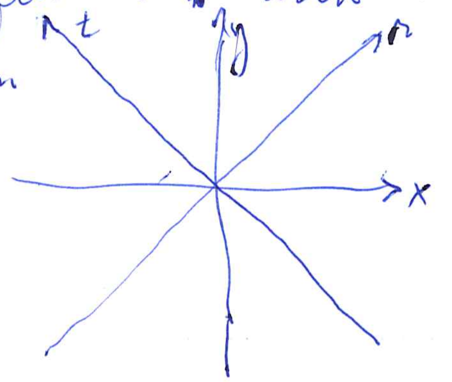
Discuss Green's ~~fn~~ function. What thoughts come to mind? You need Hilb space completion for the geometric series. You have a spectral parameter λ for u, want eigenvectors for u corresp to λ. Patterns: Splitting into half spaces connected to ~~an~~ construction of the Green's function. ~~[illegible]~~ You want to replace grid space by eigenfunctions. Better, to develop the theory of eigenfunctions. An eigenfn for the eigenvalue λ is a linear fnl on grid space E which kills $(\lambda-u)E$, i.e. it's a solution of the grid equations ~~such that~~ where ~~the operator~~ u is given by multiplication by λ. Now you know that the space of eigenfns for λ is 2 diml, because E is a rank 2 module over $\mathbb{C}[u,u^{-1}]$. It seems you should understand growth of an eigenfunction. You expect ~~[illegible]~~ the space $V_\lambda = (E/(\lambda-u)E)^*$ of λ-eigenfunctions to have distinguished lines called decaying to the right (resp. left), and that these lines furnish the boundary conditions required for the Green's function. Example? ~~Scattering~~ Free case: All $h_n = 0$. Then $p_n = z^n p_0$ $q_n = q_0$ $\forall n$ in E, corresp. V_λ consists of ψ with $\psi_n = \begin{pmatrix} \lambda^n & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \psi_0^1 \\ \psi_0^2 \end{pmatrix}$ ~~So~~ it's 2 diml. How do you define growth? How ~~does~~ is the pre-Hilbert space structure on E ~~become~~ felt ~~[illegible]~~ by eigenfunctions.

Return to wave eqn. to answer question: How is the pos. def. herm. product felt by eigenfunctions? You want to review grid space picture

Green's function with λ. Take simple wave equation



$$\partial_x = \partial_r - \partial_t \qquad r = x+y$$
$$\partial_y = \partial_r + \partial_t \qquad t = -x+y$$

$$\partial_x\psi^1 = (\partial_r - \partial_t)\psi^1 = 0 \qquad \psi^1 \text{ function of } y = \frac{r+t}{2}$$
$$\partial_y\psi^2 = (\partial_r + \partial_t)\psi^2 = 0. \qquad \psi^2 \text{ ——— } x = \frac{r-t}{2}$$

What do you mean by G fn? Fundl soln.

Introduce $e^{\lambda t}$ time dep.

$$(\partial_r - \lambda)\psi^1 = 0 \qquad \psi^1(r,t) = e^{\lambda(r+t)}\cdot\text{const}$$
$$(\partial_r + \lambda)\psi^2 = 0 \qquad \psi^2(r,t) = e^{\lambda(-r+t)}\cdot\text{const.}$$

Green's function should solve.

$$(\partial_r - \lambda)\psi^1 = \delta(r)$$
$$(\partial_r + \lambda)\psi^2 = \delta(r)$$

$$(\partial_r - \partial_t)\psi^1 = \psi^2$$
$$(\partial_r + \partial_t)\psi^2 = \psi^1$$

$$\partial_t\psi = \begin{pmatrix}\partial_r & -1\\ 1 & -\partial_r\end{pmatrix}\psi$$

Consider case with ~~potential~~ $m(r)$ of compact support (better decaying rapidly)

$$\partial_t \psi = \begin{pmatrix} \partial_r & -m \\ \bar{m} & -\partial_r \end{pmatrix} \psi$$

Compare what you know from ODE theory with Hilbert space picture! The ODE theory ~~yields a transfer~~ take e^{st} time dep.

$$\boxed{s\psi = \begin{pmatrix} \partial_r & -m \\ \bar{m} & -\partial_r \end{pmatrix} \psi}$$

get transfer matrix

assuming decay, you get the asymptotics.

$$\begin{pmatrix} e^{rs} & 0 \\ 0 & e^{-rs} \end{pmatrix}\begin{pmatrix} \\ \\ \end{pmatrix} \xleftarrow{r \to -\infty} \psi(r) \xrightarrow{r \to +\infty} \begin{pmatrix} e^{rs} & 0 \\ 0 & e^{-rs} \end{pmatrix}\begin{pmatrix} \\ \\ \end{pmatrix}$$

$$\partial_r \begin{pmatrix} \psi^1 \\ \psi^2 \end{pmatrix} = \begin{pmatrix} s & m \\ \bar{m} & -s \end{pmatrix} \psi$$

$$\partial_r \begin{pmatrix} e^{-rs}\psi^1 \\ e^{rs}\psi^2 \end{pmatrix} = \begin{pmatrix} \cancel{e^{-rs} s}\ 0 & e^{-rs} m \\ e^{rs}\bar{m} & \cancel{-e^{rs} s}\ 0 \end{pmatrix} \psi + \begin{pmatrix} \cancel{-s e^{-rs}\psi^1} \\ \cancel{s e^{rs}\psi^2} \end{pmatrix}$$

$$\boxed{\partial_r \begin{pmatrix} e^{-rs}\psi^1 \\ e^{rs}\psi^2 \end{pmatrix} = \begin{pmatrix} 0 & e^{-2rs} m(r) \\ e^{2rs}\bar{m}(r) & 0 \end{pmatrix} \begin{pmatrix} e^{-rs}\psi^1 \\ e^{rs}\psi^2 \end{pmatrix}}$$

leads to transfer matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ from $r = -\infty$ to $r = +\infty$.

At this point you have ~~this~~ done something — maybe, what? Review. You started with the wave equation $\partial_t \psi = \begin{pmatrix} \partial_r & -m \\ +\bar{m} & -\partial_r \end{pmatrix} \psi$ where $m(r)$ decays rapidly enough. Assoc. to the wave equation is a Hilbert space ~~consisting~~ consisting of $\psi(r) = \begin{pmatrix} \psi^1(r) \\ \psi^2(r) \end{pmatrix}$ with usual L^2 norm $\|\psi\|^2 = \int_{-\infty}^{\infty} \psi^* \psi \, dr$, and a 1-par. unitary gp gen. by the skew adjoint differential op. $\begin{pmatrix} \partial_r & -m \\ \bar{m} & -\partial_r \end{pmatrix}$. You can analyze all this by using F.T. in time, the Laplace transform might be better, since it allows one to handle the Cauchy problem on $t=0$. Everything should be very explicit. You get the eigenvalue problem

$$s\psi = \begin{pmatrix} \partial_r & -m \\ \bar{m} & -\partial_r \end{pmatrix} \psi$$

For each s this has a 2 dim'l space of solutions. For the LT. you proceed as follows: to solve IVP $\quad \partial_t \psi = D\psi \quad \psi_0(r)$ given

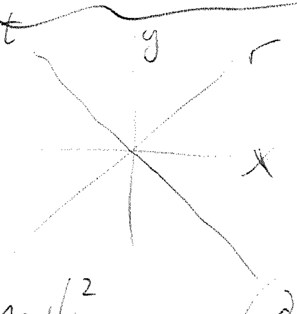
Transf. $\quad -\psi_0 + s\mathcal{L}\psi = D\mathcal{L}\psi$

$$\therefore \quad \mathcal{L}\psi = \frac{1}{s-D}\psi_0$$

$$\psi = \int_{-i\infty}^{i\infty} \frac{ds}{2\pi i} e^{st} \frac{1}{s-D} \psi_0(r)$$

What do you mean by $\frac{1}{s-D}$ here? D is skew-adjoint so the Green's op. or resolvent should be defined via Hilbert space unbdd. operator theory. So there is a definite Green's op. $(s-D)^{-1}$ for $\mathrm{Re}(s) \neq 0$, which should be accessible to you, the usual procedure where you have decaying solutions

---



$$\partial_x = \partial_r - \partial_t \qquad r = x+y$$
$$\partial_y = \partial_r + \partial_t \qquad t = -x+y$$

$$\partial_x \psi^1 = m\psi^2 \qquad (\partial_r - \partial_t)\psi^1 = m\psi^2$$
$$\partial_y \psi^2 = \bar{m}\psi^1 \qquad (\partial_r + \partial_t)\psi^2 = \bar{m}\psi^1$$

$$\partial_t \psi = \begin{pmatrix} \partial_r & -m \\ \bar{m} & -\partial_r \end{pmatrix} \psi$$

skew adj. D

Cauchy problem on $t=0$

$$\psi(r,t) = \int_{-i\infty}^{i\infty} e^{st} \frac{1}{s-D} \psi_0 \frac{ds}{2\pi i} = H(t) e^{tD} \psi_0$$

So you need $\frac{1}{s-D}$ Green's function for an ordinary DE. But you compare

$$\frac{1}{s-D} = \frac{1}{s-D_0} + \frac{1}{s-D_0} V \frac{1}{s-D_0} + \cdots$$

What's your philosophy here? To construct $G_s(r,r')$. ~~One should be able~~ This should be equivalent to the Birkhoff factorization.

Repeat: Begin with $\partial_t \psi = \begin{pmatrix} \partial_r & -m \\ +\bar{m} & -\partial_r \end{pmatrix} \psi$

wave equation in space time ~ there's a Hilbert space of finite energy Cauchy data, 1-parameter unitary group generated by a skew. op. D, which is a pert.

of D_0. Now one transforms

Discuss philosophy. You begin with a wave equation in 2 dimensions (cont. or disc.). This leads to a Hilbert space of finite energy solutions together with a 1-param. unitary group giving time translation. The generator is a skew-adjoint operator (unbdd) on the Hilbert space. This leads to a spectral decomposition of the Hilbert space indexed by an eigenvalue parameter λ ($\lambda \in i\mathbb{R}$ cont case, $\lambda \in S^1$ disc case). For each

---

Discuss philosophy. You begin with a wave equation in 2 dimensions:

$$\partial_t \psi = \begin{pmatrix} \partial_x & -m \\ \bar{m} & -\partial_x \end{pmatrix} \psi$$

(Ultimate goal: to understand its solutions by constructing an appropriate universal solution, with values in a "grid space". Done in the discrete case. For the moment you restrict attention to Hilbert space aspects.)

The wave equation Associated to is a Hilbert space of finite energy Cauchy data on the line $t=0$. The diff op $D = \begin{pmatrix} \partial_x & -m \\ \bar{m} & -\partial_x \end{pmatrix}$ is formally skew-adjoint, solving the wave equation for finite energy Cauchy data should be the same (by Hilbert space theory – von Neumann's unbdd operators) as constructing the resolvent $(\lambda - D)^{-1}$ on the Hilbert space for $\lambda \notin i\mathbb{R}$.

Here you encounter a concrete analytical problem, namely take an $m(r)$ and construct a bounded operator $(\lambda - D)^{-1}$ on $L^2(\mathbb{R}, \mathbb{C}^2)$. This should be easy by perturbation from $(\lambda - D_0)^{-1}$ where

$$D_0 = \begin{pmatrix} \partial_r & 0 \\ 0 & -\partial_r \end{pmatrix}$$

$$(s - D_0)\, G_0(r, r') = \begin{pmatrix} s - \partial_r & 0 \\ 0 & s + \partial_r \end{pmatrix} G_0(r,r') \qquad G_0(r,r') = \begin{pmatrix} H(r'-r)\, e^{s(r-r')} & \\ & H(r-r')\, e^{-s(r-r')} \end{pmatrix}$$

$\mathrm{Re}(s) > 0$

$$\frac{1}{\lambda - \partial_r}\,\delta(r) = \frac{1}{\lambda - \partial_r} \int_{a-i\infty}^{a+i\infty} \frac{e^{sr}}{2\pi i}\, ds = \int \frac{e^{sr}}{\lambda - s} \frac{ds}{2\pi i}$$

$$= \begin{cases} 0 & \mathrm{Re}(\lambda) < 0 \\ -e^{\lambda r} & \mathrm{Re}(\lambda) > 0 \end{cases}$$

Point is that $s - D_0 = \begin{pmatrix} s - \partial_r & \\ & s + \partial_r \end{pmatrix}$ has homog. solns. $\begin{pmatrix} e^{sr} & 0 \\ 0 & e^{-sr} \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix}$ so you choose to decay appropriately as $r \to -\infty$, or $+\infty$

Your aim: Derive factorization of the S matrix from the Green's function.

So you fix s, $\quad D = \begin{pmatrix} \partial_r & -m \\ \bar{m} & -\partial_r \end{pmatrix} = D_0 + V$

Also you take $r' = 0$

$$\phi = \begin{pmatrix} e^{sr} & 0 \\ 0 & e^{-sr} \end{pmatrix}^{-1} \psi = \begin{pmatrix} e^{-sr} \psi^1(r) \\ e^{sr} \psi^2(r) \end{pmatrix}$$

$$\cancel{\partial_r \phi = \begin{pmatrix} -s\phi^1 \\ s\phi^2 \end{pmatrix} + \begin{pmatrix} e^{sr} & 0 \\ 0 & e^{-sr} \end{pmatrix} \begin{pmatrix} \partial_r \psi^1 - m\psi^2 \\ -\partial_r \psi^2 + \bar{m}\psi^1 \end{pmatrix}}$$

$$\cancel{\begin{pmatrix} e^{-sr} & 0 \\ 0 & e^{sr} \end{pmatrix} \begin{pmatrix} \partial_r & -m \\ \bar{m} & -\partial_r \end{pmatrix} \begin{pmatrix} e^{sr} & 0 \\ 0 & e^{-sr} \end{pmatrix} \phi = \begin{pmatrix} s+\partial_r & 0 \\ 0 & s-\partial_r \end{pmatrix} \phi}$$

$$\psi = \begin{pmatrix} e^{sr} & 0 \\ 0 & e^{-sr} \end{pmatrix} \phi$$

$$\begin{pmatrix} e^{-sr} & 0 \\ 0 & e^{sr} \end{pmatrix} D \begin{pmatrix} e^{sr} & 0 \\ 0 & e^{-sr} \end{pmatrix} = \begin{pmatrix} e^{-sr} & 0 \\ 0 & e^{sr} \end{pmatrix} \begin{pmatrix} \partial_r & 0 \\ 0 & -\partial_r \end{pmatrix} \begin{pmatrix} e^{sr} & 0 \\ 0 & e^{-sr} \end{pmatrix}$$

$$+ \begin{pmatrix} e^{-sr} & 0 \\ 0 & e^{sr} \end{pmatrix} \begin{pmatrix} 0 & -m \\ \bar{m} & 0 \end{pmatrix} \begin{pmatrix} e^{sr} & 0 \\ 0 & e^{-sr} \end{pmatrix}$$

$$= \begin{pmatrix} \partial_r + s & -e^{-2sr} m \\ e^{2sr} \bar{m} & -\partial_r + s \end{pmatrix} = s + D_0 + \begin{pmatrix} 0 & -e^{-2sr} m \\ e^{2sr} \bar{m} & 0 \end{pmatrix}$$

$$D\psi = s\psi \qquad g_s = \begin{pmatrix} e^{sr} & \\ & e^{-sr} \end{pmatrix}$$

$$g_s^{-1}(D-s)g_s\phi = \left(D_0 + \begin{pmatrix} 0 & -e^{-2sr}m \\ e^{2sr}\bar{m} & 0 \end{pmatrix}\right)\phi$$

it's OK but confusing

Repeat: You study $s\psi = D\psi = \begin{pmatrix} \partial_r & -m \\ \bar{m} & -\partial_r \end{pmatrix}\psi$

and ~~change~~ use variation of constants, let

$\psi = g_s\phi \qquad g_s = \begin{pmatrix} e^{sr} & 0 \\ 0 & e^{-sr} \end{pmatrix} \qquad D_0 g_s = s g_s$

Then the gen. soln of $D\psi = s\psi$ is

$\psi = g_s\phi$ where $\left[D_0 + \begin{pmatrix} 0 & -e^{-2sr}m \\ e^{2sr}\bar{m} & 0 \end{pmatrix}\right]\phi = 0$

in particular $\phi(r)$ constant for $r \gg 0$ and for $r \ll 0$ assuming $m(r)$ ~~[illegible]~~ compact supp.

Now you want the Green's function at $r=0$, the Green's function of $s-D$. What's important is the transfer matrix. This concerns the ϕ equation

$$\left[D_0 + \begin{pmatrix} 0 & -e^{-2sr}m \\ e^{2sr}\bar{m} & 0 \end{pmatrix}\right]\phi = 0$$

$\Phi(r,s)$ satisfies the above and $\Phi(r,s) = I$ for $r \ll 0$.

A slightly simpler version is

~~$\partial_r \phi +$~~

$$\left[\begin{pmatrix} \partial_r & 0 \\ 0 & +\partial_r \end{pmatrix} + \begin{pmatrix} 0 & -e^{-2sr}m \\ -e^{2sr}\bar{m} & 0 \end{pmatrix}\right]\phi = 0$$

or

$$\boxed{\partial_r \phi = \begin{pmatrix} 0 & e^{-2sr}m \\ \bar{m}e^{2sr} & 0 \end{pmatrix}\phi}$$

analogous to

$$\begin{pmatrix} z^{-n}p_n \\ q_n \end{pmatrix} = \frac{1}{k_n}\begin{pmatrix} 1 & h_n z^{-n} \\ \bar{h}_n z^{n} & 1 \end{pmatrix}\begin{pmatrix} z^{-n+1}p_{n-1} \\ q_{n-1} \end{pmatrix}$$

Now you have

$$\Phi_0 = T\exp\left\{\int_{-\infty}^{0}\begin{pmatrix} 0 & e^{-2sr'}m \\ e^{+2sr'}\bar{m} & 0 \end{pmatrix}dr'\right\} = \begin{pmatrix} a^\ell & b^\ell \\ c^\ell & d^\ell \end{pmatrix}$$

Next: the Green's function. [redacted]

~~What~~ You need to what diff operator to invert.

$$(s - D)G(r) = \delta(r)$$

$$\Phi_{-\infty}^{\infty} = \Phi_0^{\infty}\,\Phi_{-\infty}^{0}$$

~~What~~ You should construct the G function with the appropriate boundary conditions — to be in L^2. Suppose $\mathrm{Re}(s) > 0$

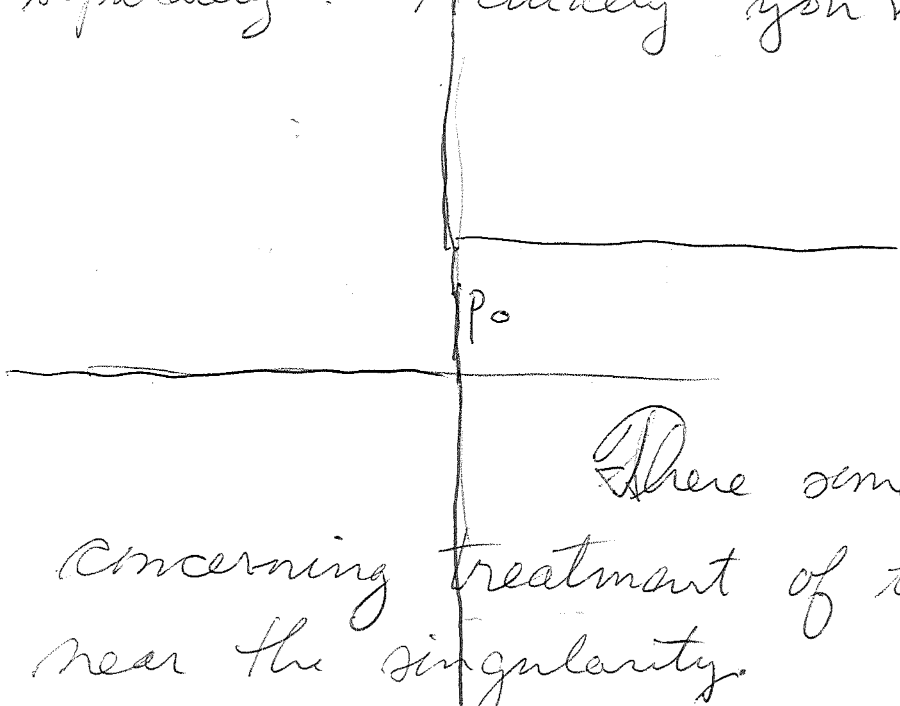
Discuss the discrete case

Aim: Use the Green's function, which $\exists$ by Hilbert space theory, to ~~construct~~ factor the S matrix.

What is the Green's function? It's assoc. to a "site" where the δ function occurs and is a solution - really eigenfunction - elsewhere. I should have said solution satisfying the boundary conditions. Solution ~~is~~ should be some sort of linear functional so you want something like

$$\begin{pmatrix} p_0 \\ q_0 \end{pmatrix}^* \frac{1}{\lambda - u} (?)$$ for the Green's function associated the position $n=0$. This gives a matrix function

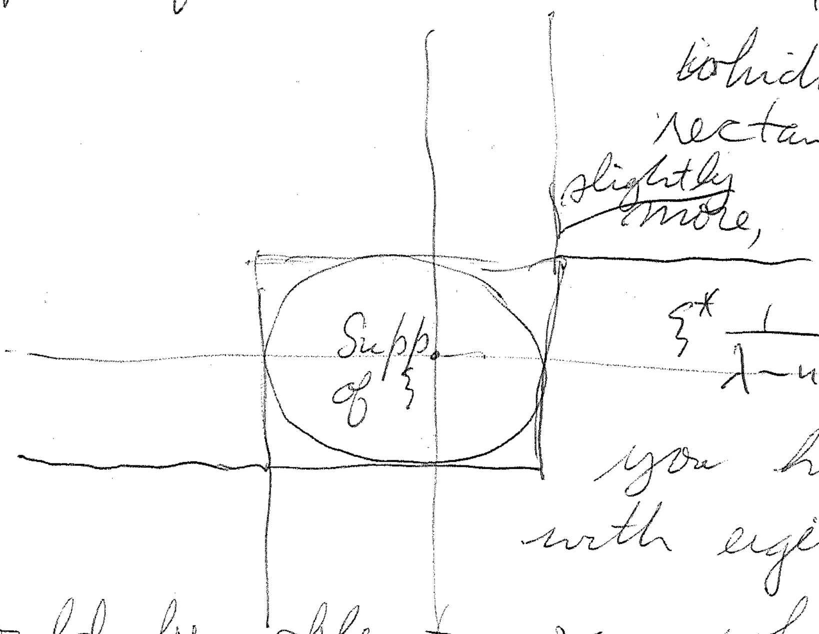
$$\psi_n = \begin{pmatrix} p_0 \\ q_0 \end{pmatrix}^* \frac{1}{\lambda - u} \begin{pmatrix} p_n \\ q_n \end{pmatrix}$$ perhaps it would be better to treat p_0^*, q_0^* separately. Actually you may know something.



just that $p_0^* \frac{1}{\lambda - u}$ kills $(\lambda - u) \cdot p_0^{\perp}$

There some obvious problem concerning treatment of the Green's function near the singularity. However it should be possible to determine the decaying boundary conditions. Picture ~~of~~ $\xi^* \frac{1}{\lambda - u} = (\xi | \frac{1}{\lambda - u} | ?)$ which is a linear function on the grid space, hence is a solution of the grid difference ~~eq~~ equations

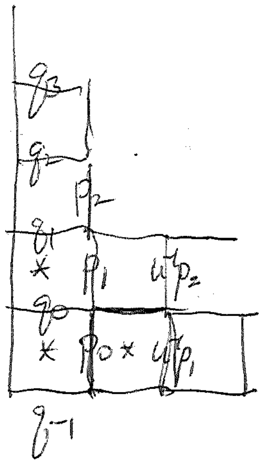
Suppose you are far away from the support of ξ:



Then $\perp$ to Supp ξ, which is a well-defined rectangle, and maybe slightly more, so that ~~Supp(ξ)~~ $\xi * \frac{1}{\lambda-u}$ kills $(\lambda-u)X$

you have an eigenfunction with eigenvalue λ. You should be able to say when this eigenfunction decays.

Calculation: suppose you have a first quadrant system with say h_0, h_1 nonzero at most.



Consider the ascending staircase basis

$$q_{-1}, p_0, u^{-1}q_1, u^{-1}p_2, u^{-2}q_3, u^{-2}p_4$$

You have $u^{-1}p_1 = u^{-2}p_2 = \cdots$

that is $u^{-n}p_n = u^{-n-1}p_{n+1}$ for $n \geq 1$

$$\boxed{u p_n = p_{n+1} \quad \text{for } n \geq 1}$$

You have $q_1 = q_2 = \cdots$

$$\boxed{q_n = q_{n+1} \quad \text{for } n \geq 1}$$

Suppose f is an eigenfunction for the eigenvalue λ. ~~[illegible]~~ You want to know when f is in L^2, i.e. ~~the~~ f applied to the orthonormal sequence

$$q_{-1}, p_0, u^{-1}q_1, \underset{\substack{\| \\ p_1}}{u^{-1}p_2}, u^{-2}q_3, \underset{\substack{\| \\ p_2}}{u^{-2}p_4}$$

~~In particular~~ Since q_n is const for large n $= \xi_-$ we have

$$f\left(u^{-k} q_{2k-1}\right) = \lambda^{-k} f(\xi_-) \quad \text{for}$$

large k, so if $|\lambda| < 1$, then $f(\xi_-) = 0$.

Similarly $f\left(u^{-k} p_{2k}\right) = f\left(u^{+k} u^{-2k} p_{2k}\right)$

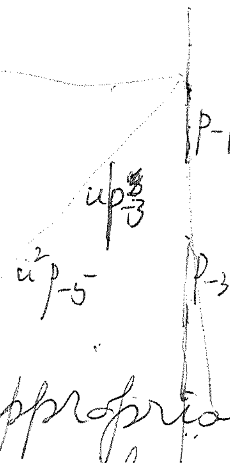
$= \lambda^k f(\xi_+)$ for $k \gg 0$ which presents no condition on $f(\xi_+)$ for $|\lambda| < 1$. So if we stick to $|\lambda| < 1$ it seems that the appropriate boundary condition for $\xi^* \frac{1}{\lambda - u} = f$ is for $f(\xi_-) = 0$.

Consider $\xi'_- = \lim_{n \to -\infty} u^{-n} p_n$ $\qquad \xi'_- = u^{-n} p_n \quad n \ll 0$

~~$f(\ldots) = \lambda^{-n} f(p_n)$~~ You want the sequence

$$f\left(u^{+k} p_{-2k-1}\right) = f\left(u^{-(k+1)} u^{2k+1} p_{-2k-1}\right)$$

$$= \lambda^{-(k+1)} f(\xi'_-)$$

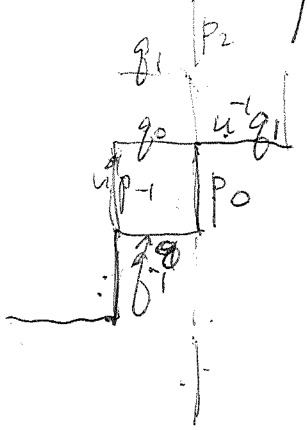


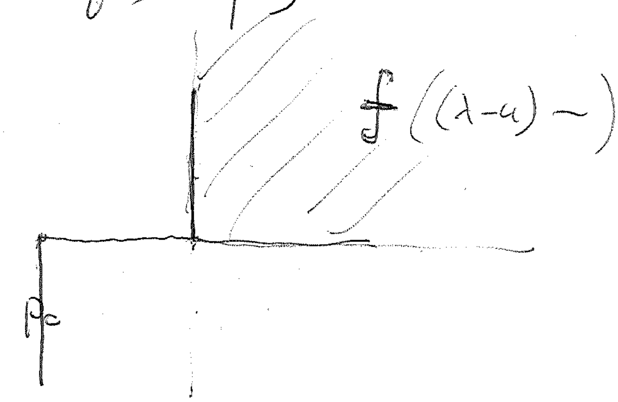
to be in l^2; if $|\lambda| < 1$, this implies $f(\xi'_-) = 0$.

So the appropriate boundary conditions for $\xi^*(\lambda - u)^{-1}$ when $|\lambda| < 1$, is $f(\xi'_-) = f(\xi_-) = 0$

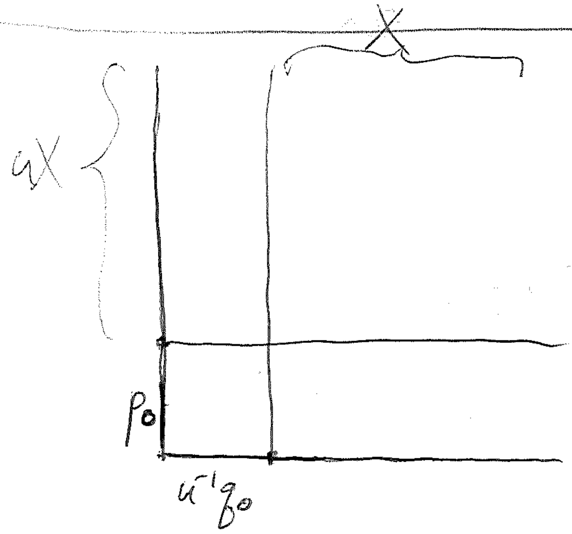
Continue. I think you now want to look at $p_0^*(\lambda - u)^{-1}$. This is a linear functional on E i.e. a solution of the grid equation. ~~can also say~~ This $p_0^*(\lambda - u)^{-1}$ is part of the Green's function "at $n=0$". This linear functional should be describable ~~in terms~~ by its values on the ascending staircase bases.

$$u p_{-2},\ \xi_{-1},\ p_0,\ u^{-1} q_1,\ u^{-1} p_2, \ldots$$





$$|\lambda| < 1. \qquad (\lambda-u)^{-1}\xi = -u^{-1}(1-\lambda u^{-1})^{-1}\xi = \sum_{n\geq 0} \lambda^n u^{-(n+1)}\xi$$



Y = whole quadrant

$X = (u^{-1}q_0)^{\perp} \qquad \xi_+ = u^{-1}q_0$

$uX = (p_0)^{\perp} \qquad \xi_- = p_0$

So look at the eigenvector equation for a λ $|\lambda| < 1$.

$$\underbrace{(\lambda-u)}\, x = -v^+ + v^-$$

$$(\lambda a - b)\, x = \qquad\qquad \text{apply } b^* \text{ gives}$$

$$(\lambda a b^* - 1)\, x = -b^* v^+$$

$$\begin{pmatrix} b & \xi_- \end{pmatrix}\begin{pmatrix} b^* \\ \xi_-^* \end{pmatrix} = 1$$

$$Y \xrightarrow{\begin{pmatrix} b^* \\ \xi_-^* \end{pmatrix}} \begin{matrix} X \\ + \\ \mathbb{C} \end{matrix} \xrightarrow{\begin{pmatrix} b^* & \xi_-^* \end{pmatrix}} Y$$

$$\begin{pmatrix} b-\lambda a & \xi_- \end{pmatrix}\begin{pmatrix} b^* \\ \xi_-^* \end{pmatrix} = 1 - \lambda a b^*$$

$$\begin{matrix} X \\ \oplus \\ \mathbb{C} \end{matrix} \xrightarrow{(b-\lambda a\ \ \xi_-)} Y \xrightarrow{(1-\lambda a b^*)^{-1}} Y \xrightarrow{\begin{pmatrix} b^* \\ \xi_-^* \end{pmatrix}} \begin{matrix} X \\ \oplus \\ \mathbb{C} \end{matrix}$$

$$\begin{matrix} X \\ \oplus \\ \mathbb{C} \end{matrix} \xrightarrow{(b-\lambda a\ \ \xi_-)} Y \xrightarrow{\begin{pmatrix} b^* \\ \xi_-^* \end{pmatrix}(1-\lambda ab^*)^{-1}} \begin{matrix} X \\ \oplus \\ \mathbb{C} \end{matrix}$$

You want to obtain $y \longmapsto \underbrace{\xi_-^*(1-\lambda ab^*)^{-1}y}_{=\tilde{y}(\lambda)}$

such that $y - \tilde{y}(\lambda)\xi_- \in (b-\lambda a)X$

If I take $y \longmapsto \begin{pmatrix} b^* \\ \xi_-^* \end{pmatrix}(1-\lambda ab^*)^{-1}y$

Then the claim is that

$$\xi_-\underbrace{\left(\xi_-^*(1-\lambda ab^*)^{-1}y\right)}_{\tilde{y}(\lambda)} + (b-\lambda a)\underbrace{b^*(1-\lambda ab^*)^{-1}y}_{x}$$

$$= y \qquad Y \in S.$$

In the end you want to know about

$$p_0^*\frac{1}{\lambda-u}y = p_0^*\frac{1}{\lambda-u}\left(p_0\tilde{y}(\lambda) + (u-\lambda)ab^*(1-\lambda ab^*)^{-1}y\right)$$

$$= \left(p_0^*\frac{1}{\lambda-u}p_0\right)\tilde{y}(\lambda) + p_0^*ab^*(1-\lambda ab^*)^{-1}y$$

Cont case $\quad S\psi = \begin{pmatrix} \partial_r & -m \\ \bar{m} & -\partial_r \end{pmatrix}\psi = (\varepsilon\partial_r + V)\psi$

so $\quad \partial_r\psi = (\varepsilon S - \varepsilon V)\psi = \begin{pmatrix} S & m \\ \bar{m} & -S \end{pmatrix}\psi$

You want ~~$(S-D)^{-1}\ \delta(r)$~~

$$(S - \varepsilon\partial_r - V)G = \delta(r)$$

$$\partial_r - \varepsilon S + \varepsilon V$$

$$s\psi = \begin{pmatrix} \partial_r & -m \\ \bar{m} & -\partial_r \end{pmatrix}\psi = (\varepsilon\partial_r + V)\psi$$

$$-\varepsilon s\psi = -\partial_r\psi - \varepsilon V\psi$$

$$(\partial_r - \varepsilon s + \varepsilon V)\psi = \delta$$

$$D = \begin{pmatrix} \partial_r & -m \\ \bar{m} & -\partial_r \end{pmatrix} = \varepsilon\partial_r + V$$

To solve where $V^* = -V$

$$(s - D)\,G = \delta(r)$$

$$\Vert$$

$$(s - \varepsilon\partial_r - V)\,G = \delta(r) \qquad \text{mult by } -\varepsilon$$

$$(-\varepsilon s + \partial_r + \varepsilon V)\,G = -\varepsilon\,\delta(r)$$

$$\boxed{(\partial_r - \varepsilon s + \varepsilon V)\,G = -\varepsilon\,\delta(r)}$$

Take $V = 0$ $\qquad \mathrm{Re}(s) > 0.$

$$\begin{pmatrix} e^{rs}\,\text{const} \\ 0 \end{pmatrix} \underset{r<0}{=} G(r) = \begin{pmatrix} 0 \\ e^{-rs}\cdot\text{const} \end{pmatrix} \qquad r > 0$$

$$G(0+) - G(0-) = -\varepsilon = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

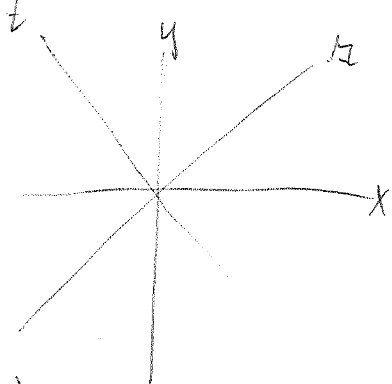
$$G(r) = \begin{pmatrix} H(-r)\,e^{rs} \\ H(r)\,e^{-rs} \end{pmatrix}$$

Idea In the scattering situation, grid space is pretty clear – it's a pair of functions of s. The Hilbert space completion $\bar{E}$ will be two copies of $L^2(\mathbb{R}, \frac{dp}{2\pi})$ depending on the bases you choose.

Analogy: In the case of $\partial_x \psi^1 = \psi^2$, $\partial_y \psi^2 = \psi^1$?

General case: General solution with time dep. e^{st} is
$\psi(r,s)e^{st}$ where $(s-D)\psi = 0$.

$$\begin{pmatrix} \partial_r - s & -m \\ \bar{m} & -\partial_r - s \end{pmatrix} \psi(r,s) = 0$$



$$\begin{pmatrix} \gamma e^{rs} \\ \delta e^{-rs} \end{pmatrix} \xleftarrow{r \to -\infty} \psi(r,s) \xrightarrow{r \to +\infty} \begin{pmatrix} \alpha e^{rs} \\ \beta e^{-rs} \end{pmatrix}$$

$\partial_x = \partial_r - \partial_t$
$\partial_y = \partial_r + \partial_t$
$r = x + y$
$t = -x + y$

The b. cond at $r \to -\infty$ is $\delta = 0$
_____ $r \to +\infty$ is $\alpha = 0$.

assuming $\mathrm{Re}(s) > 0$.

What are $\xi_\pm$ $\xi'_\pm$?

Start with wave equation $\begin{pmatrix} \partial_r - \partial_t & -m \\ -\bar{m} & \partial_r + \partial_t \end{pmatrix} \psi = 0$
where $m = m(r)$ is indep of t.
Problem: to solve Cauchy problem at $t=0$ via Laplace transform in t, converting ∂_t to mult by s.

$$\begin{pmatrix} -\partial_r + s & +m \\ -\bar{m} & \partial_r + s \end{pmatrix} \psi = 0$$

$$s - \begin{pmatrix} \partial_r & -m \\ \bar{m} & -\partial_r \end{pmatrix} = s - D$$

$$\partial_r \psi = \begin{pmatrix} s & m \\ \bar{m} & -s \end{pmatrix} \psi \qquad \partial_r \phi = \begin{pmatrix} 0 & m e^{-2rs} \\ \bar{m} e^{2rs} & 0 \end{pmatrix} \phi$$

$$\psi = \begin{pmatrix} e^{rs} & 0 \\ 0 & e^{-rs} \end{pmatrix} \phi$$

Think: to solve the wave equation via ~~the~~ L.T., you get $(s - D)(\mathcal{L}\psi)(r) = \psi_0(r)$ $\quad (\mathcal{L}\psi)(r,s) = \frac{1}{s-D}\psi_0(r)$

and the inverse Laplace transform gives

$$\psi(r,t) = \int_{-i\infty}^{i\infty} e^{st} \frac{1}{s-D} \psi_0(r) \frac{ds}{2\pi i}$$

which formally says $\psi(r,t) = e^{tD}\psi_0(r)$, somehow operator theory justifies this in some apparently non trivial way. It's probable that once you make sense of the ~~function~~ $\left(\frac{1}{s-D}\psi_0\right)(r)$, that then the contour integral above yields the correct $\psi(r,t)$.

Now as for $\frac{1}{s-D}\psi_0$, this should reduce to the Green's function for $\underbrace{\text{a}}_{\text{first order}}$ ODE.

What seems to happen is that the Green function jumps ~~as one~~ on $Re(s) = 0$ and the jump at $i\rho$ is the spectral projection.

Think: solutions of $\partial_t \psi = D_r \psi$ originally, now $(s-D)\psi = 0$. Take $Re(s) > 0$. You what the the Green's function at position r_0 looks like. You have decaying solutions on the left and on the right

$$\psi_\ell(r) = \begin{pmatrix} e^{rs} \\ 0 \end{pmatrix} \quad \text{for } r \ll 0$$

and $\quad \psi_r(r) = \begin{pmatrix} 0 \\ e^{-rs} \end{pmatrix} \quad$ for $r \geq 0$.

Take compact support $m(r)$ to simplify. You want ~~s to be~~ the limiting case where $s \in i\mathbb{R}$. Let's tentatively use $\pm$ to mean $\mathrm{Re}(s)\pm$. Then you have 4 solutions

$$\psi_{l,+} \underset{r\to-\infty}{\sim} \begin{pmatrix} e^{rs} \\ 0 \end{pmatrix} \qquad \psi_{rt,+} \underset{r\to+\infty}{\sim} \begin{pmatrix} 0 \\ e^{-rs} \end{pmatrix}$$

$$\psi_{l,-} \underset{r\to-\infty}{\sim} \begin{pmatrix} 0 \\ e^{-rs} \end{pmatrix} \qquad \psi_{rt,-} \sim \begin{pmatrix} e^{rs} \\ 0 \end{pmatrix}$$

these are specific solutions to $(s-D)\psi = 0$.

Convert from ψ to ϕ defined by $\psi = \begin{pmatrix} e^{rs} & \\ & e^{-rs} \end{pmatrix}\phi$

$r \ll 0 \qquad\qquad r \gg 0$

$$\phi_{l,+} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \phi_{rt,+} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\phi_{l,-} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad \phi_{rt,-} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Describe in words what you want to do. The aim is to construct $G(r) = \frac{1}{s-D}\delta(r)$. This means you give

μ, ν const. row vectors

means $\mathrm{Re}(s) \geq 0^+$

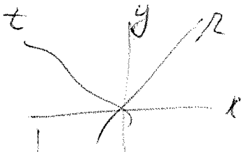
$$G_+(r) = \psi_{l,+}(r)\,\mu \qquad r < 0$$

$$= \psi_{rt,+}(r)\,\nu \qquad r > 0$$

At $r = 0$ you want

$$\psi_{l,+}(0)\mu - \psi_{rt,+}(0)\nu = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \quad -s ?$$

Go back to $\partial_x \psi^1 = \psi^2 = (\partial_r - \partial_t)\psi^1$
$\partial_y \psi^2 = \psi^1 = (\partial_r + \partial_t)\psi^2$



time dependence e^{st}

$$\partial_t \psi = \underbrace{\begin{pmatrix} \partial_r & -1 \\ 1 & -\partial_r \end{pmatrix}}_{D} \psi \qquad \Bigg| \qquad \partial_r \psi = \begin{pmatrix} \partial_t & 1 \\ 1 & -\partial_t \end{pmatrix} \psi$$

Construct G for fun

$$(s - D)\,G = \delta(r) \qquad \left(s - \begin{pmatrix} ik & -1 \\ 1 & -ik \end{pmatrix} \right) \hat{G} = 1$$

$s = i\rho$

$$\left[\rho - \underbrace{\begin{pmatrix} k & i \\ -i & -k \end{pmatrix}}_{A_k} \right]^{-1} = \frac{1}{\rho - \omega_k} \left(\frac{\omega_k + A_k}{2\omega_k} \right) + \frac{1}{\rho + \omega_k} \left(\frac{\omega_k - A_k}{2\omega_k} \right)$$

$$A_k^2 = (k^2 + 1) = \omega_k^2$$

This seems misguided — too hard

Go back to

$$\partial_x \psi^1 = (\partial_r - \partial_t)\psi^1 = m\psi^2$$
$$\partial_y \psi^2 = (\partial_r + \partial_t)\psi^2 = \bar{m}\psi^1$$

$$\partial_r \psi = \begin{pmatrix} \partial_t & m \\ \bar{m} & -\partial_t \end{pmatrix} \psi$$

Assume time dependence e^{st}

$m = m(r)$
comp. support.

$$\psi(r, s) = \begin{pmatrix} e^{sr} & 0 \\ 0 & e^{-sr} \end{pmatrix} \phi(r, s)$$

ϕ constant in r for $r \gg 0$ or $r \ll 0$.

For each $s \in \mathbb{C}$ you get a 2 dim'l solution space W_s, four natural coords on W_s, namely the values of ϕ^1, ϕ^2 for $r = \pm\infty$. You now want to get control of the scattering and the G function.

To simplify suppose s purely imag. Instead of decay in space look for incoming + outgoing waves. Decay in space is easy i.e. for $\mathrm{Re}(s) > 0$

you want $\psi = \begin{pmatrix} e^{sr} \\ 0 \end{pmatrix}$ const on the left and $\begin{pmatrix} 0 \\ e^{-sr} \end{pmatrix}$ const on the right.

But ~~...~~ consider time dependence e^{st}, where s purely imag. On the left we have a wave

$$\begin{pmatrix} e^{s(r+t)} \\ 0 \end{pmatrix}$$ which is outgoing

and ~~on the right~~ also $\begin{pmatrix} 0 \\ e^{-sr+st} \end{pmatrix}$ incoming.

On the right $\begin{pmatrix} 0 \\ e^{-sr+st} \end{pmatrix}$ outgoing

and $\begin{pmatrix} e^{s(r+t)} \\ 0 \end{pmatrix}$ incoming. Thus spatial decay seems linked to outgoing. The condition that $Re(s) = 0^+$ (I mean you decide decay by picking the RHP and then allow s to go to boundary), this condition amounts to outgoing waves

Wrong convention probably because radiative modes have exp time decay and exp growth in space.
Can change time dependence to $e^{-i\omega t}$

Repeat: $\partial_t \psi = \begin{pmatrix} \partial_r & -m \\ \bar{m} & -\partial_r \end{pmatrix} \psi \qquad s\psi = D\psi$

$$\partial_t \psi = (\varepsilon \partial_r + V)\psi \qquad \partial_r \psi = (\varepsilon \partial_t - \varepsilon V)\psi$$

$$\partial_r \psi = \begin{pmatrix} \partial_t & m \\ \bar{m} & -\partial_t \end{pmatrix} \psi \qquad \partial_r \psi = \begin{pmatrix} s & m \\ \bar{m} & -s \end{pmatrix} \psi \qquad g_s = \begin{pmatrix} e^{sr} & 0 \\ 0 & e^{-sr} \end{pmatrix}$$

$$\left(\partial_r + \begin{pmatrix} s & 0 \\ 0 & -s \end{pmatrix}\right)\phi = g_s^{-1} \partial_r g_s \phi = \begin{pmatrix} s & m e^{-2rs} \\ m e^{2rs} & -s \end{pmatrix} \phi$$

$$\partial_r\psi = \begin{pmatrix} s & m \\ \bar{m} & -s \end{pmatrix}\psi \qquad \partial_r\phi = \begin{pmatrix} 0 & me^{-2sr} \\ \bar{m}e^{2sr} & 0 \end{pmatrix}\phi$$

2diml space W_s eigenf.

m decays hyp. $\Longrightarrow$ $\phi(\pm\infty)$ exist.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \Phi(\infty,-\infty) = \Phi(\infty,0)\,\Phi(0,-\infty) = \begin{pmatrix} a^r & b^r \\ c^r & d^r \end{pmatrix}\begin{pmatrix} a^l & b^l \\ c^l & d^l \end{pmatrix}$$

for any ~~solution~~ eigenfn: $\phi(\infty) = \Phi(\infty,-\infty)\,\phi(-\infty)$.

$$\begin{pmatrix} \phi^1(\infty) \\ \phi^2(\infty) \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} \phi^1(-\infty) \\ \phi^2(-\infty) \end{pmatrix}$$

$$\begin{pmatrix} \phi^1(\infty) \\ \phi^2(-\infty) \end{pmatrix} = \begin{pmatrix} \frac{1}{d} & \frac{b}{d} \\ -\frac{c}{d} & \frac{1}{d} \end{pmatrix}\begin{pmatrix} \phi^1(-\infty) \\ \phi^2(\infty) \end{pmatrix}$$

It seems that your ξ'_-, ξ_- are the linear fns on eigenfunctions $\phi \longmapsto \phi^1(-\infty)$, $\phi \longmapsto \phi^2(\infty)$.

Now look at decay, incoming & outgoing. Consider e.f. $\psi(r,s) \sim \begin{pmatrix} e^{sr} & 0 \\ 0 & e^{-sr} \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$

equiv. wave eqn. soln. $\psi(r,s) \sim \begin{pmatrix} e^{s(r+t)} & 0 \\ 0 & e^{s(-r+t)} \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$

Then ψ^1 is left moving
ψ^2 — rt —.

For the Green's fn. G you want decaying eigenfunctions on left + rt. Look at $r \gg 0$

and $\psi(r,s) = \begin{pmatrix} e^{sr} & 0 \\ 0 & e^{-sr} \end{pmatrix}\begin{pmatrix} \phi^1(\infty) \\ \phi^2(\infty) \end{pmatrix}$ If $s = 0^+ + i\omega$

then e^{sr} grows
e^{-sr} decays.

Table ~~[illegible]~~

$r \ll 0$

$$\begin{pmatrix} e^{s(r+t)} \\ 0 \end{pmatrix} \underbrace{\phi^1(-\infty)}_{=\phi(\xi'_-)}$$ left moving (outgoing) decaying spatially for $Re(s)>0$

$$\begin{pmatrix} 0 \\ e^{s(-r+t)} \end{pmatrix} \underbrace{\phi^2(-\infty)}_{=\phi(\xi'_+)}$$ right moving (incoming) decaying spatially for $Re(s)<0$

$r \gg 0$

$$\begin{pmatrix} e^{s(r+t)} \\ 0 \end{pmatrix} \underbrace{\phi^1(+\infty)}_{=\phi(\xi_+)}$$ left moving (incoming) decaying spatially for $Re(s) < 0$

$$\begin{pmatrix} 0 \\ e^{s(-r+t)} \end{pmatrix} \underbrace{\phi^2(+\infty)}_{=\phi(\xi_-)}$$ right moving (outgoing) decays spatially for $Re(s) > 0$

~~So it appears that~~

Repeat : $\partial_t \psi = \begin{pmatrix} \partial_r & -m \\ \bar{m} & -\partial_r \end{pmatrix} \psi \qquad \partial_r \psi = \begin{pmatrix} s & m \\ \bar{m} & -s \end{pmatrix} \psi$

$$\psi = g_s \phi \qquad g_s = \begin{pmatrix} e^{sr} & 0 \\ 0 & e^{-sr} \end{pmatrix} \qquad \partial_r \phi = \begin{pmatrix} 0 & m e^{-2sr} \\ \bar{m} e^{2sr} & 0 \end{pmatrix} \phi$$

$$-\infty \longleftarrow r \longmapsto +\infty$$

$$\begin{pmatrix} e^{sr} \phi^1(-\infty) \\ e^{-sr} \phi^2(-\infty) \end{pmatrix} \sim \psi(r) \sim \begin{pmatrix} e^{sr} \phi^1(\infty) \\ e^{-sr} \phi^2(\infty) \end{pmatrix}$$

$$\begin{pmatrix} \phi^1(-\infty) \\ \phi^1(-\infty) \end{pmatrix} \longleftarrow \phi(r) \longrightarrow \begin{pmatrix} \phi^1(\infty) \\ \phi^2(\infty) \end{pmatrix}$$

$$\phi(\infty) = \Phi(\infty, 0)\, \Phi(0, -\infty)\, \phi(-\infty)$$

$$\Phi(+\infty, -\infty) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

4 dist. linear functionals on the eigenspace W_s $\phi \mapsto \phi^i(\pm\infty)$

$$\begin{pmatrix} \phi^1(+\infty) \\ \phi^2(+\infty) \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \phi^1(-\infty) \\ \phi^2(-\infty) \end{pmatrix} \qquad \begin{pmatrix} \xi_+ \\ \xi_- \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \xi'_- \\ \xi'_+ \end{pmatrix}$$

$\xi_+ : \phi \longmapsto \phi^1(\infty)$ $\qquad \xi'_- : \phi \longmapsto \phi^1(-\infty)$

$\xi_- : \phi \longmapsto \phi^2(+\infty)$ $\qquad \xi'_+ : \phi \longmapsto \phi^2(-\infty)$

You are thinking of grid space as a space of dual sections of a vector bundle of eigenspaces.

Table: If you use time dependence e^{st}:

$$\psi(r,t) = e^{st}\psi(r,s) = e^{st}\begin{pmatrix} e^{sr} & 0 \\ 0 & e^{-sr} \end{pmatrix}\phi(r,s) = \begin{pmatrix} e^{s(t+r)} & 0 \\ 0 & e^{s(t-r)} \end{pmatrix}\phi$$

Far out ψ^1 is ~~incom~~ left-moving
ψ^2 ——— rt - ——

You also have notions of decay spatially

$r \to -\infty$ $\begin{pmatrix} e^{sr} \\ 0 \end{pmatrix}$ decays for Re(s) > 0 — left-moov outgoing to left

$r \to -\infty$ $\begin{pmatrix} 0 \\ e^{-sr} \end{pmatrix}$ ——— Re(s) < 0 — right moov incoming left

$r \to +\infty$ $\begin{pmatrix} e^{sr} \\ 0 \end{pmatrix}$ decays — Re(s) < 0 — left moov inc. rt

$r \to -\infty$ $\begin{pmatrix} 0 \\ e^{-sr} \end{pmatrix}$ ——— Re(s) > 0 — rt moov out rt

What is the problem? You begin with the wave eqn. $\boxed{\partial_t \psi = \begin{pmatrix} \partial_r & -m \\ \bar{m} & -\partial_r \end{pmatrix}\psi}$

$$\partial_t \psi = \varepsilon \partial_r \psi + V\psi \qquad \partial_r \psi = +\varepsilon \partial_t \psi - \varepsilon V \psi$$

assume time dep. $\psi(r,t) = \psi(r,s)e^{st}$ $\boxed{\partial_r \psi = \begin{pmatrix} \partial_t & m \\ \bar{m} & -\partial_t \end{pmatrix}\psi.}$

$$\partial_r \psi = \begin{pmatrix} s & m \\ \bar{m} & -s \end{pmatrix}\psi, \quad \psi = g_s \phi, \quad \partial_r \phi = \begin{pmatrix} 0 & me^{-2sr} \\ \bar{m}e^{2sr} & 0 \end{pmatrix}\phi$$

assume $m(r)$ decays suff. for good asymptotics $\phi(+\infty)$, $\phi(-\infty)$ to exist, at least for s suff near to $i\mathbb{R}$. ~~significant~~

You are ~~interested in~~ concerned with the Green's fn. $G = \frac{1}{s-D}\delta(r)$. This defined for $\mathrm{Re}(s) \neq 0$. Constructed using "the" eigenfunctions $\in W_s$ which decays as $r \to -\infty$ and the one decaying as $r \to +\infty$.

This decay idea, which is defined for $\mathrm{Re}(s) \neq 0$, is linked to incoming + outgoing notions

incoming on left	$\psi(r,t) \sim \begin{pmatrix} 0 \\ e^{s(-r+t)} \end{pmatrix} \phi^2(-\infty)$	decays ($r \to -\infty$) for $\mathrm{Re}(s) < 0$
outgoing on left	$\psi(r,t) \sim \begin{pmatrix} e^{s(r+t)} \\ 0 \end{pmatrix} \phi^1(-\infty)$	——— $\mathrm{Re}(s) > 0$
~~outgoing~~ outgoing on right	$\psi(r,t) \sim \begin{pmatrix} 0 \\ e^{s(-r+t)} \end{pmatrix} \phi^2(+\infty)$	decays ($r \to +\infty$) for $\mathrm{Re}(s) > 0$
incoming on right	$\psi(r,t) \sim \begin{pmatrix} e^{s(r+t)} \\ 0 \end{pmatrix} \phi^1(+\infty)$	——— $\mathrm{Re}(s) < 0$

outgoing means spatial decay for $\mathrm{Re}(s) > 0$

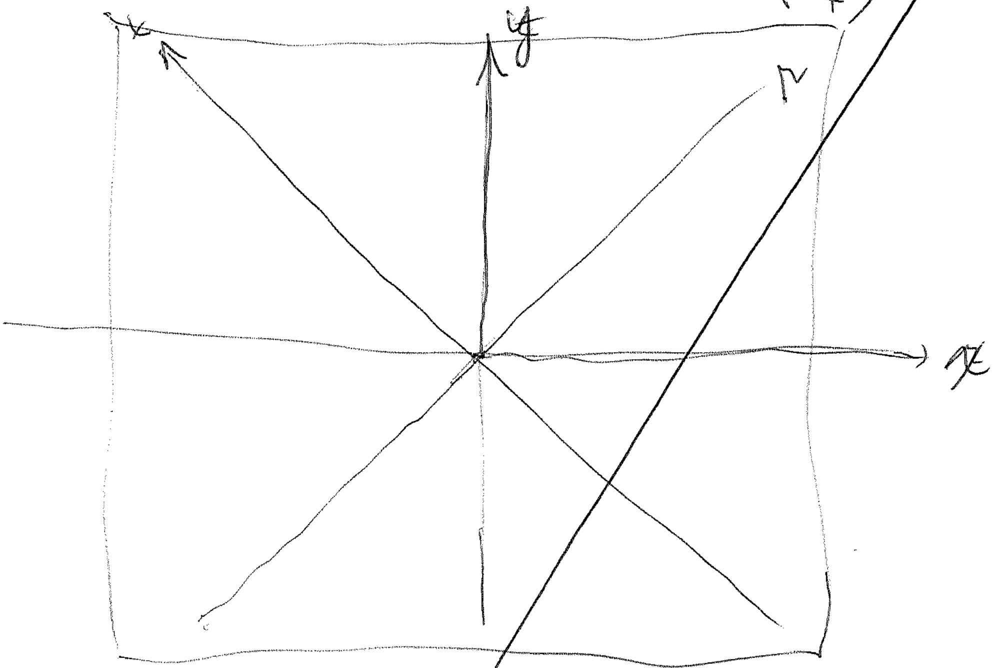
transfer: $\partial_r \phi = \begin{pmatrix} 0 & m e^{-2sr} \\ \bar m e^{2sr} & 0 \end{pmatrix}\phi$

> Point: There's a pairing between elements of grid space and solutions of the wave equation.

$$\phi(+\infty) = \underbrace{\Phi(\infty,-\infty)}\,\phi(-\infty)$$

$$\begin{pmatrix}\xi_+\\ \xi_-\end{pmatrix} = \begin{pmatrix} a & b\\ c & d\end{pmatrix}\begin{pmatrix}\xi'_-\\ \xi'_+\end{pmatrix}$$

$$\begin{pmatrix}\xi_+\\ \xi_-\end{pmatrix}\phi = \begin{pmatrix}\phi^1(\infty)\\ \phi^2(\infty)\end{pmatrix} \qquad \begin{pmatrix}\xi'_-\\ \xi'_+\end{pmatrix}\phi = \begin{pmatrix}\phi^1(-\infty)\\ \phi^2(-\infty)\end{pmatrix}$$



$$\begin{pmatrix} z^{-n}p_n \\ q_n\end{pmatrix} = \frac{1}{R_n}\begin{pmatrix}1 & h_n z^{-n}\\ \bar h_n z^{n} & 1\end{pmatrix}\begin{pmatrix} z^{-n+1}p_{n-1}\\ q_{n-1}\end{pmatrix}$$

certainly this gives

$$\begin{pmatrix}\xi_+\\ \xi_-\end{pmatrix} = \begin{pmatrix} a & b\\ c & d\end{pmatrix}\begin{pmatrix}\xi'_-\\ \xi'_+\end{pmatrix}$$

$\xi_+ = \lim_{n\to+\infty} z^{-n}p_n$

$\xi'_- = \lim_{n\to-\infty} z^{-n}p_n$

apply a linear fnl. ϕ killing $(S-z)E$ to get

$$\begin{pmatrix}\phi(\xi_+)\\ \phi(\xi_-)\end{pmatrix} = \begin{pmatrix} & \\ & \end{pmatrix}\begin{pmatrix}\phi(\xi'_-)\\ \phi(\xi'_+)\end{pmatrix}$$

$\phi^1(+\infty)$

What is the problem? Something is not working. You have $\psi \in W_s$ i.e.

$$\partial_r \psi = \begin{pmatrix} s & m \\ \bar{m} & -s \end{pmatrix} \psi \quad \text{or} \quad \partial_r \phi = \begin{pmatrix} 0 & me^{-2sr} \\ \bar{m}e^{2sr} & 0 \end{pmatrix} \phi$$

You look at the 2 diml space of eigenfns. $\psi(r)$ satisfying (s fixed) equiv the 2 diml sp. of $\phi(r)$ sat

~~Hat Wave grid~~

Move to discrete case. Generators for grid space are $z^{-n}p_n$, q_n $n \in \mathbb{Z}$. An element of W_λ is a solution ϕ_n of

$$\phi_n = \frac{1}{k_n}\begin{pmatrix} 1 & h_n\lambda^{-n} \\ \bar{h}_n\lambda^n & 1 \end{pmatrix} \phi_{n-1}$$

Four linear functionals $\phi^i_{\pm\infty}$

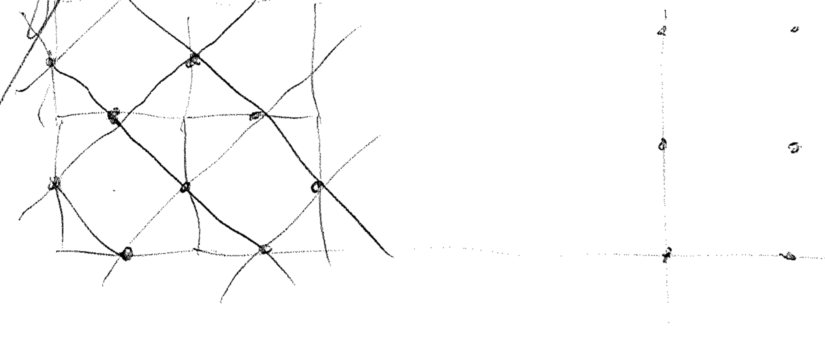
$$\begin{pmatrix} \phi^1_\infty \\ \phi^2_\infty \end{pmatrix} = \begin{pmatrix} a(\lambda) & b(\lambda) \\ c(\lambda) & d(\lambda) \end{pmatrix} \begin{pmatrix} \phi^1_{-\infty} \\ \phi^2_{-\infty} \end{pmatrix}$$

$$\phi_\infty = \phi\begin{pmatrix} \xi_+ \\ \xi_- \end{pmatrix}$$

$$\phi_{-\infty} = \phi\begin{pmatrix} \xi'_- \\ \xi'_+ \end{pmatrix}$$

Problem: To construct understand the G fn.

You problem is to construct, understand the Green's function. In the cont case $G = \frac{1}{s-D}\delta_{r_0}$



the reason you avoided this is that only half the squares give relations return to cont. case. $\partial_t \psi = \begin{pmatrix} \partial_r & -m \\ \bar{m} & -\partial_r \end{pmatrix} \psi$

$\partial_t \psi = \varepsilon \partial_r \psi + V\psi$

$\partial_r \psi = \varepsilon \partial_t \psi - \varepsilon V \psi$ $\qquad \partial_r \psi = \begin{pmatrix} \partial_t & m \\ \bar{m} & -\partial_t \end{pmatrix} \psi$

t-dependence $\psi(r,t) = e^{st}\psi(r,s)$ yields

$\partial_r \psi = \begin{pmatrix} s & m \\ \bar{m} & -s \end{pmatrix} \psi$, subs. $\psi = \begin{pmatrix} e^{sr} & 0 \\ 0 & e^{-sr} \end{pmatrix} \phi$ to get

$$\star \quad \boxed{\partial_r \phi = \begin{pmatrix} 0 & m e^{-2sr} \\ \bar{m} e^{2sr} & 0 \end{pmatrix} \phi}$$

But you want G fn. which involves boundary conditions. To work more with ϕ than before. Focus on the G function at $\underline{s, r=0}$. ~~[illegible]~~

This is constructed from two solutions of $\star$ which satisfy appropriately decaying b/c. as $r \to +\infty$ or $-\infty$ with a jump at $r=0$.

Left solution $\qquad$ right solution

$\phi = \Phi(r, -\infty)\begin{pmatrix} ? \\ ? \end{pmatrix}$ $\qquad$ $\begin{pmatrix} ? \\ ? \end{pmatrix}$ must satisfy the b.c. at $r = -\infty$.

~~$\psi(0) = \begin{pmatrix} e^{sr} & 0 \\ 0 & e^{-sr} \end{pmatrix}\begin{pmatrix} a^l & b^l \\ c^l & d^l \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$~~

$$\psi(r) = \begin{pmatrix} e^{sr} & 0 \\ 0 & e^{-sr} \end{pmatrix} \Phi(r, -\infty) \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \xrightarrow{r \to -\infty} \sim \begin{pmatrix} e^{sr} \\ 0 \end{pmatrix} \text{const} \quad Re(s) > 0$$

$$\phi(-\infty) = \begin{pmatrix} a^l & b^l \\ c^l & d^l \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{const.}$$

right solution

$$\phi(r) = \Phi(r,\infty)\begin{pmatrix}\gamma\\ \delta\end{pmatrix} \quad (?) \quad \text{must sat b.c. at } r=+\infty.$$

$$\psi(r) = \begin{pmatrix}e^{sr} & 0\\ 0 & e^{-sr}\end{pmatrix}\phi(r) = \begin{pmatrix}e^{sr} & 0\\ 0 & e^{-sr}\end{pmatrix}\Phi(r,\infty)\begin{pmatrix}\gamma\\ \delta\end{pmatrix} \overset{r\to+\infty}{\sim} \begin{pmatrix}0\\ e^{-sr}\end{pmatrix}\text{const}$$

$$\phi(+\infty) = \begin{pmatrix}d^{rt} & -b^{rt}\\ -c^{rt} & a^{rt}\end{pmatrix}\begin{pmatrix}\gamma\\ \delta\end{pmatrix} = \begin{pmatrix}0\\ 1\end{pmatrix}\text{const.}$$

Again $\partial_r\psi = (\varepsilon S + V)\psi = \begin{pmatrix}s & m\\ \bar m & -s\end{pmatrix}\psi \qquad \psi = g_s\phi = \begin{pmatrix}e^{sr} & 0\\ 0 & e^{-sr}\end{pmatrix}\phi$

$$\partial_r\phi = \begin{pmatrix}0 & me^{-2sr}\\ \bar m e^{2sr} & 0\end{pmatrix}\phi \qquad D = \begin{pmatrix}\partial_r & -m\\ \bar m & -\partial_r\end{pmatrix} = \varepsilon\partial_r + V$$

For G you want eigenfunctions ψ which decay

at $r=-\infty$: $\qquad \psi_<(r) = \begin{pmatrix}e^{sr} & 0\\ 0 & e^{-sr}\end{pmatrix}\Phi(r,-\infty)\begin{pmatrix}*\\ 0\end{pmatrix}$

at $r=+\infty$ $\qquad \psi_>(r) = \begin{pmatrix}e^{sr} & 0\\ 0 & e^{-sr}\end{pmatrix}\Phi(r,\infty)\begin{pmatrix}0\\ *\end{pmatrix}$ assuming $\mathrm{Re}(s)>0$

$$\left(s - \varepsilon\partial_r - V\right)G = \delta(r)\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} \therefore -\varepsilon\left(G(0^+)-G(0^-)\right) = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$$

$$\left\{\underbrace{\Phi(0,\infty)}_{\begin{pmatrix}d^{r} & -b^{r}\\ -c^{r} & a^{r}\end{pmatrix}}\begin{pmatrix}0\\ *\end{pmatrix} - \underbrace{\Phi(0,-\infty)}_{\begin{pmatrix}a^{\ell} & b^{\ell}\\ c^{\ell} & d^{\ell}\end{pmatrix}}\begin{pmatrix}*\\ 0\end{pmatrix}\right\} = \begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}$$

$$\begin{pmatrix}p_0\\ q_0\end{pmatrix} = \begin{pmatrix}a^{\ell} & b^{\ell}\\ c^{\ell} & d^{\ell}\end{pmatrix}\begin{pmatrix}\xi_-'\\ \xi_+'\end{pmatrix} = \begin{pmatrix}d^{r} & -b^{r}\\ -c^{r} & a^{r}\end{pmatrix}\begin{pmatrix}\xi_+\\ \xi_-\end{pmatrix}$$

$$= \begin{pmatrix}a^{\ell} & -b^{r}\\ c^{\ell} & a^{r}\end{pmatrix}\begin{pmatrix}\xi_+\\ \xi_+'\end{pmatrix} = \frac{1}{d}\begin{pmatrix}d^{r} & b^{\ell}\\ -c^{r} & d^{\ell}\end{pmatrix}\begin{pmatrix}\xi_-'\\ \xi_-\end{pmatrix}$$

So

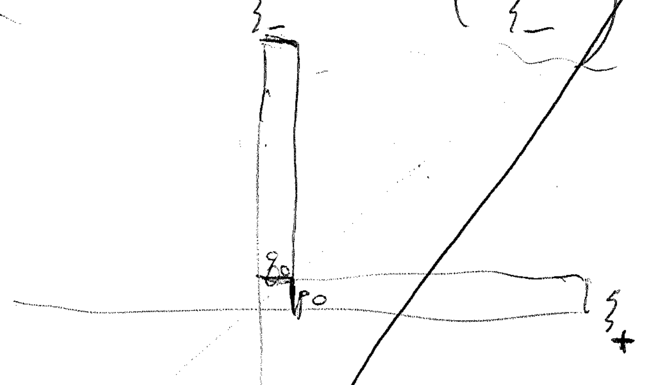
$$-G_<(0) = -\Phi(0,-\infty)\begin{pmatrix} x_1 & x_2 \\ 0 & 0 \end{pmatrix} = -\begin{pmatrix} a^\ell & b^\ell \\ c^\ell & d^\ell \end{pmatrix}\begin{pmatrix} x_1 & x_2 \\ 0 & 0 \end{pmatrix}$$

$$G_>(0) = \Phi(0,\infty)\begin{pmatrix} 0 & 0 \\ y_1 & y_2 \end{pmatrix} = \begin{pmatrix} d^r & -b^r \\ -c^r & a^r \end{pmatrix}\begin{pmatrix} 0 & 0 \\ y_1 & y_2 \end{pmatrix}$$

$$G_>(0) - G_<(0) = \begin{pmatrix} a^\ell & -b^r \\ c^\ell & a^r \end{pmatrix}\begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

Center things at 0.

You want
$$\begin{pmatrix} \xi_+ \\ \xi_- \end{pmatrix} = \begin{pmatrix} a^r & b^r \\ c^r & d^r \end{pmatrix}\begin{pmatrix} p_0 \\ q_0 \end{pmatrix}$$



$$\xi_- = \underset{zH_+}{\hat{c^r}} p_0 + \underset{H_+}{\hat{d^r}} q_0$$

$$\xi_+ = \underset{zH_-}{\hat{a^r}} p_0 + \underset{H_-}{\hat{b^r}} q_0$$

So your notation seems to be lousy. Ignore it. Go over what you know.

Basically you have $\overset{\text{Diff O}}{D} = \varepsilon\partial_r + V$ $\quad V = \begin{pmatrix} 0 & -m \\ m & 0 \end{pmatrix}$
operating on functions $\phi(r)$, formally self-adj.

and you want to ~~or~~ construct the resolvent so as to do its spectral decomposition. Do by eigenfunctions. $s\psi = D\psi = (\varepsilon\partial_r + V)\psi$, write

$$\partial_r \psi = (\varepsilon s - \varepsilon V)\psi = \begin{pmatrix} s & m \\ \bar{m} & -s \end{pmatrix}\psi, \text{ ~~substitute~~ } \psi = \underbrace{\begin{pmatrix} e^{sr} & 0 \\ 0 & e^{-sr} \end{pmatrix}}_{g_s}\phi$$

$$\partial_r \phi = \begin{pmatrix} 0 & me^{-2sr} \\ \bar{m}e^{2sr} & 0 \end{pmatrix}\phi.$$ ~~Now you want~~

Have propagator $\Phi(r, r')$ with limits at $r = \pm\infty$.

You solve $(s-D)G = \delta(r)$ G satisfies decaying b.c.

$$(s - \varepsilon\partial_r - V)G = \delta(r) \qquad (\partial_r - \varepsilon s + \varepsilon V)G = -\varepsilon\delta(r)$$

$$G_< = \begin{pmatrix} e^{sr} & 0 \\ 0 & e^{-sr} \end{pmatrix}\Phi(r, -\infty)\begin{pmatrix} x_1 & x_2 \\ 0 & 0 \end{pmatrix}$$

$$G_> = \begin{pmatrix} & \\ & \end{pmatrix}\Phi(r, \infty)\begin{pmatrix} 0 & 0 \\ y_1 & y_2 \end{pmatrix}$$

because $G_<(r) \sim \begin{pmatrix} e^{sr} & 0 \\ 0 & e^{-sr} \end{pmatrix}\begin{pmatrix} x_1 & x_2 \\ 0 & 0 \end{pmatrix}$ decays as $r \to -\infty$ when $Re(s) > 0$.

$G_>(r) \sim \begin{pmatrix} e^{sr} & \\ & e^{-sr} \end{pmatrix}\begin{pmatrix} 0 & 0 \\ y_1 & y_2 \end{pmatrix}$ decays as $r \to +\infty$ for $Re(s) > 0$

Now we know $\Phi(r, -\infty) = g_s\begin{pmatrix} a^l & b^l \\ c^l & d^l \end{pmatrix}$ l means left of 0

Describe $\Phi(r, -\infty)$ better

$$= \begin{pmatrix} e^{sr} & 0 \\ 0 & e^{-sr} \end{pmatrix}\Phi(r, -\infty)$$

You want

go over the ~~[illegible]~~ construction of $G = (s-D)^{-1}\delta(r) I$ $\qquad (s - \varepsilon\partial_r - V) G = \delta(r)$

$$G_<(r) = \begin{pmatrix} e^{sr} & 0 \\ 0 & e^{-sr} \end{pmatrix} \Phi(r,-\infty)\, M_< \sim \begin{pmatrix} e^{sr} & 0 \\ 0 & e^{-sr} \end{pmatrix} M_< \quad r \to -\infty$$

$$G_>(r) = \begin{pmatrix} & \\ & \end{pmatrix} \Phi(r,\infty)\, M_> \sim \begin{pmatrix} & \\ & \end{pmatrix} M_> \quad r \to +\infty$$

If $\operatorname{Re}(s) > 0$, then decay on the left means no e^{-sr}
rt e^{sr}

so $\quad M_< = \begin{pmatrix} x_1 & x_2 \\ 0 & 0 \end{pmatrix} \qquad M_> = \begin{pmatrix} 0 & 0 \\ y_1 & y_2 \end{pmatrix}$

$$G_<(0^-) = \Phi(0,-\infty)\begin{pmatrix} x_1 & x_2 \\ 0 & 0 \end{pmatrix}$$

$$G_>(0^+) = \Phi(0,\infty)\begin{pmatrix} 0 & 0 \\ y_1 & y_2 \end{pmatrix}$$

$$G_>(0^+) - G_>(0^-) = -\varepsilon = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= -\Phi(0,-\infty)\begin{pmatrix} x_1 & x_2 \\ 0 & 0 \end{pmatrix} + \Phi(0,\infty)\begin{pmatrix} 0 & 0 \\ y_1 & y_2 \end{pmatrix}$$

$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} d^r & -b^r \\ -c^r & a^r \end{pmatrix}\begin{pmatrix} 0 & 0 \\ y_1 & y_2 \end{pmatrix} - \begin{pmatrix} a^l & b^l \\ c^l & d^l \end{pmatrix}\begin{pmatrix} x_1 & x_2 \\ 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & -b^r \\ 0 & a^r \end{pmatrix}\begin{pmatrix} 0 & 0 \\ y_1 & y_2 \end{pmatrix} - \begin{pmatrix} a^l & 0 \\ c^l & 0 \end{pmatrix}\begin{pmatrix} x_1 & x_2 \\ 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} -b^r y_1 & -b^r y_2 \\ a^r y_1 & a^r y_2 \end{pmatrix} - \begin{pmatrix} a^l x_1 & a^l x_2 \\ c^l x_1 & c^l x_2 \end{pmatrix}$$

$$= \left(\begin{pmatrix} -b^r & a^l \\ a^r & c^l \end{pmatrix}\begin{pmatrix} y_1 \\ x_1 \end{pmatrix} \quad \begin{pmatrix} -b^r & a^l \\ a^r & c^l \end{pmatrix}\begin{pmatrix} y_2 \\ x_2 \end{pmatrix} \right) = \begin{pmatrix} -b^r & a^l \\ a^r & c^l \end{pmatrix}\begin{pmatrix} y_1 & y_2 \\ x_1 & x_2 \end{pmatrix}$$

Green's function. $D = \varepsilon \partial_r + V$, $(s - \varepsilon\partial_r - V)\psi = 0$ 410

$G = (s-D)^{-1}\delta(r)I.$ $\partial_r \psi = (\varepsilon s - \varepsilon V)\psi = \begin{pmatrix} s & -m \\ \bar m & -s \end{pmatrix}\psi$

$\psi = \begin{pmatrix} e^{sr} & 0 \\ 0 & e^{-sr} \end{pmatrix}\phi$, $\partial_r \phi = \begin{pmatrix} 0 & m e^{-2sr} \\ \bar m e^{2sr} & 0 \end{pmatrix}\phi.$

$$G_< = \begin{pmatrix} e^{sr} & 0 \\ 0 & e^{-sr} \end{pmatrix} \Phi(r,-\infty) M_< \qquad \mathrm{Im}(M_<) \subset \begin{pmatrix} * \\ 0 \end{pmatrix}$$

$$G_> = \begin{pmatrix} & \\ & \end{pmatrix} \Phi(r,\infty) M_> \qquad \mathrm{Im}(M_>) \subset \begin{pmatrix} 0 \\ * \end{pmatrix}$$

$$G_>(0^+) - G_<(0^-) = -\varepsilon$$

why not ~~describe~~ introduce the columns.

$$\Phi(r,-\infty) = \begin{pmatrix} a_< & b_< \\ c_< & d_< \end{pmatrix}$$

$$\Phi(\infty,-\infty) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Given ψ an eigenfunction, ϕ corresp., we know

$$\phi(\infty) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\phi(-\infty)$$

so $\begin{pmatrix} \xi_+ \\ \xi_- \end{pmatrix} : \phi \longmapsto \begin{pmatrix} \phi^1(\infty) \\ \phi^2(\infty) \end{pmatrix}$

so what are you trying to say? The point is that $\xi_\pm$ can be viewed as lin. fnls on the space of W_s of eigenfns.

$$W_s = \{\psi \mid (s-D)\psi = 0\} \xleftarrow[\sim]{g_s} \left\{\phi \,\middle|\, \partial_r\phi = \begin{pmatrix} 0 & \times \\ \times & 0 \end{pmatrix}\phi\right\}$$

you want the lines in W_s decaying to the left and the right. Vanishing $\xi_+^1 = 0$ $\phi^2(-\infty) = 0$

$\xi_+ = 0$: $\phi^1(\infty) = 0$

There is a notational hurdle. You needs to specify the decaying lines for $\mathrm{Re}(s) > 0$.

on left $\sim \begin{pmatrix} e^{sr} \\ 0 \end{pmatrix} *$ on right $\sim \begin{pmatrix} 0 \\ e^{-sr} \end{pmatrix} *$

Pick a specific basis. $\varphi_{\pm}^{<,>}$

$r = -\infty$

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \longleftarrow \Phi(r,-\infty)\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a_< \\ c_< \end{pmatrix}$$

$r = +\infty$

$$\Phi(r,\infty)\begin{pmatrix} 0 \\ 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} -b_> \\ a_> \end{pmatrix} = \begin{pmatrix} d_> & -b_> \\ c_> & a_> \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

So you have two basic solutions decaying

$\begin{pmatrix} a_< \\ c_< \end{pmatrix}$ decays on left

$\begin{pmatrix} -b_> \\ a_> \end{pmatrix}$ ——— right

~~So on~~

---

need to overcome paralysis. You have a 2 dim'l space W_s of solutions to $\partial_r \phi = \begin{pmatrix} 0 & m e^{-2sr} \\ \bar{m} e^{2sr} & 0 \end{pmatrix} \phi$, and four coord functions $\phi^i(\pm\infty)$, $i = 1,2$ related by

$$\begin{pmatrix} \phi^1(\infty) \\ \phi^2(\infty) \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} \phi^1(-\infty) \\ \phi^2(-\infty) \end{pmatrix} \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \Phi(\infty,-\infty)$$

$$\partial_r \Phi = \begin{pmatrix} 0 & x \\ x & 0 \end{pmatrix} \Phi$$

You have this interpretation of these linear functions on W_s as elements $\xi_\pm$ $\xi'_\pm$

Now you need boundary conditions for the Green's function. ~~Look at $r \longrightarrow +\infty$.~~

$$\cancel{\psi(r,t) = \begin{pmatrix} e^{s(r+t)}\phi^1(r) \\ e^{s(-r+t)}\phi^2(r) \end{pmatrix} \sim \begin{pmatrix} e^{s(r+t)}\phi^1(\infty) \\ e^{s(-r+t)}\phi^2(\infty) \end{pmatrix}}$$

~~You have spatial decay as $r \to \infty$ when $Re(s) > 0$ and $\phi^1(\infty) = 0$~~

~~the wave is purely outgoing~~

Start again. You want bdry conds for G, specifically decay as $r \longrightarrow +\infty$ or $-\infty$. This linked to incoming + outgoing. Decay seems to be linked to incoming + outgoing. How?

$$\psi(r,t) = e^{st}\psi(r) \sim \begin{pmatrix} e^{s(r+t)}\,\phi^1(r) \\ e^{s(-r+t)}\,\phi^2(r) \end{pmatrix} \quad \text{far out}$$

upper amp left moving wave
lower " right " wave

Fix $Re(s) > 0$. Decaying $\psi(r)$ are

$$\psi(r) \sim \begin{pmatrix} e^{sr} & 0 \\ 0 & e^{-sr} \end{pmatrix} \begin{pmatrix} 0 \\ * \end{pmatrix} \qquad r \longrightarrow \infty$$

corresp wave outgoing to right

i.e. $\phi^1(+\infty) = 0$

$\boxed{\xi_+(\phi) = 0}$

$$\psi(r) \sim \begin{pmatrix} e^{sr} & 0 \\ 0 & e^{-sr} \end{pmatrix} \begin{pmatrix} * \\ 0 \end{pmatrix} \qquad r \longrightarrow -\infty$$

corresp wave is outgoing to left

i.e. $\phi^2(-\infty) = 0$

$\boxed{\xi'_+(\phi) = 0}$

$Re(s) < 0$

$r \to -\infty$ $\quad \psi(r) \sim \begin{pmatrix} e^{sr} & 0 \\ 0 & e^{-sr} \end{pmatrix} \begin{pmatrix} 0 \\ * \end{pmatrix}$ $\qquad r \longrightarrow -\infty$

i.e. $\phi^1(-\infty) = 0$, incom. from left

$\boxed{\xi'_-(\phi) = 0}$

$r \to +\infty$ $\quad \psi(r) \sim \begin{pmatrix} e^{sr} & 0 \\ 0 & e^{-sr} \end{pmatrix} \begin{pmatrix} * \\ 0 \end{pmatrix}$

i.e. $\phi^2(\infty) = 0$, incoming from Right

$\boxed{\xi_-(\phi) = 0}$

So what we learn is that $Re(s) > 0$ implies spatial decay is equivalent to outgoing and $Re(s) < 0 \Longrightarrow$ spatial decay equivalent to incoming wave. Next: The Green's function.

$$(s-D)G = \delta(r)I \qquad (s-D)\,G(r,r') = \delta(r-r')I$$

so $G(r,r')$ is a solution on $r<r'$ and on $r>r'$

$$G_< = \begin{pmatrix} e^{sr} & 0 \\ 0 & e^{-sr} \end{pmatrix} \Phi(r,-\infty)\, M_<$$

where $M_<$ must yield decay as $r \longrightarrow -\infty$, i.e. $\begin{pmatrix} 0 & e^{-sr} \end{pmatrix} \underbrace{\Phi(-\infty,-\infty)}_{I} M_< = 0 \quad \therefore \begin{pmatrix} 0 & 1 \end{pmatrix} M_< = 0$

$$\therefore M_< = \begin{pmatrix} 1 \\ 0 \end{pmatrix} (x_1 \;\; x_2)$$

Put another way first column of $\Phi(r,-\infty) = \begin{pmatrix} a_< & b_< \\ c_< & d_< \end{pmatrix}$

$$\boxed{G_<(r) = \begin{pmatrix} e^{sr} & 0 \\ 0 & e^{-sr} \end{pmatrix} \begin{pmatrix} a_< \\ c_< \end{pmatrix} (x_1 \;\; x_2)}$$

$$G_> = \begin{pmatrix} e^{sr} & 0 \\ 0 & e^{-sr} \end{pmatrix} \Phi(r,\infty)\, M_>$$

$$\begin{pmatrix} e^{sr} & 0 \end{pmatrix} M_> = 0$$

$$M_> = \begin{pmatrix} 0 \\ 1 \end{pmatrix} (y_1 \;\; y_2)$$

2nd col of $\Phi(r,\infty) = \begin{pmatrix} d_> & -b_> \\ -c_> & a_> \end{pmatrix}$

$$\boxed{G_>(r) = \begin{pmatrix} e^{sr} & 0 \\ 0 & e^{-sr} \end{pmatrix} \begin{pmatrix} -b_> \\ a_> \end{pmatrix} (y_1 \;\; y_2)}$$

$$-\varepsilon = G_>(0) - G_<(0) = -\begin{pmatrix} a_< \\ c_< \end{pmatrix} (x_1 \;\; x_2) + \begin{pmatrix} -b_> \\ a_> \end{pmatrix} (y_1 \;\; y_2)$$

what does this mean?

Review the philosophy. First you have W_s the space of eigenfunctions: $D\phi = s\phi$ for $D = \varepsilon\partial_r + V$

What do you need, How do you construct $G(r,r')$. Put it all into words. You have a DE $(s-D)\psi = 0$ for a 2-component function ψ on $\mathbb{R}$. There's an assoc inhomog. DE $(s-D)\psi = f$ whose solution is unique up to $\mathrm{Ker}(s-D)$ which is 2-diml. Boundary conditions are needed for a unique solution. Use fact that D is a perturbation $D = \varepsilon\partial_r + V$ with V decaying to get asymptotics for any solution $\psi \in W_s = \{\psi \mid D\psi = s\psi\} \xrightarrow{\sim} \{\phi \mid \partial_r\phi = \begin{pmatrix}0 & * \\ * & 0\end{pmatrix}\psi\}$.

So you end up with ~~a picture left and r~~ left and right pictures of W_s

$$\phi(-\infty) \in \mathbb{C}^2 \longleftarrow\!\!\shortmid \ (\phi \leftrightarrow \psi) \ \longmapsto \phi(+\infty) \in \mathbb{C}^2$$

Issue of spatial decay (since you want L^2 resolvent you need $G(r,r')$ to be in L^2). $\mathrm{Re}(s) > 0$

$$\begin{array}{ccc} -\infty \leftarrow r & & r \to +\infty \\ \begin{pmatrix} e^{sr} & 0 \\ 0 & e^{-sr}\end{pmatrix}\begin{pmatrix} * \\ 0\end{pmatrix} \longleftarrow & G(r,r') & \sim \begin{pmatrix} e^{sr} & 0 \\ 0 & e^{-sr}\end{pmatrix}\begin{pmatrix} 0 \\ * \end{pmatrix}\end{array}$$

spatial decay for $\mathrm{Re}(s) > 0 \iff$ only outgoing asymptotics

Fibre at ∞ has outgoing + incoming lines $\left\{ \text{measured by } \phi^2(\infty), \phi^1(\infty) \right.$

$-\infty$ $\left\{ \text{measured by } \phi^1(-\infty), \phi^2(-\infty) \right.$

$\psi \in$ outgoing line as $r \to \infty \iff \phi^1(\infty) = 0$

$\psi \in$

$$\phi\begin{pmatrix}\xi_+ \\ \xi_-\end{pmatrix} = \begin{pmatrix} a & b \\ c & d\end{pmatrix}\phi\begin{pmatrix}\xi'_- \\ \xi'_+\end{pmatrix}$$

There a notation problem to be sorted out.
Think using the grid space, solutions are linear functionals on grid space E which kill $(\lambda - z)E$, in the cont case this becomes $(\partial_t - s)E$. Too hard

---

Begin again. Study $(s-D)G = \delta(r)I$ $\qquad Re(s) > 0$

$$G_<(r) = \begin{pmatrix} e^{sr} & 0 \\ 0 & e^{-sr} \end{pmatrix} \Phi(r, -\infty) \begin{pmatrix} x_1 & x_2 \\ 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} e^{sr} & 0 \\ 0 & e^{-sr} \end{pmatrix} \begin{pmatrix} a^<(r) \\ c^<(r) \end{pmatrix} (x_1 \;\; x_2)$$

$$= \begin{pmatrix} e^{sr} a^<(r) \\ e^{-sr} c^<(r) \end{pmatrix} (x_1 \;\; x_2)$$

$$G_>(r) = \begin{pmatrix} e^{sr} & 0 \\ 0 & e^{-sr} \end{pmatrix} \Phi(r, \infty) \begin{pmatrix} 0 & 0 \\ y_1 & y_2 \end{pmatrix}$$

$$= \begin{pmatrix} e^{sr} & 0 \\ 0 & e^{-sr} \end{pmatrix} \begin{pmatrix} -b^> \\ a^> \end{pmatrix} (y_1 \;\; y_2)$$

$$G_>(r) = \begin{pmatrix} e^{sr} b^>(r) \\ e^{-sr} a^>(r) \end{pmatrix} (y_1 \;\; y_2)$$

$$-\varepsilon = G_>(0) - G_<(0) = \begin{pmatrix} -b^>(0) \\ a^>(0) \end{pmatrix} (y_1 \;\; y_2) - \begin{pmatrix} a^<(0) \\ c^<(0) \end{pmatrix} (x_1 \;\; x_2)$$

this must amount to an inverse matrix relation

$$\begin{pmatrix} -b^{>} \\ a^{>} \end{pmatrix} y_1 - \begin{pmatrix} a^{<} \\ c^{<} \end{pmatrix} x_1 = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -b^{>} \\ a^{>} \end{pmatrix} y_2 - \begin{pmatrix} a^{<} \\ c^{<} \end{pmatrix} x_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} -b^{>} & a^{<} \\ a^{>} & c^{<} \end{pmatrix} \begin{pmatrix} y_1 \\ -x_1 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$$

$$\boxed{\begin{pmatrix} a^{<} & -b^{>} \\ c^{<} & a^{>} \end{pmatrix} \begin{pmatrix} -x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}}$$

$$\boxed{\begin{pmatrix} a^{<} & -b^{>} \\ c^{<} & a^{>} \end{pmatrix} \begin{pmatrix} -x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}}$$

$$\begin{pmatrix} p_0 \\ q_0 \end{pmatrix} = \begin{pmatrix} a^{\ell} & b^{\ell} \\ c^{\ell} & d^{\ell} \end{pmatrix} \begin{pmatrix} \xi'_- \\ \xi'_+ \end{pmatrix} = \begin{pmatrix} d^{r} & -b^{r} \\ -c^{r} & a^{r} \end{pmatrix} \begin{pmatrix} \xi_+ \\ \xi_- \end{pmatrix}$$

$$= \frac{1}{a} \begin{pmatrix} a^{\ell} & \overset{H_-}{-b^{r}} \\ c^{\ell} & a^{r} \end{pmatrix} \begin{pmatrix} \xi_+ \\ \xi'_+ \end{pmatrix} = \frac{1}{d} \begin{pmatrix} d^{r} & b^{\ell} \\ -c^{r} & d^{\ell} \end{pmatrix} \begin{pmatrix} \xi'_- \\ \xi_- \end{pmatrix}$$

embarassed about H_+ being used with (ξ'_-, ξ_-)

$$\begin{pmatrix} \xi_+ \\ \xi_- \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \xi'_- \\ \xi'_+ \end{pmatrix} \qquad \frac{q}{b}$$

$$\begin{pmatrix} \xi_+ \\ \xi_- \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \xi'_- \\ \xi'_+ \end{pmatrix} \qquad \begin{pmatrix} \xi_+ \\ \xi'_+ \end{pmatrix} = \begin{pmatrix} \frac{1}{d} & \frac{b}{d} \\ -\frac{c}{d} & \frac{1}{d} \end{pmatrix} \begin{pmatrix} \xi'_- \\ \xi_- \end{pmatrix}$$

$$\begin{pmatrix} \xi'_- \\ \xi'_+ \end{pmatrix} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \begin{pmatrix} \xi_+ \\ \xi_- \end{pmatrix} \qquad \begin{pmatrix} \xi'_- \\ \xi_- \end{pmatrix} = \begin{pmatrix} \frac{1}{a} & -\frac{b}{a} \\ +\frac{c}{a} & \frac{1}{a} \end{pmatrix} \begin{pmatrix} \xi_+ \\ \xi'_+ \end{pmatrix}$$

$$\begin{pmatrix} p_0 \\ q_0 \end{pmatrix} = \begin{pmatrix} a^l & b^l \\ c^l & d^l \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\frac{c}{d} & \frac{1}{d} \end{pmatrix} \begin{pmatrix} \xi'_- \\ \xi_- \end{pmatrix} = \underbrace{\begin{pmatrix} \frac{a^l d - b^l c}{d} & \frac{b^l}{d} \\ \frac{c^l d - d^l c}{d} & \frac{d^l}{d} \end{pmatrix}} \begin{pmatrix} \xi'_- \\ \xi_- \end{pmatrix}$$

$$\begin{pmatrix} d^r & -b^r \\ -c^r & a^r \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a^l & b^l \\ c^l & d^l \end{pmatrix} \qquad \varphi \qquad \frac{1}{d} \begin{pmatrix} d^r & b^l \\ -c^r & d^l \end{pmatrix} \begin{pmatrix} \xi'_- \\ \xi_- \end{pmatrix}$$

$$\begin{pmatrix} d^r & -b^r \\ -c^r & a^r \end{pmatrix} = \begin{pmatrix} a^l & b^l \\ c^l & d^l \end{pmatrix} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \qquad \frac{1}{a} \begin{pmatrix} a^l & -b^r \\ c^l & a^r \end{pmatrix}$$

$$\begin{pmatrix} p_0 \\ q_0 \end{pmatrix} = \begin{pmatrix} d^r & -b^r \\ -c^r & a^r \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \frac{c}{a} & \frac{1}{a} \end{pmatrix} \begin{pmatrix} \xi_+ \\ \xi'_+ \end{pmatrix} = \begin{pmatrix} \frac{d^r a - b^r c}{a} & -\frac{b^r}{a} \\ \frac{-c^r a + a^r c}{a} & \frac{a^r}{a} \end{pmatrix} \begin{pmatrix} \xi_+ \\ \xi'_+ \end{pmatrix}$$

Continue constructing G. $\qquad v = \phi \begin{pmatrix} \xi'_- \\ \xi'_+ \end{pmatrix}$ $\parallel$

Gen Solution of $(s-D)\psi = 0$ is

$$\psi = \begin{pmatrix} e^{sr} & 0 \\ 0 & e^{-sr} \end{pmatrix} \Phi(r,-\infty)\, v \qquad v = (g_s^{-1}\psi)(-\infty) = \begin{pmatrix} \phi^1(-\infty) \\ \phi^2(-\infty) \end{pmatrix}$$

$\parallel$

$$\psi = \begin{pmatrix} e^{sr} & 0 \\ 0 & e^{-sr} \end{pmatrix} \Phi(r,\infty)\, w \qquad w = (g_s^{-1}\psi)(+\infty)$$

$$W = \Phi(\infty,r)\,\Phi(r,-\infty)\,v = \begin{pmatrix} a & b \\ c & d \end{pmatrix} v$$

Forget grid spaces. General solution of $(s-D)\psi = 0$ is $\psi = g_s \Phi(r,-\infty) v$ where $v = (g_s^{-1}\psi)(-\infty) = \phi(-\infty)$. ψ decays as $r \to -\infty$ when $\mathrm{Re}(s) > 0$ iff $v = \phi(-\infty) \in \begin{pmatrix} 1 \\ 0 \end{pmatrix} \mathbb{C}$

Similarly the gen. soln is $\psi = g_s \Phi(r, +\infty) w$ with $w = (g_s^{-1}\psi)(+\infty) = \phi(+\infty) = \Phi(\infty, -\infty) v = \begin{pmatrix} a & b \\ c & d \end{pmatrix} v$

ψ decays as $r \to \infty$ when $\mathrm{Re}(s) > 0$ iff $w = \phi(\infty) \in \begin{pmatrix} 0 \\ 1 \end{pmatrix} \mathbb{C}$

~~We now know that there are~~

We now have the ingredient for G namely "the" two decaying solutions at ∞

$$\psi = g_s \Phi(r,-\infty)\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \psi = g_s \Phi(r,\infty)\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$= g_s \begin{pmatrix} a^l(r) \\ c^l(r) \end{pmatrix}, \qquad g_s \begin{pmatrix} -b^r(r) \\ a^r(r) \end{pmatrix}$$

$\psi = g_s \begin{pmatrix} a^l \\ c^l \end{pmatrix}$ "the" solution decaying at $-\infty$

$\psi = g_s \begin{pmatrix} -b^r \\ a^r \end{pmatrix}$ "the" ——————— $+\infty$

$$G^{<}(x,x') = g_s(x) \underbrace{\begin{pmatrix} a^l(x) \\ c^l(x) \end{pmatrix}}_{\to \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ as } x \to -\infty} \overset{\text{row vector dep on } x'}{(\text{fn of } x')}, \qquad G^{>}(x,x') = g_s(x) \underbrace{\begin{pmatrix} -b^r(x) \\ a^r(x) \end{pmatrix}}_{\text{as } x \to +\infty \ \begin{pmatrix} 0 \\ 1 \end{pmatrix}} \overset{\text{row vector dep on } x'}{(?)}$$

$$-\varepsilon = G^{>}(x',x') - G^{<}(x',x')$$

$$= g_s(x') \left[\begin{pmatrix} a^l \\ c^l \end{pmatrix} (c_1 \ c_1') - \begin{pmatrix} -b^r \\ a^r \end{pmatrix} (c_2 \ c_2') \right]$$

There is some algebra here that's confusing. Namely representing a matrix in a peculiar form. Given two ind. vector $\xi_1, \xi_2 \in \mathbb{C}^2$, then you can solve.

$$\cancel{v_1 t_1 + v_2 t_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}}$$

$$\cancel{v_1 u_1 + v_2 u_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}}$$

$$v_1 c_1 + v_2 c_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

with $c_i, c_i' \in \mathbb{C}$.

$$v_1 c_1' + v_2 c_2' = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Are the c's coeff of the inverse matrix?

$$\begin{pmatrix} v_1 & v_2 \end{pmatrix} \begin{pmatrix} c_1 & c_1' \\ c_2 & c_2' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

2×2 inv. matrix

$$\begin{pmatrix} a^l \\ c^l \end{pmatrix}$$

You want

$$\begin{pmatrix} a^l \\ c^l \end{pmatrix} \begin{pmatrix} c_1 & c_1' \end{pmatrix} - \begin{pmatrix} -b^r \\ a^r \end{pmatrix} \begin{pmatrix} c_2 & c_2' \end{pmatrix} = M$$

$$\begin{pmatrix} a^l & -b^r \\ c^l & a^r \end{pmatrix} \begin{pmatrix} c_1 & c_1' \\ -c_2 & -c_2' \end{pmatrix} = M$$

$$\frac{\begin{pmatrix} a^l & -b^r \\ c^l & a^r \end{pmatrix}(x) \begin{pmatrix} a^r & b^r \\ -c^l & a^l \end{pmatrix}(x')}{a(x')}$$

What is the Greens function of $\partial_t \xi = A\xi$ $0 \le t \le 1$.

Consider $\partial_t \xi = A \xi$ $\quad A = A(t)$ on an interval say $0 \le t \le 1$. I recall there a various possibilities for b.c. There's a transfer $\xi(t) = \Phi(t,0)\,\xi(0)$. the transfer matrix gives a $\Gamma_T \subset V_0 \times V_1$ and space of b.c. is a subspace $W \subset V_0 \times V_1$ ~~transver~~ complementary to Γ_T.

$$(\partial_t - A)\,G(t,t') = \delta(t-t')\,I$$

$$G^{<}(t,t')v = \Phi(t,0)\,v_0$$

$$G^{>}(t,t')v = \Phi(t,1)\,v_1$$

this is a solution for $t \neq t'$ and it's b.v.s are $(v_0, v_1) \in V_0 \times V_1$, which is to lie in W.

Need to check that

$$v = G^{>}(t',t')v - G^{<}(t',t')v \qquad \text{i.e.}$$

$$v = \Phi(t',1)\,v_1 - \Phi(t',0)\,v_0$$

need $\forall v$ a unique $(v_0, v_1) \in W$ such that

$$v = \Phi(t', 1) v_1 - \Phi(t', 0) v_0$$

equiv. $\underbrace{\Phi(1, t')v}_{\text{arb in } V} = v_1 - \Phi(1,0) v_0$

$$W \subset V_0 \times V_1 \xrightarrow{(-T \quad 1)} V \qquad \text{clear.}$$

Go back to $D = \varepsilon \partial_x + V \qquad V = \begin{pmatrix} 0 & -m \\ \bar{m} & 0 \end{pmatrix}$

$$(s - D) G = \delta(x) \qquad \begin{pmatrix} s & m \\ \bar{m} & -s \end{pmatrix}$$

$$(s - \varepsilon \partial_x - V)\psi = 0 \qquad \partial_x \psi = (\varepsilon s - \varepsilon V)\psi$$

$$\psi = g_s \phi = \begin{pmatrix} e^{sx} & 0 \\ 0 & e^{-sx} \end{pmatrix} \phi \qquad g_s^{-1} \partial_x \psi = g_s^{-1}(\varepsilon s - \varepsilon V) g_s \phi$$

$$(\partial_x + \varepsilon s)\phi = (\cancel{\varepsilon s} - g_s^{-1} \varepsilon V g_\varepsilon)\phi$$

$$G^<(x) = g_s \Phi(x, -\infty) \begin{pmatrix} 1 \\ 0 \end{pmatrix} (\quad) \qquad = \begin{pmatrix} 0 & m e^{-2sx} \\ \bar{m} e^{2sx} & 0 \end{pmatrix} \phi$$

$$\begin{pmatrix} e^{sx} & 0 \\ 0 & e^{-sx} \end{pmatrix} \begin{pmatrix} a^l(x) & b^l(x) \\ c^l(x) & d^l(x) \end{pmatrix}$$

$$G^<(x) = \begin{pmatrix} e^{sx} a^l(x) \\ e^{-sx} c^l(x) \end{pmatrix} (\alpha_1 \quad \alpha_2) \qquad \begin{pmatrix} d^r & -b^r \\ -c^r & a^r \end{pmatrix}$$

$$G^>(x) = g_s \Phi(x, \infty) \begin{pmatrix} 0 \\ 1 \end{pmatrix} (\beta_1 \quad \beta_2)$$

$$= \begin{pmatrix} e^{sx}(-b^r)(x) \\ e^{-sx} a^r(x) \end{pmatrix} (\beta_1 \quad \beta_2)$$

$$-\varepsilon = G^>(0) - G^<(0) = \begin{pmatrix} a^l & -b^r \\ c^l & a^r \end{pmatrix} \begin{pmatrix} -\alpha_1 & -\alpha_2 \\ \beta_1 & \beta_2 \end{pmatrix}$$

422

So $\begin{pmatrix} a^l & -b^l \\ c^l & a^r \end{pmatrix}\begin{pmatrix} -\alpha_1 & -\alpha_2 \\ \beta_1 & \beta_2 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$$\begin{pmatrix} -\alpha_1 & -\alpha_2 \\ \beta_1 & \beta_2 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = \frac{1}{a}\begin{pmatrix} a^r & b^l \\ -c^l & a^l \end{pmatrix}$$

$$G^<(x) = \begin{pmatrix} e^{sx} & \\ & e^{-sx} \end{pmatrix} \underbrace{\Phi(x,-\infty)}_{} \begin{pmatrix} 1 \\ 0 \end{pmatrix} (\alpha_1 \;\; \alpha_2)$$

$\begin{pmatrix} a^l & b^l \\ c^l & d^l \end{pmatrix}$

$$G^>(x) = \begin{pmatrix} & \\ & e^{-sx} \end{pmatrix} \Phi(x,\infty) \begin{pmatrix} 0 \\ 1 \end{pmatrix} (\beta_1 \;\; \beta_2)$$

$\begin{pmatrix} d^r & -b^r \\ -c^r & a^r \end{pmatrix}$

$$G^>(x) = \begin{pmatrix} e^{sx} & \\ & e^{-sx} \end{pmatrix}\begin{pmatrix} -b^r \\ a^r \end{pmatrix}(\beta_1 \;\; \beta_2)$$

$$-G^<(x) = \begin{pmatrix} & \\ & \end{pmatrix}\begin{pmatrix} a^l \\ c^l \end{pmatrix}(-\alpha_1 \; -\alpha_2)$$

$$-\varepsilon = \begin{pmatrix} a^l & -b^r \\ c^l & a^r \end{pmatrix}\begin{pmatrix} -\alpha_1 & -\alpha_2 \\ \beta_1 & \beta_2 \end{pmatrix}$$

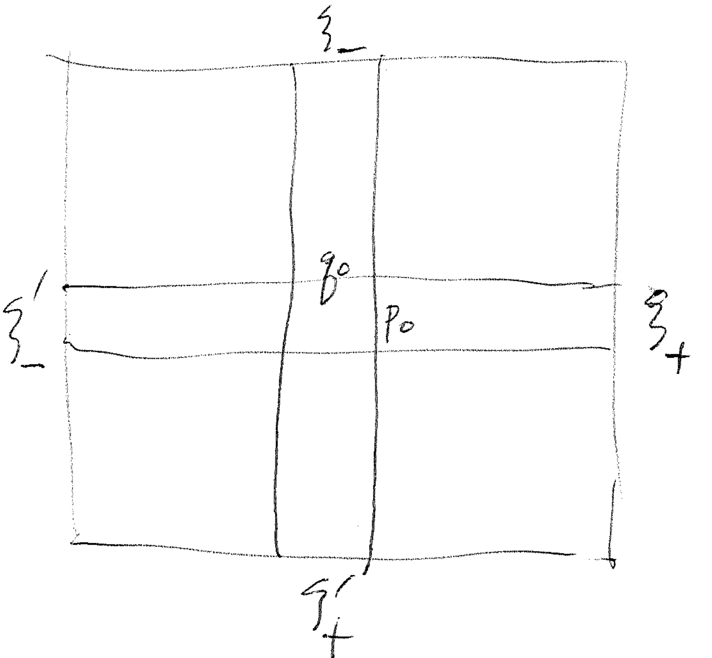
$$\begin{pmatrix} +\alpha_1 & -\alpha_2 \\ -\beta_1 & \beta_2 \end{pmatrix} = \frac{1}{a}\begin{pmatrix} a^r & +b^r \\ -c^l & a^l \end{pmatrix}$$

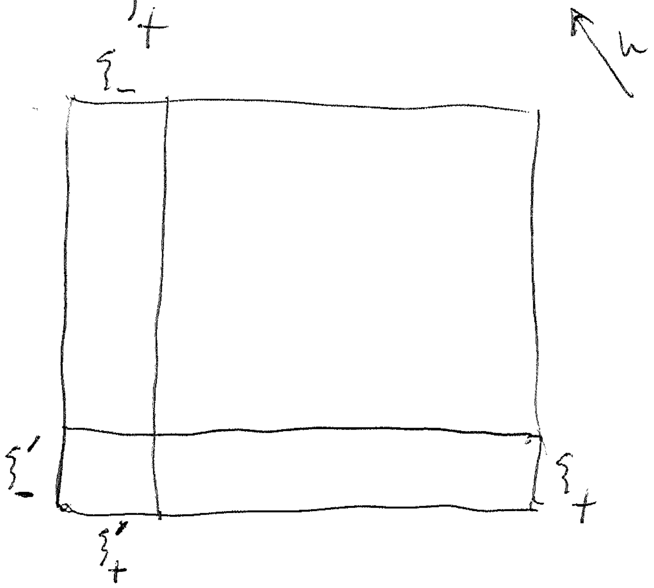
$$\boxed{\begin{pmatrix} \alpha_1 & \alpha_2 \\ \beta_1 & \beta_2 \end{pmatrix} = \frac{1}{a}\begin{pmatrix} a^r & -b^r \\ c^l & a^l \end{pmatrix}}$$

$$G^<(x,x') = \begin{pmatrix} e^{sx} & 0 \\ 0 & e^{-sx} \end{pmatrix}\begin{pmatrix} a^l(x) \\ c^l(x) \end{pmatrix}\frac{1}{a}(a^r \;\; -b^r)(x')$$

$$-G^>(x) = \begin{pmatrix} e^{sx} & \\ & e^{-sx} \end{pmatrix}\begin{pmatrix} -b^r(x) \\ a^r(x) \end{pmatrix}\frac{1}{a}(c^l \;\; a^l)(x')$$

Look at discrete case





$$X = \xi_+ H_+ + \xi_- H_-$$

$$uX = \xi_+ zH_+ + \xi_- zH_-$$

$$Y = \xi_+ H_+ + \xi_+ zH_-$$

~~Put~~ λ is given $\quad X \oplus \mathbb{C}\xi'_+ = uX \oplus \mathbb{C}\xi'_-$

$$(\lambda - u)x = -v_1 \xi'_+ + v_2 \xi'_-$$

$$\begin{tikzcd}
 & \mathbb{C}\xi'_- \arrow[d] & \\
X \arrow[r, "\lambda a - b"] \arrow[dr, "1-\lambda b^* a"', "S"] & Y \arrow[r] \arrow[d, "-b^*"] & Y/(\lambda a - b)X \\
 & X &
\end{tikzcd}$$

$$y - \tilde{y}(\lambda)\xi'_- \in (\lambda - u)X$$

$$(\lambda a - b)(1 - \lambda b^* a)^{-1}(-b^*)y = (-\lambda a + b)b^*(1 - \lambda ab^*)^{-1}y$$

$$y - \underline{\qquad\qquad} = \left[(1 - \lambda ab^*) - (-\lambda a + b)b^*\right](\qquad)^{-1}y$$

$$(1 - bb^*)(1 - \lambda ab^*)^{-1}y$$

$$\|$$

so $\tilde{y}(\lambda) = \xi_-'^{*}(1-\lambda ab^*)^{-1} y$

$$\begin{tikzcd} Y \arrow[r, "\begin{pmatrix} b^* \\ \xi_-'^* \end{pmatrix}"] \arrow[rr, bend left, "1-\lambda ab^*"] & X \oplus \mathbb{C} \arrow[r, "(b-\lambda a{,}\ \xi'_-)"] & Y \end{tikzcd}$$

inverse of $(b-\lambda a \quad \xi'_-)$ is $y \mapsto \begin{pmatrix} b^* \\ \xi_-'^* \end{pmatrix}(1-\lambda ab^*)^{-1} y$

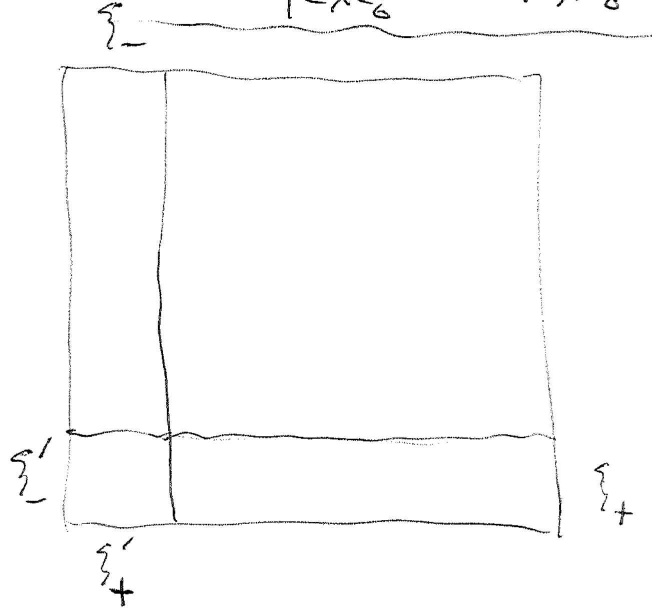
maybe $(1-\lambda ab^*)^{-1} = (1-\lambda u)^{-1}$

$$(1-\lambda c_h^*)^{-1} = \frac{1}{1-\lambda c_0^* - \lambda \delta c^*}$$

$c_h = ba^* + \xi_+'^{*} h \xi_+$

$c_h^* = ab^* + \xi_+$

$$= \frac{1}{1-\lambda c_0^*} + \frac{1}{1-\lambda c_0^*}\lambda \delta c^* \frac{1}{1-\lambda c_h^*}$$



$X = \xi_+ H_+ + \xi_- H_-$

$aX = \xi_+ z H_+ + \xi_- z H_-$

$Y = \xi_+ \cancel{z} H_+ + \xi_- z H_-$

$Y = aX \oplus \mathbb{C}\xi'_+$

$= \underbrace{uaX}_{\mathcal{B}} \oplus \mathbb{C}\xi'_-$

~~offsetter~~ You have a partial unitary

What do you need.

$$\begin{tikzcd} Y \arrow[r, "\begin{pmatrix} b^* \\ \xi_-'^* \end{pmatrix}"] \arrow[rr, bend right, "1-\lambda ab^*"'] & X \oplus \mathbb{C} \arrow[r, "(b-\lambda a\ \ \xi'_-)"] & Y \end{tikzcd}$$

$$\begin{tikzcd} Y \arrow[r, "\begin{pmatrix} b^* \\ \xi_-'^* \end{pmatrix}(1-\lambda ab^*)^{-1}", "\sim"'] & X \oplus \mathbb{C} \end{tikzcd}$$

$$\underbrace{Y \xrightarrow{\begin{pmatrix} a^* \\ \xi_+'^* \end{pmatrix}} \begin{matrix} X \\ \oplus \\ \mathbb{C} \end{matrix} \xrightarrow{(a - \lambda^{-1}b \;\; \xi_+')} Y}_{1-\lambda^{-1}ba^*}$$

$$Y \xrightarrow{\begin{pmatrix} a^* \\ \xi_+'^* \end{pmatrix}(1-\lambda^{-1}ba^*)^{-1}} \begin{matrix} X \\ \oplus \\ \mathbb{C} \end{matrix}$$

What you are doing is ~~...~~ to transform $y \in Y$ into a pair of functions

$$\tilde{y}(\lambda) = \xi_+'^*(1-\lambda^{-1}ba^*)^{-1}y \qquad \xi_-'^*(1-\lambda ab^*)^{-1}y$$

$$|\lambda| > 1 \qquad |\lambda| < 1.$$

$$y = \xi_+'\tilde{y}(\lambda) + (a-\lambda^{-1}b)\underset{\text{some}}{x} \qquad y = \xi_-'\tilde{y}(\lambda) + (\lambda a - b)\underset{\text{some}}{x}$$

But what you want is something involving a.

~~...~~ Question: You feel that the Green function should tell you the factorization of the S matrix somehow. In the ~~...~~ discrete case the Green's function is $\xi^* \frac{1}{1-\lambda u}$ where ξ is a grid vector. In the cont case the Green's function is $\frac{1}{s-D}$

You are too confused. ~~...~~ There are different kinds of G fns. 2 dims ~~...~~

So what?

$$\Phi(\infty,-\infty) = \Phi(\infty,x)\,\Phi(x,-\infty)$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a^r & b^r \\ c^r & d^r \end{pmatrix}\begin{pmatrix} a^l & b^l \\ c^l & d^l \end{pmatrix}$$

$$G^<(x,x') = \begin{pmatrix} e^{sx}\,a^l(x) \\ e^{-sx}\,c^l(x) \end{pmatrix}\begin{pmatrix} \alpha_1 & \alpha_2 \end{pmatrix} \qquad \Phi(x,\infty) = \begin{pmatrix} d^r & -b^r \\ -c^r & a^r \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$G^>(x,x') = \begin{pmatrix} -e^{sx}\,b^r(x) \\ e^{-sx}\,a^r(x) \end{pmatrix}\begin{pmatrix} \beta_1 & \beta_2 \end{pmatrix}$$

$$-\Sigma = G^>(x',x') - G^<(x',x')$$

$$= \begin{pmatrix} -e^{sx}\,b^r(x) \\ e^{-sx}\,a^r(x) \end{pmatrix}\begin{pmatrix} \beta_1 & \beta_2 \end{pmatrix} + \begin{pmatrix} e^{sx}\,a^l(x) \\ e^{-sx}\,c^l(x) \end{pmatrix}\begin{pmatrix} -\alpha_1 & -\alpha_2 \end{pmatrix}$$

$$= \begin{pmatrix} e^{sx'} & 0 \\ 0 & e^{-sx'} \end{pmatrix}\begin{pmatrix} a^l(x') & -b^r(x') \\ c^l(x') & a^r(x') \end{pmatrix}\begin{pmatrix} -\alpha_1 & -\alpha_2 \\ \beta_1 & \beta_2 \end{pmatrix}$$

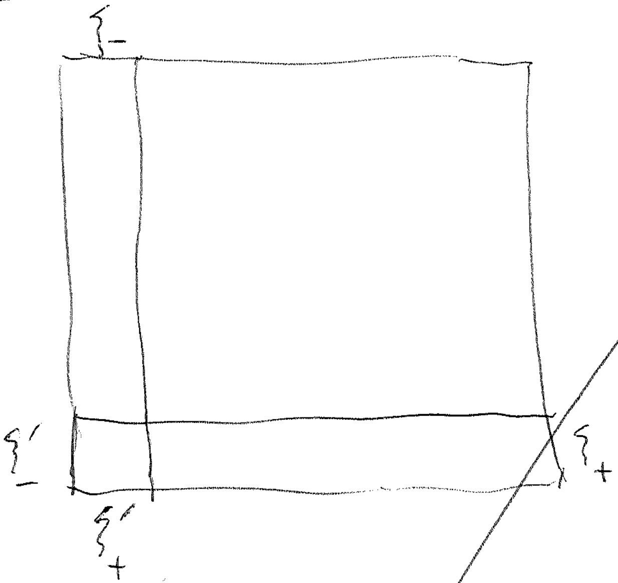
$$\begin{pmatrix} -\alpha_1 & -\alpha_2 \\ \beta_1 & \beta_2 \end{pmatrix} = \frac{1}{a}\begin{pmatrix} a^r & +b^r \\ -c^l & a^l \end{pmatrix}\begin{pmatrix} e^{-sx'} & 0 \\ 0 & e^{sx'} \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} \alpha_1 & \alpha_2 \\ \beta_1 & \beta_2 \end{pmatrix} = \frac{1}{a}\begin{pmatrix} +a^r & -b^r \\ +c^l & a^l \end{pmatrix}\begin{pmatrix} e^{-sx'} & 0 \\ 0 & e^{sx'} \end{pmatrix}$$

$$G^<(x,x') = \begin{pmatrix} e^{sx} & 0 \\ 0 & e^{-sx} \end{pmatrix} \begin{pmatrix} a^l(x) \\ c^l(x) \end{pmatrix} \frac{1}{a(x')} \begin{pmatrix} a^r(x') & -b^r(x') \end{pmatrix} \begin{pmatrix} e^{-sx'} & 0 \\ 0 & e^{sx'} \end{pmatrix}$$

$$G^>(x,x') = \begin{pmatrix} & \\ & \end{pmatrix} \begin{pmatrix} -b^r(x) \\ a^r(x) \end{pmatrix} \frac{1}{a(x')} \begin{pmatrix} c^l(x') & a^l(x') \end{pmatrix} \begin{pmatrix} e^{-sx'} & 0 \\ 0 & e^{sx'} \end{pmatrix}$$

$$\begin{pmatrix} a^l & -b^r \\ c^l & a^r \end{pmatrix} \begin{pmatrix} +a^r & +b^r \\ -c^l & a^l \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$



$$X = \xi_+ H_+ + \xi_- H_-$$
$$uX = \xi_+ zH_+ + \xi_- zH_-$$
$$Y = \xi_+ \cancel{z}H_+ + \xi_- zH_-$$

~~Why not look~~ What does $\frac{1}{1-\lambda^{-1}u}$

Recover factorization from G-fn.

$$\partial_x \phi = \begin{pmatrix} 0 & m e^{-2sx} \\ \bar{m} e^{2sx} & 0 \end{pmatrix} \phi$$

$$\Phi(\infty, -\infty) = \Phi(\infty, x)\, \Phi(x, -\infty)$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a^r(x) & b^r(x) \\ c^r(x) & d^r(x) \end{pmatrix} \begin{pmatrix} a^l(x) & b^l(x) \\ c^l(x) & d^l(x) \end{pmatrix}$$

You probably still need some more about G-function.

$$\partial_x \phi = \begin{pmatrix} 0 & m(x)e^{-2sx} \\ \bar{m}(x)e^{2sx} & 0 \end{pmatrix} \phi .$$

$\forall x$.

$$\Phi(\infty,-\infty) = \Phi(\infty, x)\,\Phi(x,-\infty)$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a^r & b^r \\ c^r & d^r \end{pmatrix}_x \begin{pmatrix} a^l & b^l \\ c^l & d^l \end{pmatrix}_x$$

Suppose you know G

$$\tilde{G}^<(x,x') = \begin{pmatrix} a^l \\ c^l \end{pmatrix}_x \frac{1}{a}\begin{pmatrix} a^r & -b^r \end{pmatrix}_{x'}$$

$$\tilde{G}^>(x,x') = \begin{pmatrix} -b^r \\ a^r \end{pmatrix}_x \frac{1}{a}\begin{pmatrix} c^l & a^l \end{pmatrix}_{x'}$$

$$(\tilde{G}^> - \tilde{G}^<)(x,x) = \begin{pmatrix} a^l & -b^r \\ c^l & a^r \end{pmatrix} \frac{1}{a} \begin{pmatrix} -a^r & +b^r \\ c^l & a^l \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

What does this mean? Namely you have

$$\begin{pmatrix} a^l \\ c^l \end{pmatrix} \qquad ?$$

Let's try to understand the resolvent better

~~the resol~~ There two things one can consider

First

$$Y = aX \oplus \mathbb{C}\xi_+ = bX \oplus \mathbb{C}\xi_-$$

$$c_h = ba^* + \xi_- h \xi_+^*$$

$$\xi_+^*(1-\lambda^{-1}c_h)^{-1} = \xi_+^*(1-\lambda^{-1}c_0)^{-1} + \xi_+^*(1-\lambda^{-1}c_0)^{-1}(-\lambda^{-1}\xi_- h \xi_+^*)(1-\lambda^{-1}c_h)^{-1}$$

$$\xi_+^*\frac{1}{\lambda-c_h} = \xi_+^*\frac{1}{\lambda-c_0} + \underbrace{\xi_+^*\frac{1}{\lambda-c_0}\xi_-}_{S_0} h \xi_+^*\frac{1}{\lambda-c_h}$$

$$S_h^- = S_0^- + S_0^- h S_h^- \qquad S_h^-(\lambda) = \xi_+^*\frac{1}{\lambda-c_h}\xi_-$$

$$S_h^- = \frac{S_0^-}{1-S_0^- h}$$

$$\xi_-^*\frac{1}{1-\lambda c_h^*}\xi_+ = \xi_-^*\frac{1}{1-\lambda c_0^*}\xi_+ + \xi_-^*\frac{1}{1-\lambda c_0^*}\lambda\xi_+\bar{h}\xi_-^*\frac{1}{1-\lambda c_h^*}\xi_+$$

$$S_h^+ = S_0^+ + S_0^+\lambda\bar{h}S_h^+$$

$$S_h^- = \xi_+^*\frac{1}{\lambda-c_h}\xi_- \qquad S_h^+ = \xi_-^*\frac{1}{1-\lambda c_h^*}\xi_+$$

$$(S_h^+(\lambda))^* = \xi_+^*\frac{1}{1-\bar{\lambda}c_h}\xi_- = \frac{1}{\lambda}\xi_+^*\frac{1}{\bar{\lambda}^{-1}-c_h}\xi_-$$

$$\left(\frac{1}{1-\lambda c_h^*}\right)^* = \frac{1}{1-\bar{\lambda}c_h} = \frac{\lambda^*}{\lambda^*-c_h} \qquad \lambda^* = \frac{1}{\lambda}$$

$$\left(\frac{1}{1-\lambda^{-1}c_h^*}\right)^* = \frac{1}{1-\lambda^* c_h^*}$$

$$S_h^-(\lambda) \qquad \frac{1}{1-\lambda^{-1}c_h} = S_h^+(\lambda^*)^*$$

$$\xi_+^* \frac{1}{\lambda - c_h} \xi_- = S_h(\lambda) \qquad \text{defined for } |\lambda| > 1$$

$$S_h(\lambda) = S_0 + S_0 h S_h \qquad S_h = \frac{S_0}{1 - S_0 h}$$

$$\boxed{1 + h S_h = \frac{1}{1 - S_0 h}}$$

$$S_1 = \xi_+^* \frac{1}{\lambda - u} u \xi_+ = \xi_+^* \left(\sum_{n \geq 0} \frac{u^{n+1}}{\lambda^{n+1}} \right) \xi_+$$

$$1 + S_1 = \underbrace{\xi_+^* \left(\sum_{n \geq 0} \frac{u^n}{\lambda^n} \right) \xi_+}_{} = \frac{1}{1 - S_0}$$

$$\int \frac{1}{1 - \frac{z}{\lambda}} d\mu = \lambda \int \frac{d\mu(z)}{\lambda - z}$$

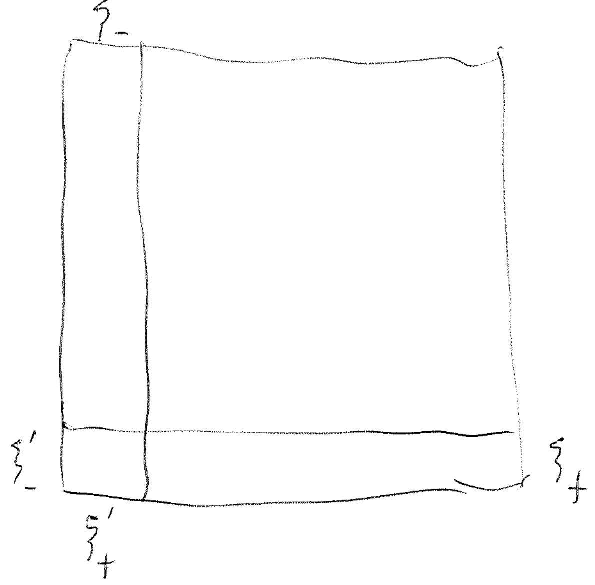
$$S_1 = \xi_+^* \frac{1}{\lambda - u} u (\xi_+) = \xi_+^* \frac{u}{\lambda} \frac{1}{1 - \frac{u}{\lambda}} \xi_+$$

$$= \sum_{n \geq 1} \lambda^{-n} \xi_+^* u^n \xi_+$$

$$1 + S_1 = \xi_+^* \frac{1}{1 - \lambda^{-1} u} \xi_+ = \frac{1}{1 - S_0}$$

$$\frac{1}{2} + S_1 = \frac{1}{1 - S_0} - \frac{1}{2} = \frac{2 - 1 + S_0}{2(1 - S_0)} = \frac{1 + S_0}{2(1 - S_0)}$$

$$\frac{1 + S_0}{1 - S_0} = 1 + 2 S_1$$



$$\begin{pmatrix}\xi_+\\ \xi_-\end{pmatrix} = \begin{pmatrix} a^n & b^n \\ c^n & d^n\end{pmatrix} \frac{1}{k_0}\begin{pmatrix}1 & h_0\\ \bar h_0 & 1\end{pmatrix}\begin{pmatrix}\xi'_+\\ \xi'_-\end{pmatrix}$$

take $h_0 = 0$.

simple case $\begin{pmatrix}\xi_+\\ \xi_-\end{pmatrix} = \frac{1}{k}\begin{pmatrix}1 & hz^{-1}\\ \bar h z & 1\end{pmatrix} \begin{pmatrix}\xi'_-\\ \xi'_+\end{pmatrix}$

$$\xi'^{*}_{+} \frac{1}{\lambda - ba^*}\xi'_- \qquad \xi'^{*}_{+} \frac{1}{\lambda - u}\xi'_-$$

Pursue Green's functions. Return to $(E/(\lambda-U)E)^*$ the 2 diml space of eigenfunctions, call it W_λ, elts are $(\phi_n)_{n\in\mathbb{Z}}$ $\ni$ $\phi_n = \frac{1}{k_n}\begin{pmatrix}1 & h_n z^{-n}\\ \bar h_n z^n & 1\end{pmatrix}\phi_{n-1}$. This is a simple recursion relation leading to propagators $\Phi(n,n')$, ~~perhaps~~ to certain asymptotic behavior when the (h_n) decays enough. What do you mean by the Green's functions in this case? ~~...~~ In the continuous case ~~you have a specific op.~~ your discussion was a bit muddy. How? mainly assuming $\mathrm{Re}(s) \neq 0$ is possible. In general incoming and outgoing ~~make sense~~ waves make more sense that your decay condition

Start with disc. case. Then given λ you have eigenfunctions $\varphi \in W_\lambda$, same as linear fnls on $E/(\lambda-u)E$. You also have the resolvent operator $\frac{1}{\lambda-u}$ on $\bar{E}$ for $|\lambda| \neq 1$.

$$\xi \in \bar{E} \qquad \frac{1}{\lambda-u}\xi = \sum_{n\geq 0} \lambda^{-n-1}u^n\xi \qquad \text{for } |\lambda|>1$$

$$= -\sum_{n\geq 0} \lambda^n u^{-n-1}\xi \qquad \text{for } |\lambda|<1.$$

Is there any relation between the two, namely eigenfunctions with eigenvalue λ and the resolvent operator $\frac{1}{\lambda-u}$ applied to grid vectors? In the continuous case ~~you~~ there is an analog.

$$\frac{1}{s-D}\xi = \int_0^\infty e^{-st}e^{tD}\xi\,dt$$

and the same ~~problem arises relation occurs~~ question arises: link between the operator $\frac{1}{s-D}$ on $\mathcal{H}$ and $W_s = \operatorname{Ker}(s-D)$, pairing?

There should be something here.

Look at the free case – shift on $L^2(S^1)$. $\mathbb{Z}$

Simplest case $\mathcal{H} = \ell^2(\mathbb{Z})$ with u = shift. Compare resolvent $\frac{1}{\lambda-u}$ which is a bdd operator on $\mathcal{H}$ for $\lambda \notin S^1$, ~~and its delta function~~ and the space $W_\lambda = \operatorname{Ker}(\lambda-u)$

Strange as all this seems there should be a link. Between eigenfunctions for $D(u)$ and the resolvent operator $\frac{1}{\lambda - u}$ $\frac{1}{s-D}$

Example: $\partial_t \psi = D\psi$, $D = \begin{pmatrix} \partial_x & -m \\ \bar{m} & -\partial_x \end{pmatrix}$

$$\psi = g_s \phi \qquad g_s(x) = \begin{pmatrix} e^{sx} & 0 \\ 0 & e^{-sx} \end{pmatrix}$$

$m = m(x)$

$$\psi(x,t) = e^{st}\psi(x,s)$$

$$(s-D)\psi = (s - \varepsilon\partial_x - V)\psi = 0$$

$$\partial_x \psi = (\varepsilon s - \varepsilon V)\psi = \begin{pmatrix} s & m \\ \bar{m} & -s \end{pmatrix}\psi$$

$$\partial_x \phi = \begin{pmatrix} 0 & m e^{-2sx} \\ \bar{m} e^{2sx} & 0 \end{pmatrix}\phi$$

You want Green's function for $s-D$:

$$(s-D)\, G(x,x') = \delta(x-x')\, I$$

$$(s-D)\, g_s(x)\, G^\phi(x,x')\, g_s(x')^{-1} = \delta(x-x')\, I$$

$$(s-D)\, g_s(x) G^\phi(x,x') = \delta(x-x') I\, g_s(x')$$

$$= g_s(x)\, \delta(x-x') I$$

$$g_s^{-1}(s-D)\, g_s\, G^\phi(x,x') = \delta(x-x')$$

$$s - D = s - \varepsilon\partial_x - V$$

$$g_s^{-1} D g_s = g_s^{-1}(\varepsilon\partial_x - V) g_s$$

$$= \varepsilon\partial_x + \varepsilon s - V$$

want $(s-D)\,G(x,x') = \delta(x-x')\,I$

$$g_s^{-1}(s-D)g_s \quad \underbrace{g_s^{-1}G(x,x')\,g_s(x')}_{G^\phi(x,x')} = \delta(x-x')\,I$$

$$g_s^{-1}(s-\varepsilon\partial_x - V)g_s = s - \varepsilon\, \overbrace{g_s^{-1}\partial_x g_s}^{\varepsilon s + \partial_x} - g_s^{-1}Vg_s$$

$$= \cancel{s}\cancel{s} - \varepsilon\partial_x - g_s^{-1}Vg_s$$

$$g_s^{-1}(s-\varepsilon\partial_x - V)g_s = s - \varepsilon(\partial_x + \varepsilon s) - g_s^{-1}Vg_s$$

$$= -\varepsilon\partial_x - g_s^{-1}Vg_s = -\varepsilon\left(\partial_x + \underbrace{\varepsilon g_s^{-1}Vg_s}\right)$$

$$\begin{pmatrix} 0 & me^{-2sx} \\ -\bar{m}e^{2sx} & 0 \end{pmatrix} \quad ($$

$$G^<(x,x') = g_s(x)\,\Phi(x,-\infty)$$

Start again $\quad \partial_t^{s}\psi = D\psi = \begin{pmatrix} \partial_x & -m \\ \bar{m} & -\partial_x \end{pmatrix}\psi$

$$\partial_t^{s}\psi = (\varepsilon\partial_x + V)\psi$$

$$\partial_x\psi = (\varepsilon\partial_t^{s} - \varepsilon V)\psi = \begin{pmatrix} s & m \\ \bar{m} & -s \end{pmatrix}\psi$$

$$\|$$

$$g_s^{-1}\partial_x g_s(x)\phi = g_s^{-1}\begin{pmatrix} s & m \\ \bar{m} & -s \end{pmatrix} g_s\phi$$

$$(\partial_x + \cancel{\varepsilon s})\phi = \left(\cancel{\varepsilon s} + \begin{pmatrix} 0 & me^{-2sx} \\ \bar{m}e^{2sx} & 0 \end{pmatrix}\right)\phi$$

$$\partial_x\phi = \begin{pmatrix} 0 & * \\ * & 0 \end{pmatrix}\phi$$

$$(s-D)\, G(x,x') = \delta(x-x')\, I$$

$$(s-\varepsilon\partial_x - V)\, g_s(x) G^{\phi}(x,x')\, g_s(x')^{-1} = \delta(x-x')\, I$$

$$\underbrace{g_s^{-1}(s-\varepsilon\partial_x - V)\, g_s}\quad G^{\phi}(x,x') = \delta(x-x')\, I$$

$$\cancel{s} - \varepsilon(\partial_x + \cancel{\varepsilon s}) - \begin{pmatrix} 0 & -me^{-2sx} \\ \bar{m}e^{2sx} & 0 \end{pmatrix} =$$

$$\begin{pmatrix} -\partial_x & me^{2sx} \\ -\bar{m}e^{2sx} & \partial_x \end{pmatrix} G^{\phi}(x,x') = \delta(x-x')\, I$$

$$\begin{pmatrix} \partial_x & -me^{2sx} \\ -\bar{m}e^{2sx} & \partial_x \end{pmatrix} G^{\phi}(x,x') = -\varepsilon\,\delta(x-x')\, I$$

$$\boxed{\partial_x G^{\phi} - \begin{pmatrix} 0 & me^{-2sx} \\ \bar{m}e^{2sx} & 0 \end{pmatrix} G^{\phi} = -\varepsilon\,\delta(x-x')\, I}$$

corresp. to $\partial_x \phi = \begin{pmatrix} 0 & me^{-2sx} \\ +\bar{m}e^{+2sx} & 0 \end{pmatrix}\phi$.

Repeat. $(s-D)\, G(x,x') = \delta(x-x')\, I$

$$\underline{g_s^{-1}(s-D)\, g_s}\quad \underline{g_s^{-1}(x)\, G(x,x')\, g_s(x')} = \delta(x-x')\, I$$

$$s - \varepsilon(\underline{g_s^{-1}\cdot\partial_x\cdot g_s}) - g_s^{-1} V g_s \quad G^{\phi}(x,x') \qquad -\begin{pmatrix} 0 & me^{-2sx} \\ \bar{m}e^{2sx} & 0 \end{pmatrix}$$

$\partial_x + \varepsilon s$

$$(\cancel{s} - \varepsilon\partial_x \cancel{-s} - g_s^{-1} V g_s)\, G^{\phi}(x,x') = \delta(x-x')\, I$$

$$(\partial_x + \varepsilon g_s^{-1} V g_s)\, G^{\phi}(x,x') = (-\varepsilon)\,\delta(x-x')\, I$$

What's happened is that changed Green's function to $G_\phi(x,x')$. What is your aim?

The problem is to link the resolvent $\frac{1}{\lambda - u}$ or $(s-D)^{-1}$ on $\mathcal{H}$ to the 2 diml space of eigenfunctions. I think this is the missing ingredient. Your problem is to ~~constitute~~ find some non-computational idea that explains the phenomena. At the moment you have an $\mathcal{H}$.

Better: Examine approaches.

cont. case wave equation $\rightsquigarrow$ $\mathcal{H} = L^2$ Cauchy data

D skew-adj diff op. There's the von Neumann ~~algebra~~ analysis constructing the 1-parameter unitary group e^{tD}. The problem

---

~~At~~ Go back to $\partial_t \psi = \begin{pmatrix} \partial_x & -m \\ \bar{m} & -\partial_x \end{pmatrix} \psi$ m constant.

---

discrete case: Have E, n relations

$$\begin{pmatrix} u^{-n} p_n \\ q_n \end{pmatrix} = \frac{1}{k} \begin{pmatrix} 1 & h_n u^{-n} \\ \bar{h}_n u^{n} & 1 \end{pmatrix} \begin{pmatrix} u^{-n+1} p_{n-1} \\ q_{n-1} \end{pmatrix}$$

$\psi \in W_\lambda =$ [redacted] $(E/(\lambda-u)E)^*$ yields $\psi_n = \psi\begin{pmatrix} u^{-n} p_n \\ q_n \end{pmatrix}$

$$\psi_n = \frac{1}{k_n} \begin{pmatrix} 1 & h_n \lambda^{-n} \\ \bar{h}_n \lambda^{n} & 1 \end{pmatrix} \psi_{n-1}$$

equivalence between eigenfunctions + such (ψ_n)

~~Now if you want the resolvent~~

You want ~~to~~ link [redacted] W_λ to $\frac{1}{\lambda - u}$

So what do you expect? You expect W_λ to have decaying lines to the left and to the right $l_\lambda^<$, $l_\lambda^>$. Is there a Green's function? What should the Green's function be in the discrete case? Nothing is clear!

Look at all $h_n = 0$. Recursion relations are $\psi_n = \psi_{n-1}$ $\forall n$. Should ~~be~~ maybe be a Heaviside function. $\psi_n = \psi_{n-1}$. Boundary conditions should make $u^{-n/2} p_n$, $u^{n/2} q_n$

Consider ~~[illegible]~~ $\mathbb{C}[z, z^{-1}]$, u = mult by z, nice 1-diml "grid space". Obvious Hilb. completion to $\mathcal{H} = \ell^2(\mathbb{Z}) \simeq L^2(S^1)$. Then have operator $\frac{1}{\lambda - u}$ defined on $\mathcal{H}$ for $\lambda \notin S^1$ by the geometric series, YES. Now look at ~~[illegible]~~ eigenfunctions, i.e. $\psi = (\psi_n)$: But first label grid vectors p_n ~~$up_n = p_{n+1}$~~. $p_n = up_{n-1}$

$\psi_n = \psi(p_n)$ ~~[illegible]~~ $\psi_n = \psi(up_{n-1}) = \lambda\psi_{n-1}$. So you end up with the difference eqn. $\boxed{\psi_n = \lambda\psi_{n-1}}$

To solve inhomog. $\psi_n - \lambda\psi_{n-1} = f_n$,

$$\psi_n = f_n + \lambda\psi_{n-1} = f_n + \lambda f_{n-1} + \lambda^2\psi_{n-2}$$

$$= \sum_{k \geq 0} \lambda^k f_{n-k} \qquad \text{if} \quad \lambda^k \psi_{n-k} \to 0$$

~~Confused by recursion relation~~
~~Because~~ ~~to visualize what are you~~

Repeat. $E = \mathbb{C}[z, z^{-1}] = \bigoplus_{n \in \mathbb{Z}} \mathbb{C} p_n \qquad u p_{n-1} = p_n$

$\psi \in (E/(\lambda - u)E)^* \qquad \psi_n = \psi(p_n) = \psi(u p_{n-1}) = \lambda \psi_{n-1}$

inhom. equ. $\qquad \psi_n - \lambda \psi_{n-1} = f_n$

G fn. method

$$G(n, n') - \lambda G(n-1, n') = \delta_{nn'}$$

$$G(-1,0) = \lambda G(-2,0)$$

$$G^{<}(n, 0) = \lambda^n a \qquad G(0,0) - \lambda G(-1,0) = 1$$

$$G^{\geq}(n, 0) = \lambda^n b \qquad G(1,0) - \lambda G(0,0) = 0$$

assume $|\lambda| < 1$. Then $a = 0$, $b = 1$.

$$G(n, n') = \begin{cases} \lambda^{n-n'} & n \geq n' \\ 0 & \end{cases}$$

$$\psi_n = \sum_{n'} G(n, n') f_{n'} = \sum_{n' \leq n} \lambda^{n-n'} f_{n'}$$

$$\sum_n \psi_n t^n = \sum_n t^n \sum_{\substack{n'+n''=n \\ n'' \geq 0}} \lambda^{n''} f_{n'}$$

$$= \sum_{n'' \geq 0} t^{-n''} \lambda^{n''} \sum_{n'} t^{-n'} f_{n'}$$

$$\hat{\psi}(t) = \frac{1}{1 - \frac{\lambda}{t}} \hat{f}(t)$$

$|\lambda| > 1$. $\qquad b = 0 \qquad G(-1, 0) = -\frac{1}{\lambda}$

$$G(n, 0) = \begin{cases} -\frac{1}{\lambda^n} & n < 0 \\ 0 & n \geq 0 \end{cases}$$

$$\sum_{n<0} (-1) \lambda^{-n} t^{-n}$$

$$\sum_{n \geq 1} -1 (\lambda t)^n = -\frac{\lambda t}{1 - \lambda t}$$

$$\psi_n - \lambda \psi_{n-1} = f_n \qquad \textit{iterate}$$

$$\psi_n = f_n + \lambda \psi_{n-1} = f_n + \lambda f_{n-1} + \lambda^2 \psi_{n-2} = \sum_{k \geq 0} \lambda^k f_{n-k}$$

$$\hat{\psi}(t) = \frac{1}{1-\lambda t} \hat{f}(t) = \sum_{k \geq 0} \lambda^k t^k \hat{f}(t)$$

$$= \frac{(\lambda t)^{-1}}{(\lambda t)^{-1} - 1} \hat{f}(t) = -\sum_{k \geq 1} \lambda^{-k} t^{-k} \hat{f}(t)$$

$$-\lambda \psi_n = f_n - \psi_{n+1}$$

$$\psi_n = -\lambda^{-1} f_n + \lambda^{-1} \psi_{n+1}$$

$$= -\lambda^{-1} f_n + (-\lambda)^{-2} f_{n+1} + \lambda^{-2} \psi_{n+2}$$

$$\psi_n = (-\lambda^{-1}) f_n + \lambda^{-1} \psi_{n+1}$$

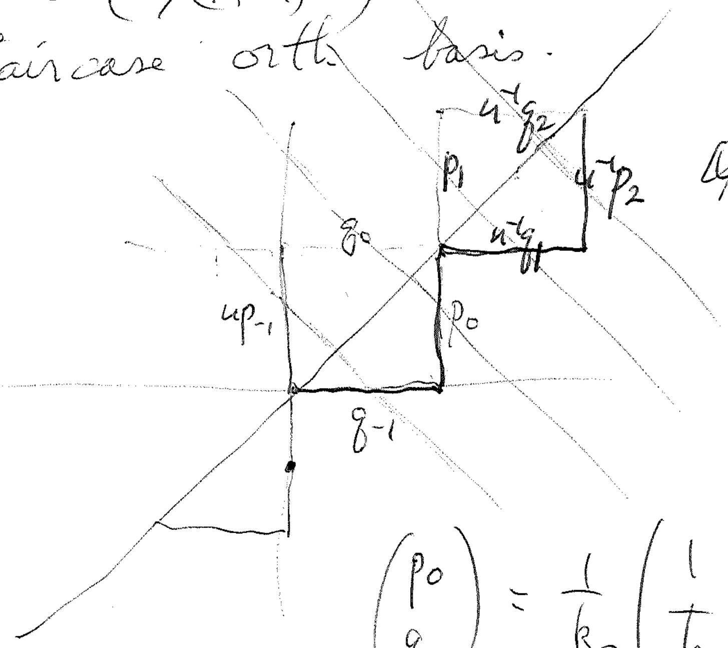
$$= -\lambda^{-1} f_n + \lambda^{-1}\left(-\lambda^{-1} f_{n+1} + \lambda^{-1} \underbrace{\psi_{n+2}}_{}\right)$$

$$-\lambda^{-1} f_{n+2} + \lambda^{-1} \psi_{n+3}$$

$$= -\lambda^{-1} f_n - \lambda^{-2} f_{n+1} - \lambda^{-3} f_{n+2} + \lambda^{-3} \psi_{n+3}$$

So now you understand G fn. for the disc line and shift

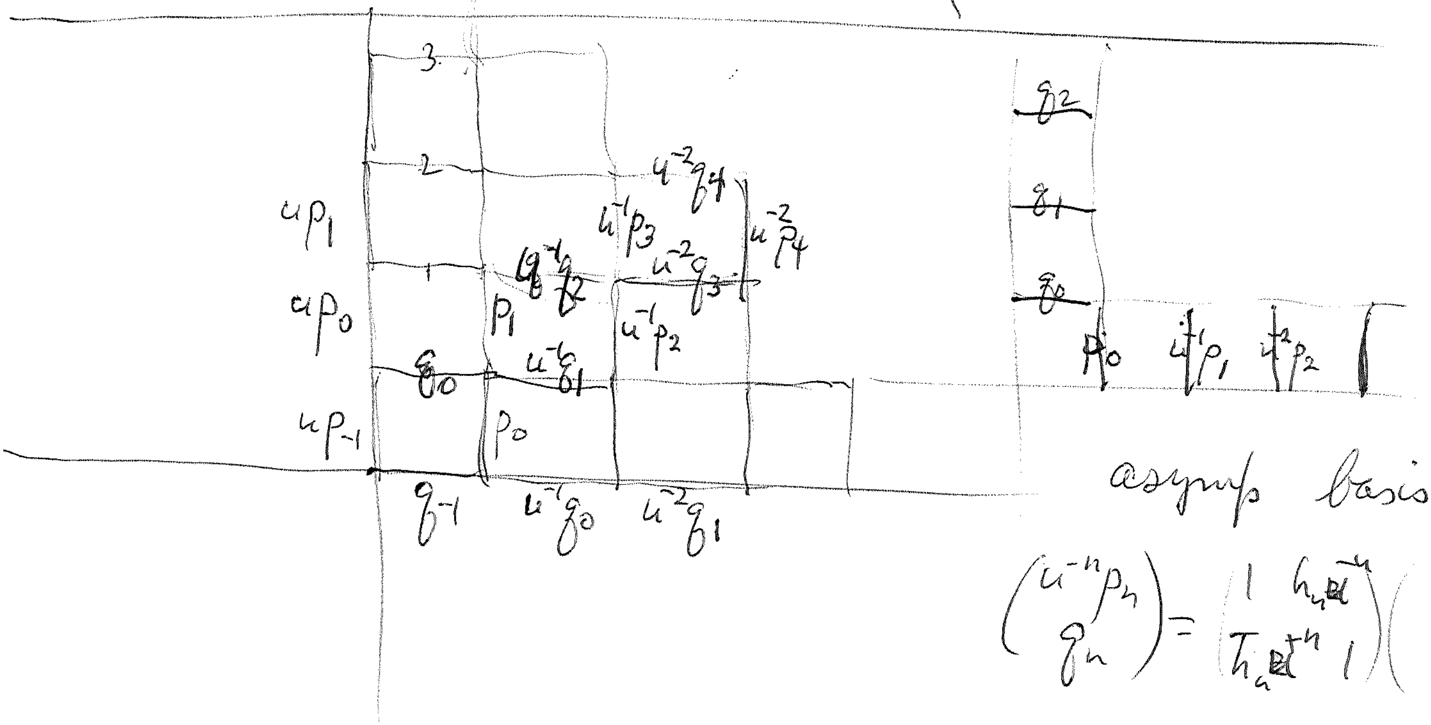
Problem: Let E be the grid space with u and h_n decaying enough so that one has scattering type asymptotics. You know is a free rank 2 module over $\mathbb{C}[u,u^{-1}]$, so for each λ there is a 2 diml space of eigenfns. $\psi \in (E/(\lambda-u)E)^*$. Pick ~~a~~ convenient asc. staircase orth basis.



$\psi_n \neq \frac{1}{k_n}$

$$\begin{pmatrix} p_0 \\ q_0 \end{pmatrix} = \frac{1}{k_0}\begin{pmatrix} 1 & h_0 \\ \bar h_0 & 1 \end{pmatrix}\begin{pmatrix} up_{-1} \\ q_{-1} \end{pmatrix}$$

$$\begin{pmatrix} p_1 \\ q_1 \end{pmatrix} = \frac{1}{k_1}\begin{pmatrix} 1 & h_1 \\ \bar h_1 & 1 \end{pmatrix}\begin{pmatrix} up_0 \\ \end{pmatrix}$$



asymp basis

$$\begin{pmatrix} u^{-n}p_n \\ q_n \end{pmatrix} = \begin{pmatrix} 1 & h_n u^{-n} \\ \bar h_n u^{n} & 1 \end{pmatrix}\Bigg($$

basis $\begin{array}{c|c|} p_0 & u^{-1}p_2 \\ q_0 & u^{-1}q_2 \end{array}$

$$\begin{pmatrix} p_2 \\ q_2 \end{pmatrix} = \frac{1}{k_2}\begin{pmatrix} 1 & h_2 \\ \bar{h}_2 & 1 \end{pmatrix}\begin{pmatrix} up_1 \\ q_1 \end{pmatrix}$$

$$= \frac{1}{k_2}\begin{pmatrix} 1 & h_2 \\ \bar{h}_2 & 1 \end{pmatrix}\begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} p_1 \\ q_1 \end{pmatrix}$$

$$\begin{pmatrix} u^{-1}p_2 \\ u^{-1}q_2 \end{pmatrix} = \frac{1}{k_2}\begin{pmatrix} 1 & h_2 \\ \bar{h}_2 & 1 \end{pmatrix}\begin{pmatrix} u & 0 \\ 0 & u^{-1} \end{pmatrix}\frac{1}{k_1}\begin{pmatrix} 1 & h_1 \\ \bar{h}_1 & 1 \end{pmatrix}\begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} p_0 \\ q_0 \end{pmatrix}$$

$$= \frac{1}{k_2}\begin{pmatrix} & \\ & \end{pmatrix}\begin{pmatrix} u^{1/2} & 0 \\ 0 & u^{-1/2} \end{pmatrix}\frac{1}{k_1}\begin{pmatrix} & \\ & \end{pmatrix}\begin{pmatrix} u^{1/2} & 0 \\ 0 & u^{-1/2} \end{pmatrix}\begin{pmatrix} p_0 \\ q_0 \end{pmatrix}$$

From the viewpoint of ~~the eigenvalues~~ eigenfunctions no problem, ~~no problem~~

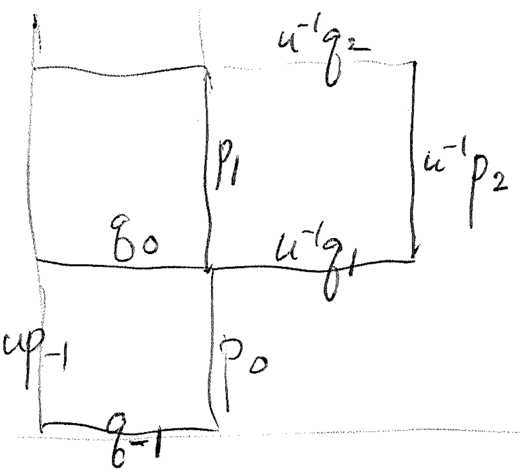
What might be interesting would be to do the descent. Basically you mean making the base extension ~~[illegible]~~ adjoining $u^{1/2}$ – get a new grid space but half the h's are zero.

Anyway now you have a space W_λ of eigenfunctions, ~~[illegible]~~ 2 diml, presumably, it has decaying lines, ~~[illegible]~~ for $|\lambda| \neq 1$. This should be transparent from the ~~scattering~~ asymptotics

Where are you? ~~[illegible]~~ An eigenfunction ψ has ~~[illegible]~~ 2-components at each site n = space position.

$$\begin{pmatrix} u^{-n}p_{2n} \\ u^{-n} q_{2n}\end{pmatrix} = \frac{1}{k_{2n}} \begin{pmatrix} 1 & h_{2n} \\ \bar h_{2n} & 1\end{pmatrix}\begin{pmatrix} u & 0 \\ 0 & u^{-1}\end{pmatrix}\begin{pmatrix} u^{-n+1}p_{2n-1}\\ u^{-n+1} q_{2n-1}\end{pmatrix}$$

$$u^{-n+1}\begin{pmatrix}p_{2n-1}\\q_{2n-1}\end{pmatrix} = \frac{1}{k_{2n-1}}\begin{pmatrix}1 & h_{2n-1}\\ \bar h_{2n-1} & 1\end{pmatrix}\begin{pmatrix}u&0\\0&1\end{pmatrix}\begin{pmatrix}u^{-n+1}p_{2n-2}\\u^{-n+1}q_{2n-2}\end{pmatrix}$$



$$\begin{pmatrix}p_0\\q_0\end{pmatrix} = \frac{1}{k_0}\begin{pmatrix}1 & h_0\\ \bar h_0 & 1\end{pmatrix}\begin{pmatrix}1&0\\0&u^{-1}\end{pmatrix}\begin{pmatrix}u p_{-1}\\u q_{-1}\end{pmatrix}$$

$$u^{-1}\begin{pmatrix}p_2\\q_2\end{pmatrix} = \frac{1}{k_2}\begin{pmatrix}1 & h_2\\ \bar h_2 & 1\end{pmatrix}\begin{pmatrix}u&0\\0&u^{-1}\end{pmatrix}\begin{pmatrix}p_1\\q_1\end{pmatrix}$$

$$\begin{pmatrix}p_1\\q_1\end{pmatrix} = \frac{1}{k_1}\begin{pmatrix}1 & h_1\\ \bar h_1 & 1\end{pmatrix}\begin{pmatrix}u&0\\0&1\end{pmatrix}\begin{pmatrix}p_0\\q_0\end{pmatrix}$$

So what happens with G? ~~[illegible]~~

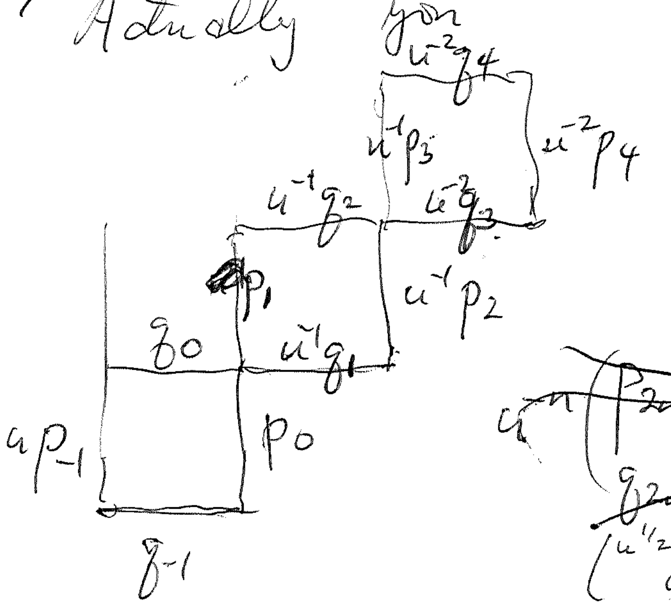
At this point you have ~~[illegible]~~ good control over eigenfunctions. You've chosen ~~[illegible]~~ a basis at each n $\begin{pmatrix}u^{-\frac{n}{2}}p_n\\ u^{-\frac{n}{2}}q_n\end{pmatrix}$. The ratio $\left(\frac{p_n}{q_n}\right)$

should be a well defined response function. It should be a simple exercise to get the Green's function at position n.

<u>Repeat</u>: Consider the grid space E with parameters $(h_n)_{n\in\mathbb{Z}}$. ~~You know those~~ You know E is a rank 2 free module over $\mathbb{C}[u,u^{-1}]$, so for each $\lambda\in\mathbb{C}^\times$ get a 2-dim'l space W_λ of eigenfunctions. Fix $n=0$ use the bases

$u^{-n}\begin{pmatrix}p_{2n}\\ q_{2n}\end{pmatrix}$ for E over $\mathbb{C}[u,u^{-1}]$.

Actually you



$$\cancel{u^{-n}\begin{pmatrix}p_{2n}\\ q_{2n}\end{pmatrix} = \frac{1}{k_{2n}}\begin{pmatrix}1 & h_{2n}\\ \bar h_{2n} & 1\end{pmatrix}(\quad)(}$$

$$u^{-n}\begin{pmatrix}p_{2n}\\ q_{2n}\end{pmatrix} = \frac{1}{k_{2n}}\begin{pmatrix}1 & h_{2n}\\ \bar h_{2n} & 1\end{pmatrix}\overbrace{\begin{pmatrix}1 & 0\\ 0 & u^{-1}\end{pmatrix}\begin{pmatrix}u^{-n+1}p_{2n-1}\\ u^{-n+1}q_{2n-1}\end{pmatrix}}^{\begin{pmatrix}u^{1/2} & \\ & u^{-1/2}\end{pmatrix}u^{-n+\frac12}\begin{pmatrix}p_{2n-1}\\ q_{2n-1}\end{pmatrix}}$$

$$u^{-n+\frac12}\begin{pmatrix}p_{2n-1}\\ q_{2n-1}\end{pmatrix} = \frac{1}{k_{2n-1}}\begin{pmatrix}1 & h_{2n}\\ \bar h_{2n-1} & 1\end{pmatrix}\underbrace{\begin{pmatrix}u & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}u^{-n+1/2}p_{2n-2}\\ u^{-n+1/2}q_{2n-2}\end{pmatrix}}_{\begin{pmatrix}u^{1/2} & \\ & u^{-1/2}\end{pmatrix}u^{-n+1}\begin{pmatrix}p_{2n-2}\\ q_{2n-2}\end{pmatrix}}$$

~~a~~ For eigenvectors you choose $\lambda^{1/2}$ and put $\psi_n = \psi\left(u^{-n/2}\begin{pmatrix}p_n\\ q_n\end{pmatrix}\right) = \lambda^{-n/2}\begin{pmatrix}\psi(p_n)\\ \psi(q_n)\end{pmatrix}$?,

and you find the recursion relation

$$\psi_n = \lambda^{-n/2}\frac{1}{k_n}\begin{pmatrix}1 & h_n\\ \bar{h}_n & 1\end{pmatrix}\begin{pmatrix}\lambda & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}\psi(p_{n-1})\\ \psi(q_{n-1})\end{pmatrix}$$

$$= \frac{1}{k_n}\begin{pmatrix} & \\ & \end{pmatrix}\begin{pmatrix}\lambda^{1/2} & 0\\ 0 & \lambda^{-1/2}\end{pmatrix}\begin{matrix}\lambda^{\frac{-n+1}{2}}\psi(p_{n-1})\\ \lambda^{\frac{-n+1}{2}}\psi(q_{n-1})\end{matrix} \quad ?\ \text{OK}$$

The $\psi_{2n} = \lambda^{-n}\begin{pmatrix}\psi(p_{2n})\\ \psi(q_{2n})\end{pmatrix} = \frac{1}{k_{2n}}\begin{pmatrix} & \\ & \end{pmatrix}\begin{pmatrix}\lambda^{1-n} & 0\\ 0 & \lambda^{-n}\end{pmatrix}\begin{pmatrix}\psi(p_{2n-1})\\ \psi(q_{2n-1})\end{pmatrix}$

$$= \frac{1}{k_{2n}}\begin{pmatrix} & \\ & \end{pmatrix}\begin{pmatrix}\lambda^{1/2} & 0\\ 0 & \lambda^{-1/2}\end{pmatrix}\begin{pmatrix}\lambda^{\frac{1}{2}-n}\psi(p_{2n-1})\\ \lambda^{\frac{1}{2}-n}\psi(q_{2n-1})\end{pmatrix}$$

Try again. ~~Choose~~ Given $\psi \in \left(E/(\lambda-u)E\right)^*$, choose $\lambda^{1/2}$,

put $\psi_n = \lambda^{-n/2}\begin{pmatrix}\psi(p_n)\\ \psi(q_n)\end{pmatrix}$ then $= \frac{1}{k_n}\begin{pmatrix}1 & h_n\\ \bar{h}_n & 1\end{pmatrix}\underbrace{\begin{pmatrix}\lambda & 0\\ 0 & 1\end{pmatrix}\lambda^{-\frac{n}{2}}}_{\begin{pmatrix}\lambda^{1/2} & 0\\ 0 & \lambda^{-1/2}\end{pmatrix}\lambda^{\frac{-n+1}{2}}}\begin{pmatrix}\psi(p_{n-1})\\ \psi(q_{n-1})\end{pmatrix}$

$$= g(h_n)\begin{pmatrix}\lambda^{1/2} & 0\\ 0 & \lambda^{-1/2}\end{pmatrix}\psi_{n-1}$$

So now you understand eigenfunctions, namely they are solutions of ~~the~~

$$\boxed{\psi_n = g(h_n)\begin{pmatrix}\lambda^{1/2} & 0\\ 0 & \lambda^{-1/2}\end{pmatrix}\psi_{n-1}}$$

Now we understand W_λ and we can look for the lines in W_λ generated by eigenfunctions decaying to the right (resp. left).

Schur expansion

$$\psi_n = \begin{pmatrix} 1 & h_n \\ \bar{h}_n & 1 \end{pmatrix} \begin{pmatrix} \\ \lambda \psi_{n-1} \end{pmatrix}$$

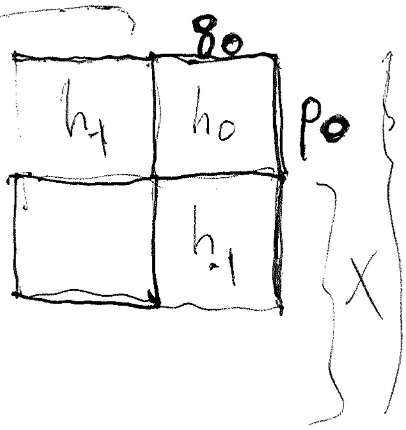
Point: Schur expansion says you get these decaying lines quite generally. Agrees with u is unitary in general.

You want ~~the Green's function~~ to understand assoc. to W_λ

You want the G-fn and its relation to $\frac{1}{\lambda - u}$ on $\mathcal{H}$

So how does this work?

First uX



$$\psi_0 = g(h_0) \begin{pmatrix} \lambda^{1/2} & 0 \\ 0 & \lambda^{-1/2} \end{pmatrix} \psi_{-1}$$

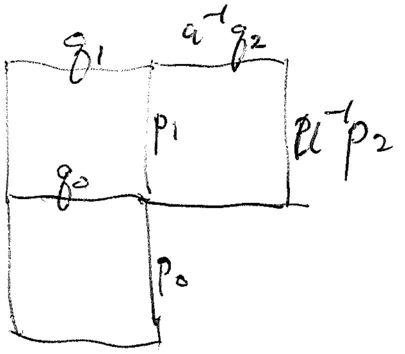
$$= g(h_0) \begin{pmatrix} & \\ & \end{pmatrix} g(h_{-1}) \begin{pmatrix} & \\ & \end{pmatrix} \psi_{-2}$$

looks good $\quad Y = X \oplus \mathbb{C} p_0 = uX \oplus \mathbb{C} q_0$

Problem: Given grid space E you have the space $W_\lambda = (E/(\lambda - u)E)^*$ of eigenfunctions ~~and~~ which you want to relate ~~the~~ to the resolvent $\frac{1}{\lambda - u}$ on $\mathcal{H}_E$. ~~Obvious~~

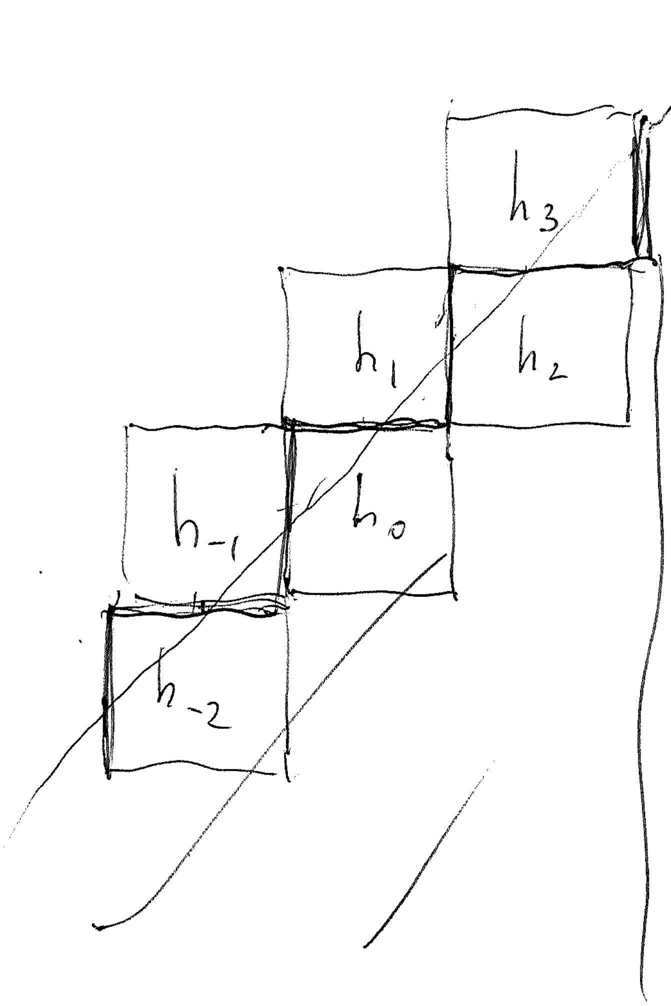
~~[illegible] Now~~ Obvious thing to hope for,

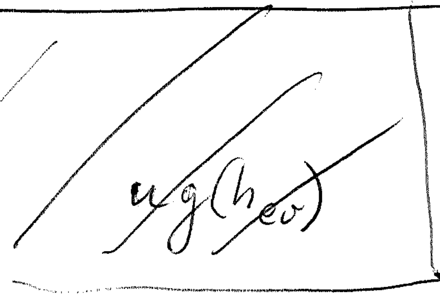
~~Sy~~ to do, is that $\frac{1}{\lambda-u}$ applied?
Say it simply. If $\xi \in E$, then $(\xi|\frac{1}{\lambda-u}$, the resolvent applied to ξ^* should be an eigenfunction far out, it should be a decaying eigfn. Also given an eigenfunction ψ ~~and a position n it should be possible to~~ decaying to the right say and a position n you should be able to form an element of $\mathcal{H}_{\geq n}$ from the components of ψ.



$$\begin{pmatrix} u^{-1}p_2 \\ u^{-1}q_2 \end{pmatrix} = g(h_2)\begin{pmatrix} 1 & 0 \\ 0 & u^{-1} \end{pmatrix}\begin{pmatrix} p_1 \\ q_1 \end{pmatrix}$$

$$= g(h_2)\begin{pmatrix} 1 & 0 \\ 0 & u^{-1} \end{pmatrix} g(h_1)\begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} p_0 \\ q_0 \end{pmatrix}$$





Recall how u is a product of two unitaries, each a direct sum of 2×2 blocks.