Can I calculate these? What is the idea. You want to ~~represent~~ start with $\psi(r,0)$ given, then write the solution $\psi(r,t)$ in the form.

$$\psi(r,t) = \int e^{xs+ys^{-1}} \begin{pmatrix} 1 \\ s \end{pmatrix} f(s) \frac{ds}{2\pi i s}$$

some "measure"

$$\boxed{\psi(r,t) = \int e^{r\left(\frac{s+s^{-1}}{2}\right) + t\left(\frac{-s+s^{-1}}{2}\right)} \begin{pmatrix} 1 \\ s \end{pmatrix} f(s) \frac{ds}{2\pi i s}}$$

$$ik = \frac{s+s^{-1}}{2} \qquad k = \frac{\rho-\rho^{-1}}{2}$$

$$-i\omega = \frac{s-s^{-1}}{2} \qquad \omega = \frac{\rho+\rho^{-1}}{2}$$

$\partial_x = \partial_r - \partial_t$ $\partial_y = \partial_r + \partial_t$	$(\partial_r - \partial_t)\psi^1 = \psi^2$ $(\partial_r + \partial_t)\psi^2 = \psi^1$	$\partial_t \psi = \begin{pmatrix} \partial_r & -1 \\ 1 & -\partial_r \end{pmatrix} \psi$	$\partial_r \psi = \begin{pmatrix} \partial_t & 1 \\ 1 & -\partial_t \end{pmatrix} \psi$

You want to solve

$$\psi(r,0) = \int e^{r(ik)} \begin{pmatrix} 1 \\ s \end{pmatrix} f(s) \frac{ds}{2\pi i s}$$

$$\psi(r,0) = \int \frac{dk}{2\pi} e^{ikr} \hat{\psi}_0(k)$$

$$k = \frac{\rho-\rho^{-1}}{2} \quad \omega = \frac{\rho+\rho^{-1}}{2}$$

$$dk = \frac{1+\rho^{-2}}{2} d\rho = \omega \frac{d\rho}{\rho}$$

$$\psi(r,0) = \int_{\rho=0}^{\rho=+\infty} e^{ikr} \begin{pmatrix} 1 \\ i\rho \end{pmatrix} f(i\rho) \frac{d\rho}{2\pi\rho}$$

k arise from ρ and $-\rho^{-1}$

$$= \int e^{ikr} \left(\begin{pmatrix} 1 \\ i\rho \end{pmatrix} f(i\rho) + \begin{pmatrix} 1 \\ -i\rho^{-1} \end{pmatrix} f(-i\rho^{-1}) \right) \frac{dk}{\omega}$$

$\forall k$ ~~want~~ have two values of s, namely $i\rho$ and $(i\rho)^{-1} = -i\rho^{-1}$

$$\psi(r,t) = \exp\left\{t\begin{pmatrix}\partial_r & -1\\ 1 & -\partial_r\end{pmatrix}\right\}\psi(r,0)$$

$$= \int e^{ikr} \underbrace{\exp it\begin{pmatrix}k & i\\ -i & -k\end{pmatrix}}_{A_k} \hat{\psi}(k,0)\,\frac{dk}{2\pi}$$

$A_k^2 = (k^2+1)I$

$\omega = \sqrt{k^2+1}$

$$e^{i\omega t}\,\frac{\omega + A_k}{2\omega} + e^{-i\omega t}\,\frac{-\omega + A_k}{-2\omega}$$

$$= \frac{1}{2\omega}\left(e^{i\omega t}\begin{pmatrix}\omega+k & i\\ -i & \omega-k\end{pmatrix} + e^{-i\omega t}\begin{pmatrix}\omega-k & -i\\ i & \omega+k\end{pmatrix}\right)$$

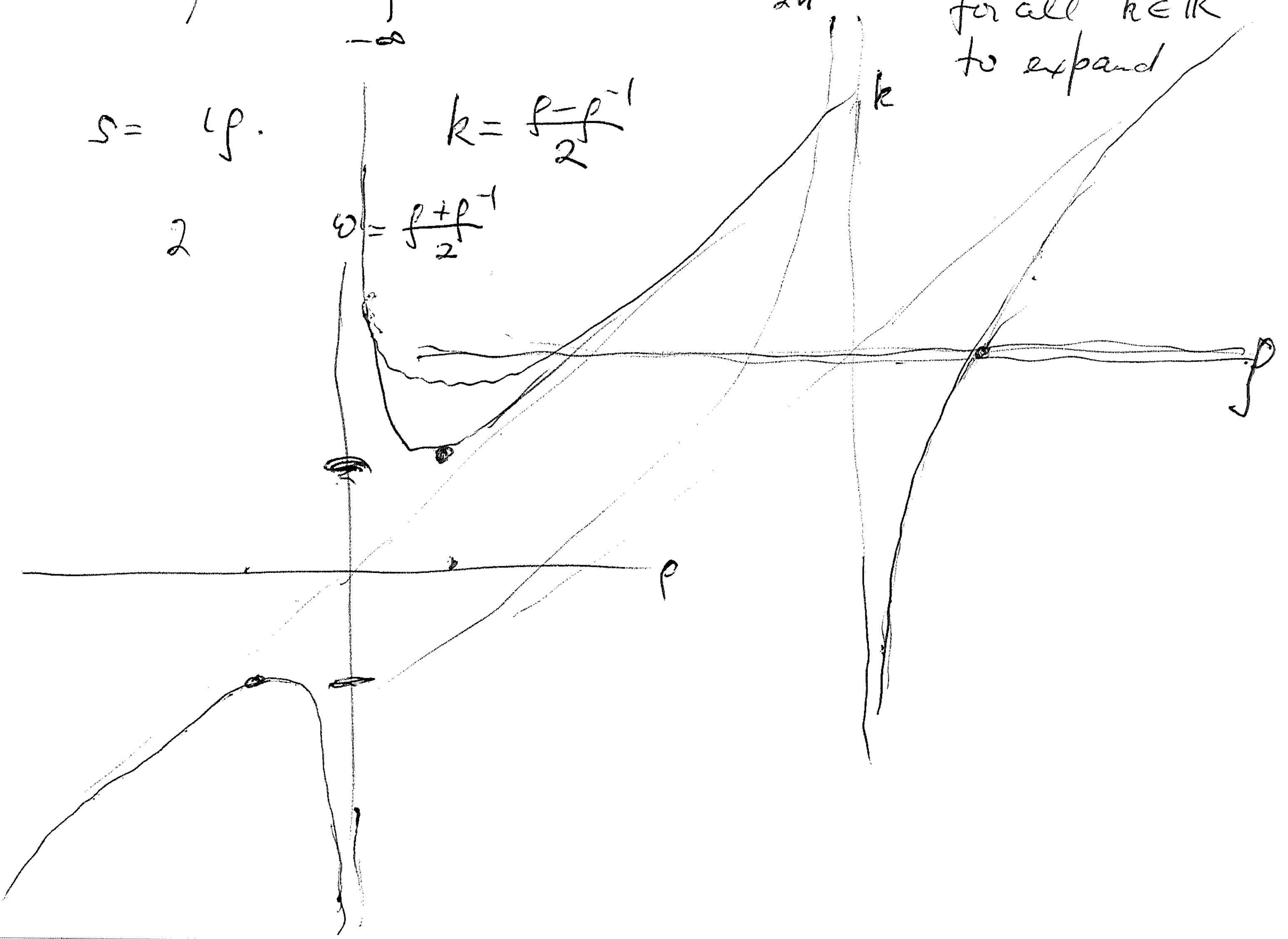
If you want to ~~write~~ find $f(s)$ + contour so that

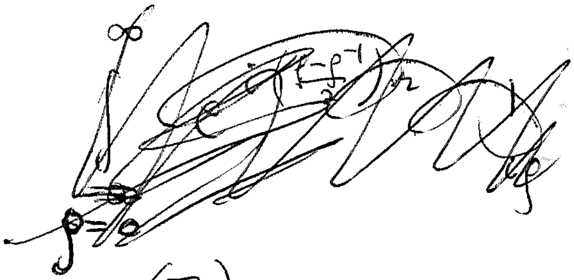
$$\psi(r,0) \overset{?}{=} \int_{\infty} e^{ikr}\begin{pmatrix}1\\ s\end{pmatrix} f(s)\,\frac{ds}{2\pi i s} \qquad ik = \frac{s+s^{-1}}{2}$$

know $\psi(r,0) = \int_{-\infty} e^{ikr}\,\hat{\psi}(k,0)\,\frac{dk}{2\pi}$ need ~~all~~ e^{ikr} for all $k \in \mathbb{R}$ to expand

$s = i\rho.$ $\qquad k = \frac{\rho - \rho^{-1}}{2}$

2 $\qquad \omega = \frac{\rho + \rho^{-1}}{2}$





$$d\left(\begin{pmatrix} z \\ 1 \end{pmatrix} f(z)\right) \wedge \begin{pmatrix} z \\ 1 \end{pmatrix} f(z) = \begin{vmatrix} (dz) f & (dz) f \\ 0 & f \end{vmatrix} = (dz) f^2$$

$$\begin{pmatrix} z \\ 1 \end{pmatrix} c \longmapsto c^2 dz = c^2 e^{i\theta} i\, d\theta$$

$\in i\mathbb{R}_{\geq 0}$ when $c e^{i\theta/2} = z^{1/2} c \in \mathbb{R}$

$z^{-1/2}\begin{pmatrix} z \\ 1 \end{pmatrix}$ is real so

$$\left(f(z)\begin{pmatrix} z \\ 1 \end{pmatrix}\right)^* = \left(\left(f(z) z^{\frac{1}{2}}\right) z^{-\frac{1}{2}}\begin{pmatrix} z \\ 1 \end{pmatrix}\right)^* = \overline{f(z) z^{\frac{1}{2}}}\; z^{-\frac{1}{2}}\begin{pmatrix} z \\ 1 \end{pmatrix}$$

$$= \left(\overline{f(z)}\, z^{-1}\right)\begin{pmatrix} z \\ 1 \end{pmatrix}$$

$$\left\| f(z)\begin{pmatrix} z \\ 1 \end{pmatrix}\right\|^2 = dz\, f \bar{f} \frac{1}{z} = |f|^2 \frac{dz}{z}$$

Write up $\mathcal{O}(-1)$ stuff

Riemann sphere $\mathbb{C} \cup \infty \simeq \mathbb{CP}^1$ = space of lines $\begin{pmatrix} z \\ 1 \end{pmatrix}\mathbb{C}$ in $\mathbb{C}^2$. $\mathcal{O}(-1)$ canonical sub-line bundle of $\mathcal{O} \otimes \mathbb{C}^2$, sections $s(z) = f(z)\begin{pmatrix} z \\ 1 \end{pmatrix}$.

Action of $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2, \mathbb{C})$ $\quad g(z) = \begin{pmatrix} az+b \\ cz+d \end{pmatrix}$

right action on functions

$$f(z) \longrightarrow f(gz) = f\left(\frac{az+b}{cz+d}\right)$$

on sections s of $\mathcal{O}(-1)$: $s \mapsto g^{-1} s g$

$$f(z)\begin{pmatrix} z \\ 1 \end{pmatrix} \longmapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} f\left(\frac{az+b}{cz+d}\right)\begin{pmatrix} \frac{az+b}{cz+d} \\ 1 \end{pmatrix}$$

$$f\left(\frac{az+b}{cz+d}\right)\frac{1}{cz+d}\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1}\begin{pmatrix} az+b \\ cz+d \end{pmatrix} \neq \begin{pmatrix} z \\ 1 \end{pmatrix}.$$

on differentials

$$f(z)\,dz \longmapsto f\left(\frac{az+b}{cz+d}\right) d\left(\frac{az+b}{cz+d}\right) = f\left(\frac{az+b}{cz+d}\right)\frac{(ad-bc)}{(cz+d)^2}$$

Comparison of action on $\mathcal{O}(-1)$ and Ω^1 indicates there is a natural symmetric pairing

$$\left(f_1(z)\begin{pmatrix} z \\ 1\end{pmatrix},\ f_2(z)\begin{pmatrix} z \\ 1\end{pmatrix}\right) \longmapsto f_1(z)f_2(z)\,dz \qquad \mathcal{O}(-1)\otimes_{\mathcal{O}}\mathcal{O}(-1) \to \Omega^1$$

provided one restricts to $SL(2,\mathbb{C})$.

Def.

$$s_1, s_2 \longmapsto ds_1 \wedge s_2$$

$$\left| df_1\begin{pmatrix} z \\ 1\end{pmatrix} \ \ f_2\begin{pmatrix} z \\ 1\end{pmatrix}\right| = \begin{vmatrix} (df_1)z + f_1\,dz & f_2 z \\ df_1 & f_2\end{vmatrix} = f_1 f_2\,dz$$

$\therefore$ a non-zero tangent vector t at z yields nonzero quadratic function $c\begin{pmatrix} z \\ 1\end{pmatrix} \longmapsto c^2(dz,t)$ on $\mathcal{O}(-1)_z \longrightarrow \mathbb{C}$, whence a real line in $\mathcal{O}(-1)_z$ where $c^2(dz,t) \geq 0$.

extends to sections over an oriented embedded closed curve,

Example $\mathbb{R}\cup\{\infty\} \subset \mathbb{C}\cup\{\infty\}$.

$c\begin{pmatrix} x \\ 1\end{pmatrix}$ real when $c^2\,\underline{dx} \geq 0$ wrt orientation (increasing x)

i.e. $c^2 \geq 0$ $\quad c \in \mathbb{R}$. $\qquad \left(c\begin{pmatrix} x \\ 1\end{pmatrix}\right)^* = \bar{c}\begin{pmatrix} x \\ 1\end{pmatrix}$

$$\left\| f\begin{pmatrix} x \\ 1\end{pmatrix}\right\|^2 = \int \bar{f} f\,dx$$

$^+HS \oplus {}^+H = $

S

$^+H \oplus {}^-H = L^2(\mathbb{R})$

$S^1 = \{z \mid |z| = 1\}$ $\quad z = e^{i\theta}$ $\quad$ orient increasing θ.

$$c\binom{z}{1} \longmapsto c^2\, dz = c^2 e^{i\theta} i\, d\theta,$$

define $c\binom{z}{1}$ to be real when $c^2 e^{i\theta} \geq 0$.

i.e. $cz^{1/2}$ real. Thus [illegible]

[illegible]

$$z^{-1/2}\binom{z}{1} = \binom{z^{1/2}}{z^{-1/2}} \text{ is real, so } \left(c\binom{z}{1}\right)^* =$$

$$\left(cz^{1/2}\binom{z^{1/2}}{z^{-1/2}}\right)^* = \overline{cz^{1/2}}\binom{z^{1/2}}{z^{-1/2}} = \bar{c}z^{-1/2}\binom{z^{1/2}}{z^{-1/2}} = \bar{c}z^{-1}\binom{z}{1}$$

$$\left\| f\binom{z}{1} \right\|^2 = \int \overline{f(z)}\, z^{-1} f(z)\, \frac{dz}{i} = \int |f(z)|^2 \frac{dz}{iz}$$

Note $\binom{z^{1/2}}{z^{-1/2}}$ is anti-periodic, so the real line bundle over S^1 inside $\mathcal{O}(-1)$ is Mobius bundle.

$$L^2(S^1, \mathcal{O}(-1)) \longrightarrow L^2(\mathbb{R} \cup \infty, \mathcal{O}(-1))$$

$$f(z)\binom{z}{1} \longmapsto f\left(\frac{1+ix}{1-ix}\right)\frac{1}{1-ix}\binom{x}{1}$$

$$\int |f(z)|^2 \frac{dz}{iz} \qquad \int \left| f\left(\frac{1+ix}{1-ix}\right)\right|^2 \frac{dx}{1+x^2}$$

$2\pi \qquad \pi$

$f(z)\binom{z}{1}$ $\uparrow$ $f(z)$ $\qquad$ $f(x)\binom{x}{1}$ $\uparrow$ $f(x)$ $\qquad$ $\begin{vmatrix} i & 1 \\ -i & 1 \end{vmatrix} = 2i$

$z = \dfrac{1+ix}{1-ix}$

$$L^2\left(S^1, \frac{dz}{2\pi i z}\right) \qquad L^2(\mathbb{R}, dx)$$

$$f(z) \longmapsto f\left(\frac{1+ix}{1-ix}\right)\frac{1}{1-ix}$$

$$1,\ z^{-1} \qquad , \frac{1}{1-ix},\ \frac{1}{1+ix},$$

$E = E_+ \oplus S E_-$ Why should this yield the Birkhoff factorization?

Take a simple case. $L^2 = H_- \oplus S H_+$ where S is a loop. Look at $zH_- \cap SH_+$ which is 1 dimensional, i.e. $\exists\, 0 \neq f_+ \in H_+$ such that $Sf_+ \in zH_- = \mathbb{C} + H_-$. Then $\boxed{Sf_+ = f_-}$. But now you have to show $f_\pm$ invertible [illegible] on $D_\pm$, maybe you need boundedness of some sort.

$$E = E_- \oplus SE_+$$

Same argument $zE_- \cap SE_+ = V$

whence

E_- contains $\bigoplus_{n<0} z^n V$

SE_+ —— $\bigoplus_{n\geq 0} z^n V$

Your ~~splitting~~ should give you V.

$$\begin{array}{ccccc} & & E_\pm & & \\ & & \downarrow S_{f_\pm} & & \\ E_+ & \longrightarrow & E & \xrightarrow{f_+^*} & E_\pm \end{array}$$

Take S a loop and assume

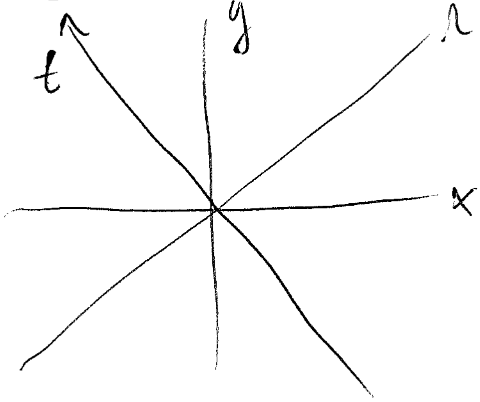
$L^2 = H_- \oplus SH_+$ $\quad \mathbb{C}v = zH_- \cap SH_+$ $\quad \|v\| = 1$.

Is v a cyclic vector?

$$\cdots \oplus H_- \oplus (\mathbb{C}v + \mathbb{C}zv + \cdots + \mathbb{C}z^N v) \oplus \underbrace{Sz^{N+1}H_+}_{\text{go to zero}}$$

The idea is that you have these opposing filtrations. You want to show that v is a cyclic vector, in fact you probably want it to be given by a bounded measurable function.

Back to your wave equation calculations.



$$r = x + y \qquad t = -x + y$$

$$\partial_x = \partial_r - \partial_t \qquad \partial_y = \partial_r + \partial_t$$

$$\partial_t \psi = \begin{pmatrix} \partial_r & i \\ i & -\partial_r \end{pmatrix} \psi$$

$$(\partial_t - \partial_r)\psi^1 = i\psi^2$$
$$(\partial_t + \partial_r)\psi^2 = i\psi^1$$

$$-\partial_x \psi^1 = i\psi^2$$
$$\partial_y \psi^2 = i\psi^1$$

$$\psi(x,t) = e^{i(x\xi + y\eta)} \begin{pmatrix} v^1 \\ v^2 \end{pmatrix}$$

$$-i\xi v^1 = i v^2 \qquad \xi\eta = -1$$
$$i\eta v^2 = i v^1 \qquad \eta = -\xi^{-1}$$

$$\psi(x,t) = e^{i(x\xi - y\xi^{-1})} \begin{pmatrix} 1 \\ -\xi \end{pmatrix} \text{const.} \qquad v^2 = -\xi v^1$$

$$\partial_t \psi = \begin{pmatrix} \partial_r & i \\ i & -\partial_r \end{pmatrix} \psi \qquad \psi(r,t) = \exp\left\{ t \begin{pmatrix} \partial_r & i \\ i & -\partial_r \end{pmatrix} \right\} \psi(r,0)$$

$$= \int_{-\infty}^{\infty} e^{ikr} \underbrace{\exp\left\{ it \begin{pmatrix} k & 1 \\ 1 & -k \end{pmatrix} \right\}} \hat{\psi}_0(k) \frac{dk}{2\pi}$$

$$\omega = \pm\sqrt{k^2+1}$$

$$\left(e^{i\omega t} \frac{\omega + A}{2\omega} + e^{-i\omega t} \frac{\omega - A}{2\omega} \right)$$

$$= \int_{-\infty}^{\infty} \left\{ e^{i(kr+\omega t)} \begin{pmatrix} \omega+k & 1 \\ 1 & \omega-k \end{pmatrix} + e^{i(kr-\omega t)} \begin{pmatrix} \omega-k & -1 \\ -1 & \omega+k \end{pmatrix} \right\} \frac{\hat{\psi}_0(k)}{2\omega} \frac{dk}{2\pi}$$

$$kr-\omega t = k(x+y) - \omega(-x+y) = (\omega+k)x - (\omega-k)y = \rho x - \rho^{-1} y$$

For each $k = \frac{\rho-\rho^{-1}}{2}$ there are two ρ values related by $\rho \longrightarrow -\rho^{-1}$.

$$kr+\omega t = k(x+y) + \omega(-x+y) = -\rho^{-1}x + \rho y$$

$$\psi(r,t) = \int_{k=-\infty}^{\infty} e^{i(\rho x-\rho^{-1}y)} \begin{pmatrix} \rho^{-1} & -1 \\ -1 & \rho \end{pmatrix} \hat{\psi}_0(k) \frac{dk}{4\pi\omega}$$

$$\psi(r,t) = \int_{-\infty}^{\infty} \Bigg[\underbrace{\frac{e^{i(\rho x - \rho^{-1} y)}}{\rho+\rho^{-1}} \begin{pmatrix} \rho^{-1} & -1 \\ -1 & \rho \end{pmatrix}}_{F(\rho)} + \underbrace{\frac{e^{i(-\rho^{-1}x+\rho y)}}{+\rho+\rho^{-1}} \begin{pmatrix} +\rho & +1 \\ +1 & +\rho^{-1} \end{pmatrix}}_{F(-\rho^{-1})} \Bigg] \hat{\psi}_0(k) \frac{dk}{2\pi}$$

$$= \int_{k=-\infty}^{k=\infty} \left(F(\rho) + F(-\rho^{-1})\right) \hat{\psi}_0\left(\frac{\rho-\rho^{-1}}{2}\right)\left(\frac{\rho+\rho^{-1}}{2}\right)\frac{d\rho}{\rho}$$

Thus for each $-\infty < k < \infty$, can choose unique $\rho > 0 \ni \frac{\rho-\rho^{-1}}{2} = k$. $dk = \omega \frac{d\rho}{\rho}$

$$\int_{\rho=0}^{\infty} \left(F(\rho) + F(-\rho^{-1})\right) \hat{\psi}_0\left(\frac{\rho-\rho^{-1}}{2}\right) \left(\frac{\rho+\rho^{-1}}{2}\right)\frac{d\rho}{\rho\, 2\pi}$$

$$= \left(\int_{\rho=-\infty}^{0} + \int_{\rho=0}^{\infty} \right) \left(e^{i(\rho x - \rho^{-1} y)} \begin{pmatrix} \rho^{-1} & -1 \\ -1 & \rho \end{pmatrix} \right) \hat{\psi}_0\left(\frac{\rho-\rho^{-1}}{2}\right) \frac{d\rho}{4\pi\rho}$$

What to do next? To understand details of Birkhoff decomp. Take $L^2 = H_- \oplus SH_+$. These are closed subspaces giving rise to complementary filtrations. You know that $z^n H_- \cap SH_+$ has dim n $n \geq 0$ and codim $-n$ for $n<0$. Let v be a unit vector in $zH_- \cap SH_+$. You want to show v is a cyclic vector and that the measure is equiv. to Lebesgue measure.

$v = f_- = Sf_+$ with $f_- \in \frac{1}{z} H_-$, $f_+ \in H_+$

Since S is unitary, i.e. $|S(z)| = 1$ for $|z|=1$ you should have $|f_-| = |f_+|$. ~~Something going on?~~

~~What next~~. You get this vector $v \in H_+$

Natural question: What is the closure of $\mathbb{C}[z^{\pm}]v$?

Point: v is an L^2 function, an elt of L^2, and the cyclic repn gen. by v should be the measure $|f_-|^2 \frac{d\theta}{2\pi} = |f_+|^2 \frac{d\theta}{2\pi}$. Because things are taking place in L^2 the measure is abs. cont wrt Lebesgue measure

---

~~go back~~ Repeat to the case $L^2 = H_- \oplus SH_+$

then $f_- = Sf_+$ in $zH_- \cap SH_+$. ~~This vector~~ make into a unit vector v. Then look at the subrepresentation gen. by v. Should be the measure $|f_-|^2 \frac{d\theta}{2\pi} = |f_+|^2 \frac{d\theta}{2\pi}$.

Conversely suppose given $d\mu$ $\rho \frac{d\theta}{2\pi}$, form $L^2(S^1, \rho\frac{d\theta}{2\pi})$, go through orthogonal poly stuff of Szego. Need Szegö alternative, . statement

~~Construct~~ $E = L^2(d\mu) \supset E_+ = H^2_+(d\mu)$

span of $1, z, z^2, \ldots$

Start with $L^2(d\mu)$ form $H^2_+(d\mu)$ closure of $\mathbb{C}[z]$, have! $g_n \in \mathbb{R}_{>0} + zF_{n-1} \perp zF_{n-1}$, unit vector, ~~and can~~ let $n \to \infty$. $g_n \to g_\infty$ equiv. to $\sum |h_n|^2 < \infty$, bad case is $\|g_n\| \to 0$ means $zH_2^+ = H_2^+$.

Better: $\lim_{n\to\infty} g_n \ \exists \iff \sum |h_n|^2 < \infty$.

In this case $z^n g_\infty$ is an orth sequence, $L^2(d\mu)$ splits $L^2(\frac{d\theta}{2\pi})g_\infty \oplus L^2(d\mu_{sing})$

Start again with $L^2(d\mu) \supset H_+(d\mu) = \overline{\mathbb{C}[z]}$

$\left(\begin{array}{l} g_n \in \mathbb{R}_{>0} + zF_{n-1}, \quad \|g_n\| = 1 \\ \perp zF_{n-1} \end{array} \right.$

$\lim g_n = g_\infty$ exists $\iff \sum |h_n|^2 < \infty$.

$g_\infty \in \mathbb{R}_{>0} + zH_+$

$\|g_\infty\| = 1$ $\quad g_\infty \in H_+ \ominus zH_+$, so $z^n g_\infty$ $n \in \mathbb{Z}$ is orth seq $\Rightarrow L^2(S^1, \frac{d\theta}{2\pi})g_\infty$ splits off, leaving a comp. gen. by $1 - \tilde{g}_\infty$ of the form $L^2(d\mu')$ satisfying $zH_+ = H_+$.

Szegö theory $L^2(S^1, d\mu) \supset H_+ = \overline{\mathbb{C}[z]}$

$q_n \in (\mathbb{R}_{>0} + zF_{n-1}) \cap (zF_{n-1})^\perp$, $\lim_{n} q_n = q_\infty \ \exists$

$\iff \sum |h_n|^2 < \infty$, in this case $q_\infty \in \mathbb{R}_{>0} \cap zH_+$

$\cap (zH_+)^\perp$, so get $L^2(S^1, \frac{d\theta}{2\pi}) q_\infty \subset L^2(d\mu)$, so

$d\mu = |q_\infty|^2 \frac{d\theta}{2\pi} + d\mu'$, where $d\mu'$ such that

$zH_+ = H_+$. Point: $d\mu'$ ~~...~~ can have an absolutely continuous part wrt $\frac{d\theta}{2\pi}$. Why? Look at $q_\infty = \lim q_n$. Zeros of q_n outside S^1!

~~...~~ $\log(q_\infty)$ defined on D up to $2\pi i\mathbb{Z}$,

~~...~~ here I assume some sort of uniform convergence on $|z| \leq r < 1$. What's reasonable to assume? reasonable relation between q_∞ on S^1 and interior

Clearly $q_\infty \in H_+$ and $\|q_\infty\| = 1$. But since $q_\infty(z) \neq 0$ for $z \in D$, ~~says smooth~~ yields $\log(q_\infty)$ on D up to $2\pi i\mathbb{Z}$. $\mathrm{Re}(\log(q_\infty)) = \log|q_\infty|$ harmonic function on disk, look at boundary.

~~...~~ What must happen is that the behavior of this function must be restricted. Meaning? The Szegö det. formula says $\log(q_\infty(0)) = \prod(1-|h_n|^2)^{1/2}$

$= \frac{1}{2}\int_{|z|=1} \log(q_\infty(z)) \frac{d\theta}{2\pi}$, so $\log|q_\infty|$ should be ~~...~~ integrable

Review logic: $L^2(S^1, d\mu) \supset H_+(d\mu) = \overline{\mathbb{C}[z]}$

$g_n \in F_n \ominus zF_{n+1}$ $g_n(0) > 0$, $\|g_n\| = 1$

$g_\infty = \lim g_n \; \exists \iff \sum |h_n|^2 < \infty \iff zH_+ < H_+$

in this case then $L^2(d\mu) = L^2\left(|g_\infty|^2 \frac{d\theta}{2\pi}\right) \oplus L^2(d\mu')$

where $H_+(d\mu') = zH_+(d\mu')$. What properties should g_∞ have. $g_\infty \in H_+$ $\log(g_\infty)$ analytic in disk

$$\log(g_\infty(0)) = \prod (1 - |h_n|^2)^{1/2} = \frac{1}{2}\int \log|g_\infty| \frac{d\theta}{2\pi}$$

Therefore given $d\mu = \rho \frac{d\theta}{2\pi}$, ~~with~~ the condition for $zH_+^{(d\mu)} < H_+(d\mu)$, is that $\log \rho$ be integrable. Obvious for $\rho \geq 1$ part, so condition should be

$\int [\log(1/\rho)]^+ < \infty$. And because $\log(\rho)$ integrable its Fourier series is defined

$$\log(\rho) = \sum a_n z^n = g(z) + \overline{g(\bar{z})}$$

so $\rho = |e^g|^2$

OKAY. Again - go over the idea.

Go back to $L^2 = H_+ \oplus S H_+$, $f_- = S f_+$

S unitary in L^∞, $f_- \in zH_- = \mathbb{R}_{>0} + H_-$, $f_+ \in H_+$

$\|f_-\| = \|f_+\| = 1$. Look at $\overline{\mathbb{C}[z,z^{-1}]f_-} =$

$L^2(S^1, |f_-|^2 \frac{d\theta}{2\pi})$, $|f_-|^2 \in L^1$, f_- (should be non vanishing) analytic on D_-; $\log f_-$ defined up to $2\pi i\mathbb{Z}$, $\log|f_-| = \mathrm{Re}\log(f_-)$ harmonic in D_-.

Concern about zeroes of f_- on the bdry.

Good case: one sided harmonic function.

A harmonic function ≥ 0 on D same as pos. measure on boundary

You have $\log|f_+|$ harmonic on D

$\log|f_-|$ ——— D_-

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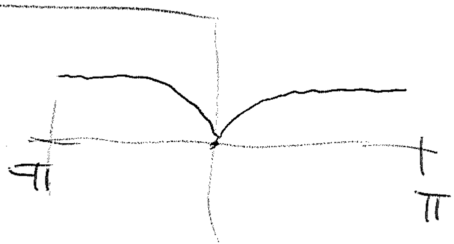
example

$$\rho(\theta) = \left|2\sin\left(\frac{\theta}{2}\right)\right|$$



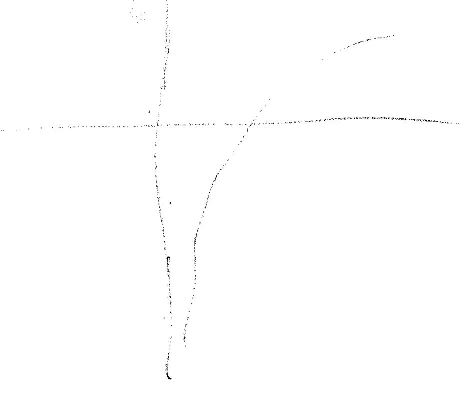
$\rho(\theta) \geq 0$ on circle

$\rho(0) = 0$ $\quad$ $\log\rho$ integrable

$\rho(\theta) = |\theta|^a$ near 0.

$\log\rho(\theta) = a\log|\theta|$

$$\int_0^1 (\log x)\,dx = \left[x\log x - x\right]_0^1 = -1$$



So what to do? To construct int. examples of S.

Simplify logic for constructing example.

Before: start with $S \ni L^2 = H_- \oplus SH_+$

get $f_- = Sf_+$, then $\rho = |f_-|^2 = |f_+|^2$. $\quad S \mapsto \rho$

Conversely given ρ form $\log(\rho)$, take its F.S.

$$\log \rho = \sum a_n z^n = h(z) + \overline{h(\bar z)} \qquad h = \frac{a_0}{2} + \sum_{n\geq 1} a_n z^n$$

$$f_- = e^{\overline{h(\bar z)}} \quad (h^*) \qquad f_+ = e^h, \text{ then } S = e^{h^*-h} = e^{-i \operatorname{Im} h}$$

$$S = e^{h^*-h} \qquad \rho = e^{h^*+h} \qquad\qquad f_- = e^{h^*} \quad f_+ = e^h$$

What happens if $h(z) = 1-z$ ~~nothing~~

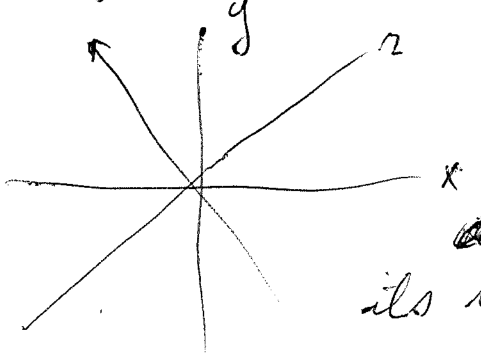
$$h(z) = 1-z \qquad h^*(z) = 1-z^{-1}$$

$$S = \frac{1-z}{1-z^{-1}} = z\frac{1-z}{z-1} = -z$$

This raises an interesting point, namely $S(z)$ as an operator on L^2 does not ~~see whether~~ tell you whether $S(z)$ is continuous. NO something is wrong. You must

At some stage

Digress. Back to $\partial_x \psi^1 = \psi^2$, $\partial_y \psi^2 = \psi^1$ $\qquad \psi(x,y) = e^{xs+ys^{-1}}\binom{1}{s}$ "universal solution".



$r = x + y$
$t = -x + y$

Consider an arb. soln $\psi(x,y)$. Can you ~~describe~~ express $\psi(x,y)$ in terms of its values on the line ~~t~~ $= 0$

$$\partial_x = \partial_r - \partial_t \qquad (\partial_r - \partial_t)\psi^1 = \psi^2 \qquad \partial_t \psi = \begin{pmatrix} \partial_r & -1 \\ 1 & -\partial_r \end{pmatrix}\psi$$
$$\partial_y = \partial_r + \partial_t \qquad (\partial_r + \partial_t)\psi^2 = \psi^1$$

As $\psi(r,0) = \int e^{ikr}\, \hat{\psi}_0(k)\frac{dk}{2\pi}$ then

$$\psi(r,t) = \int e^{ikr} \exp\left\{\begin{pmatrix} ik & -1 \\ 1 & -ik \end{pmatrix} t\right\} \underbrace{\hat{\psi}_0(k)}\frac{dk}{2\pi}$$

$$\int e^{-ikr'}\psi(r',0)\,dr'$$

So what's important is

$$K(r,t) = \int e^{ikr} \exp\begin{pmatrix} ik & -1 \\ 1 & -ik \end{pmatrix} t \;\frac{dk}{2\pi}$$

the kernel (or Green fn) that will give the contribution to $\psi(r,t)$ of $\psi(0,0)$.

---

go over what you did! You have $\partial_x \psi^1 = \psi^2$, $\partial_y \psi^2 = \psi^1$ with initial conditions $\psi^1(0,y)$, $\psi^2(x,0)$ given

Use L.T. $\int_0^\infty e^{-xs}\psi(x,y)\,dx = \hat{\psi}(s,y)$.

$$\hat{\psi}^2(s,y) = \widehat{\partial_x \psi^1}(s,y) = -\psi^1(0,y) + s\hat{\psi}^1(s,y) = \hat{\psi}^2(s,y)$$

$$\partial_y \hat{\psi}^2 = \widehat{\partial_y \psi^2} = \hat{\psi}^1 \qquad \hat{\psi}^1(s,y) = \partial_y \hat{\psi}^2(s,y)$$

$$\partial_y \hat{\psi}^2 = \hat{\psi}^1 = \frac{\psi^1(0,y) + \hat{\psi}^2}{s}$$

$$\partial_y \hat{\psi}^2 = s^{-1}\hat{\psi}^2 + s^{-1}\psi^1(0,y)$$

$$\left(\partial_y - s^{-1}\right)\hat{\psi}^2(s,y) = s^{-1}\psi^1(0,y)$$

$$\hat{\psi}^2(s,y) = \int_0^y e^{(y-y')s^{-1}} s^{-1}\psi^1(0,y')\,dy' + e^{ys^{-1}}\hat{\psi}^2(s,0)$$

$$\hat{\psi}^2(s,0) = \int e^{-x's}\psi^2(x',0)\,dx'$$

$$\psi^2(x,y) = \int_0^y \left(\int_{a-i\infty}^{a+i\infty} e^{xs+(y-y')s^{-1}} \frac{ds}{2\pi i s}\right)\psi^1(0,y')\,dy'$$

$$+ \int_0^\infty \left(\int_{a-i\infty}^{a+i\infty} e^{(x-x')s+ys^{-1}} \frac{ds}{2\pi i}\right)\psi^2(x',0)\,dx'$$

$$\int_{0^+-i\infty}^{0^++i\infty} e^{xs+(y)s^{-1}} \frac{ds}{2\pi i s} = H(x)\,J_0(x(y))$$

$$\int_{0^+-i\infty}^{0^++i\infty} e^{xs+ys^{-1}} \frac{ds}{2\pi i} = \delta(x) + H(x)\,y\,J_1(xy)$$

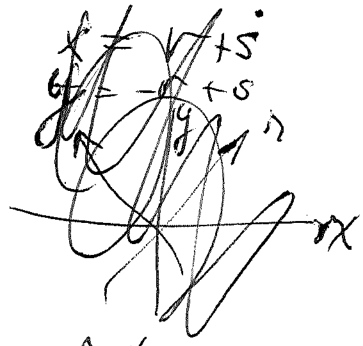
$$\psi^2(x,y) = \int_0^y J_0(x(y-y'))\,\psi^1(0,y')\,dy'$$

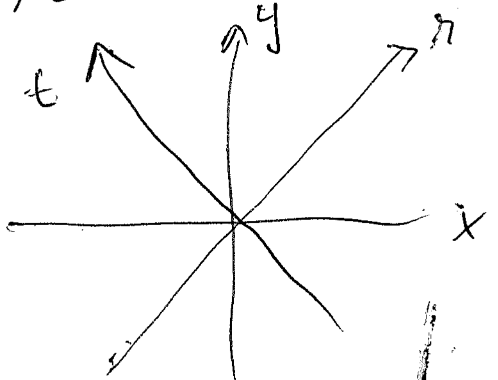
$$+ \psi^2(x,0) + \int_0^x y\,J_1((x-x')y)\,\psi^2(x',0)\,dx'$$

So what happens. Think of grid space

Can you express the universal solution

$\psi(x,y) = \int e^{xs+ys^{-1}} \begin{pmatrix}1\\ s\end{pmatrix}$ in terms of the Cauchy data $\psi(r,0)$ or $\psi(0,t)$





$r = x + y$ $\quad \frac{r+t}{2} = y \quad \frac{r-t}{2} = x$

$t = -x + y$

$$\psi(r,t) = e^{\frac{r-t}{2}s + \frac{r+t}{2}s^{-1}} \begin{pmatrix}1\\ s\end{pmatrix}$$

$$= e^{r\left(\frac{s+s^{-1}}{2}\right) + t\left(\frac{-s+s^{-1}}{2}\right)} \begin{pmatrix}1\\ s\end{pmatrix}$$

Want to express the functions

$e^{xs+ys^{-1}}$, $e^{xs+ys^{-1}} s^{-1}$ as linear combos

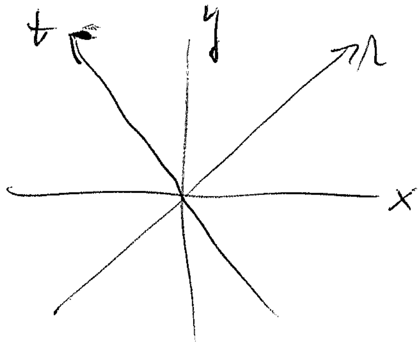
of the functions $e^{r\left(\frac{s+s^{-1}}{2}\right)}$, $e^{r\left(\frac{s+s^{-1}}{2}\right)} s^{-1}$

---

to express $e^{r\left(\frac{s+s^{-1}}{2}\right) + t\left(\frac{-s+s^{-1}}{2}\right)} \begin{pmatrix}1\\ s\end{pmatrix}$

in terms, as lin comb. of $e^{r\left(\frac{s+s^{-1}}{2}\right)}$, $e^{r\left(\frac{s+s^{-1}}{2}\right)} s^{-1}$

e



$$r = x + y \qquad \frac{r-t}{2} = x$$
$$t = -x + y \qquad \frac{r+t}{2} = y$$

$$e^{xs + ys^{-1}} = e^{\left(\frac{r-t}{2}\right)s + \left(\frac{r+t}{2}\right)s^{-1}}$$
$$= e^{r\left(\frac{s+s^{-1}}{2}\right) + t\left(\frac{-s+s^{-1}}{2}\right)}$$

What do you want to do? To express the function

$$\psi(r,t) = e^{r\left(\frac{s+s^{-1}}{2}\right) + t\left(\frac{-s+s^{-1}}{2}\right)} \begin{pmatrix} 1 \\ s \end{pmatrix}$$

in terms of the function $\psi(r,0)$ e.g. (Yes!)

$$\psi(r,t) = \exp t\begin{pmatrix} \partial_r & -1 \\ 1 & -\partial_r \end{pmatrix} \psi(r,0)$$

Check that $\partial_t \psi(r,t) = e^{r\left(\frac{s+s^{-1}}{2}\right) + t\left(\frac{-s+s^{-1}}{2}\right)} \left(\frac{-s+s^{-1}}{2}\right) \begin{pmatrix} 1 \\ s \end{pmatrix}$

$$\begin{pmatrix} \partial_r & -1 \\ 1 & -\partial_r \end{pmatrix} \psi(r,t) = e^{r(\;)+t(\;)} \begin{pmatrix} \frac{s+s^{-1}}{2} & -1 \\ 1 & -\frac{s+s^{-1}}{2} \end{pmatrix} \begin{pmatrix} 1 \\ s \end{pmatrix}$$

$$1 - \frac{s^2+1}{2} = \frac{1-s^2}{2}$$

it's very simple, or should be,

$$ik = \frac{s+s^{-1}}{2} \qquad -i\omega = \frac{-s+s^{-1}}{2}$$

$$i(k+\omega) = s \qquad i(k-\omega) = s^{-1}$$

$$ip = s \qquad i(-p^{-1}) = s^{-1}$$

so

$$\psi(r,t) = e^{i(kr-\omega t)} \begin{pmatrix} 1 \\ i(\omega+k) \end{pmatrix}$$

$$e^{ikr} \exp it\begin{pmatrix} k & +i \\ -i & -k \end{pmatrix} \begin{pmatrix} 1 \\ i(\omega+k) \end{pmatrix}$$

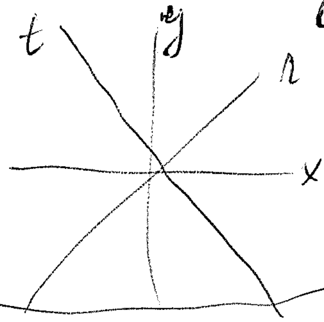
$$e^{ikr}\left(e^{-i\omega t}\,\frac{\omega + Ak}{2\omega} + e^{+i\omega t}\,\frac{\omega - Ak}{2\omega} \right)$$

$$= e^{ikr}\left(e^{-i\omega t}\begin{pmatrix} \omega+k & i \\ -i & \omega-k \end{pmatrix} + e^{i\omega t}\begin{pmatrix} \omega-k & -i \\ +i & \omega+k \end{pmatrix}\right)\frac{1}{2\omega}$$

Problem: To express $\psi(x,y) = e^{xs+ys^{-1}}\begin{pmatrix}1\\s\end{pmatrix}$ in terms of $\psi\left(\frac{r}{2},\frac{r}{2}\right) = e^{r\left(\frac{s+s^{-1}}{2}\right)}\begin{pmatrix}1\\s\end{pmatrix}$

---

think grid space ~~with generator~~ generated appropriately by $\psi(x,y) = e^{xs+ys^{-1}}\begin{pmatrix}1\\s\end{pmatrix}$, ~~also~~ but also by ~~r, t~~ $e^{r\left(\frac{s+s^{-1}}{2}\right)}\begin{pmatrix}1\\s\end{pmatrix}$ and by $e^{t\left(\frac{-s+s^{-1}}{2}\right)}\begin{pmatrix}1\\s\end{pmatrix}$



$r = x+y$
$t = x+y$

$$xs + ys^{-1} = \left(\frac{r-t}{2}\right)s + \left(\frac{r+t}{2}\right)s^{-1}$$
$$= r\left(\frac{s+s^{-1}}{2}\right) + t\left(\frac{-s+s^{-1}}{2}\right)$$

In some sense it should be true that that $e^{r\left(\frac{s+s^{-1}}{2}\right)}$, $e^{r\left(\frac{s+s^{-1}}{2}\right)}s^{-1}$ these functions for r real form an increasing staircase basis for grid space.

$$e^{r\left(\frac{s+s^{-1}}{2}\right)+t\left(\frac{-s+s^{-1}}{2}\right)}\begin{pmatrix}1\\s\end{pmatrix}$$

$f(r)$ matrix

$$e^{t\left(\frac{-s+s^{-1}}{2}\right)}\begin{pmatrix}1\\s\end{pmatrix} \overset{?}{=} \int e^{r\left(\frac{s+s^{-1}}{2}\right)}\begin{pmatrix}1\\s\end{pmatrix} f(r)\,dr$$

$$e^{t\left(\frac{-s+s^{-1}}{2}\right)}\begin{pmatrix}1\\ s\end{pmatrix} = \int F(r,t)\begin{pmatrix}1\\ s\end{pmatrix} e^{r\left(\frac{s+s^{-1}}{2}\right)}dr$$

$$e^{t\left(\frac{-s+s^{-1}}{2}\right)}\begin{pmatrix}1\\ s\end{pmatrix} = \int F(r,t)\begin{pmatrix}1\\ s\end{pmatrix} e^{r\left(\frac{s+s^{-1}}{2}\right)}dr \qquad F \text{ matrix}$$

right side is the Fourier transform of $F(r,t)$ with respect to r.

$$\left(\int F(t,r)\, e^{r\left(\frac{s+s^{-1}}{2}\right)}dr\right)\begin{pmatrix}1\\ s\end{pmatrix}$$

t const

To solve

$$e^{t\left(\frac{-s+s^{-1}}{2}\right)}\begin{pmatrix}1\\ s\end{pmatrix} = \left(\underbrace{\int F(t,r)\, e^{r\left(\frac{s+s^{-1}}{2}\right)}dr}_{\text{matrix}}\right)\begin{pmatrix}1\\ s\end{pmatrix}$$

put $\omega_s = \frac{-s+s^{-1}}{2}$ inv under $s \leftrightarrow s^{-1}$

$$\begin{pmatrix}1 & 1\\ s & s^{-1}\end{pmatrix}\begin{pmatrix}e^{t\omega_s} & 0\\ 0 & e^{-t\omega_s}\end{pmatrix} = \hat{F}\begin{pmatrix}1 & 1\\ s & s^{-1}\end{pmatrix}$$

so
$$\hat{F} = \begin{pmatrix}e^{t\omega} & e^{-t\omega}\\ se^{t\omega} & s^{-1}e^{-t\omega}\end{pmatrix}\begin{pmatrix}s^{-1} & -1\\ -s & 1\end{pmatrix}\frac{1}{s^{-1}-s}$$

$$= \left\{\frac{e^{t\omega_s}}{2\omega_s}\begin{pmatrix}s^{-1} & -1\\ 1 & -s\end{pmatrix} + \frac{e^{-t\omega_s}}{2\omega_s}\begin{pmatrix}-s & 1\\ -1 & s^{-1}\end{pmatrix}\right\}$$

Consider grid space realized as a space of entire L series, and functions of $s \in \mathbb{C}^\times$, with "universal soln" $\psi(x,y) = e^{xs+ys^{-1}}\begin{pmatrix}1\\s\end{pmatrix}$.

These generators restricted to $t=0$, $t=-x+y$ should form a basis for E. Thus there should be a ! expression

$$e^{xs+ys^{-1}}\begin{pmatrix}1\\s\end{pmatrix} = \text{lin comb of } e^{r\left(\frac{s+s^{-1}}{2}\right)}\begin{pmatrix}1\\s\end{pmatrix}$$

$$e^{r\left(\frac{s+s^{-1}}{2}\right)+t\left(\frac{-s+s^{-1}}{2}\right)}\begin{pmatrix}1\\s\end{pmatrix} = \int F(r',t)\, e^{r'\left(\frac{s+s^{-1}}{2}\right)}\begin{pmatrix}1\\s\end{pmatrix} dr'$$

$$e^{t\left(\frac{-s+s^{-1}}{2}\right)}\begin{pmatrix}1\\s\end{pmatrix} = \int F(r'+r,t)\, e^{(r')\left(\frac{s+s^{-1}}{2}\right)}\begin{pmatrix}1\\s\end{pmatrix} dr'$$

Thus you want a representation

$$e^{t\left(\frac{-s+s^{-1}}{2}\right)}\begin{pmatrix}1\\s\end{pmatrix} = \hat{F}\left(\frac{s+s^{-1}}{2}, t\right)\begin{pmatrix}1\\s\end{pmatrix}$$

$$\Rightarrow e^{t\left(\frac{-s^{-1}+s}{2}\right)}\begin{pmatrix}1\\s^{-1}\end{pmatrix} = \hat{F}(\quad, t)\begin{pmatrix}1\\s^{-1}\end{pmatrix}$$

$$\begin{pmatrix} e^{\omega_s t} & e^{-\omega_s t} \\ e^{\omega_s t} s & e^{-\omega_s t} s^{-1} \end{pmatrix}\begin{pmatrix}1 & 1\\ s & s^{-1}\end{pmatrix}^{-1} \qquad \begin{pmatrix} s^{-1} & -1 \\ -s & 1\end{pmatrix}\frac{1}{s^{-1}-s}$$

$$= \frac{e^{\omega_s t}}{2\omega_s}\begin{pmatrix}s^{-1} & -1\\ 1 & -s\end{pmatrix} + \frac{e^{-\omega_s t}}{-2\omega_s}\begin{pmatrix}+s & -1\\ +1 & -s^{-1}\end{pmatrix}$$

So we have $\omega_s = \frac{-s+s^{-1}}{2}$ $k_s = \frac{s+s^{-1}}{2}$

$$e^{t\omega_s}\begin{pmatrix}1\\s\end{pmatrix} = \underbrace{\hat{F}(k_s, t)\begin{pmatrix}1\\s\end{pmatrix}}$$

$$\int F(r', t)\, e^{r'k}\, dr'$$

$$= \int F(r', t) \begin{pmatrix}1\\s\end{pmatrix} e^{r'k}\, dr'$$

$$\hat{F}(k_s, t) = \frac{e^{\omega_s t}}{2\omega_s}\begin{pmatrix}s^{-1} & -1\\ 1 & -s\end{pmatrix} + \frac{e^{-\omega_s t}}{+2\omega_s}\begin{pmatrix}-s & +1\\ -1 & +s^{-1}\end{pmatrix}$$

It's not immediately obvious that the R.S. is function of $k_s = \frac{s+s^{-1}}{2}$.

$$\frac{e^{\omega_s t}s^{-1} - e^{-\omega_s t}s}{-s+s^{-1}} \qquad \frac{-e^{\omega_s t} + e^{-\omega_s t}}{2\omega_s}$$

$$\frac{e^{\omega_s t} - e^{-\omega_s t}}{2\omega_s} \qquad \frac{-e^{\omega_s t}s + e^{-\omega_s t}s^{-1}}{-s+s^{-1}}$$

What is the problem: To express $\psi(x,y) = e^{xs+ys^{-1}}\begin{pmatrix}1\\s\end{pmatrix}$ in terms of the "basis" arising from the line $r = x+y = 0$, namely

$$e^{t\left(\frac{-s+s^{-1}}{2}\right)}\begin{pmatrix}1\\s\end{pmatrix}$$

as t varies. So you want

$$e^{r\overbrace{\left(\frac{s+s^{-1}}{2}\right)}^{k_s}}\begin{pmatrix}1\\s\end{pmatrix} = \int F(r,t)\,e^{t\overbrace{\left(\frac{-s+s^{-1}}{2}\right)}^{\omega_s}}\begin{pmatrix}1\\s\end{pmatrix}dt$$

$\frac{r+t}{2} = y \qquad \frac{r-t}{2} = x$

$r = x+y \qquad t = -x+y$

$$xs+ys^{-1} = \left(\frac{r-t}{2}\right)s + \left(\frac{r+t}{2}\right)s^{-1} = r\left(\frac{s+s^{-1}}{2}\right) + t\left(\frac{-s+s^{-1}}{2}\right)$$

i.e. $$e^{r\overbrace{\left(\frac{s+s^{-1}}{2}\right)}^{k_s}}\begin{pmatrix}1\\s\end{pmatrix} = \hat{F}(r,\omega_s)\begin{pmatrix}1\\s\end{pmatrix}$$

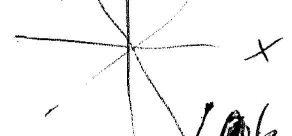
$\omega_s = \frac{-s+s^{-1}}{2}$ unchgd. by $s \mapsto -s^{-1}$

$$\begin{pmatrix}1 & 1\\ s & -s^{-1}\end{pmatrix}\begin{pmatrix}e^{rk_s} & 0\\ 0 & e^{-rk_s}\end{pmatrix}\begin{pmatrix}+s^{-1} & +1\\ +s & -1\end{pmatrix}\frac{1}{s+s^{-1}}$$

$$= \frac{e^{rk_s}}{s+s^{-1}}\begin{pmatrix}s^{-1} & 1\\ 1 & s\end{pmatrix} + \frac{e^{-rk_s}}{s+s^{-1}}\begin{pmatrix}s & -1\\ -1 & +s^{-1}\end{pmatrix}$$

YOU WANT TO RECOGNIZE THIS AS e^{rB}

So instead of $\int F(r,t)\,e^{t\omega_s}\begin{pmatrix}1\\s\end{pmatrix}$, you have the above.



$\partial_x\psi^1 = (\partial_r - \partial_t)\psi^1 = \psi^2$

$\partial_y\psi^2 = (\partial_r + \partial_t)\psi^2 = \psi^1$

$$\partial_t\psi = \begin{pmatrix}\partial_r & -1\\ 1 & -\partial_r\end{pmatrix}\psi$$

$$itA_k = i\begin{pmatrix}k & +i\\ -i & -k\end{pmatrix}t$$

$$\partial_r\psi = \begin{pmatrix}\partial_t & 1\\ 1 & -\partial_t\end{pmatrix}\psi$$

$$B = \frac{k_s}{s+s^{-1}}\begin{pmatrix} s^{-1} & 1 \\ 1 & s \end{pmatrix} + \frac{-k_s}{s+s^{-1}}\begin{pmatrix} s & -1 \\ -1 & s^{-1} \end{pmatrix}$$

$$= \frac{k_s}{s+s^{-1}}\begin{pmatrix} s^{-1}-s & 2 \\ 2 & s-s^{-1} \end{pmatrix}$$

$k_s = \frac{s+s^{-1}}{2}$

$\omega_s = \frac{-s+s^{-1}}{2}$

$$= \begin{pmatrix} \frac{-s+s^{-1}}{2} & 1 \\ 1 & \frac{s-s^{-1}}{2} \end{pmatrix} = \begin{pmatrix} \omega_s & 1 \\ 1 & -\omega_s \end{pmatrix}$$

$$A = \frac{1}{2}\begin{pmatrix} s^{-1}+s & -2 \\ 2 & -s-s^{-1} \end{pmatrix} = \begin{pmatrix} k_s & -1 \\ 1 & -k_s \end{pmatrix}$$

Apparently then

~~$\hat{F}(k_s, t) = e^{t(\ )}$~~

$$\hat{F}(r, \omega_s) = e^{r\begin{pmatrix} \omega_s & 1 \\ 1 & -\omega_s \end{pmatrix}}$$

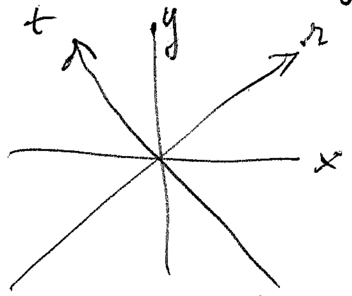
$$\boxed{F(r,t) = e^{r\begin{pmatrix} \partial_t & 1 \\ 1 & -\partial_t \end{pmatrix}}}$$

means $F(r,t)$ is the kernel of the operator on the r.t.

while $\hat{F}(k_s, t) = e^{t\begin{pmatrix} k_s & -1 \\ 1 & -k_s \end{pmatrix}}$ or

$$\hat{F}(r, t) = e^{t\begin{pmatrix} \partial_r & -1 \\ 1 & -\partial_r \end{pmatrix}}$$, kernel of

Repeat: Problem is to express $\psi(x,y) = e^{xs+ys^{-1}}\begin{pmatrix}1\\s\end{pmatrix}$
in terms of the "basis" $e^{r\left(\frac{s+s^{-1}}{2}\right)}\begin{pmatrix}1\\s\end{pmatrix}$



$r = x+y \qquad x = \frac{r-t}{2}$

$t = -x+y \qquad y = \frac{r+t}{2}$

$xs + ys^{-1} = \left(\frac{r-t}{2}\right)s + \left(\frac{r+t}{2}\right)s^{-1}$

$= r\underbrace{\left(\frac{s+s^{-1}}{2}\right)}_{k_s} + t\underbrace{\left(\frac{-s+s^{-1}}{2}\right)}_{\omega_s}$

$\partial_x\psi^1 = (\partial_r - \partial_t)\psi^1 = \psi^2$

$\partial_y\psi^2 = (\partial_r + \partial_t)\psi^2 = \psi^1$

$k_s^2 - \omega_s^2 = 1$

$\partial_r\psi = \begin{pmatrix}\partial_t & 1\\ 1 & -\partial_t\end{pmatrix}\psi \qquad \partial_t\psi = \begin{pmatrix}\partial_r & -1\\ 1 & -\partial_r\end{pmatrix}\psi$

$\psi(r,t) = e^{r\begin{pmatrix}\partial_t & 1\\ 1 & -\partial_t\end{pmatrix}}\psi(0,t)$

Apply to the universal solution

$\psi(r,t) = e^{r\underbrace{\left(\frac{s+s^{-1}}{2}\right)}_{k_s} + t\underbrace{\left(\frac{-s+s^{-1}}{2}\right)}_{\omega_s}}\begin{pmatrix}1\\s\end{pmatrix} \qquad B_s,\ B^2 = \omega_s^2+1$

$\psi(r,t) = e^{r\begin{pmatrix}\partial_t & 1\\ 1 & -\partial_t\end{pmatrix}}e^{t\omega_s}\begin{pmatrix}1\\s\end{pmatrix} = e^{t\omega_s}e^{r\begin{pmatrix}\omega_s & 1\\ 1 & -\omega_s\end{pmatrix}}\begin{pmatrix}1\\s\end{pmatrix}$

$k_s \sim B$

$= e^{t\omega_s}\left\{\frac{e^{rk_s}}{2k_s}\begin{pmatrix}k_s+\omega_s & 1\\ 1 & k_s-\omega_s\end{pmatrix} + \frac{e^{-rk_s}}{2k_s}\begin{pmatrix}k_s-\omega_s & -1\\ -1 & k_s+\omega_s\end{pmatrix}\right\}\begin{pmatrix}1\\s\end{pmatrix}$

$= e^{t\omega_s}\left(\cosh(rk_s)I + \frac{\sinh(rk_s)}{k_s}\begin{pmatrix}\omega_s & 1\\ 1 & -\omega_s\end{pmatrix}\right)\begin{pmatrix}1\\s\end{pmatrix}$

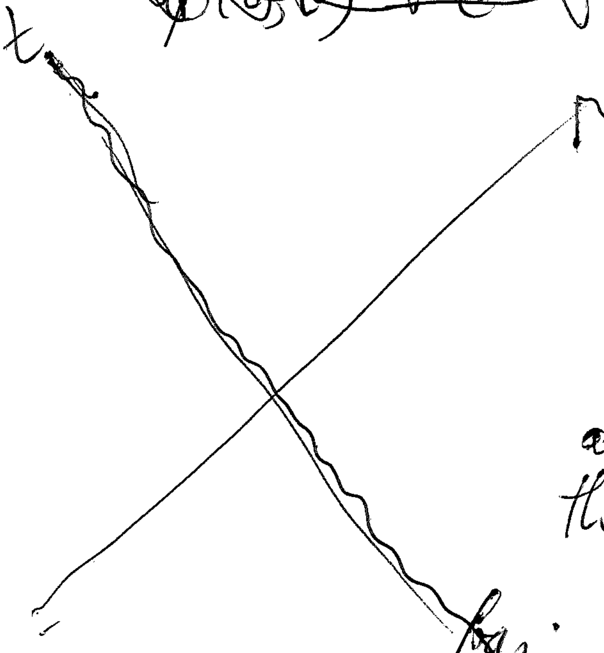
Does this help? What you want is

$\psi(r,t) = e^{r\left(\frac{s+s^{-1}}{2}\right) + t\left(\frac{-s+s^{-1}}{2}\right)}\begin{pmatrix}1\\s\end{pmatrix}$ expressed as

an integral $\int F(r,t')\,\psi(0,t')\,dt'$

Basis is $e^{t\left(\frac{-s+s^{-1}}{2}\right)}\begin{pmatrix}1\\s\end{pmatrix} = \psi(0,t)$

~~Ultimately you want $\psi(0,t) = e^{t\left(\frac{-s+s^{-1}}{2}\right)}$~~



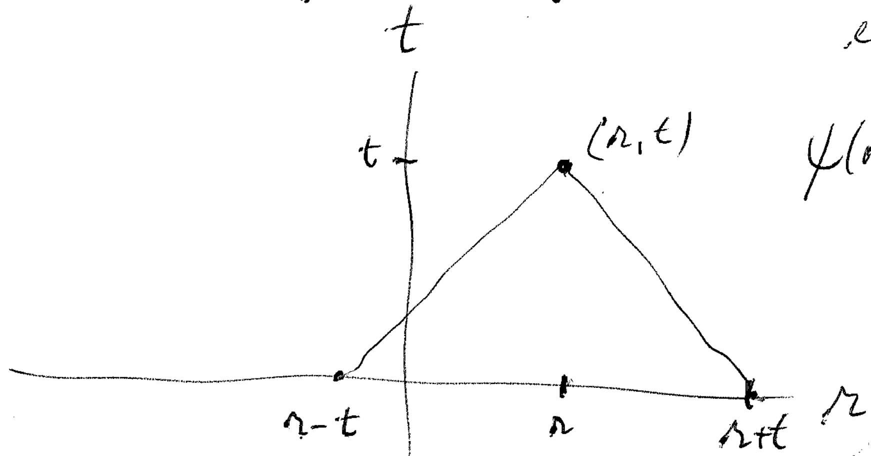
What do you want? to express

$$\psi(r,t) = e^{r\left(\frac{s+s^{-1}}{2}\right)+t\left(\frac{-s+s^{-1}}{2}\right)}\begin{pmatrix}1\\ s\end{pmatrix}$$

as a linear combination of the "basis"

$$\psi(0,t') = e^{t'\left(\frac{-s+s^{-1}}{2}\right)}\begin{pmatrix}1\\ s\end{pmatrix}$$

what range of t' is needed? So you expect



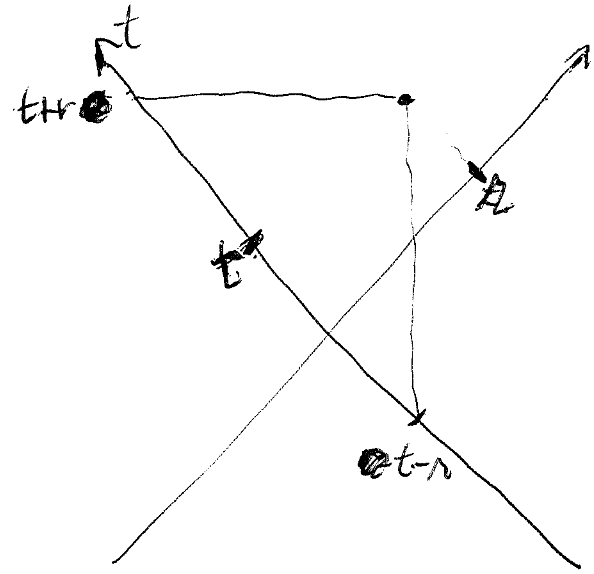
$$\psi(r,t) = \int_{r-t}^{r+t} dt'\,(\ ?\ )\,\psi(0,t')$$

NO

$$e^{rk_s+t\omega_s}\begin{pmatrix}1\\ s\end{pmatrix} = \int_{r-t}^{r+t} F(r,t,t')\,e^{t'\omega_s}\begin{pmatrix}1\\ s\end{pmatrix}dt'$$

~~$$e^{rk_s}\begin{pmatrix}1\\ s\end{pmatrix} = \int_{r-t}^{r+t} F(r,t,t')\,e^{(-t+t')\omega_s}\begin{pmatrix}1\\ s\end{pmatrix}dt'$$~~

$$= \int_{r}$$



$$e^{rk_s+t\omega_s}\begin{pmatrix}1\\s\end{pmatrix} = \int_{t-r}^{t+r} F(r,t,t')\, e^{t'\omega_s}\begin{pmatrix}1\\s\end{pmatrix} dt'$$

Let $t' = t + t''$

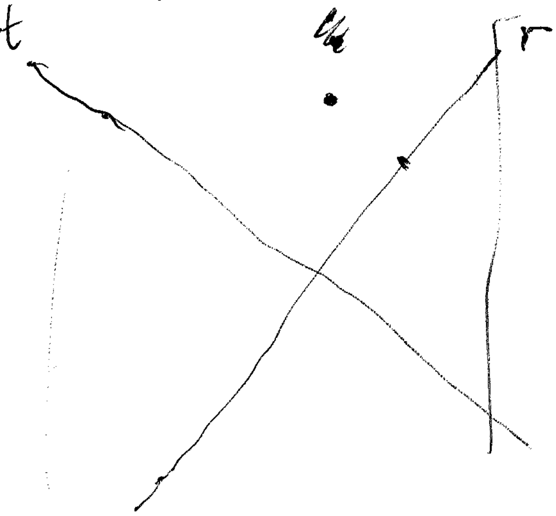
$$= \int_{-r}^{r} F(r,t,t+t'')\, e^{(t+t'')\omega_s}\begin{pmatrix}1\\s\end{pmatrix} dt''$$

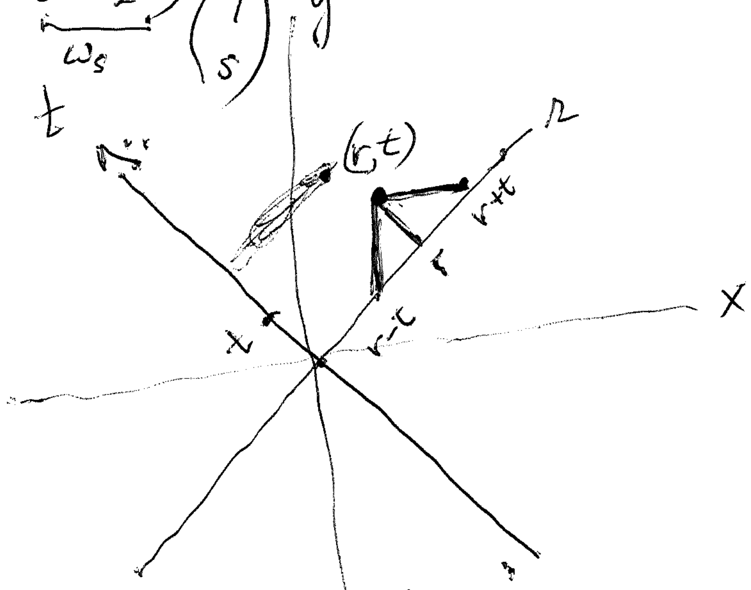
so

$$\underbrace{e^{rk_s}\begin{pmatrix}1\\s\end{pmatrix}}_{} = \int_{-r}^{r} F(r,t')\, \underbrace{e^{t'\omega_s}\begin{pmatrix}1\\s\end{pmatrix}}_{\psi(0,t')} dt'$$

Repeat what we learned. The problem is to express $\psi(x,y) = e^{xs+ys^{-1}}\begin{pmatrix}1\\s\end{pmatrix}$ in terms of the "basis" given by ~~the~~ components along $t=0$ (and also $r=0$). Use (r,t) coords

$$\psi(r,t) = e^{r\underbrace{\left(\frac{s+s^{-1}}{2}\right)}_{k_s} + t\underbrace{\left(\frac{-s+s^{-1}}{2}\right)}_{\omega_s}}\begin{pmatrix}1\\s\end{pmatrix}$$





You want $\psi(r,t) = e^{rk_s+t\omega_s}\begin{pmatrix}1\\s\end{pmatrix}$ written as

$\int_{r-t<r'<r+t}$ better $\int_{r-t}^{r+t} \left(\qquad\right) e^{r'k_s}\begin{pmatrix}1\\s\end{pmatrix} dr'$

enough to have

$$e^{t\omega_s}\begin{pmatrix}1\\s\end{pmatrix} = \int_{-t}^{t} (\quad) e^{r'k_s}\begin{pmatrix}1\\s\end{pmatrix} dr'$$

~~So what?~~ Something to get straight: the contours

$$\psi(r,t) = e^{t\begin{pmatrix}\partial_r & -1\\ 1 & -\partial_r\end{pmatrix}} \psi(r,0)$$

$$= \int_{-i\infty}^{i\infty} e^{rk_s} e^{t\begin{pmatrix}k_s & -1\\ 1 & -k_s\end{pmatrix}} \hat{\psi}_0(k_s) \frac{dk_s}{2\pi i}$$

$$\hat{\psi}_0(k_s) = \int_{-\infty}^{\infty} e^{-r'k_s}\psi(r',0)\,dr'$$

$$= \int K(r,t;r',0)\,\psi(r',0)\,dr'$$

where $K(r,t;r',0) = \int_{-i\infty}^{i\infty} e^{(r-r')k_s} e^{t\begin{pmatrix}k_s & -1\\ 1 & -k_s\end{pmatrix}} \frac{dk_s}{2\pi i}$

too confused spectral picture.

Szego. $L^2 = H_- \oplus S H_+$ $\qquad f_- = S f_+$

$f_- \in zH_-$
$f_-(0) > 0$
$\|f_-\| = 1.$

$\rho = |f_-|^2 = |f_+|^2$

to start with ~~[illegible]~~

$f_+(z) =$ ~~[illegible]~~ $1-z$ analytic in D

$\log f_+(z) = -\sum_{n\geq 1} \frac{z^n}{n}$ $\qquad (f_+)^*(z) = 1-z^{-1} = f_-(z).$

Then $\frac{f_-}{f_+} = \frac{1-z^{-1}}{1-z} =$

The pattern you want. Equivalence between various things. i) smooth strictly pos. measure $d\mu = \rho \frac{d\theta}{2\pi}$ $\rho > 0$

(ii) inv. and function on $\bar{D}$ extending smoothly to S^1.

(iii) smooth degree 0, loop in $U(1)$.

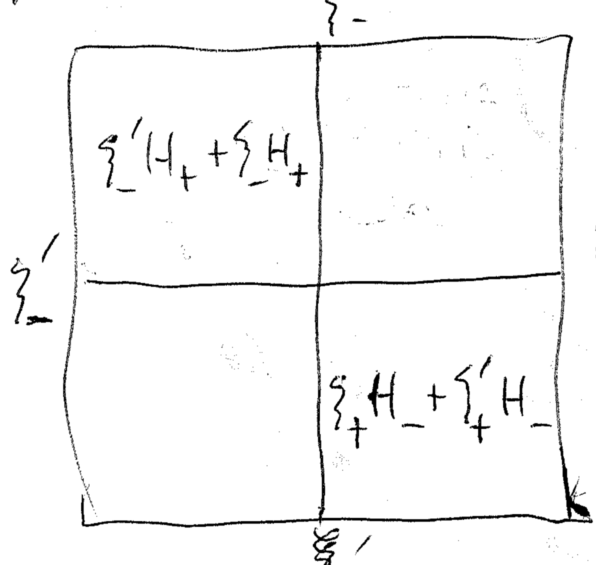
$$-\log \rho = \sum a_n z^n \qquad \bar{a}_n = a_{-n}$$

$$= h + \bar{h} \qquad h(z) = \frac{a_0}{2} + \sum_{n \geq 1} a_n z^n$$

$$g(z) = e^{h(z)} \qquad |g|^2 = \frac{1}{\rho}$$

$$p(z) = g^*(z) \qquad S = \frac{p}{g} \qquad Sg = p.$$

Review inverse scattering



loop in $SU(1,1)$

$$\begin{pmatrix} \xi_+ \\ \xi_- \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \xi'_- \\ \xi'_+ \end{pmatrix}$$

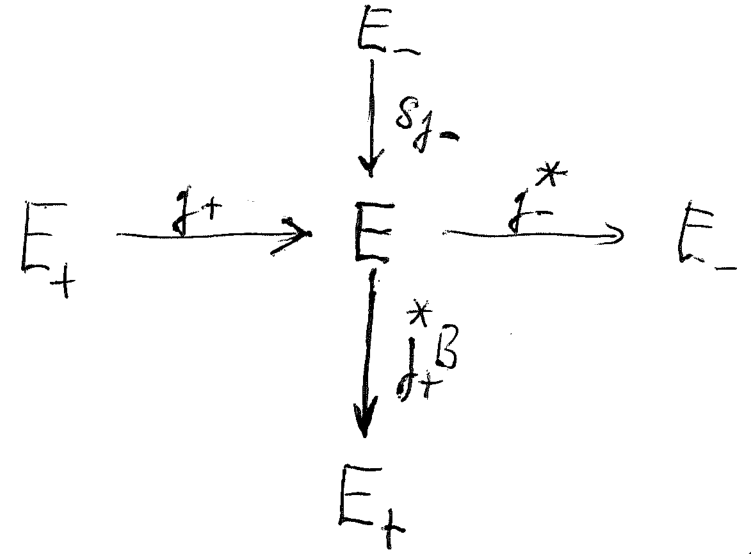
$$\begin{pmatrix} \xi_+ \\ \xi'_+ \end{pmatrix} = \frac{1}{d} \begin{pmatrix} 1 & b \\ -c & 1 \end{pmatrix} \begin{pmatrix} \xi'_- \\ \xi_- \end{pmatrix}$$

loop in $U(2)$ denoted S^t

$$S = \frac{1}{d} \begin{pmatrix} 1 & -\bar{b} \\ b & 1 \end{pmatrix}$$

Claim $(\xi'_- H_+ + \xi_- H_+) \oplus (\xi_+ H_- + \xi'_+ H_-) = \xi'_- L^2 + \xi_- L^2$

$$\begin{pmatrix} H_+ \\ H_+ \end{pmatrix} \oplus S \begin{pmatrix} H_- \\ H_- \end{pmatrix} = \begin{pmatrix} L^2 \\ L^2 \end{pmatrix}$$



vertical exactness identifies the B-orthog comp of E_+ with SE_-.

$$B = \begin{pmatrix} 1 & \bar b \\ b & -1 \end{pmatrix} \qquad \frac{1}{d}\begin{pmatrix} 1 & -\bar b \\ b & 1 \end{pmatrix} = S$$

$$B^2 = (1+|b|^2) I = |d|^2 I$$

$$j_+^* B \frac{1}{d} B \varepsilon j_- = j_+^* \bar d j_- \varepsilon$$

$$\bar d j_- = j_- \bar d$$

Conversely $\xi \in E$ in $\operatorname{Ker} j_+^* B$

$$0 = j_+^* B \xi \implies B\xi \in E_- \implies \xi \in B^{-1} E_-$$

$$\begin{pmatrix} 1 & \bar b \\ b & -1 \end{pmatrix} \xi \in E_- \qquad B^{-1} = \frac{1}{|d|^2} B$$

$$\implies \frac{1}{-1-|b|^2}\begin{pmatrix} -1 & -\bar b \\ -b & 1 \end{pmatrix} E_- = \frac{1}{d}\begin{pmatrix} 1 & \bar b \\ b & -1 \end{pmatrix} \frac{1}{\bar d} E_-$$

Suggests possibility of improvement

$B^2 = |d|^2$ so $\frac{1}{d} B$ is a projector

and $S = \frac{1}{d} B \varepsilon$

Is it clear that $j_-^* S j_-$ is invertible?

$$\bar{d} j_- = j_- \bar{d}$$

$$j_-^* S j_- \text{ inv.} \Longleftrightarrow j_-^* S \bar{d}^{-1} j_-$$

$$\frac{1}{d} S = \frac{1}{|d|^2}\begin{pmatrix} 1 & -\bar{b} \\ b & 1 \end{pmatrix}.$$

At the moment the only point

$$\mathrm{Ker}(j_+^* B) \qquad j_+^* B \xi = 0$$

$$\Longleftrightarrow B\xi \in E_-$$

$$\Longleftrightarrow \xi \in B^{-1} E_- = \frac{1}{|d|^2} B E_- = \frac{1}{d} B \frac{\xi}{\bar{d}} E_-$$

(above the circled $\frac{1}{d}B$: S)

$$\mathrm{Ker}(j_+^* B) = S j_-(E_-)$$

Now you've done the review, next step is to get the factorization + differentiation.

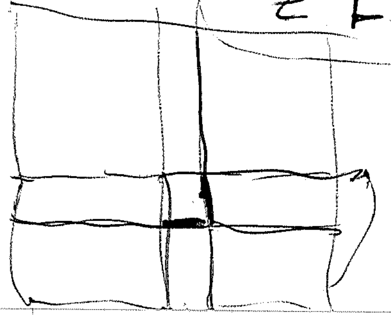
$$\begin{matrix} E_- \\ \oplus \\ E_+ \end{matrix} \xrightarrow[\sim]{(S j_- \;\; j_+)} E$$

$$W = \underline{(\xi_-' \;\; \xi_-)}\begin{pmatrix} H_+ \\ H_+ \end{pmatrix} \quad \text{comp. is}$$

$$z^{-1} E_+ \cap S E_-$$

two dimensional

$$z E_- \supset E_-$$
$$z E_+ \subset E_+$$



Now you have to concentrate on the important details required. You've settled the projection aspects.

Recover the potential. Replace b by $z^x b$

The situation: Instead of z^n on $L^2(S^1)$ you want to consider e^{ikx} where

$$z = \frac{1+ik}{1-ik} = \frac{-k+i}{k+i} = \begin{pmatrix} -1 & i \\ 1 & i \end{pmatrix}(k)$$

$$k = \begin{pmatrix} i & -i \\ -1 & -1 \end{pmatrix} z = \frac{i(z-1)}{-(z+1)} = \frac{1}{i}\frac{z-1}{z+1}$$

$$k = \begin{pmatrix} i & 1 \\ -i & 1 \end{pmatrix}^{-1}(z) = \begin{pmatrix} 1 & -1 \\ i & i \end{pmatrix}(z) = \frac{z-1}{i(z+1)}$$

$z=0 \to k=i$

$$\frac{d}{dx}\tan^2(\pi x) = 2\tan(\pi x)\sec^2(\pi x)\pi$$

[redacted] Have #200. S. Kass, #200.

$D^n \sin(ax) =$

x^x

$$\frac{d}{dx}(\ln(\sin x)) = [redacted] \frac{\cos x}{\sin(x)} = \cot(x)$$

$$x^x(\ln(x)+1)$$

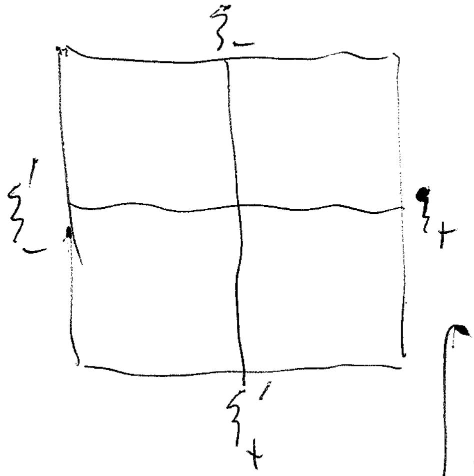
$$e^{x\ln(x)}\left(1\ln(x)+x\frac{1}{x}\right)$$

$$X^x = \left(e^{\ln x}\right)^x = e^{x\ln(x)}$$

So you need to understand how to handle k, as in e^{ikx}

Go back to $\quad j_+^* B j_+ = \begin{pmatrix} 1 & T^* \\ T & -1 \end{pmatrix}$ $\qquad T = j_+^* b j_+$

$j_+ : H_+ \hookrightarrow L^2$



$$p = \xi_-'(1-f) + \xi_-(-g)$$
$$q = \xi_-'(-\phi) + \xi_-(1-\psi)$$

$$\int \begin{pmatrix} H_+ \\ H_+ \end{pmatrix}^* \begin{pmatrix} 1 & \bar{b} \\ b & -1 \end{pmatrix} \begin{pmatrix} 1-f & -\phi \\ -g & 1-\psi \end{pmatrix} = 0$$

$$\begin{pmatrix} 1 & T^* \\ T & -1 \end{pmatrix} \begin{pmatrix} f & +\phi \\ g & \psi \end{pmatrix} = \begin{pmatrix} 0 & j_+^*(\bar{b}) \\ j_+^*(b) & 0 \end{pmatrix}$$

now you propose to vary b to bz^*

$$\begin{pmatrix} f & \phi \\ g & \psi \end{pmatrix} = \begin{pmatrix} 1 & T^* \\ T & -1 \end{pmatrix} \begin{pmatrix} 0 & j_+^*(\bar{b}) \\ j_+^*(b) & 0 \end{pmatrix}$$

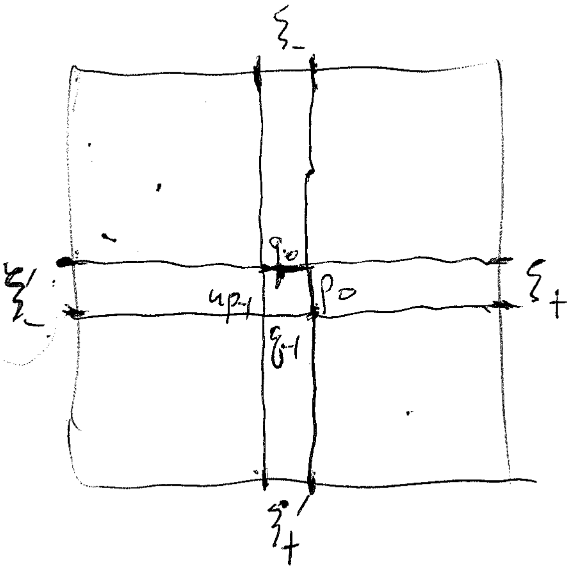
You want to vary ^{b} + differentiate δb.
Is there a geometric interpn. of these equations, say involving the Grassmannian in some form.

$$e^{ikx} = e^{ix\frac{z+1}{i(z-1)}} = e^{x\left(\frac{z+1}{z-1}\right)}, = e^{ix\frac{\cos(\theta/2)}{\sin(\theta/2)}}$$

Is this a smooth function of $z \in S^1$. Take

$$\frac{z+1}{i(z-1)} = \frac{e^{i\theta/2} + e^{-i\theta/2}}{i\left(e^{i\theta/2} - e^{-i\theta/2}\right)} = \frac{2\cos\theta/2}{i\,2i\sin\theta/2} = -\cot(\theta/2)$$

You somehow need to find examples, or ways to understand all this, particularly the variation. Where to begin? discrete case



$$\begin{pmatrix} \xi_+ \\ \xi'_+ \end{pmatrix} = \begin{pmatrix} \frac{1}{d} & \frac{b}{d} \\ -\frac{c}{d} & \frac{1}{d} \end{pmatrix} \begin{pmatrix} \xi'_- \\ \xi_- \end{pmatrix}$$

$$\begin{pmatrix} \xi'_- \\ \xi_- \end{pmatrix} = \begin{pmatrix} \frac{1}{a} & -\frac{b}{a} \\ \frac{c}{a} & \frac{1}{a} \end{pmatrix} \begin{pmatrix} \xi_+ \\ \xi'_+ \end{pmatrix}$$

$$\xi'_- = \xi_+\left(\frac{1}{a}\right) + \xi'_+\left(-\frac{b}{a}\right)$$

$$\tilde{q}_0 = \xi'_-(-\phi) + \xi_-(1-\psi) \quad \perp\left(\xi'_- zH_+ + \xi_- zH_+\right)$$

$IH(\xi'_-, \tilde{q}_0)$

page 479–495 for expressions giving h_0 from the Birkhoff factorization.

Idea: $L^2 = H_- \oplus SH_+$ assume

then $zH_- \cap SH_+$ dim 1, let v denote the unit vector in $zH_- \cap SH_+$ with $v(0) > 0$. $v \in zH_- = \mathbb{C} + H_-$

$f_+ = S^{-1}v \in H_+$. $f_- = Sf_+$. Why is f_+ non vanishing for $|z| < 1$. If $f_+(a) = 0$ $|a| < 1$. then f_+ is divisible by $z-a$. point is $\frac{1}{z-a}$ bdd op on L^2

$$f_+ = (z-a)g_+ \qquad \boxed{g_+ \in H_+}$$

$$f_- = Sf_+ = (z-a)Sg_+$$

$$\frac{1}{z-a} f_- = Sg_+ \in H_- \cap SH_+ = 0.$$

matrix version $E = (\mathbb{C}^2)^n = E_- \oplus SE_+$

$V = zE_- \cap SE_+$ dim n. V is an $n \times n$ matrix of elts in zH_-. You want to know ~~whether~~ that V is non singular ~~inside~~ for z outside S^1.

Back to rank 1 $\quad \boxed{f_- = Sf_+} \in zH_- \cap SH_+$

Consider ~~cyclic~~ subrep gen. by f_- in L^2

$\overline{\mathbb{C}[z,z^{-1}]f_-} \subset L^2(S^1)$, projection

$\uparrow S$

$L^2(S^1, |f_-|^2 \frac{d\theta}{2\pi})$

$$L^2 = H_- \oplus SH_+$$

$$V = zH_- \cap SH_+ \qquad \text{dim } 1$$

$$f_- = Sf_+ \in V \quad , \quad \|f_-\| = 1, \quad f_-(\infty) \geq 0$$

Cons cyclic rep gen. by f_+: [redacted]

$\varphi f_- \quad \overline{\mathbb{C}[z,z^{-1}]f_-} \subset L^2(S^1, \frac{d\theta}{2\pi})$

$\uparrow \qquad \uparrow S \qquad\qquad$ projection

$\varphi \in L^2(S^1, d\mu) \qquad d\mu = |f_-|^2 \frac{d\theta}{2\pi}$

In general $\rho = |f|^2$

$\cup$

$L^2(S^1, |f|^2 \frac{d\theta}{2\pi}) \xrightarrow{\sim} \overline{C(S^1)f} < L^2$

$W = \overline{\mathbb{C}[z^{-1}]f_-} \subset zH_-$

$\varphi \longmapsto \varphi f$

$\cup$

inverse is

$z^{-1}W \qquad H_-$

$\varphi[\frac{1}{f}] \longleftarrow \varphi$

where $[\frac{1}{f}] = f^{-1}$ defined where $f \neq 0$ extended by 0.

Note $f[f^{-1}] = \chi$ subset where $f \neq 0$.

To see what happens

$$\mathbb{L}^2\left(S^1, |f_-|^2 \frac{d\theta}{2\pi}\right) \overset{1 \mapsto f_-}{=} \overline{\mathbb{C}[z,z^{-1}]f_-} \subset L^2$$

$$zH_-^2(\quad\text{—}\quad) = \overline{\mathbb{C}[z^{-1}]f_-} = W \subset zH_-$$

$$\qquad\qquad\qquad\qquad\qquad \vee \qquad \vee$$

$$\overline{z^{-1}\mathbb{C}[z^{-1}]f} = z^{-1}W \subset H_-$$

$$p \in W \ominus z^{-1}W, \quad \|p\| = 1$$

Let $p = \varphi f_-$, $\varphi \in L^2(S^1, |f_-|^2\frac{d\theta}{2\pi})$. Then $|\varphi f_-| = 1$

Also φ extends analytically outside D.

So conclude f_- invertible.

Find better notation:

$$V = \overline{\mathbb{C}[z,z^{-1}]f_-} \subset L^2 \qquad \bigcup z^n W = V$$

$$W = \overline{\mathbb{C}[z^{-1}]f_-} \subset zH_- \qquad \bigcap z^{-n}W \subset \bigcap z^{-n}H_- = 0$$

$$z^{-1}W = \overline{z^{-1}\mathbb{C}[z^{-1}]f_-} \subset H_-$$

$p \in W$, $p \perp z^{-1}W$, $\|p\| = 1$.

Then

$$V \xleftarrow[\cdot p]{\sim} L^2\left(S^1, \frac{d\theta}{2\pi}\right)$$

$$z^n p \longleftarrow z^n$$

$$W \xleftarrow{\sim} zH_-$$

Let $gp = f_-$ (with $g \in zH_-$, $p \in zH_-$)

OK

$\therefore$

$$L^2 \xrightarrow{\sim} \overbrace{\overline{\mathbb{C}[z,z^{-1}]f_-}}^{V} \subset L^2$$

$$zH_- \xrightarrow{\sim} W \subset zH_-$$

$$1 \qquad\quad f_- \qquad\quad f_-$$

Get clearer. $f_- = Sf_+ \in (zH_-) \cap SH_+$

$$L^2 \xrightarrow[\sim]{\cdot p} V = \overline{\mathbb{C}[z,z^{-1}]f_-} \subset L^2$$

$$zH_- \xrightarrow[\sim]{\cdot p} W = \overline{\mathbb{C}[z^{-1}]f_-} \subset zH_-$$

$$z^{-1}W = \overline{z^{-1}\mathbb{C}[z^{-1}]f_-} \subset H_-$$

Thus you have a $p \in L^2$ such that $|p|^2 = 1$ on S^1 also $L^2 \xrightarrow{\cdot p} L^2$ is isometry.

In simple terms: $f_- = Sf_+ \in (zH_-) \cap SH_+$

Let

$$\begin{matrix} W = \overline{\mathbb{C}[z^{-1}]f_-} & \subset & zH_- \\ \cup & & \cup \\ z^{-1}W = \overline{z^{-1}\mathbb{C}[z^{-1}]f_-} & \subset & H_- \end{matrix}$$

$W > z^{-1}W$ as $f_- \in W$ and $f_- \notin H_-$

$W = \mathbb{C}f_- + z^{-1}W$, let $p \in W$, $p \perp z^{-1}W$, $\|p\| = 1$.

Then $(p|z^n p) = \delta_n$ so $|p|^2 = 1$ as well as $p \in zH_-$. You certainly know then that

$$L^2(S^1)p \subset V \subset L^2(S^1)$$

so $f_- = g_- p$

Assume similarly $f_+ = g_+ q$

$g_- p = g_+ S q$

$$f_- = S f_+ \in zH_- \cap SH_+$$
$$V = \overline{\mathbb{C}[z,z^{-1}]f_-} \subset L^2$$
$$W = \overline{\mathbb{C}[z^{-1}]f_-} \subset zH_-$$
$$\rho \quad V \qquad\qquad V$$
$$z^{-1}W \qquad\qquad \subset H_-$$

$$\Rightarrow W = \overset{(2)}{\bigoplus_{n\le 0}} \mathbb{C} z^n p$$

Thus $W \subset zH_-$ ~~[illegible]~~ closed under z^{-1}!

$(zH_-)p$ $\qquad f_- = g_- p$ some ~~[illegible]~~ g_-

which should be invertible analytic ~~[illegible]~~ on D_-.

Other side

Begin again with $L^2 = H_- \oplus SH_+$, then $zH_- \cap SH_+$ is 1 diml, whence $f_- = Sf_+$ with $f_- \in zH_-$, $f_+ \in H_+$ nonzero unique up to a scalar. Can Assume $\|f_-\| = 1$, whence $\rho = |f_-|^2 = |f_+|^2$ is an integrable density of mass 1. Better maybe to say we need inv. of $f_\pm$ on the appropriate disks for Birkhoff decomp, so consider $d\mu = \rho \frac{d\theta}{2\pi}$ $\qquad \rho = |f_-|^2 = |f_+|^2 \ge 0$

invoke ~~[illegible]~~ Szego theory essentially.

$$f_- = Sf_+ \in zH_- \cap SzH_+ = z(\underbrace{H_- \cap SH_+}_{0})$$

$f_+ \in H_+$, ~~[illegible]~~ $\notin zH_+$

$$V = \overline{\mathbb{C}[z,z^{-1}]f_+} \subset L^2$$
$$\cup$$
$$W = \overline{\mathbb{C}[z]f_+} \subset H_+$$
$$zW = \overline{z\mathbb{C}[z]f_+} \subset zH_+$$

$W = \mathbb{C}f_+ + zW$

$zW \neq W$.

find $\tilde{g}_\infty \in 1 + zW$

Again $W = \overline{\mathbb{C}[z, z^{-1}] f_+} \subset L^2$

$W = \overline{\mathbb{C}[z] f_+} \subset H_+$

codim 1 $\cup$

$zW = \overline{z\,\mathbb{C}[z] f_+} \subset zH_+$

Let q be a unit vector in $W \perp$ to zW. Then

$q \perp z^n q \quad n \neq 0$, so $q \in H_+$, $|q|^2 = 1$ ae

Then $H_+ q \xrightarrow{\sim} W$ So get $g_+ \in H_+ \ni$

$g_+ q = f_+$

Repeat stuff Assume S [redacted] $|S| = 1$ s.t.

$L^2 = H_- \oplus SH_+$. Then $zH_- \cap SH_+$ spanned by $f_- = Sf_+$. The assumption that S unitary is unnecessary probably. Outer + inner functions theory.

[redacted] Consider new approach. First consider $f \in H_+$, $f \neq 0$. Let [redacted]

$$V = \overline{\mathbb{C}[z, z^{-1}] f} \subset L^2$$

$$W = \overline{\mathbb{C}[z] f} \subset H_+$$

$\bigcap z^n H_+ = 0$, so there is a [redacted] largest n s.t. $W \subset z^n H_+$. Can assume [redacted] $W \subset H_+$ but not in zH_+. $\therefore zW \subsetneq W$ Choose q [redacted] a unit vector in $W \perp z$

Again

Take $f \in H_+$, $f \neq 0$ let n be largest so that $f \in z^n H_+$, replace f by $z^{-n} f$ can suppose $f \in H_+$, $f \notin z H_+$. $W = \overline{\mathbb{C}[z] f} \subset H_+$

$$zW \subset zH_+$$

$f + zW = W$. ~~Then you find~~ Let $\mathfrak{g}$ be a unit vector in W $\perp$ to zW. Then

$f = g\mathfrak{g}$ — g outer, $\mathfrak{g}$ inner

$\rho = |f|^2 = |g|^2$ $\log \rho$ known to be in L^1 by Szegö alternative. $\log \rho = \sum_1 a_n z^n = h + h^*$

$h = \frac{a_0}{2} + \sum_{n \geq 1} a_n z^n$. $g = e^h$

~~[illegible]~~
~~Work out the details~~

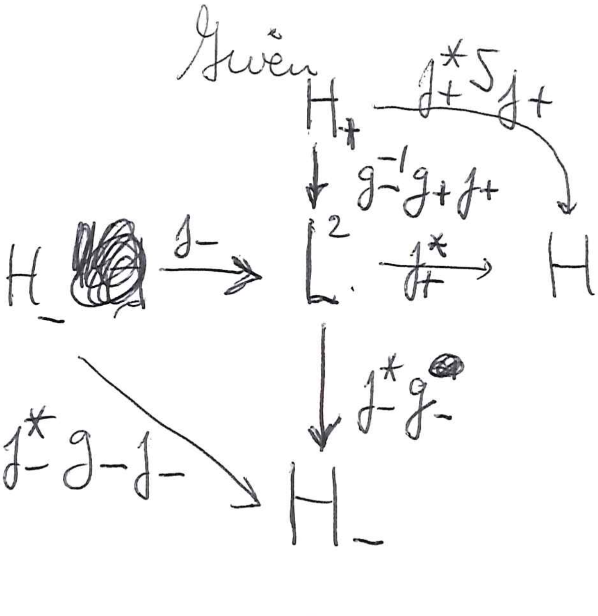
What to say? You need examples.

converse. Suppose $S = g_-^{-1} g_+$ do you get splitting: $L^2 \overset{?}{=} H_- \oplus S H_+ = H_- \oplus g_-^{-1} H_+$

$\xrightarrow[\sim]{g_-}$ $g_- H_- \oplus H_+ = H_- + H_+$, seems to work

$H_- \oplus H_+$.

Given



$j_-^* g_- \xi = 0$

$\Leftrightarrow g_- \xi \in j_+ H_+ = g_+ j_+ H_+$

$\Leftrightarrow \xi \in g_-^{-1} g_+ j_+ H_+$

$S = g_-^{-1} g_+$ $\qquad$ $g_+ H_+ = H_+$

$$L^2 \xleftarrow{\text{answer?}} H_- \oplus SH_+ = H_- \oplus g_-^{-1} H_+ \xleftarrow{\begin{pmatrix} g_- & 0 \\ 0 & g_- \end{pmatrix}} H_- \oplus H_+$$

~~You want to start with~~

You have these isoms. to sort out.

I am confused – there seems to be something interesting happening. You have $L^2 = H_+ \oplus H_-$ and "triangular" subgroups G_+, G_- of a group G of autos of L^2. How to organize?

Discuss how a factorization $g = g_-^{-1} g_+$ yields a splitting $L^2 = H_- \oplus gH_+$, and conversely. I can do this via

$$\begin{array}{ccccc} & & H_+ & & \\ & & \downarrow{\scriptstyle gj_+} & & \\ H_- & \xrightarrow{j_-} & L^2 & \xrightarrow{j_+^*} & H_+ \\ & \searrow & \downarrow{\scriptstyle j_-^* g_-} & & \\ & & H_- & & \end{array}$$

rest of g_- to H_- $= j_-^* g_- j_-$

Check exactness of column.

$j_-^* g_- g j_+ = j_-^* g_+ j_+ = 0$ since $g_+ H_+ = H_+$

conv. $j_-^* g_- \xi = 0$

$\Leftrightarrow g_- \xi \in H_+ = g_+ H_+$

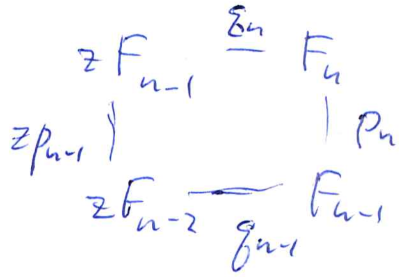
$\Leftrightarrow \xi \in g_-^{-1} g_+ H_+ = g j_+ H_+$

Stone von Neumann thm: Normally this means the unique of the CCR $[p,q] = \frac{\hbar}{i}$, but there appears to be a discrete form involving a unitary operator u and a closed subspace W such that $uW \subset W$.

need progress on smooth case.
at some point you need to ~~do~~ organize the smooth Szegö stuff. Do this rapidly.

$\mu = \rho \frac{d\theta}{2\pi}$ ρ smooth and >0.

$F_n = \mathbb{C} + \mathbb{C}z + \cdots + \mathbb{C}z^n$



p_n, q_n

$$\begin{pmatrix} p_n \\ q_{n-1} \end{pmatrix} = \underbrace{\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}}_{\in U(2)} \begin{pmatrix} zp_{n-1} \\ q_n \end{pmatrix}$$

$q_{n-1}(0) = \delta q_n(0)$ $\therefore \delta > 0$

$$\begin{pmatrix} p_n \\ q_n \end{pmatrix} = \begin{pmatrix} \frac{\alpha\delta - \gamma\beta}{\delta} & \frac{\beta}{\delta} \\ -\frac{\gamma}{\delta} & \frac{1}{\delta} \end{pmatrix} \begin{pmatrix} zp_{n-1} \\ q_{n-1} \end{pmatrix}$$

lead. coeff of p_n, zp_{n-1} $>0 \Rightarrow \frac{\alpha\delta - \gamma\beta}{\delta} > 0$
$\therefore \alpha\delta - \beta\gamma > 0$ but also $|\alpha\delta - \beta\gamma| = 1$

$\therefore \alpha\delta - \gamma\beta = 1$

$$\begin{pmatrix} \bar\alpha & \bar\gamma \\ \bar\beta & \bar\delta \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}^{-1} = \begin{pmatrix} \delta & -\beta \\ -\gamma & \alpha \end{pmatrix} \Longrightarrow \alpha = \bar\delta, \ \bar\gamma = -\beta, \text{ whence}$$

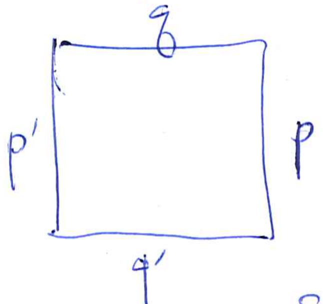
$$\begin{pmatrix} p_n \\ q_{n-1} \end{pmatrix} = \begin{pmatrix} k & h \\ -\bar h & k \end{pmatrix} \begin{pmatrix} zp_{n-1} \\ q_{n-1} \end{pmatrix} \qquad \begin{pmatrix} p_n \\ q_n \end{pmatrix} = \frac{1}{k} \begin{pmatrix} 1 & h \\ \bar h & 1 \end{pmatrix} \begin{pmatrix} zp_{n-1} \\ q_{n-1} \end{pmatrix}.$$

---

there should be a simple wedge calculation.

$$\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} \frac{\alpha\delta - \beta\gamma}{\delta} & \frac{\beta}{\delta} \\ -\gamma/\delta & \frac{1}{\delta} \end{pmatrix} \begin{pmatrix} p' \\ q' \end{pmatrix}$$

~~$p_n \wedge q_{n-1} = (\alpha\delta - \beta\gamma)(z$~~

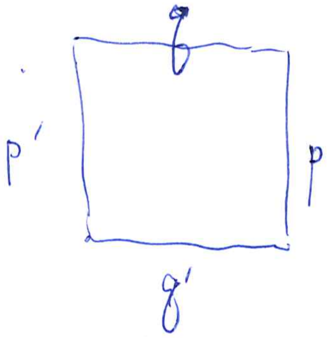


$$\begin{pmatrix} p \\ q' \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} p' \\ q \end{pmatrix}$$

$q \equiv (>0) q' \mod p'$

$p \equiv (>0) p' \mod q'$

$p' \wedge q' = (p' \wedge q) \overset{\delta}{(>0)}$
$\parallel$
$(p \wedge q')(>0)$ — $\frac{\alpha\delta - \beta\gamma}{\delta}$



$$\begin{pmatrix} p \\ q' \end{pmatrix} = \underbrace{\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}}_{\in U(2)} \begin{pmatrix} p' \\ q \end{pmatrix}$$

assume p, p' pos. related mod q'
q', q ——— mod p'

Then $p \wedge q'$ pos. related to $p' \wedge q'$ pos. rel to $p' \wedge q$.

$\therefore \begin{vmatrix} \alpha & \beta \\ \gamma & \delta \end{vmatrix} > 0 \qquad \therefore = 1.$

$p \equiv$ pos. mult of p' (mod q')
$q \equiv$ ——— q' (mod p')

$p \wedge q' =$ pos. mult. of $p' \wedge q' =$ pos mult of $p' \wedge q$

$$\frac{p_n}{q_n} = \begin{pmatrix} 1 & h \\ \bar{h} & 1 \end{pmatrix} \begin{pmatrix} \frac{z p_{n-1}}{q_{n-1}} \end{pmatrix}$$

so by induction
$|z| < 1 \implies \left|\frac{p_n}{q_n}\right| < 1.$
$= 1 \qquad\qquad = 1$
$> 1 \qquad\qquad > 1.$

zeroes of q_n outside S^1.

What are results, find good exposition

$F_{n-1} \xrightarrow[z]{1} F_n$ is a partial unitary with

$V_+ = F_{n-1}^{\perp} = \mathbb{C} p_n$
$V_- = (zF_{n-1})^{\perp} = \mathbb{C} q_n$

scattering op is $S_n = \frac{p_n}{q_n}$

$$y = u^{-n} a^{*}(ba^*)^n y + \sum_{k=0}^{n} u^{-k} \pi_+ (ba^*)^k y$$

$$\rightarrow (1-aa^*)\left(\frac{1}{1-z^{-1}ba^*}\right) y$$

$\det(1-zab^*)$

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topics

Contraction ops. + scattering

real h's change $\begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix}$ to $\begin{pmatrix} 1 & s \\ s & 1 \end{pmatrix}$
scrunching, ℓ^2 eigenvectors as $n \to +\infty$, say always exist for $|z| \neq 1$.

acyclic coeff. system on a tree (June 98)

Composing partial unitaries (or glueing)

Lagrangian subbundles.

can discuss boundary conditions at n, also p_n is a characteristic poly, should be the char. poly of z on $H_+/SH_+ = H_+/p_nH_+$. What sort of questions to ask? kernel function (Bergman) reproducing kernel, i.e. projector

$$L^2(d\mu) = F_{n-1} \oplus S_n H_+$$ Are these orthogonal

What to say? where to start?

objects.

measure $d\mu$ (moments) g_∞

$(h_n)_{n\geq 1}$ Schur parameters.

Pick function

You really want to start with S, or equivalently g_∞ see what happens. S smooth loop with values in $U(1)$ of degree 0. ~~$\log S = i\sum_{n\in\mathbb{Z}} a_n z^n$ $\overline{a_n} = +a_n$~~

~~$f(z) - \overline{f(z)}$ where $f(z) = \frac{a_0}{2} + \sum_{n\geq 1} a_n z^n$~~

~~$\log S = f - \bar{f}$~~ Let $\log(S) = \sum_{n\in\mathbb{Z}} b_n z^n$.

$\log(S)$ unique up to $+2\pi i\mathbb{Z}$, Also $\log(S)$ purely imaginary, Also S is probably unique up to a scalar in $U(1)$, can assume $\int \log(S)\frac{d\theta}{2\pi} = 0$.

From S you get $g = e^{g}$

smooth Szegö stuff. ~~[crossed out]~~

You can begin with $\rho, q, \text{or } S$ which are nearly equivalent (i) ρ smooth >0 on S^1, (ii) S smooth degree 0 unitary loop, (iii) q non-vanishing smooth function on $\bar{D}$ analytic in D.

$$\log \rho = f + \bar{f} \qquad f \text{ analytic on } \bar{D}$$
unique up to an imag. constant.

$$\log S = f - \bar{f} \qquad f \text{ analytic on } \bar{D}$$
unique up to a real constant.

$$\log(q) = f \qquad f \text{ unique up to } 2\pi i \mathbb{Z}$$

---

begin with $\rho > 0$ smooth, form $L^2(S^1, \rho \frac{d\theta}{2\pi})$, equiv. "the" Hilbert space H with unitary op u and cyclic vector ξ such that $(\xi | z^n \xi) = \int z^n \rho \frac{d\theta}{2\pi}$, form sequences $\tilde{p}_n, \tilde{q}_n$, p_n, q_n, get h_n

$$\begin{pmatrix} \tilde{p}_n \\ \tilde{q}_n \end{pmatrix} = \begin{pmatrix} 1 & h_n \\ \bar{h}_n & 1 \end{pmatrix} \begin{pmatrix} z\tilde{p}_{n-1} \\ \tilde{q}_{n-1} \end{pmatrix}$$

Q: Show (h_n) rapidly decreasing when ρ is smooth >0. Can you use ∂_θ?

Other idea: Remove ξ from H to get a partial unitary, and a two sided situation with $h_n = 0$ for $n \leq 0$.

A massive review is now necessary.

Start with a smooth Szegö situation and "remove" the cyclic vector, find the scattering data. The smooth Szegö situation gives orth. poly system

$$\begin{pmatrix} p_n \\ q_n \end{pmatrix} = \frac{1}{k_n}\begin{pmatrix} 1 & h_n \\ \bar{h}_n & 1 \end{pmatrix}\begin{pmatrix} z p_{n-1} \\ q_{n-1} \end{pmatrix} \quad \text{for } n \geq 1, \text{ where } p_0 = q_0 = \xi.$$

Situation (H, u, ξ). $\quad X = \xi^\perp \quad Y = H$

$a : X \hookrightarrow H, \quad b = ua : X \longrightarrow H$

$$H = aX \oplus \mathbb{C}\xi^+ \qquad \xi^+ = \xi$$

$$= \mathbb{C}\xi^- \oplus bX \qquad \xi^- = u(\xi)$$

$$bX = uX$$

Contraction $c_h = ba^* \oplus \xi_- h \xi_+^*$, this is almost unitary, it's natural to dilate a contraction. Given $c : Y \longrightarrow Y$ you ask for $j : Y \hookrightarrow H$ and u unitary on Y such that

$$j^* u^n j = \begin{cases} c^n & n \geq 0 \\ (c^*)^{-n} & n \leq 0. \end{cases}$$

This is a pos. def. function on the gp $\mathbb{Z}$ with operator values.

What do you know

$$\overline{(jy_1 | ujy_2) = (y_1 | cy_2)}$$

Look at $Z = jY + ujY = jY \oplus V^+$

$$\| jy_1 + ujy_2 \|^2 = \left(\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \middle| \begin{pmatrix} 1 & c \\ c^* & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \right)$$

$$= \|y_1\|^2 + (y_1 | cy_2) + (cy_2 | y_1) + \|cy_2\|^2 + (y_2 | (1 - c^*c) y_2)$$

V^+ = completion of Y wrt norm $(y | (1 - c^*c) y)$

$$c_h = ba^* + \xi_- h \xi_+^* \; : aX \oplus V_+^{\oplus} \longrightarrow bX \oplus V_-$$

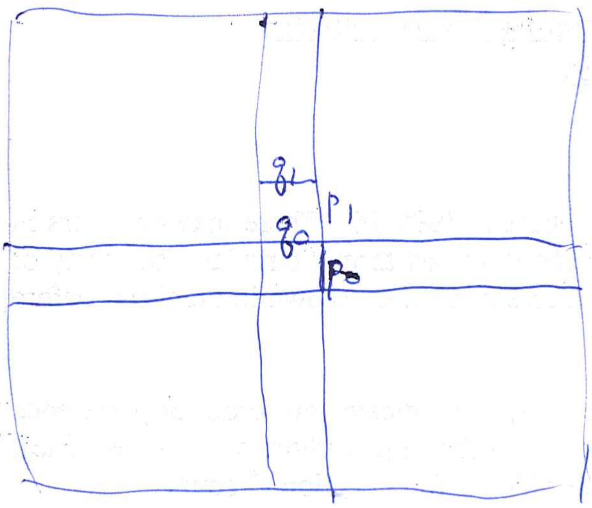
For $|h| < 1$ this is not unitary.

To Achieve an understanding.

Start again ~~consider~~ with Szegö (H, u, ξ) which should lead (in some way) to $S = \frac{\xi_+}{\xi_-}$

take simple example: only $h_1 \neq 0$.

$$\begin{pmatrix} p_1 \\ q_1 \end{pmatrix} = \frac{1}{k_1}\begin{pmatrix} 1 & h_1 \\ h_1 & 1 \end{pmatrix}\begin{pmatrix} z \\ 1 \end{pmatrix} = \frac{1}{k_1}\begin{pmatrix} z + h_1 \\ 1 + h_1 z \end{pmatrix}$$



Go over the procedure. You have ρ which yields the $h_1, h_2, \ldots\ldots$, and the boundary condition $p_0 = q_0$

Maybe simpler to start with the h-sequence zero after h_n.

Suppose given $h_1, \ldots\ldots, h_n$ ~~finitely many $\neq 0$~~

Then
$$\begin{pmatrix} p_n \\ q_n \end{pmatrix} = \frac{1}{k_n}\begin{pmatrix} 1 & h_n \\ \bar{h}_n & 1 \end{pmatrix}\begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix} \cdots \frac{1}{k_1}\begin{pmatrix} 1 & h_1 \\ \bar{h}_1 & 1 \end{pmatrix}\begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} p_0 \\ q_0 \end{pmatrix}$$

$$\begin{pmatrix} \xi_+ \\ \xi_- \end{pmatrix} = \begin{pmatrix} z^{-n} p_n \\ q_n \end{pmatrix} = \begin{pmatrix} a^n & b^n \\ c^n & d^n \end{pmatrix}\begin{pmatrix} p_0 \\ q_0 \end{pmatrix}$$

in fact this is the scattering matrix for the system with $h_i = 0$ $\quad i \leq 0$.

Your idea when you form c_h is to ~~change~~ introduce $\begin{pmatrix} p_0 \\ q_0 \end{pmatrix} = \; ?$

Fix $h_1, h_2, \ldots$ Ifl

You should be able to take an (H, a, ξ), and remove the boundary condition $h_0 = 1$, or $p_0 = q_0$ and put in $|h_0| < 1$.

You should calculate a simple example. Suppose $h_n = 0$ $n \geq 1$. Then

$$\begin{pmatrix} a^n & b^n \\ c^n & d^n \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\boxed{\begin{aligned} \xi_+ &= p_0 \\ \xi_- &= q_0 \end{aligned}}$$

Adjust h_0

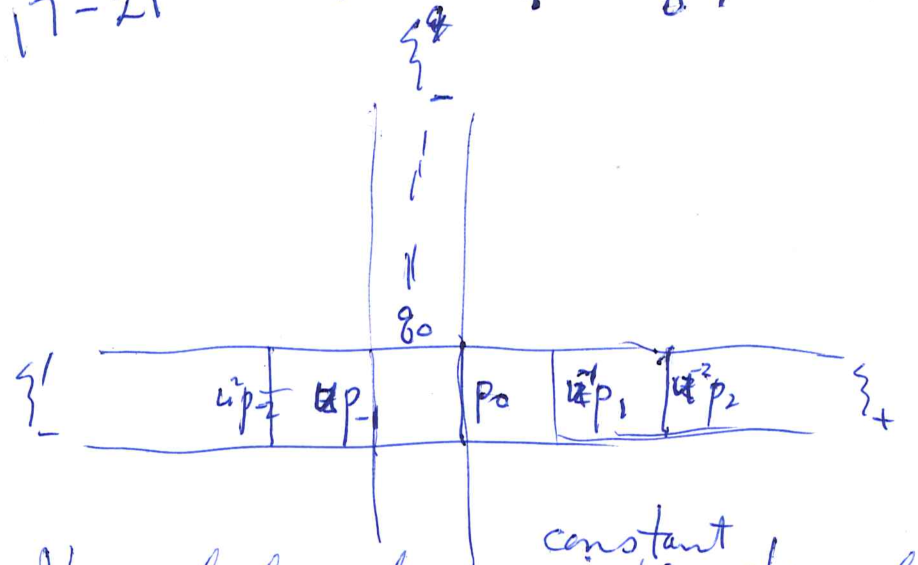
$\xi'_- = k p_{-1}$

$\xi'_+ = q_{-1}$

17–21st July

$$\begin{pmatrix} \xi_+ \\ \xi_- \end{pmatrix} = \frac{1}{k} \begin{pmatrix} 1 & h \\ \bar{h} & 1 \end{pmatrix} \begin{pmatrix} \xi'_- \\ \xi'_+ \end{pmatrix}$$

$$\begin{pmatrix} p_0 \\ q_0 \end{pmatrix} = \frac{1}{k} \begin{pmatrix} 1 & h \\ \bar{h} & 1 \end{pmatrix} \begin{pmatrix} k p_{-1} \\ q_{-1} \end{pmatrix}$$



I'm looking at a ~~simple~~ constant transfer matrix

$$\frac{1}{k} \begin{pmatrix} 1 & h \\ \bar{h} & 1 \end{pmatrix}$$

Given ρ you get $h_1, h_2, \ldots$, Green's fn idea. Is factorization equivalent to the Green's fn? This would explain δ functions in the cont case, what is best way to proceed? Start with a grid space E, i.e. sequence $(h_n)_{n \in \mathbb{Z}}$. On E you have a ~~[illegible]~~ preserving two hermitian forms. On the Hilb space completion $\mathcal{I} - \mathcal{U}$ is invertible for $|\lambda| \neq 1$. Why for $|\lambda|<1$:

$$(\lambda-u)^{-1} = \frac{1}{\lambda-u} = \frac{u^{-1}}{\lambda u^{-1}-1} = -u^{-1}\sum_{n\geq 0}\lambda^n u^{-n} = -\sum_{n\geq 0}\lambda^n u^{-n-1}$$

$|\lambda|>1$. $$\frac{1}{\lambda-u} = \frac{1}{\lambda}\frac{1}{1-u\lambda^{-1}} = \sum_{n\geq 0} u^n\lambda^{-n-1}.$$

An idea from yesterday: Given (H, u, ξ), adjoin $\xi\xi^*$ to functions of u. This is close to $QA = A*A$. More interesting might be adjoining the Hilbert transform to $A = C(S^1)$. Does ΩA stuff yield anything.

[Focus on the Green's function idea]

What meaning for G fn. in the grid space context. Linear functionals on E = solutions of the grid DE. G fn is a kind of inverse. Be more precise

E is a rank 2 free module over $\mathbb{C}[u,u^{-1}]$.

$$X \overset{a}{\underset{b}{\rightrightarrows}} Y \quad \cdots \oplus u^{-1}V_{*} \oplus \underbrace{aX \oplus V_{*}}_{Y} \oplus uV_{*} \oplus \cdots$$

$$V_{-} \oplus bX$$

$$y = aa^*y + \pi_+ y$$

$$uy = ba^*y + u\pi_+ y$$

$$= aa^* ba^*y + \pi_+ ba^*y + u\pi_+ y$$

$$u^2y = aa^*(ba^*)^2 y + \pi_+(ba^*)^2y + u\pi_+ ba^*y + u^2\pi_+ y$$

$$u^3y = aa^*(ba^*)^3 y + u\pi_+(ba^*)^2 y + u^2\pi_+(ba^*)y + u^3\pi_+ y$$

$$u^0\pi_+(ba^*)^3 y$$

Try to recall how you handled contractions and partial unitaries last year.

① An operator $c: Y \to Y'$ between Hilb spaces is called a contraction when $\|c\| \leq 1$; $c^*: Y' \to Y$ is also a contraction, and $A = \begin{pmatrix} 0 & c^* \\ c & 0 \end{pmatrix}$ is an odd self-adjoint contraction from $\begin{matrix} Y \\ \oplus \\ Y' \end{matrix}$ to itself. On the kernel of $1 - A^2 = \begin{pmatrix} 1-c^*c & 0 \\ 0 & 1-cc^* \end{pmatrix}$, c and c^* are unitary, inverses of each other. On the orthogonal complement of the kernel, one has $\|c\xi\| < \|\xi\|$ and $\|c^*\xi'\| < \|\xi'\|$ for $\xi, \xi' \neq 0$. In this way a contraction splits into a unitary part and a strictly contractive part. To be more precise $\begin{pmatrix} Y \\ \oplus \\ Y' \end{pmatrix} = \begin{pmatrix} X \\ \oplus \\ X' \end{pmatrix} \oplus \begin{pmatrix} Z \\ Z' \end{pmatrix}$ where

$X \underset{c}{\overset{c^*}{\rightleftarrows}} X'$ are inverse and unitary, and $Z \underset{c}{\overset{c^*}{\rightleftarrows}} Z'$ are strictly contractive.

A partial unitary between Y and Y' consists of closed subspaces $X \subset Y$, $X' \subset Y'$ and a unitary operator $X \xrightarrow{c} X'$. A partial unitary yields a contraction by extending c to be zero on the orth complements Z, Z' of X, X'. In this way partial unitaries between Y and Y' can be identified with contractions $c: Y \to Y'$ such that $cc^*c = c$. Note this implies $c^*cc^* = c^*$ whence $A = \begin{pmatrix} 0 & c^* \\ c & 0 \end{pmatrix}$ is self-adjoint satisfying $A^3 = A$. Then $A^4 = A^2$ so A^2 is idempotent, yielding a splitting where $A^2 = I$ and $A^2 = 0$. Note that $A^4 = A^2 \implies A^3 = A$ because $(A - A^3)^2 = A^2 - 2A^4 + A^6 = 0 \implies A - A^3 = 0$ as it's s.a.

Next dilating a contraction $C : Y \to Y'$ TO A UNITARY

Again form $A = \begin{pmatrix} 0 & C^* \\ C & 0 \end{pmatrix}$ ~~self-adj~~ odd contraction on $\begin{matrix} Y \\ \oplus \\ Y' \end{matrix}$

You want to add $Z' = \overline{\sqrt{1-C^*C}\, Y}$ to Y' and $Z = \overline{\sqrt{1-CC^*}\, Y'}$ to Y. Here $\overline{\sqrt{1-C^*C}\, Y}$ is the closure in Y of the image of $j : y \mapsto (1-C^*C)^{1/2} y$, alternatively $j : Y \to Z$ is the completion of Y wrt the inner product $\|jy\|^2 = \|y\|^2 - \|Cy\|^2 = (y|(1-C^*C)y)$.

Then

$$\begin{pmatrix} j & C^* \\ -C & j' \end{pmatrix} : \begin{pmatrix} Y \\ Z' \end{pmatrix} \longrightarrow \begin{pmatrix} Z \\ Y' \end{pmatrix}$$

should be unitary

$$\begin{pmatrix} j^* & -C^* \\ C & j'^* \end{pmatrix}\begin{pmatrix} j & C^* \\ -C & j' \end{pmatrix} = \begin{pmatrix} j^*j + C^*C & j^*C^* - C^*j' \\ Cj - j'^*C & j'^*j' + CC^* \end{pmatrix}$$

($j^*j = 1-C^*C$, $j'^*j' = 1-CC^*$)

$Cj = C\sqrt{1-C^*C}$
$j'^*C = \sqrt{1-CC^*}\, C$

Correct way. $A = \begin{pmatrix} 0 & C^* \\ C & 0 \end{pmatrix}$ on $\begin{pmatrix} Y \\ Y' \end{pmatrix}$

~~$A + \sqrt{1-A^2}$~~

A s.a. contraction on W – dilate?

$j = \sqrt{1-A^2} : W \longrightarrow \overline{\sqrt{1-A^2}\, W}$

~~To combine~~ Form $W \oplus \overline{\sqrt{1-A^2}\, W}$

$$\begin{pmatrix} A & \sqrt{1-A^2} \\ \sqrt{1-A^2} & -A \end{pmatrix}^2 = I$$

$$\begin{array}{c|cc|cc|c} Y & 0 & c^* & \sqrt{1-c^*c} & 0 & Y \\ Y' & c & 0 & 0 & \sqrt{1-cc^*} & Y' \\ \sqrt{1-c^*c}\,Y & \sqrt{1-c^*c} & 0 & 0 & -c^* & \sqrt{1-c^*c}\,Y \\ \sqrt{1-cc^*}\,Y' & 0 & \sqrt{1-cc^*} & -c & 0 & \sqrt{1-cc^*}\,Y' \end{array}$$

$A = \begin{pmatrix} 0 & c^* \\ c & 0 \end{pmatrix}$ on $\begin{pmatrix} Y \\ Y' \end{pmatrix}$

$$\begin{pmatrix} \sqrt{1-c^*c} & +c^* \\ -c & \sqrt{1-cc^*} \end{pmatrix}\begin{pmatrix} \sqrt{1-c^*c} & -c^* \\ +c & \sqrt{1-cc^*} \end{pmatrix}$$

$$\begin{pmatrix} \sqrt{1-c^*c} & -c^* \\ +c & \sqrt{1-cc^*} \end{pmatrix} : \begin{pmatrix} Y \\ Y' \end{pmatrix} \xrightarrow{\sim} \begin{pmatrix} Y \\ Y' \end{pmatrix}$$

Return to the interesting case. What do you want?? Suppose given (H, u, ξ). You want the partial unitary given by restricting u (or maybe u^{-1}) to $\xi^\perp$

$$f(z) = \sum_{n\geq 0} a_n z^n = \sum_{n \geq 0} z^n \oint \frac{f(\zeta)}{\zeta^n} \frac{d\zeta}{2\pi i \zeta}$$

$$= \oint \frac{f(\zeta)}{\left(1 - \frac{z}{\zeta}\right)\zeta} \frac{d\zeta}{2\pi i} = \oint \frac{f(\zeta)}{\zeta - z} \frac{d\zeta}{2\pi i}$$

$$\overline{f(\bar{z}^{-1})} = \sum \bar{a}_n z^{-n} \qquad \text{analytic outside } S^1$$

$$\therefore \oint \frac{\overline{f(\bar{\zeta}^{-1})}}{\zeta - z} \frac{d\zeta}{2\pi i} = \bar{a}_0$$

$\overline{f(\zeta)}$ for $|\zeta| = 1$.

$$\therefore \quad f(z) - \bar{a}_0 = \oint \frac{2i \,\mathrm{Im} f(\zeta)}{\zeta - z} \frac{d\zeta}{2\pi i}$$

$$\frac{a_0 - \bar{a}_0}{2i} = \int \mathrm{Im} f(\zeta) \frac{d\zeta}{2\pi i \zeta} \quad \frac{d\theta}{2\pi}$$

$\mathrm{Im} f(z)$ harm.

$$f(z) - \bar{a}_0 = \int \frac{\mathrm{Im} f(\zeta)}{\zeta - z} \frac{d\zeta}{\pi}$$

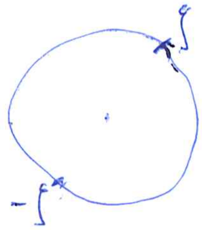
$$\frac{a_0 - \bar{a}_0}{2i} = \int \frac{\mathrm{Im} f(\zeta)}{\zeta} \frac{d\zeta}{2\pi i}$$

$$\therefore \quad f(z) - \left(\frac{a_0 + \bar{a}_0}{2}\right) = \int \left(\frac{-1}{\zeta} + \frac{2}{\zeta - z}\right) \mathrm{Im} f(\zeta) \frac{d\zeta}{2\pi}$$

$$i \frac{\zeta + z}{\zeta - z} \mathrm{Im} f(\zeta) \frac{d\zeta}{2\pi i \zeta}$$

$$\boxed{f(z) = \mathrm{Re} f(0) + \int_0^{2\pi} i \frac{\zeta + z}{\zeta - z} \mathrm{Im} f(\zeta) \frac{d\theta}{2\pi}}$$

$$\frac{1}{i} \frac{z + \zeta}{z - \zeta}$$



Review Poisson kernel

$$u(z) = \sum_{n \in \mathbb{Z}} a_n r^{|n|} e^{in\theta}$$

real harmonic $\bar{a}_n = a_{-n}$

~~$= \sum a_n r^{|n|} \int e^{-in\theta} u(e^{i\theta}) \frac{d\theta}{2\pi}$~~

$$= \sum r^{|n|} e^{in\theta} \int \zeta^{-n} u(\zeta) \frac{d\zeta}{2\pi i \zeta}$$

$$\sum_{n \geq 0} z^n \zeta^{-n} + \sum_{n \geq 1} \bar{z}^n \zeta^n = \frac{1}{1 - z\zeta^{-1}} + \frac{\bar{z}\zeta}{1 - \bar{z}\zeta} = \frac{1 - |z|^2}{|1 - z\zeta^{-1}|^2}$$

~~What ...~~

$$Y = aX + \mathbb{C}\xi_+ = bX + \mathbb{C}\xi_-$$

$$c_h = ba^* + \xi_- h \xi_+^* = c_0 + c_1$$

$$\frac{1}{z-c_h} = \frac{1}{z-c_0} + \frac{1}{z-c_0} \, c_1 \, \frac{1}{z-c_h} \qquad (c_1 = \xi_- h \xi_+^*)$$

$$\xi_+^* \frac{1}{z-c_h} = \xi_+^* \frac{1}{z-c_0} + \underbrace{\xi_+^* \frac{1}{z-c_0} \xi_-}_{S_0(z)} h \, \xi_+^* \frac{1}{z-c_h}$$

$$(1 - S_0 h)\, \xi_+^* \frac{1}{z-c_h} = \xi_+ \frac{1}{z-c_0} \qquad (1-S_0 h) S_h = S_0$$

$$S_h h = \frac{S_0 h}{1 - S_0 h} \qquad 1 + S_h h = \frac{1}{1 - S_0 h}$$

Start with smooth $d\mu = \rho \frac{d\theta}{2\pi}$.

$$1 + S_1 = \frac{1}{1 - S_0} \qquad S_1 = \left(\xi_+^* \middle| \frac{1}{z-u} u \xi_+\right)$$

$$S_1 = \int \frac{\zeta}{z - \zeta} \, d\mu \qquad \int d\mu = \xi_+^* \xi_+ = 1$$

$$\frac{1}{2} + S_1 = \int \left(\frac{1}{2} + \frac{\zeta}{z-\zeta}\right) d\mu = \int \frac{1}{2} \frac{z+\zeta}{z-\zeta} \, d\mu$$

$$(-i)(1 + 2S_1) = \int \frac{1}{i} \frac{z+\zeta}{z-\zeta} \, d\mu(\zeta)$$

$$S_h = \xi_+^* \frac{1}{z - c_h} \xi_-$$

$$S_1 = \xi_+^* \frac{1}{z-u} u \xi_+ = \int \frac{\zeta}{z-\zeta} \, d\mu$$

$$1 + S_1 = \int \left(1 + \frac{\zeta}{z-\zeta}\right) d\mu = \int \frac{z - \zeta + \zeta}{z - \zeta} \, d\mu$$

$$\boxed{1 + S_1 = z \int \frac{1}{z-\zeta} \, d\mu}$$

$$1+S_1 = \frac{1}{1-S_0} \qquad 1-S_0 = \frac{1}{1+S_1}$$

$$S_0 = 1 - \frac{1}{1+S_1} = \frac{S_1}{1+S_1}$$

$$S_h = \xi_+^* \frac{1}{z-c_h} \xi_- = \frac{1}{1-S_0 h} S_0$$

$$1+S_h h = \frac{1}{1-S_0 h}$$

Do $|z|<1$. $\qquad c_h^* = (c_0 + \xi_- h \xi_+^*)^* = c_0^* + \xi_+ \bar{h} \xi_-^*$

$$\xi_-^* \frac{1}{1-zc_h^*} = \xi_-^* \frac{1}{1-zc_0^*} + \xi_-^* \frac{1}{1-zc_0^*} z\xi_+ \bar{h} \xi_-^* \frac{1}{1-zc_h^*}$$

$$T_h = \xi_-^* \frac{1}{1-zc_h^*} \xi_+$$

$$T_h = T_0 + zT_0 \bar{h} T_h$$

$$= T_0 (1+z\bar{h} T_h)$$

$$T_0 = \frac{T_h}{1+z\bar{h}T_h}$$

$$(1-zT_0\bar{h})T_h = T_0$$

$$T_h = \frac{T_0}{1-zT_0\bar{h}}$$

$$T_0 = \begin{pmatrix} 1 & 0 \\ z\bar{h} & 1 \end{pmatrix} (T_h)$$

$$T_h = \begin{pmatrix} 1 & \\ -z\bar{h} & 1 \end{pmatrix} (T_0) = \frac{T_0}{1-z\bar{h}T_0}$$

$$\boxed{T_1 = \frac{T_0}{1-zT_0}} \quad \boxed{T_0 = \frac{T_1}{1+zT_0}}$$

$c_h = ba^* + \xi_- h \xi_+^*$ $\qquad Y = aX \oplus \mathbb{C}\xi_+ = bX \oplus \mathbb{C}\xi_-$

$aa^* + \xi_+\xi_+^* = 1. \qquad bb^* + \xi_-\xi_-^* = 1$

there are two representations, outgoing + incoming

$$y = aa^*y + \underbrace{\xi_+\xi_+^*}_{\pi_+}y$$

$$uy = aa^*(ba^*)y + \pi_+(ba^*)y + u\pi_+y$$

$$u^2y = aa^*(ba^*)^2y + \pi_+(ba^*)^2y + u\pi_+(ba^*)y + u^2\pi_+y$$

$$y = u^{-n}aa^*c_0^ny + \sum_{k=0}^{n} u^{-k}\pi_+c_0^ky$$

If $c_0^ny \to 0$ get norm pres. isom. emb.

$$y \longmapsto \xi_+^*\sum_{k\geq 0} z^{-k}c_0^ky = \xi_+^*\frac{1}{1-z^{-1}c_0}y$$

$$\left\|\xi_+^*\frac{1}{1-z^{-1}c_0}y\right\|^2 = \left(\frac{1}{1-z^{-1}c_0}y \,\Big|\, \xi_+\xi_+^*\frac{1}{1-z^{-1}c_0}y\right) \qquad 1-aa^* = 1-c_0^*c_0$$

$$= \sum_n \left(c_0^ny \,\big|\, (1-c_0^*c_0)c_0^ny\right) = \|y\|^2 - \lim_{n\to\infty}\|c_0^ny\|^2$$

$$S_0 = \xi_+^*\frac{1}{1-z^{-1}c_0}\xi_- \qquad |z|>1.$$

Put $\quad T_0 = \xi_-^*\frac{1}{1-zc_0^*}\xi_-$ $\quad$ essentially S_0^* — def. for $|z|<1$.

$$T_h = \xi_-^*\frac{1}{1-zc_h^*}\xi_+ = \xi_-^*\left(\frac{1}{1-zc_0^*} + \frac{1}{1-zc_0^*}z(\xi_+\bar{h}\xi_-^*)\frac{1}{1-zc_h^*}\right)\xi_+$$

$$\boxed{T_h = T_0 + T_0 z\bar{h}T_h} \qquad \boxed{T_h = \frac{1}{1-T_0z\bar{h}}T_0}$$

$$1+z\bar{h}T_h = \frac{1}{1-z\bar{h}T_0} \qquad 1+zT_1 = \frac{1}{1-zT_0}$$

$$1 + zT_1 = 1 + z\xi_-^* \frac{1}{1-zu^{-1}}\xi_+ \qquad \xi_+ = \xi, \quad \xi_- = u\xi$$

$$= 1 + z\xi^* u^{-1}\frac{1}{1-zu^{-1}}\xi$$

$$= \xi^*\left(1 + \frac{zu^{-1}}{1-zu^{-1}}\right)\xi = \xi^* \frac{1}{1-zu^{-1}}\xi = \int \frac{1}{1-z\zeta^{-1}}\,d\mu$$

$$T_0 = \xi_-^* \frac{1}{1-z\zeta_0^*}\xi_+ \qquad T_1 = \xi_-^*\frac{1}{1-zu^*}\xi_+$$

$$1 + zT_1 = \boxed{\frac{1}{1-zT_0} = \int \frac{1}{1-z\zeta^{-1}}\,d\mu} = \xi^*\frac{1}{1-zu^*}\xi$$

So there's a puzzle why this works. What to do? ~~[illegible]~~ —

Problem: How to get further. Where to begin. Puzzle — what ~~[illegible]~~ can be significant. Describe in words. Discuss ~~philosophy.~~ W

Maybe go over the result many times. Maybe ~~the~~ generalize your projection op.

Suppose you take a (H, u, ξ). ~~[illegible]~~ Can you find S_0 at least in simple cases, e.g. where the measure is given by a polynomial.

$$d\mu = \frac{1}{|g|^2}\frac{d\theta}{2\pi} \qquad g \text{ poly with roots outside}$$

S^1. ~~[illegible]~~ This should be simple algebra. For example take $g = 1 - \bar{h}z$

$$\begin{pmatrix} p_1 \\ q_1 \end{pmatrix} = \frac{1}{k_1}\begin{pmatrix} 1 & h_1 \\ \bar{h}_1 & 1 \end{pmatrix}\begin{pmatrix} z \\ 1 \end{pmatrix}$$

Consider $L^2(S^1, \rho \frac{d\theta}{2\pi})$. Can you solve the eigenvalue equation leading to S_0

Assume $\int \rho \frac{d\theta}{2\pi} = 1$. ~~of course ρ smooth > 0.~~

eigenvector equation is $(z - \zeta) x = -v_+ + v_-$

$\zeta = e^{i\theta}$ $\quad v_+ \in \mathbb{C}\xi_+ = \mathbb{C}1$, ~~so v_+~~

(Repeat, recall: $H = aX \oplus \mathbb{C}\xi_+ = bX \oplus \mathbb{C}\xi_-$

$\xi_+ =$ the cyclic vector $\zeta = 1$, $\xi_- =$ the vector $u(\zeta) = \zeta$

so that $uax = bx$. $\quad c_h = ba^* + \xi_- h \xi_+$, $c_h \xi_+ = h\xi_-$

You want to solve $(z - \zeta)x = -1 + S_0(z)\zeta$

with $x \in \mathrm{Ker}(a^*) = \perp$ of $\mathbb{C}\xi_+ = \mathbb{C}$.

$$(z - \zeta)\, x(\zeta) = S_0(z)\zeta - 1$$

In other words you seek $f(\zeta)$ function on S^1 such that

(i) $\int f(\zeta)\, d\mu = 0$

(ii) $f(\zeta) = \dfrac{\sigma\zeta - 1}{\lambda - \zeta}$?

<u>Is there a way</u> to generate $f(\zeta) \perp 1$

$$H = \mathbb{C}\xi_+ \oplus X \simeq \mathbb{C} \oplus X$$
$$= \mathbb{C}u\xi \oplus uX = \mathbb{C}\zeta \oplus \zeta X$$

$\zeta = e^{i\theta}$

$$X \xrightarrow{az-b} Y$$

$$X \xrightarrow{\binom{a}{b}} \begin{pmatrix} Y \\ \oplus \\ Y \end{pmatrix}$$

(arrow $Y \downarrow$ into the column above, and $\downarrow$ below it)

$$\begin{array}{ccc} & & Y \\ & & \Big\downarrow \begin{pmatrix}1\\ z\end{pmatrix} \\ W^\circ = \begin{pmatrix}a\\ b\end{pmatrix} X + \begin{pmatrix}0\\ \operatorname{Ker} b^*\end{pmatrix} & \subset & Y \oplus Y \\ & & \Big\downarrow (z\ \ -1) \\ & & Y \end{array}$$

---

eigenvalue equation Given (H, u) and $Y \subset H$ put $X = u^{-1} Y \cap Y$, then $H = X \oplus V_+ \oplus Y^\perp = V_- \oplus uX \oplus Y^\perp$; Given $u\xi = \lambda\xi$ let $\xi = x_1 + v_+ + \eta_1 = v_- + u(x_2) + \eta_1$, project

$$u(\xi) = u(x_1) + \underbrace{u(v_+ + \eta_1)}_{\in u(X^\perp)} \quad \text{and} \quad \lambda\xi = u(\lambda x_2) + \underbrace{\lambda v_- + \lambda\eta_1}_{\in (uX)^\perp}$$

onto uX to get $u(x_1) = u(\lambda x_2) \Rightarrow x_1 = \lambda x_2$.

Put $x = x_2$, get $\lambda x + v_+ = v_- + ux$ in Y

or $\boxed{(\lambda - u)x = -v_+ + v_-}$ other form $(za - b)x = -v_+ + v_-$

For $|z| < 1$ can solve:

$$\begin{array}{ccccc} & & \mathbb{C}\xi_- & & \\ & & \downarrow & & \\ X & \xrightarrow{az-b} & Y & \longrightarrow & Y/(az-b)X \\ & \underset{1-zb^*a}{\overset{\sim}{\searrow}} & \Big\downarrow -b^* & & \\ & & X & & \end{array}$$

so ξ_- generates or trivializes the line bundle $z \mapsto Y/(az-b)X$.

$$Y \xrightarrow{\begin{pmatrix} b^* \\ j_-^* \end{pmatrix}} \begin{pmatrix} X \\ V_- \end{pmatrix} \xrightarrow{(b \;\; j_-)} Y$$ gives splitting $Y = bX + j_- V_-$

Consider the perturbation $b - az$ of b.

$$(b - az \;\; j_-) \begin{pmatrix} b^* \\ j_-^* \end{pmatrix} = bb^* + jj_-^* - zab^* = 1 - zab^*$$

so when $1 - zab^*$ is invertible one gets

$$(b - az \;\; j_-)^{-1} = \begin{pmatrix} b^* \\ j_-^* \end{pmatrix} (1 - zab^*)^{-1}$$

i.e. $$y = (b - az)x + j_-(v_-) \implies v_- = j_-^* (1 - zab^*)^{-1} y$$

So back to problem: ~~[illegible]~~

(H, u, ξ_0) $\quad H = \mathbb{C}\xi_0 \oplus X \quad X = \{\xi \in H \,|\, (\xi_0|\xi) = 0\}$

$\|\xi_0\| = 1$ assume. Take a ~~specific~~ simple ρ,

~~[illegible]~~ namely where $\begin{pmatrix} p_1 \\ q_1 \end{pmatrix} = \frac{1}{k} \begin{pmatrix} 1 & h \\ \bar{h} & 1 \end{pmatrix} \begin{pmatrix} z \\ 1 \end{pmatrix}$

So you have this cyclic vector $\xi_0 = p_0 = q_0$

The measure should be $\rho = \frac{1}{|q_1|^2}$ $\quad q_1 = \frac{\bar{h}z + 1}{k}$

$$\rho = \frac{k^2}{|1 + \bar{h}z|^2}$$ Check $$\int \rho \frac{d\theta}{2\pi} = \oint \frac{k^2}{(1 + \bar{h}z)(z + h)} \frac{dz}{2\pi i}$$

$$= \frac{k^2}{1 - \bar{h}h} = 1.$$ Now look at the geometry

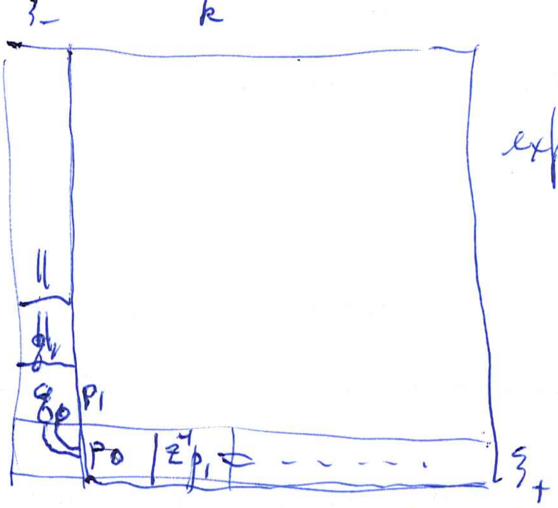
H consists of $f(z)$ on $|z| = 1$. What do you want? You have (H, u, ξ_0), you want to understand the partial unitary arising ~~[illegible]~~ by: $Y = H$, $X = \xi_0^\perp$

$a : X \longrightarrow H$ inclusion $\quad b = z : X \longrightarrow H$.

You have a scattering picture of (H, u, ξ_0) perhaps, Yes this should be clear from the h-sequence where $h_n = 0$ $n \geq 2$. To understand this angle better:

$$\xi_+ = z^{-1} p_1 \qquad \xi_- = q_1 = \frac{1}{k}(1+\bar{h} z)$$

$$\xi_+ = z^{-1}\frac{z+h}{k} = \frac{1+hz^{-1}}{k} \qquad \delta = \frac{\xi_+}{\xi_-} = \frac{1+hz^{-1}}{1+\bar{h}z}$$



How can I get more explicit?

What sort of description of H can you give

Span of $p_0, p_1, \dots$ is $\overline{\mathbb{C}[z]}$

$z^{-1}q_0, z^{-2}q_1, \dots$ is $\overline{z^{-1}\mathbb{C}[z^{-1}]}$

this (should be) a decreasing staircase basis ~~ff~~

Actually your approach might improve if you start with the grid space (rank 2) i.e. define the grid space using the h_n $n \geq 1$ from the orthog polys and $h_n = 0$ $n \leq 0$.

This idea suggests varying h_n, treating a single h_n as a variable to understand the Green's fn.

Do you have enough information?

~~[scribble]~~ Something natural about taking (H, u, ξ_0), put $X = \xi_0^\perp$, splitting $H = X + \mathbb{C}\xi_0$ $\quad 1 = aa^* + \xi_0\xi_0^*$

$$Y \xrightarrow{\begin{pmatrix} a^* \\ \xi_0^* \end{pmatrix}} \begin{pmatrix} X \\ \oplus \\ \mathbb{C} \end{pmatrix} \xrightarrow{(a \ \xi_0)} Y$$

then perturbing a to $a - \lambda b$

$$(a - \lambda b \ \ \xi_0)\begin{pmatrix} a^* \\ \xi_0^* \end{pmatrix} = 1 - \lambda b a^*$$

so $\quad (a - \lambda b \ \ \xi_0)^{-1} = \begin{pmatrix} a^* \\ \xi_0^* \end{pmatrix}(1 - \lambda b a^*)^{-1}$

<u>Idea</u> instead of decreasing H to X increase it, add a line to H.

What is your aim? Review what you tried yesterday. The problem: Start with a ~~[scribble]~~ smooth measure $\rho \frac{d\theta}{2\pi}$, ~~[scribble]~~ form $L^2(S^1, d\mu)$, orthog. polys $\begin{pmatrix} p_n \\ q_n \end{pmatrix}$, get h_n $n \geq 1$. You want to extend the sequence by 0 to get a scattering situation. So you should have a way to go from q_∞ to an S matrix.

$$\begin{pmatrix} \xi_+ \\ \xi_- \end{pmatrix} = \begin{pmatrix} a^n & b^n \\ c^n & d^n \end{pmatrix}\begin{pmatrix} p_0 \\ q_0 \end{pmatrix} \qquad \begin{pmatrix} p_0 \\ q_0 \end{pmatrix} = \begin{pmatrix} \xi'_- \\ \xi'_+ \end{pmatrix}$$

~~[scribble]~~ This transfer matrix is such that $c^n \in zH_+$

$$\begin{pmatrix} z^{-1}p_1 \\ q_1 \end{pmatrix} = \frac{1}{k_1}\begin{pmatrix} 1 & z^{-1}h_1 \\ \bar{h}_1 z & 1 \end{pmatrix}\begin{pmatrix} p_0 \\ q_0 \end{pmatrix} \qquad S = \frac{1}{d}\begin{pmatrix} 1 & b \\ -c & 1 \end{pmatrix}$$

$$\begin{pmatrix}\bar{q}\\ q\end{pmatrix} = \begin{pmatrix}\bar{d} & \bar{c}\\ c & d\end{pmatrix}\begin{pmatrix}1\\ 1\end{pmatrix}$$

Given q inv. analytic on D can you find c, d with $c \in zH_+$, d inv. analytic, $|c|^2 + |d|^2 = 1$ on S^1, $q = c + d$. ? already what does this look like for q rational with roots outside S^1.

Suppose given (Y, u, ξ_0) to adjoin a new vector to H and introduce coupling. No you do what you did before, namely form c_h and dilate

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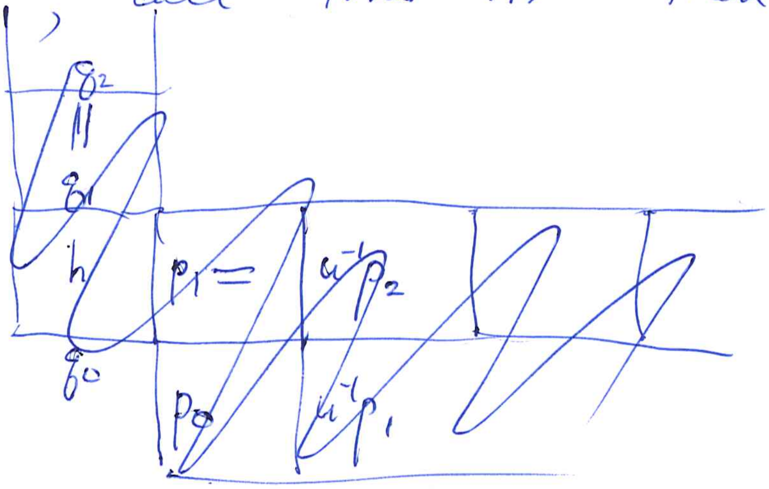
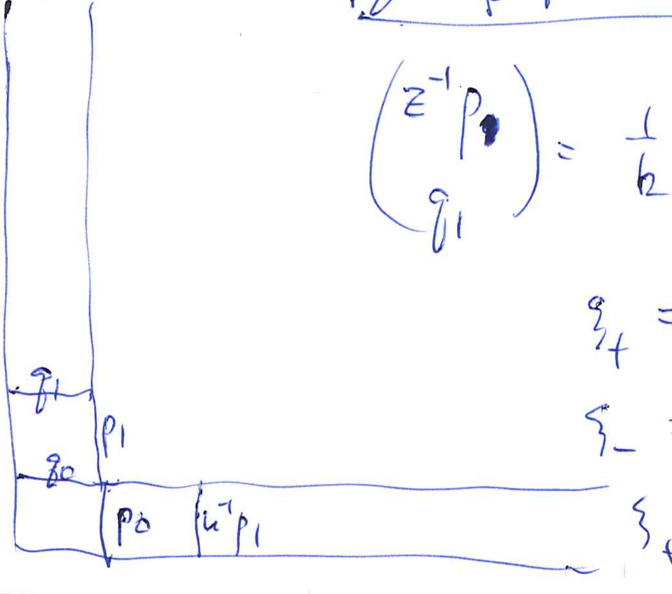
Suppose you have (H, u, ξ_0) $\|\xi_0\| = 1$.

Specific calc. Take one h non zero namely h_{11}, call this h. Then





$$\begin{pmatrix} z^{-1}p_1 \\ \xi_1 \end{pmatrix} = \frac{1}{k}\begin{pmatrix} 1 & z^{-1}h \\ \bar{h}z & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\xi_+ = z^{-1}p_1 = \frac{1 + z^{-1}h}{k}$$

$$\xi_- = \xi_1 = \frac{1 + \bar{h}z}{k}$$

What are you calculating?

first you begin with measure

$$d\mu = \frac{1}{|q_1|^2}\frac{d\theta}{2\pi} = \frac{k^2}{(1+\bar{h}z)(1+hz^{-1})}\frac{dz}{2\pi i z}$$

$$d\mu = \frac{k^2}{(1+\bar{h}z)(z+h)}\frac{dz}{2\pi i}$$ The measure gives you

$$\begin{pmatrix} p_1 \\ q_1 \end{pmatrix} = \frac{1}{k}\begin{pmatrix} 1 & h \\ \bar{h} & 1 \end{pmatrix}\begin{pmatrix} z \\ 1 \end{pmatrix} =$$

$$\xi_+ = \lim z^{-n} p_n = z^{-1} p_1 \qquad \text{for } n \geq 1.$$

$$= \frac{1+hz^{-1}}{k}$$

$$\begin{pmatrix} z^{-1}p_1 \\ q_1 \end{pmatrix} = \frac{1}{k}\begin{pmatrix} 1 & hz^{-1} \\ \bar{h}z & 1 \end{pmatrix}\begin{pmatrix} p_0 \\ q_0 \end{pmatrix}$$

~~So the scat data is either~~

All you've done so far: Start with the orth sys with only $h_1 = h \neq 0$. Get $\begin{pmatrix} p_n \\ q_n \end{pmatrix}$

$$= \begin{pmatrix} z^{n-1}p_1 \\ q_1 \end{pmatrix}$$

$$\begin{pmatrix} p_n \\ q_n \end{pmatrix} = \begin{pmatrix} z^{-1}p_1 \\ q_1 \end{pmatrix} \qquad n \geq 1.$$

$$= \frac{1}{k}\begin{pmatrix} 1 & hz^{-1} \\ \bar{h}z & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad n \geq 1$$

so the relevant "scattering" or asymptotic info is the poly $q_1 = \frac{1+\bar{h}z}{k}$. Transfer matrix

Somehow the asymptotics of the orth poly system lead to $S = \frac{z^{-n} p_n}{q_n}$ a rational loop. formulate better.

~~Find the rotang~~

Take $h_1 \neq 0$ only. $\begin{pmatrix} z^{-1} p_1 \\ q_1 \end{pmatrix} = \frac{1}{k} \begin{pmatrix} 1 & h z^{-1} \\ \bar{h} z & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

Your problem is to relate q_∞ to the ~~S matrix~~. S or transfer matrix.

$$\begin{pmatrix} \xi_+ \\ \xi_- \end{pmatrix} = \underbrace{\frac{1}{k} \begin{pmatrix} 1 & h z^{-1} \\ \bar{h} z & 1 \end{pmatrix}}_{\begin{pmatrix} a & b \\ c & d \end{pmatrix}} \begin{pmatrix} p_0 \\ q_0 \end{pmatrix}$$

Supposedly this comes from $d\mu = \frac{1}{|q_1|^2} \frac{d\theta}{2\pi} = \frac{k^2}{(1+\bar{h}z)(z+h)} \frac{dz}{2\pi i}$

What's going on. Assume

$$\begin{pmatrix} \xi_+ \\ \xi_- \end{pmatrix} = \begin{pmatrix} a^n & b^n \\ c^n & d^n \end{pmatrix} \begin{pmatrix} p_0 \\ q_0 \end{pmatrix}$$

Assume $\begin{pmatrix} a^n & b^n \\ c^n & d^n \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

i.e. $\frac{\xi_+}{\xi_-} = \frac{a+b}{c+d} = \frac{\bar{d}+\bar{c}}{c+d}$ Your problem concerns writing $q_\infty = c'z + d$ where c', d ~~[illegible]~~ analytic for $|z| < 1$ d non vanishing $|d|^2 - |c|^2 = 1$.

Look at this algebraically in the simple case $g_\infty = \frac{1 + \bar{h} z}{k}$

Green's function $\frac{1}{\lambda - u}$ $\frac{1}{1 - \lambda u^{-1}}$

How do I use this? I need examples.

Given $d\mu$ a S^1 not finite supp. (H, u, ξ_0)

$\int d\mu$.

Form $(\lambda - u)^{-1} \xi_0 = \sum_{n \geq 0} \lambda^{-n-1} u^n \xi_0$ $|\lambda| > 1$

$= -\sum_{n \geq 0} \lambda^n u^{-n-1} \xi_0$ $|\lambda| < 1$.

Is an element of the L^2 completion of grid space

i.e. $L^2(S^1, d\mu)$. Grid space is $\mathbb{C}[z, z^{-1}]$

$\xi_0^* \frac{1}{\lambda - u}$ is a linear functional on grid space so you get $\begin{pmatrix} p_n(\lambda) \\ q_n(\lambda) \end{pmatrix}$

$$\psi_n = \xi_0^* \frac{1}{\lambda - u} \begin{pmatrix} p_n \\ q_n \end{pmatrix} = \frac{1}{k_n} \begin{pmatrix} 1 & h_n \\ \bar{h}_n & 1 \end{pmatrix} \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix} \text{ ?}$$

~~boundary~~ initial condition $\psi_0^1 = \psi_0^2 = \xi_0^* \frac{1}{\lambda - u} \xi_0$

Say $h_n = 0$ $n \gg 0$, then have asymptotics

~~Green's fn.~~ Linear functional $\xi_0^* \frac{1}{\lambda - u}$ on $L^2(S^1, d\mu)$

You have a linear fnl on grid space, which kills

$(\lambda - u)\underbrace{(\mathrm{Ker}\, \xi_0^*)}_{X}$. X spanned by p_n, q_n $n \geq 1$.

so that $u =$

Linear fnl $\xi_0^* \frac{1}{\lambda - u}$ kills $(\lambda-u)\,\underline{\mathrm{Ker}\,\xi_0^*}$

$X = \{p_n, q_n\}\ n \geq 1$ NO

Linear fnl kills $\xi\ (\lambda-u)\ \begin{matrix} p_n \\ q_n \end{matrix}$ $n \geq 1$.

so that $$\begin{pmatrix} p_n \\ q_n \end{pmatrix} = \frac{1}{k_n}\begin{pmatrix} 1 & h_n \\ \bar{h}_n & 1 \end{pmatrix}\begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} p_{n-1} \\ q_{n-1} \end{pmatrix}$$

should hold for $n \geq 2$.

$\xi_0^* \frac{1}{\lambda - u}$ kills $(\lambda-u)\,\underline{\mathrm{Ker}\,\xi_0^*}$ — X = span of p_n, q_n

so that $$\psi_n(\lambda) = \xi_0^* \frac{1}{\lambda-u}\begin{pmatrix} p_n \\ q_n \end{pmatrix} = \xi_0^* \frac{1}{\lambda-u}\frac{1}{k_n}\begin{pmatrix} 1 & h_n \\ \bar{h}_n & 1 \end{pmatrix}\begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix}\begin{matrix} p_{n-1} \\ q_{n-1} \end{matrix}$$

$$= \frac{1}{k_n}\begin{pmatrix} 1 & h_n \\ \bar{h}_n & 1 \end{pmatrix}\begin{pmatrix} \lambda & 0 \\ 0 & 0 \end{pmatrix}\psi_{n-1}(\lambda)$$ for $n \geq 1$.

But what about $$\begin{pmatrix} p_1 \\ q_1 \end{pmatrix} = \frac{1}{k_1}\begin{pmatrix} 1 & h_1 \\ \bar{h}_1 & 1 \end{pmatrix}\begin{pmatrix} u\xi_0 \\ \xi_0 \end{pmatrix}$$

$\xi_0^* \frac{u}{\lambda-u}\xi_0 = \xi_0^* \frac{\lambda}{\lambda-u}\xi_0 - 1$

$R = \xi_0^* \frac{1}{\lambda-u}\xi_0$

$$\xi_0^* (\lambda-u)^{-1} \frac{1}{k_1}\begin{pmatrix} 1 & -h_1 \\ -\bar{h}_1 & 1 \end{pmatrix}\begin{pmatrix} p_1 \\ q_1 \end{pmatrix}$$

$$\|$$

$$\frac{1}{k_1}\begin{pmatrix} 1 & -h_1 \\ -\bar{h}_1 & 1 \end{pmatrix}\psi_1(\lambda) = \xi_0^* \frac{1}{\lambda-u}\begin{pmatrix} u\xi_0 \\ \xi_0 \end{pmatrix} = \begin{pmatrix} \lambda R - 1 \\ R \end{pmatrix}$$

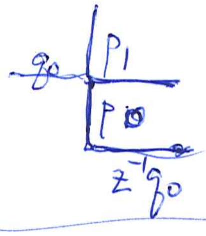
$$\psi_1 = \frac{1}{k_2}\begin{pmatrix} 1 & -h_2 \\ -\bar{h}_2 & 1 \end{pmatrix}$$

~~but I ...~~

$$\psi_{n-1}(\lambda) = \begin{pmatrix} \lambda^{-1} & 0 \\ 0 & 1 \end{pmatrix}\frac{1}{k_n}\begin{pmatrix} 1 & -h_n \\ -\bar{h}_n & 1 \end{pmatrix}\psi_n(\lambda)$$

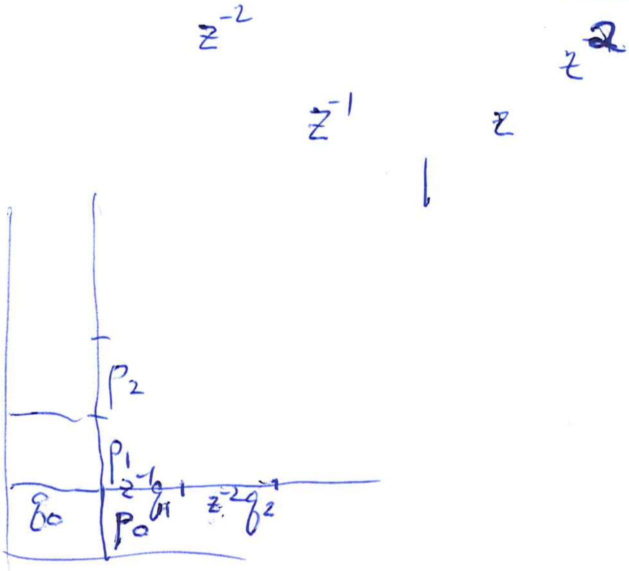
Start again. Consider $H = L^2(S^1; d\mu)$, $\int d\mu = 1$, inf. supp. 368

$\xi_0 = 1$, $u = z\cdot$, $\xi_0^* \frac{1}{\lambda - z}$ linear fnl. on H vanishes on $(\lambda - z)X$ where $X = \xi_0^\perp$. X includes $p_1, p_2, \dots$ and $z^{-1}q_0, z^{-2}q_1, \dots\dots$??



$$\tilde{p}_n \in (z^n + F_{n-1}) \cap (F_{n-1})^\perp$$

$$\tilde{q}_n \in (1 + zF_{n-1}) \cap (zF_{n-1})^\perp$$



$$\tilde{q}_1 \in (1 + \mathbb{C}z) \cap (\mathbb{C}z)^\perp$$

$$z^{-1}\tilde{q}_1 \in (z^{-1} + \mathbb{C}) \cap \mathbb{C}^\perp$$

$$\tilde{q}_n \in (1 + zF_{n-1}) \cap (zF_{n-1})^\perp$$

$$z^{-n}\tilde{q}_n \in (z^{-n} + \underbrace{z^{-n+1}F_{n-1}}_{F_{n-1}}) \cap (z^{-n+1}F_{n-1})^\perp$$

X has the basis p_n, $z^{-n}q_n$, for $n \geq 1$.

To understand $\xi_0^* \frac{1}{\lambda - u}$ (vanishes on $(\lambda - u)X$ $\quad X = \operatorname{Ker} \xi_0^*$

recursion relations.

$$\begin{pmatrix} p_n \\ q_n \end{pmatrix} = \gamma(h_n) \begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} p_{n-1} \\ q_{n-1} \end{pmatrix}$$

$$\begin{pmatrix} p_n \\ z^{-n}q_n \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & z^{-n} \end{pmatrix} \frac{1}{k_n} \begin{pmatrix} 1 & h_n \\ \bar{h}_n & 1 \end{pmatrix} \begin{pmatrix} z & 0 \\ 0 & z^{n-1} \end{pmatrix} \begin{pmatrix} p_{n-1} \\ z^{-n+1}q_{n-1} \end{pmatrix}$$

$$= \frac{1}{k_n} \begin{pmatrix} z & h_n z^{n-1} \\ z^{-n+1}\bar{h}_n & z^{-1} \end{pmatrix} \begin{pmatrix} p_{n-1} \\ z^{-n+1}q_{n-1} \end{pmatrix}$$

So there are obvious elements of X namely $z^j \xi_+^*$, $z^j \xi_-$ for $j \geq 1$.

$\xi_- \perp z\mathbb{C}[z]$ for $j \geq 1$.

so $z^{-j}\xi_- \perp z^{-j} z^j = 1$

and you get a ~~nice~~ basis

So $X = (p_0)^\perp = \xi_+ z H_+ + \xi_- H_-$

What next? To understand

$\xi_0^* \frac{1}{\lambda - u}$. What does to understand mean?

This is an important piece of the resolvent of u on H.

Recall the problem. Start with a smooth ρ equivalently the "determinant" fn. g.

To find the dilation of the partial unitary. You want to go from a g to a b.

What's going on? take simple case

Start say with $g = e^f$ f smooth on $\bar{D}$ analytic in D

$S = \frac{\bar{g}}{g}$, $L^2 = H_- \oplus SH_+$?

because $H_- \oplus H_+ \longrightarrow L^2$. We've been through this many times. Why?

$$\begin{matrix} H_+ \\ \oplus \\ H_- \end{matrix} \xrightarrow{\begin{pmatrix} \frac{1}{g} & 0 \\ 0 & \# \end{pmatrix}} \begin{matrix} H_+ \\ \oplus \\ H_- \end{matrix} \xrightarrow{(f_+ \ f_-)} L^2 \xrightarrow{\bar{g}\,\#} L^2$$

$$\bar{g}\,\#\,(f_+ \ f_-)\begin{pmatrix} \frac{1}{g} & 0 \\ 0 & \# \end{pmatrix} \quad \text{is an isom.}$$

$$(\bar{g}f_+ g^{-1} \quad \bar{g}f_-) : \begin{pmatrix} H_+ \\ H_- \end{pmatrix} \xrightarrow{\sim} L^2$$

image of first comp is $\frac{\bar{g}}{g} H_+ = SH_+$

What is $zH_- \cap SH_+$? it's generated by

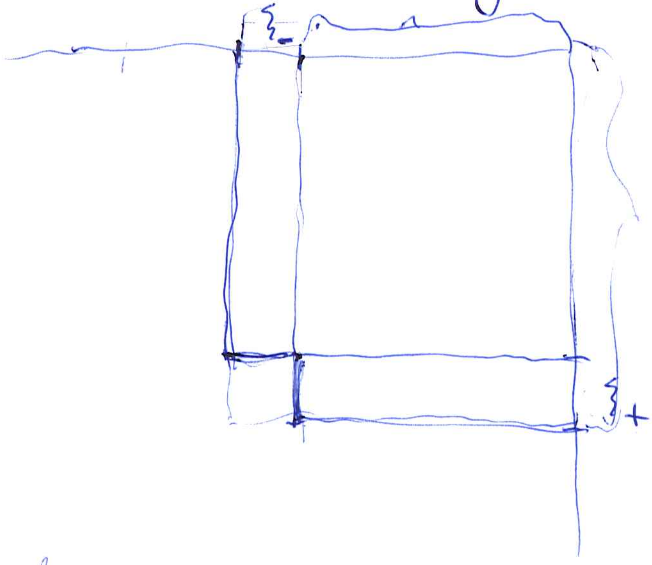
$$\bar{g} = Sg$$

So ~~what~~ next? Recall the situation. You start with

What are you trying to do? First repeat: You start with $g = e^f$ invertible smooth on $\bar{D}$ analytic inside, form $L^2\left(S^1, \frac{1}{|g|^2}\frac{d\theta}{2\pi}\right) = H$, $u = z\cdot$, $\xi_0 = 1$ assume g normalized so that $\int d\mu = 1$, $g(0) > 0$. Then get $\xi_- = g$, $\xi_+ = \bar{g}$ $\quad S\xi_- = \xi_+ \quad S(z) = \frac{\bar{g}}{g}$

Point here maybe is that $\xi_0^\perp = \xi_+ zH_+ + \xi_- H_-$

How much do you know? You know something about intersections



$\xi_+ = S\xi_-$

$\xi_+ H_+ + \xi_- H_-$

$S H_+ \oplus H_- = L^2$

$zH_- \cap SH_+ \qquad \xi_- = S\xi_+$

$z^n H_- \cap SH_+ \qquad$ dim n.

So you need to straighten this out.

$$z^n H_- \cap \overset{\bar{g}/g}{S} H_+ = z^n H_- \cap \frac{\bar{g}}{g} H_+ \xleftarrow{\;\bar{g}/g\;} \frac{z^n H_- \cap H_+}{F_{n-1}} \quad \left(\frac{F_{n-1}}{\bar{g}}\right)$$

$$F_{n-1} \longrightarrow z^n H_- \cap SH_+$$

polys of degree $< n$ in z

Your aim is to understand $\xi_0^* \frac{1}{\lambda - u} \xi_0$.

$\mathbb{C}\xi_0 =$

$\xi_- H_- \oplus \xi_+ H_+ \overset{?}{=} H$

$H_- \oplus SH_+ = L^2 \qquad$ Yes.

$\xi_- zH_- \oplus \xi_+ H_+ \longrightarrow L^2(d\mu)$

$\xi_- zH_- \cap \xi_+ H_+ = \underline{zH_- \cap SH_+} \quad \ni \left(\bar{g} = Sg\right)$

So given g $\qquad d\mu = \frac{1}{|g|^2}\frac{d\theta}{2\pi}$ $\qquad \int d\mu = 1$

$$H = L^2(S^1, d\mu) \qquad \xi_0 = 1 \qquad \xi_- = g \qquad \xi_+ = \bar{g}$$

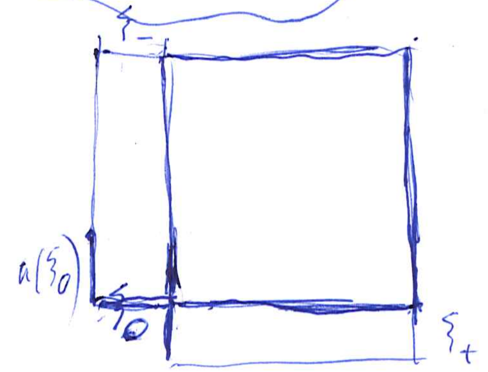
$$L^2(S^1) \xrightarrow{\xi_- = g\cdot} H \xleftarrow{\xi_+ = \bar{g}\cdot} L^2(S^1)$$

$$S = \frac{\bar{g}}{g} \qquad S\xi_- = \xi_+ .$$

$$\xi_0^* \frac{1}{\lambda - u}\xi_0$$

You are working in $L^2(S^1)$ with $\xi_- = 1$, $\xi_0 = g^{-1}$

$$\xi_+ = \frac{\bar{g}}{g}$$



$$\xi_0^* \frac{1}{\lambda - u}\xi_0 = \int \frac{1}{\lambda - z}\, d\mu$$

$$u^{-1}Y = \xi_+ H_+ + \xi_- H_- = \xi_-(S H_+ \oplus H_-)$$

$$X = \xi_+ z H_+ + \xi_- H_-$$

$$Y = \xi_+ z H_+ + \xi_- z H_-$$

$$\mathrm{Ker}(a^*) = X^\perp = \xi_0$$

$$\mathrm{Ker}(b^*) = aX^\perp = u\xi_0$$

Repeat. given $d\mu = \rho\frac{d\theta}{2\pi}$ $\quad \rho$ smooth >0 $\quad \int d\mu = 1.$

Let $\log\rho = \sum_{n\in\mathbb{Z}} a_n z^n = f + \bar{f}$ $\qquad f(z) = \frac{a_0}{2} + \sum_{n\geq 1} a_n z^n$

$$g = e^{-f} \qquad |g|^2 d\mu = |g|^2\rho\frac{d\theta}{2\pi} = \frac{d\theta}{2\pi}$$

$$H = L^2(S^1, d\mu) \qquad \xi_0 = 1, \quad \xi_- = g, \quad \xi_+ = \bar{g}$$

just $L^2(S^1, \frac{d\theta}{2\pi})$ with different (1).

~~[illegible]~~

$$L^2 \xrightarrow[\sim]{1 \mapsto g = \xi_-} L^2(d\mu) \xleftarrow[\sim]{\bar g = \xi_+ \mapsfrom 1} L^2$$

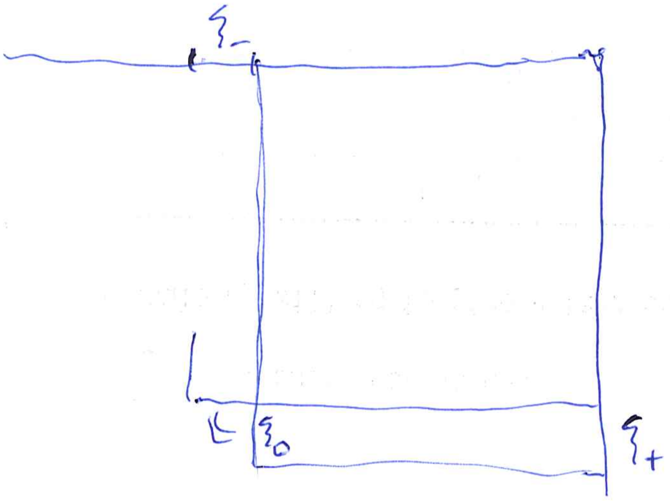
$$S = \frac{\bar g}{g}$$

$$H_- \oplus S H_+ = L^2 \qquad \xi_- H_- \oplus \xi_+ H_+ = L^2(d\mu)$$

$$g H_- \oplus \bar g H_+ = L^2(d\mu)$$

$$\begin{pmatrix} H_+ \\ H_- \end{pmatrix} \xrightarrow[\sim]{\begin{pmatrix} g^{-1} & 0 \\ & 1 \end{pmatrix}} \begin{pmatrix} H_+ \\ H_- \end{pmatrix} \xrightarrow[\sim]{(g_+ \ \ g_-)} L^2 \xrightarrow[\sim]{\bar g} L^2$$

$$\begin{pmatrix} f_+ \\ f_- \end{pmatrix} \longmapsto \bar g g_+ g^{-1} f_+ + \bar g g_- f_- = \underbrace{S f_+}_{SH_+} + \underbrace{\bar g f_-}_{H_-}$$



important is that

$$L^2(d\mu) = \mathbb{C} + (H_- g + z H_+ \bar g)$$

$$L^2 = \mathbb{C} g^{-1} + H_- + S z H_+$$

$E =$

$$g = \frac{1 + \bar h z}{k} \qquad \int \frac{d\theta}{2\pi} \frac{1}{|g|^2} \frac{d\theta}{2\pi} = \frac{k^2}{(1+\bar h z)(z+h)} \frac{dz}{2\pi i}$$

$|h| < 1$.

$$\int \rho \frac{d\theta}{2\pi} = \mathrm{Res}\left(\frac{k^2}{(1+\bar h z)(z+h)} \right) = \frac{k^2}{1-|h|^2} = 1.$$

~~[illegible]~~

$$\xi_0^* \frac{1}{\lambda - u} \xi_0 = \oint \frac{1}{\lambda - z} \frac{k^2}{(1+\bar h z)(z+h)} \frac{dz}{2\pi i}$$

$$= \frac{1}{\lambda + h} \quad \text{if} \quad |\lambda| > 1.$$

more generally $\quad \xi_0^* \frac{1}{\lambda - u} f = \int \frac{f(z)}{\lambda - z} \frac{k^2}{(1+\bar h z)(z+h)} \frac{dz}{2\pi i}$

$$= \frac{f(-h)}{\lambda + h} \quad \text{if } f \text{ ~~[illegible]~~ analytic in the disk}$$

$g\,\xi_0 = \xi_-$ $\quad\therefore\ \xi_0 = \frac{1}{g}$ in the ξ_- rep.

$$\xi_0^* \frac{1}{\lambda - u} f\, \xi_0 = \int \frac{1}{\lambda - z} f(z) \frac{1}{|g|^2} \frac{d\theta}{2\pi}$$

Consider the linear fnl $\xi_0^* \frac{1}{\lambda - u}$ on the subspace $H_+ \xi_0 = H_+ \xi_-$ includes $\mathbb{C}[z]$ i.e. p_n, q_n

It seems that provided I deal with polys in z you can remove the value at λ, the result is div by $\lambda - z$, so the value counts. If true then $\begin{pmatrix} \psi_n^1 \\ \psi_n^2 \end{pmatrix} = \xi_0^* \frac{1}{\lambda - u} \begin{pmatrix} p_n \\ q_n \end{pmatrix}$ should satisfy the rec. relns.

---

idea Consider example only $h_1 \neq 0$.

$$\begin{pmatrix} \xi_+ \\ \xi_- \end{pmatrix} = \frac{1}{k} \begin{pmatrix} 1 & hz^{-1} \\ \bar{h}z & 1 \end{pmatrix} \begin{pmatrix} \xi'_- \\ \xi'_+ \end{pmatrix}$$

Check $\begin{pmatrix} z^{-1}p_1 \\ q_1 \end{pmatrix} = \frac{1}{k} \begin{pmatrix} 1 & hz^{-1} \\ \bar{h}z & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$$\begin{cases} p_1 = \dfrac{z+h}{k} \\ q_1 = \dfrac{1+\bar{h}z}{k} \end{cases}$$

$c = \frac{\bar{h}}{k} z$

In general $\begin{pmatrix} \xi_+ \\ \xi_- \end{pmatrix} = \begin{pmatrix} d & \bar{c} \\ c & d \end{pmatrix} \begin{pmatrix} \xi'_- \\ \xi'_+ \end{pmatrix}$

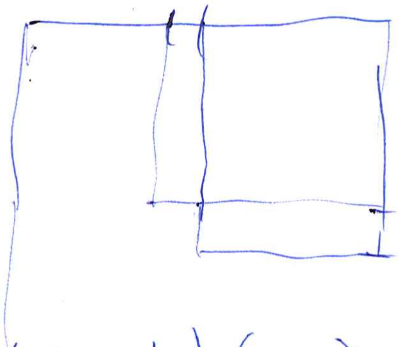
analytic in D, d invertible, $c(z) = 0$

$1 + |c|^2 = |d|^2$.

Response fn. – brings in λ.

Green's function

Make an effort



Start with $\begin{pmatrix}\xi_+\\ \xi_-\end{pmatrix} = \begin{pmatrix} a & b\\ c & d\end{pmatrix}\begin{pmatrix}p_0\\ q_0\end{pmatrix}$, $\begin{pmatrix}p_0\\ q_0\end{pmatrix} = \begin{pmatrix}\xi_-'\\ \xi_+'\end{pmatrix}$

In principle the Hilbert space is $\xi_+ H_+ + \xi_- H_-$

$$\frac{\bar q}{q} H_+ \oplus H_- \xrightarrow{\simeq ?} L^2$$

---

idea that $\xi_+ H_+ + \xi_- H_-$, on this, u is simple, a simple shift with a carryover. to express ξ_- as an element of this subspace. So why is 1 is $SH_+ + H_-$? You know there's a formula involving q.

$$\frac{\bar q}{q} H_+ \oplus H_- = L^2$$

$$\frac{1}{q} H_+ \oplus \frac{1}{\bar q} H_- = L^2$$

$$\frac{\bar q}{q} f_+ + f_- = 1$$

$$\frac{1}{q} f_+ + \frac{1}{\bar q} f_- = \frac{1}{\bar q}$$

$$\frac{1}{q} f_+ = \frac{1}{\bar q}(1 - f_-) = \text{const. } c$$

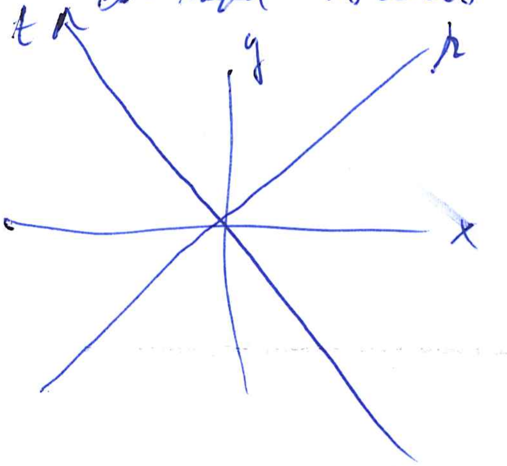
$f_+ = cq$

$f_- = 1 - c\bar q$

adjust c so that $1 - c\bar q(0) = 0$.

You have problem of understanding Green's functions for grid space. What does this mean?

~~Green's~~ Green's functions for $\begin{cases} \partial_x \psi^1 = \psi^2 \\ \partial_y \psi^2 = \psi^1 \end{cases}$



~~$x = r + t$~~
~~$y = r + t$~~
~~$r = x + y$~~
~~$t = -x + y$~~

~~$\partial_x f = \partial_r f + \partial_t f$~~
~~$\partial_y f = \partial_r f \; \partial_t f$~~

$$\partial_x f = \partial_r f \frac{\partial r}{\partial x} + \partial_t f \frac{\partial t}{\partial x}$$
$$\partial_y f = \partial_r f \frac{\partial r}{\partial y} + \partial_t f \frac{\partial t}{\partial y}$$

(annotations: $\frac{\partial r}{\partial x} = 1$, $\frac{\partial t}{\partial x} = -1$, $\frac{\partial r}{\partial y} = 1$)

$$r = x + y$$
$$t = -x + y$$

$$\partial_x = \partial_r - \partial_t \qquad (\partial_r - \partial_t)\psi^1 = \psi^2$$
$$\partial_y = \partial_r + \partial_t \qquad (\partial_r + \partial_t)\psi^2 = \psi^1$$

$$\partial_t \psi = \begin{pmatrix} \partial_r & -1 \\ 1 & -\partial_r \end{pmatrix} \psi \qquad \partial_r \psi = \begin{pmatrix} \partial_t & 1 \\ 1 & -\partial_t \end{pmatrix} \psi$$

So you need to understand Green's fn., response function. This means eigenvectors. Grid space E is a rank 2 free module over $\mathbb{C}[u,u^{-1}]$. So $E/(\lambda-u)E$ has dim 2, for any λ. ~~But you want to come But your~~ So linear functionals ϕ on $E/(\lambda-u)E$ form a two dim'l space. But E has the two herm. inner products, but let's fix attention on the pos. def one. You have

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Consider finite support potential

$$Y = aX + \mathbb{C}\xi_+$$

$$Y \xrightarrow{\begin{pmatrix} a^* \\ \xi_+^* \end{pmatrix}} X \oplus \mathbb{C} \xrightarrow{(a \;\; \xi_+)} Y$$

$$b = ua$$

$$aa^* + \xi_+\xi_+^* = 1.$$

$$(\;\; \xi_+)\begin{pmatrix} a^* \\ \xi_+^* \end{pmatrix} =$$

$$\begin{pmatrix} a^* \\ \xi^* \end{pmatrix}(a \;\; \xi) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

---

partial ~~unitary~~ isometry on Y

$$X \underset{b}{\overset{a}{\rightrightarrows}} Y \qquad a^*a = 1, \quad b^*b = 1$$

$$Y = aX \oplus V_+ = bX \oplus V_-$$

H Hilb. space, u ~~~~ on H,

Y closed subspace. $\quad X = u^{-1}Y \cap Y \underset{b = ua}{\overset{a = \text{inc}}{\rightrightarrows}} Y$

$$H = \underbrace{aX \oplus V_+}\oplus Y^\perp \qquad \xi = ax_1 + v_+ + \eta$$

$$= bX \oplus V_- \oplus Y^\perp \qquad = bx_2 + v_- + \eta$$

assume $u(\xi) = \lambda\xi \qquad u(\xi) = bx_1 + u(v_+ + \eta) \in u((aX)^\perp)$