The aim now is to ~~find~~ understand the ~~Green~~ Green's fn. $G_\lambda(x,y)$ sat $\left(\partial_x - V_\lambda(x)\right) G_\lambda(x,y) = \delta(x-y)$ and the appropriate boundary conditions. The boundary conditions requires $Re(\lambda) \neq 0$, ~~involving incom~~ or if you ~~stick~~ to $Re(\lambda) = 0$ something involving incoming ~~and~~ or outgoing waves. ~~Another ingredient is~~ Why? Because the ~~~~ Green's fn. is essentially the resolvent of a ~~self~~ skew-adjoint operator on Hilbert space. Take $h=0$.

IDEA: Does $SL(2,\mathbb{Z})$ appear in your discrete DE? Symmetries of space-time?

Take $h=0$. Then

$\partial_x G_\lambda(x,y) = \delta(x-y)$. So G_λ is independent of λ.

Actually you ~~~~ can't see the significance of $Re(\lambda)$ on this level

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Review: To understand the Green's function $G_\lambda(x,y)$ defined by

$$\left(\partial_x - V_\lambda(x)\right) G_\lambda(x,y) = \delta(x-y)$$

together with boundary condition at $x = -\infty, \infty$. In the case $h=0$ you have $V_\lambda = 0$, ind of λ so $G_\lambda(x,y)$ is constant in x to the left (resp right) of y with a unit jump at y. So you have to ~~understand~~ pick a boundary condition.

Review yesterdays construction of Green's function

$$\partial_x \begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} \lambda & h \\ h & 0 \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix} \qquad \begin{pmatrix} \tilde{p} \\ \tilde{q} \end{pmatrix} = z^{-x/2} \begin{pmatrix} p \\ q \end{pmatrix}$$

Review construction of Green's function

$$\partial_x \begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} \lambda & h \\ \bar{h} & 0 \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix} \qquad \begin{pmatrix} \tilde{p} \\ \tilde{q} \end{pmatrix} = z^{-x/2} \begin{pmatrix} p \\ q \end{pmatrix}$$

$$\underbrace{\begin{pmatrix} \partial_x & -h \\ +\bar{h} & -\partial_x \end{pmatrix}} \begin{pmatrix} \tilde{p} \\ \tilde{q} \end{pmatrix} = \frac{\lambda}{2} \begin{pmatrix} \tilde{p} \\ \tilde{q} \end{pmatrix} \qquad z^x = e^{\lambda x}$$

Use skew adjoint to get boundary conditions. Idea here is that Hilbert space theory tells you that the resolvent $\left(\frac{\lambda}{2} - D\right)^{-1}$ is defined for $Re(\lambda) \neq 0$, so the Green's function should have L^2 eigen functions away from the singularity. Another point: If h decays fast enough, then in the ODE

$$\partial_x \begin{pmatrix} z^{-x} p_x \\ q_x \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & h e^{-\lambda x} \\ \bar{h} e^{\lambda x} & 0 \end{pmatrix}}_{V_\lambda(x)} \begin{pmatrix} z^{-x} p_x \\ q_x \end{pmatrix}$$

because ______ $V_\lambda(x)$ is continuous in x and bounded analytic in any strip $|Re(\lambda)| < \text{const}$, you should be able to use ODE in a suitable Banach space to prove convergence

$$\begin{pmatrix} \xi'_- \\ \xi'_+ \end{pmatrix} \xleftarrow{-\infty \leftarrow x} \begin{pmatrix} z^{-x} p_x \\ q_x \end{pmatrix} \xrightarrow{x \to \infty} \begin{pmatrix} \xi_+ \\ \xi_- \end{pmatrix}$$

of entire functions of λ. Assume this holds (e.g. h bdd support) Then

$$\begin{pmatrix} z^{x/2} \xi'_- \\ z^{-x/2} \xi'_+ \end{pmatrix} \xleftarrow[\text{as } x \to -\infty]{\text{asympt}} \begin{pmatrix} z^{-x/2} p_x \\ z^{-x/2} q_x \end{pmatrix} \xrightarrow[\text{as } x \to +\infty]{\text{asympt}} \begin{pmatrix} z^{x/2} \xi_+ \\ z^{-x/2} \xi_- \end{pmatrix}$$

For $Re(\lambda) < 0$ $\quad z^{x/2} = e^{\lambda x/2}$ grows as $x \to +\infty$

$z^{-x/2} = e^{-\lambda x/2}$ ______ $x \to -\infty$

So the L^2 boundary conditions are $\xi_+ = 0$ and $\xi'_+ = 0$.

~~[illegible]~~ Conclude that for $\mathrm{Re}(\lambda) < 0$ the boundary conditions are "outgoing", resp. "incoming" for $\mathrm{Re}(\lambda) > 0$. ~~Have the a~~ ~~Try some more~~ Now we want to find $G_\lambda = G_\lambda(x,0)$ satisf. $(\partial_x - V_\lambda(x))G = \delta(x)$ and the appropriate b.c. ~~For $x<0$~~ have

$$G_\lambda(x) = \begin{pmatrix} a^\ell_x & b^\ell_x \\ c^\ell_x & d^\ell_x \end{pmatrix} \cancel{\begin{pmatrix} \xi' \\ 0 \end{pmatrix}} \begin{pmatrix} 0 \\ \xi'_+ \end{pmatrix} \qquad x<0$$

$$= \begin{pmatrix} d^r_x & -b^r_x \\ -c^r_x & a^r_x \end{pmatrix} \cancel{\begin{pmatrix} 0 \\ \xi_- \end{pmatrix}} \begin{pmatrix} \xi_+ \\ 0 \end{pmatrix} \qquad x>0$$

?

ξ'_-, ξ_- such that

$$G_\lambda(0^+) - G_\lambda(0^-) = \begin{pmatrix} d^r_0 \\ -c^r_0 \end{pmatrix}$$

The preceding is confused because $G_\lambda(x)$ is a 2×2 matrix. We know that

$$G_\lambda(x) = \begin{pmatrix} a^\ell_x & b^\ell_x \\ c^\ell_x & d^\ell_x \end{pmatrix} \begin{pmatrix} r & s \\ t & u \end{pmatrix} \qquad \text{for } x<0$$

and that if $\mathrm{Re}(\lambda) < 0$, then ?

$$\cancel{G_\lambda(x) \sim} \begin{pmatrix} e^{\lambda x/2}\xi' \\ e^{-\lambda x/2}\xi'_+ \end{pmatrix} \leftarrow \begin{pmatrix} z^{-x/2} p \\ z^{-x/2} q \end{pmatrix}$$

$$\begin{pmatrix} z^{-x} p \\ q \end{pmatrix} = \begin{pmatrix} a^\ell_x & b^\ell_x \\ c^\ell_x & d^\ell_x \end{pmatrix} \begin{pmatrix} 0 \\ \xi'_+ \end{pmatrix}$$

So $G_\lambda(x) = \begin{pmatrix} a^l_x & b^l_x \\ c^l_x & d^l_x \end{pmatrix}\begin{pmatrix} 0 & 0 \\ r & s \end{pmatrix}$ $\quad x<0$

$$G_\lambda(x) = \begin{pmatrix} d^r_x & -b^r_x \\ -c^r_x & a^r_x \end{pmatrix}\begin{pmatrix} t & u \\ 0 & 0 \end{pmatrix} \qquad x>0$$

$$\begin{pmatrix} d^r_0 t & d^r_0 u \\ -c^r_0 t & -c^r_0 u \end{pmatrix} - \begin{pmatrix} b^l_0 r & b^l_0 s \\ d^l_0 r & d^l_0 s \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

But you've encountered $\begin{pmatrix} d^r & b^l \\ -c^r & d^l \end{pmatrix}$ before:

$$\begin{pmatrix} \frac{1}{d} & \frac{b}{d} \\ -\frac{c}{d} & \frac{1}{d} \end{pmatrix} = \begin{pmatrix} a^r & b^r \\ -c^l & a^l \end{pmatrix}\frac{1}{d}\begin{pmatrix} d^r & b^l \\ -c^r & d^l \end{pmatrix}$$

Review again $\quad \partial_x\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} \lambda & h \\ \bar{h} & 0 \end{pmatrix}\begin{pmatrix} p \\ q \end{pmatrix} \qquad \begin{pmatrix} \tilde{p} \\ \tilde{q} \end{pmatrix} = z^{-x/2}\begin{pmatrix} p \\ q \end{pmatrix}$

$$\begin{pmatrix} \partial_x & -h \\ \bar{h} & -\partial_x \end{pmatrix}\begin{pmatrix} \tilde{p} \\ \tilde{q} \end{pmatrix} = \frac{\lambda}{2}\begin{pmatrix} \tilde{p} \\ \tilde{q} \end{pmatrix}$$

$$\boxed{\partial_x\begin{pmatrix} z^{-x}p \\ q \end{pmatrix} = \begin{pmatrix} 0 & hz^{-x} \\ \bar{h}z^{x} & 0 \end{pmatrix}\begin{pmatrix} z^{-x}p \\ q \end{pmatrix}} \Longrightarrow \begin{pmatrix} \xi'_- \\ \xi'_+ \end{pmatrix} \leftarrow \begin{pmatrix} z^{-x}p \\ q \end{pmatrix} \rightarrow \begin{pmatrix} \xi_+ \\ \xi_- \end{pmatrix}$$

blowsup

$$\begin{pmatrix} z^{x/2}\xi'_- \\ z^{-x/2}\xi'_+ \end{pmatrix} \longleftarrow \begin{pmatrix} \tilde{p} \\ \tilde{q} \end{pmatrix} = \begin{pmatrix} z^{x/2}z^{-x}p \\ z^{-x/2}q \end{pmatrix} \longrightarrow \begin{pmatrix} z^{x/2}\xi_+ \\ z^{-x/2}\xi_- \end{pmatrix}$$

blowsup for $Re(\lambda)<0$, $|z|<1$

Anyway we are now ready to construct $G = G_\lambda(x,0)$. $\forall v \in V_0 = \mathbb{C}^2$, $G(x)v$ is the solution of $\left(\partial_x - V_\lambda(x)\right)G(x)v = \delta(x)v$ satisfying b.c. at $\pm\infty$.

Let $v = \begin{pmatrix}p_0\\ q_0\end{pmatrix}$

$$G(0^-)v = \begin{pmatrix}a^l & b^l\\ c^l & d^l\end{pmatrix}\begin{pmatrix}0\\ \xi_+'\end{pmatrix}$$

$$G(0^+)v = \begin{pmatrix}d^r & -b^r\\ -c^r & a^r\end{pmatrix}\begin{pmatrix}\xi_+\\ 0\end{pmatrix}$$

$$\boxed{\begin{pmatrix}p_0\\ q_0\end{pmatrix} = v = G(0^+)v - G(0^+)v = \begin{pmatrix}d^r & b^l\\ -c^r & d^l\end{pmatrix}\begin{pmatrix}\xi_+\\ \xi_+'\end{pmatrix}}$$

Check this carefully.

$$\begin{pmatrix}\xi_+\\ \xi_-\end{pmatrix} = \begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}\xi_-'\\ \xi_+'\end{pmatrix} = \begin{pmatrix}a^r & b^r\\ c^r & d^r\end{pmatrix}\overbrace{\begin{pmatrix}a^l & b^l\\ c^l & d^l\end{pmatrix}\begin{pmatrix}\xi_-'\\ \xi_+'\end{pmatrix}}^{\begin{pmatrix}p_0\\ q_0\end{pmatrix}}$$

$$\begin{pmatrix}p_0\\ q_0\end{pmatrix} = \begin{pmatrix}d^r & -b^r\\ -c^r & a^r\end{pmatrix}\begin{pmatrix}\xi_+\\ \xi_-\end{pmatrix}$$

$$\begin{pmatrix}\xi_-'\\ \xi_+'\end{pmatrix} = \begin{pmatrix}d & -b\\ -c & a\end{pmatrix}\begin{pmatrix}\xi_+\\ \xi_-\end{pmatrix}$$

$$= \begin{pmatrix}d^r & -b^r\\ -c^r & a^r\end{pmatrix}\begin{pmatrix}1 & 0\\ \frac{c}{a} & \frac{1}{a}\end{pmatrix}\begin{pmatrix}\xi_+\\ \xi_+'\end{pmatrix}$$

$$\begin{pmatrix}\xi_-'\\ \xi_-\end{pmatrix} = \begin{pmatrix}\frac{1}{a} & -\frac{b}{a}\\ \frac{c}{a} & \frac{1}{a}\end{pmatrix}\begin{pmatrix}\xi_+\\ \xi_+'\end{pmatrix}$$

$$= \begin{pmatrix}d^r - b^r\frac{c}{a} & -\frac{b^r}{a}\\ -c^r + a^r\frac{c}{a} & \frac{a^r}{a}\end{pmatrix}\begin{pmatrix}\xi_+\\ \xi_+'\end{pmatrix}$$

$$\begin{pmatrix}d^r & -b^r\\ -c^r & a^r\end{pmatrix}\begin{pmatrix}a & b\\ c & d\end{pmatrix} = \begin{pmatrix}a^l & b^l\\ c^l & d^l\end{pmatrix}$$

$$\begin{pmatrix}d^r a - b^r c & d^r b - b^r d\\ -c^r a + a^r c & -c^r b + a^r d\end{pmatrix}$$

$$= \frac{1}{a}\begin{pmatrix}a^l & -b^r\\ c^l & a^r\end{pmatrix}\begin{pmatrix}\xi_+\\ \xi_+'\end{pmatrix}$$

You probably can reconcile things by shifting to $\mathrm{Re}(\lambda) > 0$. In this case you want $\xi_+ = \xi_+' = 0$

$$v = \begin{pmatrix} p_0 \\ q_0 \end{pmatrix} \qquad G(0^-)v = \begin{pmatrix} a^l & b^l \\ c^l & d^l \end{pmatrix}\begin{pmatrix} \xi_-' \\ 0 \end{pmatrix}$$

$$G(0^+)v = \begin{pmatrix} d^r & -b^r \\ -c^r & a^r \end{pmatrix}\begin{pmatrix} 0 \\ \xi_- \end{pmatrix}$$

$$\boxed{v = \begin{pmatrix} p_0 \\ q_0 \end{pmatrix} \simeq G(0^+)v - G(0^-)v = \begin{pmatrix} a^l & -b^r \\ c^l & a^r \end{pmatrix}\begin{pmatrix} \xi_-' \\ \xi_- \end{pmatrix}}$$

Recover the potential, how? to the first order in h the scattering is the F.T. of the potential

$$\begin{pmatrix} a^l & b^l \\ c^l & d^l \end{pmatrix} = T \exp \int_{-\infty}^{0} \begin{pmatrix} 0 & h_x e^{-\lambda x} \\ \bar{h}_x e^{\lambda x} & 0 \end{pmatrix} dx$$

You have some idea of using asymptotics in λ

$$\lambda \quad \begin{pmatrix} \partial_x & -h \\ \bar{h} & -\partial_x \end{pmatrix} \qquad \Bigg| \qquad \begin{pmatrix} \partial_x & \\ & \partial_x \end{pmatrix}\begin{pmatrix} z^{-x} p_x \\ q_x \end{pmatrix} = \begin{pmatrix} \lambda & h \\ \bar{h} & 0 \end{pmatrix}\begin{pmatrix} \\ \end{pmatrix}$$

$$\begin{matrix} \lambda - \partial_x & h \\ \bar{h} & -\partial_x \end{matrix}$$

elle

$$\begin{pmatrix} \partial_x - \lambda & 0 \\ 0 & \partial_x \end{pmatrix}\begin{pmatrix} \psi \end{pmatrix} = \begin{pmatrix} 0 & h \\ \bar{h} & 0 \end{pmatrix}\psi$$

$$(D - V)\psi = 0 \qquad D = \begin{pmatrix} \partial_x - \lambda & 0 \\ 0 & \partial_x \end{pmatrix}$$

$$\frac{1}{D - V} = \frac{1}{1 - D^{-1}V} D^{-1} = D^{-1} + D^{-1}VD^{-1} + \cdots\cdots$$

You are trying to recover the potential from the λ asymptotics, which is maybe a wave equation characteristics, i.e. Fourier Integral Operator approach. Let's carry this out in the present case.

$$\frac{1}{\partial_x - \lambda} = \left(e^{\lambda x}\partial_x e^{-\lambda x}\right)^{-1} = \left(\partial_x - \lambda\right)^{-1}$$

$$\parallel$$

$$(\partial_x - \lambda)^{-1}(x, x') = e^{\lambda x} H(x-x') e^{-\lambda x'}$$

$$\boxed{B} \quad G_{fr}(x, x') = \begin{pmatrix} e^{\lambda(x-x')}H(x-x') & 0 \\ 0 & -H(-x+x') \end{pmatrix}$$

$$G(x-x') = G_{fr}(x, x') + \int dy\, G_{fr}(x,y)\, V(y)\, G_{fr}(y, x')$$

$$= \begin{pmatrix} e^{\lambda(x-x')}H(x-x') & 0 \\ 0 & -H(-x+x') \end{pmatrix} + \int dy \begin{pmatrix} 0 & -e^{\lambda(x-y)}H(x-y)\, h_y\, H(-y+x') \\ -H(-x+y)\bar{h}_y\, e^{\lambda(y-x')}H(y-x') & 0 \end{pmatrix}$$

What is $H(x-x')\int_{x'}^{x} dy \; e^{\lambda(x-y)} h_y$?

You are interested in the singularity at $x = x'$, maybe the asymptotics as $\lambda \to \infty$, $Re(\lambda) = 0$.

Digress. Suppose $Re(\lambda) < 0$ so that $e^{\lambda(x-x')}H(x-x')$ is the L^2 Green's function for $\partial_x - \lambda$. The L^2 Green's function for ∂_x should be ?

$$\hat{\varphi} = \begin{matrix} z^{-x/2}p_x = z^{x/2}(z^{-x}p_x) \longrightarrow z^{x/2}\xi_+ \\ z^{-x/2} q_x \longrightarrow z^{-x/2}\xi_- \end{matrix}$$

$$\frac{\lambda}{2}\tilde{\varphi} = \begin{pmatrix} \partial_x & -h \\ \bar{h} & -\partial_x \end{pmatrix}\tilde{\varphi} \qquad \tilde{\varphi} = \begin{pmatrix} z^{x/2} & 0 \\ 0 & z^{-x/2} \end{pmatrix}\underbrace{\begin{pmatrix} z^{-x}p_x \\ q_x \end{pmatrix}}_{\varphi^{\#}}$$

Idea: Work in the picture $\psi^{\#} = \begin{pmatrix} z^{-x} p_x \\ q_x \end{pmatrix}$ but use "Feynman" boundary conditions.

$$G^{\#}_{free}(x,x') = \begin{pmatrix} \cancel{e^{\lambda(x-x')}} H(x-x') & 0 \\ 0 & -H(-x+x') \end{pmatrix}$$

L^2 picture $\quad \tilde{\psi} = \begin{pmatrix} z^{x/2} & 0 \\ 0 & z^{-x/2} \end{pmatrix} \psi^{\#} = \begin{pmatrix} z^{-x/2} p_x \\ z^{-x/2} q_x \end{pmatrix}$

$$\psi^{\#} = G^{\#}_{free} f^{\#}$$

$$\overbrace{\begin{pmatrix} z^{x/2} & \\ & z^{-x/2} \end{pmatrix} \psi^{\#}}^{\tilde{\psi}} = \underbrace{\begin{pmatrix} z^{x/2} & \\ & z^{-x/2} \end{pmatrix} G^{\#} \begin{pmatrix} z^{-x/2} & \\ & z^{+x/2} \end{pmatrix}} \overbrace{\begin{pmatrix} z^{-x/2} & 0 \\ 0 & z^{x/2} \end{pmatrix} f^{\#}}^{\tilde{f}}$$

$$\tilde{G}_{free} = \begin{pmatrix} z^{x/2} & 0 \\ 0 & z^{-x/2} \end{pmatrix} \begin{pmatrix} \cancel{e^{\lambda(x-x')}} H(x-x') & 0 \\ 0 & -H(-x+x') \end{pmatrix} \begin{pmatrix} z^{-x'/2} & 0 \\ 0 & z^{x'/2} \end{pmatrix}$$

$$= \begin{pmatrix} e^{\lambda/2(x-x')} H(x-x') & \\ & -e^{-\frac{\lambda}{2}(x-x')} H(-x+x') \end{pmatrix}$$

Wrong sign, start again.

$$\psi = \begin{pmatrix} p \\ q \end{pmatrix} \qquad \partial_x \psi = \begin{pmatrix} \lambda & h \\ \bar{h} & 0 \end{pmatrix} \psi$$

$$\psi^{\#} = \begin{pmatrix} z^{-x} p \\ q \end{pmatrix} \qquad \partial_x \psi^{\#} = \begin{pmatrix} 0 & h z^{-x} \\ \bar{h} z^{x} & 0 \end{pmatrix} \psi^{\#}$$

$$\tilde{\psi} = \begin{pmatrix} z^{-x/2} p \\ z^{-x/2} q \end{pmatrix} \qquad \partial_x \tilde{\psi} = \begin{pmatrix} \lambda/2 & h \\ \bar{h} & -\lambda/2 \end{pmatrix} \tilde{\psi} \qquad \tilde{\psi} = \begin{pmatrix} z^{x/2} & 0 \\ 0 & z^{-x/2} \end{pmatrix} \psi^{\#}$$

$$\tilde{G}_{free} = \begin{pmatrix} e^{\frac{\lambda}{2}(x-x')} \overset{\neq 0 \;\; x > x'}{H(x-x')} & \\ 0 & -e^{\frac{\lambda}{2}(-x+x')} \underset{\neq 0 \text{ when } -x + x' > 0}{H(-x+x')} \end{pmatrix}$$

$$G^{\#}_{free}(x,x') = \begin{pmatrix} H(x-x') & 0 \\ 0 & -H(-x+x') \end{pmatrix}$$

$$G^{\#} = G^{\#}_{fr} + \underbrace{G^{\#}_{fr} V G^{\#}_{fr}} + \cdots$$

$$\int dy \begin{pmatrix} H(x-y) & 0 \\ 0 & -H(-x+y) \end{pmatrix} \begin{pmatrix} 0 & h_y e^{-\lambda y} \\ \bar{h}_y e^{\lambda y} & 0 \end{pmatrix} \begin{pmatrix} H(y-x') & \\ & -H(-y+x') \end{pmatrix}$$

$$\int dy \; H(x-y)\, h(y) e^{-\lambda y} (-1) H(-y+x')$$

$$= (-1) \int_{y < x, x'} dy \; h(y) e^{-\lambda y}$$

---

begin again $\quad \psi^{\#} = \begin{pmatrix} z^{-x} p_x \\ q_x \end{pmatrix} \quad \partial_x \psi^{\#} = \underbrace{V(x)} \psi^{\#}$

$$V(x) = \begin{pmatrix} 0 & h_x z^{-x} \\ \bar{h}_x z^{x} & 0 \end{pmatrix}$$

use $G^{\#}_{fr}(x,x') = \begin{pmatrix} H(x-x') & 0 \\ 0 & -H(-x+x') \end{pmatrix}$

Assume h comp support, then $\big(\partial_x - V(x)\big) G^{\#}(x,x') = \delta(x-x')$

To ~~solve~~ $\quad (\partial_x - V)\psi = f$

$(D-V)\psi = f \qquad G_0 + G_0 V G_0 + \cdots$

Let $D_0 G_0 = 1$. Then

$$(D_0 - V)(G_0 + G_0 V G_0 + \cdots)$$

$$= (1 + VG_0 + (VG_0)^2 + \cdots) - V(G_0 + G_0 V G_0 + \cdots)$$

$$= 1. \qquad D_0 = \partial_x \qquad V(x) = \begin{pmatrix} 0 & h_x z^{-x} \\ \overline{h}_x z^x & 0 \end{pmatrix}$$

so

$$(G_0 V G_0)(x,x') = \int dy \begin{pmatrix} H(x-y) & 0 \\ 0 & -H(y-x) \end{pmatrix} \begin{pmatrix} 0 & h_y z^{-y} \\ \overline{h}_y z^y & 0 \end{pmatrix} \begin{pmatrix} H(y-x') & 0 \\ 0 & -H(x'-y) \end{pmatrix}$$

$$\int dy\, (-1)\, H(y-x)\, \overline{h(y)}\, z^y\, H(y-x') \;\Big|\; \int dy\, H(x-y) h(y) z^{-y} (-1) H(x'-y)$$

$$= -\int_{y \geq \max\{x,x'\}} \overline{h(y)}\, e^{\lambda y}\, dy$$

~~Any~~ If $x = x'$ you get

$$-\int_x^\infty \overline{h(y)}\, e^{\lambda y}\, dy$$

Actually I seem to get

$$G(x,x') = \begin{pmatrix} H(x-x') & 0 \\ 0 & -H(-x+x') \end{pmatrix} + (-1) \begin{pmatrix} 0 & \int_{-\infty}^{x \wedge x'} dy\, h(y) e^{-\lambda y} \\ \int_{x \vee x'}^{\infty} dy\, \overline{h(y)}\, e^{\lambda y} & 0 \end{pmatrix}$$

~~Heaviside~~

$$\cancel{G(x,x)} = (G_0 V G_0)(x,x) = (-1) \begin{pmatrix} 0 & \int_{-\infty}^{x} dy\, h(y) e^{-\lambda y} \\ \int_x^\infty dy\, \overline{h(y)}\, e^{\lambda y} & 0 \end{pmatrix}$$

Start again $\psi^{\#} = \begin{pmatrix} z^{-x} p \\ q \end{pmatrix}$ $\partial_x \psi^{\#} = \begin{pmatrix} 0 & h z^{-x} \\ \bar{h} z^{x} & 0 \end{pmatrix} \psi^{\#}$

$$G^{\#} = G_0^{\#} + G_0^{\#} V G_0^{\#} + \cdots$$

$$G^{\#}(x,x') = H(x-x') + \int dy\, H(x-y) V(y) H(y-x')$$

$$+ \int dy_1\, dy_2\, H(x-y_1) V(y_1) H(y_1-y_2) V(y_2) H(y_2-x') + \cdots$$

Go back to your Dirac equation

$$\partial_x \begin{pmatrix} z^{-x} p \\ q \end{pmatrix} = \begin{pmatrix} 0 & h z^{-x} \\ \bar{h} z^{x} & 0 \end{pmatrix} \begin{pmatrix} z^{-x} p \\ q \end{pmatrix}$$

make assumption that h decays so that one has nice convergence of functions of λ.

$$\begin{pmatrix} \xi'_- \\ \xi'_+ \end{pmatrix} \xleftarrow{x \to -\infty} \begin{pmatrix} z^{-x} p \\ q \end{pmatrix} \xrightarrow{x \to +\infty} \begin{pmatrix} \xi_+ \\ \xi_- \end{pmatrix}$$

Basic formulas are

$$\begin{pmatrix} p_0 \\ q_0 \end{pmatrix} = \begin{pmatrix} a^l & b^l \\ c^l & d^l \end{pmatrix} \begin{pmatrix} \xi'_- \\ \xi'_+ \end{pmatrix} = \begin{pmatrix} d^r & -b^r \\ -c^r & a^r \end{pmatrix} \begin{pmatrix} \xi_+ \\ \xi_- \end{pmatrix}$$

$$\begin{pmatrix} a^l & b^l \\ c^l & d^l \end{pmatrix} = \begin{pmatrix} d^r & -b^r \\ -c^r & a^r \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

You are interested in the ~~one~~ Green's function defined by $\left(\partial_x - V(x)\right) G(x,x') = \delta(x-x')$ and sat certain boundary conditions as $x \to +\infty$ and $-\infty$.

Exam~~pl~~ine two cases: $\xi_- = 0 = \xi'_-$ and $+$.

Given $v \in \mathbb{C}^2$ at $x=0$. You want $G(x)v$ to satisfy $\left(\partial_x - V(x)\right) G(x) v = \delta(x) v$ and $\begin{pmatrix} 0 \\ * \end{pmatrix} \longleftarrow G(x)v \longrightarrow \begin{pmatrix} * \\ 0 \end{pmatrix}$

So $G(0^-)v = \begin{pmatrix} b^\ell \\ d^\ell \end{pmatrix} \xi'_+(v)$, $G(0^+)v = \begin{pmatrix} d^r \\ -c^r \end{pmatrix} \xi_+(v)$

where ξ_+ and ξ'_+ to be chosen so that

$$\underbrace{\begin{pmatrix} d^r & b^\ell \\ -c^r & d^\ell \end{pmatrix}}\begin{pmatrix} \xi_+(v) \\ -\xi'_+(v) \end{pmatrix} = v$$

have encountered this before, expressing $\begin{pmatrix} p_0 \\ q_0 \end{pmatrix}$ in terms of $\begin{pmatrix} \xi_+ \\ \xi'_+ \end{pmatrix}$ or $\begin{pmatrix} \xi'_- \\ \xi_- \end{pmatrix}$

$$\cancel{\begin{pmatrix} \xi_+ \\ \xi_- \end{pmatrix} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}\begin{pmatrix} \xi'_- \\ \xi'_+ \end{pmatrix}}$$

$$\begin{pmatrix} p_0 \\ q_0 \end{pmatrix} = \begin{pmatrix} d^r & -b^r \\ -c^r & a^r \end{pmatrix}\begin{pmatrix} \xi_+ \\ \xi_- \end{pmatrix}$$

$$\cancel{\begin{pmatrix} \xi_+ \\ \xi'_+ \end{pmatrix} = \begin{pmatrix} \frac{1}{a} & -\frac{b}{a} \\ \frac{c}{a} & \frac{1}{a} \end{pmatrix}\begin{pmatrix} \xi'_- \\ \xi_- \end{pmatrix}}$$

$$\begin{pmatrix} \xi'_- \\ \xi'_+ \end{pmatrix} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}\begin{pmatrix} \xi_+ \\ \xi_- \end{pmatrix} \qquad \begin{pmatrix} \xi'_- \\ \xi_- \end{pmatrix} = \begin{pmatrix} \frac{1}{a} & -\frac{b}{a} \\ \frac{c}{a} & \frac{1}{a} \end{pmatrix}\begin{pmatrix} \xi_+ \\ \xi'_+ \end{pmatrix} \qquad \cancel{\begin{pmatrix} \xi'_- \\ \xi'_+ \end{pmatrix} = \begin{pmatrix} & \\ & \end{pmatrix}\begin{pmatrix} \xi_+ \\ \xi_- \end{pmatrix}}$$

$$\begin{pmatrix} \xi_+ \\ \xi_- \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} \xi'_- \\ \xi'_+ \end{pmatrix} \qquad \begin{pmatrix} \xi_+ \\ \xi'_+ \end{pmatrix} = \begin{pmatrix} \frac{1}{d} & \frac{b}{d} \\ -\frac{c}{d} & \frac{1}{d} \end{pmatrix}\begin{pmatrix} \xi'_- \\ \xi_- \end{pmatrix} \qquad \boxed{\cancel{\begin{pmatrix} a^\ell & b^\ell \\ c^\ell & d^\ell \end{pmatrix}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}} = \begin{pmatrix} a^r & b^r \\ c^r & d^r \end{pmatrix}}$$

$$\cancel{\begin{pmatrix} p_0 \\ q_0 \end{pmatrix}} \qquad \begin{pmatrix} p_0 \\ q_0 \end{pmatrix} = \begin{pmatrix} a^\ell & b^\ell \\ c^\ell & d^\ell \end{pmatrix}\begin{pmatrix} \xi'_- \\ \xi'_+ \end{pmatrix} = \begin{pmatrix} a^\ell & b^\ell \\ c^\ell & d^\ell \end{pmatrix}\begin{pmatrix} \frac{1}{a} & -\frac{b}{a} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \xi_+ \\ \xi'_+ \end{pmatrix}$$

$$= \frac{1}{a}\begin{pmatrix} a^\ell & \overset{-b^r}{-ba^\ell + ab^\ell} \\ c^\ell & \underset{d^r}{-bc^\ell + ad^\ell} \end{pmatrix}$$

$$\begin{pmatrix} a^\ell & b^\ell \\ c^\ell & d^\ell \end{pmatrix} \cdot \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \begin{pmatrix} d^r & -b^r \\ -c^r & a^r \end{pmatrix}$$

$$\begin{pmatrix} p_0 \\ q_0 \end{pmatrix} = \begin{pmatrix} a^l & b^l \\ c^l & d^l \end{pmatrix}\begin{pmatrix} \frac{1}{a} & -\frac{b}{a} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \xi_+ \\ \xi'_+ \end{pmatrix}$$

$$= \frac{1}{a}\begin{pmatrix} a^l & -a^l b + b^l a \\ c^l & -c^l b + d^l a \end{pmatrix}\begin{pmatrix} " \end{pmatrix} = \frac{1}{a}\begin{pmatrix} a^l & -b^r \\ c^l & a^r \end{pmatrix}\begin{pmatrix} \xi_+ \\ \xi'_+ \end{pmatrix}$$

$$\begin{pmatrix} p_0 \\ q_0 \end{pmatrix} = \begin{pmatrix} a^l & b^l \\ c^l & d^l \end{pmatrix}\begin{pmatrix} \xi'_- \\ \xi'_+ \end{pmatrix} = \begin{pmatrix} a^l & b^l \\ c^l & d^l \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -\frac{c}{d} & \frac{1}{d} \end{pmatrix}\begin{pmatrix} \xi'_- \\ \xi_- \end{pmatrix}$$

$$= \frac{1}{d}\begin{pmatrix} +a^l d - b^l c & b^l \\ c^l d - d^l c & d^l \end{pmatrix}\begin{pmatrix} \cdot \end{pmatrix} = \frac{1}{d}\begin{pmatrix} d^r & b^l \\ -c^r & d^l \end{pmatrix}\begin{pmatrix} \xi'_- \\ \xi_- \end{pmatrix}$$

$$\begin{pmatrix} a^r & b^r \\ c^r & d^r \end{pmatrix}\begin{pmatrix} a^l & b^l \\ c^l & d^l \end{pmatrix}$$

$$\begin{pmatrix} p_0 \\ q_0 \end{pmatrix} = \begin{pmatrix} a^l & b^l \\ c^l & d^l \end{pmatrix}\begin{pmatrix} \xi'_- \\ \xi'_+ \end{pmatrix} = \begin{pmatrix} a^l & b^l \\ c^l & d^l \end{pmatrix}\begin{pmatrix} \frac{1}{a} & -\frac{b}{a} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \xi_+ \\ \xi'_+ \end{pmatrix}$$

$$= \frac{1}{a}\begin{pmatrix} a^l & -a^l b + b^l a \\ c^l & -c^l b + d^l a \end{pmatrix}\begin{pmatrix} \xi_+ \\ \xi'_+ \end{pmatrix}$$

$$\begin{pmatrix} d^r & -b^r \\ -c^r & a^r \end{pmatrix}\cancel{\begin{pmatrix} a & b \\ c & d \end{pmatrix}} = \cancel{\begin{pmatrix} a^r & b^r \\ c^r & d^r \end{pmatrix}}\begin{pmatrix} a^l & b^l \\ c^l & d^l \end{pmatrix}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

$$\begin{pmatrix} d^r & -b^r \\ -c^r & a^r \end{pmatrix} = \begin{pmatrix} a^l & b^l \\ c^l & d^l \end{pmatrix}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

$$\begin{pmatrix} p_0 \\ q_0 \end{pmatrix} = \frac{1}{a}\begin{pmatrix} a^l & -b^r \\ c^l & a^r \end{pmatrix}\begin{pmatrix} \xi_+ \\ \xi'_+ \end{pmatrix}$$

Two ways to split the space $\mathbb{C}^2 = V_0$. First is Green's function, i.e. $(\partial_x - V(x))\, G(x) v = \delta(x) v$; where $G(x) v$ $x<0$

$$G(x) v = \begin{pmatrix} a^l_x & b^l_x \\ c^l_x & d^l_x \end{pmatrix} \begin{pmatrix} 0 \\ \xi'_+ \end{pmatrix} \qquad x<0$$

$$G(x) v = \begin{pmatrix} d^r_x & -b^r_x \\ -c^r_x & a^r_x \end{pmatrix} \begin{pmatrix} \xi_+ \\ 0 \end{pmatrix} \qquad x>0$$

and $v = G(0^+) v - G(0^-) v = \boxed{\begin{pmatrix} d^r_0 & b^l_0 \\ -c^r_0 & d^l_0 \end{pmatrix} \begin{pmatrix} \xi_+ \\ \xi^r_+ \end{pmatrix} = \begin{pmatrix} p_0 \\ q_0 \end{pmatrix}}$

If $v = \begin{pmatrix} p_0 \\ q_0 \end{pmatrix}$ then you get →

2nd is to take the ~~solution~~ eigenfunction with initial values v at $x=0$ and to split it into outgoing left right components.

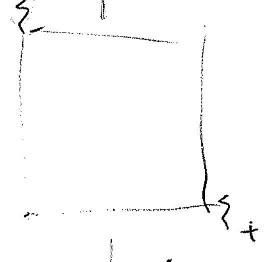
This yields $\begin{pmatrix} p_0 \\ q_0 \end{pmatrix} = \dfrac{1}{a} \begin{pmatrix} a^l & -b^r \\ c^l & a^r \end{pmatrix} \begin{pmatrix} \xi_+ \\ \xi'_+ \end{pmatrix}$

With incoming components you get scattering

$$\begin{pmatrix} p_0 \\ q_0 \end{pmatrix} = \frac{1}{d} \begin{pmatrix} d^r & b^l \\ -c^r & d^l \end{pmatrix} \begin{pmatrix} \xi'_- \\ \xi_- \end{pmatrix}$$

and $v = \begin{pmatrix} p_0 \\ q_0 \end{pmatrix} = G(0^+) v - G(0^-) v = \begin{pmatrix} a^l & -b^r \\ c^l & a^r \end{pmatrix} \begin{pmatrix} \xi_+ \\ \xi'_+ \end{pmatrix}$

Next project – recover h to first order. The idea is that $h \mapsto \beta = \int_{-\infty}^{\infty} h(x) e^{-\lambda x}\, dx$, maybe there is a Hilbert space projection method for splitting β. What is the linearization of ~~proj~~



$$p_0 = d^r \xi_+ - b^r \xi_-$$

$$(u^x \xi_+ \mid p_0) = (z^x \mid d^r \cancel{\beta} - b^r \bar{\beta}) = 0 \qquad x>0$$

$$(u^y \xi_- \mid p_0) = (z^y \mid d^r \beta - b^r) = 0 \qquad y<0$$

you solve by passing to the Toeplitz operator to first order? $\quad p_0 = \xi_+ + d\xi_+ - b\xi_-$

$$(u^y\xi_- | p_0) = (z^y | d\beta_+ - b) = 0 \qquad y < 0$$

$$(u^x\xi_+ | p_0) = (z^x | d - b\overline{\beta}) = 0 \qquad x > 0.$$

Here β is a function of $k = \frac{1}{i}\lambda$. Want $d \in H_+, b \in H_-$

Work to first order in β. d will be second order. ~~If~~ b is first order, so you get simply $(z^y | \beta - b) = 0 \quad y < 0$

so $\quad b \in H_- \qquad \beta - b \in H_+$.

Reconstruct potential. Given $\beta = \frac{b}{d}$ you do the integral equation stuff to find the factorization

$$\begin{pmatrix} \frac{1}{d} & +\frac{b}{d} \\ -\frac{c}{d} & \frac{1}{d} \end{pmatrix} = \begin{pmatrix} a_x^r & b_x^r \\ -c_x^l & a_x^l \end{pmatrix} \frac{1}{d} \begin{pmatrix} d_x^r & b_x^l \\ -c_x^r & d_x^l \end{pmatrix}$$

derive this again.

$$\begin{pmatrix} \cancel{z}^{-x} p_x \\ q_x \end{pmatrix} = \begin{pmatrix} a_x^l & b_x^l \\ c_x^l & d_x^l \end{pmatrix} \begin{pmatrix} \xi'_- \\ \xi'_+ \end{pmatrix}$$

$$= \begin{pmatrix} a^l & b^l \\ c^l & d^l \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\frac{c}{d} & \frac{1}{d} \end{pmatrix} \begin{pmatrix} \xi'_- \\ \xi_- \end{pmatrix}$$

$$= \frac{1}{d} \begin{pmatrix} \overset{d^r}{a^l d - b^l c} & b^l \\ \underset{-c^r}{c^l d - d^l c} & d^l \end{pmatrix} \begin{pmatrix} \xi'_- \\ \xi_- \end{pmatrix}$$

$$\begin{pmatrix} a^l & b^l \\ c^l & d^l \end{pmatrix} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \begin{pmatrix} d^r & -b^r \\ -c^r & a^r \end{pmatrix}$$

$$\begin{pmatrix} \xi_+ \\ \xi_- \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \xi'_- \\ \xi'_+ \end{pmatrix}$$

$$\begin{pmatrix} \xi'_- \\ \xi'_+ \end{pmatrix} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \begin{pmatrix} \xi_+ \\ \xi_- \end{pmatrix}$$

$$\begin{pmatrix} \xi_+ \\ \xi'_+ \end{pmatrix} = \begin{pmatrix} \frac{1}{d} & \frac{b}{d} \\ -\frac{c}{d} & \frac{1}{d} \end{pmatrix} \begin{pmatrix} \xi'_- \\ \xi_- \end{pmatrix}$$

$$\begin{pmatrix} \xi'_- \\ \xi_- \end{pmatrix} = \begin{pmatrix} \frac{1}{a} & -\frac{b}{a} \\ \frac{c}{a} & \frac{1}{a} \end{pmatrix} \begin{pmatrix} \xi_+ \\ \xi'_+ \end{pmatrix}$$

$$\begin{pmatrix} z^{-x}p_x \\ q_x\end{pmatrix} = \frac{1}{d}\begin{pmatrix} d^r_x & b^l_x \\ -c^r_x & d^l_x\end{pmatrix}\begin{pmatrix}\xi'_- \\ \xi_-\end{pmatrix} = \frac{1}{a}\begin{pmatrix} a^l_x & -b^r_x \\ c^l_x & a^r_x\end{pmatrix}\begin{pmatrix}\xi_+\\ \xi'_+\end{pmatrix}$$

$$\begin{pmatrix} \xi_+ \\ \xi'_+\end{pmatrix} = \begin{pmatrix} \frac{1}{d} & \frac{b}{d} \\ -\frac{c}{d} & \frac{1}{d}\end{pmatrix}\begin{pmatrix}\xi'_- \\ \xi_-\end{pmatrix} = \begin{pmatrix} a^r_x & b^r_x \\ -c^l_x & a^l_x\end{pmatrix}\begin{pmatrix} d^r_x & b^l_x \\ -c^r_x & d^l_x\end{pmatrix}\frac{1}{d}\begin{pmatrix}\xi'_- \\ \xi_-\end{pmatrix}$$

What is the way to think?
You fell, or felt, that it was best to factor the transfer matrix.

~~what the~~

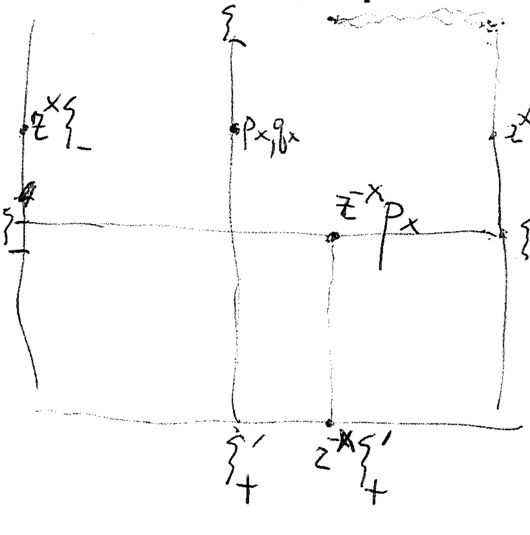
$$\begin{pmatrix} d^r & -b^r \\ -c^r & a^r\end{pmatrix}\begin{pmatrix} a & b \\ c & d\end{pmatrix} = \begin{pmatrix} a^l & b^l \\ c^l & d^l\end{pmatrix}$$

$$\begin{pmatrix} d^r & -b^r \\ -c^r & a^r\end{pmatrix}\begin{pmatrix} 1 & \frac{b}{d} \\ \frac{c}{a} & 1\end{pmatrix} = \begin{array}{cc} \tilde{H}_- & H_+ \\ \multicolumn{2}{c}{\begin{pmatrix} \frac{a^l}{a} & \frac{b^l}{d} \\ \frac{c^l}{a} & \frac{d^l}{d}\end{pmatrix}} \\ H_- & \tilde{H}_+ \end{array}$$

Do this: Go $S \longmapsto \begin{pmatrix} a^r & b^r \\ -c^l & a^l\end{pmatrix}\frac{1}{d}\begin{pmatrix} d^r & b^l \\ -c^r & d^l\end{pmatrix}$

and conjugate S so as to handle $x \neq 0$.

$$\begin{pmatrix} z^{-x}p_x \\ q_x\end{pmatrix} = \begin{array}{cc} \tilde{H}_- & z^{-x}H_+ \\ \multicolumn{2}{c}{\begin{pmatrix} a^l_x & b^l_x \\ c^l_x & d^l_x\end{pmatrix}} \\ z^xH_- & \tilde{H}_+ \end{array}\begin{pmatrix}\xi'_- \\ \xi'_+\end{pmatrix}$$



$$\begin{pmatrix} z^{-x}p_x \\ q_x\end{pmatrix} = \begin{array}{cc} \tilde{H}_+ & z^{-x}H_- \\ \multicolumn{2}{c}{\begin{pmatrix} d^r_x & -b^r_x \\ -c^r_x & a^r_x\end{pmatrix}} \\ z^{+x}H_+ & \tilde{H}_- \end{array}\begin{pmatrix}\xi_+ \\ \xi_-\end{pmatrix}$$

$$\begin{pmatrix} \frac{1}{d} & \frac{b}{d} \\ -\frac{c}{d} & \frac{1}{d} \end{pmatrix} = \begin{pmatrix} a^r & b^r \\ -c^l & a^l \end{pmatrix} \frac{1}{d} \begin{pmatrix} d^r & b^l \\ -c^r & d^l \end{pmatrix}$$

$$= \begin{array}{c} \begin{array}{cc} \tilde{H}_- & z^{-x}H_- \end{array} \\ \begin{pmatrix} a^r_x & b^r_x \\ -c^l_x & a^l_x \end{pmatrix} \\ \begin{array}{cc} z^x H_- & \tilde{H}_- \end{array} \end{array} \frac{1}{d} \begin{array}{c} \begin{array}{cc} \tilde{H}^+ & z^{-x}H_+ \end{array} \\ \begin{pmatrix} d^r_x & b^l_x \\ -c^r_x & d^l_x \end{pmatrix} \\ \begin{array}{cc} z^x H_+ & \tilde{H}_+ \end{array} \end{array}$$

$$\boxed{\begin{pmatrix} \frac{1}{d} & \frac{b}{d} z^x \\ -\frac{c}{d} z^{-x} & \frac{1}{d} \end{pmatrix} = \begin{array}{c} \begin{array}{cc} \tilde{H}_- & H_- \end{array} \\ \begin{pmatrix} a^r_x & z^x b^r_x \\ -z^{-x} c^l_x & a^l_x \end{pmatrix} \\ \begin{array}{cc} H_- & \tilde{H}_- \end{array} \end{array} \frac{1}{d} \begin{array}{c} \begin{array}{cc} \tilde{H}_+ & H_+ \end{array} \\ \begin{pmatrix} d^r_x & z^x b^l_x \\ -z^{-x} c^r_x & d^l_x \end{pmatrix} \\ \begin{array}{cc} H_+ & \tilde{H}_+ \end{array} \end{array}}$$

Thus you conjugate $S \longmapsto \begin{pmatrix} z^x & 0 \\ 0 & 1 \end{pmatrix} S \begin{pmatrix} z^{-x} & 0 \\ 0 & 1 \end{pmatrix}$, but you still haven't found $h(x)$.

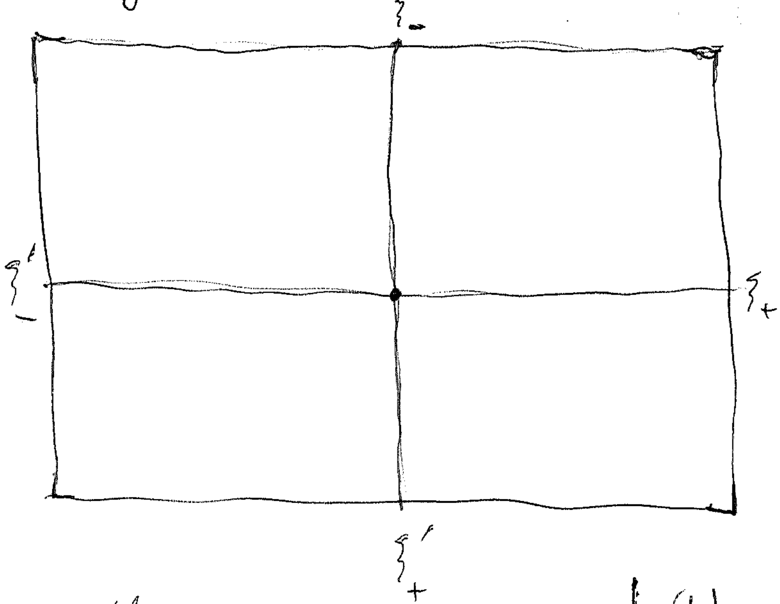
$$\begin{pmatrix} 1 & bz^x \\ -cz^{-x} & 1 \end{pmatrix} = \underbrace{\begin{pmatrix} a^r_x & z^x b^r_x \\ -z^{-x} c^l_x & a^l_x \end{pmatrix}}_{\begin{pmatrix} \tilde{H}_- & H_- \\ H_- & \tilde{H}_- \end{pmatrix}} \underset{\begin{pmatrix} \tilde{H}_+ & H_+ \\ H_+ & \tilde{H}_+ \end{pmatrix}}{\begin{pmatrix} d^r_x & z^x b^l_x \\ -z^{-x} c^r_x & d^l_x \end{pmatrix}}$$

$1 - bc = ad$

$$\begin{pmatrix} a & z^x b \\ z^{-x} c & d \end{pmatrix} = \begin{array}{c} \begin{array}{cc} \tilde{H}_- & H_- \end{array} \\ \begin{pmatrix} a^r_x & z^x b^r_x \\ z^{-x} c^r_x & d^r_x \end{pmatrix} \\ \begin{array}{cc} H_+ & \tilde{H}_+ \end{array} \end{array} \begin{array}{c} \begin{array}{cc} \tilde{H}_- & H_+ \end{array} \\ \begin{pmatrix} a^l_x & z^x b^l_x \\ z^{-x} c^l_x & d^l_x \end{pmatrix} \\ \begin{array}{cc} H_- & \tilde{H}_+ \end{array} \end{array}$$

You need to calculate the variation. You have a factorization $g = g^r g^l$ which is <u>somehow</u> linked to a splitting. $S = S_- S_+$

basic diagram



what is the splitting? $b(\lambda)$ $|a(\lambda)|^2 = 1 + |b(\lambda)|^2$

$$\delta g = \delta g^r g^l + g^r \delta g^l$$

$$g^{-1}\delta g = (g^l)^{-1}(g^r)^{-1}\delta g^r g^l + (g^l)^{-1}\delta g^l$$

$$(g^r)^{-1}\delta g (g^l)^{-1} = (g^r)^{-1}\delta g^r + (\delta g^l)(g^l)^{-1}$$

$$g = xy^{-1}$$

$$\delta g = \delta x\, y^{-1} - x y^{-1}\delta y\, y^{-1}$$

$$\boxed{x^{-1}\delta g\, y = x^{-1}\delta x - y^{-1}\delta y}$$

$$g = x^{-1}y$$

$$\delta g = -x^{-1}\delta x\, x^{-1} y + x^{-1}\delta y$$

$$x\,\delta g\, y^{-1} = -\delta x\, x^{-1} + \delta y\, y^{-1}$$

$$\begin{pmatrix} z^{-x}p_x \\ q_x \end{pmatrix} = \begin{pmatrix} a_x^l & b_x^l \\ c_x^l & d_x^l \end{pmatrix} \begin{pmatrix} \xi'_- \\ \xi'_+ \end{pmatrix}$$

$$\begin{pmatrix} p_x \\ q_x \end{pmatrix} = \begin{pmatrix} a_x^l & z^x b_x^l \\ z^{-x} c_x^l & d_x^l \end{pmatrix} \begin{pmatrix} z^x \xi'_- \\ \xi'_+ \end{pmatrix} = \begin{pmatrix} d_x^r & -z^x b_x^r \\ -z^{-x} c_x^r & a_x^r \end{pmatrix} \begin{pmatrix} z^x \xi_+ \\ \xi_- \end{pmatrix}$$

$$\begin{pmatrix} a & z^\varepsilon b \\ z^{-\varepsilon} c & d \end{pmatrix} = \begin{pmatrix} a_\varepsilon^r & z^\varepsilon b_\varepsilon^r \\ z^{-\varepsilon} c^r & d^r \end{pmatrix} \begin{pmatrix} a^l & z^\varepsilon b^l \\ z^{-\varepsilon} c^l & d^l \end{pmatrix}$$

$$z^\varepsilon = e^{\lambda \varepsilon} = 1 + \lambda \varepsilon$$

$$\lambda\varepsilon \begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix} = \begin{pmatrix} 0 & \lambda\varepsilon b^r \\ -\lambda\varepsilon c^r & 0 \end{pmatrix} \begin{pmatrix} a^l & b^l \\ c^l & d^l \end{pmatrix} + \begin{pmatrix} a^r & b^r \\ c^r & d^r \end{pmatrix} \begin{pmatrix} 0 & \lambda\varepsilon b^l \\ -\lambda\varepsilon c^l & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & z^x b \\ -z^{-x} c & 1 \end{pmatrix} = \begin{pmatrix} a_x^r & z^x b_x^r \\ -z^{-x} c_x^l & a_x^l \end{pmatrix} \begin{pmatrix} d_x^r & z^x b_x^l \\ -z^{-x} c_x^r & d_x^l \end{pmatrix}$$

Take? $x = \varepsilon$

$$\begin{pmatrix} 0 & \lambda b \\ +\lambda c & 0 \end{pmatrix} = \begin{pmatrix} \delta a^r & \delta b^r + \lambda b^r \\ -\delta c^l + \lambda c^l & \delta a^l \end{pmatrix} \begin{pmatrix} d^r & b^l \\ -c^r & d^l \end{pmatrix}$$

$$+ \begin{pmatrix} a^r & b^r \\ -c^l & a^l \end{pmatrix} \begin{pmatrix} \delta d^r & \delta b^l + \lambda b^l \\ -\delta c^r + \lambda c^r & \delta d^l \end{pmatrix}$$

coeff of λ automatic.

$$b = b^r d^l + a^r b^l \checkmark \qquad \cancel{-b^r c^r + b^r c^r}$$

$$+c = +c^l d^r + a^l c^r \checkmark \qquad c^l b^l \cancel{-c^l b^l}$$

$$g = g_- g_+^{-1}$$

$$0 = \delta g_- g_+^{-1} - g_- g_+^{-1} \delta g_+ g_+^{-1}$$

$$0 = g_-^{-1} \delta g_- - g_+^{-1} \delta g_+$$

$g_-^{-1} \delta g_- = g_+^{-1} \delta g_+$ must be constant in x matrix.

Last point yesterday. Given a smooth $b(\lambda)$ on $Re(\lambda)=0$, ~~you construct~~ decaying sufficiently, you ~~could~~ construct transfer matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Go over what you need. $c = \bar{b}$ $\quad 1+|b|^2 = ad$ $\quad a = \bar{d}$ $\quad d \in 1 + H_+$. $1+|b|^2$ is smooth ≥ 1, so its logarithm is smooth, now use ~~exp~~ Hilbert transform, Fourier transform. $\log(1+|b|^2) = \int_{-\infty}^{\infty} e^{\lambda x} f \, dx$ where f is Schwartz.

Actually what's going is pretty general. Suppose you have a matrix function $T = b(z)$, not necessarily square matrix, then use graph construction, Cayley Transform Γ_T. At some point you form $1+T^*T$ $\quad 1+TT^*$ and their pos. square roots. $X = \begin{pmatrix} 0 & -T^* \\ T & 0 \end{pmatrix}$ $\quad 1 - X^2 = \begin{pmatrix} 1+T^*T & 0 \\ 0 & 1+TT^* \end{pmatrix}$

$$\frac{1+X}{\sqrt{1-X^2}} \text{ is unitary} \quad \left(\frac{1+X}{\sqrt{1-X^2}}\right)^2 = \frac{(1+X)^2}{(1+X)(1-X)} = \frac{1+X}{1-X} = g$$

$$g\varepsilon(1+X) = g(1-X)\varepsilon = (1+X)\varepsilon$$

$$g\varepsilon\begin{pmatrix} 1 & -T \\ T & 1 \end{pmatrix} = \begin{pmatrix} 1 & -T \\ T & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \varepsilon g \varepsilon = g^{-1}$$
$$(g\varepsilon)^2 = 1$$

You tried to do a lot with this. The point now is that in the circle case you have a different square root around. ~~Say $\rho(z)\,d\Theta$ is~~

Suppose $\rho(z) = 1 + b(z)^* b(z)$ pos. def. matrix fn on S^1. Form $L^2(S^1, \rho \frac{d\Theta}{2\pi})$, completion of vector fns. $f(z)$ wrt norm $\int f(z)^* \rho(z) f(z) \frac{d\Theta}{2\pi}$. ~~Pos. def.~~ But assuming $0 < c \leq \rho(z) \leq C$ this should be $L^2(S^1, \frac{d\Theta}{2\pi})$ with

ρ norm. Then $V = H^2(S^1, \rho \frac{d\theta}{2\pi})$ is outgoing

i.e. $zV \subset V$ $\quad \bigcap z^n V = 0 \quad \overline{\bigcup z^n V} = L^2$

so the orth comp. $V \ominus zV$ gives the desired

$d(z)$ holom. in $|z|<1$. $\quad 1 + b^* b = d^* d$

OK. What about replacing S^1 by $i\mathbb{R}$?

Somehow this should be similar. ~~Replace~~ | Katrina

~~Finite measure support n points~~

Consider n dim'l Hilb. space H with s.a. A mult. 1 spectrum, ξ cyclic vector. Consider family of cyclic vectors $e^{tA}\xi$

It seems that

Go back to your factorizing of the scattering matrix.

$$\begin{pmatrix} 1 & b \\ -c & 1 \end{pmatrix} = \begin{pmatrix} a^r & b^r \\ -c^l & a^l \end{pmatrix} \begin{pmatrix} d^r & b^l \\ -c^r & d^l \end{pmatrix}$$

$$\begin{pmatrix} \tilde H_- & H_- \\ H_- & \tilde H_- \end{pmatrix} \begin{pmatrix} \tilde H_+ & H_+ \\ H_+ & \tilde H_+ \end{pmatrix}$$

So what are you doing? We will alter b by $e^{i\lambda x} = z^x$, doesn't affect d You need.

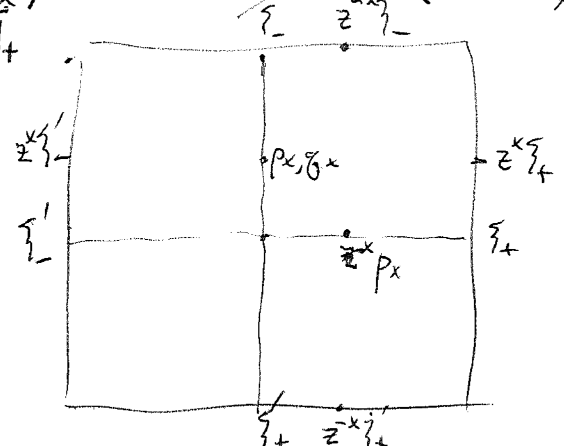
$$\begin{array}{cc} \tilde H_- & z^{-x} H_- \\ \end{array} \quad \begin{array}{cc} \tilde H_+ & z^{-x} H_+ \end{array}$$

$$\begin{pmatrix} \frac{1}{d} & \frac{b}{d} \\ -\frac{c}{d} & \frac{1}{d} \end{pmatrix} = \begin{pmatrix} a^r_x & b^r_x \\ -c^l_x & a^l_x \end{pmatrix} \begin{pmatrix} d^r_x & b^l_x \\ -c^r_x & d^l_x \end{pmatrix} \frac{1}{d}$$

$$\begin{array}{cc} z^x H_- & \tilde H_- \end{array} \quad \begin{array}{cc} z^x H_+ & \tilde H_+ \end{array}$$

$$\begin{pmatrix} z^{-x} p_x \\ q_x \end{pmatrix} = \begin{pmatrix} d^r_x & -b^r_x \\ -c^r_x & a^r_x \end{pmatrix} \begin{pmatrix} \xi_+ \\ \xi_- \end{pmatrix}$$

$$\begin{pmatrix} \tilde H_+ & z^{-x} H_- \\ z^x H_+ & \tilde H_- \end{pmatrix}$$



$$\begin{pmatrix} z^{-x}p_x \\ q_x \end{pmatrix} = \begin{array}{cc} H_- & z^{-x}H_+ \\ \begin{pmatrix} a^\ell_x & b^\ell_x \\ c^\ell_x & d^\ell_x \end{pmatrix} & \\ z^x H_- & \tilde{H}_+ \end{array} \begin{pmatrix} \xi'_- \\ \xi'_+ \end{pmatrix}$$

So what you have is

$$\begin{pmatrix} 1 & z^x b \\ -z^{-x}c & 1 \end{pmatrix} = \begin{pmatrix} a^r_x & z^x b^r_x \\ -z^{-x}c^\ell_x & a^\ell_x \end{pmatrix} \begin{pmatrix} d^r_x & z^x b^\ell_x \\ -z^{-x}c^r_x & d^\ell_x \end{pmatrix}$$

$$g = g_- g_+$$

This ~~is~~ is not as convenient

You want

$$\begin{pmatrix} 1 & b \\ -c & 1 \end{pmatrix} = \begin{pmatrix} a^r_x & b^r_x \\ -c^\ell_x & a^\ell_x \end{pmatrix} \begin{pmatrix} d^r_x & b^\ell_x \\ -c^r_x & d^\ell_x \end{pmatrix}$$

$$g = g_- g_+$$

$$0 = \dot{g}_- g_+ + g_- \dot{g}_+$$

$$0 = g_-^{-1}\dot{g}_- + \dot{g}_+ g_+^{-1}$$

$$\frac{1}{a}\begin{pmatrix} a^\ell_x & -b^\ell_x \\ +c^\ell_x & a^r_x \end{pmatrix} \begin{pmatrix} \dot{a}^r_x & \dot{b}^r_x \\ -\dot{c}^\ell_x & \dot{a}^\ell_x \end{pmatrix}$$

$$\begin{pmatrix} \tilde{H}_- & z^{-x}H_- \\ z^x H_- & \tilde{H}_- \end{pmatrix} \Bigg($$

~~doesn't work.~~

$$\begin{pmatrix} 1 & z^x b \\ -z^{-x}c & 1 \end{pmatrix} =$$

$$\begin{pmatrix} \xi'_- \\ \xi'_+ \end{pmatrix} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \begin{pmatrix} \xi_+ \\ \xi_- \end{pmatrix}$$

$$\begin{pmatrix} \xi'_- \\ \xi_- \end{pmatrix} = \begin{pmatrix} \frac{1}{a} & -\frac{b}{a} \\ \frac{c}{a} & \frac{1}{a} \end{pmatrix} \begin{pmatrix} \xi_+ \\ \xi'_+ \end{pmatrix}$$

$$\begin{pmatrix} \xi_+ \\ \xi'_+ \end{pmatrix} = \begin{pmatrix} \frac{1}{d} & \frac{b}{d} \\ -\frac{c}{d} & \frac{1}{d} \end{pmatrix} \begin{pmatrix} \xi'_- \\ \xi_- \end{pmatrix}$$

Consider
$$\begin{pmatrix} z^{-x}p_x \\ q_x \end{pmatrix} = \begin{pmatrix} a^\ell_x & b^\ell_x \\ c^\ell_x & d^\ell_x \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\frac{c}{d} & \frac{1}{d} \end{pmatrix} \begin{pmatrix} \xi'_- \\ \xi_- \end{pmatrix}$$

$$= \frac{1}{d} \begin{pmatrix} d^r_x & b^\ell_x \\ -c^r_x & d^\ell_x \end{pmatrix} \begin{pmatrix} \xi'_- \\ \xi_- \end{pmatrix}$$

$$\begin{pmatrix} b^\ell_x \\ d^\ell_x \end{pmatrix} = \begin{pmatrix} a^\ell_x & b^\ell_x \\ c^\ell_x & d^\ell_x \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\partial_x \begin{pmatrix} b^\ell_x \\ d^\ell_x \end{pmatrix} = \begin{pmatrix} 0 & hz^{-x} \\ \bar{h}z^{+x} & 0 \end{pmatrix} \begin{pmatrix} b^\ell_x \\ d^\ell_x \end{pmatrix}$$

$$\partial_x \begin{pmatrix} d^r_x & -b^r_x \\ -c^r_x & a^r_x \end{pmatrix} = \begin{pmatrix} 0 & hz^{-x} \quad\quad h_x z^{-x} \end{pmatrix} \quad ?$$

You want to use the $\tilde{H}_+ \cap \tilde{H}_- = 1$ idea to recover the potential h. ~~Let us cons~~ So you need to understand λ - nature of functions

$$\begin{pmatrix} z^{-x}p_x \\ q_x \end{pmatrix} = \begin{pmatrix} a^\ell_x & b^\ell_x \\ c^\ell_x & d^\ell_x \end{pmatrix} \begin{pmatrix} \xi'_- \\ \xi'_+ \end{pmatrix} = \begin{pmatrix} d^r_x & -b^r_x \\ -c^r_x & a^r_x \end{pmatrix} \begin{pmatrix} \xi_+ \\ \xi_- \end{pmatrix}$$

$$\cap \begin{pmatrix} \tilde{H}_- & z^{-x}H_+ \\ z^x H_- & \tilde{H}_+ \end{pmatrix} \qquad\qquad \cap \begin{pmatrix} \tilde{H}_+ & z^{-x}H_- \\ z^x H_+ & \tilde{H}_- \end{pmatrix}$$

$$\partial_x \begin{pmatrix} a^\ell_x & b^\ell_x \\ c^\ell_x & d^\ell_x \end{pmatrix} = \begin{pmatrix} 0 & z^{-x}h_x \\ z^x\bar{h}_x & 0 \end{pmatrix} \begin{pmatrix} a^\ell_x & b^\ell_x \\ c^\ell_x & d^\ell_x \end{pmatrix}$$

$$\partial_x \begin{pmatrix} d^r_x & -b^r_x \\ -c^r_x & a^r_x \end{pmatrix} = \begin{pmatrix} 0 & z^{-x}h_x \\ z^x h_x & 0 \end{pmatrix} \begin{pmatrix} d^r_x & -b^r_x \\ -c^r_x & a^r_x \end{pmatrix}$$

You want to get the principle straight.

$$\partial_x \begin{pmatrix} p_x \\ q_x \end{pmatrix} = \partial_x \begin{pmatrix} z^x & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} z^{-x}p_x \\ q_x \end{pmatrix}$$

$$= \begin{pmatrix} \lambda z^x & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} z^{-x}p_x \\ q_x \end{pmatrix} + \begin{pmatrix} z^x & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & z^{-x}h_x \\ \cancel{z^x}\bar{h}_x & 0 \end{pmatrix}\begin{pmatrix} \cancel{z^{-x}}p_x \\ q_x \end{pmatrix}$$

$$= \begin{pmatrix} \lambda & h_x \\ \bar{h}_x & 0 \end{pmatrix}\begin{pmatrix} p_x \\ q_x \end{pmatrix}$$

$$\partial_x \overbrace{\begin{pmatrix} z^x & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a^\ell_x & b^\ell_x \\ c^\ell_x & d^\ell_x \end{pmatrix}\begin{pmatrix} z^{-x} & 0 \\ 0 & 1 \end{pmatrix}}$$

$$\cancel{\partial_x \begin{pmatrix} \lambda & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} a^\ell_x & zb^\ell_x \\ z^{-x}c^\ell_x & d^\ell_x \end{pmatrix}}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} \lambda & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

$$= \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} \lambda d & -\lambda b \\ 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} a\lambda d & -a\lambda b \\ c\lambda d & -c\lambda b \end{pmatrix}$$

$$= \begin{pmatrix} \lambda & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} z^x & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} & \\ & \end{pmatrix}\begin{pmatrix} z^{-x} & 0 \\ 0 & 1 \end{pmatrix}$$

$$+ \begin{pmatrix} z^x & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & z^{-x}h_x \\ \bar{h}_x z^x & 0 \end{pmatrix}\begin{pmatrix} a^\ell_x & \\ & \end{pmatrix}\begin{pmatrix} z^{-x} & 0 \\ 0 & 1 \end{pmatrix}$$

$$+ \underbrace{\begin{pmatrix} z^x & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a^\ell_x & \\ & \end{pmatrix}\begin{pmatrix} z^{-x} & 0 \\ 0 & 1 \end{pmatrix}}_{g_x}\begin{pmatrix} -\lambda & 0 \\ 0 & 0 \end{pmatrix}$$

$$\partial_x g^\ell_x = \begin{pmatrix} \lambda & 0 \\ 0 & 0 \end{pmatrix} g^\ell_x + \begin{pmatrix} & h \\ \bar{h} & \end{pmatrix} g^\ell_x + g^\ell_x \begin{pmatrix} -\lambda & 0 \\ 0 & 0 \end{pmatrix}$$

$$\partial_x \begin{pmatrix} a^\ell_x & z^x b^\ell_x \\ z^{-x}c^\ell_x & d^\ell_x \end{pmatrix} = ($$

$$\partial_x \begin{pmatrix} a & z^x b \\ z^{-x} c & d \end{pmatrix} = \begin{pmatrix} 0 & \lambda z^x b \\ -\lambda z^{-x} c & 0 \end{pmatrix} + \begin{pmatrix} z^x & 0 \\ 0 & 1 \end{pmatrix} \partial_x \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} z^{-x} & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} & z^x h \\ z^{-x} \bar{h} & \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$\boxed{\partial_x \begin{pmatrix} a & z^x b \\ z^{-x} c & d \end{pmatrix} = \begin{pmatrix} \cancel{z^x} & \lambda z^x b + h_x \\ -\lambda z^{-x} c + \bar{h}_x & 0 \end{pmatrix} \begin{pmatrix} a & z^x b \\ z^{-x} c & b \end{pmatrix}}$$

$$\begin{pmatrix} 1 & b \\ -c & 1 \end{pmatrix} = \begin{pmatrix} a^r_x & b^r_x \\ -c^l_x & a^l_x \end{pmatrix} \begin{pmatrix} d^r_x & b^l_x \\ -c^r_x & d^l_x \end{pmatrix}$$

$$\begin{array}{cc} \cap & \cap \\ \begin{pmatrix} \tilde{H}_- & z^{-x} H_- \\ z^{+x} H_- & \tilde{H}_- \end{pmatrix} & \begin{pmatrix} \tilde{H}_+ & z^{-x} H_+ \\ z^x H_+ & \tilde{H}_+ \end{pmatrix} \end{array}$$

$$\begin{pmatrix} 1 & z^x b \\ -z^{-x} c & 1 \end{pmatrix} = \begin{pmatrix} a^r_x & z^x b^r_x \\ -z^{-x} c^l_x & a^l_x \end{pmatrix} \begin{pmatrix} d^r_x & z^x b^l_x \\ -z^{-x} c^r_x & d^l_x \end{pmatrix}$$

$$\partial_x \begin{pmatrix} a^l_x & b^l_x \\ c^l_x & d^l_x \end{pmatrix} = \begin{pmatrix} 0 & h z^{-x} \\ \bar{h} z^{+x} & 0 \end{pmatrix} \begin{pmatrix} a^l_x & b^l_x \\ c^l_x & d^l_x \end{pmatrix}$$

$$\partial_x \begin{pmatrix} d^r_x & -b^r_x \\ -c^r_x & a^r_x \end{pmatrix} = \begin{pmatrix} 0 & h z^{-x} \\ \bar{h} z^x & 0 \end{pmatrix} \begin{pmatrix} d^r_x & -b^r_x \\ -c^r_x & a^r_x \end{pmatrix}$$

$$\partial_x \begin{pmatrix} a^r_x & b^r_x \\ c^r_x & d^r_x \end{pmatrix} = \begin{pmatrix} -b^r_x \bar{h} z^x & -a^r_x h z^{-x} \\ -d^r_x \bar{h} z^x & -c^r_x h z^{-x} \end{pmatrix} = -\begin{pmatrix} a^r_x & b^r_x \\ c^r_x & d^r_x \end{pmatrix} \begin{pmatrix} 0 & h z^{-x} \\ \bar{h} z^x & 0 \end{pmatrix}$$

so ? $\partial_x \begin{pmatrix} a^r & z^x b^r \\ -z^{-x} c^l & d^l \end{pmatrix} = \begin{pmatrix} -b^r \bar{h} z^x & \lambda z^x b^r + z^x(-a^r h z^{-x}) \\ +\lambda z^{-x} c^l + z^{-x} \bar{h} z^x a^l & \bar{h} z^x b^l \end{pmatrix}$

$$0 = \begin{pmatrix} \partial_x a^r & \partial_x b^r \\ \partial_x c^r & \partial_x d^r \end{pmatrix} \begin{pmatrix} a^l & b^l \\ c^l & d^l \end{pmatrix} + \begin{pmatrix} a^r & b^r \\ c^r & d^r \end{pmatrix} \begin{pmatrix} \partial_x a^l & \partial_x b^l \\ \partial_x c^l & \partial_x d^l \end{pmatrix}$$

$$\begin{pmatrix} a_x^r & b_x^r \\ c_x^r & d_x^r \end{pmatrix} \in \begin{pmatrix} \tilde{H}_- & z^{-x} H_- \\ z^x H_+ & \tilde{H}_+ \end{pmatrix}$$ not a subgroup.

$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_n \\ y_n \end{pmatrix}$ Fibonacci $\lambda^2 - \lambda - 1 = 0$

$$\lambda = \frac{1 \pm \sqrt{5}}{2}$$

$\begin{pmatrix} 1-\lambda & 1 \\ 1 & -\lambda \end{pmatrix} ($

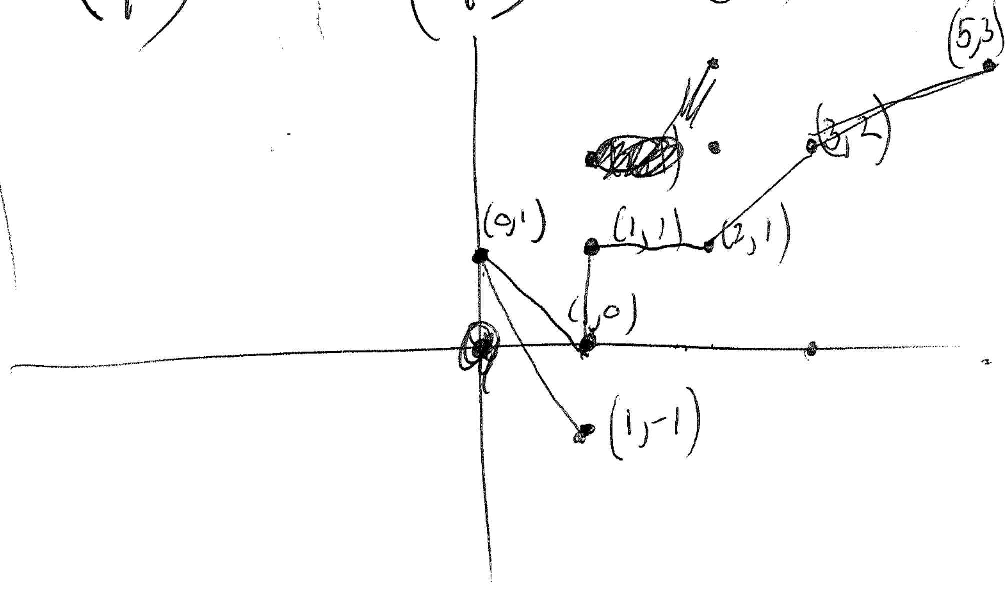
0 1 1 2 3 5

$\begin{pmatrix} 0 \\ 1 \end{pmatrix} \to \begin{pmatrix} 1 \\ 0 \end{pmatrix} \to \begin{pmatrix} 1 \\ 1 \end{pmatrix} \to \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

$\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} x+y \\ x \end{pmatrix} \to \begin{pmatrix} 2x+y \\ x+y \end{pmatrix}$ $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}$

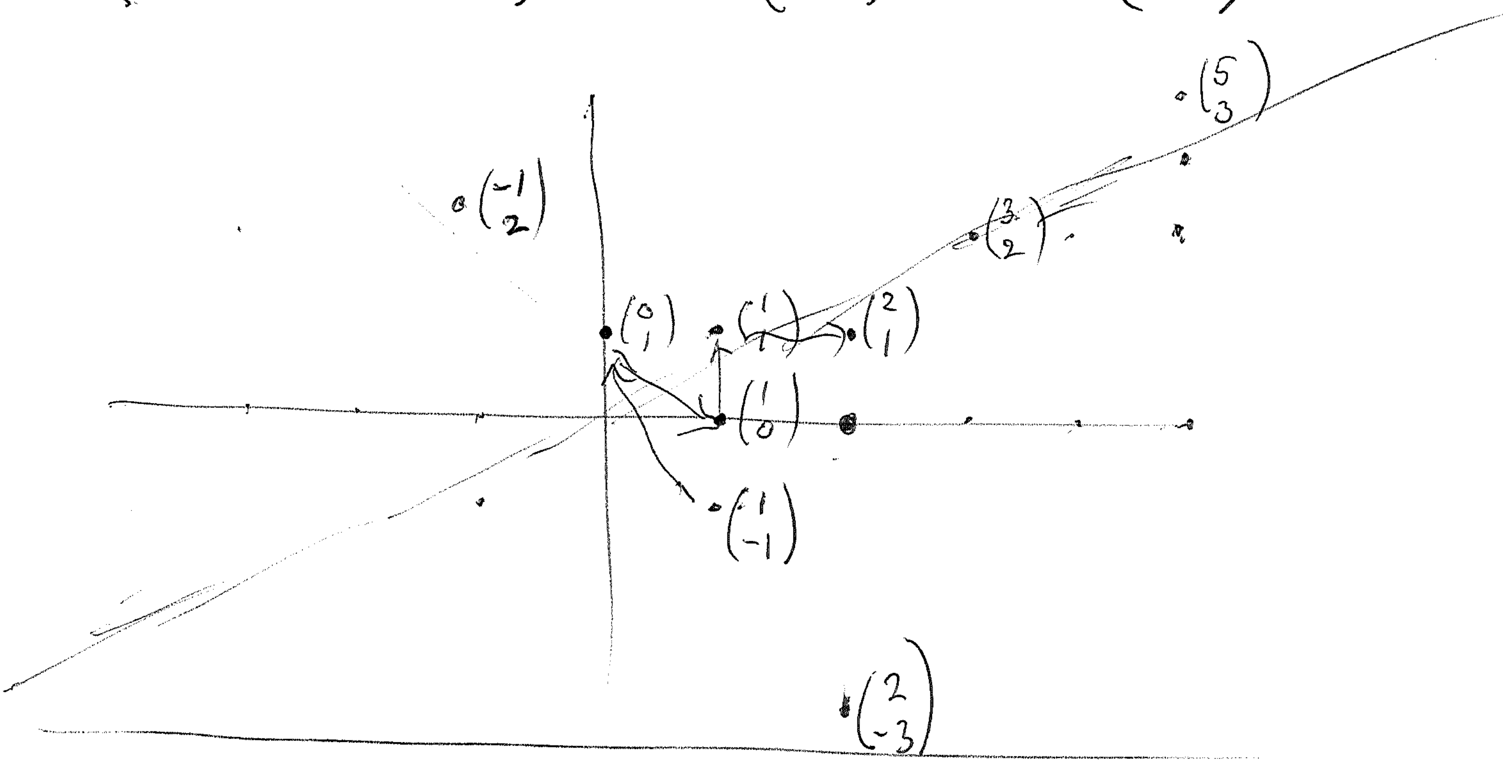
$\begin{pmatrix} 0 \\ 1 \end{pmatrix} \to \begin{pmatrix} 1 \\ 0 \end{pmatrix} \to \begin{pmatrix} 1 \\ 1 \end{pmatrix} \to \begin{pmatrix} 2 \\ 1 \end{pmatrix} \to \begin{pmatrix} 3 \\ 2 \end{pmatrix} \to \begin{pmatrix} 5 \\ 3 \end{pmatrix}$

$\begin{pmatrix} -1 \\ 2 \end{pmatrix} \to \begin{pmatrix} 1 \\ -1 \end{pmatrix} \to \begin{pmatrix} 0 \\ 1 \end{pmatrix}$



$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \qquad \begin{aligned} y' &= x \\ x' &= x+y \end{aligned}$$

$$\begin{tikzcd}
\begin{pmatrix} 0 \\ 1 \end{pmatrix} \arrow[r] & \begin{pmatrix} 1 \\ 0 \end{pmatrix} \arrow[r] & \begin{pmatrix} 1 \\ 1 \end{pmatrix} \arrow[r] & \begin{pmatrix} 2 \\ 1 \end{pmatrix} \arrow[r] & \begin{pmatrix} 3 \\ 2 \end{pmatrix} \arrow[r] & \begin{pmatrix} 5 \\ 3 \end{pmatrix} \\
\begin{pmatrix} 1 \\ -1 \end{pmatrix} \arrow[u] & \begin{pmatrix} -1 \\ 2 \end{pmatrix} \arrow[l] & \begin{pmatrix} 2 \\ -3 \end{pmatrix} \arrow[l] & \begin{pmatrix} -3 \\ 5 \end{pmatrix} \arrow[l] & &
\end{tikzcd}$$



take a $g \in SL_2(\mathbb{Z})$ $\qquad \begin{vmatrix} a & b \\ c & d \end{vmatrix} = 1$

$$\lambda^2 - (a+d)\lambda + 1 = 0.$$

$$\lambda = \frac{\pm(a+d) \pm \sqrt{(a+d)^2 - 4}}{2} \qquad |a+d| > 2.$$

$$\begin{matrix} 2 & 3 \\ 1 & 2 \end{matrix} \qquad \begin{pmatrix} a-\lambda & b \\ c & d-\lambda \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0$$

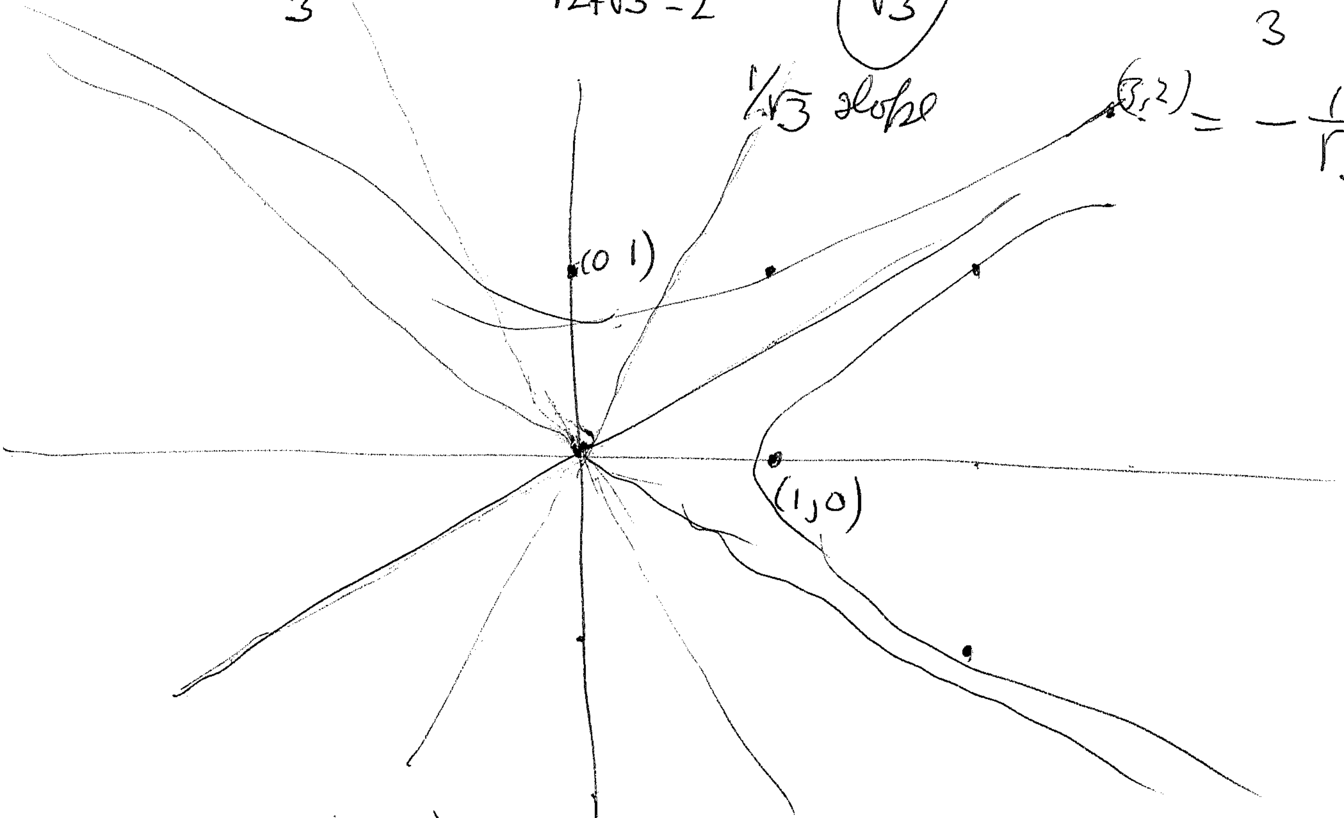
$$(a-\lambda)x + by = 0 \qquad \frac{x}{y} = \frac{b}{\lambda - a}$$

$$\frac{y}{x} = \frac{\lambda - a}{b} = \frac{c}{\lambda - d} \qquad \lambda = +2 \pm \sqrt{3}$$

$$\shortparallel$$

$$\frac{+2+\sqrt{3}-2}{3} = \frac{1}{+2+\sqrt{3}-2} = \frac{1}{\sqrt{3}}$$

$$\frac{2-\sqrt{3}-2}{3} = \frac{1}{2-\sqrt{3}-2}$$



$$\begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix} \qquad \begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 7 \\ -4 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 7 \\ 4 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ \cancel{4} \\ 2 \end{pmatrix}$$

Idea: Look at $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$

Recover the potential.

$$\partial_x \begin{pmatrix} a^l & b^l \\ c^l & d^l \end{pmatrix} = \begin{pmatrix} 0 & h z^{-x} \\ \bar h z^x & 0\end{pmatrix}\begin{pmatrix} a^l & b^l\\ c^l & d^l\end{pmatrix}$$

$$\partial_x \begin{pmatrix} d^r & -b^r \\ -c^r & a^r\end{pmatrix} = \begin{pmatrix} 0 & z^{-x}h \\ z^x\bar h & 0\end{pmatrix}\begin{pmatrix} d^r & -b^r\\ -c^r & a^r\end{pmatrix}$$

Yesterday I learned that you probably ~~cant~~ have to use orthogonality (in some way) to show that $(\partial_x g^l)(g^l)^{-1}$ has the form $\begin{pmatrix}0 & z^{-x}h_x\\ z^x\bar h_x & 0\end{pmatrix}$. The idea is that

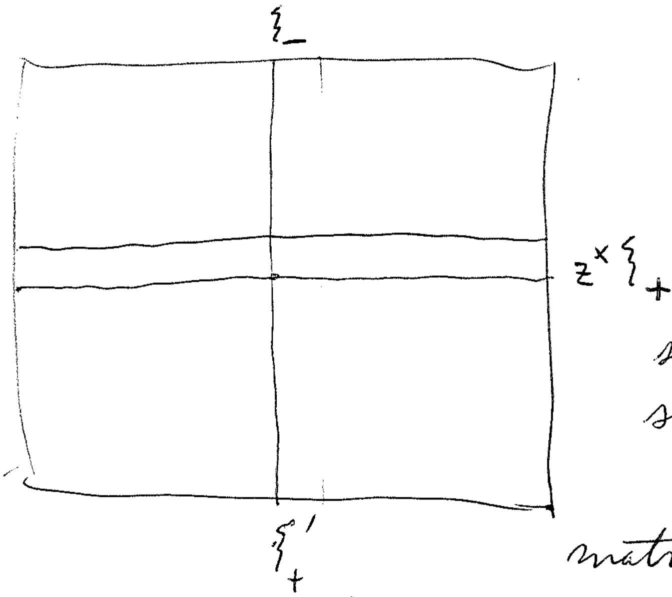
$$g^l_{x+\Delta x} = T\exp\left\{\int_{-\infty}^{x+\Delta x}\begin{pmatrix}0 & z^{-x'}h\\ z^{x'}\bar h & 0\end{pmatrix}dx'\right\}$$

$$= T\exp\left\{\int_x^{x+\Delta x}\cdots\right\}\quad T\exp\left\{\int_{-\infty}^x \cdots\right\}$$

~~Does~~ You should be able to prove that

$$g^l_{x+\Delta x}(g^l_x)^{-1} \in \begin{pmatrix}\tilde H & [z^{-x}, z^{-(x+\Delta x)}]\\ [z^x, z^{x+\Delta x}] & \tilde H_+\end{pmatrix}$$

where $[z^x, z^{x+\Delta x}] = z^x H_+ \ominus z^{x+\Delta x} H_+$



~~So you have to look at~~ ~~filch~~ ~~You want to look at~~

In principle this should work, but you don't ~~quite~~ yet see how smoothness of β should enter.

For now see if scattering matrix gives a better picture

$$\begin{pmatrix}1 & b\\ -c & 1\end{pmatrix} = \begin{pmatrix}a^r_x & b^r_x\\ -c^l_x & a^l_x\end{pmatrix}\begin{pmatrix}d^r_x & b^l_x\\ -c^r_x & d^l_x\end{pmatrix} \in \begin{pmatrix}\tilde H_- & z^{-x}H_-\\ z^xH_- & \tilde H_-\end{pmatrix}\begin{pmatrix}\tilde H_+ & z^{-x}H_+\\ z^xH_+ & \tilde H_+\end{pmatrix}$$

$$\begin{pmatrix} \xi_+ \\ \xi_+' \end{pmatrix} = \begin{pmatrix} \frac{1}{d} & \frac{b}{d} \\ -\frac{c}{d} & \frac{1}{d} \end{pmatrix} \begin{pmatrix} \xi_-' \\ \xi_- \end{pmatrix} =$$

$$\begin{pmatrix} z^{-x} p_x \\ q_x \end{pmatrix} = \frac{1}{a} \begin{pmatrix} a_x^l & -b_x^r \\ c_x^l & a_x^r \end{pmatrix} \begin{pmatrix} \xi_+ \\ \xi_+' \end{pmatrix} = \frac{1}{d} \begin{pmatrix} d_x^r & b_x^l \\ -c_x^r & d_x^l \end{pmatrix} \begin{pmatrix} \xi_-' \\ \xi_- \end{pmatrix}$$

$\cancel{\partial_x \begin{pmatrix} d_x^r & b_x^l \\ -c \end{pmatrix}}$ Cindy's passport

$$\begin{pmatrix} 1 & z^x b \\ -z^{-x} c & 1 \end{pmatrix} = \begin{pmatrix} a_x^r & z^x b_x^r \\ -z^{-x} c_x^l & a_x^l \end{pmatrix} \begin{pmatrix} d_x^r & z^x b_x^l \\ -z^{-x} c_x^r & d_x^l \end{pmatrix}$$

$$\partial_x \begin{pmatrix} d_x^r & b_x^l \\ -c_x^r & d_x^l \end{pmatrix} = \begin{pmatrix} 0 & h_x z^{-x} \\ z^x \bar{h}_x & 0 \end{pmatrix} \begin{pmatrix} d_x^r & b_x^l \\ -c_x^r & d_x^l \end{pmatrix}$$

$$\partial_x \begin{pmatrix} a_x^r & b_x^r \\ -c_x^l & a_x^l \end{pmatrix} = - \begin{pmatrix} a_x^r & b_x^r \\ -c_x^l & a_x^l \end{pmatrix} \begin{pmatrix} & z^{-x} h_x \\ z^x \bar{h}_x & \end{pmatrix}$$

$$\partial_x \begin{pmatrix} a_x^r & z^x b_x^r \\ -z^{-x} c_x^l & a_x^l \end{pmatrix} = \begin{pmatrix} -z^x \bar{h} b_x^r & \lambda z^x b_x^r - z^x (z^{-x} h a_x^r) \\ \lambda z^{-x} c_x^l - z^{-x} (\bar{h} z^x a_x^l) & h z^{-x} c_x^l \end{pmatrix}$$

$$= \begin{pmatrix} -\bar{h}\, z^x b_x^r & -h a_x^r + \lambda z^x b_x^r \\ -\bar{h} a_x^l + \lambda z^{-x} c_x^l & h z^{-x} c_x^l \end{pmatrix}$$

$$\cancel{= \begin{pmatrix} a_x^r & z^x b_x^r \\ -z^{-x} c_x^l & a_x^l \end{pmatrix} \begin{pmatrix} 0 & -h \\ -\bar{h} & 0 \end{pmatrix}}$$

$$\partial_x \begin{pmatrix} a^r_x & z^x b^r_x \\ -z^{-x} c^l_x & a^l_x \end{pmatrix} = \left[\begin{pmatrix} \lambda & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} a^r_x & z^x b^r_x \\ -z^{-x} c^l_x & a^l_x \end{pmatrix} \right]$$

$$- \begin{pmatrix} a^r_x & z^x b^r_x \\ -z^{-x} c^l_x & a^l_x \end{pmatrix} \begin{pmatrix} 0 & h \\ \bar{h} & 0 \end{pmatrix}$$

Lorentz transf. $\begin{pmatrix} x' \\ t' \end{pmatrix} = \underbrace{\begin{pmatrix} a & b \\ c & d \end{pmatrix}}_{g} \begin{pmatrix} x \\ t \end{pmatrix}$ preserving

$x^2 - t^2 = \begin{pmatrix} x \\ t \end{pmatrix}^t \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ t \end{pmatrix}$.

$$\begin{pmatrix} x' \\ t' \end{pmatrix}^t \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x' \\ t' \end{pmatrix} = \begin{pmatrix} x \\ t \end{pmatrix}^t g^t \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} g \begin{pmatrix} x \\ t \end{pmatrix} = \begin{pmatrix} x \\ t \end{pmatrix}^t \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ t \end{pmatrix}$$

$$g^t \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} g = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \implies \det(g)^2 = 1$$

$\therefore \det(g) = \pm 1.$

$$g^t = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} g^{-1} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\begin{pmatrix} a & c \\ b & d \end{pmatrix} = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} d & b \\ c & a \end{pmatrix}$$

$\therefore \quad a = d \qquad b = c. \qquad g = \begin{pmatrix} d & c \\ c & d \end{pmatrix}$

$$\begin{pmatrix} d & c \\ c & d \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} d+c & 0 \\ 0 & d-c \end{pmatrix} \qquad (d+c)(d-c) = 1$$

Good coord system is thus ~~x+t, x-t,~~ given by eiginvectors $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$.

$$\begin{pmatrix} x' \\ t' \end{pmatrix} = \begin{pmatrix} d & c \\ c & d \end{pmatrix} \begin{pmatrix} x \\ t \end{pmatrix} \implies x' + t' = \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} d & c \\ c & d \end{pmatrix} \begin{pmatrix} x \\ t \end{pmatrix}$$

$$= (d+c)(x+t)$$

$$x' - t' = dx + ct - cx - dt$$

$$= (d-c)(x-t)$$

Standard form.

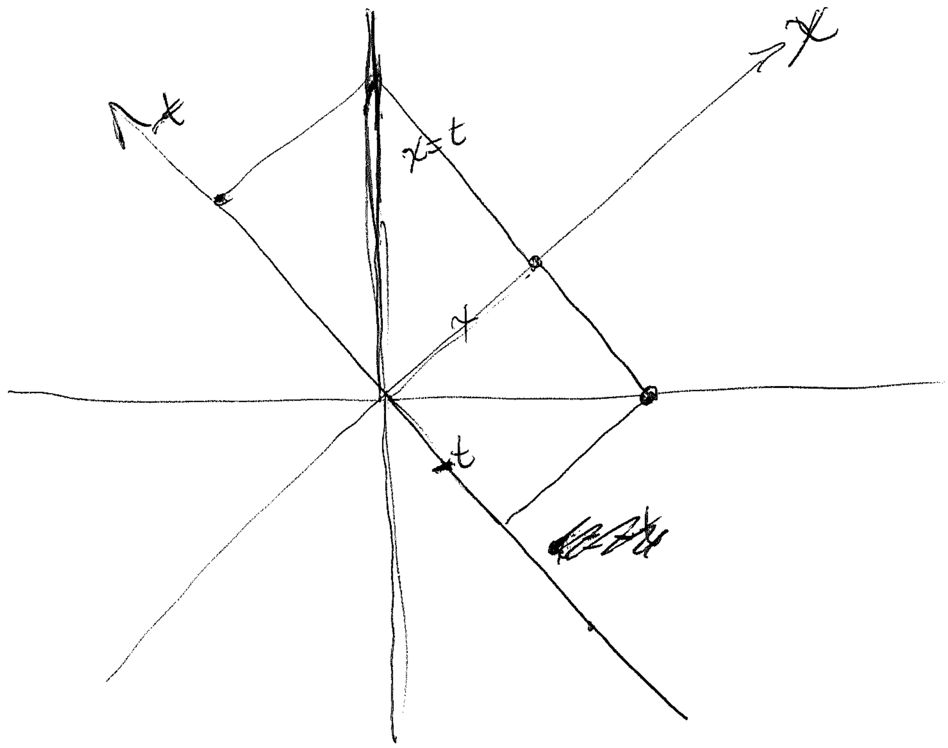
$$\begin{pmatrix} x' \\ t' \end{pmatrix} = \begin{pmatrix} d & c \\ c & d \end{pmatrix} \begin{pmatrix} x \\ t \end{pmatrix} \qquad d^2 - c^2 = 1$$

$$\frac{c}{d} = v \qquad 1 - v^2 = 1 - \frac{c^2}{d^2} = \frac{1}{d^2} \qquad d = \frac{1}{\sqrt{1-v^2}} \qquad c = \frac{v}{\sqrt{1-v^2}}$$

$$\begin{pmatrix} x' \\ t' \end{pmatrix} = \frac{1}{\sqrt{1-v^2}} \begin{pmatrix} 1 & v \\ v & 1 \end{pmatrix} \begin{pmatrix} x \\ t \end{pmatrix}$$

$$x' = \frac{x + tv}{\sqrt{1-v^2}}$$

$$t' = \frac{vx + t}{\sqrt{1-v^2}}$$



Think mathematics.

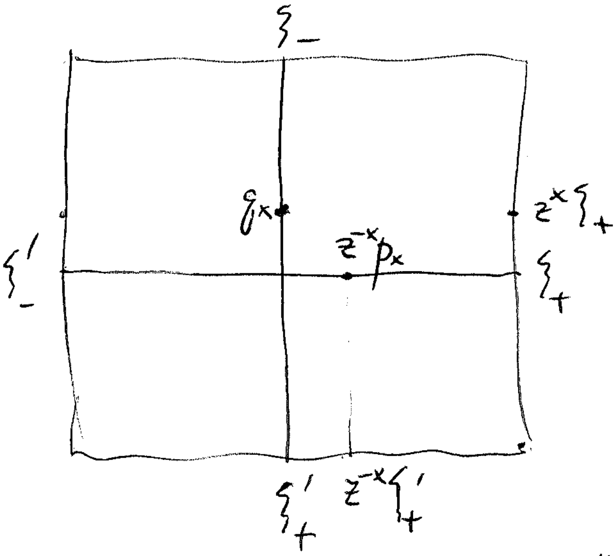
$$\begin{pmatrix} z^{-x}p_x \\ q_x \end{pmatrix} = \begin{pmatrix} a^l & b^l \\ c^l & d^l \end{pmatrix}\begin{pmatrix} \xi'_- \\ \xi'_+ \end{pmatrix} = \begin{pmatrix} d^r & -b^r \\ -c^r & a^r \end{pmatrix}\begin{pmatrix} \xi_+ \\ \xi_- \end{pmatrix}$$

four columns

satisfy $\quad \partial_x \psi = \begin{pmatrix} 0 & h z^{-x} \\ \bar{h} z^x & 0 \end{pmatrix} \psi.$

Look at the "integral" equations

$$\begin{array}{rcll} & & \overset{1+H_-}{\Downarrow} & \overset{z^x H_+}{\Downarrow} \\ z^{-x}p_x & = & a^l \xi'_- & +\ b^l \xi'_+ \\ q_x & = & c^l \xi'_- & +\ d^l \xi'_+ \\ & & \underset{z^x H_-}{\Uparrow} & \underset{1+H_+}{\Uparrow} \end{array}$$



what are the ~~integral~~ orth relations

$$\begin{array}{rcll} & & \overset{1+H_+}{\Downarrow} & \overset{z^{-x}H_-}{\Downarrow} \\ z^{-x}p_x & = & d^l \xi_+ & -\ b^l \xi_- \\ q_x & = & -c^l \xi_+ & +\ a^l \xi_- \\ & & \underset{z^x H_+}{\Uparrow} & \underset{1+H_-}{\Uparrow} \end{array}$$

work with other

The orth relations result from the factorization

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a^r & b^r \\ c^r & d^r \end{pmatrix}\begin{pmatrix} a^l & b^l \\ c^l & d^l \end{pmatrix}$$

$$\begin{pmatrix} d^r & -b^r \\ -c^r & a^r \end{pmatrix} = \begin{pmatrix} a^l & b^l \\ c^l & d^l \end{pmatrix}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

$$\begin{pmatrix} d^r & -b^r \\ -c^r & a^r \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a^l & b^l \\ c^l & d^l \end{pmatrix}$$

$$\begin{pmatrix} d^r & -b^r \\ -c^r & a^r \end{pmatrix}\begin{pmatrix} 1 & \frac{b}{d} \\ \frac{c}{a} & 1 \end{pmatrix} = \begin{pmatrix} \frac{a^l}{a} & \frac{b^l}{d} \\ \frac{c^l}{a} & \frac{d^l}{d} \end{pmatrix} \in \begin{pmatrix} 1+H_- & z^{-x}H_+ \\ z^x H_- & 1+H_+ \end{pmatrix}$$

So you get for the Nth time.

$$\underbrace{(d^r \quad -b^r)}_{1+H_+ \quad z^{-x}H_-}\begin{pmatrix} 1 & \beta \\ \bar\beta & 1 \end{pmatrix} \approx \underbrace{\left(\frac{a^\ell}{a} \quad \frac{b^\ell}{d}\right)}_{1+H_- \quad z^{-x}H_+}$$

Conjugate

$$\begin{pmatrix} d^r & -z^x b^r \\ -z^{-x}c^r & a^r \end{pmatrix}\begin{pmatrix} 1 & z^x\frac{b}{d} \\ z^{-x}\frac{c}{a} & 1 \end{pmatrix} = \begin{pmatrix} \frac{a^\ell}{a} & \frac{z^x b^\ell}{b} \\ \frac{z^{-x}c^\ell}{a} & \frac{d^\ell}{d} \end{pmatrix} \in \begin{pmatrix} 1+H_- & H_+ \\ H_- & 1+H_+ \end{pmatrix}$$

$$\begin{pmatrix} d^r & -z^x b^r \\ -z^{-x}c^r & a^r \end{pmatrix} \in \begin{pmatrix} 1+H_+ & H_- \\ H_+ & 1+H_- \end{pmatrix}$$

So you end up with the equations

$$\cancel{\begin{pmatrix} d & \\ & \end{pmatrix}\begin{pmatrix} 1 & \beta \\ \bar\beta & 1 \end{pmatrix}}$$

$$1+\hat d^r - z^x b^r z^{-x}\bar\beta \in 1+H_-$$

$$(1+\hat d^r)z^x\beta - z^x b^r \in H_-$$

$$\boxed{\begin{aligned} &\hat d^r - b^r\bar\beta \in H_- \\ &(1+\hat d^r)z^x\beta - z^x b^r \in H_- \end{aligned}}$$

$$\hat d^r = \pi_+(b^r\bar\beta)$$

$$z^x\beta$$

Set $x=0$

$$1+\hat d^r - b^r\bar\beta \in 1+H_- \qquad \hat d^r = \pi_+(b^r\bar\beta)$$

$$(1+\hat d^r)\beta - b^r \in H_+ \qquad b^r = (1-\pi_-\beta\pi_+\bar\beta)^{-1}\pi_-\beta$$

$$\pi_-(1+\hat d^r)\beta - b^r = 0. \qquad \pi_-\beta + \pi_-\beta\,\pi_+\bar\beta\, b^r = b^r$$

$$\begin{pmatrix} 1 & \bar\beta \\ \beta & 1 \end{pmatrix}\begin{pmatrix} d^r \\ -b^r \end{pmatrix} \in \begin{pmatrix} 1+H_- \\ H_+ \end{pmatrix}$$

$$1+\hat d^r - \bar\beta b^r \in 1+H_- \iff \hat d^r = \pi_+ \bar\beta b^r$$

$$\beta \underbrace{d^r}_{1+\hat d^r} - b^r \in H_+ \iff \pi_-\beta 1 + \pi_-\beta \hat d^r - b^r = 0.$$

$$\Downarrow$$

$$\pi_-\beta 1 + \pi_-\beta\pi_+\bar\beta\, b^r = b^r$$

$$\therefore \boxed{b^r = (1-\pi_-\beta\pi_+\bar\beta)^{-1}\pi_-\beta 1}$$

$$d^r = 1 + \pi_+\bar\beta(1-\pi_-\beta\pi_+\bar\beta)^{-1}\pi_-\beta 1$$

$$= (1-\pi_+\bar\beta\pi_-\beta)^{-1}1$$

Put in x. $\beta_x = z^x\beta$

$\cancel{1+\hat d^r_x = \bar\beta}$

Clean up a bit

$$\begin{pmatrix} d^r & -z^x b^r \\ -z^{-x}c^r & a^r \end{pmatrix} = \begin{pmatrix} a^l & z^x b^l \\ z^{-x}c^l & d^l \end{pmatrix}\begin{pmatrix} d & -z^x b \\ -z^{-x}c & a \end{pmatrix}$$

$$\begin{pmatrix} 1+H_+ & H_- \\ H_+ & 1+H_- \end{pmatrix} \qquad \begin{pmatrix} 1+H_- & H_+ \\ H_- & 1+H_+ \end{pmatrix}$$

$$\begin{pmatrix} a^r & z^x b^r \\ z^{-x} c^r & d^r \end{pmatrix} = \begin{pmatrix} a & z^x b \\ z^{-x} c & d \end{pmatrix} \begin{pmatrix} d^l & -z^x b^l \\ -z^{-x} c^l & a^l \end{pmatrix}$$

$$\begin{pmatrix} 1+H_- & H_- \\ H_+ & 1+H_+ \end{pmatrix} \qquad \qquad \begin{matrix} \cap \\ \begin{pmatrix} 1+H_+ & H_+ \\ H_- & 1+H_- \end{pmatrix} \end{matrix}$$

$$(1+\hat{d}^l) - \frac{z^x b}{a}(z^{-x} c^l) \in 1+H_-$$

$$\hat{d}^l - \left(\frac{z^x b}{a}\right)(z^{-x} c^l) \in H_-$$

$$\boxed{\hat{d}^l = \pi_+ \, z^x \frac{b}{a} \, (z^{-x} c^l)}$$

$$z^{-x}\frac{c}{d}(1+\hat{d}^l) - (z^{-x} c^l) \in H_+$$

$$\boxed{\pi_-\left(z^{-x}\frac{c}{d}1\right) + \pi_- \, z^{-x}\frac{c}{d} \, \pi_+ \, z^x \frac{b}{a} \, z^{-x} c^l = z^{-x} c^l}$$

$$\boxed{z^{-x} c^l = \left(1 - \pi_- \, z^{-x}\frac{c}{d} \, \pi_+ \, z^x \frac{b}{a}\right)^{-1} \pi_-\left(z^{-x}\frac{c}{d}1\right)}$$

$$\boxed{\hat{d}^l = \left(1 - \pi_+ \, z^x \frac{b}{a} \, \pi_- \, z^{-x}\frac{c}{d}\right)^{-1} 1}$$

Those Neumann series should have a Grassmannian interpretation.

Aim for a better understanding

I need a better understanding of the basic equations. Suppose h given & nice

$$\begin{pmatrix} z^{-x}p \\ g \end{pmatrix} = \begin{pmatrix} a^l & b^l \\ c^l & d^l \end{pmatrix}\begin{pmatrix} \xi'_- \\ \xi'_+ \end{pmatrix} = \begin{pmatrix} d^r & -b^r \\ -c^r & a^r \end{pmatrix}\begin{pmatrix} \xi_+ \\ \xi_- \end{pmatrix}$$

$$= \frac{1}{a}\begin{pmatrix} a^l & -b^r \\ c^l & a^r \end{pmatrix}\begin{pmatrix} \xi_+ \\ \xi'_+ \end{pmatrix} = \frac{1}{d}\begin{pmatrix} d^r & b^l \\ -c^r & d^l \end{pmatrix}\begin{pmatrix} \xi'_- \\ \xi_- \end{pmatrix}$$

$\tilde{H}_- \quad z^{-x}H_-$ $\qquad\qquad$ $\tilde{H}_+ \quad z^{-x}H_+$

$z^{x}H_- \quad \tilde{H}_-$ $\qquad\qquad$ $z^{x}H_+ \quad \tilde{H}_+$

Can you find something that will ~~[illegible]~~ organize things?

Can you adapt ~~[illegible]~~ the orth. conditions to the scattering picture?

$$\cancel{\frac{1}{a}\begin{pmatrix} a^l & -b^r \\ c^l & a^r \end{pmatrix} = \frac{1}{d}\begin{pmatrix} d^r & b^l \\ -c^r & d^l \end{pmatrix}\begin{pmatrix} \frac{1}{d} & \frac{b}{d} \\ -\frac{c}{d} & \frac{1}{d} \end{pmatrix}}$$

Check formulas.

$$\begin{pmatrix} \frac{1}{d} & \frac{b}{d} \\ \frac{-c}{d} & \frac{1}{d} \end{pmatrix} = \begin{pmatrix} a^r & b^r \\ -c^l & a^l \end{pmatrix}\frac{1}{d}\begin{pmatrix} d^r & b^l \\ -c^r & d^l \end{pmatrix}$$

need new approach - you should be able to ~~[illegible]~~ easily translate existence of factorization from transfer to scattering setting. Recall the existence proof. Given $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ you want to factor

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a^r & b^r \\ c^r & d^r \end{pmatrix}\begin{pmatrix} a^l & b^l \\ c^l & d^l \end{pmatrix}$$

$$\begin{pmatrix} \tilde{\hat{H}}_- & \hat{H}_- \\ H_+ & \tilde{H}_+ \end{pmatrix} \begin{pmatrix} \tilde{\hat{H}}_- & H_+ \\ H_- & \tilde{H}_+ \end{pmatrix}$$

method

$$\begin{pmatrix} d^r & -b^r \\ -c^r & a^r \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a^l & b^l \\ c^l & d^l \end{pmatrix}$$

$$\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}\begin{pmatrix} a^r & b^r \\ c^r & d^r \end{pmatrix} = \begin{pmatrix} d^l & -b^l \\ -c^l & a^l \end{pmatrix}$$

$$\begin{pmatrix} 1 & \overset{-\beta}{-\frac{b}{d}} \\ \underset{-\bar{\beta}}{-\frac{c}{a}} & 1 \end{pmatrix}\begin{pmatrix} a^r & b^r \\ c^r & d^r \end{pmatrix} = \begin{pmatrix} \frac{d^l}{d} & -\frac{b^l}{d} \\ -\frac{c^l}{a} & \frac{a^l}{a} \end{pmatrix} \in \begin{pmatrix} \tilde{H}_+ & H_+ \\ H_- & \tilde{H}_- \end{pmatrix}$$

$$b^r - \beta(1+\tilde{d}^r) \in H_+$$

$$-\bar{\beta} b^r + (1+\tilde{d}^r) \in \tilde{H}_- = 1 + H_-$$

$$b^r = \pi_-\beta(1+\tilde{d}^r)$$

$$\tilde{d}^r = \pi_+\bar{\beta}\, b^r$$

$$\pi_-\beta 1 = b^r - \pi_-\beta\pi_+\bar{\beta}\, b^r$$

$$b^r = (1-\pi_-\beta\pi_+\bar{\beta})^{-1}\pi_-\beta 1$$

$$\tilde{d}^r = (1-\pi_+\bar{\beta}\pi_-\beta)^{-1} 1$$

$$\begin{pmatrix} \pi_- & 0 \\ 0 & \pi_+ \end{pmatrix} \begin{pmatrix} 1 & -\beta \\ -\bar\beta & 1 \end{pmatrix} \begin{pmatrix} a^r & b^r \\ c^r & d^r \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -\pi_-\beta \\ -\pi_+\bar\beta & 1 \end{pmatrix} \begin{pmatrix} a^r & b^r \\ c^r & d^r \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -\pi_-\beta \\ -\pi_+\bar\beta & 1 \end{pmatrix}^{-1} = (I - X)^{-1} = \frac{1+X}{1-X^2}$$

$$= \begin{pmatrix} 1 & +\pi_-\beta \\ +\pi_+\bar\beta & 1 \end{pmatrix} \begin{pmatrix} 1-\pi_-\beta\pi_+\bar\beta & 0 \\ 0 & 1-\pi_+\bar\beta\pi_-\beta \end{pmatrix}^{-1}$$

$$\therefore \begin{pmatrix} a^r & b^r \\ c^r & d^r \end{pmatrix} = \begin{pmatrix} (1-\pi_-\beta\pi_+\bar\beta)^{-1}1 & \pi_-\beta(1-\pi_+\bar\beta\pi_-\beta)^{-1}1 \\ \pi_+\bar\beta(1-\pi_-\beta\pi_+\bar\beta)^{-1}1 & (1-\pi_+\bar\beta\pi_-\beta)^{-1}1 \end{pmatrix}$$

So things improve. Now go for

$$\begin{pmatrix} 1 & b \\ -\bar b & 1 \end{pmatrix} = \underset{\begin{pmatrix} 1+H_- & H_- \\ H_- & 1+H_- \end{pmatrix}}{\underset{\cap}{\begin{pmatrix} a^r & b^r \\ -c^l & d^l \end{pmatrix}}} \underset{\begin{pmatrix} 1+H_+ & H_+ \\ H_+ & 1+H_+ \end{pmatrix}}{\underset{\cap}{\begin{pmatrix} d^l & b^l \\ -c^r & d^l \end{pmatrix}}}$$

$$\cancel{\begin{pmatrix} d^l & -b^l \\ c^l & d^l \end{pmatrix}} \cancel{(1/d)} \qquad \cancel{\begin{pmatrix} \frac{1}{d} & \frac{b}{d} \\ -\frac{b}{d} & \frac{1}{d} \end{pmatrix}}$$

$$\begin{pmatrix} \frac{1}{d} & \frac{b}{d} \\ -\frac{\bar b}{d} & \frac{1}{d}\end{pmatrix} = \begin{pmatrix} a^r & b^r \\ -c^l & a^l\end{pmatrix} \frac{1}{d}\begin{pmatrix} d^r & b^l \\ -c^r & d^l\end{pmatrix}$$

$$\begin{pmatrix} \frac{1}{d} & \frac{b}{d} \\ -\frac{\bar b}{d} & \frac{1}{d}\end{pmatrix}\begin{pmatrix} d^l & -b^l \\ c^r & d^r\end{pmatrix} = \begin{pmatrix} a^r & b^r \\ -c^l & a^l\end{pmatrix} \in \begin{pmatrix} 1+H_- & H_- \\ H_- & 1+H_-\end{pmatrix}$$

$$\begin{pmatrix} \pi_+ & \text{[redacted]} \\ 0 & \pi_+\end{pmatrix}\begin{pmatrix} \frac{1}{d} & \frac{b}{d} \\ -\frac{\bar b}{d} & \frac{1}{d}\end{pmatrix}\underbrace{\begin{pmatrix} d^l & -b^l \\ c^r & d^r\end{pmatrix}}_{\overset{?}{\in} \begin{pmatrix} 1+H_+ & H_+ \\ H_+ & 1+H_+\end{pmatrix}} = \begin{pmatrix} 1 & 0 \\ 0 & 1\end{pmatrix}$$

$$\underbrace{\begin{pmatrix} \pi_+ & 0 \\ 0 & \pi_+\end{pmatrix}\begin{pmatrix} 1 & b \\ -\bar b & 1\end{pmatrix}}_{\begin{pmatrix} \pi_+ & \pi_+ b \\ -\pi_+\bar b & \pi_+\end{pmatrix}\begin{pmatrix} & \\ & \end{pmatrix}}\begin{pmatrix} d^l/d & -b^l/d \\ c^r/d & d^r/d\end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1\end{pmatrix}$$

You are working in the ring $\mathbb{C}\,\mathrm{Id} \oplus \begin{pmatrix} H_+ & H_+ \\ H_+ & H_+\end{pmatrix}$ $\subset \mathbb{C}\,\mathrm{Id} \oplus \begin{pmatrix} L^2 & L^2 \\ L^2 & L^2\end{pmatrix}$. So it appears that

$$\frac{1}{d}\begin{pmatrix} d^l & -b^l \\ c^r & d^r\end{pmatrix} = \left[\begin{pmatrix} 1 & 0 \\ 0 & 1\end{pmatrix} + \begin{pmatrix} 0 & \pi_+ b \\ -\pi_+\bar b & 0\end{pmatrix}\right]^{-1}$$

$$\begin{pmatrix} 1 & b \\ -c & 1 \end{pmatrix} = \begin{pmatrix} a^r & b^r \\ -c^l & a^l \end{pmatrix} \cancel{*} \begin{pmatrix} d^r & b^l \\ -c^r & d^l \end{pmatrix}$$

$$\begin{matrix} \cap & \qquad & \cap \\ \begin{pmatrix} \tilde{H}_- & H_- \\ H_- & \tilde{H}_- \end{pmatrix} & & \begin{pmatrix} \tilde{H}_+ & H_+ \\ H_+ & \tilde{H}_+ \end{pmatrix} \end{matrix}$$

$$\begin{pmatrix} 1 & b \\ -c & 1 \end{pmatrix} \begin{pmatrix} d^l & -b^l \\ c^r & d^r \end{pmatrix} \frac{1}{d} \in \begin{pmatrix} \tilde{H}_- & H_- \\ H_- & \tilde{H}_- \end{pmatrix}$$

$$\underset{\displaystyle \begin{pmatrix} \frac{1}{d} & \frac{b}{d} \\ -\frac{c}{d} & \frac{1}{d} \end{pmatrix}}{S} = \underset{\displaystyle \begin{pmatrix} a^r & b^r \\ -c^l & a^l \end{pmatrix}}{\begin{pmatrix} \xi_+ \\ \xi_+ \end{pmatrix}} \underset{\displaystyle \frac{1}{d}}{\begin{pmatrix} 2^{-x} p \\ \xi \end{pmatrix}} \underset{\displaystyle \begin{pmatrix} d^r & b^l \\ -c^r & d^l \end{pmatrix}}{\begin{pmatrix} \xi_- \\ \xi_- \end{pmatrix}}$$

$$S = g_-^{-1} g_+ \qquad S g_+^{-1} = g_-^{-1}$$

$$\begin{pmatrix} \frac{1}{d} & \frac{b}{d} \\ -\frac{c}{d} & \frac{1}{d} \end{pmatrix} \begin{pmatrix} d^l & -b^l \\ c^r & d^r \end{pmatrix} = \begin{pmatrix} a^r & b^r \\ -c^l & a^l \end{pmatrix}$$

$$\begin{pmatrix} 1 & b \\ -\bar{b} & 1 \end{pmatrix} \begin{pmatrix} d^l & -b^l \\ c^r & d^r \end{pmatrix} \frac{1}{d}$$

$$\underbrace{\begin{pmatrix} \pi_+ & \pi_+ b \\ -\pi_+ \bar{b} & \pi_+ \end{pmatrix}}_{\begin{pmatrix} 1 & \pi_+ b \\ -\pi_+ \bar{b} & 1 \end{pmatrix}} \begin{pmatrix} d^l/d & -b^l/d \\ c^r/d & d^r/d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

have Toeplitz operator $\pi_+ b$ on H_+ and its adjoint $\pi_+ \bar{b}$

$$\begin{pmatrix} 1 & -T^* \\ T & 1 \end{pmatrix}^{-1} = (1+X)^{-1} = \frac{1-X}{1-X^2}$$

$$= \begin{pmatrix} 1 & T^* \\ -T & 1 \end{pmatrix} \begin{pmatrix} (1+T^*T)^{-1} & \\ & (1+TT^*)^{-1} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{1+T^*T} & T^* \frac{1}{1+TT^*} \\ -T/1+T^*T & \frac{1}{1+TT^*} \end{pmatrix}$$

$$\begin{pmatrix} 1 & b \\ -c & 1 \end{pmatrix} = \begin{pmatrix} a^r & b^r \\ -c^l & a^l \end{pmatrix} \begin{pmatrix} d^r & b^l \\ -c^r & d^l \end{pmatrix}$$

$$\begin{pmatrix} a^l & -b^r \\ c^l & a^r \end{pmatrix} \begin{pmatrix} \frac{1}{a} & \frac{b}{a} \\ -\frac{c}{a} & \frac{1}{a} \end{pmatrix} = \begin{pmatrix} d^r & b^l \\ -c^r & d^l \end{pmatrix}$$

$$\begin{pmatrix} \frac{1}{a} & -\frac{c}{a} \\ \frac{b}{a} & \frac{1}{a} \end{pmatrix} \begin{pmatrix} a^l & c^l \\ -b^r & a^r \end{pmatrix} = \begin{pmatrix} d^r & -c^r \\ b^l & d^l \end{pmatrix}$$

$$\begin{pmatrix} 1 & -\pi_-\bar{b} \\ \pi_- b & 1 \end{pmatrix} \begin{pmatrix} \frac{a^l}{a} & \frac{c^l}{a} \\ -\frac{b^r}{a} & \frac{a^r}{a} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Review yesterday, where you saw how to directly ~~solve~~ do the factorization of the scattering matrix.

$$\begin{pmatrix} \frac{1}{d} & \frac{b}{d} \\ -\frac{c}{d} & \frac{1}{d} \end{pmatrix} = \begin{pmatrix} a^r & b^r \\ -c^l & a^l \end{pmatrix} \frac{1}{d} \begin{pmatrix} d^r & b^l \\ -c^r & d^l \end{pmatrix}$$

$$\begin{pmatrix} \tilde{H}_- & H_- \\ H_- & \tilde{H}_- \end{pmatrix} \quad \begin{pmatrix} \tilde{H}_+ & H_+ \\ H_+ & \tilde{H}_+ \end{pmatrix}$$

$$\begin{pmatrix} \frac{1}{d} & \frac{b}{d} \\ -\frac{c}{d} & \frac{1}{d} \end{pmatrix} \begin{pmatrix} d^l & -b^l \\ c^r & d^r \end{pmatrix} = \begin{pmatrix} a^r & b^r \\ -c^l & a^l \end{pmatrix}$$

$$\boxed{\begin{aligned} &\begin{pmatrix} 1 & b \\ -c & 1 \end{pmatrix} = \begin{pmatrix} a^r & b^r \\ -c^l & a^l \end{pmatrix} \begin{pmatrix} d^r & b^l \\ -c^r & d^l \end{pmatrix} \\ &\begin{pmatrix} 1 & b \\ -c & 1 \end{pmatrix} \begin{pmatrix} \frac{d^r}{d} & \frac{b^l}{d} \\ -\frac{c^r}{d} & \frac{d^l}{d} \end{pmatrix} = \begin{pmatrix} a^r & b^r \\ -c^l & a^l \end{pmatrix} \end{aligned}}$$

$$\begin{pmatrix} \pi_+ & \pi_+ b \\ -\pi_+ \bar{b} & \pi_+ \end{pmatrix} \begin{pmatrix} & \\ & \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} \pi_+ \alpha & \pi_+ \beta \\ \pi_+ \gamma & \pi_+ \delta \end{pmatrix} \begin{pmatrix} d^r & b^l \\ -c^r & d^l \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$\pi_+ S$ | You want $S g_+ = g_-$

This $\Rightarrow$ $\pi_+ S g_+ = 1$

In our case $\pi_+ \alpha$ on $\tilde{H}_+$ is the identity

Write $g_+ = 1 + \hat{g}_+$ $\qquad \hat{g}_+ \in M_2(H_+)$

$$\begin{pmatrix} \pi_+\alpha & \pi_+\beta \\ \pi_+\gamma & \pi_+\delta \end{pmatrix}\begin{pmatrix} 1+\hat{d}^{\wedge} & b^{\ell} \\ -c^{\ell} & 1+\hat{d}^{\ell} \end{pmatrix}$$

<u>Troubled</u> by π_+ acting on $\mathbb{C} + L^2$

You want $\quad S g_+ = g_-$. $\quad \Rightarrow \quad \pi_+(S g_+) = 1$

$$\pi_+ S \pi_+^* = \tau(S)$$

Toeplitz operator associated to S

$$S g_+ = g_- \qquad g_+^* S^* = g_-^* \qquad g_+^* = g_-^* S$$

π_+ unclear.

$$S g_+ = \begin{pmatrix} 1 & b \\ -\bar{b} & 1 \end{pmatrix} d^{-1} g_+ = g_-$$

$$\pi_+ S g_+ = \pi_+\left(1 + \begin{pmatrix} 0 & b \\ -\bar{b} & 0 \end{pmatrix}\right)\left(1 + \widehat{d^{-1} g_+}\right)$$

$$\cancel{= \dots 1 + \pi_+\begin{pmatrix} 0 & b \\ -\bar{b} & 0 \end{pmatrix}}$$

$$= \pi_+\left(1 + \begin{pmatrix} 0 & b \\ -\bar{b} & 0 \end{pmatrix} + \widehat{d^{-1} g_+} + \begin{pmatrix} 0 & b \\ -\bar{b} & 0 \end{pmatrix}\widehat{d^{-1} g_+}\right)$$

$$= 1 + \widehat{d^{-1} g_+} + \begin{pmatrix} 0 & \pi_+ b \\ -\pi_+\bar{b} & 0 \end{pmatrix}\left(1 + \widehat{d^{-1} g_+}\right)$$

$$= d^{-1} g_+ + \begin{pmatrix} 0 & \pi_+ b \\ -\pi_+\bar{b} & 0 \end{pmatrix}\left(d^{-1} g_+\right)$$

So the point is that because the [redacted] diagonal part of $S = \begin{pmatrix} \frac{1}{d} & \frac{b}{d} \\ -\frac{c}{d} & \frac{1}{d} \end{pmatrix}$ is in $\tilde{H}_+$ you have

$$\pi_+ \frac{1}{d} f_+ = \frac{1}{d} f_+$$

So we have $Sg_+ = g_- \implies \pi_+ S g_+ = 1$

$$\pi_+ S g_+ = \pi_+ \begin{pmatrix} 1 & b \\ -\bar{b} & 1 \end{pmatrix} d^{-1} g_+ = \begin{pmatrix} 1 & \pi_+ b \\ -\pi_+\bar{b} & 1 \end{pmatrix} d^{-1} g_+$$

$$\therefore\ d^{-1} g_+ = \begin{pmatrix} 1 & \pi_+ b \\ -\pi_+\bar{b} & 1 \end{pmatrix}^{-1} \text{ applied to } \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

You are working inside $M_2(\tilde{H}_+)$

The situation to understand, to examine.

You are going to ~~[illegible]~~ combine a loop in $U(2)$ with an F, i.e. ~~[illegible]~~ setting is part of Connes', also part of A-S proof of periodicity.

Start with $\begin{pmatrix} 1 & b \\ -\bar{b} & 1 \end{pmatrix}$ functions on S^1 acting by multiplication. Model

$$g = \frac{1+X}{1-X} = \frac{(1+X)^2}{1-X^2}$$

$$\begin{pmatrix} 1 & -T^* \\ T & 1 \end{pmatrix} = 1 + X \qquad g^{1/2} = \frac{1+X}{\sqrt{1-X^2}}$$

$$\underbrace{g\varepsilon}_{F} \begin{pmatrix} 1 & -T^* \\ T & 1 \end{pmatrix} = g\varepsilon(1+X) = g(1-X)\varepsilon = (1+X)\varepsilon$$
$$= \begin{pmatrix} 1 & -T^* \\ T & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

idea: Given $-\bar{b}(\lambda)$ have assoc. unitary

$$\cancel{\begin{pmatrix} 1 & b \\ -\bar{b} & 1 \end{pmatrix}} \quad \text{C.T. of } \overset{X=}{\begin{pmatrix} & b \\ -\bar{b} & \end{pmatrix}} \text{ is } \frac{1+X}{1-X} = \frac{\begin{pmatrix} 1 & b \\ -\bar{b} & 1 \end{pmatrix}}{\left(1+|b|^2\right.}$$

C.T. of $X = \begin{pmatrix} 0 & b \\ -\bar{b} & 0 \end{pmatrix}$ is $\dfrac{1+X}{1-X} = \dfrac{(1+X)^2}{1-X^2}$

Actually you seem to be interested in $\dfrac{1+X}{\sqrt{1-X^2}}$

$= \begin{pmatrix} 1 & b \\ -\bar{b} & 1 \end{pmatrix} \dfrac{1}{\sqrt{1+|b|^2}}$ — there's a choice of square roots here. Things are more subtle than the C.T. This is the loop side. You have b given then get a, d

<u>Next bring in</u> $H_+ \oplus H_-$ Connes' F.

~~Work~~ Go back over two approaches to unify.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \underset{H_+}{\overset{H_-}{\begin{pmatrix} a^r & b^r \\ c^r & d^r \end{pmatrix}}} \underset{H_-}{\overset{H_+}{\begin{pmatrix} a^l & b^l \\ c^l & d^l \end{pmatrix}}}$$

$$\begin{pmatrix} d^r & -b^r \\ -c^r & a^r \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a^l & b^l \\ c^l & d^l \end{pmatrix}$$

$$\begin{pmatrix} d^r & -b^r \\ -c^r & a^r \end{pmatrix} \begin{pmatrix} 1 & \frac{b}{d} \\ \frac{c}{a} & 1 \end{pmatrix} = \begin{pmatrix} \frac{a^l}{a} & \frac{b^l}{d} \\ \frac{c^l}{a} & \frac{d^l}{d} \end{pmatrix}$$

$$\begin{pmatrix} 1 & \frac{c}{a} \\ \frac{b}{d} & 1 \end{pmatrix} \begin{pmatrix} d^r & -c^r \\ -b^r & a^r \end{pmatrix} = \begin{pmatrix} \frac{a^l}{a} & \frac{c^l}{a} \\ \frac{b^l}{d} & \frac{d^l}{d} \end{pmatrix}$$

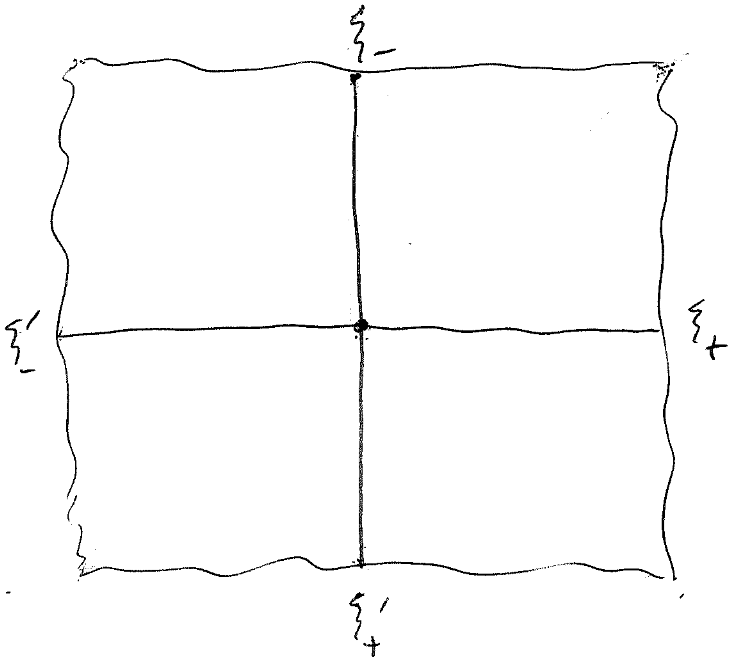
$\pi_+\left(d^r - \frac{c}{a} b^r\right) = 1$ $\qquad \pi_+\left(-c^r + \frac{c}{a} a^r\right) = 0$

$\pi_-\left(\frac{b}{d} d^r - b^r\right) = 0$ $\qquad \pi_{\cancel{+}}\left(-\frac{b}{a} c^r + a^r\right) = 1$

You want an understanding rather than a calculation. So how to proceed?

$$\begin{pmatrix} p_0 \\ q_0 \end{pmatrix} = \begin{pmatrix} a^l & b^l \\ c^l & d^l \end{pmatrix} \begin{pmatrix} \xi'_- \\ \xi'_+ \end{pmatrix} = \begin{pmatrix} d^r & -b^r \\ -c^r & a^r \end{pmatrix} \begin{pmatrix} \xi_+ \\ \xi_- \end{pmatrix}$$

Draw picture



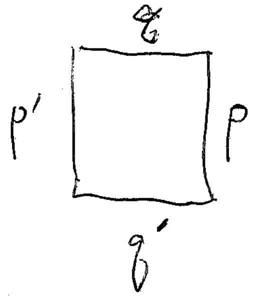
$$p_0 \in (1+H_-)\xi'_- + (H_+)\xi'_+ \;,\quad (1+H_+)\xi_+ + H_-\xi_-$$

$$q_0 \in (H_-)\xi'_- + (1+H_+)\xi'_+ \;,\quad H_+\xi_+ + (1+H_-)\xi_-$$

$$\begin{pmatrix} p_0 \\ q_0 \end{pmatrix} = \begin{pmatrix} a^l & b^l \\ c^l & d^l \end{pmatrix} \begin{pmatrix} \xi'_- \\ \xi'_+ \end{pmatrix} = \begin{pmatrix} d^r & -b^r \\ -c^r & a^r \end{pmatrix} \begin{pmatrix} \xi_+ \\ \xi_- \end{pmatrix}$$

$$= \frac{1}{a} \begin{pmatrix} a^l & -b^r \\ c^l & a^r \end{pmatrix} \begin{pmatrix} \xi_+ \\ \xi'_+ \end{pmatrix} = \frac{1}{d} \begin{pmatrix} d^r & b^l \\ -c^r & d^l \end{pmatrix} \begin{pmatrix} \xi'_- \\ \xi_- \end{pmatrix}$$

$$\begin{pmatrix} \frac{1}{d} & \frac{b}{d} \\ -\frac{c}{d} & \frac{1}{d} \end{pmatrix} = \begin{pmatrix} a^r & b^r \\ -c^l & a^l \end{pmatrix} \frac{1}{d} \begin{pmatrix} d^r & b^l \\ -c^r & d^l \end{pmatrix}$$



Four diml space with coords p, q, p', q' and herm. form

$$|p|^2 - |q|^2 - \left(|p'|^2 - |q'|^2\right)$$

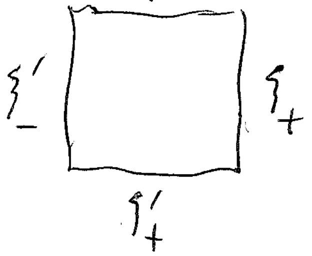
$$= |p|^2 + |q'|^2 - |q|^2 - |p'|^2$$

2 diml isotropic subspace described by

$$\begin{pmatrix} p \\ q \end{pmatrix} = \frac{1}{k}\begin{pmatrix} 1 & h \\ \bar{h} & 1 \end{pmatrix}\begin{pmatrix} p' \\ q' \end{pmatrix} \qquad \begin{pmatrix} p' \\ q' \end{pmatrix} = \frac{1}{k}\begin{pmatrix} 1 & -h \\ -\bar{h} & 1 \end{pmatrix}\begin{pmatrix} p \\ q \end{pmatrix}$$

$$\begin{pmatrix} p \\ q' \end{pmatrix} = \begin{pmatrix} k & h \\ -\bar{h} & k \end{pmatrix}\begin{pmatrix} p' \\ q \end{pmatrix} \qquad \begin{pmatrix} p' \\ q \end{pmatrix} = \begin{pmatrix} k & -h \\ \bar{h} & k \end{pmatrix}\begin{pmatrix} p \\ q' \end{pmatrix}$$

scattering situation. Think of $\xi_\pm$ $\xi'_\pm$ as coordinates, **i.e.** maps **from the** space under consideration to functions on the circle commuting with the action of $z^\times$. I guess you want hermitian form. ?? Let A = nice functions on the circle. You want a subspace of $A^{\oplus 4}$ equipped with $|\xi_+|^2 - |\xi_-|^2 - \left(|\xi'_-|^2 - |\xi'_+|^2\right)$



On A itself you have the map $a \mapsto |a|^2 = \bar{a}a$ which polarizes to $(a|a') = \bar{a}a'$. Now look at rank 2 subspace described by

$$\begin{pmatrix} \xi_+ \\ \xi_- \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} \xi'_- \\ \xi'_+ \end{pmatrix} \quad \text{equiv.} \quad \begin{pmatrix} \xi_+ \\ \xi'_+ \end{pmatrix} = \begin{pmatrix} \frac{1}{d} & \frac{b}{d} \\ -\frac{c}{d} & \frac{1}{d} \end{pmatrix}\begin{pmatrix} \xi'_- \\ \xi_- \end{pmatrix}$$

check this is isotropic

$$|\xi_+|^2 - |\xi_-|^2 = \begin{pmatrix}\xi_+ \\ \xi_-\end{pmatrix}^* \begin{pmatrix}1 & 0 \\ 0 & -1\end{pmatrix}\begin{pmatrix}\xi_+ \\ \xi_-\end{pmatrix}$$

$$= \begin{pmatrix}\xi_-' \\ \xi_+'\end{pmatrix}^* \underbrace{\begin{pmatrix}a & b \\ c & d\end{pmatrix}^* \begin{pmatrix}1 & 0 \\ 0 & -1\end{pmatrix}\begin{pmatrix}a & b \\ c & d\end{pmatrix}}\begin{pmatrix}\xi_-' \\ \xi_+'\end{pmatrix}$$

so

$$\begin{pmatrix}\bar a & \bar c \\ \bar b & \bar d\end{pmatrix}\begin{pmatrix}a & b \\ -c & -d\end{pmatrix} = \begin{pmatrix}d & \bar b \\ \bar b & \bar d\end{pmatrix}\begin{pmatrix}\bar d & \bar c \\ -c & -d\end{pmatrix}$$

preservation $\iff$

$$\cancel{g^*\begin{pmatrix}1 & 0 \\ 0 & -1\end{pmatrix} = g^{-1}}$$

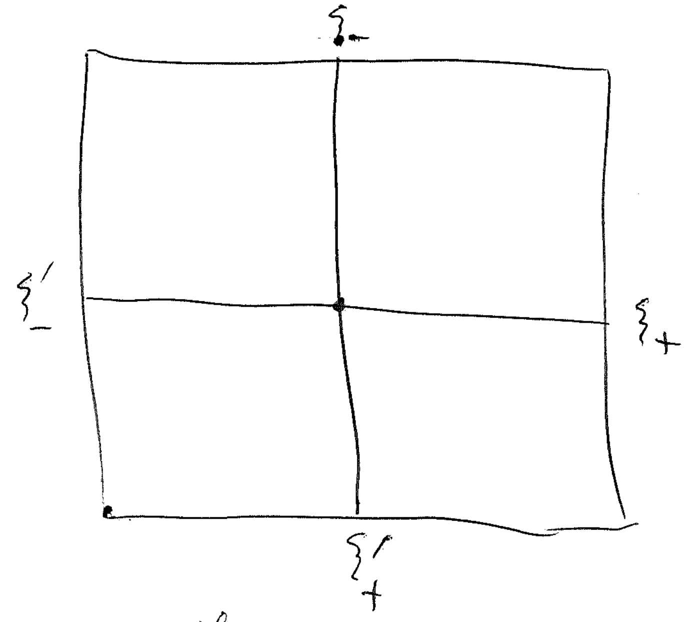
$$\cancel{\begin{pmatrix}\bar a & -\bar c \\ +\bar b & -\bar d\end{pmatrix} = \begin{pmatrix}d & -b \\ -c & a\end{pmatrix} \quad \text{if } \det \neq 1.}$$

$$g^*\begin{pmatrix}1 & 0 \\ 0 & -1\end{pmatrix} g = \begin{pmatrix}1 & 0 \\ 0 & -1\end{pmatrix}$$

$$\underbrace{\begin{pmatrix}1 & 0 \\ 0 & -1\end{pmatrix}\begin{pmatrix}\bar a & \bar c \\ \bar b & \bar d\end{pmatrix}\begin{pmatrix}1 & 0 \\ 0 & -1\end{pmatrix}} = \begin{pmatrix}d & -b \\ -c & a\end{pmatrix} \quad \text{if } \det = 1.$$

$$\begin{pmatrix}\bar a & -\bar c \\ -\bar b & \bar d\end{pmatrix} \qquad \bar a = d \quad \bar c = b$$

You want now to understand the equivalences of two factorizations. Some formulation in Krein situation reducing to the two types. How do we handle H_+ H_- ?



$$\begin{pmatrix} p_0 \\ q_0 \end{pmatrix} \in \begin{pmatrix} \tilde{H}_+ & H_- \\ H_+ & \tilde{H}_- \end{pmatrix} \begin{pmatrix} \xi_+ \\ \xi_- \end{pmatrix}$$

$$\in \begin{pmatrix} \tilde{H}_- & H_+ \\ H_- & \tilde{H}_+ \end{pmatrix} \begin{pmatrix} \xi'_- \\ \xi'_+ \end{pmatrix}$$

this gives subspaces $H_+\xi_+ + H_-\xi_-$, $H_-\xi'_- + H_+\xi'_+$ which are $\perp$ complements

Also have
$$\begin{pmatrix} p_0 \\ q_0 \end{pmatrix} \in \begin{pmatrix} \tilde{H}_+ & H_+ \\ H_+ & \tilde{H}_+ \end{pmatrix} \begin{pmatrix} \xi'_- \\ \xi_- \end{pmatrix}$$

$$\in \begin{pmatrix} \tilde{H}_- & H_- \\ H_- & \tilde{H}_- \end{pmatrix} \begin{pmatrix} \xi_+ \\ \xi'_+ \end{pmatrix}$$

this gives the subspaces $H_+\xi'_- + H_+\xi_-$, $H_-\xi_+ + H_-\xi'_+$

Q. where does the Hilbert space structure arise? Somehow this "space" is isotropic for the (pseudo) scalar product.

Recall that $\underbrace{W}_{\text{isotropic}} \subset \underbrace{V}_{\text{Krein}} \xrightarrow{\sim} A^4$

so if you split V Pick a non-deg. rank 2 subspace Z so that $V = Z \oplus Z^0$, then W becomes the graph of $Z \to Z^0$.

Lorentz transf. $\underbrace{\begin{pmatrix}1&0\\0&-1\end{pmatrix}\begin{pmatrix}a&b\\c&d\end{pmatrix}^t\begin{pmatrix}1&\\&-1\end{pmatrix}}_{\begin{pmatrix}a&-c\\-b&d\end{pmatrix}} = \begin{pmatrix}d&-b\\-c&a\end{pmatrix}$ if det $=1$.

$g^t\begin{pmatrix}1&0\\0&-1\end{pmatrix}g = \begin{pmatrix}1&0\\0&-1\end{pmatrix}$

$a=d \quad b=c.$

$a^2-b^2=1.$

$$g = \begin{pmatrix}d&c\\c&d\end{pmatrix}$$

$$g\begin{pmatrix}1&-1\\1&1\end{pmatrix} = \begin{pmatrix}d+c&-d+c\\c+d&-c+d\end{pmatrix} = \begin{pmatrix}1&-1\\1&1\end{pmatrix}\begin{pmatrix}d+c&0\\0&d-c\end{pmatrix}$$

$$\begin{pmatrix}x'\\t'\end{pmatrix} = \begin{pmatrix}d&c\\c&d\end{pmatrix}\begin{pmatrix}x\\t\end{pmatrix} = \begin{pmatrix}1&-1\\1&1\end{pmatrix}\begin{pmatrix}d+c&\\&d-c\end{pmatrix}\begin{pmatrix}1&1\\-1&1\end{pmatrix}\frac{1}{2}\begin{pmatrix}x\\t\end{pmatrix}$$

$$\begin{pmatrix}1&1\\-1&1\end{pmatrix}\begin{pmatrix}x'\\t'\end{pmatrix} = \begin{pmatrix}d+c&\\&d-c\end{pmatrix}\begin{pmatrix}1&1\\-1&1\end{pmatrix}\begin{pmatrix}x\\t\end{pmatrix}$$

$$\begin{pmatrix}d&c\\c&d\end{pmatrix} = \begin{pmatrix}1&v\\v&1\end{pmatrix}\frac{1}{\sqrt{1-v^2}}$$

$$\begin{pmatrix}x'+t'\\-x'+t'\end{pmatrix} = \begin{pmatrix}d+c&\\&d-c\end{pmatrix}\begin{pmatrix}x+t\\-x+t\end{pmatrix}$$

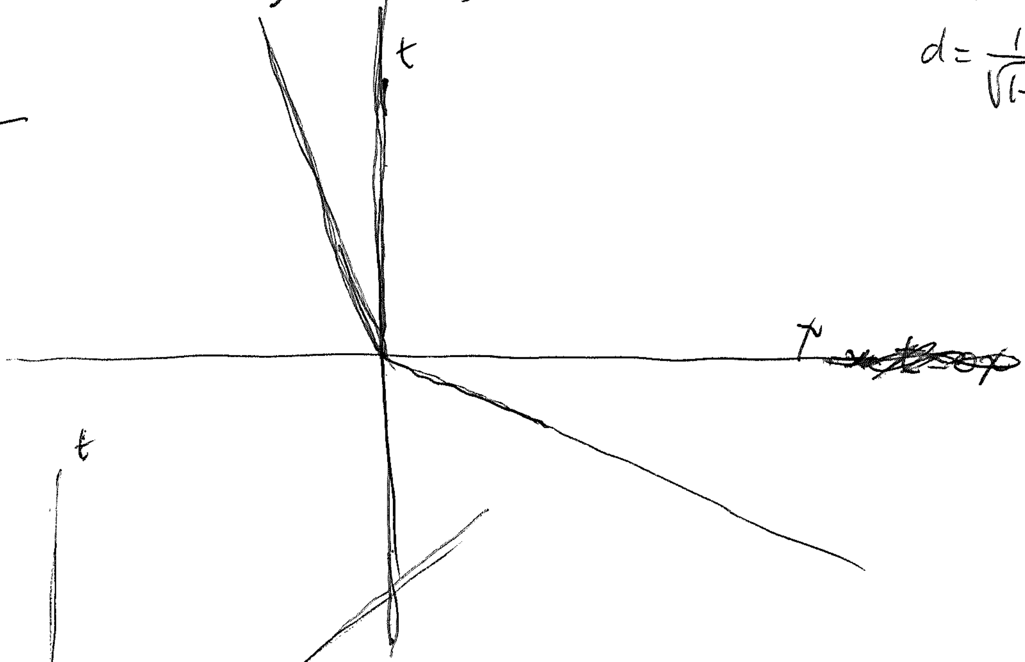
$v = \frac{c}{d} \qquad d^2-c^2=1.$

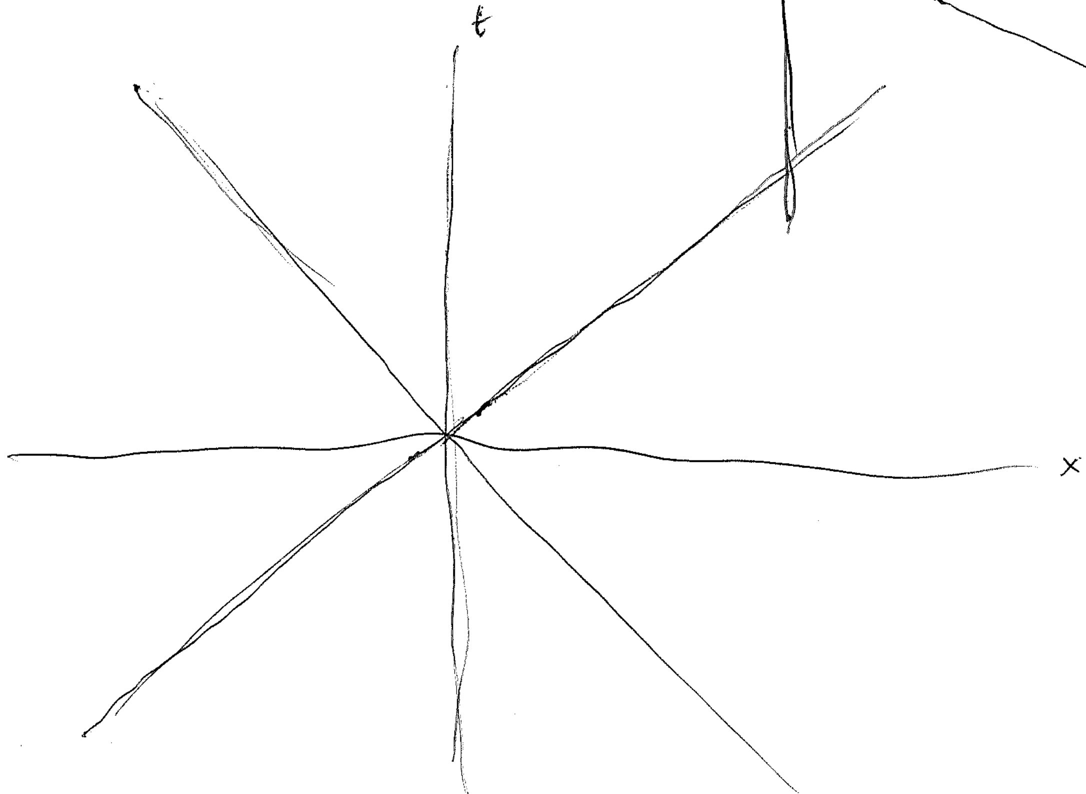
$(1-v^2)d^2=1$

$d = \frac{1}{\sqrt{1-v^2}}$

$$x' = \frac{x+vt}{\sqrt{1-v^2}}$$

$$t' = \frac{vx+t}{\sqrt{1-v^2}}$$





~~Problem. Subspace Thm~~

C^* module. A C^* algebra eg $C(X)$

C^*-module E is a right A-module

duality: dual pair P_A $\ {}_AQ$ $\qquad {}_AQ \times P_A \xrightarrow{\langle q,p\rangle} A$

allows ~~to~~ form $P \otimes_A Q$ $\qquad \begin{pmatrix} A & Q \\ P & P\otimes_A Q \end{pmatrix}$ Mor. context.

Hilbert C^*-module over A ~~is such a pair~~

A has an involution $*$, anti-linear on $\mathbb{C}$.

So E becomes a left A-module, can ask for

$E \longrightarrow \mathrm{Hom}_{A^o}(E, A)$ i.e. pairing $E \times E \longrightarrow A$

$\langle \xi a, \xi' a' \rangle = a^* \langle \xi, \xi' \rangle a'$. Another condition is positivity and completeness. Example: hermitian vector bundle.

So where to start? Hilbert space + unitary op.

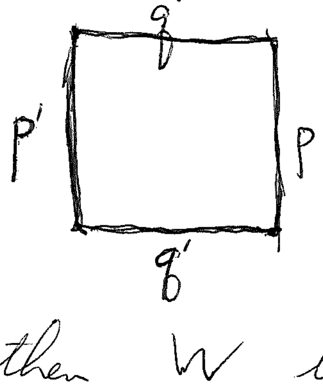
What do you want? A theoretical explanation of why the factorizations

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a^r & b^r \\ c^r & d^r \end{pmatrix} \begin{pmatrix} a^l & b^l \\ c^l & d^l \end{pmatrix} \in \begin{pmatrix} \tilde H_- & H_- \\ H_+ & \tilde H_+ \end{pmatrix} \begin{pmatrix} \tilde H_- & H_+ \\ H_- & \tilde H_+ \end{pmatrix}$$

$$\begin{pmatrix} \frac{1}{d} & \frac{b}{d} \\ -\frac{c}{d} & \frac{1}{d} \end{pmatrix} = \begin{pmatrix} a^r & b^r \\ -c^l & a^l \end{pmatrix} \frac{1}{d} \begin{pmatrix} d^r & b^l \\ -c^r & d^l \end{pmatrix} \in \begin{pmatrix} \tilde H_- & H_- \\ H_- & \tilde H_- \end{pmatrix} \begin{pmatrix} \tilde H_+ & H_+ \\ H_+ & \tilde H_+ \end{pmatrix}$$

are equivalent.

Simple example:



4 dim space $V = \mathbb{C}^4$ with coords p, q, p', q'

Krein form ~~2 dim subspace~~ $|p|^2 - |q|^2 - (|p'|^2 - |q'|^2)$

W 2 dim isotropic.

Point: If you split $V = V' \times V''$ then W becomes ~~a~~ the graph of corresp. between V' and V''.

Question: You have this rank 2 module W over the functions on S^1, ~~...~~ isotropic for the Krein form $|\xi_+|^2 - |\xi_-|^2 - |\xi_-'|^2 + |\xi_+'|^2$ ~~...~~ There are two obvious ways to get a hermitian form on W:

$$\text{pos. def.} \quad |\xi_+|^2 + |\xi_+'|^2 = |\xi_-'|^2 + |\xi_-|^2$$

$$\text{indef} \quad |\xi_+|^2 - |\xi_-|^2 = |\xi_-'|^2 - |\xi_+'|^2$$

~~...~~ Go over it again. For each $z \in S^1$ you get a 2 dim'l subspace of solutions of the D.E.

$$\partial_x \begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} \lambda & h \\ \bar{h} & 0 \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix}$$

and you get 4 numbers from the asymptotics.

$$\begin{pmatrix} \xi_-' \\ \xi_+' \end{pmatrix} \xleftarrow{-\infty \leftarrow x} \begin{pmatrix} z^{-x} p_x \\ q^x \end{pmatrix} \xrightarrow{x \to +\infty} \begin{pmatrix} \xi_+ \\ \xi_- \end{pmatrix}$$

We know that $|z^{-x} p_x|^2 - |q_x|^2 = |p_x|^2 - |q_x|^2$ is independent of x.

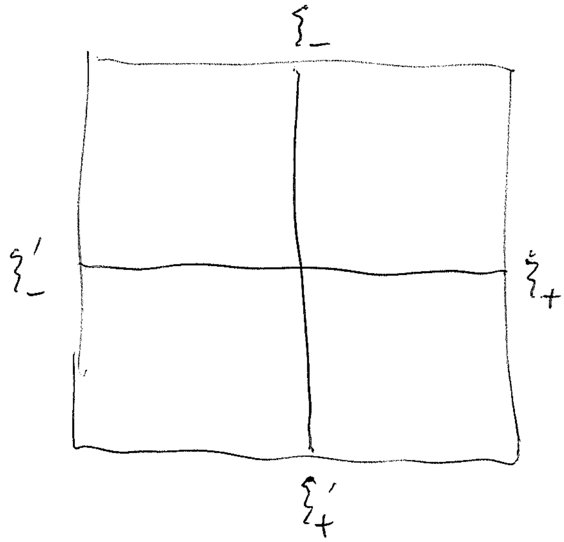
$$\begin{pmatrix} z^{-1} p_x \\ q_x \end{pmatrix} = \begin{pmatrix} a^l & b^l \\ c^l & d^l \end{pmatrix} \begin{pmatrix} \xi_-' \\ \xi_+' \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} g^* \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \cancel{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}} g^{-1}$$

$$\begin{pmatrix} \bar{a} & -\bar{c} \\ -\bar{b} & \bar{d} \end{pmatrix} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \qquad \text{since } \det = 1$$

Idea: Consider the discrete case with (h_n) finite support. Then you an obvious ring $A = \mathbb{C}[z, z^{-1}]$. and W is a rank 2 free A-module. Then W has a pos. def inner product and an indef inner product. W is the space of finite vectors – finite lin. comb. of grid vectors.

How to handle $H_\pm$?



$$H_- \xi_+ + H_- \xi'_+$$
$$H_+ \xi'_- + H_+ \xi_-$$

$$|\xi'_-|^2 - |\xi'_+|^2 = |\xi_+|^2 - |\xi_-|^2$$

$$\begin{pmatrix} f_- & g_- \end{pmatrix} \begin{pmatrix} \xi_+ \\ \xi'_+ \end{pmatrix} = \begin{pmatrix} f_- & g_- \end{pmatrix} \begin{pmatrix} \frac{1}{d} & \frac{b}{d} \\ -\frac{c}{d} & \frac{1}{d} \end{pmatrix} \begin{pmatrix} \xi'_- \\ \xi_- \end{pmatrix}$$

$$= \begin{pmatrix} f_- \frac{1}{d} - g_- \frac{c}{d} & f_- \frac{b}{d} + g_- \frac{1}{d} \end{pmatrix} \begin{pmatrix} \xi'_- \\ \xi_- \end{pmatrix}$$

$$\int \left(\overline{f_+} \left(f_- \frac{1}{d} - g_- \frac{c}{d} \right) - \overline{g_+} \left(f_- \frac{b}{d} + g_- \frac{1}{d} \right) \right)$$

This doesn't seem to work.

~~Lorentz transformations + Lorentz invariance of Dirac equation~~

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a^r & b^r \\ c^r & d^r \end{pmatrix}\begin{pmatrix} a^l & b^l \\ c^l & d^l \end{pmatrix}$$

$$\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}\begin{pmatrix} a^r & b^r \\ c^r & d^r \end{pmatrix} = \begin{pmatrix} d^l & -b^l \\ -c^l & a^l \end{pmatrix}$$

$$\begin{pmatrix} 1 & -\beta \\ -\bar\beta & 1 \end{pmatrix}\begin{pmatrix} a^r & b^r \\ c^r & d^r \end{pmatrix} = \begin{pmatrix} \frac{d^l}{d} & -\frac{b^l}{d} \\ -\frac{c^l}{a} & \frac{a^l}{a} \end{pmatrix}$$

$$\begin{pmatrix} \pi_- & -\pi_-\beta \\ -\pi_+\bar\beta & \pi_+ \end{pmatrix}\begin{pmatrix} a^r & b^r \\ c^r & d^r \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} 1+H_+ & H_+ \\ H_- & 1+H_- \end{pmatrix}$$

$$B = \begin{pmatrix} 0 & +\beta \\ \bar\beta & 0 \end{pmatrix} \qquad \begin{pmatrix} a^r & b^r \\ c^r & d^r \end{pmatrix} = 1+Y$$

$$\pi = \begin{pmatrix} \pi_- & 0 \\ 0 & \pi_+ \end{pmatrix} \qquad Y \in \begin{pmatrix} H_- & H_- \\ H_+ & H_+ \end{pmatrix}$$

$$(1-B)(1+Y) \in 1 + \begin{pmatrix} H_+ & H_+ \\ H_- & H_- \end{pmatrix} \qquad \pi Y = Y$$

$$\cancel{1} - B + Y - BY \in \cancel{1} + \quad ''$$

$$-\pi B + Y - \pi B Y = 0$$

$$(1-\pi B)\,Y = \pi B 1 \qquad Y = \pi B 1 + (\pi B)^2 1 + \cdots$$

$$1+Y = 1 + \pi B 1 + (\pi B)^2 1 + \cdots$$

---

alternative $\beta = \frac{b}{d}$, $\gamma = -\frac{c}{d}$

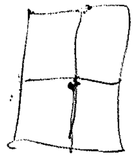
$$\begin{pmatrix} 1 & \frac{b}{a} \\ \frac{c}{d} & 1 \end{pmatrix}\begin{pmatrix} d^l & -b^l \\ -c^l & a^l \end{pmatrix} = \begin{pmatrix} \frac{a^r}{a} & \frac{b^r}{a} \\ \frac{c^r}{d} & \frac{d^r}{d} \end{pmatrix} \in \begin{pmatrix} 1+H_- & H_- \\ H_+ & 1+H_+ \end{pmatrix}$$

$$\begin{pmatrix} 1 & -\pi_+\gamma \\ -\pi_-\bar\gamma & 1 \end{pmatrix}\begin{pmatrix} & '' & \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

~~Further work~~ Next on the scattering picture

$$\begin{pmatrix} d & b \\ -\bar b & d \end{pmatrix} = \begin{pmatrix} a^r & b^r \\ -c^l & a^l \end{pmatrix}_- \begin{pmatrix} d^r & b^l \\ -c^r & d^l \end{pmatrix}_+$$

$$\begin{pmatrix}1 & b\\ -b & 1\end{pmatrix}\begin{pmatrix}d^l & -b^l\\ +c^r & d^r\end{pmatrix}\frac{1}{d} = \begin{pmatrix}a^r & b^r\\ -c^l & a^l\end{pmatrix}$$



~~So you~~ Anyway you have a mess of matrices and no understanding. Look at the scattering picture.

$$\begin{pmatrix}p_0\\ q_0\end{pmatrix} = \frac{1}{d}\begin{pmatrix}d^r & b^l\\ -c^r & d^l\end{pmatrix}\begin{pmatrix}\xi'_-\\ \xi_-\end{pmatrix} = \frac{1}{a}\begin{pmatrix}a^l & -b^r\\ c^l & a^r\end{pmatrix}\begin{pmatrix}\xi_+\\ \xi'_+\end{pmatrix}$$

$$\begin{pmatrix}1 & b\\ -c & 1\end{pmatrix} = \begin{pmatrix}a^r & b^r\\ -c^l & a^l\end{pmatrix}\begin{pmatrix}d^r & b^l\\ -c^r & d^l\end{pmatrix}$$

One thing you might understand now is why the S matrix $S=\begin{pmatrix}\frac{1}{d} & \frac{b}{d}\\ -\frac{c}{d} & \frac{1}{d}\end{pmatrix}$ has a Birkhoff factorization

The reason should be that the Toeplitz operator

$$\pi_+ S : H^2_+ \longrightarrow H^2_+$$ is invertible.

$$\boxed{\underbrace{\begin{pmatrix}\frac{1}{d} & \frac{b}{d}\\ -\frac{c}{d} & \frac{1}{d}\end{pmatrix}}_{S}\underbrace{\begin{pmatrix}d^l & -b^l\\ -c^r & d^r\end{pmatrix}}_{g_+} = \underbrace{\begin{pmatrix}a^r & b^r\\ -c^l & a^l\end{pmatrix}}_{g_-}}$$

Question: Why solvable. $\delta g = g-1.$

$$\underbrace{(1+\delta S)\,(1+\delta g_+)} = (1+\delta g_-)$$

$$1+\delta S+\delta g_+ + \delta S\,\delta g_+ = 1+\delta g_-$$

$$1+\pi_+\delta S + \delta g_+ + \pi_+\delta S\,\delta g_+ = 1$$

$$\pi_+ S\, g_+ = 1$$

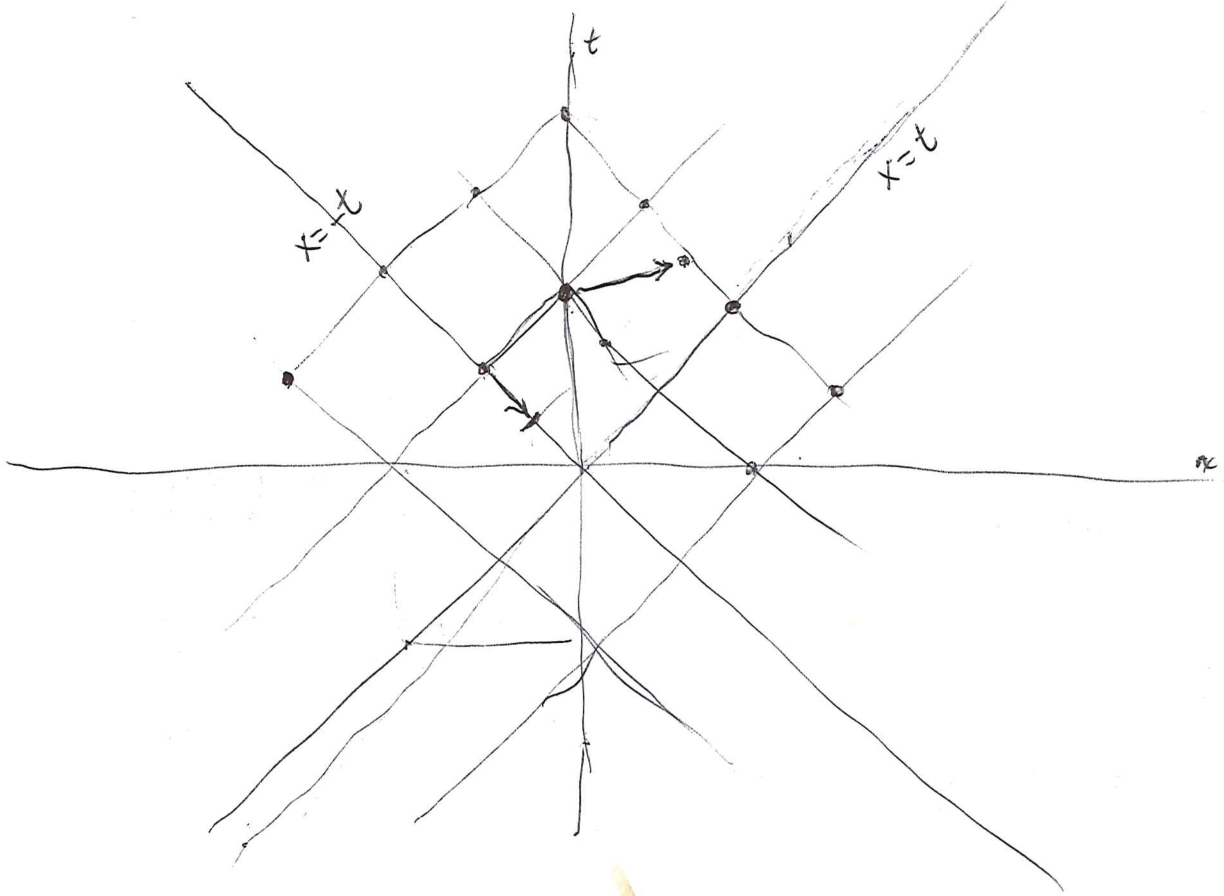
You seem to be involved with F, g

Can you use the Poincaré group = Lorentz ⋉ translations. ~~Poincaré~~

Begin with Hilbert space ~~simplest model~~

$\ell^2\{\mathbb{Z}\}_+ \oplus \ell^2\{\mathbb{Z}\}_-$

Need general formula for Lorentz transformations.



$$\begin{pmatrix}x\\t\end{pmatrix} = \frac{1}{2}\begin{pmatrix}1&-1\\1&1\end{pmatrix}\begin{pmatrix}1&1\\-1&1\end{pmatrix}\begin{pmatrix}x\\t\end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix}1&-1\\1&1\end{pmatrix}\begin{pmatrix}x+t\\-x+t\end{pmatrix}$$

$\longrightarrow$

$$\begin{pmatrix}x'\\t'\end{pmatrix} = \frac{1}{2}\begin{pmatrix}1&-1\\1&1\end{pmatrix}\begin{pmatrix}\alpha&0\\0&\alpha^{-1}\end{pmatrix}\begin{pmatrix}x+t\\-x+t\end{pmatrix}$$

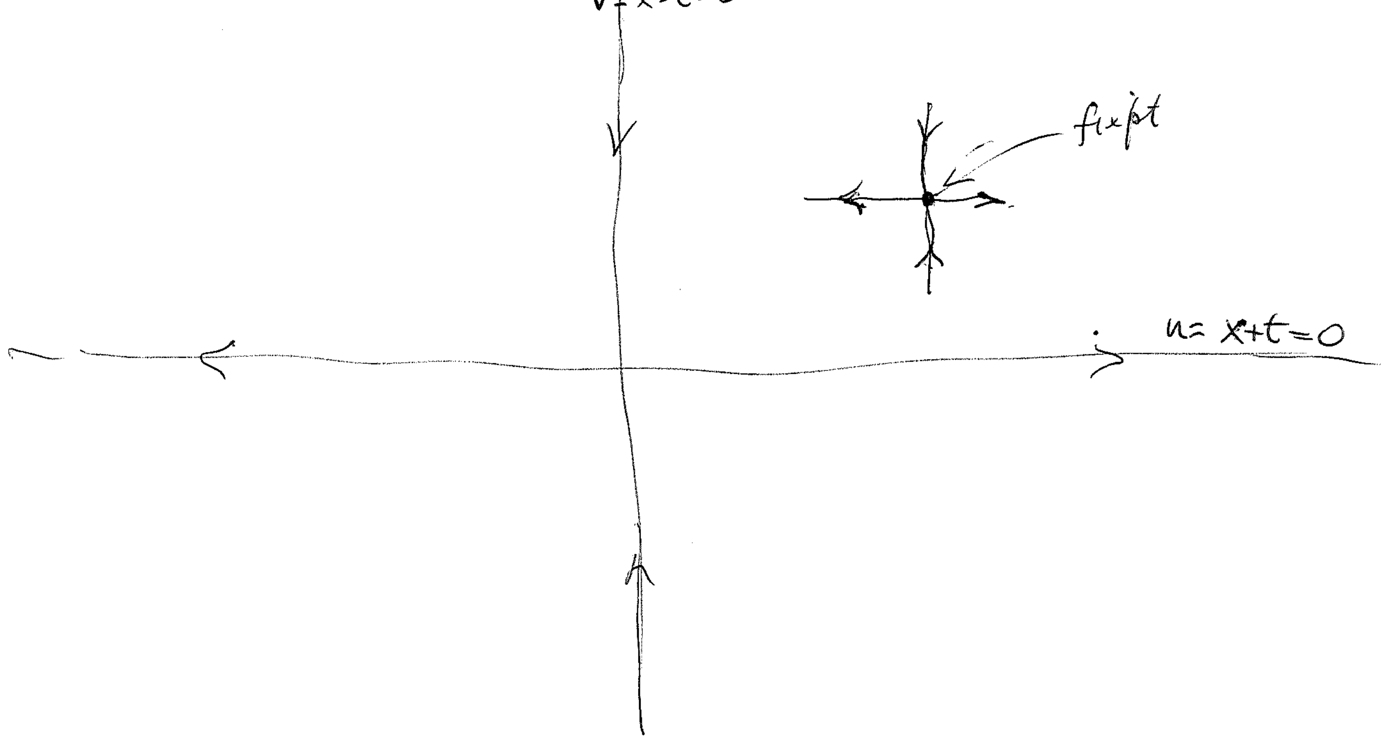
$$\begin{pmatrix} x' \\ t' \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} x \\ t \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} \alpha & \alpha \\ -\alpha^{-1} & \alpha^{-1} \end{pmatrix} = \begin{pmatrix} \frac{\alpha+\alpha^{-1}}{2} & \frac{\alpha-\alpha^{-1}}{2} \\ \frac{\alpha-\alpha^{-1}}{2} & \frac{\alpha+\alpha^{-1}}{2} \end{pmatrix}\begin{pmatrix} x \\ t \end{pmatrix}$$

$$\begin{pmatrix} x' \\ t' \end{pmatrix} = \begin{pmatrix} d & c \\ c & d \end{pmatrix}\begin{pmatrix} x \\ t \end{pmatrix} \quad \text{where } d^2 - c^2 = 1.$$

$$\begin{pmatrix} x' \\ t' \end{pmatrix} = \begin{pmatrix} \quad & \quad \\ \quad & \quad \end{pmatrix}\begin{pmatrix} x \\ t \end{pmatrix} + \begin{pmatrix} x_0 \\ t_0 \end{pmatrix}$$

$$\begin{pmatrix} x'+t' \\ -x'+t' \end{pmatrix} = \begin{pmatrix} \overset{\alpha}{d+c} & 0 \\ 0 & \underset{\alpha^{-1}}{d-c} \end{pmatrix}\begin{pmatrix} x+t \\ -x+t \end{pmatrix} + \begin{pmatrix} x_0+t_0 \\ -x_0+t_0 \end{pmatrix}$$



$$u' = \alpha u + u_0$$
$$v' = \alpha^{-1} v + v_0$$
$$u' - u_f = \alpha(u - u_f)$$
$$v' - v_f = \alpha^{-1}(v - v_f)$$

fixpoint $u_f = \alpha u_f + u_0$

$$u_f = \frac{1}{1-\alpha} u_0$$

$$v_f = \alpha^{-1} v_f + v_0 \qquad v_f = \frac{1}{1-\alpha^{-1}} v_0$$

Consider $v \longmapsto Av + v_0$ bij on $\mathbb{Z}^2$

then $v_0 \in \mathbb{Z}^2$ and A auto on $\mathbb{Z}^2$ say $\det A = 1$.

Fixpt. $v_f = Av_f + v_0$

$v_f = (1-A)^{-1} v_0$ Q: is $v_f \in \mathbb{Z}^2$?

$A^2 - \overbrace{tr(A)}^{t} A + 1 = 0$ $\lambda = \frac{+t \pm \sqrt{t^2-4}}{2}$

$$\begin{array}{r|l} & A + (-t+1) \\ \hline A-1 & A^2 - tA + 1 \\ & A^2 - A \\ \hline & (-t+1)A + 1 \\ & (-t+1)A - (-t+1) \\ \hline & -t+2 \end{array} \qquad \begin{array}{r|l} & A + (1-t) \\ \hline A-1 & A^2 - tA + 1 \\ & A^2 - A \\ \hline & (1-t)A + 1 \\ & (1-t)A - (1-t) \\ \hline & 2-t \end{array}$$

$(A-1)(A+1-t) = (A^2-tA+1) - (2-t)$

for this matrix you have

$(A-1)(A+1-t) = t-2$

$\therefore (A-1)^{-1} = \frac{A+1-t}{t-2}$

want $\gamma \in SL_2(\mathbb{Z})$ $\qquad \lambda^2 - t\lambda + 1 = 0 \qquad t = \mathrm{tr}(\gamma)$.

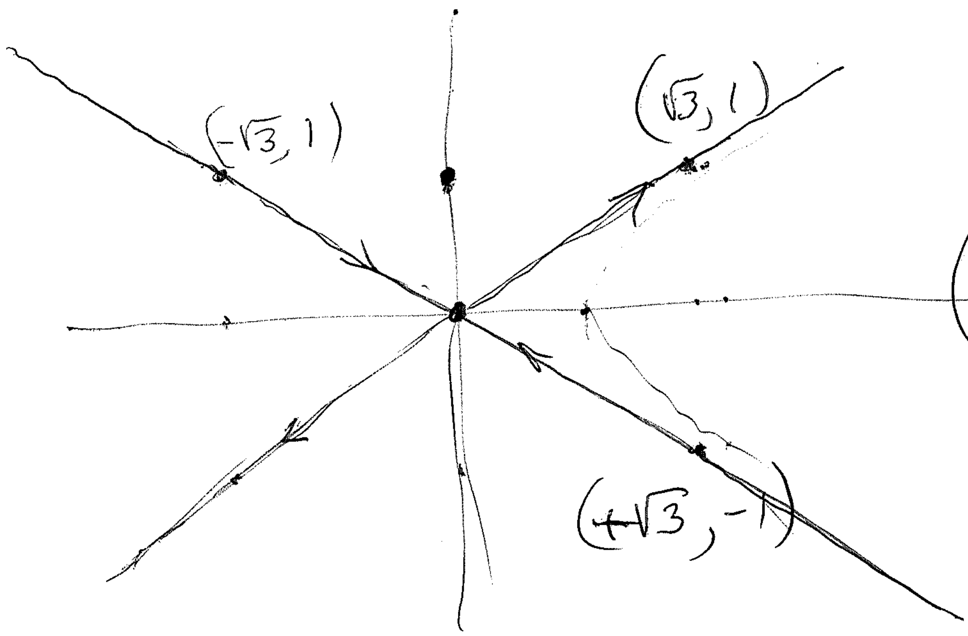
$t = 3 \quad \lambda = \dfrac{3 \pm \sqrt{5}}{2} \qquad \lambda = \dfrac{t \pm \sqrt{t^2-4}}{2}$

$$\begin{pmatrix} a-\lambda & b \\ c & d-\lambda \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0 \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} & \\ & \end{pmatrix}$$

$t = 4. \quad \lambda = 2 \pm \sqrt{3} \qquad \gamma = \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}$

$\lambda = 2+\sqrt{3} \quad \begin{pmatrix} -\sqrt{3} & 3 \\ 1 & -\sqrt{3} \end{pmatrix}\begin{pmatrix} \sqrt{3} \\ 1 \end{pmatrix} = 0 \qquad \lambda = 2+\sqrt{3} \longrightarrow \begin{pmatrix} \sqrt{3} \\ 1 \end{pmatrix}$

$\lambda = 2-\sqrt{3} \longrightarrow \begin{pmatrix} -\sqrt{3} \\ 1 \end{pmatrix}$



$$\begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} \sqrt{3} \\ -1 \end{pmatrix} = \begin{pmatrix} 2\sqrt{3}-3 \\ \sqrt{3}-2 \end{pmatrix} = (2-\sqrt{3})\begin{pmatrix} \sqrt{3} \\ -1 \end{pmatrix}$$

$$(A - \cancel{1})^{-1} = \begin{pmatrix} 1 & 3 \\ 1 & 1 \end{pmatrix}^{-1} = \frac{1}{-2}\begin{pmatrix} 1 & -3 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} & \frac{3}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix}$$

These eigenlines are the characteristics?.

What you want next is affine transf. $v \mapsto Av + v_0$ preserving $\mathbb{Z}^2$, i.e. $v_0 \in \mathbb{Z}^2$. Get $v_f = Av_f + v_0$ or $v_f = (1-A)^{-1}v_0$. want $v_f \in \mathbb{Z}^2$.

Recall $\gamma = \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix} \in SL_2(\mathbb{Z})$ acts on $\mathbb{Z}^2 \subset \mathbb{R}^2$

characteristics?

Back to ~~all~~ int equations

$$\begin{pmatrix} p_0 \\ q_0 \end{pmatrix} = \begin{pmatrix} a^l & b^l \\ c^l & d^l \end{pmatrix} \begin{pmatrix} \xi'_- \\ \xi'_+ \end{pmatrix} = \begin{pmatrix} d^r & -b^r \\ -c^r & a^r \end{pmatrix} \begin{pmatrix} \xi_+ \\ \xi_- \end{pmatrix}$$

$$= \frac{1}{d} \begin{pmatrix} d^r & b^l \\ -c^r & d^l \end{pmatrix} \begin{pmatrix} \xi'_- \\ \xi_- \end{pmatrix} = \frac{1}{a} \begin{pmatrix} a^l & -b^r \\ c^l & a^r \end{pmatrix} \begin{pmatrix} \xi_+ \\ \xi'_+ \end{pmatrix}$$

Constructing ~~~~ the factorization

$$\begin{pmatrix} 1 & b \\ -\bar{b} & 1 \end{pmatrix} = \begin{pmatrix} a^r & b^r \\ -c^l & a^l \end{pmatrix} \begin{pmatrix} d^r & b^l \\ -c^r & d^l \end{pmatrix} \qquad S = g_- g_+^{-1}$$

~~~~

$$\begin{pmatrix} 1 & b \\ -\bar{b} & 1 \end{pmatrix} \underbrace{\frac{1}{d} \begin{pmatrix} d^l & -b^l \\ c^r & d^r \end{pmatrix}} = \begin{pmatrix} a^r & b^r \\ -c^l & a^l \end{pmatrix}$$

$$\in \begin{pmatrix} 1+H_+ & H_+ \\ H_+ & 1+H_+ \end{pmatrix} \qquad \begin{pmatrix} 1+H_- & H_- \\ H_- & 1+H_- \end{pmatrix}$$

~~~~

First look at

$$\begin{pmatrix} \frac{1}{d} & \frac{b}{d} \\ -\frac{\bar{b}}{d} & \frac{1}{d} \end{pmatrix} \begin{pmatrix} d^l & -b^l \\ c^r & d^r \end{pmatrix} = \begin{pmatrix} a^r & b^r \\ -c^l & a^l \end{pmatrix}$$

$$S \quad g_+ \quad = \quad g_-$$

~~what~~

$$\Pi_+ S g_+ = 1$$

you need to find a clean setting, notation for handling this. 1 is misleading.

~~$\frac{1}{d}\pi_+ \, f_+$~~

$$\frac{1}{d}\pi_+\begin{pmatrix}1 & b\\ -\bar{b} & 1\end{pmatrix} g_+ = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$$

YES

OKAY

$\pi_+ \, d f_-$ $\qquad\qquad d f_- = g_+ + g_-$

$$\begin{pmatrix}p_0\\ q_0\end{pmatrix} = \begin{pmatrix}a^\ell & b^\ell\\ c^\ell & d^\ell\end{pmatrix}\begin{pmatrix}\xi_-'\\ \xi_+'\end{pmatrix} = \begin{pmatrix}d^r & -b^r\\ -c^r & a^r\end{pmatrix}\begin{pmatrix}\xi_+\\ \xi_-\end{pmatrix}$$

$$= \frac{1}{d}\begin{pmatrix}d^r & b^\ell\\ -c^r & d^\ell\end{pmatrix}\begin{pmatrix}\xi_-'\\ \xi_-\end{pmatrix} = \frac{1}{a}\begin{pmatrix}a^\ell & -b^r\\ c^\ell & a^r\end{pmatrix}\begin{pmatrix}\xi_+\\ \xi_+'\end{pmatrix}$$

$$\begin{pmatrix}\xi_-'\\ \xi_-\end{pmatrix} = \begin{pmatrix}d^\ell & -b^\ell\\ c^r & d^r\end{pmatrix}\begin{pmatrix}p_0\\ q_0\end{pmatrix} \qquad \begin{pmatrix}\xi_+\\ \xi_+'\end{pmatrix} = \begin{pmatrix}a^r & b^r\\ -c^\ell & a^\ell\end{pmatrix}\begin{pmatrix}p_0\\ q_0\end{pmatrix}$$

$$\begin{pmatrix}\frac{1}{d} & \frac{b}{d}\\ -\frac{c}{d} & \frac{1}{d}\end{pmatrix} = \begin{pmatrix}a^r & b^r\\ -c^\ell & a^\ell\end{pmatrix}\frac{1}{d}\begin{pmatrix}d^r & b^\ell\\ -c^r & d^\ell\end{pmatrix}$$

$$\boxed{\begin{pmatrix}\frac{1}{d} & \frac{b}{d}\\ -\frac{c}{d} & \frac{1}{d}\end{pmatrix}\begin{pmatrix}d^\ell & -b^\ell\\ c^r & d^r\end{pmatrix} = \begin{pmatrix}a^r & b^r\\ -c^\ell & a^\ell\end{pmatrix}}$$

$$S \quad g_+ = g_-$$

You want to show certain things are equiv.

existence of factorization

splitting of E into $H_+\xi_-' + H_+\xi_-$ and $H_-\xi_+ + H_-\xi_+'$

$$H_+\xi_-' + H_+\xi_- = (H_+ \; H_+)\begin{pmatrix} d^\ell & -b^\ell \\ c^r & a^r\end{pmatrix}\begin{pmatrix} p_0 \\ q_0\end{pmatrix}$$

$$H_-\xi_+ + H_-\xi_+' = (H_- \; H_-)\begin{pmatrix} a^r & b^r \\ -c^\ell & a^\ell\end{pmatrix}\begin{pmatrix} p_0 \\ q_0\end{pmatrix}$$

$(H_+ \; H_+)\, g_+ = (H_+ \; H_+)$ so the complementarity is clear. $\therefore$ $\exists$ Birkhoff fact of $S \Longrightarrow$ desired splitting of E.

Conversely assume complements.

$$(H_- \; H_-)\, S \oplus (H_+ \; H_+) = (L^2 \; L^2)$$

Better: Why not use the argument that gives Birkhoff fact.

~~Birkhoff fact. in Whitehall.~~

Distinguish between operators and the space on which they act.

~~You start You work inside~~

Problem. Given b on the circle ~~and~~ get

$S = \begin{pmatrix} \frac{1}{d} & \frac{b}{d} \\ -\frac{c}{d} & \frac{1}{d}\end{pmatrix}$. and you want to construct the factorization

$$\begin{pmatrix} \frac{1}{d} & \frac{b}{d} \\ -\frac{c}{d} & \frac{1}{d}\end{pmatrix}\begin{pmatrix} d^\ell & -b^\ell \\ c^r & d^r\end{pmatrix} = \begin{pmatrix} a^r & b^r \\ -c^\ell & a^\ell\end{pmatrix}$$

$$S\, g_+ = g_-$$

Take first column, i.e. $\begin{pmatrix} p_0 \\ q_0\end{pmatrix} = \begin{pmatrix} 1 \\ 0\end{pmatrix}$ right mult by.

To solve

$$\begin{pmatrix} \frac{1}{d} & \frac{b}{d} \\ -\frac{c}{d} & \frac{1}{d}\end{pmatrix}\begin{pmatrix} d^\ell \\ c^r\end{pmatrix} = \begin{pmatrix} a^r \\ -c^\ell\end{pmatrix}$$

with $\begin{pmatrix} d^\ell \\ c^r\end{pmatrix} \equiv \begin{pmatrix} 1 \\ 0\end{pmatrix} \mod H_+$

etc.

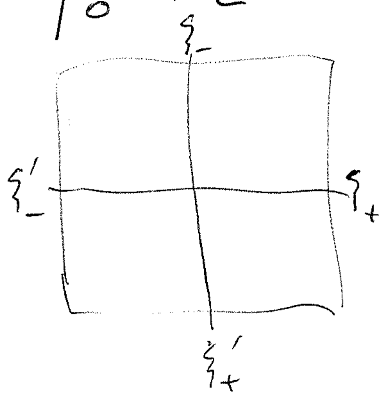
first method: to solve

$$\begin{pmatrix} 1 & b \\ -\bar{b} & 1 \end{pmatrix}\begin{pmatrix} d^\ell/d \\ c^r/d \end{pmatrix} = \begin{pmatrix} a^r \\ -c^\ell \end{pmatrix}$$

Do these equations have a useful interpretation?

Yes you are trying to construct

$p_0 \in E$



$$p_0 \in (1+H_-)\xi_+ + H_-\xi'_+$$
$$\in (1+H_+)\xi'_- + H_+\xi_-$$

$$\xi_+ a^r + \xi'_+(-c^\ell)$$
$$\xi'_- d^\ell + \xi_- c^r$$

You are looking at the outgoing picture

ξ'_-

$$p_0 = a^r\xi_+ - c^\ell\xi'_+ \in (1+H_-)\xi_+ + H_-\xi'_+$$
$$= d^\ell\xi'_- + c^r\xi_- \in (1+H_+)\xi'_- + H_+\xi_-$$

$$p_0 = \begin{pmatrix} d^\ell & c^r \end{pmatrix}\begin{pmatrix} \xi'_- \\ \xi_- \end{pmatrix} = \begin{pmatrix} d^\ell & c^r \end{pmatrix}\begin{pmatrix} \frac{1}{a} & -\frac{b}{a} \\ \frac{c}{a} & \frac{1}{a} \end{pmatrix}\begin{pmatrix} \xi_+ \\ \xi'_+ \end{pmatrix}$$

$$= \begin{pmatrix} a^r & -c^\ell \end{pmatrix}\begin{pmatrix} \xi_+ \\ \xi'_+ \end{pmatrix}$$

$$\begin{pmatrix} \frac{1}{a} & \frac{c}{a} \\ -\frac{b}{a} & \frac{1}{a} \end{pmatrix}\begin{pmatrix} d^\ell \\ c^r \end{pmatrix} = \begin{pmatrix} a^r \\ -c^\ell \end{pmatrix} \quad ?$$

$$p_0 = \begin{pmatrix} a^r & -c^\ell \end{pmatrix}\begin{pmatrix} \xi_+ \\ \xi'_+ \end{pmatrix} = \begin{pmatrix} a^r & -c^\ell \end{pmatrix}\begin{pmatrix} \frac{1}{d} & \frac{b}{d} \\ -\frac{c}{d} & \frac{1}{d} \end{pmatrix}\begin{pmatrix} \xi'_- \\ \xi_- \end{pmatrix} = \begin{pmatrix} d^\ell & c^r \end{pmatrix}\begin{pmatrix} \xi'_- \\ \xi_- \end{pmatrix}$$

$$\begin{pmatrix} d^\ell \\ c^r \end{pmatrix} = \begin{pmatrix} \frac{1}{d} & -\frac{c}{d} \\ \frac{b}{d} & \frac{1}{d} \end{pmatrix}\begin{pmatrix} a^r \\ -c^\ell \end{pmatrix} \quad ?$$

$$\begin{pmatrix} p_0 \\ q_0 \end{pmatrix} = \begin{pmatrix} a^l & b^l \\ c^l & d^l \end{pmatrix}\begin{pmatrix} \xi'_- \\ \xi'_+ \end{pmatrix} = \begin{pmatrix} d^r & -b^r \\ -c^r & a^r \end{pmatrix}\begin{pmatrix} \xi_+ \\ \xi_- \end{pmatrix}$$

$$= \frac{1}{d}\begin{pmatrix} d^r & b^l \\ -c^r & d^l \end{pmatrix}\begin{pmatrix} \xi'_- \\ \xi_- \end{pmatrix} = \frac{1}{a}\begin{pmatrix} a^l & -b^r \\ c^l & a^r \end{pmatrix}\begin{pmatrix} \xi_+ \\ \xi'_+ \end{pmatrix}$$

Check this.

$$\begin{pmatrix} \xi_+ \\ \xi'_+ \end{pmatrix} = \begin{pmatrix} \frac{1}{d} & \frac{b}{d} \\ -\frac{c}{d} & \frac{1}{d} \end{pmatrix}\begin{pmatrix} \xi'_- \\ \xi_- \end{pmatrix}$$

$$\begin{pmatrix} \xi'_- \\ \xi_- \end{pmatrix} = \begin{pmatrix} \frac{1}{a} & -\frac{b}{a} \\ \frac{c}{a} & \frac{1}{a} \end{pmatrix}\begin{pmatrix} \xi_+ \\ \xi'_+ \end{pmatrix}$$

$$\begin{pmatrix} p_0 \\ q_0 \end{pmatrix} = \begin{pmatrix} a^l & b^l \\ c^l & d^l \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -\frac{c}{d} & \frac{1}{d} \end{pmatrix}\begin{pmatrix} \xi'_- \\ \xi_- \end{pmatrix}$$

$$\begin{pmatrix} d^r & -b^r \\ -c^r & d^r \end{pmatrix} = \begin{pmatrix} a^l & b^l \\ c^l & d^l \end{pmatrix}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

$$= \frac{1}{d}\begin{pmatrix} \overset{d^r}{a^l d - b^l c} & b^l \\ \underset{-c^r}{c^l d - d^l c} & d^l \end{pmatrix}\begin{pmatrix} \xi'_- \\ \xi_- \end{pmatrix}$$

$$p_0 = \frac{d^r}{d}\xi'_- + \frac{b^l}{d}\xi_- = \frac{a^l}{a}\xi_+ - \frac{b^r}{a}\xi'_+$$

$$\frac{1}{d}\begin{pmatrix} d^r \\ +b^l \end{pmatrix}^t \begin{pmatrix} \frac{1}{a} & -\frac{b}{a} \\ \frac{c}{a} & \frac{1}{a} \end{pmatrix} = \begin{pmatrix} \frac{a^l}{a} & -\frac{b^r}{a} \end{pmatrix}$$

$$\begin{pmatrix} \frac{1}{a} & \frac{c}{a} \\ -\frac{b}{a} & \frac{1}{a} \end{pmatrix}\frac{1}{d}\begin{pmatrix} d^r \\ +b^l \end{pmatrix} = \frac{1}{a}\begin{pmatrix} a^l \\ -b^r \end{pmatrix}$$

$$\begin{pmatrix} \xi_+ \\ \xi'_+ \end{pmatrix} = \frac{1}{d}\begin{pmatrix} 1 & b \\ -c & 1 \end{pmatrix}\begin{pmatrix} \xi'_- \\ \xi_- \end{pmatrix} \qquad \begin{pmatrix} \xi_+ \\ \xi_- \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} \xi'_- \\ \xi'_+ \end{pmatrix}$$

$$\begin{pmatrix} \xi_+ & \xi'_+ \end{pmatrix}\begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} \xi'_- & \xi_- \end{pmatrix} \underbrace{\frac{1}{d}\begin{pmatrix} 1 & -\bar b \\ b & 1 \end{pmatrix}}_{S}\begin{pmatrix} f \\ g \end{pmatrix} \qquad S = \frac{1}{d} B \varepsilon$$

Claim

$$\begin{array}{c} E_+ \\ \oplus \\ E_- \end{array} \xrightarrow{\begin{pmatrix} j_+ & S j_- \end{pmatrix}} E \xrightarrow{\begin{pmatrix} (j_+^* B j_+)^{-1} j_+^* B \\ (j_-^* S)^{-1} j_-^* \end{pmatrix}} \begin{array}{c} E_+ \\ \oplus \\ E_- \end{array}$$

$$\begin{tikzcd} E_+ \arrow[r, "j_+"] \arrow[dr, "(j_+^* B j_+)"'] & E \arrow[d, "j_+^* B"] \\ & E_+ \end{tikzcd}$$

$$j_+^* B S j_- = j_+^* \underbrace{B \frac{1}{d} B}_{\frac{1+|b|^2}{d} = \bar d} j_- \varepsilon = 0$$

$$\begin{pmatrix} (j_+^* B j_+)^{-1} j_+^* B \\ (j_-^* S)^{-1} j_-^* \end{pmatrix} \begin{pmatrix} j_+ & S j_- \end{pmatrix} = \begin{pmatrix} 1_{E_+} & 0 \\ 0 & 1_{E_-} \end{pmatrix}$$

$$j_- S^* j_+ = j_- \varepsilon B \frac{1}{d} j_+$$

$$\begin{pmatrix} j_+ & S j_- \end{pmatrix} \begin{pmatrix} (j_+^* B j_+)^{-1} j_+^* B \\ (j_- S)^{-1} j_- \end{pmatrix}$$

$$\begin{tikzcd}
 & E_- \arrow[d, "S j_-"] \arrow[dr, "j_-^* S j_-", "\sim"'] & \\
E_+ \arrow[r, "j_+"] \arrow[dr, "(j_+^* B j_+)"', "\sim"] & E \arrow[r, "j_-^*"] \arrow[d, "j_+^* B"] & E_- \\
 & E_+ &
\end{tikzcd}$$

$$j_+^* B \begin{pmatrix} \phi \\ \psi \end{pmatrix} = 0$$

$$\begin{pmatrix} 1 & -\bar{b} \\ b & 1 \end{pmatrix} \begin{pmatrix} \phi \\ \psi \end{pmatrix} \in \begin{pmatrix} H_- \\ H_- \end{pmatrix}$$

$$\Downarrow$$

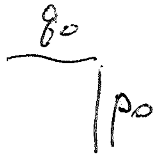
$$\begin{pmatrix} \phi \\ \psi \end{pmatrix} \in \frac{1}{d} \begin{pmatrix} 1 & \bar{b} \\ -b & 1 \end{pmatrix} \begin{pmatrix} H_- \\ H_- \end{pmatrix} \Longleftrightarrow \begin{pmatrix} \phi \\ \psi \end{pmatrix} \in \frac{1}{1+|b|^2} \begin{pmatrix} 1 & \bar{b} \\ -b & 1 \end{pmatrix} \begin{pmatrix} H_- \\ H_- \end{pmatrix}$$

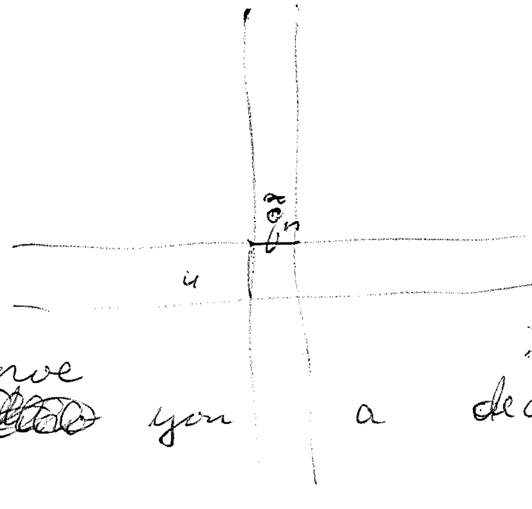
$$S = \frac{1}{d} \begin{pmatrix} 1 & -\bar{b} \\ b & 1 \end{pmatrix} \qquad S^* = \frac{1}{d} \begin{pmatrix} 1 & \bar{b} \\ -b & 1 \end{pmatrix}$$

Go back to discrete case. You problem is How ~~to construct~~ the Birkhoff factorization. You ~~can~~ construct ~~two~~ operators ~~_~~ in the algebra gen. by adjoining the Hilbert transform to functions. Suppose you construct ~~$p_0 q_0$~~

p_0, q_0 as you want.

~~What~~ you need





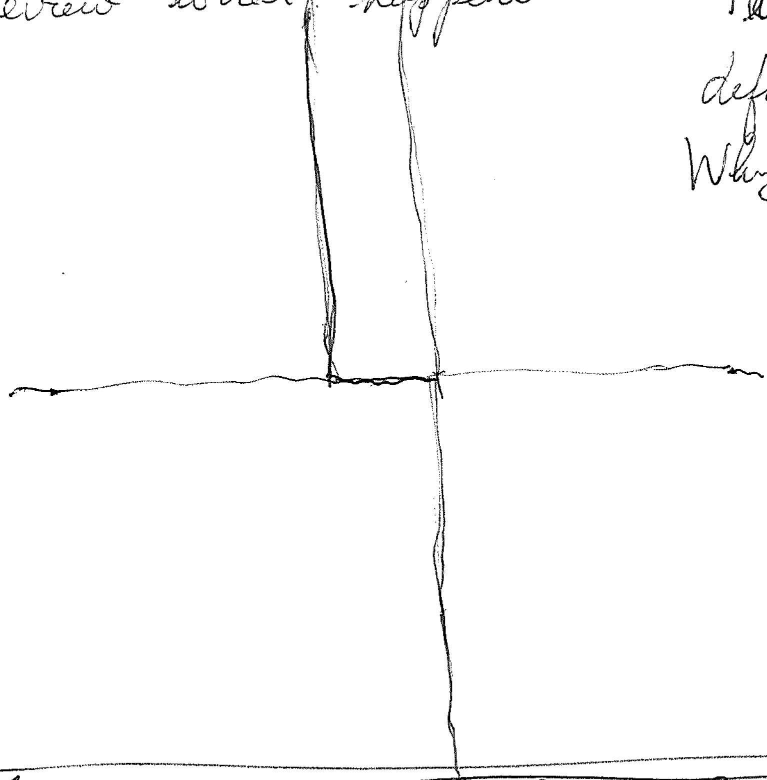
then the orthogonality should give ~~_~~ you a decreasing staircase

Certainly

$$\begin{tikzcd} & E_- \arrow[d, "s_{j_-}"] \arrow[dr, "j_-^* s_{j_-}", "\sim"'] & \\ E_+ \arrow[r, "j_+"] \arrow[dr, "j_+^* B_{j_+}"', "\sim"] & E \arrow[r, "j_-^*"] \arrow[d, "j_+^* B"] & E_- \\ & E_+ & \end{tikzcd}$$

is correct, and describes the splitting of E as well as can be expected. But you still need Birkhoff.

Review what happens?



First problem: 4 ways to define an edge by orthogonality. Why equivalent?

~~Next problem: What goes on~~

Start over again by going way back over the orthogonal projection method. Want to write up the details in a good form. You hope to avoid normalization

Review scattering formulas: b on the circle given

$$\begin{pmatrix} \xi_+ \\ \xi_- \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \xi_-' \\ \xi_+' \end{pmatrix} \qquad \begin{pmatrix} \xi_-' \\ \xi_+' \end{pmatrix} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \begin{pmatrix} \xi_+ \\ \xi_- \end{pmatrix}$$

$$\begin{pmatrix} \xi_+ \\ \xi_+' \end{pmatrix} = \begin{pmatrix} \frac{1}{d} & \frac{b}{d} \\ -\frac{c}{d} & \frac{1}{d} \end{pmatrix} \begin{pmatrix} \xi_-' \\ \xi_- \end{pmatrix} \qquad \begin{pmatrix} \xi_-' \\ \xi_- \end{pmatrix} = \begin{pmatrix} \frac{1}{a} & -\frac{b}{a} \\ \frac{c}{a} & \frac{1}{a} \end{pmatrix} \begin{pmatrix} \xi_+ \\ \xi_+' \end{pmatrix}$$

Want formulas for $(\ |\ )$ and $IH(\ ,\ )$.

$$IH\left((\xi'_-\ \xi_-)\begin{pmatrix} f \\ g\end{pmatrix}\right) \qquad \begin{pmatrix} \xi'_- \\ \xi_-\end{pmatrix} = \begin{pmatrix} d & -b \\ 0 & 1\end{pmatrix}\begin{pmatrix} \xi_+ \\ \xi_-\end{pmatrix}$$

$$= IH\left((\xi_+\ \xi_-)\begin{pmatrix} d & 0 \\ -b & 1\end{pmatrix}\begin{pmatrix} f \\ g\end{pmatrix}\right)$$

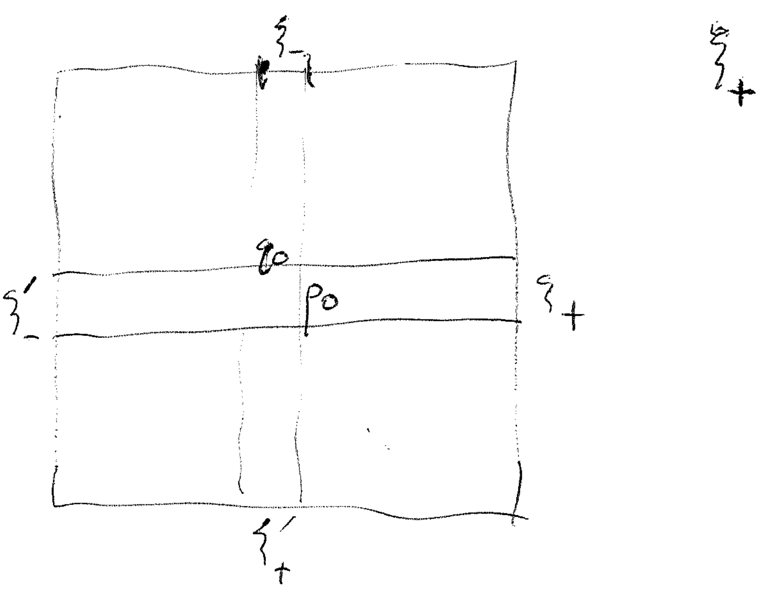
$$= IH\left((\xi_+\ \xi_-)\begin{pmatrix} df \\ -bf+g\end{pmatrix}\right) = \|df\|^2 - \|-bf+g\|^2 \qquad \begin{pmatrix} \xi_+ \\ \xi_-\end{pmatrix} = \begin{pmatrix} \frac{1}{d} & \frac{b}{d} \\ 0 & 1\end{pmatrix}\begin{pmatrix} \xi'_- \\ \xi_-\end{pmatrix}$$

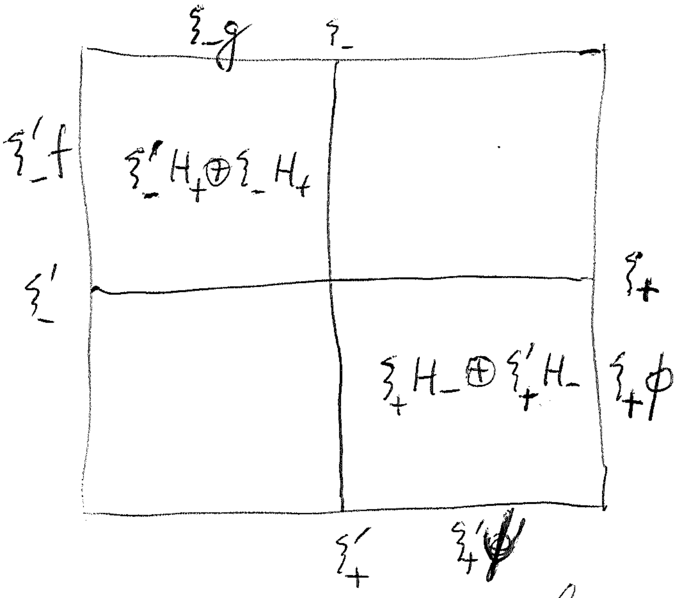
$$= \int \begin{pmatrix} f \\ g\end{pmatrix}^* \begin{pmatrix} 1 & \bar{b} \\ b & -1\end{pmatrix}\begin{pmatrix} f \\ g\end{pmatrix} \qquad (\xi_+\ \xi_-) = (\xi'_-\ \xi_-)\begin{pmatrix} \frac{1}{d} & 0 \\ \frac{b}{d} & 1\end{pmatrix}$$

$$\left\|(\xi_+\ \xi_-)\begin{pmatrix} f \\ g\end{pmatrix}\right\|^2 = \left\|(\xi'_-\ \xi_-)\begin{pmatrix} \frac{1}{d} & 0 \\ \frac{b}{d} & 1\end{pmatrix}\begin{pmatrix} f \\ g\end{pmatrix}\right\| = \left\|(\xi'_-\ \xi_-)\begin{pmatrix} \frac{1}{d}f \\ \frac{b}{d}f+g\end{pmatrix}\right\|^2$$

$$= \left\|\frac{1}{d}f\right\|^2 + \Big\|\underbrace{\frac{b}{d}}_{\beta}f+g\Big\|^2 = \int \begin{pmatrix} f \\ g\end{pmatrix}^* \begin{pmatrix} 1 & \beta^* \\ \beta & 1\end{pmatrix}\begin{pmatrix} f \\ g\end{pmatrix}$$

$$\boxed{\begin{aligned} IH(\xi'_- f + \xi_- g) &= \int \begin{pmatrix} f \\ g\end{pmatrix}^* \begin{pmatrix} 1 & \bar{b} \\ b & -1\end{pmatrix}\begin{pmatrix} f \\ g\end{pmatrix} \\ \|\xi_+ f + \xi_- g\|^2 &= \int \begin{pmatrix} f \\ g\end{pmatrix}^* \begin{pmatrix} 1 & \beta^* \\ \beta & 1\end{pmatrix}\begin{pmatrix} f \\ g\end{pmatrix} \qquad \beta = \frac{b}{d}\end{aligned}}$$





vertical vector v at the origin

$$\boxed{\begin{aligned} v_1 &= \xi_+(1-\phi) + \xi'_+(-\psi) \\ v_2 &= \xi'_-(1-f) + \xi_-(-g) \end{aligned}}$$

What to do? Suppose you can solve the orthogonality integral eqns. this means ~~...~~ certain linear functionals are bdd.

$$IH(\xi_+, -) \quad \text{bdd on } (\xi_+ \ \xi'_+)(E_-)$$

$$IH(\xi'_-, -) \quad \text{———} \quad (\xi'_- \ \xi_-)(E_+).$$

If bounded, then get well-defined elements

$$\begin{pmatrix}\phi \\ \psi\end{pmatrix} \in E_- \qquad \begin{pmatrix} f \\ g\end{pmatrix} \in E_+ .$$

You also want $v_1 = v_2$

i.e. $S\begin{pmatrix}1-\phi \\ -\psi\end{pmatrix} = \begin{pmatrix}1-f \\ -g\end{pmatrix}$.

Similarly you need the corresp. things ~~vertically~~ horizontally.

$$\begin{aligned} w_1 &= \xi'_-(-g_1) + \xi_-(1-f_1) \\ w_2 &= \xi_+(-\psi_1) + \xi'_+(1-\phi_1) \end{aligned} \qquad S\begin{pmatrix} -\psi_1 \\ 1-\phi_1\end{pmatrix} = \begin{pmatrix} -g_1 \\ 1-f_1\end{pmatrix}$$

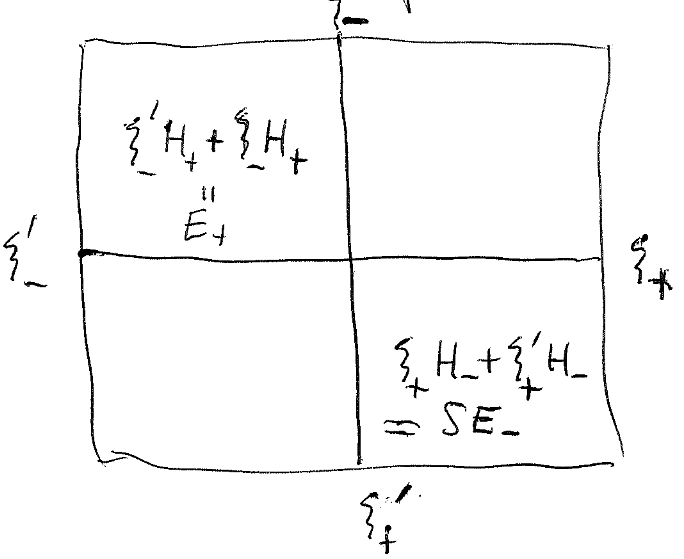
$$S\begin{pmatrix} 1-\phi & -\psi_1 \\ -\psi & 1-\phi_1\end{pmatrix} = \begin{pmatrix} 1-f & -g_1 \\ -g & 1-f_1\end{pmatrix}$$

$$IH(\xi'_+, \xi'_- f + \xi_- g) = \int \begin{pmatrix}1\\0\end{pmatrix}^* \begin{pmatrix}1 & \bar b\\ b & -1\end{pmatrix}\begin{pmatrix}f\\g\end{pmatrix}$$

$$= \int (f + \bar b g) \qquad \begin{pmatrix}f\\g\end{pmatrix} \in E_+$$

maybe $\int f = 0$ by convention.

Here's how to proceed?



We know these subspaces are complementary. You've constructed the decomposition

$$\begin{tikzcd} & E_- \arrow[d, "Sj_-"] \arrow[dr, "\sim"] & \\ E_+ \arrow[r, "j_+"] \arrow[dr, "\sim"'] & E \arrow[r, "j_-^*"] \arrow[d, "j_+^* B"] & E_- \\ & E_+ & \end{tikzcd}$$

So you have an explicit way to write any $\begin{pmatrix}f\\g\end{pmatrix} \in E$ as the sum of elts in $E_+ + SE_-$

Introduce $E \longrightarrow E_+ \times E_-$

$$(j_+^* B j_+)^{-1} j_+^* B \ , \quad (j_-^* S j_-)^{-1} j_-^*$$

Then you have that $\xi = (j_+^* B j_+)^{-1} j_+^* B \xi + S j_- (j_-^* S j_-)^{-1} j_-^* \xi$

$$\begin{cases} \partial_x \psi^1 = \psi^2 \\ \partial_y \psi^2 = \psi^1 \end{cases} \qquad \psi(x,y) = e^{xs + yu}\begin{pmatrix} v^1 \\ v^2 \end{pmatrix}$$

$$= e^{xs + ys^{-1}}\begin{pmatrix} 1 \\ s \end{pmatrix} \times \text{const}.$$

wave equation $\quad \partial_t \psi = \begin{pmatrix} \partial_r & i \\ i & -\partial_r \end{pmatrix}\psi$

$$\psi(r,t) = \exp\left(t\begin{pmatrix} \partial_r & i \\ i & -\partial_r \end{pmatrix}\right)\psi(r,0) \qquad \psi(r,0) = \int e^{ikr}\hat{\psi}_0(k)\frac{dk}{2\pi}$$

$$= \int e^{ikr} \exp\Big\{it\underbrace{\begin{pmatrix} k & 1 \\ 1 & -k \end{pmatrix}}_{A_k}\Big\}\hat{\psi}_0(k)\frac{dk}{2\pi}$$

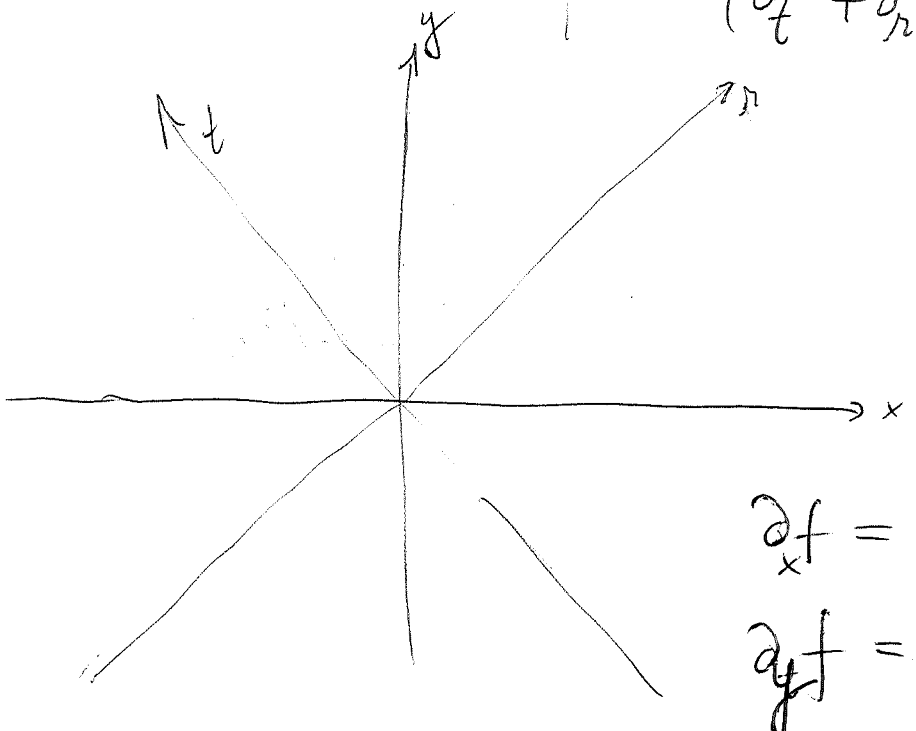
$A_k^2 = (1+k^2)I$

$\omega = \sqrt{k^2+1}$

$$= \int_{-\infty}^{\infty} e^{ikr}\left\{ e^{it\omega}\frac{\omega + A}{2\omega} + e^{-i\omega t}\frac{+\omega - A}{2\omega}\right\}\hat{\psi}_0(k)\frac{dk}{2\pi}$$

$kr + \omega t$

$$\begin{aligned} (\partial_t - \partial_r)\psi^1 &= i\psi^2 \\ (\partial_t + \partial_r)\psi^2 &= i\psi^1 \end{aligned}$$



$r = x + y$

$t = -x + y$

$f(r,t)$.

$\partial_x f = \partial_r f \, 1 + \partial_t f(-1)$

$\partial_y f = \partial_r f \, 1 + \partial_t f(1)$

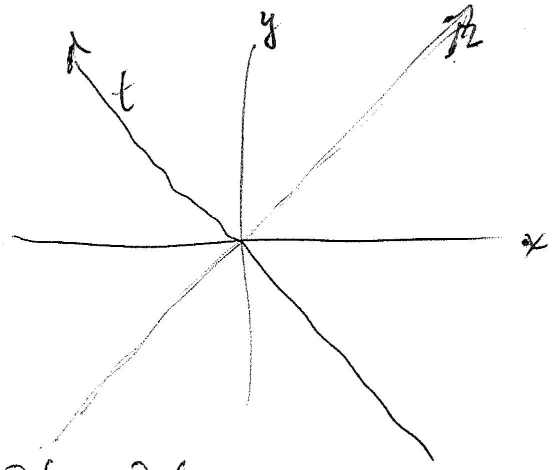
$\therefore \quad \partial_x = -\partial_t + \partial_r$

$\partial_y = \partial_t + \partial_r$

$-\partial_x \psi^1 = i\psi^2$

$\partial_y \psi^2 = i\psi^1$

$$\partial_t \psi = \begin{pmatrix} \partial_r & i \\ i & -\partial_r \end{pmatrix} \psi$$



$$\frac{1}{i}(\partial_t - \partial_r)\psi^1 = \psi^2$$
$$\frac{1}{i}(\partial_t + \partial_r)\psi^2 = \psi^1$$

$$\partial_x = -\partial_t + \partial_r \qquad \partial_x f(r,t) = \partial_r f - \partial_t f \qquad r = x+y$$
$$\partial_y = \partial_t + \partial_r \qquad \partial_y f(r,t) = \partial_r f + \partial_t f \qquad t = -x+y$$

$$\psi(r,t) = \exp\left\{ t \begin{pmatrix} \partial_r & i \\ i & -\partial_r \end{pmatrix} \right\} \underbrace{\psi(r,0)}_{\int_{-\infty}^{\infty} e^{ikr} \hat{\psi}_0(k) \frac{dk}{2\pi}}$$

$$= \int_{-\infty}^{\infty} e^{ikr} \underbrace{\exp\left\{ it \overbrace{\begin{pmatrix} k & 1 \\ 1 & -k \end{pmatrix}}^{A_k} \right\}} \hat{\psi}_0(k) \frac{dk}{2\pi}$$

$$A_k^2 = (k^2+1)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \omega = \pm\sqrt{k^2+1}$$
$$\omega^2 - A_k^2 = 0$$

$$= \int_{-\infty}^{\infty} e^{ikr} \left\{ e^{i\omega t} \frac{\omega + A_k}{2\omega} + e^{-i\omega t} \frac{\omega - A_k}{2\omega} \right\} \hat{\psi}_0(k) \frac{dk}{2\pi}$$

$$= \int_{-\infty}^{\infty} \left\{ \frac{e^{i(kr+\omega t)}}{2\omega} \begin{pmatrix} \omega + k & 1 \\ 1 & \omega - k \end{pmatrix} + \frac{e^{i(kr-\omega t)}}{2\omega} \begin{pmatrix} \omega - k & -1 \\ -1 & \omega + k \end{pmatrix} \right\} \hat{\psi}_0(k) \frac{dk}{2\pi}$$

$$kr + \omega t = k(x+y) + \omega(-x+y) = x(k-\omega) + y(k+\omega)$$
$$kr - \omega t = k(x+y) + \omega(x-y) = (k+\omega)x + (k-\omega)y$$

set $s = i(k-\omega)$ $\qquad s^{-1} = i(k+\omega)$

$$\int \underline{e^{xs+ys^{-1}}} \qquad e^{xs^{-1}+ys}$$

$$kr+\omega t = y\rho - x\rho^{-1} \qquad \rho=\omega+k \quad \rho^{-1}=\omega-k$$

$$kr-\omega t = x\rho - y\rho^{-1} \qquad \omega = \frac{\rho+\rho^{-1}}{2} \qquad \boxed{2\omega=\rho+\rho^{-1}}$$

$$\int \left\{ \frac{e^{i(y\rho-x\rho^{-1})}}{\rho+\rho^{-1}} \begin{pmatrix} \rho & 1 \\ 1 & \rho^{-1} \end{pmatrix} + \frac{e^{i(x\rho-y\rho^{-1})}}{\rho+\rho^{-1}} \begin{pmatrix} \rho^{-1} & -1 \\ -1 & \rho \end{pmatrix} \right\} \hat{\varphi}_0\left(\frac{\rho-\rho^{-1}}{2}\right) \frac{dk}{2\pi}$$

$$k=\frac{\rho-\rho^{-1}}{2}$$

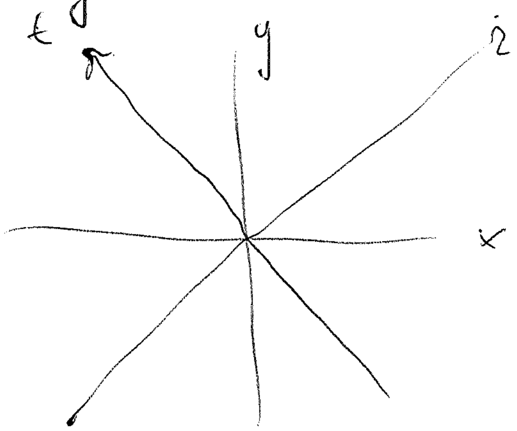
$$dk = \frac{1+\rho^{-2}}{2}\,d\rho = \omega\frac{d\rho}{\rho}$$

$$\therefore \frac{dk}{2\omega} = \frac{d\rho}{2\rho}$$

$$\int_{-\infty}^{\infty} e^{i(y\rho-x\rho^{-1})} \begin{pmatrix} \rho & 1 \\ 1 & \rho^{-1} \end{pmatrix} \hat{\varphi}_0\left(\frac{\rho-\rho^{-1}}{2}\right) \frac{d\rho}{2\rho}$$

$$\partial_x \psi^1 = \psi^2$$
$$\partial_y \psi^1 = \psi^2$$

$$\psi(x,y) = e^{xs+ys^{-1}}\begin{pmatrix} 1 \\ s \end{pmatrix} \qquad s\in\mathbb{C}-\{0\}$$



$$r = x+y \qquad y=\frac{r+t}{2}$$
$$t = -x+y \qquad x=\frac{r-t}{2}$$

$$xs+ys^{-1} = \frac{r-t}{2}s + \frac{r+t}{2}s^{-1}$$
$$= r\left(\frac{s+s^{-1}}{2}\right) + t\left(\frac{-s+s^{-1}}{2}\right)$$

~~If you take a solution~~ Cauchy problem.
~~idea is clear~~

$\int (\psi^*\psi)(r,t)\,dr$ ind. of t. $\|\psi\|^2$

$\int (\psi^*\varepsilon\psi)(r,t)\,dt$ —— r. $IH(\psi)$