\mathcal{E}_{\Sigma_F}$ is not unital. But you ought to be able to understand the example. $\mathcal{E}^{ab}_{\Sigma_F} = C(\mathbb{R})$

$$\mathcal{E}^{ab}_{\Sigma_F} \rtimes \mathbb{Z} = C(\mathbb{R}) \rtimes \mathbb{Z} = \underbrace{C(\mathbb{R}) \rtimes C(S^1)}$$

Can you construct a Hilb. repn. this is what you get by adjoining a unitary u to $C(\mathbb{R})$

~~[illegible]~~

What's the ab. $\mathcal{E}^{ab}_{\Sigma_F}$ in general. You have $h_s \geq 0$ for each $s \in \Gamma$. Say Γ finite so the relations are just $h_s\left(\sum_{t\in\Gamma} h_t - 1\right) = 0$

Suppose you have the simp. cx. $\Sigma_F = \{\emptyset \neq M < \Gamma \mid M^{-1}M \subset F\}$

better: suppose suppose $\Gamma = F$ is finite. Then

$$\mathcal{E}^{ab}_{\Sigma_\Gamma} = C^*\{h_s, s\in\Gamma \mid h_s \geq 0,\ \sum_{s\in\Gamma} h_s = 1\}$$

and the spectrum should be the full simplex with vertices $\in F$

Take the general case.

$$\mathcal{E}_{\Sigma_F} = C^*\{h_s, s\in\Gamma \mid h_s \geq 0,\ h_s h_t = 0 \text{ if } s^{-1}t \notin F\}.$$

$$h_s \sum_{t\in\Gamma} h_t = h_s$$

$\mathcal{E}^{ab}_{\Sigma_F}$: interpret h_s as number ≥ 0

Fix $s = 1$. Then $h_1 h_t = 0$ if $t \notin F$

Then $h_1\left(\sum_{t\in F} h_t - 1\right) = 0$

If $h_1 \neq 0$, then ? What do you actually have?, Look at h_s, $s\in\Gamma$ sat.

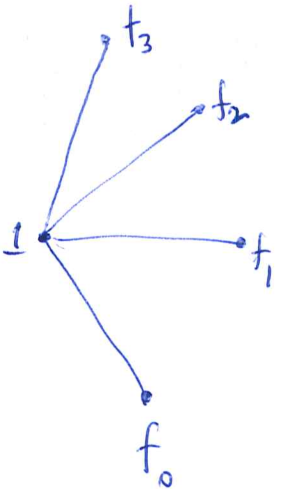
$h_s \geq 0 \quad \forall s \qquad h_s h_t = 0$ for $s^{-1}t \notin F \qquad t \notin sF$

$h_s\left(\sum_{t\in sF} h_t - 1\right) = 0 \qquad$ Assume say $h_1 > 0$

then $\sum_{t\in sF}$

Assume $h_s > 0$, since $h_s\left(\sum_{t\in sF} h_t - 1\right) = 0$

it follows that $\sum_{t\in sF} h_t = 1$.

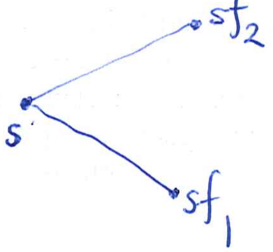


~~Start~~ again. Assume h_s, $s\in\Gamma$ are numbers ≥ 0 such that $h_s h_t = 0$ for $s^{-1}t \notin F$ and $t \notin sF$ such that

$$\sum_{t\in\Gamma} h_s h_t = h_s$$

for all $s, t \in \Gamma$. Assume $h_s > 0$, then

$$\sum_{t\in sF} h_t = 1.$$



You are confused. $M \subset \Gamma$ is a simplex when $M^{-1}M \subset F$. ~~Take an interior point in the realization of the simplex M, i.e. $\{h_s, s\in M \mid h_s > 0\}$, suppose~~

$h_s, s\in\Gamma$ is a family of numbers $\ni$ $h_s \geq 0$, $h_s h_t = 0$ $s^{-1}t \notin F$,

$$\sum_{t\in\Gamma} h_s h_t = h_s \iff \sum_{t\in sF} h_s h_t = h_s$$

Let $M = \{s \mid h_s > 0\}$. Then M is finite

$$h_s \sum_{t\in M} h_t = h_s$$

Why is M finite? $h_s \sum_{t\in M} h_t = h_s$

have to avoid $\emptyset$ simplex

Repeat: Given F finite $\subset \Gamma$, Σ_F is the simplicial complex consisting of finite $\phi \neq M \subset \Gamma$ such that $M^{-1}M \subset F$ i.e. $\forall s,t \in M$, $s^{-1}t \in F$.

Σ_F non empty $\Longleftrightarrow$ $1 \in F$. Γ has left mult action on Σ_F. Now consider $(h_s)_{s\in\Gamma}^{\in\mathbb{R}}$ $\ni$ $h_s \geq 0$, $h_s h_t = 0$ $s^{-1}t \notin F$, $h_s\left(\sum_{t\in\Gamma} h_t - 1\right) = 0$.

Exclude case $h_s = 0$ $\forall s$. Assume $\exists s$ such that $h_s > 0$. Then there only finitely many $t \in \Gamma$ st. $h_t > 0$ (because $h_s h_t \neq 0 \Longrightarrow s^{-1}t \in F \Longrightarrow t \in sF$ finite set.) Let

$M = \{t \mid h_t > 0\}$.

Then $\underbrace{h_s\left(\sum_{t\in\Gamma} h_t\right)}_{\substack{\parallel \\ h_s}} = h_s\left(\sum_{t\in M} h_t\right)$ finite sum $\Longrightarrow \sum_{t\in M} h_t = 1$

What gives?? So far you've seen that any nonzero $(h_s)_{s\in\Gamma}$ satisfying the condition has finite support M and that $\forall s,t \in M$, $s^{-1}t \in F$.

I think you have done something: established that $\mathcal{E}^{ab}_{\Sigma_F}$ is essentially the realization of the simp. complex. Σ_Γ. Examine $\mathbb{Z}$ again

$$C^*\{(h_n)_{n\in\mathbb{Z}} \mid h_n \geq 0,\ h_m h_n = 0 \text{ if } |m-n| > 1\}.$$

$$h_m = h_m \sum_n h_n = h_m(h_{m-1} + h_m + h_{m+1})$$

What's involved here? You are trying to deal with something locally compact.