Set up the scattering stuff properly.
Scattering data gives the ~~[illegible]~~ asymptotic picture from which you want to recover the potential. Scattering data is

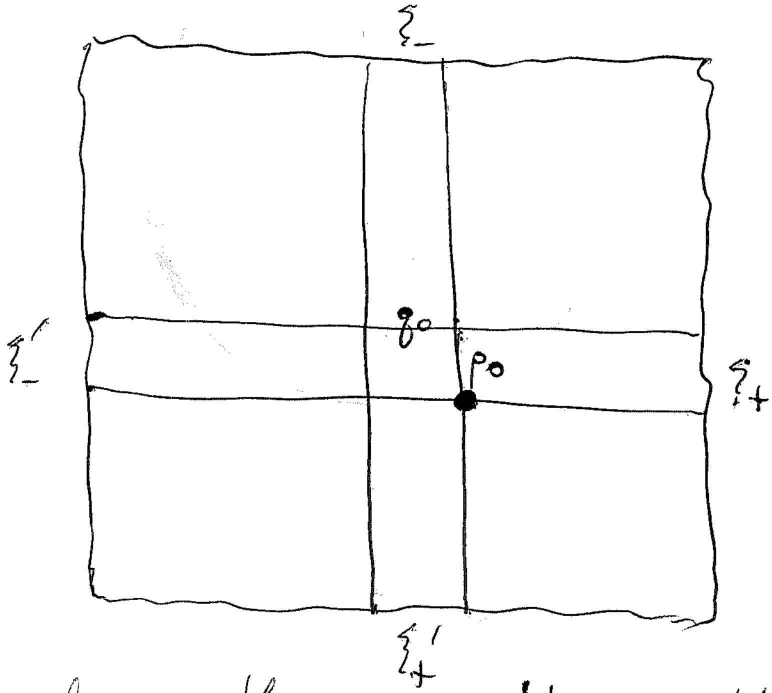
$$\begin{pmatrix}\xi_+\\ \xi_-\end{pmatrix} = \begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}\xi'_-\\ \xi'_+\end{pmatrix} \qquad \begin{pmatrix}\xi_+\\ \xi'_+\end{pmatrix} = \begin{pmatrix}\frac{1}{d} & \frac{b}{d}\\ -\frac{c}{d} & \frac{1}{d}\end{pmatrix}\begin{pmatrix}\xi'_-\\ \xi_-\end{pmatrix}$$

$$\begin{pmatrix}\xi'_-\\ \xi'_+\end{pmatrix} = \begin{pmatrix}d & -b\\ -c & a\end{pmatrix}\begin{pmatrix}\xi_+\\ \xi_-\end{pmatrix} \qquad \begin{pmatrix}\xi'_-\\ \xi_-\end{pmatrix} = \begin{pmatrix}\frac{1}{a} & -\frac{b}{a}\\ \frac{c}{a} & \frac{1}{a}\end{pmatrix}\begin{pmatrix}\xi_+\\ \xi'_+\end{pmatrix}$$

$$\begin{tikzcd} & E & \\ L^2\oplus L^2 \arrow[ur, "\sim"] \arrow[rr, "\text{S-matrix}"] & & L^2\oplus L^2 \arrow[ul, "\sim"'] \end{tikzcd}$$

light cones forward and backward. Each vertex determines a forward and backward cone

e.g.



$H_+\xi'_- + H_+\xi_-$

$H_-\xi_+ + H_-\xi'_+$

These should be complementary. Why? Because you know that $H_+p_0 + H_+q_0 = H_+\xi'_- + H_+\xi_-$

$$\begin{pmatrix}p_0\\ q_0\end{pmatrix} = \begin{pmatrix}a_0 & b_0\\ c_0 & d_0\end{pmatrix}\begin{pmatrix}\xi'_-\\ \xi'_+\end{pmatrix} = \begin{pmatrix}a_0 & b_0\\ c_0 & d_0\end{pmatrix}\begin{pmatrix}1 & 0\\ -\frac{c}{d} & \frac{1}{d}\end{pmatrix}\begin{pmatrix}\xi'_-\\ \xi_-\end{pmatrix}$$

$$\begin{pmatrix} p_0 \\ q_0 \end{pmatrix} = \frac{1}{d}\begin{pmatrix} a_0 d - b_0 c & b_0 \\ c_0 d - d_0 c & d_0 \end{pmatrix}\begin{pmatrix} \xi_-' \\ \xi_- \end{pmatrix} = \underbrace{\frac{1}{d}\begin{pmatrix} d_> & b_0 \\ -c_> & d_0 \end{pmatrix}}_{\det = \frac{1}{d}}\begin{pmatrix} \xi_-' \\ \xi_- \end{pmatrix}$$

$$\in \begin{pmatrix} H_+ & H_+ \\ zH_+ & H_+ \end{pmatrix}$$

$$\begin{pmatrix} \xi_-' \\ \xi_- \end{pmatrix} = \begin{pmatrix} d_0 & -b_0 \\ c_> & d_> \end{pmatrix}\begin{pmatrix} p_0 \\ q_0 \end{pmatrix}$$

$$\cap$$

$$\begin{pmatrix} H_+ & H_+ \\ zH_+ & H_+ \end{pmatrix}$$

$$\begin{pmatrix} p_0 \\ q_0 \end{pmatrix} = \begin{pmatrix} d_> & -b_> \\ -c_> & a_> \end{pmatrix}\begin{pmatrix} \xi_+ \\ \xi_- \end{pmatrix}$$

$$\cap$$

$$\begin{pmatrix} H_+ & H_- \\ zH_+ & zH_- \end{pmatrix}$$

$$\begin{pmatrix} p_0 \\ q_0 \end{pmatrix} = \begin{pmatrix} a_0 & b_0 \\ c_0 & d_0 \end{pmatrix}\begin{pmatrix} \xi_+ \\ \xi_+' \end{pmatrix}$$

$$\cap$$

$$\begin{pmatrix} \cancel{z}H_- & H_+ \\ zH_- & H_+ \end{pmatrix}$$

$$H_- p_0 + H_- q_0 = H_- \xi_+ + H_- \xi_+' \quad ?$$

$$\begin{pmatrix} p_0 \\ q_0 \end{pmatrix} = \begin{pmatrix} d_> & -b_> \\ -c_> & a_> \end{pmatrix}\begin{pmatrix} \xi_+ \\ \xi_- \end{pmatrix} = \begin{pmatrix} d_> & -b_> \\ -c_> & a_> \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \frac{c}{a} & \frac{1}{a} \end{pmatrix}\begin{pmatrix} \xi_+ \\ \xi_+' \end{pmatrix}$$

$$= \frac{1}{a}\begin{pmatrix} d_> a - b_> c & -b_> \\ -c_> a + a_> c & a_> \end{pmatrix}\begin{pmatrix} \xi_+ \\ \xi_+' \end{pmatrix} = \frac{1}{a}\begin{pmatrix} a_0 & -b_> \\ c_0 & a_> \end{pmatrix}\begin{pmatrix} \xi_+ \\ \xi_+' \end{pmatrix}$$

$$\cap$$

$$\begin{pmatrix} zH_- & H_- \\ zH_- & zH_- \end{pmatrix}$$

$$\begin{pmatrix} \xi_+ \\ \xi_+' \end{pmatrix} = \begin{pmatrix} a_> & +b_> \\ -c_0 & a_0 \end{pmatrix}\begin{pmatrix} p_0 \\ q_0 \end{pmatrix}$$

$$\cap$$

$$\begin{pmatrix} zH_- & H_- \\ zH_- & zH_- \end{pmatrix}$$

think $\qquad \begin{pmatrix} \xi_+ \\ \xi'_+ \end{pmatrix} = \begin{pmatrix} \frac{1}{d} & \frac{b}{d} \\ -\frac{c}{d} & \frac{1}{d} \end{pmatrix} \begin{pmatrix} \xi'_- \\ \xi_- \end{pmatrix}$

$$\begin{pmatrix} p_0 \\ q_0 \end{pmatrix} = \underbrace{\frac{1}{a}\begin{pmatrix} a_0 & -b_> \\ c_0 & a_> \end{pmatrix}}_{\epsilon\begin{pmatrix} zH_- & H_- \\ zH_- & zH_- \end{pmatrix}} \begin{pmatrix} \xi_+ \\ \xi'_+ \end{pmatrix} = \underbrace{\frac{1}{d}\begin{pmatrix} d_> & b_0 \\ -c_> & d_0 \end{pmatrix}}_{\begin{pmatrix} H_+ & H_+ \\ zH_+ & H_+ \end{pmatrix}} \begin{pmatrix} \xi'_- \\ \xi_- \end{pmatrix}$$

$$\begin{pmatrix} \xi_+ \\ \xi'_+ \end{pmatrix} = \underbrace{\begin{pmatrix} a_> & b_> \\ -c_0 & a_0 \end{pmatrix} \frac{1}{d}\begin{pmatrix} d_> & b_0 \\ -c_> & d_0 \end{pmatrix}}\begin{pmatrix} \xi'_- \\ \xi_- \end{pmatrix}$$

$$\therefore \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a_> & b_> \\ -c_0 & a_0 \end{pmatrix}\begin{pmatrix} d_0 & -b_0 \\ c_> & d_> \end{pmatrix}^{-1}$$

$$\begin{pmatrix} zH_- & H_- \\ zH_- & zH_- \end{pmatrix} \cdot \begin{pmatrix} H_+ & H_+ \\ zH_+ & H_+ \end{pmatrix}$$

$$\frac{1}{d}\begin{pmatrix} a_>d_> - b_>c_> & a_>b_0 + b_>d_0 \\ -c_0d_> - a_0c_> & -c_0b_0 + a_0d_0 \end{pmatrix}$$

$$= \frac{1}{d}\begin{pmatrix} 1 & b \\ -c & 1 \end{pmatrix}$$

$$\begin{pmatrix} a_> & b_> \\ c_> & d_> \end{pmatrix}\begin{pmatrix} a_0 & b_0 \\ c_0 & d_0 \end{pmatrix} = \begin{pmatrix} \\ \underbrace{c_>a_0 + d_>c_0}_{c} \end{pmatrix}$$

$$\begin{pmatrix} u^{-n}p_n \\ q_n \end{pmatrix} = \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix}\begin{pmatrix} \xi'_- \\ \xi'_+ \end{pmatrix}$$

$$= \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -\frac{c}{d} & \frac{1}{d} \end{pmatrix}\begin{pmatrix} \xi'_- \\ \xi_- \end{pmatrix}$$

$$= \frac{1}{d}\begin{pmatrix} a_n d - b_n c & b_n \\ c_n d - d_n c & d_n \end{pmatrix} \quad ''$$

$$\begin{pmatrix} \xi_+ \\ \xi_- \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} \xi'_- \\ \xi'_+ \end{pmatrix}$$

$$\begin{pmatrix} \xi_+ \\ \xi'_+ \end{pmatrix} = \begin{pmatrix} \frac{1}{d} & \frac{b}{d} \\ -\frac{c}{d} & \frac{1}{d} \end{pmatrix}\begin{pmatrix} \xi'_- \\ \xi_- \end{pmatrix}$$

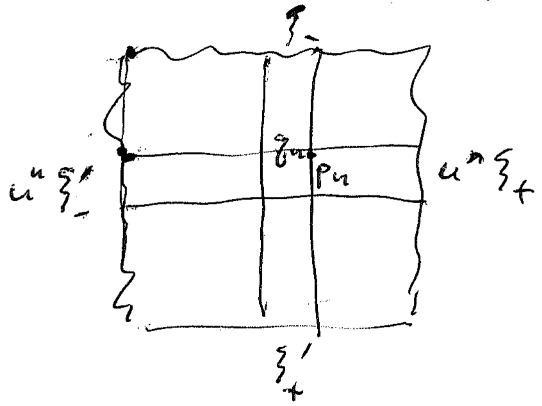
$$\begin{pmatrix} \xi'_- \\ \xi'_+ \end{pmatrix} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}\begin{pmatrix} \xi_+ \\ \xi_- \end{pmatrix}$$

$$\begin{pmatrix} \xi'_- \\ \xi_- \end{pmatrix} = \begin{pmatrix} \frac{1}{a} & -\frac{b}{a} \\ +\frac{c}{a} & \frac{1}{a} \end{pmatrix}\begin{pmatrix} \xi_+ \\ \xi'_+ \end{pmatrix}$$

$$\begin{pmatrix} a_> & b_> \\ c_> & d_> \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} d_n & -b_n \\ -c_n & a_n \end{pmatrix} = \begin{pmatrix} ad_n - bc_n & -ab_n + ba_n \\ cd_n - dc_n & -cb_n + da_n \end{pmatrix}$$

$$\boxed{\begin{pmatrix} u^{-n}p_n \\ q_n \end{pmatrix} = \frac{1}{d}\underbrace{\begin{pmatrix} d_> & b_n \\ -c_> & d_n \end{pmatrix}}\begin{pmatrix} \xi'_- \\ \xi_- \end{pmatrix}}$$

$$\in \begin{pmatrix} H_+ & z^{n}H_+ \\ z^{n+1}H_+ & H_+ \end{pmatrix}$$



$$q_n \in z^{n+1}H_+\xi'_- + H_+\xi_-$$

$$p_n \in z^{n}H_+\xi'_- + H_+\xi_-$$

$$p_n \in z^{n}H_-\xi_+ + H_-\xi'_+$$

$$q_n \in z^{n+1}H_-\xi_+ + zH_-\xi'_+$$

$$\begin{pmatrix} u^{-n}p_n \\ q_n \end{pmatrix} = \begin{pmatrix} d_> & -b_> \\ -c_> & a_> \end{pmatrix}\begin{pmatrix} \xi_+ \\ \xi_- \end{pmatrix} =$$

$$= \begin{pmatrix} d_> & -b_> \\ -c_> & a_> \end{pmatrix}\begin{pmatrix} 1 & 0 \\ +\frac{c}{a} & \frac{1}{a} \end{pmatrix}\begin{pmatrix} \xi_+ \\ \xi'_+ \end{pmatrix}$$

$$= \frac{1}{a}\begin{pmatrix} \overset{a_n}{d_>a - b_>c} & -b_> \\ \underset{c_n}{-c_>a + a_>c} & a_> \end{pmatrix}\begin{pmatrix} \xi_+ \\ \xi'_+ \end{pmatrix}$$

$$\boxed{\begin{pmatrix} u^{-n}p_n \\ q_n \end{pmatrix} = \frac{1}{a}\underbrace{\begin{pmatrix} a_n & -b_> \\ c_n & a_> \end{pmatrix}}_{\in \begin{pmatrix} zH_- & z^{-n}H_- \\ z^{n+1}H_- & zH_- \end{pmatrix}}\begin{pmatrix} \xi_+ \\ \xi'_+ \end{pmatrix}}$$

$$\begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix} = \begin{pmatrix} d_> & -b_> \\ -c_> & a_> \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} d_>a - b_>c & \\ -c_>a + a_>c & \end{pmatrix}$$

$$\begin{pmatrix} \frac{1}{d} & \frac{b}{d} \\ -\frac{c}{d} & \frac{1}{d} \end{pmatrix} = \underbrace{\begin{pmatrix} a_> & b_> \\ -c_n & a_n \end{pmatrix}}_{\cap} \frac{1}{d} \underbrace{\begin{pmatrix} d_> & b_n \\ -c_> & d_n \end{pmatrix}}_{\cap}$$

$$\begin{pmatrix} zH_- & z^{-n}H_- \\ z^{n+1}H_- & zH_- \end{pmatrix} \cdot \begin{pmatrix} H_+ & z^{-n}H_+ \\ z^{n+1}H_+ & H_+ \end{pmatrix}$$

looks like a standard factorization of

$$\begin{pmatrix} z^{+n} & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \frac{1}{d} & \frac{b}{d} \\ -\frac{c}{d} & \frac{1}{d} \end{pmatrix}\begin{pmatrix} z^{-n} & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{d} & \frac{b}{d}z^{+n} \\ -\frac{c}{d}z^{-n} & \frac{1}{d} \end{pmatrix}$$

roughly you split $\frac{b}{d}z^{+n}$ into H_- and H_+

tomorrow you want to look at the Green's fn. analogy Perfect

$$\begin{pmatrix} p_n \\ q_n \end{pmatrix} = \frac{1}{d}\begin{pmatrix} z^n d_{>n} & z^n b_{\le n} \\ -c_{>n} & d_{\le n} \end{pmatrix}\begin{pmatrix} \xi_-' \\ \xi_- \end{pmatrix}$$

solutions of the DE

constants

$$\begin{pmatrix} p_n \\ q_n \end{pmatrix} = \frac{1}{a}\begin{pmatrix} z^n a_{\le n} & -z^n b_{>n} \\ c_{\le n} & a_{>n} \end{pmatrix}\begin{pmatrix} \xi_+ \\ \xi_+' \end{pmatrix}$$

~~Anyway...~~ You want to reformulate, reinterpret in terms of solutions of the DE

When you calculate

$$\begin{pmatrix} u^{-n} p_n \\ q_n \end{pmatrix} = \frac{1}{d} \begin{pmatrix} a_{\le n} & b_{\le n} \\ c_{\le n} & d_{\le n} \end{pmatrix} \begin{pmatrix} d & 0 \\ -c & \frac{1}{d} \end{pmatrix} \begin{pmatrix} \xi_-' \\ \xi_- \end{pmatrix}$$

$$= \frac{1}{d} \begin{pmatrix} d_{>n} & b_{\le n} \\ -c_{>n} & d_{\le n} \end{pmatrix} \begin{pmatrix} \xi_-' \\ \xi_- \end{pmatrix}$$

Let's discuss specific solutions of the Dirac equation

There are 4 $\psi^{l,r}_{in,out}$

$$\psi_{out}(n,z) \sim$$

$$\begin{pmatrix} zH_- & z^{-n}H_+ \\ z^{n+1}H_- & H_+ \end{pmatrix}$$

~~continuous analogs.~~

$$c_{\le n} \in z^{n+1}H_-$$

$$c_{>n} \in z^{n+1}H_+$$

$$\begin{pmatrix} z^{-n} p_n \\ q_n \end{pmatrix} = \begin{pmatrix} a_{\le n} & b_{\le n} \\ c_{\le n} & d_{\le n} \end{pmatrix} \begin{pmatrix} \xi^l_- \\ \xi^l_+ \end{pmatrix}$$

$$= \begin{pmatrix} d_{>n} & -b_{>n} \\ -c_{>n} & a_{>n} \end{pmatrix} \begin{pmatrix} \xi^r_+ \\ \xi^r_- \end{pmatrix}$$

$$\psi(n,z) = \underbrace{\begin{pmatrix} a_{\le n} \\ c_{\le n} \end{pmatrix}, \begin{pmatrix} b_{\le n} \\ d_{\le n} \end{pmatrix}, \begin{pmatrix} d_{>n} \\ -c_{>n} \end{pmatrix}, \begin{pmatrix} -b_{>n} \\ a_{>n} \end{pmatrix}}_{\text{four important solutions}}$$

$$\xrightarrow{n \to -\infty} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \xrightarrow{n \to -\infty} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad \xrightarrow{n \to +\infty} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \xrightarrow{n \to +\infty} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Look at continuous case.

$$\partial_x \begin{pmatrix} e^{-ikx} p_x \\ q_x \end{pmatrix} = \begin{pmatrix} 0 & h_x e^{ikx} \\ \bar{h}_x e^{ikx} & 0 \end{pmatrix} \begin{pmatrix} e^{-ikx} p_x \\ q_x \end{pmatrix}$$

$$\cancel{\begin{pmatrix} -ike^{-ikx} p_x \\ 0 \end{pmatrix} + \begin{pmatrix} e^{-ikx} & 0 \\ 0 & 1 \end{pmatrix} \partial_x \begin{pmatrix} p_x \\ q_x \end{pmatrix}}$$

$$-ikx \, \cancel{e^{-ikx}} p_x + \cancel{e^{-ikx}} \partial_x p_x = h_x \cancel{e^{-ikx}} q_x$$

$$\partial_x q_x = \bar{h}_x \cancel{e^{ikx}} p_x$$

$$\partial_x \begin{pmatrix} p_x \\ q_x \end{pmatrix} = \begin{pmatrix} ik & h_x \\ \bar{h}_x & 0 \end{pmatrix} \begin{pmatrix} p_x \\ q_x \end{pmatrix}$$

$$\partial_x \begin{pmatrix} e^{-i\frac{k}{2}x} p_x \\ e^{-i\frac{k}{2}x} q_x \end{pmatrix} = -\frac{ik}{2} \begin{pmatrix} e^{-i\frac{k}{2}x} p_x \\ e^{-i\frac{k}{2}x} q_x \end{pmatrix} + e^{-i\frac{k}{2}x} \begin{pmatrix} ik & h_x \\ \bar{h}_x & 0 \end{pmatrix} \begin{pmatrix} p_x \\ q_x \end{pmatrix}$$

$$= \begin{pmatrix} \frac{ik}{2} & h_x \\ \bar{h}_x & -\frac{ik}{2} \end{pmatrix} \begin{pmatrix} e^{-i\frac{k}{2}x} p_x \\ e^{-i\frac{k}{2}x} q_x \end{pmatrix}$$

$$\partial_x \psi_x = \left(i\sigma_z \frac{k}{2} + h_R \sigma_x - h_I \sigma_y \right) \psi_x$$

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_y = \begin{pmatrix} 0 & -i \\ +i & 0 \end{pmatrix} \qquad \sigma_x \sigma_y \sigma_z = \begin{pmatrix} i & \\ & +i \end{pmatrix} = i$$

$$i\sigma_z \frac{k}{2}\psi = \partial\psi - h_R\sigma_x\psi + h_I\sigma_y\psi$$

$$\psi = \begin{pmatrix} e^{-i\frac{k}{2}x} p_x \\ e^{-i\frac{k}{2}x} q_x \end{pmatrix} \qquad \partial_x\psi = \left(\frac{ik}{2}\sigma_z \qquad \right)\psi$$

$$\partial_x \begin{pmatrix} e^{-ikx/2} p_x \\ e^{-ikx/2} q_x \end{pmatrix} = \begin{pmatrix} ik/2 & h_x \\ \bar{h}_x & -ik/2 \end{pmatrix} \underbrace{\begin{pmatrix} e^{-ikx/2} p_x \\ e^{-ikx/2} q_x \end{pmatrix}}_{\psi_x}$$

$$\frac{1}{i}\partial_x \psi_x^1 = k/2\, \psi_x^1 - i h_x \psi_x^2$$

$$-\frac{1}{i}\partial_x \psi_x^2 = + i\bar{h}_x \psi_x^1 + k/2\, \psi_x^2$$

$$\begin{pmatrix} \frac{1}{i}\partial_x & i h_x \\ -i\bar{h}_x & -\frac{1}{i}\partial_x \end{pmatrix} \psi_x = k/2\, \psi_x$$

$$\begin{pmatrix} \frac{1}{i}\partial_x & 0 \\ 0 & -\frac{1}{i}\partial_x \end{pmatrix} \begin{pmatrix} \psi_x^1 \\ \psi_x^2 \end{pmatrix} = \begin{pmatrix} \frac{k}{2} & -i h_x \\ i\bar{h}_x & +\frac{k}{2} \end{pmatrix} \begin{pmatrix} \psi_x^1 \\ \psi_x^2 \end{pmatrix}$$

$$\begin{pmatrix} \frac{1}{i}\partial_x & i h_x \\ -i\bar{h}_x & -\frac{1}{i}\partial_x \end{pmatrix} \psi_x = \frac{k}{2}\psi_x$$

Back to $\partial_x \begin{pmatrix} e^{-ikx}p_x \\ q_x \end{pmatrix} = \begin{pmatrix} ik & h_x \\ \bar{h}_x & 0 \end{pmatrix} \begin{pmatrix} e^{-ikx}p_x \\ q_x \end{pmatrix}$.

better

$$\begin{pmatrix} e^{-ikx}p_x \\ q_x \end{pmatrix} = \begin{pmatrix} a_x & b_x \\ c_x & d_x \end{pmatrix} \begin{pmatrix} \xi'_- \\ \xi'_+ \end{pmatrix}$$

$$\begin{pmatrix} \xi_+ \\ \xi'_+ \end{pmatrix} = \begin{pmatrix} \frac{1}{d} & \frac{b}{d} \\ -\frac{c}{d} & \frac{1}{d} \end{pmatrix} \begin{pmatrix} \xi'_- \\ \xi_- \end{pmatrix}$$

$$\begin{pmatrix} \xi'_- \\ \xi_- \end{pmatrix} = \begin{pmatrix} \frac{1}{a} & -\frac{b}{a} \\ \frac{c}{a} & \frac{1}{a} \end{pmatrix} \begin{pmatrix} \xi_+ \\ \xi'_+ \end{pmatrix}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a_+ & b_+ \\ c_+ & d_+ \end{pmatrix} \begin{pmatrix} a_- & b_- \\ c_- & d_- \end{pmatrix}$$

$$\boxed{\begin{pmatrix} \frac{1}{d} & \frac{b}{d} \\ -\frac{c}{d} & \frac{1}{d} \end{pmatrix} = \begin{pmatrix} a_+ & b_+ \\ -c_- & a_- \end{pmatrix} \frac{1}{d} \begin{pmatrix} d_+ & b_- \\ -c_+ & d_- \end{pmatrix}}$$

$$\begin{pmatrix} p_0 \\ q_0 \end{pmatrix} = \begin{pmatrix} a_- & b_- \\ c_- & d_- \end{pmatrix} \begin{pmatrix} \xi'_- \\ \xi'_+ \end{pmatrix} = \begin{pmatrix} d_+ & -b_+ \\ -c_+ & a_+ \end{pmatrix} \begin{pmatrix} \xi_+ \\ \xi'_- \end{pmatrix}$$

$$\| \qquad\qquad\qquad\qquad \|$$

$$\begin{pmatrix} a_- & b_- \\ c_- & d_- \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\frac{c}{d} & \frac{1}{d} \end{pmatrix} \begin{pmatrix} \xi'_- \\ \xi_- \end{pmatrix} \qquad \begin{pmatrix} d_+ & -b_+ \\ -c_+ & a_+ \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \frac{c}{a} & \frac{1}{a} \end{pmatrix} \begin{pmatrix} \xi_+ \\ \xi'_+ \end{pmatrix}$$

$$\begin{pmatrix} \xi_+ \\ \xi'_+ \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -c & a \end{pmatrix} \begin{pmatrix} a_+ & b_+ \\ c_+ & d_+ \end{pmatrix} \frac{1}{d} \begin{pmatrix} a_- & b_- \\ c_- & d_- \end{pmatrix} \begin{pmatrix} d & 0 \\ -c & 1 \end{pmatrix}$$

$$= \begin{pmatrix} a_+ & b_+ \\ -c_- & a_- \end{pmatrix} \frac{1}{d} \begin{pmatrix} d_+ & b_- \\ -c_+ & d_- \end{pmatrix}$$

This looks simple.

$$b = a_+ b_- + b_+ d_-$$

$$-c = (-c_-) d_+ + a_-(-c_+)$$

$$c = c_+ a_- + d_+ c_-$$

$$\begin{pmatrix} \frac{1}{d} & \frac{b}{d} \\ -\frac{c}{d} & \frac{1}{d} \end{pmatrix} = \underset{\begin{pmatrix} zH_- & H_- \\ zH_- & zH_- \end{pmatrix}}{\underset{\cap}{\begin{pmatrix} a_+ & b_+ \\ -c_- & a_- \end{pmatrix}}} \underset{\begin{pmatrix} H_+ & H_+ \\ zH_+ & H_+ \end{pmatrix}}{\underset{\cap}{\underbrace{\frac{1}{d}\begin{pmatrix} d_+ & b_- \\ -c_+ & d_- \end{pmatrix}}}}$$

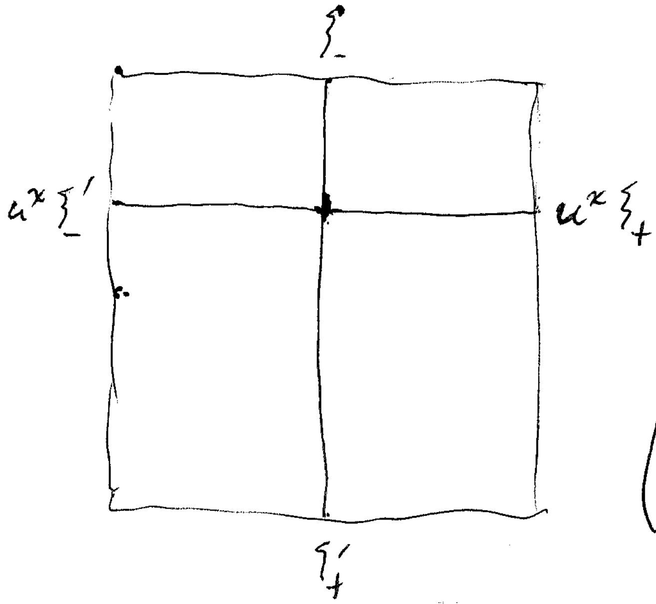
$$\begin{pmatrix} 1 & b \\ -c & 1 \end{pmatrix} = \begin{pmatrix} a_> & b_> \\ -c_< & a_< \end{pmatrix} \begin{pmatrix} d_> & b_< \\ -c_> & d_< \end{pmatrix}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a^r & b^r \\ c^r & d^r \end{pmatrix} \begin{pmatrix} a^l & b^l \\ c^l & d^l \end{pmatrix}$$

$$\begin{pmatrix} \frac{1}{d} & \frac{b}{d} \\ -\frac{c}{d} & \frac{1}{d} \end{pmatrix} = \begin{pmatrix} a^r & b^r \\ -c^l & a^l \end{pmatrix} \frac{1}{d} \begin{pmatrix} d^r & b^l \\ -c^r & d^l \end{pmatrix}$$

~~[illegible]~~ continuous analog

$$\begin{pmatrix} u^{-x}p_x \\ q_x \end{pmatrix} = \begin{pmatrix} a^l_x & b^l_x \\ c^l_x & d^l_x \end{pmatrix}\begin{pmatrix} \xi'_- \\ \xi'_+ \end{pmatrix} \in \begin{pmatrix} H_- & z^{-x}H_+ \\ z^x H_- & H_+ \end{pmatrix}\begin{pmatrix} \xi'_- \\ \xi'_+ \end{pmatrix}$$



$$p_x, q_x \in z^x H_- \xi'_- + H_+ \xi'_+$$

$$p_x, q_x \in z^x H_+ \xi_+ + H_- \xi_-$$

$$\begin{pmatrix} u^{-x}p_x \\ q_x \end{pmatrix} = \begin{pmatrix} d^r_x & -b^r_x \\ -c^r_x & a^r_x \end{pmatrix}\begin{pmatrix} \xi_+ \\ \xi_- \end{pmatrix}$$

$$\cap$$

$$\begin{pmatrix} \tilde{H}_+ & z^{-x}H_- \\ z^x H_+ & \tilde{H}_- \end{pmatrix}$$

$$\begin{pmatrix} a^l_x & b^l_x \\ c^l_x & d^l_x \end{pmatrix} \in \begin{pmatrix} \tilde{H}_- & z^{-x}H_+ \\ z^x H_- & \tilde{H}_+ \end{pmatrix}$$

$$\begin{pmatrix} a^r_x & b^r_x \\ c^r_x & d^r_x \end{pmatrix} \in \begin{pmatrix} \tilde{H}_- & z^{-x}H_- \\ z^x H_+ & \tilde{H}_+ \end{pmatrix}$$

Probably all this should be translated into ~~[illegible]~~ solutions of the DE.

You have this DE. $\partial_x \begin{pmatrix} p_x \\ q_x \end{pmatrix} = \begin{pmatrix} ik & h_x \\ \bar{h}_x & 0 \end{pmatrix}\begin{pmatrix} p_x \\ q_x \end{pmatrix}$

~~that those~~ If $\psi(x,k)$ is a solution, then you get two functions of $\psi(x,k)$

So I need to get control of the Birkhoff factorization, why it exists for any β, also its analytical properties.

It seems that ~~[illegible]~~ corresponding to a factorization of the transfer matrix is a factorization of the scattering matrix.

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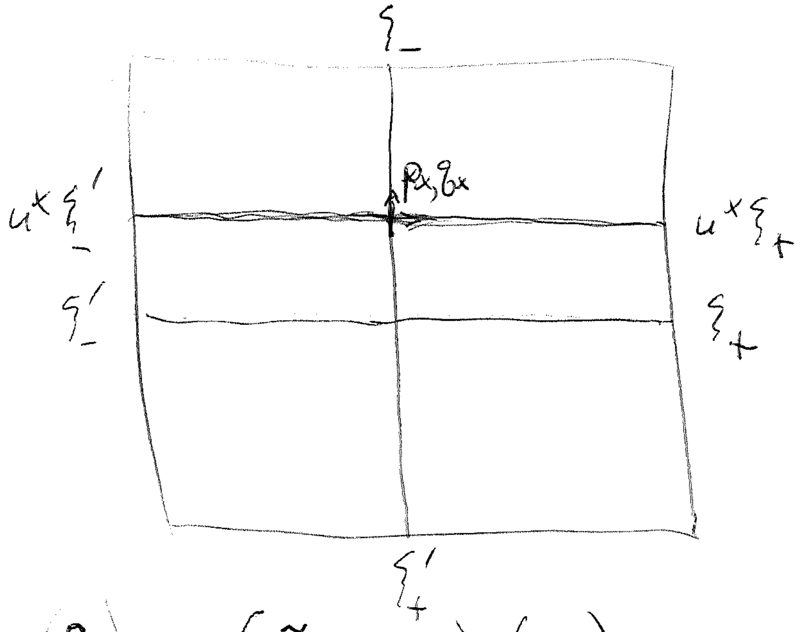
continuous case
$$\partial_x \begin{pmatrix} p_x \\ q_x \end{pmatrix} = \begin{pmatrix} ik & h_x \\ \bar{h}_x & 0 \end{pmatrix} \begin{pmatrix} p_x \\ q_x \end{pmatrix}$$

$$\partial_x \begin{pmatrix} e^{-ikx} p_x \\ q_x \end{pmatrix} = \begin{pmatrix} e^{-ikx} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \cancel{ik} & h_x \\ \bar{h}_x & 0 \end{pmatrix} \begin{pmatrix} p_x \\ q_x \end{pmatrix}$$

$$\cancel{+ \begin{pmatrix} -ike^{-ikx} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} p_x \\ q_x \end{pmatrix}}$$

$$= \begin{pmatrix} 0 & h_x e^{-ikx} \\ \bar{h}_x e^{ikx} & 0 \end{pmatrix} \begin{pmatrix} e^{-ikx} p_x \\ q_x \end{pmatrix}$$



$$\begin{pmatrix} p_x \\ q_x \end{pmatrix} \in \begin{pmatrix} u^x \tilde{H}_- & H_+ \\ u^x H_- & \tilde{H}_+ \end{pmatrix} \begin{pmatrix} \xi'_+ \\ \xi'_- \end{pmatrix}$$

$$\begin{pmatrix} a^l_x & b^l_x \\ c^l_x & d^l_x \end{pmatrix} \in \begin{pmatrix} \tilde{H}_- & u^{-x} H_+ \\ u^x H_- & \tilde{H}_+ \end{pmatrix}$$

$$\begin{pmatrix} p_x \\ q_x \end{pmatrix} \in \begin{pmatrix} u^x \tilde{H}_+ & H_- \\ u^x H_+ & \tilde{H}_- \end{pmatrix} \begin{pmatrix} \xi_+ \\ \xi_- \end{pmatrix}$$

$$\begin{pmatrix} d^r_x & -b^r_x \\ -c^r_x & a^r_x \end{pmatrix} \in \begin{pmatrix} \tilde{H}_+ & u^{-x} H_- \\ u^x H_+ & \tilde{H}_- \end{pmatrix}$$

$$p_x = u^x d_x^r \xi_+ - \underbrace{\cancel{b}\, u^x b_x^r}_{\in H_-} \xi_- \qquad d_x^r \in \tilde{H}_+$$

$$b_x^r \in u^{-x} H_-$$

$$\boxed{u^{-x} p_x = d_x^r \xi_+ - b_x^r \xi_- \qquad d_x^r \in \tilde{H}_+ \,, \quad b_x^r \in u^{-x} H_-}$$

to find d_x^r, b_x^r so that $p_x \perp u^x H_+ \xi_+ + H_- \xi_-$

$$p_x = z^x d_x^r \xi_+ - z^x b_x^r \xi_-$$

$$\overset{y<0}{0} = \left(z^y \xi_- \,\middle|\, z^x d_x^r \xi_+ - z^x b_x^r \xi_- \right)$$

$$0 = \left(f_- \xi_- \,\middle|\, z^x d_x^r \xi_+ - z^x b_x^r \xi_- \right)$$

$$= \left(f_- \,\middle|\, z^x d_x^r \beta - z^x b_x^r \right) \qquad \forall f_- \in H_-$$

$$\Longrightarrow \boxed{z^x \left(d_x^r \beta - b_x^r \right) \in H_+}$$

$$0 = \left(\cancel{z^x} f_+ \xi_+ \,\middle|\, \cancel{z^x} d_x^r \xi_+ - \cancel{z^x} b_x^r \xi_- \right)$$

$$= \left(f_+ \,\middle|\, d_x^r - b_x^r \bar{\beta} \right) \qquad \forall f_+ \in H_+$$

$$\Longrightarrow \boxed{d_x^r - b_x^r \bar{\beta} \in \tilde{H}_-}$$

Is it possible to formulate orthogonality in the scattering picture.

$b^r - d^r\beta \in H_+$ $\qquad$ $b^r \in H_-$

$d^r - b^r\overline{\beta} \in zH_-$ $\qquad$ $d^r \in H_+$

house 845 tomorrow
No milk tonight
car Monday 9am

$b^r d - d^r b \in H_+$

$d^r a - b^r c \in zH_-$

$$\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}\begin{pmatrix} b^r \\ d^r \end{pmatrix} \in \begin{pmatrix} H_+ \\ zH_- \end{pmatrix}$$

$$\cap$$

$$\begin{pmatrix} H_- \\ H_+ \end{pmatrix}$$

$$\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}\begin{pmatrix} a^r & b^r \\ c^r & d^r \end{pmatrix} = \begin{pmatrix} d^l & -b^l \\ -c^l & a^l \end{pmatrix}$$

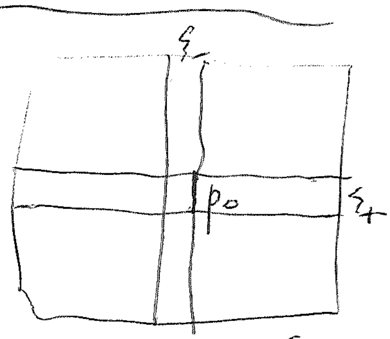
transfer picture

---

orth relations for p_0, q_0 yield

$b^r - d^r\beta \in H_+$

$d^r - b^r\overline{\beta} \in zH_-$ $\qquad$ hopeless.



$p_0 \in H_+\xi_+ + H_-\xi_-$

Instead go back to the equations.

$d \in 1+zH_+$, $b \in H_-$

$$p_0 = \sum_{(j\geq 0)} d_j u^j \xi_+ - \sum_{(k<0)} b_k u^k \xi_-$$

$\beta(z) = \sum \beta_n z^n$

$\beta_n = (z^n|\beta)$

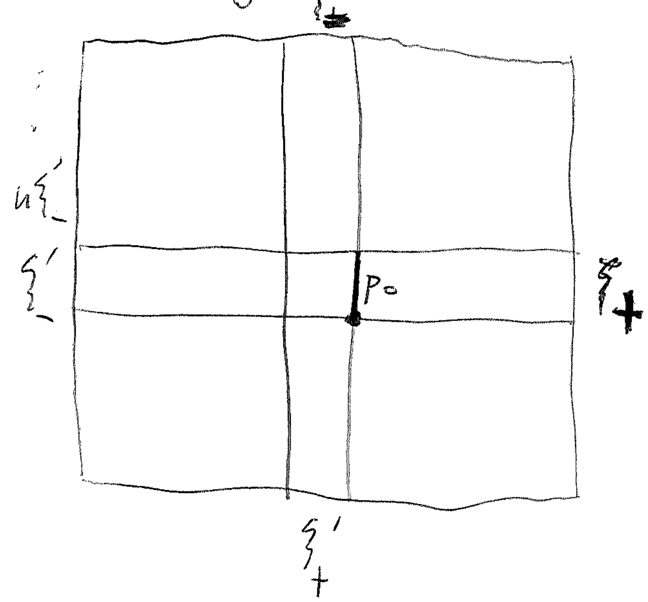
$k<0$

$$0 = (u^k\xi_-|p_0) = \sum d_j \underbrace{(u^k\xi_-|u^j\xi_+)}_{\beta_{k,j}} - b_k$$

$j\geq 0$

$$0 = (u^j\xi_+|p_0) = d_j - \sum_{k<0} b_k \underbrace{(u^j\xi_+|u_k\xi_-)}_{\overline{\beta_{k,j}}}$$

Basically what you have is a subspace



$$p_0 \in H_+\xi_-' + H_+\xi_-$$
$$\in zH_-\xi_+ + H_-\xi_+'$$

$$p_0 \in (zH_+\xi_+ + H_+\xi_+')^\perp$$
$$= zH_-\xi_+ + H_-\xi_+'$$

set up these equations

$$\begin{pmatrix} p_0 \\ q_0 \end{pmatrix} = \begin{pmatrix} a^l & b^l \\ c^l & d^l \end{pmatrix}\begin{pmatrix} \xi_-' \\ \xi_+' \end{pmatrix} = \frac{1}{d}\begin{pmatrix} d^r & b^l \\ -c^r & d^l \end{pmatrix}\begin{pmatrix} \xi_-' \\ \xi_- \end{pmatrix}$$

$$\begin{pmatrix} p_0 \\ q_0 \end{pmatrix} = \begin{pmatrix} d^r & -b^r \\ -c^r & a^r \end{pmatrix}\begin{pmatrix} \xi_+ \\ \xi_- \end{pmatrix} = \frac{1}{a}\begin{pmatrix} a^l & -b^r \\ c^l & a^r \end{pmatrix}\begin{pmatrix} \xi_+ \\ \xi_+' \end{pmatrix}$$

$$p_0 \in H_+\xi_-' + H_+\xi_- \cap (zH_+\xi_+ + H_+\xi_+')^\perp$$

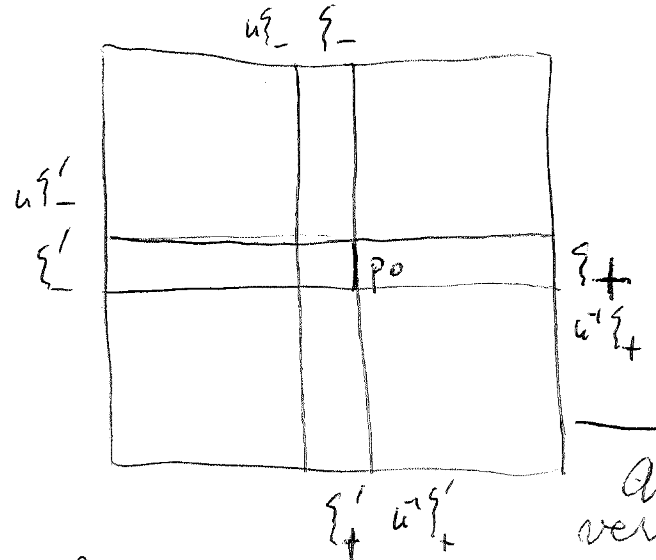
$$p_0 = s\xi_-' + t\xi_-$$

$$\underset{j>0}{0 =} (z^j\xi_+ \mid s\xi_-' + t\xi_-) = \left(z^j\left(\tfrac{1}{d}\xi_-' + \tfrac{b}{d}\xi_-\right) \middle| s\xi_-' + t\xi_-\right)$$

$$0 = \left(z^j\frac{1}{d} \middle| s\right) + \left(z^j\frac{b}{d} \middle| t\right) = \left(z^j \middle| \frac{1}{\bar{d}}s + \frac{\bar{b}}{\bar{d}}t\right)$$

$$\frac{1}{a}s + \frac{c}{a}t \in zH_- \qquad s + ct \in zH_-$$

$$\boxed{\frac{1}{d}(d^r + cb^l) \in zH_-}$$



$$\begin{pmatrix}\xi_+\\ \xi'_+\end{pmatrix} = \begin{pmatrix}\frac{1}{d} & \frac{b}{d}\\ -\frac{c}{d} & \frac{1}{d}\end{pmatrix}\begin{pmatrix}\xi'_-\\ \xi_-\end{pmatrix}$$

$$p_0 \in H_+\xi'_- + H_+\xi_-$$
$$\in zH_-$$

Aim to find the Hilbert space version of Birkhoff factorization

$$\begin{pmatrix}\xi_+\\ \xi_-\end{pmatrix} = \begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}\xi'_-\\ \xi'_+\end{pmatrix} \qquad \begin{pmatrix}\xi_+\\ \xi'_+\end{pmatrix} = \begin{pmatrix}\frac{1}{d} & \frac{b}{d}\\ -\frac{c}{d} & \frac{1}{d}\end{pmatrix}\begin{pmatrix}\xi_+\\ \xi_-\end{pmatrix}$$

$$\begin{pmatrix}\xi'_-\\ \xi'_+\end{pmatrix} = \begin{pmatrix}d & -b\\ -c & a\end{pmatrix}\begin{pmatrix}\xi_+\\ \xi_-\end{pmatrix} \qquad \begin{pmatrix}\xi'_-\\ \xi_-\end{pmatrix} = \begin{pmatrix}\frac{1}{a} & -\frac{b}{a}\\ \frac{c}{a} & \frac{1}{a}\end{pmatrix}\begin{pmatrix}\xi_+\\ \xi'_+\end{pmatrix}$$

$$p_0 = \frac{1}{d}(s\xi'_- + t\xi_-) \in H_+\xi'_- + H_+\xi_-$$

$$p_0 \in \cancel{\quad} zH_-\xi_+ + H_-\xi_-$$

$$p_0 = \frac{1}{d}\begin{pmatrix}s & t\end{pmatrix}\begin{pmatrix}\xi'_-\\ \xi_-\end{pmatrix} = \frac{1}{d}\begin{pmatrix}s & t\end{pmatrix}\begin{pmatrix}\frac{1}{a} & -\frac{b}{a}\\ \frac{c}{a} & \frac{1}{a}\end{pmatrix}\begin{pmatrix}\xi_+\\ \xi'_+\end{pmatrix}$$

$$\underbrace{\frac{1}{d}\begin{pmatrix}s & t\end{pmatrix}\begin{pmatrix}\frac{1}{a} & -\frac{b}{a}\\ \frac{c}{a} & \frac{1}{a}\end{pmatrix}}_{\frac{1}{a}\left(\frac{1}{d}s + t\frac{c}{d} \quad -\frac{bs}{d} + t\frac{1}{d}\right) \in \left(zH_- \quad H_-\right)}$$

So the condition is that

$$\underset{H_+ \quad H_+}{\underbrace{\begin{pmatrix}s & -t\end{pmatrix}}}\begin{pmatrix}\frac{1}{d} & \frac{b}{d}\\ -\frac{c}{d} & \frac{1}{d}\end{pmatrix} \in \begin{pmatrix}zH_- & H_-\end{pmatrix}$$

Can I prove this factorization. The basic claim is that given a scattering matrix $S = \begin{pmatrix} \delta & \beta \\ \gamma & \delta \end{pmatrix}$ with $\delta \in H_+$ invertible

then $S\begin{pmatrix} H_+ \\ H_+ \end{pmatrix}$ is complementary to $\begin{pmatrix} H_- \\ H_- \end{pmatrix}$.

This follows from the Birkhoff factorization. Why? because suppose ~~$S_+ S_-$ with~~

$g_\pm$ an auto of $H_\pm^{\oplus 2}$

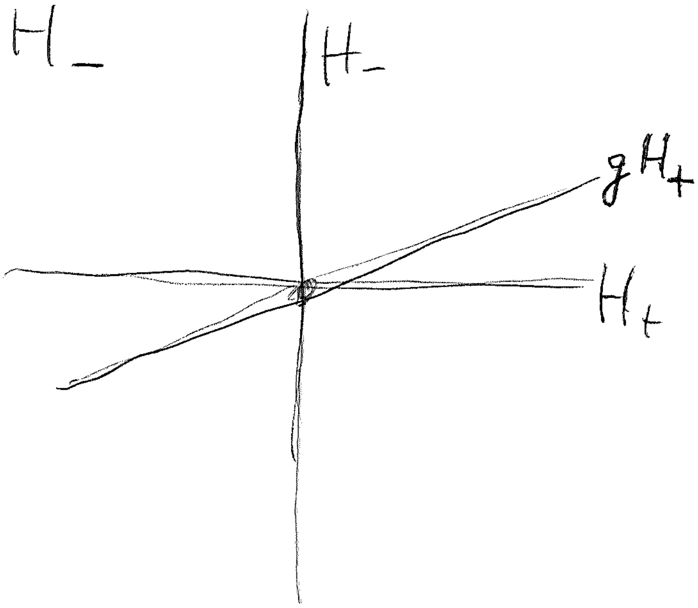
~~explain~~

$$g_+ H_+ = H_+$$

$$H_+ \oplus H_- \longrightarrow H$$

$$\begin{matrix} H_+ \\ \oplus \\ H_- \end{matrix} \xrightarrow{(g_+ \quad g_-)} H$$

idea



Your formulas probably establish equivalence between Birkhoff factorization of the scattering matrix S and the left-right factorization of the transfer matrix. One direction ~~xxxxxxxxx~~ is clear, namely

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a^r & b^r \\ c^r & d^r \end{pmatrix} \begin{pmatrix} a^l & b^l \\ c^l & d^l \end{pmatrix}$$

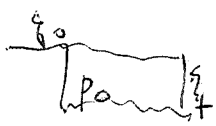
$$\qquad\qquad \begin{pmatrix} zH_- & H_- \\ zH_+ & H_+ \end{pmatrix} \quad \begin{pmatrix} zH_- & H_+ \\ zH_+ & H_+ \end{pmatrix}$$

transforms to

$$\begin{pmatrix} \frac{1}{d} & \frac{b}{d} \\ -\frac{c}{d} & \frac{1}{d} \end{pmatrix} = \begin{pmatrix} a^r & b^r \\ -c^l & a^l \end{pmatrix} \frac{1}{d} \begin{pmatrix} d^r & b^l \\ -c^r & d^l \end{pmatrix}$$

$$\qquad\qquad \begin{pmatrix} zH_- & H_- \\ zH_- & zH_- \end{pmatrix} \quad \begin{pmatrix} H_+ & H_+ \\ zH_+ & H_+ \end{pmatrix}$$

ξ_0



What does the existence amount to?

$$\begin{pmatrix} p_0 \\ q_0 \end{pmatrix} = \begin{pmatrix} d^r & -b^r \\ -c^r & a^r \end{pmatrix} \begin{pmatrix} \xi_+ \\ \xi_- \end{pmatrix} \in \begin{pmatrix} H_+ & H_- \\ zH_+ & zH_- \end{pmatrix} \begin{pmatrix} \quad \end{pmatrix}$$

existence done as follows

$$\tilde{p}_0 = \xi_+ + \sum_{j>0} d^r_j u^j \xi_+ - \sum_{k<0} b^r_k u^k \xi_-$$

$$0 = (u^k \xi_- | \tilde{p}_0) = \beta_k + \sum_{j>0} d^r_j \beta_{k-j} - b^r_k \qquad k<0$$

$$0 = (u^j \xi_+ | \tilde{p}_0) = d^r_j - \sum_{k<0} b^r_k \bar{\beta}_{k-j} \qquad j>0$$

Here $\beta(z) = \sum_{k\in\mathbb{Z}} \beta_k z^k$, $\beta_k = (z^k|\beta)$.

$d^r\beta$

$$\sum_{k\in\mathbb{Z}} \Big(\sum_{j\geq 0}{}' d^r_j \beta_{k-j} \Big) z^k = \sum_{j\geq 0}{}' d^r_j z^j \sum_{k\in\mathbb{Z}} \beta_{k-j} z^{k-j}$$

$$\therefore \boxed{d^r\beta - b^r \in H_+} \qquad \sum \bar\beta_{+j} \bar z^j$$

$$\sum_{j\in\mathbb{Z}} \sum_{k<0} b^r_k \overline{\beta_{k-j}} \, z^j = \sum_{k<0} b^r_k z^k \overbrace{\sum_{j\in\mathbb{Z}}{}' \overline{\beta}_{k-j} z^{j-k}}$$

$$= b^r \bar\beta .$$

$$\boxed{d^r - b^r\bar\beta \quad \in zH_-}$$

true because

$$\begin{pmatrix} d^r & -b^r \\ -c^r & a^r \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a^l & b^l \\ c^l & d^l \end{pmatrix}$$

$$\frac{d^r b}{d} - b^r = \frac{b^l}{d} \in \frac{H_+}{d} = H_+$$

$$\frac{d^r a - b^r c}{a} = \frac{c^l}{a} \in \frac{zH_-}{a} = zH_-$$

You should ask why these equations for d^r, b^r have a unique solution.

Use linear equations:

$$b^r = (b_k)_{k<0}$$
$$d^r = (d_j)_{j\geq 0} \qquad d_0 = 1.$$

$$b_k - \sum_{j\geq 0} d_j \beta_{k-j} = 0 \qquad \text{for } k<0$$

$$d_j - \sum_{k<0} b_k \overline{\beta_{k-j}} = 0 \qquad \text{for } j>0$$

Rewrite as

$$b_k - \sum_{j\geq 1} d_j \beta_{k,j} = \beta_{k,0} \qquad k<0.$$

$$d_j - \sum_{k<0} b_k \overline{\beta_{k,j}} = 0 \qquad j>0.$$

In terms of matrices & vectors $b = (b_k)_{<0}$
$d = (d_j)_{j>0}$. $\quad \beta = (\beta_{k,j})_{j<0,\, k>0}$

$$b - d\beta = (\beta_{k,0})_{k<0}$$
$$d - b\beta^* = 0$$

What's important is that β is a contraction.

$$b(1-\beta^*\beta) = (\beta_{k,0})_{k<0}$$

Can be solved by iteration

Inverse scattering - think, Fredholm alternative? ~~So What~~ You want to set up Birkhoff factorization as an integral equation. ~~Bettie~~ So how? Ingredients Bruhat decomposition, buildings. Contractions.

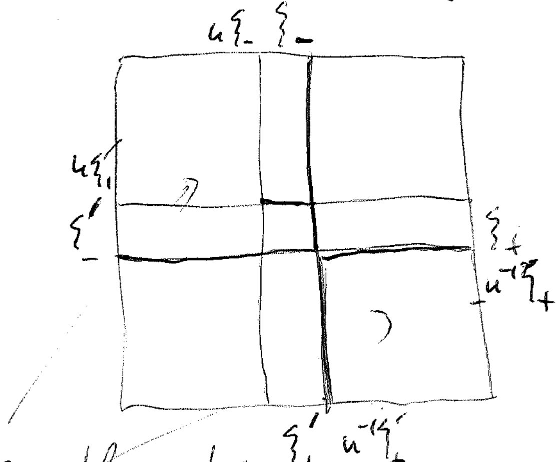
Given $S = \begin{pmatrix}\alpha & \beta \\ \gamma & \delta\end{pmatrix} = \begin{pmatrix}\frac{1}{d} & \frac{b}{d} \\ -\frac{c}{d} & \frac{1}{d}\end{pmatrix}$

intrinsic space $E = L^2\xi_+ \oplus L^2\xi'_+ = L^2\xi'_- \oplus L^2\xi_-$

$\begin{pmatrix}\xi_+ \\ \xi'_+\end{pmatrix} = S\begin{pmatrix}\xi'_- \\ \xi_-\end{pmatrix}$ ~~Yes~~. S is a unitary from $L^{2,\oplus 2}_{in}$ to $L^{2,\oplus 2}_{out}$

and you are given $H_+\xi'_- \oplus H_+\xi_-$ ~~and~~ incoming

$H_-\xi_+ \oplus H_-\xi'_+$ outgoing



complementary not orthogonal

The group ~~of~~ $GL_2(L^\infty(S^1))$ acts on incoming subspaces, there's a polar decomposition involving $U_2(L^\infty(S^1))$ and stabilizer of $H_+^{\oplus 2}$ i.e. invertibles in $M_2(H^\infty(S^1))$

In your situation ~~you~~ have $H_+\xi'_- \oplus H_+\xi$ basepoint incoming space and $(H_-\xi_+ \oplus H_-\xi'_+)^\perp = H_+\xi_+ \oplus H_+\xi'_+$ another incoming space, and S is the unitary relating the two. ~~You need to express comp~~

There are ~~so~~ math problems here. You need to explain the Birkhoff factorization of S. S is ~~completely~~ equivalent to the outgoing subspace. What is important ~~is~~ is these being complementary. I know Birkhoff fact $\Rightarrow$ complementary.

$$\begin{pmatrix}\xi_+\\ \xi'_+\end{pmatrix} = S\begin{pmatrix}\xi'_-\\ \xi_-\end{pmatrix} \qquad \begin{pmatrix}p_0\\ q_0\end{pmatrix} = g_+\begin{pmatrix}\xi'_-\\ \xi_-\end{pmatrix} = g_-\begin{pmatrix}\xi_+\\ \xi'_+\end{pmatrix}$$

$$\therefore\ S = g_-^{-1}g_+ \qquad \begin{pmatrix}\frac{1}{d} & +\frac{b}{d}\\ -\frac{c}{d} & \frac{1}{d}\end{pmatrix} = \underbrace{\begin{pmatrix}a^r & b^r\\ -c^l & d^l\end{pmatrix}}_{\begin{pmatrix}zH_- & H_-\\ zH_- & zH_-\end{pmatrix}} \frac{1}{d}\underbrace{\begin{pmatrix}d^r & b^l\\ -c^r & d^l\end{pmatrix}}_{\begin{pmatrix}H_+ & H_+\\ zH_+ & H_+\end{pmatrix}}$$

Check complementary, i.e. $E = E_+ + \quad ?$

$$E = L^2\xi_+ \oplus L^2\xi'_+ = L^2\xi'_- \underset{\cup}{\oplus} L^2\xi_-$$

$$E_+ = \quad H_+\xi'_- \oplus H_+\xi_-$$

$$E_- = H_-\xi_+ \oplus H_-\xi'_+ = (H_-\ \ H_-)\underbrace{\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}}_{S}\begin{pmatrix}\xi'_-\\ \xi_-\end{pmatrix}$$

$$\boxed{\begin{aligned} E_+ &= (H_+\ \ H_+)\begin{pmatrix}\xi'_-\\ \xi_-\end{pmatrix}\\ E_- &= (H_-\ \ H_-)\,S\begin{pmatrix}\xi'_-\\ \xi_-\end{pmatrix}\end{aligned}}$$

show these are complementary.

Check $E_+ \cap E_- = 0$. i.e.

$$\underline{(H_-\ \ H_-)S \cap (H_+\ \ H_+)} = 0.$$

this is the kernel of the Toeplitz operator

$$(f, g) \longmapsto (f, g)\,S\begin{pmatrix}\pi_- & 0\\ 0 & \pi_-\end{pmatrix}$$

better:
$$\begin{pmatrix}f\\ g\end{pmatrix} \longmapsto \begin{pmatrix}\pi_- & 0\\ 0 & \pi_-\end{pmatrix}\begin{pmatrix} & S^t & \end{pmatrix}\begin{pmatrix}f\\ g\end{pmatrix}$$

$\alpha f + \gamma g \in H_+$ with $f, g \in H_-$

$\beta f + \delta g \in H_+$

$$S^t \begin{pmatrix} f \\ g \end{pmatrix} = g_+^t (g_-^t)^{-1} \begin{pmatrix} f \\ g \end{pmatrix} \quad \text{ok.}$$

$H_+ \ H_+$ $\qquad (H_- \ H_-)$

$$(H_+ H_+) \ni (f, g) S = (f, g) g_-^{-1} g_+ \implies (H_+ \ H_+) g_+^{-1} = (f, g) g_-$$

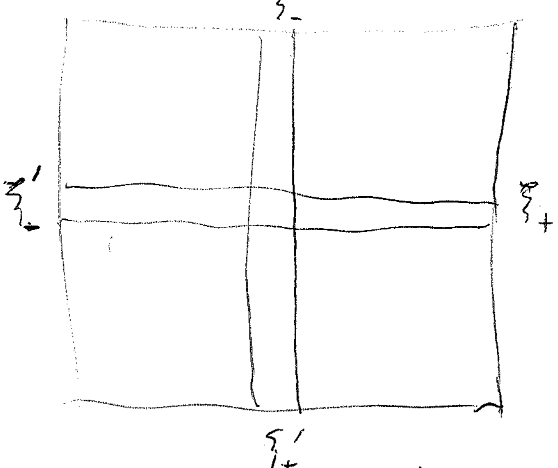
auto. Want $(H_- \ H_-) S + (H_+ \ H_+) = (L^2 \oplus L^2)$

Argument: ~~$S = g_-^{-1} g_+$ $\quad H = H_+ \oplus H_-$~~

Assume $H = H_+ \oplus H_-$, $S_\pm$ invert ops on H such that $S_\pm$ restricts to an inv. op on $H_\pm$.

Then
$$\begin{array}{ccccc} H & = & H_+ \oplus H_- & \xrightarrow[\sim]{\begin{pmatrix} S_+ & 0 \\ 0 & S_- \end{pmatrix}} & H_+ \oplus H_- = H \\ & & \xi + \eta & \longmapsto & S_+\xi + S_-\eta \qquad \Big\downarrow S_+^{-1} \\ & & & & \downarrow \qquad\qquad H \\ & & & & \xi + S_+^{-1} S_- \eta \ \in \end{array}$$

How do I proceed to handle this? Abstract the situation.



$\begin{pmatrix} \xi_+ \\ \xi'_+ \end{pmatrix} = S \begin{pmatrix} \xi'_- \\ \xi_- \end{pmatrix}$

$E_+ = H_+ \xi'_- + H_+ \xi_-$

$u E_+ \subset E_+$

$E_- = H_- \xi_+ + H_- \xi'_+$

$u E_- \supset E_-$

$$E = (\xi_+ \ \ \xi'_+) \begin{pmatrix} L^2 \\ L^2 \end{pmatrix} = (\xi'_- \ \ \xi_-) \begin{pmatrix} L^2 \\ L^2 \end{pmatrix}$$

$$E_+ = (\xi'_- \ \ \xi_-) \begin{pmatrix} H_+ \\ H_+ \end{pmatrix} \qquad E_- = \underbrace{(\xi_+ \ \ \xi'_+)}_{(\xi'_- \ \ \xi') S^t} \begin{pmatrix} H_- \\ H_- \end{pmatrix}$$

$\therefore$ You have $E_+ = \begin{pmatrix} H_+ \\ H_+ \end{pmatrix}$ and $E_- = S^t \begin{pmatrix} H_- \\ H_- \end{pmatrix}$.

You want these to be complementary:

$$\begin{pmatrix} \xi_-' & \xi_- \end{pmatrix} \begin{pmatrix} L^2 \\ L^2 \end{pmatrix} = \begin{pmatrix} \xi_-' & \xi_- \end{pmatrix} \begin{pmatrix} H_+ \\ H_+ \end{pmatrix} \oplus \underbrace{\begin{pmatrix} \xi_+ & \xi_+' \end{pmatrix} \begin{pmatrix} H_- \\ H_- \end{pmatrix}}_{\begin{pmatrix} \xi_-' & \xi_- \end{pmatrix} S^t \begin{pmatrix} H_- \\ H_- \end{pmatrix}}$$

$$\therefore \begin{pmatrix} L^2 \\ L^2 \end{pmatrix} \overset{?}{=} \begin{pmatrix} H_+ \\ H_+ \end{pmatrix} \oplus S^t \begin{pmatrix} H_- \\ H_- \end{pmatrix}$$

$$\begin{pmatrix} H_+ \\ H_+ \end{pmatrix} \cap S^t \begin{pmatrix} H_- \\ H_- \end{pmatrix} \xleftarrow{\sim} \left\{ \begin{pmatrix} f_- \\ f_-' \end{pmatrix} \middle| \, S^t \begin{pmatrix} f_- \\ f_-' \end{pmatrix} \in \begin{pmatrix} H_+ \\ H_+ \end{pmatrix} \right\}$$

$$\Updownarrow$$

$$\operatorname{Ker} \begin{pmatrix} \pi_- & 0 \\ 0 & \pi_- \end{pmatrix} S^t \text{ on } \begin{pmatrix} H_- \\ H_- \end{pmatrix} = \left\{ \begin{pmatrix} f_- \\ f_-' \end{pmatrix} \middle| \begin{pmatrix} \pi_- & 0 \\ 0 & \pi_- \end{pmatrix} S^t \begin{pmatrix} f_- \\ f_-' \end{pmatrix} = 0 \right\}$$

Toeplitz operator

~~Simplest case~~ $S = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$. Assume $\begin{pmatrix} f \\ g \end{pmatrix} \in H_-^{\oplus 2}$

and $S^t \begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix} \begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} \alpha f + \gamma g \\ \beta f + \delta g \end{pmatrix} \in H_+^{\oplus 2}$

$f, g \in H_-$ $\qquad \alpha f + \gamma g \in H_+$

$\qquad\qquad\quad \beta f + \delta g \in H_+$ $\qquad \alpha = \delta \in H_+$

$\frac{1}{d} f - \frac{c}{d} g \in H_+$ $\qquad f - cg \in H_+$

$\frac{b}{d} f + \frac{1}{d} g \in H_+$ $\qquad bf + g \in H_+$

$$\begin{pmatrix} \frac{1}{d} & \frac{b}{d} \\ -\frac{c}{d} & \frac{1}{d} \end{pmatrix} = \underbrace{\begin{pmatrix} a^r & b^r \\ -c^l & a^l \end{pmatrix}}_{\begin{pmatrix} zH_- & H_- \\ zH_- & zH_- \end{pmatrix}} \underbrace{\frac{1}{d}\begin{pmatrix} d^r & b^l \\ -c^r & d^l \end{pmatrix}}_{\begin{pmatrix} H_+ & H_+ \\ zH_+ & H_+ \end{pmatrix}}$$

$$\begin{pmatrix} f & g \end{pmatrix}\begin{pmatrix} a^r & b^r \\ -c^l & a^l \end{pmatrix} \frac{1}{d}\begin{pmatrix} d^r & b^l \\ -c^r & d^l \end{pmatrix} \in \begin{pmatrix} H_+ \\ H_+ \end{pmatrix}$$

$$\underbrace{\begin{pmatrix} f & g \end{pmatrix}\begin{pmatrix} a^r & b^r \\ -c^l & a^l \end{pmatrix}}_{\in \begin{pmatrix} H_- \\ H_- \end{pmatrix}} \in \begin{pmatrix} d^l & b^l \\ c^r & d^r \end{pmatrix}\begin{pmatrix} H_+ \\ H_+ \end{pmatrix} \subset \begin{pmatrix} H_+ \\ H_+ \end{pmatrix}$$

definitely on the right track.

$$\begin{tikzcd}
\begin{pmatrix} L^2 \\ L^2 \end{pmatrix} \arrow[r, "(\xi'_- \ \xi_-)", "\sim"'] & E & \begin{pmatrix} L^2 \\ L^2 \end{pmatrix} \arrow[l, "(\xi_+ \ \xi'_+)"', "\sim"] \arrow[ll, bend right, "S^t"'] \\
\begin{pmatrix} H_+ \\ H_+ \end{pmatrix} \arrow[u, phantom, "\cup"] \arrow[r, "\sim"] & E_+ \quad E_- \arrow[u, phantom, "\cup \ \cup"] & \begin{pmatrix} H_- \\ H_- \end{pmatrix} \arrow[u, phantom, "\cup"] \arrow[l, "\sim"']
\end{tikzcd}$$

$$\begin{pmatrix} \xi_+ \\ \xi'_+ \end{pmatrix} = S\begin{pmatrix} \xi'_- \\ \xi_- \end{pmatrix}$$

$$\begin{pmatrix} \xi'_- & \xi_- \end{pmatrix} S^t = \begin{pmatrix} \xi_+ & \xi'_+ \end{pmatrix}$$

You want $E_+ \pitchfork E_-$

$$S^t\begin{pmatrix} H_- \\ H_- \end{pmatrix} \pitchfork \begin{pmatrix} H_+ \\ H_+ \end{pmatrix}$$

equiv $\begin{pmatrix} H_- \\ H_- \end{pmatrix} \overset{\pi_-^*}{\subset} \begin{pmatrix} L^2 \\ L^2 \end{pmatrix} \xrightarrow{S^t} \begin{pmatrix} L^2 \\ L^2 \end{pmatrix} \overset{\pi_-}{\twoheadrightarrow} \begin{pmatrix} H_- \\ H_- \end{pmatrix}$ isom.

Thus want $\pi_- S^t \pi_-^* = \begin{pmatrix} \pi_- \alpha \pi_-^* & \pi_- \beta \pi_-^* \\ \pi_- \gamma \pi_-^* & \pi_- \delta \pi_-^* \end{pmatrix}$ invertible on $\begin{pmatrix} H_- \\ H_- \end{pmatrix}$.

~~all~~ always a Fredholm contraction.

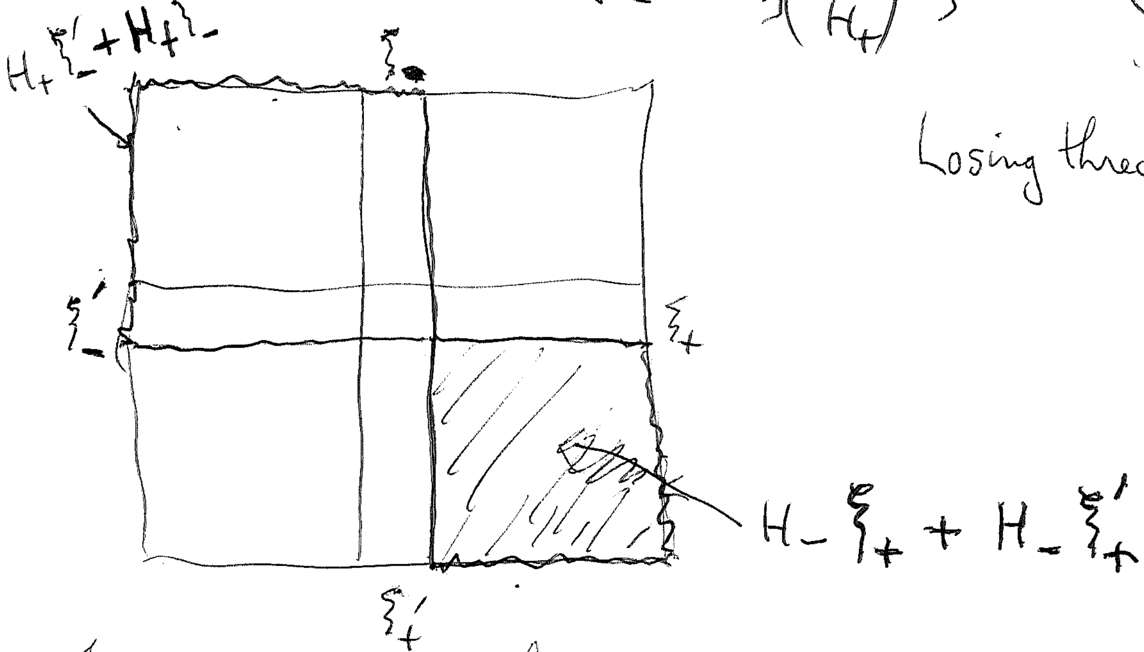
$\alpha = \frac{1}{d} \in H_+$

$\pi_- \frac{1}{d} f_-$

Idea: $E = \begin{pmatrix} \xi'_- & \xi_- \end{pmatrix}\begin{pmatrix} L^2 \\ L^2 \end{pmatrix} = \begin{pmatrix} \xi_+ & \xi'_+ \end{pmatrix}\begin{pmatrix} L^2 \\ L^2 \end{pmatrix}$

two subsp. $E_+ = \begin{pmatrix} \xi'_- & \xi_- \end{pmatrix}\begin{pmatrix} H_+ \\ H_+ \end{pmatrix}$, $E_- = \begin{pmatrix} \xi_+ & \xi'_+ \end{pmatrix}\begin{pmatrix} H_- \\ H_- \end{pmatrix}$

Losing thread.



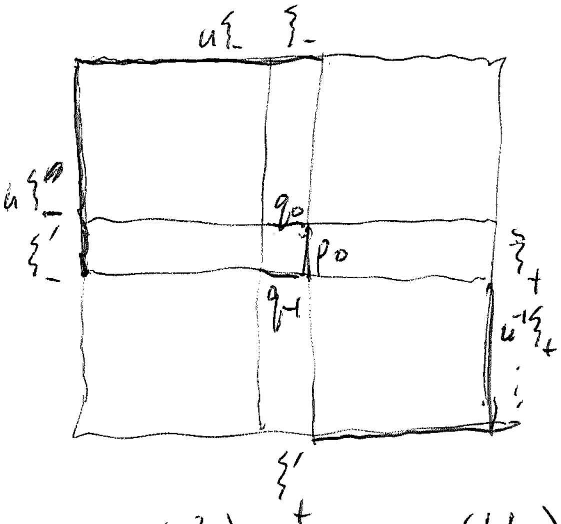
The point is that E_+ E_- are complementary but not necess. $\perp$. So.

$$\begin{tikzcd}
 & & E_- \arrow[d] \arrow[dr, "S", "\text{Toeplitz op}"'] & & \\
0 \arrow[r] & E_+ \arrow[r] \arrow[dr, "\text{induced isom.}"'] & E \arrow[r] \arrow[d] & (E_+)^\perp \arrow[r] & 0 \\
 & & E_-^\perp & &
\end{tikzcd}$$

$$\begin{tikzcd}
H_+^{\oplus 2} \arrow[r, hook] & L^{2,\oplus 2} \arrow[r, two heads, "\pi_-"] & H_-^{\oplus 2} \\
H_+^{\oplus 2} & L^{2,\oplus 2} \arrow[l, two heads] \arrow[u, "S^t"] & H_-^{\oplus 2} \arrow[l, hook'] \arrow[u, dotted]
\end{tikzcd}$$

Big puzzle. Review carefully.

$$E = (\xi'_- \ \ \xi_-)\begin{pmatrix}L^2\\ L^2\end{pmatrix} = (\xi_+ \ \ \xi'_+)\begin{pmatrix}L^2\\ L^2\end{pmatrix}$$



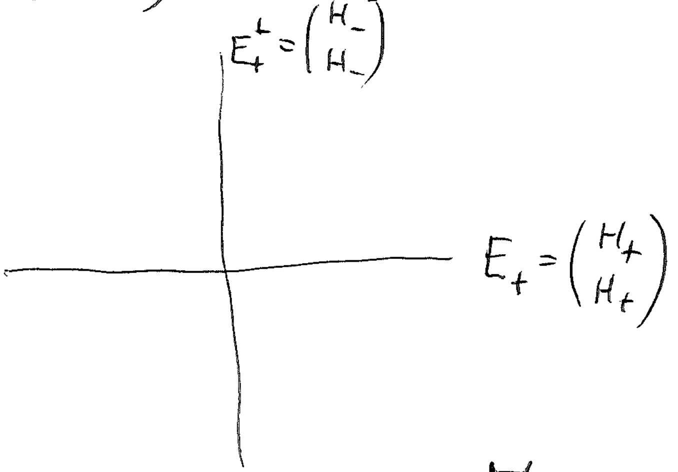
$$E_+ = (\xi'_- \ \ \xi_-)\begin{pmatrix}H_+\\ H_+\end{pmatrix}$$

$$E_- = (\xi_+ \ \ \xi'_+)\begin{pmatrix}H_-\\ H_-\end{pmatrix} = (\xi'_- \ \ \xi_-)\, S^t \begin{pmatrix}H_-\\ H_-\end{pmatrix}$$

To show E_+, E_- complementary

Use isom $(\xi'_- \ \ \xi_-)$ to assume

$$E = \begin{pmatrix}L^2\\ L^2\end{pmatrix} \qquad E_+ = \begin{pmatrix}H_+\\ H_+\end{pmatrix} \qquad E_- = S^t\begin{pmatrix}H_-\\ H_-\end{pmatrix} \qquad S^t = \begin{pmatrix}\alpha & \gamma\\ \beta & \delta\end{pmatrix} = \begin{pmatrix}\frac{1}{d} & -\frac{c}{d}\\ \frac{b}{d} & \frac{1}{d}\end{pmatrix}$$



Shift notation. V, g unitary on V, $V = V_+ \oplus V_-$ orthogonal

Q: When is gV_- complementary to V_+?

$$\begin{tikzcd}
0 \arrow[r] & V_+ \arrow[r] \arrow[d, "(g^{-1})_+"] & V \arrow[r] \arrow[dd, shift right, "g^{-1}"'] & V/V_+ \arrow[r] \arrow[d, equal] & 0 \\
 & V_+ \arrow[d, equal] & & V_- & \\
0 & V/V_- \arrow[l] & V \arrow[l] \arrow[uu, shift right, "g"'] & V_- \arrow[l] \arrow[u, "(g)_-"] & 0 \arrow[l]
\end{tikzcd}$$

$gV_- \oplus V_+ = V \iff g^{-1}V_+ \oplus V_- = V$ and in this case g ~~is the~~ isom to direct sum of $\underbrace{(g^{-1})_+}_{\text{the Toeplitz ops.}}$ on V_+ and $(g)_-$ on V_-.

But you want a factorization, but this is ok maybe since $g = (g)_- \oplus (g^{-1})_+ = \begin{pmatrix}(g)_- & \\ & 1\end{pmatrix}\begin{pmatrix}1 & \\ & (g^{-1})_+\end{pmatrix}$

so it might work, but the surprise will be why these operators commute with z.

Work out your example.

$$\begin{pmatrix}p_0\\ q_0\end{pmatrix} = \begin{pmatrix}a^\ell & b^\ell\\ c^\ell & d^\ell\end{pmatrix}\begin{pmatrix}\xi'_-\\ \xi'_+\end{pmatrix} = \begin{pmatrix}d^r & -b^r\\ -c^r & a^r\end{pmatrix}\begin{pmatrix}\xi_+\\ \xi_-\end{pmatrix}$$

$$\begin{matrix}\cap \\ \begin{pmatrix}zH_- & H_+\\ zH_- & H_+\end{pmatrix}\end{matrix} \qquad\qquad \begin{matrix}\cap \\ \begin{pmatrix}H_+ & H_-\\ zH_+ & zH_-\end{pmatrix}\end{matrix}$$

$$\begin{pmatrix}\frac{1}{d} & \frac{b}{d}\\ -\frac{c}{d} & \frac{t}{d}\end{pmatrix} = \begin{pmatrix}a^r & b^r\\ -c^\ell & a^\ell\end{pmatrix}\underbrace{\frac{1}{d}\begin{pmatrix}d^r & b^\ell\\ -c^r & d^\ell\end{pmatrix}}$$

$$\begin{matrix}\cap \\ \begin{pmatrix}zH_- & H_-\\ zH_- & zH_-\end{pmatrix}\end{matrix} \qquad \begin{pmatrix}H_+ & H_+\\ zH_+ & H_+\end{pmatrix}$$

Not thinking correctly. Think $\mathbb{P}^1$ vector bundles.

V = space of sections over S^1

$V_\pm$ = ——— holom. over $D_\pm$

g = clutching function. ~~Suppose g is coherent.~~

$$\begin{tikzcd}
0 \arrow[r] & V_- \arrow[r, "in_-"] & V \arrow[r, "pr_-"] \arrow[d, "g"] & V/V_- \arrow[r] & 0 \\
0 & V/V_+ \arrow[l] & V \arrow[l, "pr_+"'] & V_+ \arrow[l, "in_+"'] & 0 \arrow[l]
\end{tikzcd}$$

You want H^1 to be zero.

$$0 \longrightarrow \begin{matrix}\Gamma(D_+)\\ \oplus \\ \Gamma(D_-)\end{matrix} \xrightarrow{\sim} \Gamma(S^1) \longrightarrow 0$$

$$\begin{matrix}V_+\\ \oplus \\ V_-\end{matrix} \xrightarrow{(in_+ \ \ g\, in_-)} V \quad \text{is an isom.} \quad \Longleftrightarrow \quad V_- \xrightarrow[pr_+ g\, in_-]{\sim} V/V_+$$

How does this relate to factorization

$g = g_+ g_-$ in $\mathrm{Aut}(V)$.

$$\begin{matrix}V_+\\ \oplus \\ V_-\end{matrix} \xrightarrow{(in_+ \ \ g \cdot in_-)} V \ ?$$

Assume that ~~[illegible]~~ you can factor g into $g_+ g_-$. Here g is an invertible matrix function over S^1, $g_\pm$ are invertible holomorphic matrix functions over $D_\pm$.

You have a vector bundle over $\mathbb{P}^1$ obtained by glueing trivial bundles over $D_\pm$ using a clutching function. ~~Holom. fns~~ Take $\xi \in \Gamma(S^1, \mathcal{E})$. Write $g_+^{-1}\xi = \xi_- \oplus \xi_+ \in V_- \oplus V_+$?

Given $\xi \in \Gamma(S^1, \mathcal{E})$ you want to write $\xi = \xi_1 + \xi_2$ where ξ_1 extends holom. inside and ξ_2 extends holom. outside. Thus you want $\xi_1 \in V_+$ and $\xi_2 \in g V_-$. Suppose

Can do $\xi = v_+ + g v_-$ and $g_1 = g_+ g_-$

Then
$$\xi = v_+ + g_+ g_- v_-$$
$$g_+^{-1}\xi = g_+^{-1} v_+ + g_- v_- \in V_+ \oplus V_-$$

This is how to proceed: Take $\xi \in V = \Gamma(S^1, \mathcal{E})$, form $g_+^{-1}\xi$ and write $g_+^{-1}\xi = w_+ + w_- \in V_+ \oplus V_-$

Then $\xi = g_+ w_+ + g_+ g_-(g_-^{-1} w_-) \in V_+ \oplus g V_-$

~~Conversely~~ Let $\pi_\pm$ be projections onto $V_\pm$ relative to $V = V_+ \oplus V_-$. Then

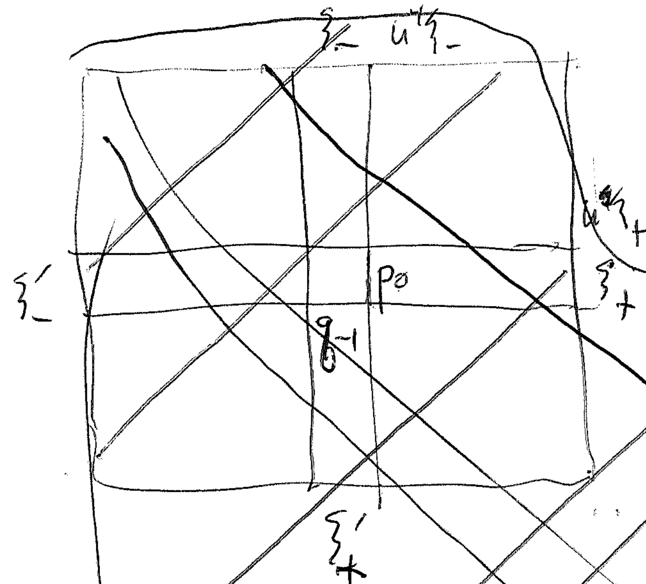
$$g_+^{-1}\xi = \pi_+ g_+^{-1}\xi + \pi_- g_+^{-1}\xi \in V_+ \oplus V_-$$

$$\xi = g_+ \pi_+ g_+^{-1}\xi + (g_+ g_-)\, g_-^{-1} \pi_- g_+^{-1}\xi$$
$$\in V_+ + g V_-$$

so conjugation by g_+ transforms the $V_+ \oplus V_-$ splitting into the $V_+ \oplus g V_-$ splitting. Now can you get the converse?

You want to assume that g has the property that $V = V_+ \oplus g V_-$. How unique is the factorization $g = g_+ g_-$? All you have used [redacted] so far is that $g_\pm$ is invertible on $V_\pm$.

$$\begin{pmatrix}\xi_+\\ \xi_-\end{pmatrix} = \begin{pmatrix} a & b\\ c & d\end{pmatrix}\begin{pmatrix}\xi'_-\\ \xi'_+\end{pmatrix} \qquad \begin{pmatrix}\xi_+\\ \xi'_+\end{pmatrix} = \underbrace{\begin{pmatrix}\frac1d & \frac bd\\ -\frac cd & \frac1d\end{pmatrix}}_{\cap\ \begin{pmatrix}H_+ & L^2\\ L^2 & H_+\end{pmatrix}}\begin{pmatrix}\xi'_-\\ \xi_-\end{pmatrix}$$



$$\begin{pmatrix}p_0\\ q_0\end{pmatrix} = \frac{1}{k_0}\begin{pmatrix} z & h_0\\ z\tilde h_0 & 1\end{pmatrix}\begin{pmatrix}p_{-1}\\ q_{-1}\end{pmatrix}$$

$$\begin{pmatrix}p_0\\ q_{-1}\end{pmatrix} = \begin{pmatrix} & \\ & \end{pmatrix}\begin{pmatrix}p_{-1}\\ q_0\end{pmatrix}$$

$$\begin{pmatrix}p_0\\ q_{-1}\end{pmatrix} = \begin{pmatrix} a_l & b_l\\ c_l & d_l\end{pmatrix}\begin{pmatrix}\xi'_-\\ \xi'_+\end{pmatrix} = \begin{pmatrix} d_r & -b_r\\ -c_r & a_r\end{pmatrix}\begin{pmatrix}\xi_+\\ \xi_-\end{pmatrix}$$

$$\cap\begin{pmatrix} \mathbb{R}_{>0}\oplus \cancel{H_+} & H_+\\ H_- & \mathbb{R}_{>0}\oplus zH_+\end{pmatrix} \qquad \cap\begin{pmatrix} \mathbb{R}_{>0}\oplus zH_+ & H_-\\ H_+ & \mathbb{R}_{>0}\oplus H_-\end{pmatrix}$$

$$\begin{pmatrix}\frac1d & +\frac bd\\ -\frac cd & \frac1d\end{pmatrix} = \begin{pmatrix} a_r & b_r\\ -c_l & a_l\end{pmatrix}\frac1d\begin{pmatrix} d_r & b_l\\ -c_r & d_l\end{pmatrix}$$

$$S^t = \begin{pmatrix}\frac1d & -\frac cd\\ \frac bd & \frac1d\end{pmatrix} = \frac1d\underbrace{\begin{pmatrix} d_r & -c_r\\ b_l & d_l\end{pmatrix}}_{g_+}\underbrace{\begin{pmatrix} a_l & -c_l\\ b_r & a_l\end{pmatrix}}_{g_-}$$

[redacted]

V_-

Anyway consider g a smooth loop of degree zero, and ~~compare~~ look at the ~~situation~~ question whether H_+ and gH_- are complementary in $L^2(S^1)$. Answer yes because you have factorization $g = e^{\log(g)} = e^{f_+}e^{f_-} = g_+g_-$. So ~~what might happen~~ what about the Toeplitz op.?

so what next?

$$\begin{tikzcd}
H_- \arrow[r] \arrow[d] & L^2 \arrow[r] \arrow[d, "g", "S"'] & H_+ & \\
H_- & L^2 \arrow[l] & H_+ \arrow[l] & 0 \arrow[l]
\end{tikzcd}$$

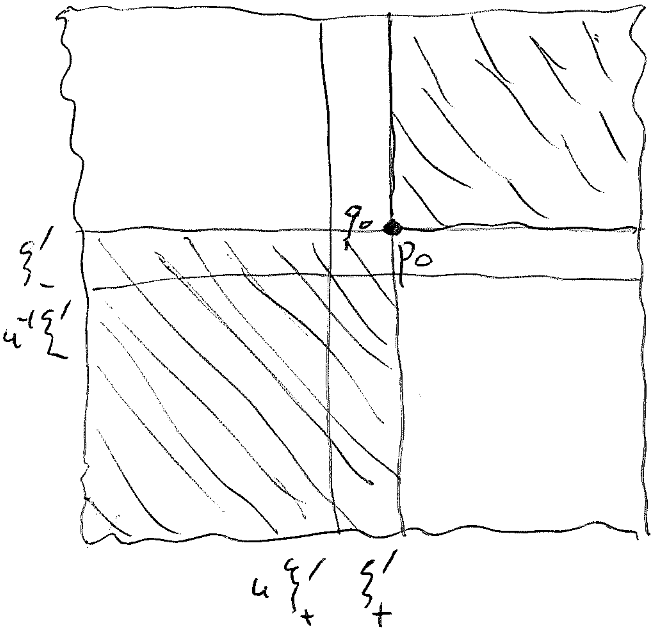
$f_- \in H_-$ $\qquad$ $gf_- = g_+ g_- f_-$

$(\bar{z}^m | g \bar{z}^n) = g_{n-m}$ $\qquad$ $m, n \geq 0$

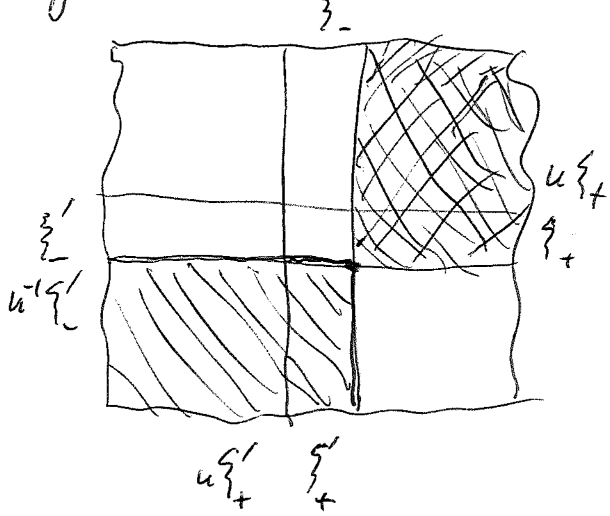
$$\begin{array}{|lll}
\hline
g_0 & g_1 & g_2 \\
g_{-1} & g_0 & \\
g_{-2} & g_{-1} & \\
& &
\end{array}$$

So far you have been unable to produce the factorization ~~for~~ the S-matrix directly. So leave ~~[illegible]~~ the S-matrix + return to the transfer matrix. It's possible that ~~[illegible]~~ splittings are not a good thing to study, to begin with.

Begin with a vertex and construct the corresp. splitting into orthogonal "space" cones.



Try instead



$$H_-\xi'_- + H_+\xi'_+ \quad , \quad H_+\xi_+ + H_-\xi_-$$

$$\parallel \qquad\qquad \parallel$$

$$\begin{pmatrix}\xi'_- & \xi'_+\end{pmatrix}\begin{pmatrix}H_- \\ H_+\end{pmatrix} \qquad \begin{pmatrix}\xi_+ & \xi_-\end{pmatrix}\begin{pmatrix}H_+ \\ H_-\end{pmatrix}$$

$$(f_-\xi'_- \mid f_+\xi_+) \qquad \begin{pmatrix}\xi_+ \\ \xi'_+\end{pmatrix} = \begin{pmatrix}\alpha & \beta \\ \gamma & \delta\end{pmatrix}\begin{pmatrix}\xi'_- \\ \xi_-\end{pmatrix}$$

$$(f_-\bar{f}_+\xi'_- \mid \alpha\xi'_- + \cancel{\beta\xi_-}) = (f_-\bar{f}_+ \mid \alpha)$$

$$H_- \quad zH_-$$

$$H_-$$

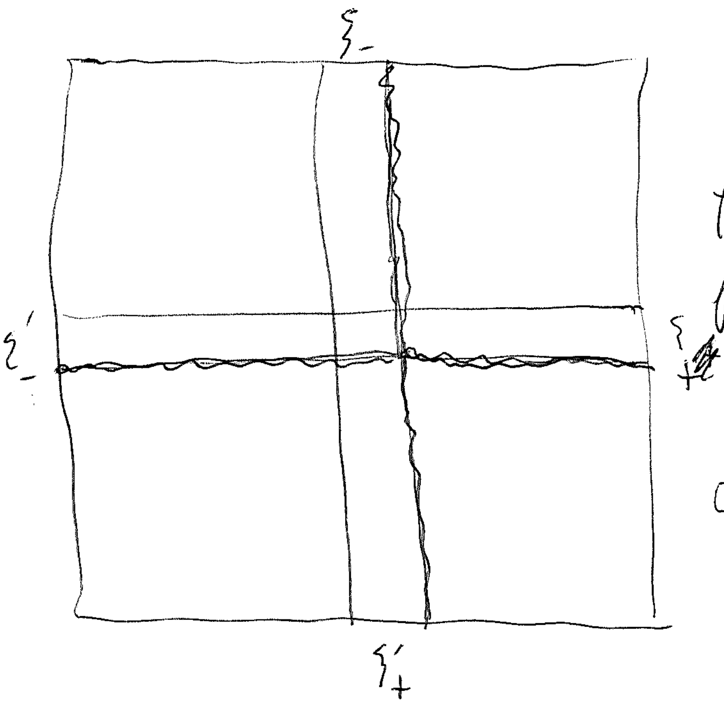
Given β form $E = L^2\xi_+ + L^2\xi_-$ with

$$(u^n\xi_- | \xi_+) = (z^n | \beta)$$

$$\| f_1\xi_+ + f_2\xi_- \|^2 = \int \begin{pmatrix}\bar f_1 \\ \bar f_2\end{pmatrix} \underbrace{\begin{pmatrix}1 & \bar\beta \\ \beta & 1\end{pmatrix}}_{\begin{pmatrix}1\\ \beta\end{pmatrix}(1 \ \ \bar\beta) + \begin{pmatrix}0 & 0 \\ 0 & 1-|\beta|^2\end{pmatrix}} \begin{pmatrix}f_1 \\ f_2\end{pmatrix} \frac{d\theta}{2\pi}$$

~~These~~ $\begin{pmatrix}L^2 \\ L^2\end{pmatrix} \xrightarrow{(\xi_+ \ \ \xi_-)} E$ bdd invertible

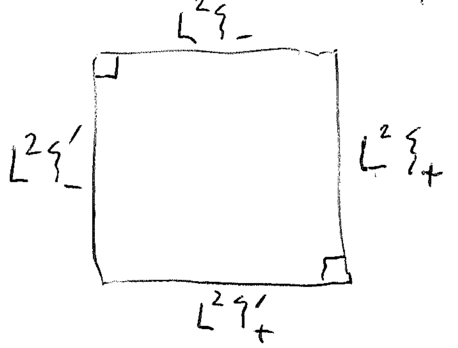
so all the subspaces $z^\mu H_+\xi_+ + z^\nu H_-\xi_-$ of E all closed. Orth. complement of $H_+\xi_+$ is $H_-\xi_+ + L^2\xi'_+$, $(H_-\xi_-)^\perp = H_+\xi_- + L^2\xi'_-$



picture of increasing bifiltration coming in from the right. You must prove that $H_+\xi_+ + H_-\xi_-$ and $H_-\xi'_- + H_+\xi'_+$ are complementary.

ξ_+

Construction from β.



$$\begin{pmatrix}\xi_+\\ \xi'_+\end{pmatrix} = \begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\begin{pmatrix}\xi'_-\\ \xi_-\end{pmatrix}$$

$\alpha = \delta$
$\in H_+^{inv.}$

Picture: Given $\beta(z)$ $\quad |\beta(z)| \leq 1-\varepsilon \quad \varepsilon > 0$

Then glue $L^2\xi_+$ $L^2\xi_-$ together

$$\left(g_1\xi_+ + g_2\xi_- \middle| f_1\xi_+ + f_2\xi_-\right) = \int \bar{g}_1 f_1 + \bar{g}_2 f_1 \beta + \bar{g}_1 f_2 \bar{\beta} + \bar{g}_2 f_2$$

$$= \int \begin{pmatrix} g_1 \\ g_2\end{pmatrix}^* \underbrace{\begin{pmatrix}1 & \bar{\beta}\\ \beta & 1\end{pmatrix}} \begin{pmatrix} f_1\\ f_2\end{pmatrix} \frac{d\theta}{2\pi}$$

$$\lambda^2 - 2\lambda + 1 - |\beta|^2 = 0$$

$$\lambda = +1 \pm \sqrt{1-(1-|\beta|^2)}$$
$$= 1 \pm |\beta|$$

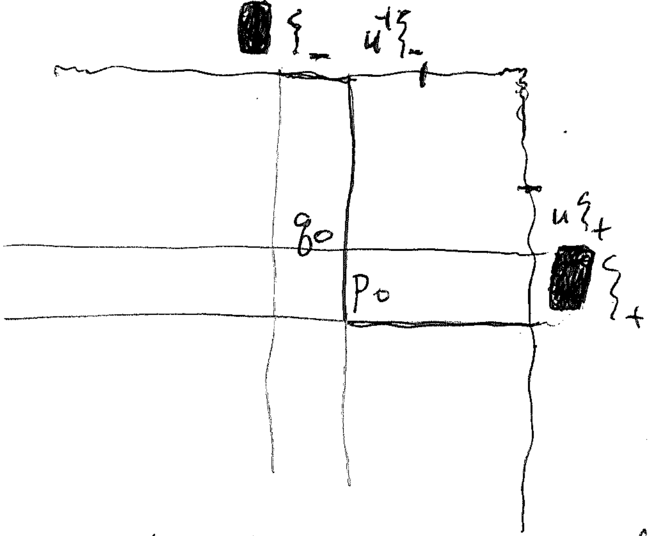
$$\begin{pmatrix}1 & \bar{\beta}\\ \beta & 1\end{pmatrix} - \begin{pmatrix}1-|\beta| & \\ & 1-|\beta|\end{pmatrix}$$

$$= \begin{pmatrix}|\beta| & \bar{\beta}\\ \beta & |\beta|\end{pmatrix} = \begin{pmatrix}|\beta| \\ \beta\end{pmatrix}\frac{1}{|\beta|}\begin{pmatrix}|\beta| & \bar{\beta}\end{pmatrix}$$

$$\begin{pmatrix}\bar{x} & \bar{y}\end{pmatrix}\begin{pmatrix}|\beta| \\ \beta\end{pmatrix}\frac{1}{|\beta|}\begin{pmatrix}|\beta| & \bar{\beta}\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}$$

$$\frac{1}{|\beta|}\left|\,|\beta| x + \bar{\beta} y\,\right|^2$$

Form



$$H_+\xi_+ + H_-\xi_-$$

$$\begin{pmatrix} H_+ & H_- \\ zH_+ & zH_- \end{pmatrix}$$

$$\Psi$$

Your construction yields $\begin{pmatrix} p_0 \\ q_0 \end{pmatrix} = \begin{pmatrix} d^r & -b^r \\ -c^r & a^r \end{pmatrix} \begin{pmatrix} \xi_+ \\ \xi_- \end{pmatrix}$

$$\tilde{p}_0 = \sum_{j(\geq 0)} d_j u^j \xi_+ - \sum_{k(<0)} b_k u^k \xi_- \qquad \begin{array}{l} d_j = 0 \quad j<0 \\ d_0 > 0 \\ b_k = 0,\ k \geq 0 \end{array}$$

$$0 = (u^k\xi_- \mid \tilde{p}_0) = \sum_{j(\geq 0)} d_j \beta_{k-j} - b_k \qquad k<0,$$

$$\underset{j>0}{0} = (u^j\xi_+ \mid \tilde{p}_0) = d_j - \sum_k b_k \bar{\beta}_{k-j} \qquad j>0$$

$$d^r\beta - b^r \in H_+$$
$$d^r - b^r\bar{\beta} \in zH_-$$

$$\tilde{q}_0 = +\sum_{k(\leq 0)} a_k u^k \xi_- - \sum_{j(>0)} c_j u^j \xi_+$$

$$\underset{j>0}{0} = (u^j\xi_+ \mid \tilde{q}_0) = \sum_k a_k \bar{\beta}_{k-j} - c_j \qquad j>0$$

$$\underset{k\leq 0}{0} = (u^k\xi_- \mid \tilde{q}_0) = a_k - \sum_{j(>0)} c_j \beta_{k-j} \qquad k \leq 0$$

$$a^r\bar{\beta} - c^r \in zH_- \qquad\qquad a^r - c^r\beta \in H_+$$

$$\begin{pmatrix} 1 & \bar{\beta} \\ \beta & 1 \end{pmatrix}\begin{pmatrix} d^r & -c^r \\ -b^r & a^r \end{pmatrix} \in \begin{pmatrix} zH_- & zH_- \\ H_+ & H_+ \end{pmatrix}$$

$$\begin{pmatrix} d^r & -b^r \\ -c^r & a^r \end{pmatrix}\begin{pmatrix} 1 & \frac{b}{d} \\ \frac{c}{a} & 1 \end{pmatrix} \; ?$$

$$d^r\frac{b}{d} - b^r \in H_+ \qquad a^r\frac{c}{a} - c^r \in zH_-$$

$$d^r - b^r\frac{b}{a} \in zH_- \qquad a^r - c^r\frac{b}{d} \in H_+$$

What to do?

$$\begin{pmatrix} d^r & -b^r \\ -c^r & a^r \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} d^r a - b^r c & d^r b - b^r d \\ -c^r a + a^r c & -c^r b + a^r d \end{pmatrix}$$

$$\cap$$

$$\begin{pmatrix} zH_- & H_+ \\ zH_- & H_+ \end{pmatrix}$$

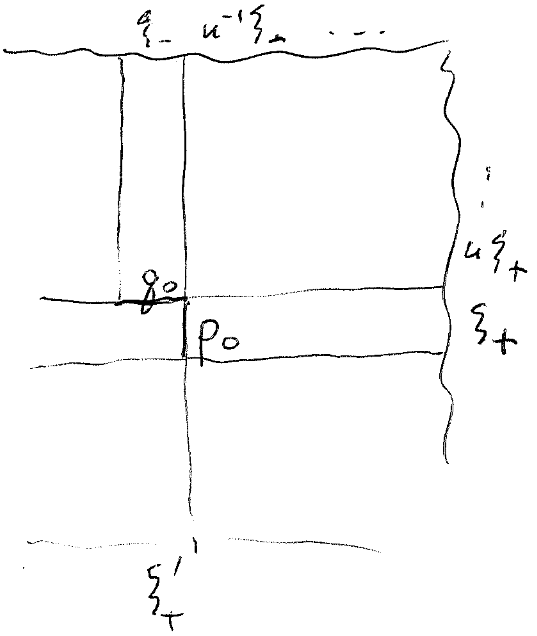
$$(q_0|p_0) = ?$$

$$\tilde{p}_0 = \xi_+ + \underbrace{f_1\xi_+ + f_2\xi_-} \qquad f_2 \in H_-$$

$$\tilde{p}_0 \perp \qquad f_1 \in zH_+$$

$$\therefore \; \|\tilde{p}_0\|^2 + \|f_1\xi_+ + f_2\xi_-\|^2 = \|\xi_+\|^2 = 1.$$

So ~~[illegible]~~ back to estimating
$h_0 = (q_0|p_0)$.



$$\begin{pmatrix} p_0\\ q_0\end{pmatrix} = \underset{\begin{pmatrix} H_+ & H_- \\ zH_+ & zH_-\end{pmatrix}}{\underset{\Uparrow}{\begin{pmatrix} d_r & -b_r \\ -c_r & a_r\end{pmatrix}}}\begin{pmatrix}\xi_+\\ \xi_-\end{pmatrix}$$

orthogonality condition says

$$\begin{pmatrix} d^r & -b_r\\ -c_r & a_r\end{pmatrix}\begin{pmatrix} a & b\\ c & d\end{pmatrix} = \begin{pmatrix} a^\ell & b^\ell \\ c^\ell & d^\ell\end{pmatrix}$$

can be rewritten

$$\boxed{\begin{pmatrix} d^r & -b_r\\ -c_r & a_r\end{pmatrix}\begin{pmatrix} 1 & \frac{b}{d}\\ \frac{c}{a} & 1\end{pmatrix} \in \begin{pmatrix} zH_- & H_+ \\ zH_- & H_+\end{pmatrix}}$$

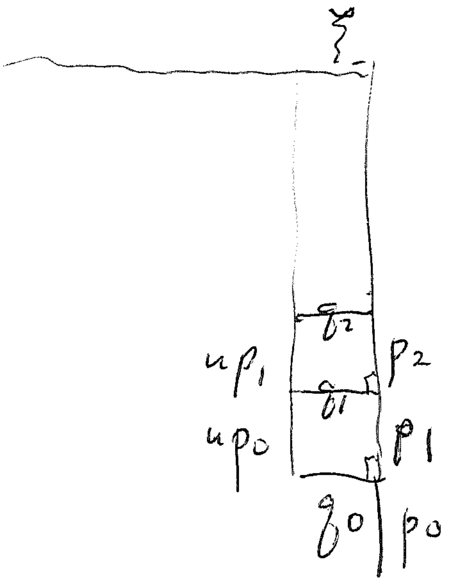
$$p_0 = \sum_{j\geq 0} d_j u^j \xi_+ - \sum_{k<0} b_k u^k \xi_-$$

$$(q_0|p_0) = (\underset{-c_r\xi_+ + a_r\xi_-}{\underset{|}{q_0}}|\, d_0 \xi_+)$$

$$p_0 = \sum_{j\geq 0} d_j u^j \xi_+ - \sum_{k<0} b_k u^k \xi_-$$

$$q_0 = -c^r \xi_+ + a^r \xi_- = -\sum_{j\geq 0} c_j u^j \xi_+ + \sum_{k\leq 0} a_k u^k \xi_-$$

$$h_0 = (q_0|p_0) = (q_0|d_0\xi_+) = (q_0|\xi_+)\, d_0$$



$$\begin{pmatrix} p_1 \\ q_1 \end{pmatrix} = \frac{1}{k_1}\begin{pmatrix} 1 & h_1 \\ \bar{h}_1 & 1 \end{pmatrix}\begin{pmatrix} up_0 \\ q_0 \end{pmatrix}$$

$$\begin{pmatrix} p_1 \\ q_0 \end{pmatrix} = \begin{pmatrix} k_1 & h_1 \\ -\bar{h}_1 & k_1 \end{pmatrix}\begin{pmatrix} up_0 \\ q_1 \end{pmatrix}$$

$$\begin{pmatrix} up_0 \\ q_1 \end{pmatrix} = \begin{pmatrix} k_1 & -h_1 \\ \bar{h}_1 & k_1 \end{pmatrix}\begin{pmatrix} p_1 \\ q_0 \end{pmatrix}$$

$$q_1 = \bar{h}_1 p_1 + k_1 q_0$$

$$q_2 = \bar{h}_2 p_2 + k_2(\bar{h}_1 p_1 + k_1 q_0)$$

$$q_3 = \bar{h}_3 p_3 + k_3 \bar{h}_2 p_2 + k_3 k_2 \bar{h}_1 p_1 + k_3 k_2 k_1 q_0$$

$$q_n = \sum_{i=1}^{n} k_n \cdots k_{i+1} \bar{h}_i p_i + k_n \cdots k_1 q_0$$

Check

$$1 = \|q_n\|^2 = \sum_{i=1}^{n} k_n^2 \cdots k_{i+1}^2 (1-k_i^2) + k_n^2 \cdots k_1^2$$

$$= -(k_n^2 \cdots k_1^2) + (k_n^2 \cdots k_2^2)$$
$$- (k_n^2 \cdots k_2^2) + (k_n^2 \cdots k_3^2)$$

Want things in terms of $|h_i|^2$

$$(1-|h_n|^2) \cdots (1-|h_{i+1}|^2)|h_i|^2$$

Somehow reconstruct $(q_0 | p_0)$

$$(q_0|p_0) = (q_0 | d_0 \xi_+) = (q_0|\xi_+)\, d_0$$

$$\begin{pmatrix} p_0 \\ q_0 \end{pmatrix} = \begin{pmatrix} d^r & -b^r \\ -c^r & a^r \end{pmatrix} \begin{pmatrix} \xi_+ \\ \xi_- \end{pmatrix}$$

$$(\xi_+ | q_0) = (\xi_+ | -c^r \xi_+ + a^r \xi_-)$$

$$= (\xi_+ | a^r \xi_-) = \overline{(a^r \xi_- | \xi_+)}$$

$$= \overline{(a^r | \beta)} = (\beta | a^r)$$

so $(q_0|p_0) = d_0 \overline{(\xi_+ | q_0)} = d_0 (a^r | \beta)$.

Clare Allen 017.84 415 6/5 29th Jan

$$p_0 = d^r \xi_+ - b^r \xi_-$$

$$q_0 = -c^r \xi_+ + a^r \xi_-$$

$$\begin{pmatrix} d^r & -b^r \\ -c^r & a^r \end{pmatrix} \begin{pmatrix} 1 & \beta \\ \bar\beta & 1 \end{pmatrix} \in \begin{pmatrix} zH_- & H_+ \\ zH_- & H_+ \end{pmatrix}$$

$$(q_0|p_0) = (-c^r \xi_+ + a^r \xi_- | d^r \xi_+ - b^r \xi_-)$$

$$= (a_0 \xi_- | p_0) = (a_0 \xi_- | d^r \xi_+)$$

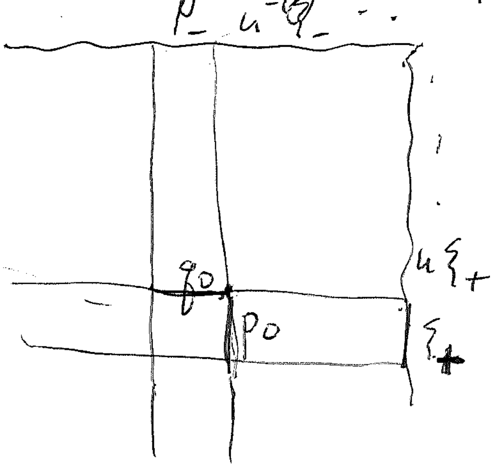
$$= \bar a_0 \int d^r \beta$$

$$\int b\ell$$

$$\|$$

$$d^r\beta - \underset{\substack{\Uparrow \\ H_1}}{b^r} \in H_+ \qquad \int d^r \beta = \int (d^r\beta - b^r) = \int \frac{d^r b - b^r d}{d}$$

Anyway repeat.



$$\begin{pmatrix} p_0 \\ q_0 \end{pmatrix} = \begin{pmatrix} d^r & -b^r \\ -c^r & a^r \end{pmatrix} \begin{pmatrix} \xi_+ \\ \xi_- \end{pmatrix}$$

$$\begin{pmatrix} d^r & -b^r \\ -c^r & a^r \end{pmatrix} \in \begin{pmatrix} H_+ & H_- \\ zH_+ & zH_- \end{pmatrix}$$

$$\begin{pmatrix} d^r & -b^r \\ -c^r & a^r \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a^l & b^l \\ c^l & d^l \end{pmatrix} \in \begin{pmatrix} zH_- & H_+ \\ zH_- & H_+ \end{pmatrix}$$

$$\Downarrow$$

$$\begin{pmatrix} d^r & -b^r \\ -c^r & a^r \end{pmatrix} \begin{pmatrix} 1 & b/d \\ c/a & 1 \end{pmatrix} = \begin{pmatrix} \frac{a^l}{a} & \frac{b^l}{d} \\ \frac{c^l}{a} & \frac{d^l}{d} \end{pmatrix}$$

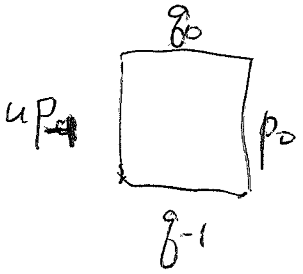
$$(q_0|p_0) = (q_0|\, d^r\xi_+ - b^r\xi_-) = (q_0|\, d^r(0)\,\xi_+)$$

$$= (\cancel{-c^r\xi_+} + a^r\xi_- \,|\, \xi_+)\, d^r(0)$$

$$= (a^r|\beta)\, d^r(0) = \Big(\underbrace{\int d^r\beta}\Big)\, d^r(0)$$

$$\int d^r\beta - b^r = \int \frac{b^l}{d} = \frac{b^l(0)}{d(0)}$$

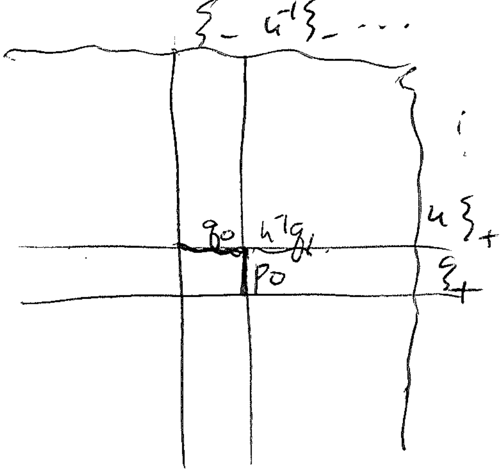
$$\therefore\ (q_0|p_0) = \frac{b^l(0)\, d^r(0)}{d(0)}$$



Let's do L^2 ~~stuff~~

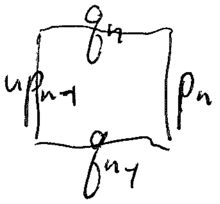
$$\frac{b^l(0)}{d^l(0)}$$

Review.



$$p_0 \in H_+\xi_+ + H_-\xi_-$$
$$q_0 \in zH_+\xi_+ + zH_-\xi_-$$

$$\begin{pmatrix} p_0 \\ q_0 \end{pmatrix} = \begin{pmatrix} d^r & -b^r \\ -c^r & a^r \end{pmatrix} \begin{pmatrix} \xi_+ \\ \xi_- \end{pmatrix}$$



$$\begin{aligned} q_n &= \bar{h}_n p_n + k_n q_{n-1} \\ &= \bar{h}_n p_n + k_n \bar{h}_{n-1} p_{n-1} + k_n k_{n-1} q_{n-2} \\ &= \underbrace{\sum_{i=1}^{n} k_n \cdots k_{i+1} \bar{h}_i p_i}_{\perp\, q_0,\ p_0} + k_n \cdots k_1 q_0 \end{aligned}$$

$p_n = h_n q_n + k_n u p_{n-1}$

$u^{-n}p_n = h_n u^{-n} q_n + k_n u^{n-1} p_{n-1}$

so $(q_0 | q_n) = k_n \cdots k_1$ $\qquad (q_0 | \xi_-) = \prod_1^\infty k_i$

$(p_0 | q_n) = k_n \cdots k_1 (p_0, q_0)$ $\qquad (p_0 | \xi_-) = \prod_1^\infty k_i \bar{h}_0$

$$\left(\begin{matrix} q_0 \\ p_0 \end{matrix} \middle| u^{-n} p_n \right) = k_n \left(\begin{matrix} q_0 \\ p_0 \end{matrix} \middle| u^{n-1} p_{n-1} \right) = k_n \cdots k_1 \left(\begin{matrix} q_0 \\ p_0 \end{matrix} \middle| p_0 \right)$$

$\therefore$ $(q_0 | \xi_+) = \left(\prod_1^\infty k_i\right) h_0$ $\qquad (p_0 | \xi_+) = \prod_1^\infty k_i$

$$(q_0 | p_0) = \left(\cancel{\quad} + a^r \xi_- \middle| d^r \xi_+ - b^r \xi_- \right)$$
$$= \underbrace{(\xi_- | p_0)}_{\prod k_i\ h_0} a^r(0) \qquad a^r \in zH_-$$

First idea. Exploit the fact that

$$d^r(0) = a^r(0) = \Big(\prod_{i\geq 1} k_i\Big)^{-1} \qquad \underbrace{\begin{pmatrix} a^l & b^l \\ c^l & d^l \end{pmatrix}}$$

Assume $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a^r & b^r \\ c^r & d^r \end{pmatrix}\begin{pmatrix} 1 & h_0 \\ \bar h_0 & 1 \end{pmatrix}\frac{1}{k_0}$

$$\cancel{\frac{b^l(0)\, d^r(0)\, k_0}{(c^r h_0 + d^r h_0)(0)}} \neq \cancel{\frac{b^l(0)\, k_0}{h_0}}$$

$$(q_0|p_0) = \frac{b^l(0)\, d^r(0)}{d(0)} = \frac{h_0\, d^r(0)}{\cancel{c^r(0)}h_0 + d^r(0)} = h_0$$

$$\begin{pmatrix} \xi_+ \\ \xi_- \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} \xi'_- \\ \xi'_+ \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} u p_{-1} \\ q_{-1} \end{pmatrix}$$

$$= \begin{pmatrix} a^r & b^r \\ c^r & d^r \end{pmatrix}\underbrace{\frac{1}{k_0}\begin{pmatrix} 1 & h_0 \\ \bar h_0 & 1 \end{pmatrix}\begin{pmatrix} u p_{-1} \\ q_{-1} \end{pmatrix}}_{\begin{pmatrix} p_0 \\ q_0 \end{pmatrix}}$$

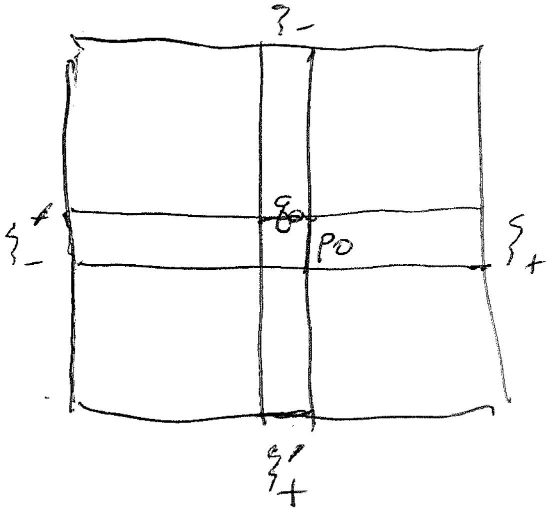
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a^r & b^r \\ c^r & d^r \end{pmatrix}\begin{pmatrix} a^l & b^l \\ c^l & d^l \end{pmatrix}$$

$$d = c^r b^l + d^r d^l \qquad\qquad d(0) = d^r(0)\, d^l(0)$$

$$\begin{pmatrix} p_0 \\ q_0 \end{pmatrix} = \begin{pmatrix} a^l & b^l \\ c^l & d^l \end{pmatrix}\begin{pmatrix} \xi'_- \\ \xi'_+ \end{pmatrix} \in \begin{pmatrix} zH_- & H_+ \\ zH_- & H_+ \end{pmatrix}$$

$$\therefore \quad (q_0 | p_0) = \frac{b^\ell(0)}{d^\ell(0)}$$

What is $\frac{b^\ell}{d^\ell} \in H_+$



$$\begin{pmatrix} p_0 \\ q_0 \end{pmatrix} = \begin{pmatrix} a^\ell & b^\ell \\ c^\ell & d^\ell \end{pmatrix} \begin{pmatrix} \xi'_- \\ \xi'_+ \end{pmatrix}$$

$$\begin{array}{c} \cap \\ \begin{pmatrix} zH_- & H_+ \\ zH_- & H_+ \end{pmatrix} \end{array}$$

$$\cancel{(q_0 | p_0) = (q_0 | a^\ell \xi'_- + b^\ell \xi'_+) = (q_0 | \xi'_-)\, a^\ell(0)}$$

$$\cancel{(c^\ell \xi'_- + d^\ell \xi'_+ | \xi'_-) = b^\ell(0)}$$

$$(q_0 | p_0) = (q_0 | \cancel{a^\ell \xi'_-} + b^\ell \xi'_+) = (q_0 | \xi'_+)\, b^\ell(0)$$

$$= (c^\ell \xi'_- + d^\ell \xi'_+ \,|\, \xi'_+)\, b^\ell(0) \qquad ?$$

$$(c^\ell \xi'_- | \xi'_+) = (c^\ell \xi'_- | -\tfrac{c}{d} \xi'_- + \cancel{\tfrac{b}{d} \xi_-}) = -\int \frac{b^\ell c}{d}$$

$$\begin{pmatrix} \xi'_- \\ \xi'_+ \end{pmatrix} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \begin{pmatrix} \xi_+ \\ \xi_- \end{pmatrix} \qquad \begin{pmatrix} \xi'_- \\ \xi_- \end{pmatrix} = \begin{pmatrix} \frac{1}{a} & -\frac{b}{a} \\ \frac{c}{a} & \frac{1}{a} \end{pmatrix} \begin{pmatrix} \xi_+ \\ \xi'_+ \end{pmatrix}$$

$$\begin{pmatrix} \xi_+ \\ \xi'_+ \end{pmatrix} = \begin{pmatrix} \frac{1}{d} & \frac{b}{d} \\ -\frac{c}{d} & \frac{1}{d} \end{pmatrix} \begin{pmatrix} \xi'_- \\ \xi_- \end{pmatrix}$$

$\cancel{c^\ell}\ \frac{1}{a}$

$$(q_0|\xi_+') = (-c^r\xi_+ + a^r\xi_- | \xi_+')$$

$$= (a^r\xi_- | \xi_+') = \frac{d^r(0)}{d(0)}$$

$-\frac{c}{d}\xi_-' + \frac{1}{d}\xi_-$ -

$$(q_0|p_0) = \frac{d^r(0)}{d(0)} b^l(0)$$

$$\cancel{\frac{b^l(0)}{d(0)} + d^l(0)\, b^l(0)}$$

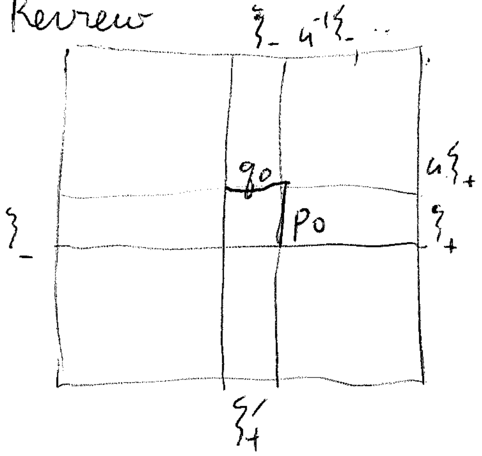
$$\begin{pmatrix}p_0\\ q_0\end{pmatrix} = \begin{pmatrix}a^l & b^l\\ c^l & d^l\end{pmatrix}\begin{pmatrix}\xi_-'\\ \xi_+'\end{pmatrix}$$

$$\frac{b^l(0)}{d^l(0)}$$

$$\|$$

$$(q_0|p_0) = (q_0|\cancel{a^l\xi_-'} + b^l\xi_+') = (q_0|\xi_+')\, b^l(0) \;//\; \frac{d^r(0)}{d(0)} b^l(0)$$

Review



$$\begin{pmatrix}d^r & -b^r\\ -c^r & a^r\end{pmatrix}\begin{pmatrix}a & b\\ c & d\end{pmatrix} = \begin{pmatrix}a^l & b^l\\ c^l & d^l\end{pmatrix}$$

$$\begin{pmatrix}d^r & -b^r\\ -c^r & a^r\end{pmatrix}\begin{pmatrix}1 & \frac{b}{d}\\ \frac{c}{a} & 1\end{pmatrix} = \begin{pmatrix}\frac{a^l}{a} & \frac{b^l}{d}\\ \frac{c^l}{a} & \frac{d^l}{d}\end{pmatrix}$$

$$d^r\beta - b^r = \frac{b^l}{d} \in H_+$$

$$d^r - b^r\bar{\beta} = \frac{a^l}{a} \in zH_-$$

$$p_0 = \overset{H_+}{d^r}\xi_+ - \overset{H_-}{b^r}\xi_- \qquad q_0 = -\overset{zH_+}{c^r}\xi_+ + \overset{zH_-}{a^r}\xi_-$$

$$(q_0|p_0) = (q_0|\xi_+)\, d^r(0)$$

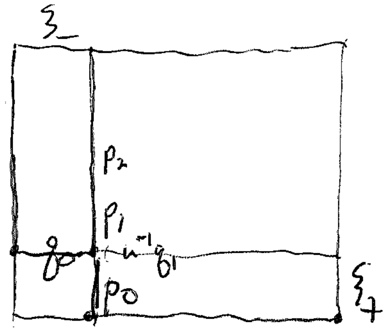
$$(q_0|p_0) = (\cancel{-c^r\xi_+} + a^r\xi_- | p_0) = \overline{a^r(0)}\,(\xi_-|p_0)$$

$$(\xi_-|p_0) = (\xi_- | d^r\xi_+ - b^r\xi_-) = \int (d^r\beta - b^r) = \frac{b^l(0)}{d(0)}$$

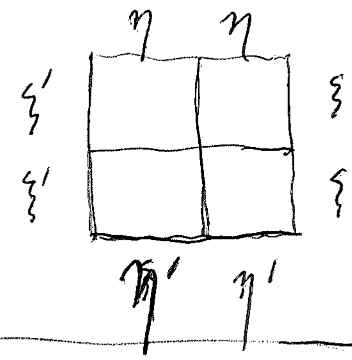
$$(q_0|p_0) = \underbrace{a^r(0)}_{d^r(0)} \frac{b^\ell(0)}{d(0)} = \frac{b^\ell(0)}{d^\ell(0)}$$

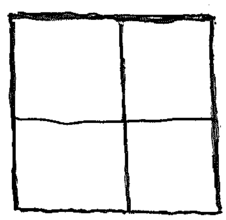
These are the best formulas I have for h_0

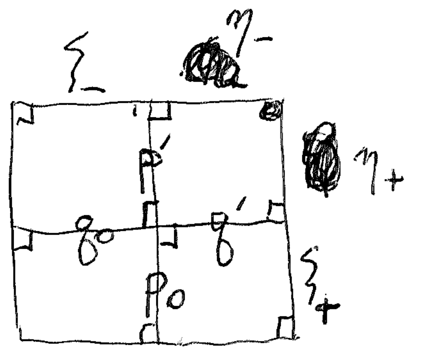
The problem is this.

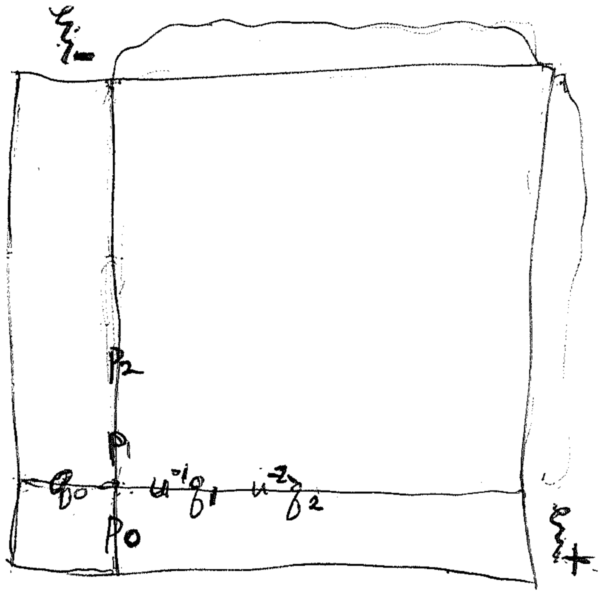


Look at the following linear algebra situation



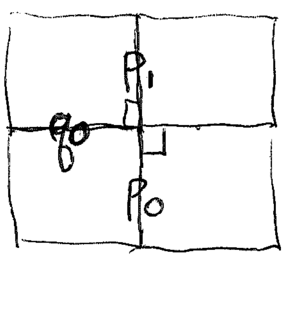


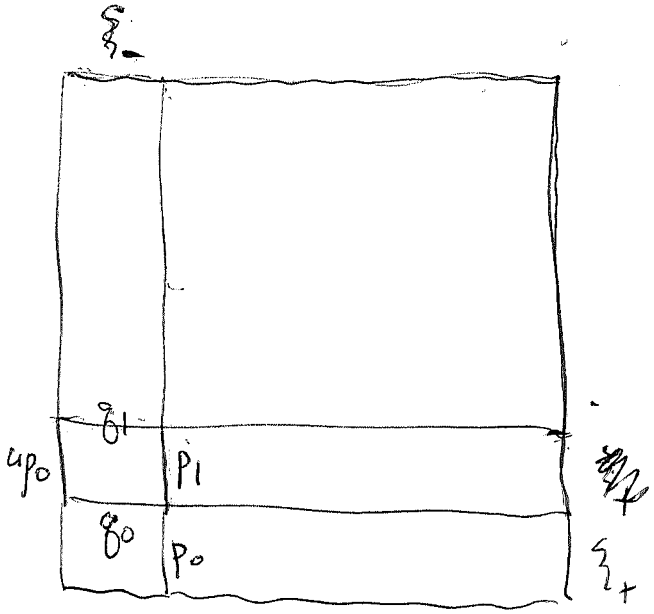




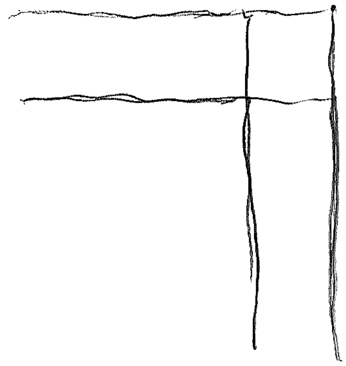
interesting geometry

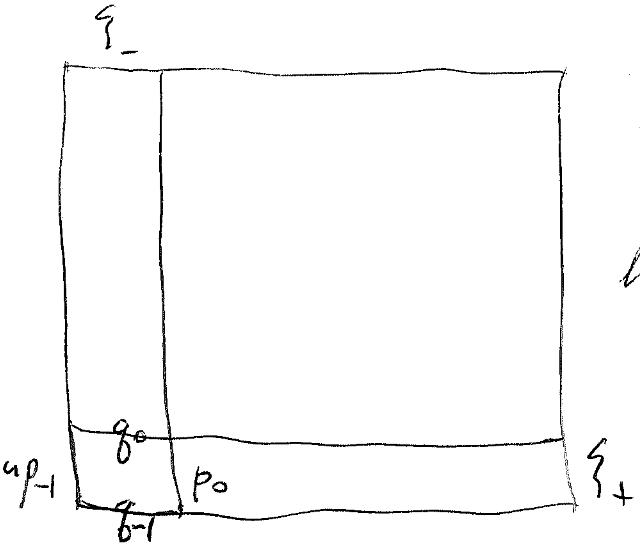
Four unit vectors





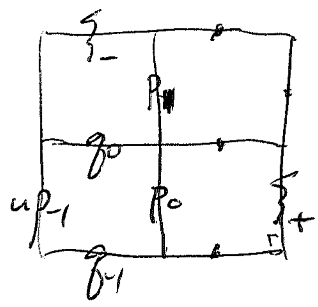
lattice of subspaces





There should be a way to analyze this.

It's clear that you have really 4 squares.

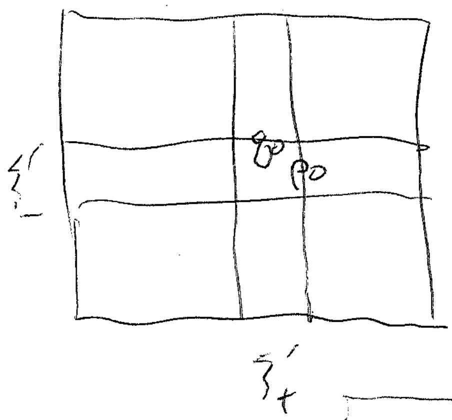


This reminds me a little of the factorization of the S-matrix.

$$\begin{pmatrix}\xi_+\\ \xi_-\end{pmatrix} = \begin{pmatrix} a & b\\ c & d\end{pmatrix}\begin{pmatrix}\xi'_-\\ \xi'_+\end{pmatrix} = \begin{pmatrix} a^r & b^r\\ c^r & d^r\end{pmatrix}\underbrace{\begin{pmatrix} a^l & b^l\\ c^l & d^l\end{pmatrix}\begin{pmatrix}\xi'_-\\ \xi'_+\end{pmatrix}}_{\begin{pmatrix}p_0\\ q_0\end{pmatrix}}$$

$$\begin{pmatrix} \frac{1}{d} & \frac{b}{d}\\ -\frac{c}{d} & \frac{1}{d}\end{pmatrix} = \begin{pmatrix} a^r & b^r\\ -c^l & a^l\end{pmatrix}\frac{1}{d}\begin{pmatrix} d^r & b^l\\ -c^r & d^l\end{pmatrix}$$

Another viewpoint.



$$\begin{pmatrix} p_0 \\ q_0 \end{pmatrix} = \begin{pmatrix} a^l & b^l \\ c^l & d^l \end{pmatrix} \begin{pmatrix} \xi'_- \\ \xi'_+ \end{pmatrix}$$

$$\cap$$

$$\begin{pmatrix} zH_- & H_+ \\ zH_- & H_+ \end{pmatrix}$$

$$\frac{b^l(z)}{d^l(z)}$$

Things for talk.

continuous limit

$$\begin{pmatrix} p_n \\ q_n \end{pmatrix} = \frac{1}{k_n} \begin{pmatrix} 1 & h_n \\ \bar{h}_n & 1 \end{pmatrix} \begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} p_{n-1} \\ q_{n-1} \end{pmatrix}$$

$$z^{dx} = e^{ikx}$$

$$z^{dx} = e^{ikdx}$$

$$\begin{pmatrix} p_{x+dx} \\ q_{x+dx} \end{pmatrix} = \cancel{\frac{1}{\sqrt{1-|h_x|^2 dx^2}}} \begin{pmatrix} 1 & h_x dx \\ \bar{h}_x dx & 1 \end{pmatrix} \begin{pmatrix} 1+ikdx & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} p_x \\ q_x \end{pmatrix}$$

$$\frac{d}{dx} \begin{pmatrix} p_x \\ q_x \end{pmatrix} = \begin{pmatrix} ik & h_x \\ \bar{h}_x & 1 \end{pmatrix} \begin{pmatrix} p_x \\ q_x \end{pmatrix}$$

$$\cancel{\partial_x \left(e^{-ikx/2}\psi\right) = e^{-ikx/2}\left(\partial_x - ik/2\right) e^{ikx/2}\, e^{-ikx/2}\psi}$$

$$\frac{1}{i}\partial_x \Big(\underbrace{e^{-ikx/2}\psi}_{\tilde{\psi}}\Big) = \begin{pmatrix} k/2 & -ih_x \\ +i\bar{h}_x & -k/2 \end{pmatrix} \Big(\underbrace{e^{-ikx/2}\psi}_{\tilde{\psi}}\Big)$$

$$\frac{1}{i}\partial_x \begin{pmatrix} \tilde{p} \\ \tilde{q} \end{pmatrix} = \begin{pmatrix} k/2 & \\ & -k/2 \end{pmatrix} \begin{pmatrix} \tilde{p} \\ \tilde{q} \end{pmatrix}$$

$$\psi_n = \begin{pmatrix} p_n \\ q_n \end{pmatrix} = \frac{1}{\sqrt{1-|h_n|^2}} \begin{pmatrix} 1 & h_n \\ \bar{h}_n & 1 \end{pmatrix} \begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix} \underbrace{\begin{pmatrix} p_{n-1} \\ q_{n-1} \end{pmatrix}}_{\psi_{n-1}}$$

$z = e^{ik}$

$z^{dx} = 1 + ik\,dx$

$$\psi_x = \frac{1}{\cancel{\sqrt{1-|h_x dx|^2}}} \begin{pmatrix} 1 & h_x dx \\ \bar{h}_x dx & 1 \end{pmatrix} \begin{pmatrix} 1+ik\,dx & 0 \\ 0 & 1 \end{pmatrix} \psi_{x-dx}$$

$\cancel{\psi_x = ((1+ik}$

$$\psi_x = \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} ik & h_x \\ \bar{h}_x & 0 \end{pmatrix} dx \right) \psi_{x-dx}$$

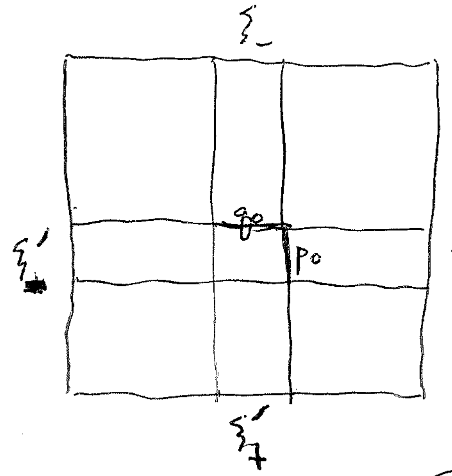
$$\boxed{\partial_x \psi_x = \begin{pmatrix} ik & h_x \\ \bar{h}_x & 0 \end{pmatrix} \psi_x}$$

$$\tilde{\psi}_x = e^{-ikx/2} \psi_x$$

$$\partial_x \tilde{\psi}_x = \begin{pmatrix} ik/2 & h_x \\ \bar{h}_x & -ik/2 \end{pmatrix} \tilde{\psi}_x$$

$$\begin{pmatrix} \partial_x & -h_x \\ \bar{h}_x & -\partial_x \end{pmatrix} \tilde{\psi}_x = \begin{pmatrix} ik/2 & \\ & +ik/2 \end{pmatrix} \tilde{\psi}_x$$

$$\boxed{\begin{pmatrix} \frac{1}{i}\partial_x & i h_x \\ -i\bar{h}_x & -\frac{1}{i}\partial_x \end{pmatrix} \tilde{\psi}_x = \frac{1}{2} k \, \tilde{\psi}_x}$$



Why not work out the link with the Dirac equation:

$$\begin{pmatrix} p_n \\ q_n \end{pmatrix} = \frac{1}{k_n}\begin{pmatrix} 1 & h_n \\ \bar{h}_n & 1 \end{pmatrix}\begin{pmatrix} z\,p_{n-1} \\ q_{n-1} \end{pmatrix}$$

assume $h_n = 0$ $|n|$ large

$$\psi_n = \begin{pmatrix} z^{-n} p_n \\ q_n \end{pmatrix} = \frac{1}{k_n}\begin{pmatrix} 1 & h_n z^{-n} \\ \bar{h}_n z^n & 1 \end{pmatrix}\begin{pmatrix} z^{-n+1} p_{n-1} \\ q_{n-1} \end{pmatrix}$$

$$\xi_-' \longleftarrow z^{-n} p_n \longrightarrow \xi_+ \qquad n \to \infty$$

$$\xi_+' \longleftarrow q_n \longrightarrow \xi_-$$

So a solution ψ of the DE can be specified by specifying the values $(\xi_-'|\psi)$ and $(\xi_+'|\psi)$.

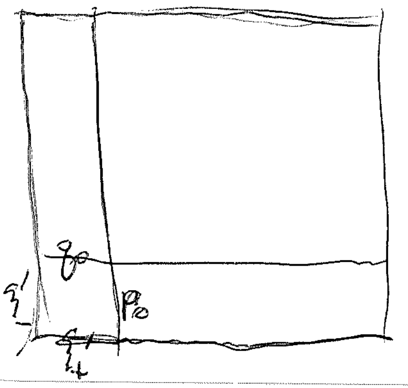
Focus on the case where $h_n = 0$ for $n \leq 0$. Then $p_0 = \xi_-'$ $q_0 = \xi_+'$. Response?

For $|z| < 1$ there should be a decaying solution

Problem: Focus on case $h_n = 0$ $n \leq 0$ where you have a partial unitary completed by a "trans. line".

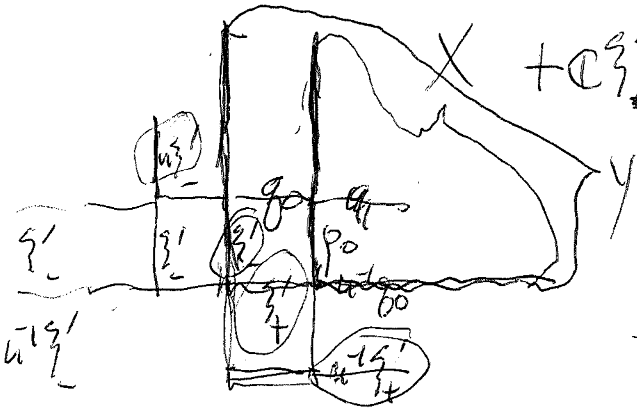
$$Y = H_+ \xi_+ + z H_- \xi_-$$

$$X = H_+ \xi_+ + H_- \xi_-$$



$$Y = X \oplus \xi_+' = u X \oplus \xi_-'$$

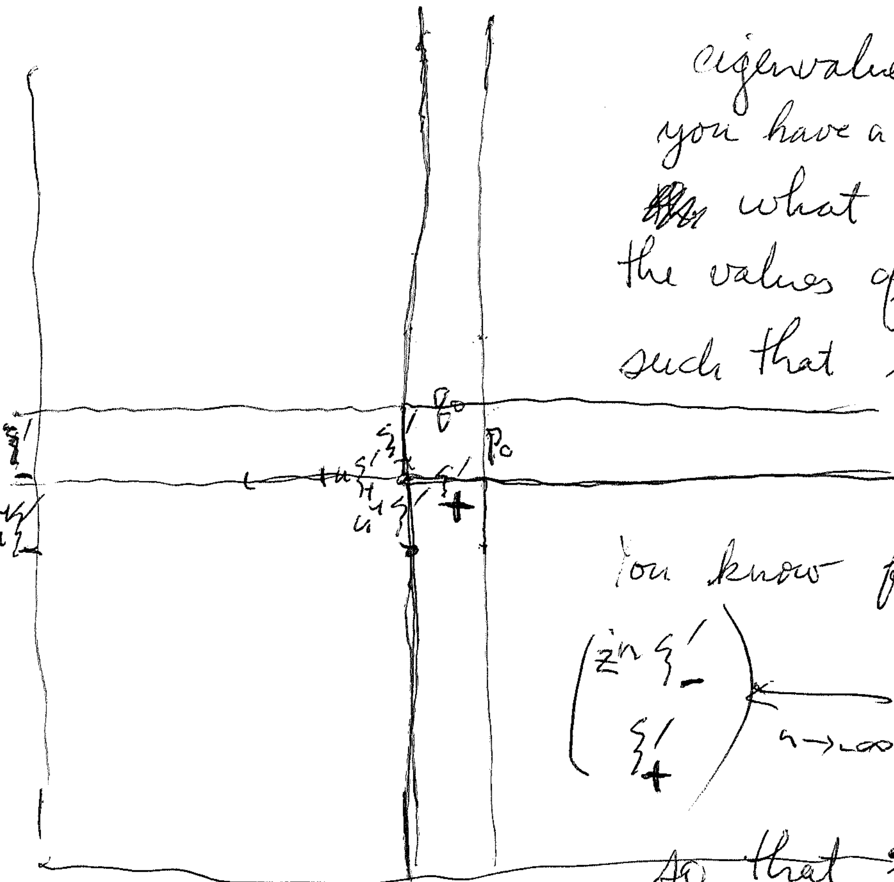
You want response — this means eigen~~vektor~~.
means solving $(az-b)x = -v_+ + v_-$



$X + \mathbb{C}\xi'_+ + \mathbb{C}u^{-1}\xi'_+ + \cdots$

$$Y = X \oplus \mathbb{C}\xi'_+ = uX + \mathbb{C}\xi'_-$$

to complete Y to E you need to add $H_-\xi'_- + zH_+\xi'_+$



eigenvalue equation — given z you have a 2dim'l space of solutions, what to ask? You want the values of $p_0 = \xi'_-$, $q_0 = \xi'_+$ such that something is to ~~be done~~ happen

You know for any solution

$$\begin{pmatrix} z^n\xi'_- \\ \xi'_+ \end{pmatrix} \xleftarrow{n\to-\infty} \begin{pmatrix} p_n \\ q_n \end{pmatrix} \xrightarrow{n\to+\infty} \begin{pmatrix} z^n\xi_+ \\ \xi_- \end{pmatrix}$$

~~so that~~ if $|z|>1$, then the "good" solution has $\xi'_+ = 0$. Check this.

$$\begin{pmatrix} p_0 \\ q_0 \end{pmatrix} = \begin{pmatrix} d^n & -b^n \\ -c^n & a^n \end{pmatrix}\begin{pmatrix} \xi_+ \\ \xi_- \end{pmatrix} \qquad \frac{p_0}{q_0} = -\frac{b^n}{a^n}$$

$\Cap$

H_-

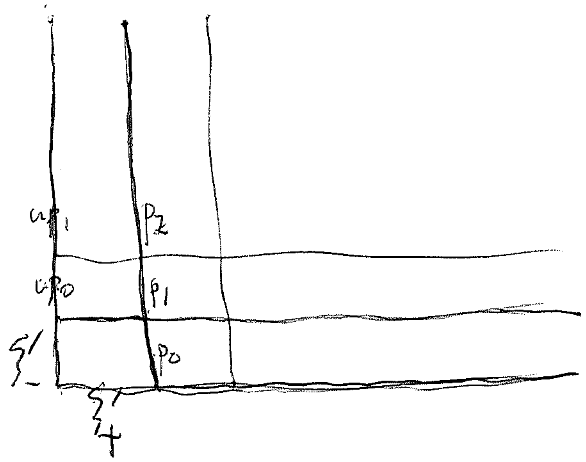
$\Uparrow$

$$\begin{pmatrix} H_+ & H_- \\ zH_+ & zH_- \end{pmatrix}$$

If $|z|<1$, then good soln has $\xi_- = 0$

$\frac{p_0}{q_0} = -\frac{d^2}{c^2}$. In general, no matter what the $(h_n)_{n>0}$ are, ~~you express the response to be~~ there is an eigenvector which is l^2 inside the first quadrant, for $|z| \neq 1$.

Idea: Take just



take up the problem of estimating h_0.

$$Y = aX + \mathbb{C}\xi'_+ = bX + \mathbb{C}\xi'_-$$

eigenvector equation

$$\oplus\, \mathbb{C}u^{-1}\xi'_- \oplus \underbrace{X \oplus \mathbb{C}\xi'_+}_{\|} \oplus \mathbb{C}u\xi'_+$$

$$\oplus\, \overbrace{\mathbb{C}\xi'_- \oplus uX} \oplus \mathbb{C}u\xi'_+ \oplus \cdots$$

$$z^{-1}u^{-2}v_-^{2} \quad z^{-1}u^{-1}v_- \quad + z^{-1}u x_1 + z^{-1}uv_+ \quad + z^{-1}uu v_+^{1} \quad + u^2 v_+^{2}$$

$$uv_-^{1} + v_- + ux_2 + uv_+^{1}$$

so you find
$$x_1 + v_+ = v_- + ux_2$$
$$z^{-1}ux_1 = ux_2 \qquad \therefore x_1 = zx_2$$

$$\boxed{(z-u)x = -v_+ + v_-}$$

$$(az-b)\,x = -v_+ + v_- \qquad \text{eigenvector equation}$$

$$(z-a^*b)\,x = +a^* v_-$$

$$x = (z-a^*b)^{-1}a^*v_- = a^*(z-ba^*)^{-1}v_-$$

$$v_+ = v_- - (az-b)\,a^*(z-ba^*)^{-1}v_-$$

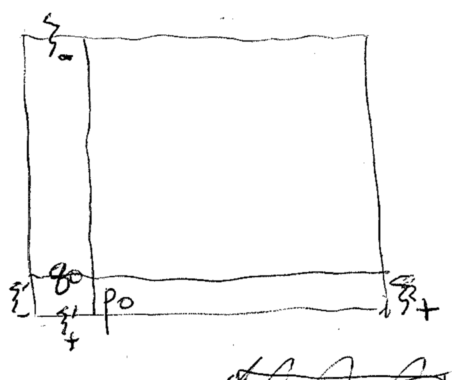
$$= \left[z-ba^* - (az-b)a^*\right](z-ba^*)^{-1}v_-$$

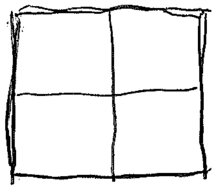
$$= (1-aa^*)(1-z^{-1}ba^*)^{-1}v_- \qquad |z|>1$$

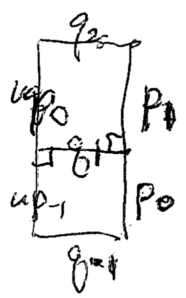
$$= \xi'_+\ \xi^*_+\,(1-z^{-1}ba^*)^{-1}\,\xi'_-$$

But what happens to the idea that you are estimating $\beta(0)$ knowing something about β.

Go back to picture

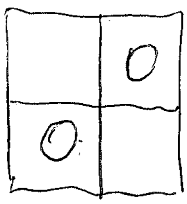


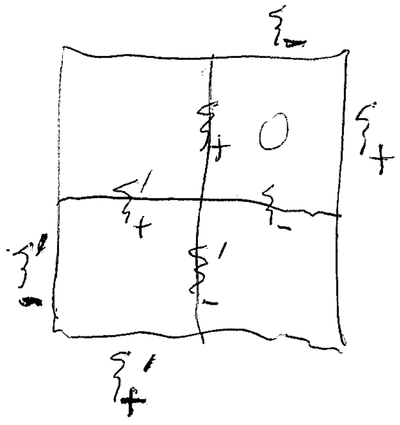




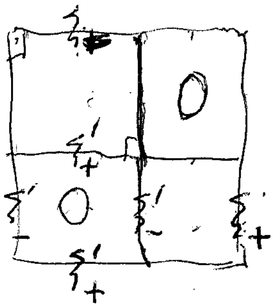
Look at

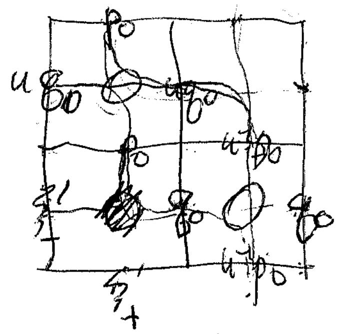
Simple case





Special cases





Reformulate. You have 4-diml

Start again: Would it help to have

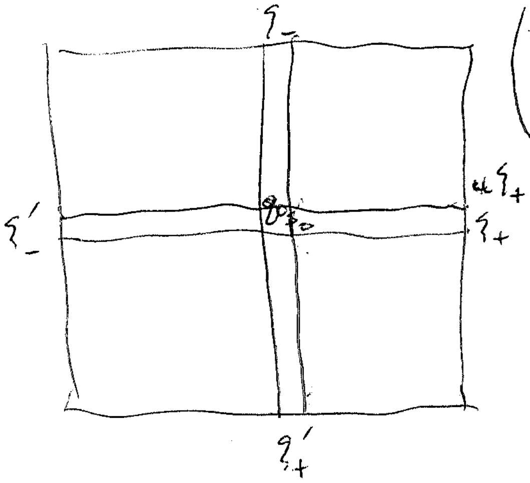
$$\begin{pmatrix}\frac{1}{d} & \frac{b}{d}\\ -\frac{c}{d} & \frac{1}{d}\end{pmatrix} = \underset{\cap}{\begin{pmatrix} a^r & b^r \\ -c^l & a^l\end{pmatrix}}\frac{1}{d}\underset{\cap}{\begin{pmatrix} +d^r & b^l \\ -c^r & d^l\end{pmatrix}}$$

$$\begin{pmatrix} zH_- & H_- \\ zH_- & zH_-\end{pmatrix} \qquad \begin{pmatrix} H_+ & H_+ \\ zH_+ & H_+\end{pmatrix}$$

85
5
425

~~85~~

1.22
.35
1.57



$$\begin{pmatrix}p_0\\ q_0\end{pmatrix} \overset{?}{=} \frac{1}{a}\begin{pmatrix} a^l & b^r \\ c^l & a^r\end{pmatrix}\begin{pmatrix}\xi_+\\ \xi_-\end{pmatrix}$$

Start again. You are given β from which you construct a scattering situation:

$$\begin{pmatrix}\xi_+\\ \xi_-\end{pmatrix} = \begin{pmatrix} a & b \\ c & d\end{pmatrix}\begin{pmatrix}\xi'_-\\ \xi'_+\end{pmatrix} \qquad \begin{pmatrix}\xi_+\\ \xi'_+\end{pmatrix} = \begin{pmatrix}\frac{1}{d} & \frac{b}{d}\\ -\frac{c}{d} & \frac{1}{d}\end{pmatrix}\begin{pmatrix}\xi'_-\\ \xi_-\end{pmatrix}$$

$$\begin{pmatrix}\xi'_-\\ \xi'_+\end{pmatrix} = \begin{pmatrix} d & -b \\ -c & a\end{pmatrix}\begin{pmatrix}\xi_+\\ \xi_-\end{pmatrix} \qquad \begin{pmatrix}\xi'_-\\ \xi_-\end{pmatrix} = \begin{pmatrix}\frac{1}{a} & -\frac{b}{a}\\ \frac{c}{a} & \frac{1}{a}\end{pmatrix}\begin{pmatrix}\xi_+\\ \xi'_+\end{pmatrix}$$

then you construct

$$\begin{pmatrix}p_0\\ q_0\end{pmatrix} = \begin{array}{c} \begin{matrix} zH_- & H_+ \end{matrix} \\ \begin{pmatrix} a_l & b_l \\ c_l & d_l\end{pmatrix} \\ \begin{matrix} zH_- & H_+ \end{matrix}\end{array}\begin{pmatrix}\xi'_-\\ \xi'_+\end{pmatrix} = \begin{array}{c} \begin{matrix} H_+ & \underset{\cup}{H_-} \end{matrix} \\ \begin{pmatrix} d^r & -b^r \\ -c^r & a^r\end{pmatrix} \\ \begin{matrix} zH_+ & zH_- \end{matrix}\end{array}\begin{pmatrix}\xi_+\\ \xi_-\end{pmatrix}$$

directly from β.

$$\begin{pmatrix} d^r & -b^r \\ -c^r & a^r \end{pmatrix}\begin{pmatrix} \cancel{1} & \frac{b}{d} \\ \frac{c}{a} & \cancel{1} \end{pmatrix} = \begin{pmatrix} \frac{a^l}{a} & \frac{b^l}{d} \\ \frac{c^l}{a} & \frac{d^l}{d} \end{pmatrix}$$

The point here is you have orthogonal projection methods allowing to solve

$$\begin{pmatrix} d^r & -b^r \\ -c^r & a^r \end{pmatrix}\begin{pmatrix} 1 & \beta \\ \bar\beta & 1 \end{pmatrix} \in \begin{pmatrix} zH_- & H_+ \\ zH_- & H_+ \end{pmatrix}$$

The way this works is that

$$d^r\beta - b^r \in H_+$$
$$d^r - b^r\bar\beta \in zH_-$$

specifies $\begin{pmatrix} b^r \\ d^r \end{pmatrix}$ up to a ~~scalar~~ scalar factor

and

$$-c^r + a^r\bar\beta \in zH_-$$
$$-c^r\beta + a^r \in H_+$$

$\begin{pmatrix} a^r \\ c^r \end{pmatrix}$

But one set goes into the other by conjugation

Also you know the leading terms $d^r(0) = a^r(0) > 0$

Maybe analyze those equations

$$\begin{cases} d^r\beta - b^r \in H_+ \\ d^r - b^r\bar\beta \in zH_- \end{cases} \qquad \begin{matrix} d^r \in H_+ \\ b^r \in H_- \end{matrix}$$

these equations depend only on β mod H_+

If $\delta\beta \in H_+$ then $d^r\delta\beta \in H_+$

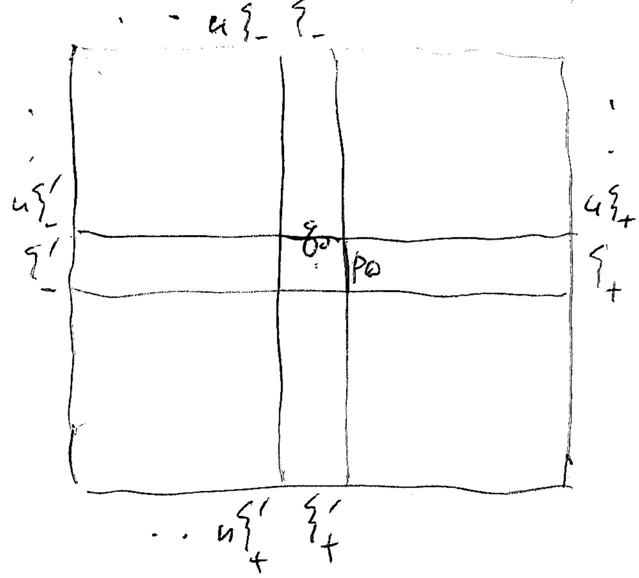
and $b^r\overline{\delta\beta} \in H_- \, z\bar H_+ = H_-$

What do you seek?

$$\begin{cases} d^r\beta - b^r = \frac{b^l}{d} \in H_+ \\ d^r - b^r\bar\beta = \frac{a^l}{a} \in zH_- \end{cases}$$

~~[scribble]~~ Review

$$\begin{pmatrix}\xi_+\\ \xi_-\end{pmatrix} = \begin{pmatrix} a^r & b^r \\ c^r & d^r\end{pmatrix}\begin{pmatrix}p_0\\ q_0\end{pmatrix} = \begin{pmatrix} a^r & b^r\\ c^r & d^r\end{pmatrix}\begin{array}{c} \begin{array}{cc} zH_- & H_+ \end{array} \\ \begin{pmatrix} a^l & b^l\\ c^l & d^l\end{pmatrix} \\ \begin{array}{cc} zH_- & H_+ \end{array}\end{array}\begin{pmatrix}\xi'_-\\ \xi'_+\end{pmatrix}$$



$$\begin{pmatrix}p_0\\ q_0\end{pmatrix} = \begin{array}{c} \begin{array}{cc} H_+ & H_- \end{array} \\ \begin{pmatrix} d^r & -b^r\\ -c^r & a^r\end{pmatrix} \\ \begin{array}{cc} zH_+ & zH_- \end{array}\end{array}\begin{pmatrix}\xi_+\\ \xi_-\end{pmatrix}$$

to reverse. Given

$$\begin{pmatrix} d^r & -b^r\\ -c^r & a^r\end{pmatrix}\begin{pmatrix}1 & \beta\\ \bar\beta & 1\end{pmatrix} = \begin{array}{c} \begin{array}{cc} zH_- & H_+ \end{array} \\ \begin{pmatrix}\frac{a^l}{a} & \frac{b^l}{d}\\ \frac{c^l}{a} & \frac{d^l}{d}\end{pmatrix} \\ \begin{array}{cc} zH_- & H_+ \end{array}\end{array}$$

You start with β, ~~[illegible]~~
form E, do your orthogonalization to obtain $\begin{pmatrix} a^r & b^r\\ c^r & d^r\end{pmatrix}$
this depends only on β mod H_+

$$d^r - b^r\bar\beta \in zH_-$$
$$d^r\beta - b^r \in H_+ \implies \beta - \frac{b^r}{d^r} \in H_+$$

$$\frac{d^r\beta - b^r}{a^r - c^r\bar\beta} = \frac{b^l}{d^l} \qquad\qquad \frac{\beta - \beta^r}{1-\overline{\beta^r}\beta} = \beta^l \quad ?$$

$$\boxed{= \frac{d^r\frac{b}{d} - b^r}{a^r - c^r\frac{b}{d}}}$$

$$\begin{pmatrix}\xi'_-\\ \xi'_+\end{pmatrix} = \begin{pmatrix} d & -b\\ -c & a\end{pmatrix}\begin{pmatrix}\xi_+\\ \xi_-\end{pmatrix}$$

Can you construct ξ'_-, ξ'_+ by orthogonalizing.

$$\xi'_- \in H_+\xi_+ + L^2\xi_-$$
$$\xi'_+ \in L^2\xi_+ + zH_-\xi_-$$

$$\xi'_- = \sum_{j\geq 0} d_j u^j\xi_+ - \sum_{k\in\mathbb{Z}} b_k u^k \xi_-$$

$$(u^k\xi_- \mid \xi'_-) = \sum_{j\geq 0} d_j\beta_{k-j} - b_k \qquad \forall k.$$

$$(u^j\xi_+|\xi'_-) \overset{j>0}{=} 0 = d_j - \sum_k b_k \underbrace{(u^j\xi_+|u^k\xi_-)}_{\beta_{k-j}}$$

so you get
$$\left.\begin{aligned} d\beta &= b \\ d - b\bar\beta &\in zH_- \end{aligned}\right) \Longrightarrow d(\underbrace{1-|\beta|^2}_{\alpha\bar\alpha}) \in zH_-$$

Guess: You know that h_0 depends only on β mod zH_+

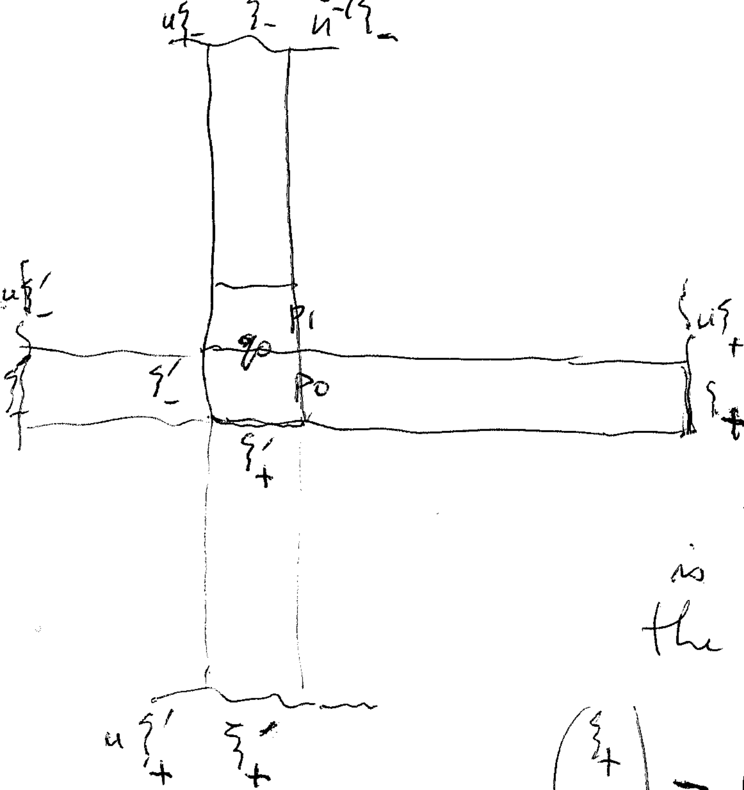
~~So constructed~~ In fact $(h_n)_{n\geq 0}$ ______.

There seems to be an error. No.

~~$d\beta - b$~~
$$d^n\beta - b^n \in H_+ \qquad\qquad d^n\beta - b^n = \frac{b^\ell}{d} \in H_+$$
$$d^n - b^n\bar\beta \in zH_- \qquad\qquad d^n - b^n\bar\beta = \frac{a^\ell}{a} \in zH_-$$

~~Look at these equations~~

~~Start again~~ Consider carefully the case



$$Y = X \oplus \mathbb{C}\xi'_+ = \mathbb{C}\xi'_- \oplus uX$$

described by

$$\begin{pmatrix}\xi_+ \\ \xi_-\end{pmatrix} = \begin{array}{cc} zH_- & zH_- \\ \multicolumn{2}{c}{\begin{pmatrix} a & b \\ c & d\end{pmatrix}} \\ H_+ & H_+ \end{array} \begin{pmatrix}\xi'_- \\ \xi'_+\end{pmatrix}$$

Important quantity here is the reflection coefficient on the left. Use scattering matrix.

$$\begin{pmatrix}\xi_+ \\ \xi'_+\end{pmatrix} = \begin{pmatrix} \frac{1}{d} & \frac{b}{d} \\ -\frac{c}{d} & \frac{1}{d}\end{pmatrix} \begin{pmatrix}\xi'_- \\ \xi_-\end{pmatrix}$$

either we want $-\frac{c}{d}$

$$\begin{pmatrix}p_0 \\ q_0\end{pmatrix} = \begin{pmatrix} a^\ell & b^\ell \\ c^\ell & d^\ell\end{pmatrix} \begin{pmatrix}\xi'_- \\ \xi'_+\end{pmatrix} = \frac{1}{k_0}\begin{pmatrix} 1 & h_0 \\ \bar h_0 & 1\end{pmatrix} \begin{pmatrix}\xi'_- \\ \xi'_+\end{pmatrix}$$

$$\xi'_- = \frac{1}{a}\xi_+ - \frac{b}{a}\xi'_+ \qquad \xi'_+ = \underset{\in H_+}{\left(-\frac{c}{d}\right)}\xi'_- + \frac{1}{d}\xi_-$$

$$\begin{pmatrix}\xi_+\\ \xi'_+\end{pmatrix} = \begin{pmatrix}\frac{1}{d} & \overset{\beta}{\frac{b}{d}}\\ \underset{\gamma}{-\frac{c}{d}} & \frac{1}{d}\end{pmatrix}\begin{pmatrix}\xi'_-\\ \xi_-\end{pmatrix}$$

You want $\gamma(0)$ in terms of β.

$$\frac{a}{d}\,\bar\gamma = -\frac{b}{a}\frac{a}{d} = -\beta$$

because $a(0) = d(0) \geq 0$.

so $-\overline{\gamma(0)} = +\beta(0)$

$\gamma(0) = -\overline{\beta(0)}$

$$\begin{pmatrix}\partial_x & -h\\ h & -\partial_x\end{pmatrix}\begin{pmatrix}u\\ v\end{pmatrix} = \lambda\begin{pmatrix}u\\ v\end{pmatrix} = \begin{pmatrix}\lambda & 0\\ 0 & \lambda\end{pmatrix}\begin{pmatrix}u\\ v\end{pmatrix}$$

$$\begin{pmatrix}\partial_x-\lambda & 0\\ 0 & -\partial_x-\lambda\end{pmatrix}\begin{pmatrix}u\\ v\end{pmatrix} = \begin{pmatrix}0 & h\\ -h & 0\end{pmatrix}\begin{pmatrix}u\\ v\end{pmatrix}$$

$$(\partial_x-\lambda)u = hv$$
$$(-\partial_x-\lambda)v = -hu$$

$$(-\partial_x^2+\lambda^2)u = (-\partial_x-\lambda)(hv)$$
$$\cancel{= h'(-\partial_x-\lambda)v}$$
$$= -h'v + h(-\partial_x-\lambda)v$$
$$= (-h'+h^2)u$$

$$(\partial_x-\lambda)u = hv$$
$$(\partial_x+\lambda)v = hu$$

$$(\partial_x^2-\lambda^2)v = (\partial_x+\lambda)(hu)$$
$$= h'u + h\underbrace{(\partial_x+\lambda)u}_{hv}$$
$$= (h'+h^2)v$$
$$-\lambda^2 v = (-\partial_x^2 + h'+h^2)v$$

$$\frac{1}{2}\begin{pmatrix}1&1\\-1&1\end{pmatrix}\left[\partial_x\begin{pmatrix}1&0\\0&-1\end{pmatrix}-h_x\begin{pmatrix}0&1\\-1&0\end{pmatrix}\right]\begin{pmatrix}1&1\\-1&1\end{pmatrix}$$

$$=\partial_x\begin{pmatrix}1&1\\-1&1\end{pmatrix}\begin{pmatrix}1&1\\1&-1\end{pmatrix}\frac{1}{2}-h_x\begin{pmatrix}-1&1\\-1&-1\end{pmatrix}\begin{pmatrix}-1&1\\1&1\end{pmatrix}\frac{1}{2}$$

$$=\partial_x\begin{pmatrix}1&0\\0&-1\end{pmatrix}-h_x\begin{pmatrix}1&0\\0&-1\end{pmatrix}$$

$$(\partial_x-h)u=\frac{ik}{2}v$$

$$-(\partial_x+h)v=\frac{ik}{2}$$

$$\begin{pmatrix}\partial_x&-h\\+h&-\partial_x\end{pmatrix}\begin{pmatrix}\tilde{\rho}\\\tilde{\sigma}\end{pmatrix}=\lambda\begin{pmatrix}\tilde{\rho}\\\tilde{\sigma}\end{pmatrix}$$

$$\cancel{\frac{1}{2}\begin{pmatrix}1&1\\-1&1\end{pmatrix}\begin{pmatrix}\partial_x&-h\\h&-\partial_x\end{pmatrix}\begin{pmatrix}1&1\\-1&1\end{pmatrix}=\frac{1}{2}\begin{pmatrix}1&1\\-1&1\end{pmatrix}\begin{pmatrix}\partial_x+h&\partial_x-h\\\partial_x+h&\partial_x+h\end{pmatrix}}$$

$$=\begin{pmatrix}\partial_x+h&0\\ & \end{pmatrix}$$

$$\partial_x u-hv=\lambda u$$
$$hu-\partial_x v=\lambda v$$

$$(\partial_x-\lambda)u=hv$$
$$(\partial_x+\lambda)v=hu$$

$$\begin{pmatrix} p_n \\ q_n \end{pmatrix} = \frac{1}{k_n}\begin{pmatrix} 1 & h_n \\ \bar{h}_n & 1 \end{pmatrix}\begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} p_{n-1} \\ q_{n-1} \end{pmatrix} \qquad z^{dx} = e^{ikdx}$$

$$\psi_x = \begin{pmatrix} p_x \\ q_x \end{pmatrix} = \frac{1}{\sqrt{1-|h_x dx|^2}}\begin{pmatrix} 1 & h_x dx \\ \bar{h}_x dx & 1 \end{pmatrix}\begin{pmatrix} e^{ikdx} & 0 \\ 0 & 1 \end{pmatrix}\psi_{x-dx}$$

$$\psi_x - \psi_{x-dx} = \begin{pmatrix} ik & h_x \\ \bar{h}_x & 0 \end{pmatrix} dx\, \psi_x$$

$$\frac{d\psi_x}{dx} = \begin{pmatrix} ik & h_x \\ \bar{h}_x & 0 \end{pmatrix}\psi_x \qquad \tilde{\psi}_x = e^{-ikx/2}\psi_x$$

$$\frac{d\tilde{\psi}_x}{dx} = \begin{pmatrix} \frac{ik}{2} & h_x \\ \bar{h}_x & -\frac{ik}{2} \end{pmatrix}\tilde{\psi}_x$$

$$\begin{pmatrix} \frac{d}{dx} & -h_x \\ +\bar{h}_x & -\frac{d}{dx} \end{pmatrix}\begin{pmatrix} \tilde{p} \\ \tilde{q} \end{pmatrix} = \begin{pmatrix} \frac{ik}{2} & h \\ -\bar{h} & +\frac{ik}{2} \end{pmatrix}\begin{pmatrix} \tilde{p} \\ \tilde{q} \end{pmatrix}$$

$$\boxed{\begin{pmatrix} \frac{d}{dx} & -h_x \\ +\bar{h}_x & -\frac{d}{dx} \end{pmatrix}\begin{pmatrix} \tilde{p} \\ \tilde{q} \end{pmatrix} = \frac{ik}{2}\begin{pmatrix} \tilde{p} \\ \tilde{q} \end{pmatrix}}$$

special case $\bar{h}_x = h_x$

$$\frac{1}{2}\begin{pmatrix} 1 & 1 \\ -1 & +1 \end{pmatrix}\begin{pmatrix} \partial_x & -h_x \\ -h_x & -\partial_x \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} \partial_x + h_x & \partial_x - h \\ -h & -h+\partial_x \end{pmatrix} = \begin{pmatrix} \partial_x & -h \\ -h & -\partial_x \end{pmatrix}$$

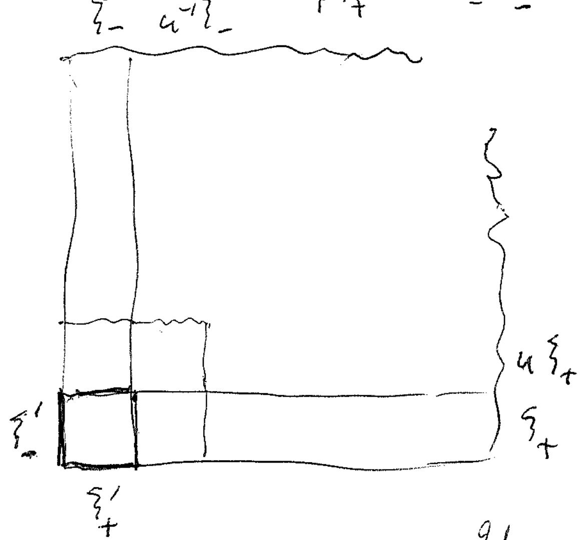
State the problem: You have $\beta(z)$ smooth

You construct E using β $\quad (u^k\xi_- | u^j\xi_+) = (z^{k-j}|\beta)_{L^2}$

You look only ~~at~~ the subspace $H_+\xi_+ + zH_-\xi_-$

i.e. $k \leq 0$ and $j \geq 0$ so that $k-j \leq 0$. Then you have the subspaces

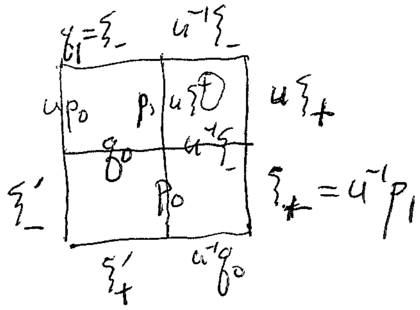
$$zH_+\xi_+ + zH_-\xi_-$$
$$\downarrow$$
$$H_+\xi_+ + zH_-\xi_- \longleftrightarrow H_+\xi_+ + H_-\xi_-$$



Here is an idea. Suppose you have $(u^k\xi_- | u^j\xi_+) = \beta_{k-j}$ equal to 0 for $k<0$ $j>0$.

This should be the case where $h_2 = h_3 = \cdots = 0$. So you have $h_0, h_1 \neq 0$.

Then you do have a square

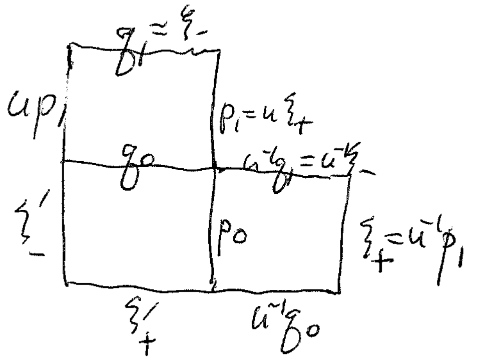


You want to control h_0, h_1 ~~[illegible]~~ in terms of ~~[illegible]~~ β_0, β_{-1}

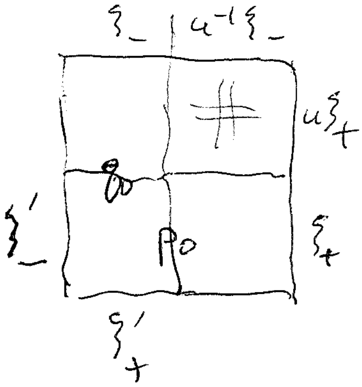
$$\begin{pmatrix} \xi_+ \\ \xi_- \end{pmatrix} = \begin{pmatrix} u^{-1}p_1 \\ q_1 \end{pmatrix} = \frac{1}{k_1}\begin{pmatrix} 1 & u^{-1}h_1 \\ \bar{h}_1 u & 1 \end{pmatrix}\begin{pmatrix} p_0 \\ q_0 \end{pmatrix}$$

$$\begin{pmatrix} \xi_+ \\ \xi_- \end{pmatrix} = \underbrace{\frac{1}{k_1}\begin{pmatrix} 1 & h_1 u^{-1} \\ \bar{h}_1 u & 1 \end{pmatrix} \frac{1}{k_0}\begin{pmatrix} 1 & h_0 \\ \bar{h}_0 & 1 \end{pmatrix}}\begin{pmatrix} \xi'_- \\ \xi'_+ \end{pmatrix}$$

$$\frac{1}{k_1 k_0}\begin{pmatrix} 1 + h_1\bar{h}_0 u^{-1} & h_0 + h_1 u^{-1} \\ \bar{h}_1 u + \bar{h}_0 & \bar{h}_1 h_0 u + 1 \end{pmatrix}$$



Given



$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \frac{1}{k_1}\begin{pmatrix} 1 & h_1 z^{-1} \\ \bar{h}_1 z & 1 \end{pmatrix}\begin{pmatrix} 1 & h_0 \\ \bar{h}_0 & 1 \end{pmatrix}\frac{1}{k_0}$$

~~you want to find~~ You have

$$\beta = \frac{b}{d} = \frac{h_0 + h_1 z^{-1}}{\bar{h}_1 h_0 \cancel{z} + 1} = \begin{pmatrix} 1 & h_1 z^{-1} \\ \bar{h}_1 z & 1 \end{pmatrix}(h_0)$$

$$\beta \doteq (h_1 z^{-1} + h_0)\left(1 - \bar{h}_1 h_0 z + (\bar{h}_1 h_0)^2 z^2 + \cdots\right)$$

$$= h_1 z^{-1} + h_0 - |h_1|^2 h_0$$

$$(u^k \xi_- | u^j \xi_+) = (z^{k-j} | \beta)$$

$$(\xi_- | \xi_+) = \beta_0 = k_1^2 h_0$$

$$(\xi_- | u\xi_+) = (u^{-1}\xi_- | \xi_+) = \beta_{-1} = h_1$$

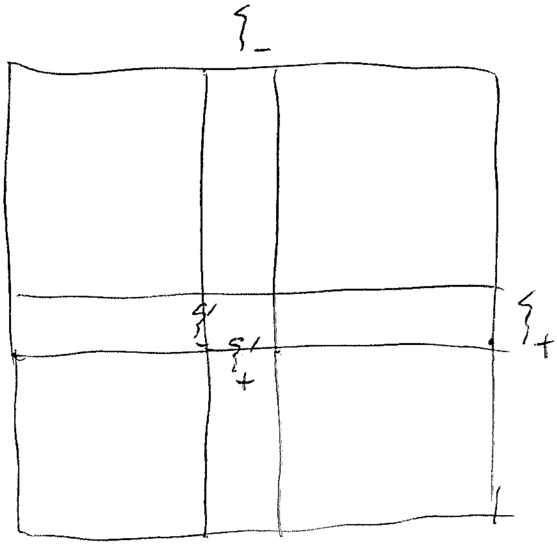
$$(u^{-1}\xi_- | u\xi_+) = \beta_{-2} = 0$$

$$\begin{pmatrix} \xi'_- \\ \xi'_+ \end{pmatrix} =$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a^r & b^r \\ c^r & d^r \end{pmatrix} \frac{1}{k_0} \begin{pmatrix} 1 & h_0 \\ \bar{h}_0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} \xi'_- \\ \xi'_+ \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \xi_+ \\ \xi_- \end{pmatrix} = \begin{pmatrix} a^r & b^r \\ c^r & d^r \end{pmatrix} \overbrace{\begin{pmatrix} a^l & b^l \\ c^l & d^l \end{pmatrix} \begin{pmatrix} \xi'_- \\ \xi'_+ \end{pmatrix}}^{\begin{pmatrix} p_0 \\ q_0 \end{pmatrix}}$$

$$\begin{pmatrix} \xi_+ \\ \xi'_+ \end{pmatrix} = \begin{pmatrix} \frac{1}{d} & \frac{b}{d} \\ -\frac{c}{d} & \frac{1}{d} \end{pmatrix} \begin{pmatrix} \xi'_- \\ \xi_- \end{pmatrix}$$



$$\begin{pmatrix} \xi'_- \\ \xi'_+ \end{pmatrix} = \frac{1}{k_0} \begin{pmatrix} 1 & -h_0 \\ -\bar{h}_0 & 1 \end{pmatrix} \begin{pmatrix} p_0 \\ q_0 \end{pmatrix}$$

$$= \frac{1}{k_0} \begin{pmatrix} 1 & -h_0 \\ -\bar{h}_0 & 1 \end{pmatrix} \begin{pmatrix} d^r & -b^r \\ -c^r & a^r \end{pmatrix} \begin{pmatrix} \xi_+ \\ \xi_- \end{pmatrix}$$

$$\begin{pmatrix} \xi'_- \\ \xi'_+ \end{pmatrix} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \begin{pmatrix} \xi_+ \\ \xi_- \end{pmatrix}$$

~~want case~~ $\cancel{\xi_+ = 0, \ \xi_- = 1.}$

want to find ξ'_+ when $\xi'_- = 1$, $\xi_- = 0$.

~~whence~~

where $\xi'_+ = -\dfrac{c}{d}$

$$\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \begin{pmatrix} d^l & -c^l \\ -b^l & +a^l \end{pmatrix} \begin{pmatrix} d^r & -b^r \\ -c^r & a^r \end{pmatrix} \quad !$$

$$\begin{pmatrix} d \\ -c \end{pmatrix} = \begin{pmatrix} d^l & -c^l \\ -b^l & a^l \end{pmatrix} \begin{pmatrix} d^r \\ c^r \end{pmatrix}$$

$$\gamma = -\frac{c}{d} = -\frac{c^r a^l + d^r c^l}{c^r b^l + d^r d^l}$$

to write this in terms of $\gamma^r = -\frac{c^r}{d^r}$

$$\gamma = -\frac{\frac{c^r}{d^r} a^l + c^l}{\frac{c^r}{d^r} b^l + d^l}$$

$$= -\frac{-\gamma^r a^l + c^l}{-\gamma^r b^l + d^l} = \frac{a^l \gamma^r - c^l}{-b^l \gamma^r + d^l}$$

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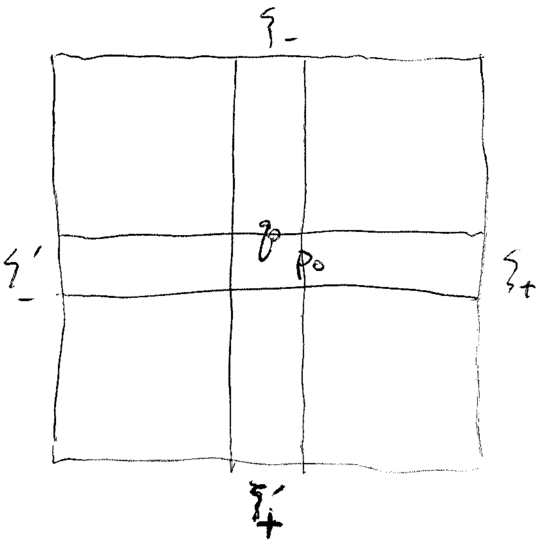
situation. $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a^r & b^r \\ c^r & d^r \end{pmatrix}\begin{pmatrix} a^l & b^l \\ c^l & d^l \end{pmatrix}$



$$\begin{pmatrix} p_0 \\ q_0 \end{pmatrix} = \begin{array}{c} \begin{array}{cc} H_+ & H_- \end{array} \\ \begin{pmatrix} d^r & -b^r \\ -c^r & a^r \end{pmatrix} \\ \begin{array}{cc} zH_+ & zH_- \end{array} \end{array} \begin{pmatrix} \xi_+^\bullet \\ \xi_-^\bullet \end{pmatrix} = \begin{array}{c} \begin{array}{cc} zH_- & H_+ \end{array} \\ \begin{pmatrix} a^l & b^l \\ c^l & d^l \end{pmatrix} \\ \begin{array}{cc} zH_- & H_+ \end{array} \end{array} \begin{pmatrix} \xi'_- \\ \xi'_+ \end{pmatrix}$$

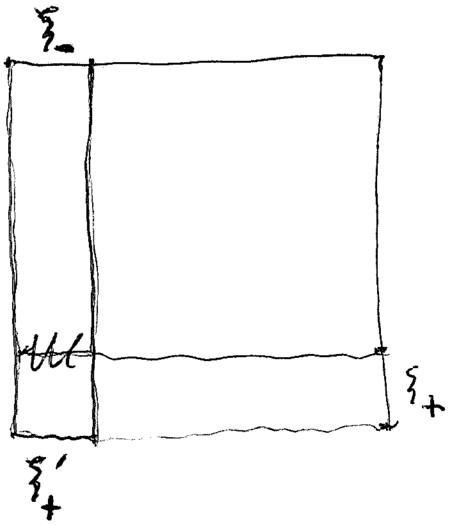
$$\begin{pmatrix} \xi_+^\bullet \\ \xi_- \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} \xi'_- \\ \xi'_+ \end{pmatrix} \qquad \begin{pmatrix} \xi'_- \\ \xi'_+ \end{pmatrix} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}\begin{pmatrix} \xi_+ \\ \xi_- \end{pmatrix}$$

$$\begin{pmatrix} \xi_+ \\ \xi'_+ \end{pmatrix} = \begin{array}{c} \begin{array}{cc} H_+ & L^2 \end{array} \\ \begin{pmatrix} \frac{1}{d} & \frac{b}{d} \\ -\frac{c}{d} & \frac{1}{d} \end{pmatrix} \\ \begin{array}{cc} L_2 & H_+ \end{array} \end{array} \begin{pmatrix} \xi'_- \\ \xi_- \end{pmatrix} \qquad \begin{pmatrix} \xi'_- \\ \xi_- \end{pmatrix} = \begin{pmatrix} \frac{1}{a} & -\frac{b}{a} \\ \frac{c}{a} & \frac{1}{a} \end{pmatrix}\begin{pmatrix} \xi_+ \\ \xi'_+ \end{pmatrix}$$

$$\left(u^k \xi_- \,\middle|\, u^j \xi_+\right) = \left(z^{k-j} \,\middle|\, \beta\right) \qquad \beta = \frac{b}{d}$$

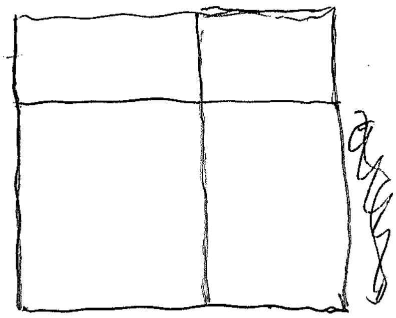
$$\left(u^k \xi'_- \,\middle|\, u^j \xi'_+\right) = \left(z^{k-j} \,\middle|\, \gamma\right) \qquad \gamma = \frac{-c}{d}$$

Situation Ultimately you have two Hilbert spaces and a contraction β between them, a glued Hilbert space $\begin{pmatrix} 1 & \beta \\ \beta^* & 1 \end{pmatrix}$, unit vectors $\xi_+ \xi_-$



which you ~~[illegible]~~ project to the ~~other~~ opposite side

It's probably better to use ~~[illegible]~~ bifiltration.

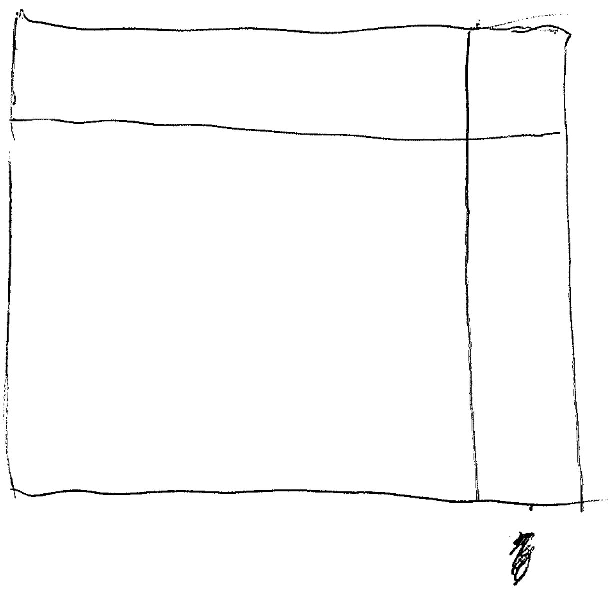


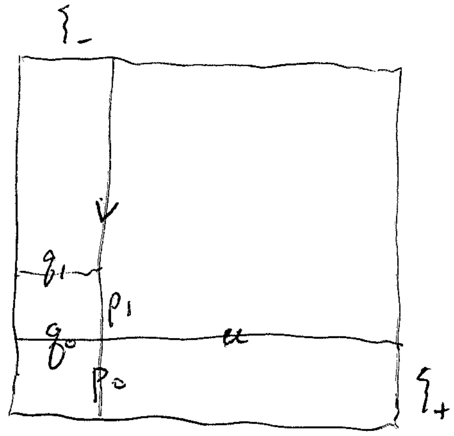
ξ'_+ is a unit vector in $(H_+ \xi_+ + z H_- \xi_-) \cap (H_+ \xi_+ + H_- \xi)^{\perp}$

$$\begin{array}{c} \qquad\qquad\qquad\qquad W^{\perp} \\ \qquad\qquad\qquad\qquad \downarrow\wr \\ \text{First case} \quad 0 \to W \to V \to V/W \to 0 \end{array}$$

$$v = w \oplus u \in W + W^{\perp}$$

General ~~[illegible]~~ question to first ask is inner product on a quotient space.





$$\xi_+ = s p_0 + u \qquad \text{orth.} \quad 1 = s^2 + \|u\|^2$$
$$\xi_- = t q_0 + v \qquad\qquad 1 = t^2 + \|v\|^2$$

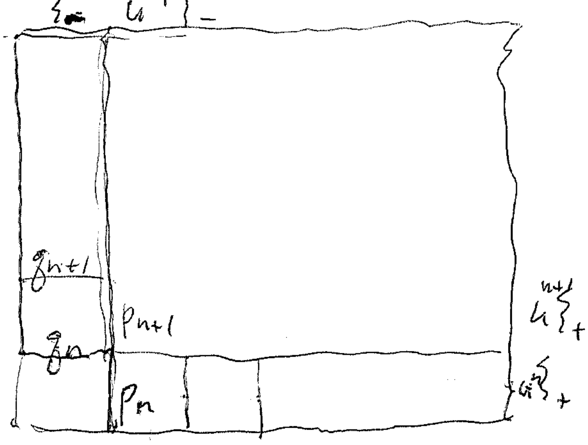
$$(\xi_- | \xi_+) = st\, h_0 + (v|u)$$

In your case ~~with~~ I think you have $s = t$ is related to $\prod_{n \geq 1}(1 - |h_n|^2)$. Do the estimates to first order in β. The idea is that $\|u\|$ $\|v\|$ are small, so s, t ~~are~~ near 1.

$$p_0 - q_1 h_1 = k_1 u p_0$$
$$q_1 - p_1 \bar{h}_1 = k_1 q_0$$
$$q_1 = \bar{h}_1 p_1 + k_1 q_0$$
$$q_2 = \bar{h}_2 p_2 + k_2 \bar{h}_1 p_1 + k_2 k_1 q_0$$
$$q_n = \sum_1^n k_n \cdots k_{i+1} \bar{h}_i p_i + k_n \cdots k_1 q_0$$
$$\xi_- = \underbrace{\sum_1^\infty \prod_{n > i} k_n \bar{h}_i p_i}_{u} + \underbrace{\prod_1^\infty k_n}_{s} q_0$$

$$(\partial_x - \lambda) u = h v \qquad \partial_x(u - v) - \lambda(u + v) = h(v - u)$$
$$(\partial_x + \lambda) v = h u \qquad (\partial_x + h)(u - v) = \lambda(v - u).$$

$$\partial_x(u + v) - \lambda(u - v) = h(u + v)$$
$$(\partial_x - h)(u + v) = \lambda(u - v)$$



latte 2.93

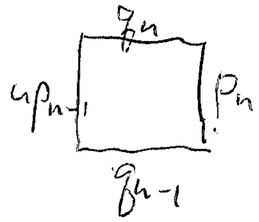
$$\begin{pmatrix} u^{-n}p_n \\ q_n \end{pmatrix} = \frac{1}{k_n}\begin{pmatrix} 1 & u^{-n}h_n \\ \bar h_n u^n & 1 \end{pmatrix}\begin{pmatrix} u^{-n+1}p_{n-1} \\ q_{n-1} \end{pmatrix}$$

$$q_{n+1} = \bar h_{n+1} p_{n+1} + k_{n+1} q_n$$

$$q_{n+2} = \bar h_{n+2} p_{n+2} + k_{n+2} k_{n+1} \bar h_{n+1} p_{n+1} + k_{n+2} k_{n+1} q_n$$

$$\xi_- = \sum_{i=n+1}^{\infty} \prod_{j>i} k_j \bar h_i p_i + \prod_{j>n} k_j q_n$$

Idea from McKean's talk of Fredholm determinants. Szego determinant? I think that this is something like $\prod(1-|h_n|^2)$. Start again

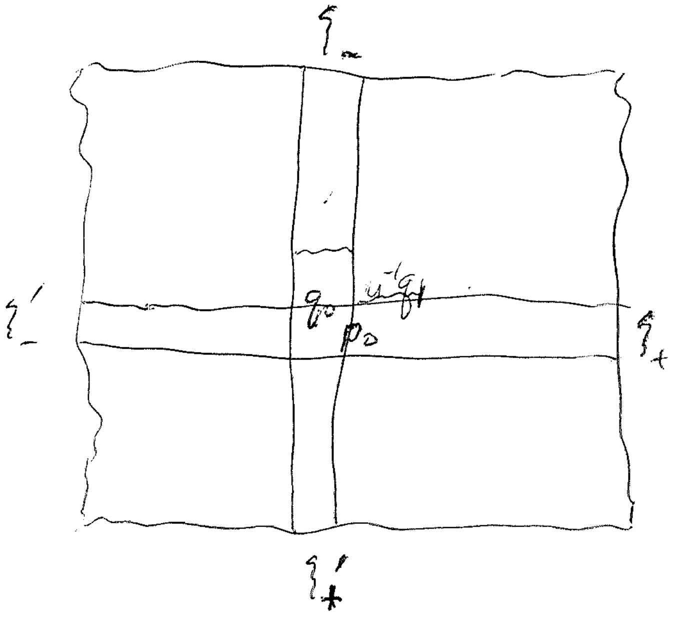


$$\begin{pmatrix} p_n \\ q_n \end{pmatrix} = \frac{1}{k_n}\begin{pmatrix} 1 & h_n \\ \bar h_n & 1 \end{pmatrix}\begin{pmatrix} up_{n-1} \\ q_{n-1} \end{pmatrix} \qquad \begin{pmatrix} up_{n-1} \\ q_{n-1} \end{pmatrix} = \frac{1}{k_n}\begin{pmatrix} 1 & -h_n \\ -\bar h_n & 1 \end{pmatrix}\begin{pmatrix} p_n \\ q_n \end{pmatrix}$$

$$\begin{pmatrix} p_n \\ q_{n-1} \end{pmatrix} = \begin{pmatrix} k_n & h_n \\ -\bar h_n & k_n \end{pmatrix}\begin{pmatrix} up_{n-1} \\ q_n \end{pmatrix} \qquad \begin{pmatrix} up_{n-1} \\ q_n \end{pmatrix} = \begin{pmatrix} k_n & -h_n \\ \bar h_n & k_n \end{pmatrix}\begin{pmatrix} p_n \\ q_{n-1} \end{pmatrix}$$

$$q_n = \bar h_n p_n + \underbrace{k_n q_{n-1}}_{k_n(\bar h_{n-1} p_{n-1} + k_{n-1} q_{n-2})}$$

$$q_n = \sum_{j=m+1}^{n} k_n \cdots k_{j+1} \bar h_j p_j + k_n \cdots k_{m+1} q_m$$



$$\begin{pmatrix}\xi_+\\ \xi_-\end{pmatrix}=\begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}\xi'_-\\ \xi'_+\end{pmatrix} \qquad \begin{pmatrix}\xi_+\\ \xi'_+\end{pmatrix}=\begin{pmatrix}\frac{1}{d} & \frac{b}{d}\\ -\frac{c}{d} & \frac{1}{d}\end{pmatrix}\begin{pmatrix}\xi'_-\\ \xi_-\end{pmatrix}$$

$$\begin{pmatrix}\xi_+\\ \xi'_+\end{pmatrix}=\begin{array}{c} L^2 \\ \begin{pmatrix}\frac{1}{d} & \frac{b}{d}\\ -\frac{c}{d} & \frac{1}{d}\end{pmatrix} \\ L^2 \quad H_+\end{array}\begin{pmatrix}\xi'_-\\ \xi_-\end{pmatrix}$$

$$\begin{pmatrix}\xi'_-\\ \xi'_+\end{pmatrix}=\begin{pmatrix}d & -b\\ -c & a\end{pmatrix}\begin{pmatrix}\xi_+\\ \xi_-\end{pmatrix}$$

$$\xi'_- = \sum_{j\geq 0} d_j u^j \xi_+ - \sum_{k\in\mathbb{Z}} b_k u^k \xi_-$$

$$(u^k\xi_-, \xi'_-) = \sum d_j \beta_{k-j} - b_k = 0 \qquad \forall k$$

$$(u^j\xi_+, \xi'_-) = d_j - \sum_k b_k \bar\beta_{k-j} = 0 \qquad j\geq 1$$

$$b = d\beta$$

$$d - b\bar\beta \in zH_- \qquad d(1-|\beta|^2)\in zH_- \qquad 1-|\beta|^2 = \alpha\bar\alpha$$

with $\alpha\in H_+$, then $d\alpha\bar\alpha \in zH_- \Rightarrow d\alpha \in H_+\cap zH_-$

$\therefore\ \alpha = \frac{1}{d}$. This you already know. ~~Whats~~

You have $\xi_- = q_\infty = \sum_{j=-\infty}^{\infty}\left(\prod_{n>j}^{\infty} k_n\right) h_j p_j + \left(\prod_{-\infty}^{\infty} k_n\right)\xi'_+$

$$\therefore\ (\xi'_+ | \xi_-) = \prod_{-\infty}^{\infty} k_n \qquad \xi'_+ = -\frac{c}{d}\xi'_- + \frac{1}{d}\xi_-$$

$$(\xi_-|\xi'_+) = \int \frac{1}{d} = \underset{\alpha(0)}{\overset{\shortparallel}{\delta(0)}}$$

So see what to do.
You basically know the arguments.

$$q_n = k_n\cdots k_{m+1}q_m + \sum_{m+1}^{n} k_n\cdots k_{i+1}\bar h_i p_i$$

$$\xi_- = \left(\prod_{m+1}^{\infty} k_i\right)q_m + \sum_{i=m+1}^{\infty}\left(\prod_{i+1}^{\infty} k_j\right)\bar h_i p_i$$

$$u^{-n}p_n = (k_n\cdots k_{m+1})u^{-m}p_m + \sum_{i>m}^{n}(k_n\cdots k_{i+1})h_i u^{-i}q_i$$

$$\xi_+ = \left(\prod k_j\right)u^{-m}p_m + \sum_{i>m}\left(\prod_{j>i} k_j\right)h_i u^{-i}q_i$$

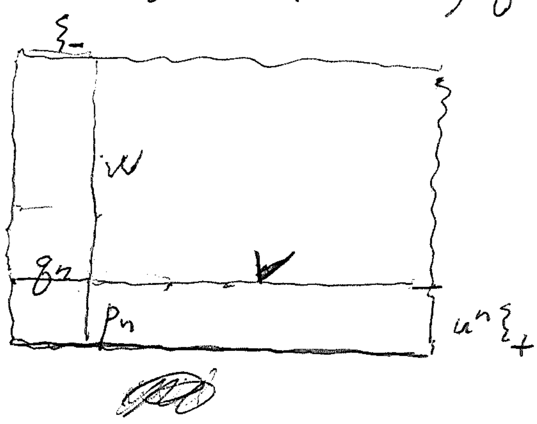
$$\begin{pmatrix}p_n\\ q_n\end{pmatrix} = \frac{1}{k_n}\begin{pmatrix}1 & h_n\\ \bar h_n & 1\end{pmatrix}\begin{pmatrix}u p_{n-1}\\ q_{n-1}\end{pmatrix} \qquad \begin{pmatrix}p_n\\ q_{n-1}\end{pmatrix} = \begin{pmatrix}k_n & h_n\\ -\bar h_n & k_n\end{pmatrix}\begin{pmatrix}u p_{n-1}\\ q_n\end{pmatrix}$$

$$\begin{pmatrix}u p_{n-1}\\ q_n\end{pmatrix} = \begin{pmatrix}k_n & -h_n\\ \bar h_n & k_n\end{pmatrix}\begin{pmatrix}p_n\\ q_{n-1}\end{pmatrix}$$

$$\boxed{q_n = \bar h_n p_n + k_n q_{n-1}}$$

$$p_n = h_n q_n + k_n u p_{n-1}$$

$$\boxed{u^{-n} p_n = h_n u^{-n} q_n + k_n u^{-n+1} p_{n-1}}$$



$$\xi_- = \Big(\prod_{j>n}^{\infty} k_j\Big) q_n + \sum_{j>n}^{\infty}\Big(\prod_{i>j} k_i\Big)\bar h_j p_j$$

$$\xi_+ = \Big(\prod_{j>n} k_j\Big) u^{-n} p_n + \sum_{j>n}^{\infty}\Big(\prod_{i>j} k_i\Big) h_j u^{-j} q_j$$

$$\xi_- = s_n q_n + W_n, \quad \xi_+ = s_n u^{-n} p_n + u^{-n} V_n, \quad u^n \xi_+ = s p_n + V_n$$

$$W = \sum_{j>n}\Big(\prod_{i>j} k_i\Big)\bar h_j p_j \qquad V = \sum_{j>n}\Big(\prod_{i>j} k_i\Big)\bar h_j u^{-j+n} q_j$$

$$1 = s_n^2 + \|V_n\|^2 = s_n^2 + \|W_n\|^2$$

$$\beta_{-n} = (\xi_- \mid u^n \xi_+) = (s q_n + W_n \mid s p_n + V_n) = s^2 h_n + (W_n \mid V_n)$$

$$s_n^2 |h_n| \le |\beta_{-n}| + |(W_n \mid V_n)| \le |\beta_{-n}| + \underbrace{\|W_n\|\,\|V_n\|}_{1-s_n^2}$$

$$|h_n| \le \frac{1}{s_n^2}|\beta_{-n}| + \frac{1-s_n^2}{s_n^2}$$ Is it true that $s_n > \varepsilon$

$s_n = \prod_{j>n} k_j$ so s_n is a decreasing sequence as n decreases bounded above zero when h_n is an ℓ^2 sequence

The continuous case. Basic difference is that you have $u^t = e^{ikt}$ instead of $u^n = z^n$. Set up the same sort of thing

$$\partial_x \begin{pmatrix} p_x \\ q_x \end{pmatrix} = \begin{pmatrix} ik & h_x \\ \bar h_x & 0 \end{pmatrix}\begin{pmatrix} p_x \\ q_x \end{pmatrix}$$

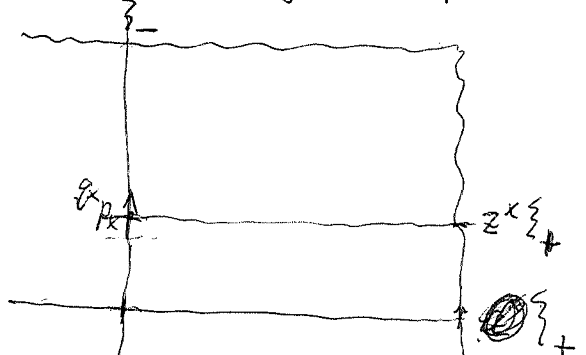
$$\partial_x \begin{pmatrix} e^{-ikx} p_x \\ q_x \end{pmatrix} = \begin{pmatrix} -ik\, e^{-ikx} p_x + e^{-ikx}\partial_x p_x \\ \partial_x q_x \end{pmatrix}$$

$$= \begin{pmatrix} -ik\, e^{-ikx} p_x + e^{-ikx}(ik p_x + h_x q_x) \\ \bar h_x p_x \end{pmatrix}$$

$$= \begin{pmatrix} e^{-ikx} h_x q_x \\ \bar h_x p_x \end{pmatrix} = \begin{pmatrix} e^{-ikx} h_x q_x \\ \bar h_x e^{ikx} e^{-ikx} p_x \end{pmatrix}$$

$$\boxed{\partial_x \begin{pmatrix} e^{-ikx} p_x \\ q_x \end{pmatrix} = \begin{pmatrix} 0 & e^{-ikx} h_x \\ \bar h_x e^{ikx} & 0 \end{pmatrix}\begin{pmatrix} e^{-ikx} p_x \\ q_x \end{pmatrix}}$$

instead of calculating, discuss what might happen. You expect the same pictures, p_x vertical, q_x horizontal



$$\begin{pmatrix} z^{-x} p_x \\ q_x \end{pmatrix} = \begin{pmatrix} d_x^r & -b_x^r \\ -c_x^r & a_x^r \end{pmatrix}\begin{pmatrix} \xi_+ \\ \xi_- \end{pmatrix}$$

First take $x = 0$.

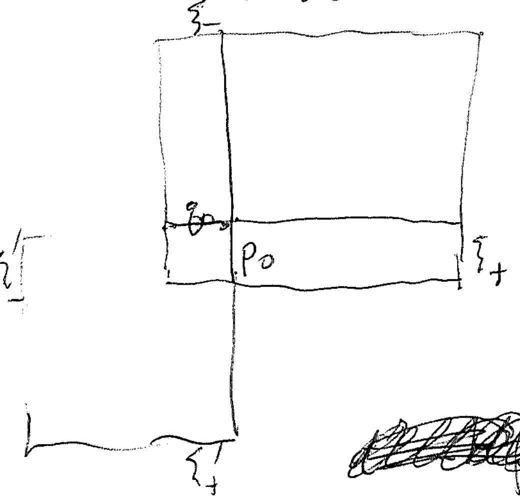
$$\begin{pmatrix} p_0 \\ q_0 \end{pmatrix} = \begin{pmatrix} d^r & -b^r \\ -c^r & a^r \end{pmatrix}\begin{pmatrix} \xi_+ \\ \xi_- \end{pmatrix} \qquad p_0 = \int_{(x>0)} d_x z^x \xi_+ - \int_{(y<0)} b_y z^y \xi_-$$

$$0 = (z^y \xi_- \mid p_0) = \int dx\, (z^y \xi_- \mid z^x \xi_+) - b_y \qquad d\beta - b \in H_+$$

for $y < 0$

$$= \int d_x \beta_{y-x} - b_y \in H_+ \qquad d - b\bar\beta \in$$

discrete case



$$\begin{pmatrix} p_0 \\ q_0 \end{pmatrix} = \begin{array}{c} \begin{array}{cc} H_+ & H_- \end{array} \\ \begin{pmatrix} d^r & -b^r \\ -c^r & +a^r \end{pmatrix} \\ \begin{array}{cc} zH_+ & zH_- \end{array} \end{array} \begin{pmatrix} \xi_+ \\ \xi_- \end{pmatrix}$$

$$\begin{pmatrix} d^r & -b^r \\ -c^r & a^r \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{array}{c} \begin{array}{cc} zH_- & H_+ \end{array} \\ \begin{pmatrix} a^l & b^l \\ c^l & d^l \end{pmatrix} \\ \begin{array}{cc} zH^- & H_+ \end{array} \end{array}$$

$$\begin{pmatrix} d^r & -b^r \\ -c^r & a^r \end{pmatrix}\begin{pmatrix} 1 & \frac{b}{d} \\ \frac{c}{a} & 1 \end{pmatrix} = \begin{array}{c} \begin{array}{cc} zH_- & H_+ \end{array} \\ \begin{pmatrix} \frac{a^l}{a} & \frac{b^l}{d} \\ \frac{c^l}{a} & \frac{d^l}{d} \end{pmatrix} \\ \begin{array}{cc} zH_- & H_+ \end{array} \end{array}$$

$$d^r\beta - b^r \in H_+$$
$$d^r - b^r\bar\beta \in zH_-$$

In the continuous case you expect to interpret H_+ as $\tilde H_+$ and zH_+ as H_+. Now what is d^r etc. functions of k. Before $d^r = \sum_{j\geq 0} d_j z^j$ with $d_0 > 0$

This should become $d^r = \int d^r_x z^x$ and you want

$d^r_x = \delta(x) + L^2$ function support $x > 0$.

so $d^r = 1$ something in H_+.

What happens to the k_n's?

What does the scattering look like? I seem to remember showing that $h_x \in L^2 \Longrightarrow$ perturbation kernel is Hilbert-Schmidt. What you can do is argue that $h_x \in L^1 \Rightarrow$ good convergence

You expect that the analog of $\prod k_n$ is 1, and the next order is interesting.

$$d^r\beta - b^r \in H_+$$
$$d^r - b^r\bar\beta \in zH$$

What are the basic facts about β?
What can you say about

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \text{time ordered } e^{\int \begin{pmatrix} 0 & h_x e^{-ikx} \\ \bar h_x e^{ikx} & 0 \end{pmatrix} dx} = \begin{pmatrix} 1 & \int h_x e^{-ikx} dx \\ \int \bar h_x e^{ikx} dx & 1 \end{pmatrix}$$

so to first order in h we have

so to first order $\beta(k) = \int h_x e^{-ikx}\,dx$. So what?

Need Fourier inversion.

Suppose $\beta(k)$ given, what properties?
Got to arise from a scattering situation

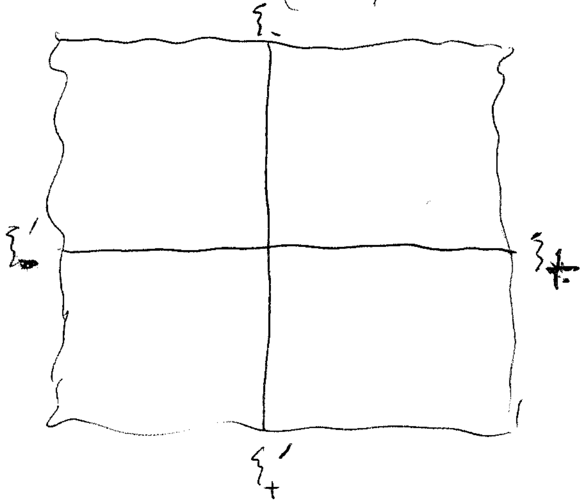
$$\partial_x \begin{pmatrix} p_x \\ q_x \end{pmatrix} = \begin{pmatrix} ik & h_x \\ \bar h_x & 0 \end{pmatrix} \begin{pmatrix} p_x \\ q_x \end{pmatrix}$$

$$\partial_x \begin{pmatrix} z^{-x} p_x \\ q_x \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & h_x e^{-ikx} \\ \bar h_x e^{ikx} & 0 \end{pmatrix}} \begin{pmatrix} z^{-x} p_x \\ q_x \end{pmatrix}$$

this is in ~~the~~ Lie alg of $SU(1,1)$.

so $$\begin{pmatrix} \xi_+ \\ \xi_- \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \xi'_- \\ \xi'_+ \end{pmatrix}$$

with $\begin{array}{ccc} & \tilde H_- & L^2 \\ & \begin{pmatrix} a & b \\ c & d \end{pmatrix} & \in SU(1,1) \\ & L^2 & \tilde H_+ \end{array}$



$$\begin{pmatrix} \xi_+ \\ \xi'_+ \end{pmatrix} = \underbrace{\begin{pmatrix} \frac{1}{d} & \frac{b}{d} \\ -\frac{c}{d} & \frac{1}{d} \end{pmatrix}}_{\text{unitary}} \begin{pmatrix} \xi'_- \\ \xi_- \end{pmatrix}$$

unitary so that $\alpha(k), \beta(k)$ modulus < 1,

$$|d|^2 - |b|^2 = 1 \qquad 1 - \left|\frac{b}{d}\right|^2 = \left|\frac{1}{d}\right|^2$$

So we know that $|\beta(k)| < 1$, ~~[illegible]~~ and $\delta = \frac{1}{d}$ is the square root of $1 - |\beta|^2$ bdd analytic in UHP.

$$(u^y \xi_- | u^x \xi_+) = \cancel{[illegible]} (u^{y-x} \xi_- | \xi_+) = \int z^{y-x} \beta$$

$$= \int e^{-iky} e^{ikx} \beta \frac{dk}{1} = \hat\beta_{y-x}$$

$$\begin{array}{ccc} \tilde H_+ & & L^2 \\ \Downarrow & & \end{array}$$

~~[illegible]~~
$$\xi'_- = d\,\xi_+ - b\,\xi_-$$
$$\xi'_+ = -c\,\xi_+ + a\,\xi_-$$

$\int dk\, e^{-iky} d(k)\beta(k)\, dk$

$$0 \overset{\forall y}{=} (u^y \xi_- | \xi'_-) = (u^y \xi_- | d\xi_+ - b\xi_-) = \int z^{-y} d\beta\, dk - \hat b(y)$$

$$\int z^{-y} (d\beta - b)\, dk = 0 \quad \text{for } \forall y \qquad \boxed{\begin{array}{ll} d\beta = b & d - b\bar\beta \in \tilde H_- \\ d(1-|\beta|^2) \in \tilde H_- & \end{array}}$$

$$0 \overset{\forall x>0}{=} (u^x \xi_+ | \xi'_-) = (u^x \xi_+ | d\xi_+ - b\xi_-) = \int z^{-x}(d - b\bar\beta)\, dk$$

So what next? How do I get h_x?

How do you get h?

$$\xi_- = \Big(\prod_{j>n} k_j\Big) g_n + \sum_{j>n}\Big(\prod_{i>j} k_i\Big)\bar h_j p_j \quad ?$$

There are problems here.

You should have analogs of $\delta(0) = \prod_{-\infty}^{\infty} k_n$

Apparently there is something subtle going on so that passing to the continuous limit is not obvious.

$$\xi_- = \Big(\prod_{j>0} k_j\Big) g_0 + \sum_{j>0}\Big(\prod_{i>j}k_i\Big)\bar h_j p_j$$

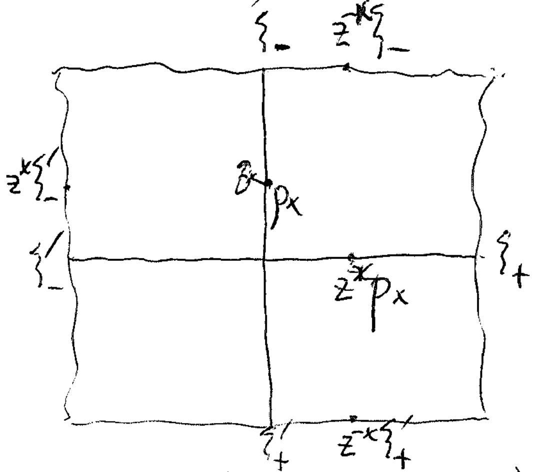
Yesterday I ~~studied~~ looked at the continuous case. Here $z^x = e^{2ikx}$ and instead of $\sum a_n z^n$ you have $\int a_x z^x \, dx$. Start again with

$$\partial_x \begin{pmatrix} p_x\\ g_x\end{pmatrix} = \begin{pmatrix} ik & h_x \\ \bar h_x & 0\end{pmatrix}\begin{pmatrix}p_x\\ g_x\end{pmatrix} \qquad \partial_x\begin{pmatrix} z^{-x}p_x\\ g_x\end{pmatrix} = \begin{pmatrix} 0 & z^{-x}h_x\\ \bar h_x z^x & 0\end{pmatrix}\begin{pmatrix} z^{-x}p_x\\ g_x\end{pmatrix}$$

$$\begin{pmatrix}a&b\\c&d\end{pmatrix} = T\exp\begin{pmatrix}0 & z^{-x}h_x\\ \bar h_x z^x & 0\end{pmatrix}$$

$$\begin{pmatrix}\xi_+\\ \xi_-\end{pmatrix} = \begin{pmatrix}a&b\\c&d\end{pmatrix}\begin{pmatrix}\xi'_-\\ \xi'_+\end{pmatrix}$$

$$\begin{pmatrix}\xi_+\\ \xi'_+\end{pmatrix} = \begin{pmatrix}\frac1d & \frac bd\\ -\frac cd & \frac1d\end{pmatrix}\begin{pmatrix}\xi'_-\\ \xi_-\end{pmatrix}$$



$$\begin{pmatrix}a&b\\c&d\end{pmatrix} = \begin{pmatrix}a^r&b^r\\c^r&d^r\end{pmatrix}\begin{pmatrix}a^\ell&b^\ell\\c^\ell&d^\ell\end{pmatrix}$$

$$\begin{pmatrix}\xi_+\\ \xi_-\end{pmatrix} = \begin{pmatrix}a^r&b^r\\c^r&d^r\end{pmatrix}\begin{pmatrix} z^{-x}p_x\\ g_x\end{pmatrix}$$

$$\begin{pmatrix} z^{-x}p_x\\ g_x\end{pmatrix} = \begin{array}{c} \begin{array}{cc}\tilde H_- & z^{-x}H_+\end{array} \\ \begin{pmatrix}a^\ell&b^\ell\\c^\ell&d^\ell\end{pmatrix} \\ \begin{array}{cc} z^x H_- & \tilde H_+\end{array}\end{array}\begin{pmatrix}\xi'_-\\ \xi'_+\end{pmatrix}$$

$$\begin{pmatrix} z^{-x}p_x\\ g_x\end{pmatrix} = \begin{array}{c} \begin{array}{cc}\tilde H_+ & z^{-x}H_-\end{array} \\ \begin{pmatrix}d^r&-b^r\\-c^r&a^r\end{pmatrix} \\ \begin{array}{cc} z^x H_+ & \tilde H_-\end{array}\end{array}\begin{pmatrix}\xi_+\\ \xi_-\end{pmatrix}$$

$$d = \underset{z^{+x}H_+\ \ z^{-x}H_+}{c^\wedge b^\ell} + \underbrace{d^\wedge d^\ell}_{\widetilde{H}_+}$$

$\underbrace{H_+ H_+}$ H_+^1

has value $= 0$. at $z=0$.

because $\int f_+ g_+ = (\bar{f}_+ | g_+) = 0$ as $H_- \perp H_+$.

Examples: h_x constant, too computational

Idea – Green's function, separating left + right, cutting. Go back to

How

$$\partial_x \begin{pmatrix} p_x \\ q_x \end{pmatrix} = \begin{pmatrix} 2ik & h_x \\ \bar{h}_x & 0 \end{pmatrix} \begin{pmatrix} p_x \\ q_x \end{pmatrix}$$

Solutions form a rank 2 v.b. over the k-plane

Your idea was to consider a specific soln.

~~I have the idea that a specific~~ You want to fix a point x_0, examine the Green's function with x_0 as singularity. The Green's [redacted] matrix jumps by the identity matrix as you pass thru x.

Look at the Green's fn. Make the operator skew adjoint

$$\begin{pmatrix} \partial_x & 0 \\ 0 & \partial_x \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \lambda & h \\ \bar{h} & -\lambda \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$$

$$\begin{pmatrix} \partial_x & -h \\ +\bar{h} & -\partial_x \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \lambda \begin{pmatrix} u \\ v \end{pmatrix}$$

Now for $\lambda \in i\mathbb{R}$ get ψ ess. unique decaying as $x \longmapsto \infty$ or $-\infty$.

e.g. $\mathrm{Re}(\lambda) > 0 \implies$

$$\begin{pmatrix} \partial_x & -h \\ \bar{h} & -\partial_x \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} = \lambda \begin{pmatrix} u \\ v \end{pmatrix}$$

solve IVP at $x=0$. You want $\Phi(x,\lambda)$ so that

$$\begin{pmatrix} u_x \\ v_x \end{pmatrix} = \Phi(x,\lambda)\begin{pmatrix} u_0 \\ v_0 \end{pmatrix}$$

This should be asymptotic as $x \longrightarrow \infty$ to

$$\begin{pmatrix} e^{\lambda x} & 0 \\ 0 & e^{-\lambda x} \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

$$\begin{pmatrix} e^{\lambda x} & 0 \\ 0 & e^{-\lambda x} \end{pmatrix}\begin{pmatrix} \partial_x & -h \\ \bar{h} & -\partial_x \end{pmatrix}\begin{pmatrix} e^{-\lambda x} & 0 \\ 0 & e^{\lambda x} \end{pmatrix} = \begin{pmatrix} e^{\lambda x}\partial_x e^{-\lambda x} & -h e^{2\lambda x} \\ \bar{h} e^{-2\lambda x} & -e^{-\lambda x}\partial e^{\lambda x} \end{pmatrix}$$

$$= \begin{pmatrix} \partial_x - \lambda & -h e^{2\lambda x} \\ \bar{h} e^{-2\lambda x} & -\partial_x - \lambda \end{pmatrix}$$

suppose

$$\begin{pmatrix} \partial_x & -h \\ \bar{h} & -\partial_x \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} = \lambda \begin{pmatrix} u \\ v \end{pmatrix}$$

$$\begin{pmatrix} e^{\lambda x} & \\ & e^{-\lambda x} \end{pmatrix}\begin{pmatrix} \partial_x & -h \\ \bar{h} & -\partial_x \end{pmatrix}\begin{pmatrix} e^{-\lambda x} & 0 \\ 0 & e^{\lambda x} \end{pmatrix}\begin{pmatrix} e^{\lambda x}u \\ e^{-\lambda x}v \end{pmatrix} = \lambda \begin{pmatrix} e^{\lambda x}u \\ e^{-\lambda x}v \end{pmatrix}$$

$$\begin{pmatrix} \partial_x & -h e^{2\lambda x} \\ \bar{h} e^{-2\lambda x} & -\partial_x \end{pmatrix}\begin{pmatrix} e^{\lambda x}u \\ e^{-\lambda x}v \end{pmatrix} = \lambda \begin{pmatrix} e^{\lambda x}u \\ e^{-\lambda x}v \end{pmatrix}$$

Review: Continuous case. $z = e^{2\lambda}$

$$\partial_x \begin{pmatrix} p_x \\ q_x \end{pmatrix} = \begin{pmatrix} 2\lambda & h \\ \bar h & 0 \end{pmatrix} \begin{pmatrix} p_x \\ q_x \end{pmatrix} \qquad \begin{pmatrix} u \\ v \end{pmatrix} = e^{-\lambda x} \begin{pmatrix} p_x \\ q_x \end{pmatrix}$$

$$\partial_x \begin{pmatrix} u \\ v \end{pmatrix} = -\lambda e^{-\lambda x} \begin{pmatrix} p \\ q \end{pmatrix} + e^{-\lambda x} \begin{pmatrix} 2\lambda & h \\ \bar h & 0 \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix}$$

$$= -\lambda \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} 2\lambda & h \\ \bar h & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \lambda & h \\ \bar h & -\lambda \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}.$$

Pert?

$$\begin{pmatrix} \partial & -h \\ +\bar h & -\partial \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \lambda \begin{pmatrix} u \\ v \end{pmatrix}$$

$$\begin{pmatrix} \lambda - \partial & 0 \\ 0 & \lambda + \partial \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 & h \\ -\bar h & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$$

Your aim is to reconstruct the ~~...~~ inverse scattering in the usual setting of functions of λ. You want to ~~impose~~ bring in the Green's function. I think this should involve contractions, at least partial unitaries. The Green's fn. ~~is a~~ involves a singularity, a cut.

You want to work out the known theory propagators from $-\infty$ to 0 and 0 to ∞.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a^r & b^r \\ c^r & d^r \end{pmatrix} \begin{pmatrix} a^l & b^l \\ c^l & d^l \end{pmatrix}$$

$$T\left\{ e^{\int_{-\infty}^{\infty} \begin{pmatrix} 0 & h_x e^{-2\lambda x} \\ \bar h_x e^{2\lambda x} & 0 \end{pmatrix} dx} \right\} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & \int h_x e^{-2\lambda x} \\ \int \bar h_x e^{2\lambda x} & 0 \end{pmatrix}$$

$$+ \int_{x_1 > x_2} \begin{pmatrix} 0 & h_{x_1} e^{-2\lambda x_1} \\ \bar h_{x_1} e^{2\lambda x_1} & 0 \end{pmatrix} \begin{pmatrix} 0 & h_{x_2} e^{-2\lambda x_2} \\ \bar h_{x_2} e^{2\lambda x_2} & 0 \end{pmatrix}$$

$$d(\lambda) = 1 + \int_{x_1 > x_2} dx_1\, dx_2\, \bar{h}_{x_1} e^{2\lambda(x_1 - x_2)} h_{x_2}$$ to 3rd order.

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} (\lambda - \partial)^{-1} & 0 \\ 0 & (\lambda + \partial)^{-1} \end{pmatrix} \begin{pmatrix} 0 & h \\ \bar{h} & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$$

$$\lambda \psi = (D_0 + V)\psi$$

$$(\lambda - D_0)\psi = V\psi \qquad \psi = (\lambda - D_0)^{-1} V \psi$$

$$\lambda G_0 \psi = (1 + G_0 V)\psi$$

$$\lambda (1 + G_0 V)^{-1} G_0 \psi = \psi$$

$$\partial_x \begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} 2\lambda & h \\ \bar{h} & 0 \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix}$$

$$\partial_x \begin{pmatrix} e^{-2\lambda x} p \\ q \end{pmatrix} = \begin{pmatrix} \cancel{-2\lambda e^{-2\lambda x}} p \\ 0 \end{pmatrix} + \begin{pmatrix} e^{-2\lambda x} (\cancel{2\lambda p} + hq) \\ \bar{h} p \end{pmatrix}$$

$$= \begin{pmatrix} h e^{-2\lambda x} q \\ \bar{h} e^{2\lambda x} e^{-2\lambda x} p \end{pmatrix} = \begin{pmatrix} 0 & h e^{-2\lambda x} \\ \bar{h} e^{2\lambda x} & 0 \end{pmatrix} \begin{pmatrix} e^{-2\lambda x} p \\ q \end{pmatrix}$$

$$\boxed{\partial_x \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 & h e^{-2\lambda x} \\ \bar{h} e^{2\lambda x} & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}}$$

$$\begin{pmatrix} u \\ v \end{pmatrix}\Bigg|_{-\infty}^{x} = \int_{-\infty}^{x} dx_1 \overbrace{\begin{pmatrix} 0 & he^{-2\lambda x_1} \\ \bar{h}e^{2\lambda x_1} & 0 \end{pmatrix}}^{V(x_1)} \begin{pmatrix} u \\ v \end{pmatrix}(x_1)$$

$$\psi(x) = \psi(-\infty) + \int_{-\infty}^{x} dx_1 \, V(x_1)\psi(x_1)$$

$$= \psi(-\infty) + \int_{-\infty}^{x} dx_1 \, V(x_1)\psi(-\infty)$$

$$+ \int_{-\infty}^{x} dx_1 \, V(x_1) \int_{-\infty}^{x_1} dx_2 \, V(x_2) \psi(x_2)$$

$$\psi(x) = \psi(-\infty) + \int_{-\infty}^{x} dx_1 \, V(x_1) \, \psi(-\infty)$$

$$+ \int_{-\infty}^{x} dx_1 \, V(x_1) \int_{-\infty}^{x_1} dx_2 \, V(x_2) \, \psi(-\infty)$$

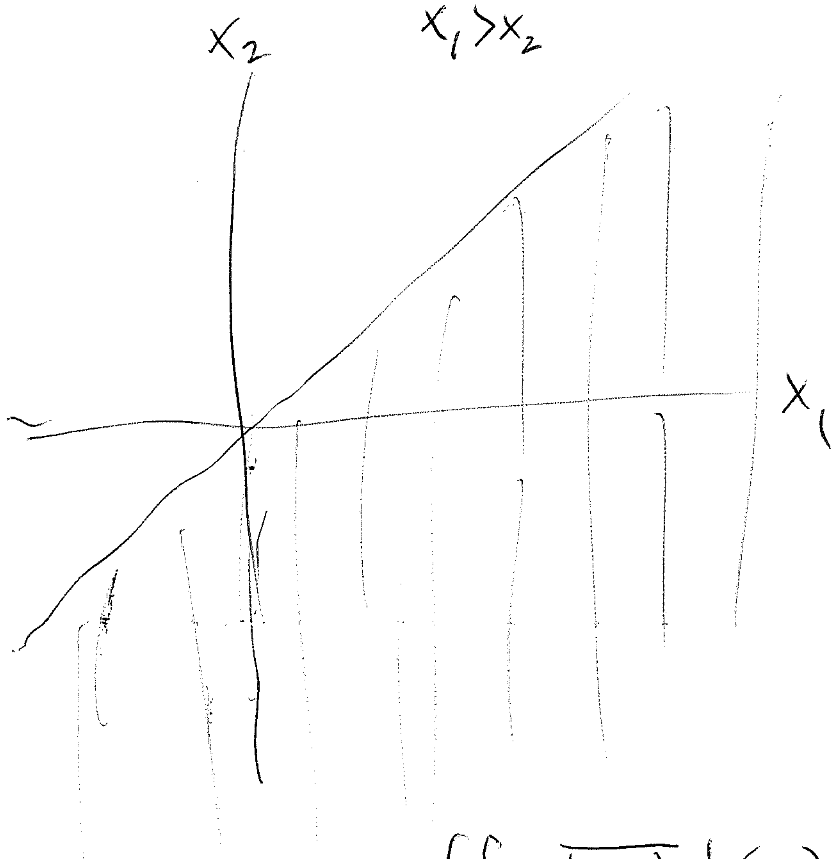
$$+ \int_{-\infty}^{x} dx_1 \, V(x_1) \int_{-\infty}^{x_1} dx_2 \, V(x_2) \int_{-\infty}^{x_2} dx_3 \, V(x_3) \psi(-\infty) + \cdot$$

$$\psi(0) = \psi(-\infty) + \int_{-\infty}^{0} dx_1 \, V(x_1) \, \psi(-\infty)$$

$$+ \int_{-\infty < x_2 < x_1 < 0} dx_1 \, dx_2 \; V(x_1) \, V(x_2) \; \psi(-\infty)$$

$$\int_{-\infty < x_2 < x_1 < 0} dx_1 \, dx_2 \begin{pmatrix} & \\ \bar{h}_{x_1} e^{2\lambda x_1} & \end{pmatrix}\begin{pmatrix} & h_{x_2} e^{-2\lambda x_2} \\ & \end{pmatrix} = \int_{x_1 > x_2} dx_1 dx_2 \, \bar{h}_{x_1} e^{2\lambda(x_1 - x_2)} h_{x_2}$$

what is $\displaystyle\iint_{x_1>x_2} dx_1 dx_2\ \bar{h}_{x_1} h_{x_2}\, e^{2\lambda(x_1-x_2)}$?



what is $\displaystyle\iint_{-\infty<x_2<x_1<\infty} \overline{h(x_1)}\, h(x_2)\, e^{2\lambda(x_1-x_2)}$

$y = x_1 - x_2$

$x_1 = y + x_2$

$$\| $$

$$\iint_{y\geq 0} \overline{h(y+x_2)}\, h(x_2)\, e^{2\lambda y}$$

$$\int_0^\infty dy\, e^{2\lambda y} \underbrace{\int_{-\infty}^{\infty} \overline{h(y+x)}\, h(x)\, dx}_{}$$

L^2 ~~notat~~ inner product $(u_y h | h)$

So if $h \in L^2$, then you should be able to analyze in terms of a measure.

$$\int_{-\infty}^{\infty} \overline{u_y h}\, h\, dx = \int dx \int \frac{dk}{2\pi} e^{-ik(y+x)}\overline{\hat{h}(k)} \int \frac{dk}{2\pi} e^{ikx}\hat{h}(k)$$

$h(x) = \int \frac{dk}{2\pi} e^{ikx}\hat{h}(k)$

$$= \int e^{-iky} |\hat{h}(k)|^2 \frac{dk}{2\pi}$$

$$\int_0^\infty dy\, e^{2\lambda y} \int e^{-iky} |\hat{h}(k)|^2 \frac{dk}{2\pi}$$

$$= \int \frac{1}{-2\lambda + ik} |\hat{h}(k)|^2 \frac{dk}{2\pi}$$ Stieltjes transf of spectral measure

---

$$\partial_x \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 & h e^{-2\lambda x} \\ \bar{h} e^{2\lambda x} & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$$

$$\partial_x \psi = V \psi \qquad \psi(x) = \psi(-\infty) + \int_{-\infty}^{x} V(x_1)\psi(x_1)\,dx_1$$

~~$\psi(x) = \psi(-\infty) + \int_{-\infty}^{x} V(x_1)\psi(x_1)\,dx_1 + \int_{-\infty}^{x}$~~

$$\psi(x) = \psi(-\infty) + \int_{-\infty}^{x} dx_1\, V(x_1)\, \psi(-\infty)$$

$$+ \int_{-\infty}^{x} dx_1\, V(x_1) \int_{-\infty}^{x_1} dx_2\, V(x_2)\, \psi(-\infty) + \cdots$$

$$\boxed{\int_{-\infty}^{\infty} dx_1\, V(x_1) \int_{-\infty}^{x_1} dx_2\, V(x_2)} = \int_{-\infty}^{\infty} dx_1 \int_{-\infty}^{x_1} dx_2 \begin{pmatrix} & h_{x_1} e^{-2\lambda x_1} \\ \bar{h}_{x_1} e^{2\lambda x_1} & \end{pmatrix} \begin{pmatrix} & h_{x_2} e^{-2\lambda x_2} \\ \bar{h}_2 e^{2\lambda x_2} & \end{pmatrix}$$

d coeff is $\displaystyle\int_{-\infty}^{\infty} dx_1 \int_{-\infty}^{x_1} dx_2\, \bar{h}_{x_1}\, e^{2\lambda(x_1 - x_2)} h_{x_2}$

$$= \int_{-\infty}^{\infty} dx_1 \int_{-\infty}^{\infty} dx_2\, \overline{h(x_2)}\, \underbrace{K(x_2 - x_1)}\, h(x_1)$$

$$\begin{cases} e^{-2\lambda(x_2 - x_1)} & x_1 > x_2 \\ 0 & x_1 < x_2 \end{cases}$$

formulas for FT. $h(x) = \int \frac{dk}{2\pi} e^{ikx} \hat{h}(k)$

$$\hat{h}(k) = \int e^{-ikx} h(x)\,dx$$

$$\int_{-\infty}^{0} e^{-ikx} e^{-2\lambda x}\,dx = \frac{1}{-ik-2\lambda} \qquad Re(\lambda)<0$$

So the d-term is

$$\int \frac{-1}{ik+2\lambda} |\hat{h}(k)|^2 \frac{dk}{2\pi}$$

analytic for $Re(\lambda)<0$ corresp to $|z| = |e^{2\lambda}| < 1$.

So the ~~scattering~~ transfer matrix to 2nd order is

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \cdot & \int_{-\infty}^{\infty} h_x e^{-2\lambda x} dx \\ \int_{-\infty}^{\infty} \bar{h}_x e^{+2\lambda x} dx & 1 + \int \frac{-1}{ik+2\lambda} |\hat{h}(k)|^2 \frac{dk}{2\pi} \end{pmatrix}$$

So this doesn't get us very far. Still need to recover the potential from the scattering data.

For this you ~~would~~ probably want the Green's function with singularity at $x=0$.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a^r & b^r \\ c^r & d^r \end{pmatrix} \begin{pmatrix} a^l & b^l \\ c^l & d^l \end{pmatrix}$$

$$\int_{-\infty}^{\infty} dx_1\, V(x_1) \int_{-\infty}^{x_1} dx_2\, V(x_2) \int_{-\infty}^{x_2} dx_3\, V(x_3) \int_{-\infty}^{x_4} dx_4\, V(x_4)$$

$$\int dx_1 \cdots dx_4 \quad \overline{h(x_1)}\, e^{2\lambda(x_1-x_2)}\, h(x_2)\, \overline{h(x_3)}\, e^{2\lambda(x_3-x_4)}\, h(x_4)$$

$\infty > x_1 > x_2 > x_3 > x_4$

Start again $\quad \partial_x \psi(x) = \begin{pmatrix} 0 & h(x) e^{-2\lambda x} \\ \overline{h(x)} e^{+2\lambda x} & 0 \end{pmatrix} \psi(x)$

$$\partial_x \psi = V(x) \psi$$

$$\psi(x) = \psi(-\infty) + \int_{-\infty}^{x} V(x_1) \psi(x_1)\, dx_1$$

$$= \psi(-\infty) + \int_{-\infty}^{x} dx_1\, V(x_1) \psi(-\infty) + \int_{-\infty}^{x} dx_1\, V(x_1) \int_{-\infty}^{x_1} dx_2\, V(x_2) \psi(-\infty)$$

Look at

$$\int_{-\infty}^{\infty} dx_1 \int_{-\infty}^{x_1} dx_2\, \underbrace{\overbrace{\begin{pmatrix} 0 & h_{x_1} e^{-2\lambda x_1} \\ \overline{h}_{x_1} e^{2\lambda x_1} & 0 \end{pmatrix}}^{V(x_1)} \overbrace{\begin{pmatrix} 0 & h_{x_2} e^{-2\lambda x_2} \\ \overline{h}_{x_2} e^{2\lambda x_2} & 0 \end{pmatrix}}^{V(x_2)}}_{\begin{pmatrix} h(x_1) e^{2\lambda(-x_1+x_2)} \overline{h(x_2)} & 0 \\ 0 & \overline{h(x_1)} e^{2\lambda(x_1-x_2)} h(x_2) \end{pmatrix}}$$

$$\boxed{\begin{gathered} \int dx_1 dx_2\ \overline{h(x_1)}\, K(x_1-x_2)\, h(x_2) \\ \| \\ \int \frac{dk}{2\pi} \frac{1}{ik-2\lambda} \left|\hat{h}(k)\right|^2 \end{gathered}}$$

$$K(x) = \begin{cases} e^{2\lambda x} & x > 0 \\ 0 & x < 0 \end{cases}$$

$$\int_0^{\infty} e^{-ikx} \overset{e^{2\lambda x}}{K(x)}\, dx = \frac{1}{ik-2\lambda}$$

$$\begin{cases} 1 & x_2 > x_3 \\ 0 & x_2 < x_3 \end{cases}$$

$$\int dx_1 dx_2 dx_3 dx_4\ \overline{h(x_1)}\, K(x_1-x_2)\, h(x_2)\, H(x_2-x_3)\, \overline{h(x_3)}\, K(x_3-x_4)\, h(x_4)$$

this can be replaced by ~~and~~ an integral, ~~over~~ sum ~~[illegible] $\hat{h}(k_i)$~~ over momenta.

You need to reconstruct the potential. List ideas to organize. Important is the Green's function idea, because you think it will allow you to handle contractions and partial unitaries. Your Hilbert space + 1 param. unitary group concerns the homogeneous D.E. You want to put in the singularity.

$$\partial_x \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 & h e^{-2\lambda x} \\ \bar{h} e^{2\lambda x} & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$$

$$\begin{pmatrix} a^r & b^r \\ c^r & d^r \end{pmatrix} = T\exp \int_0^\infty V(x)\,dx$$

First discuss scattering

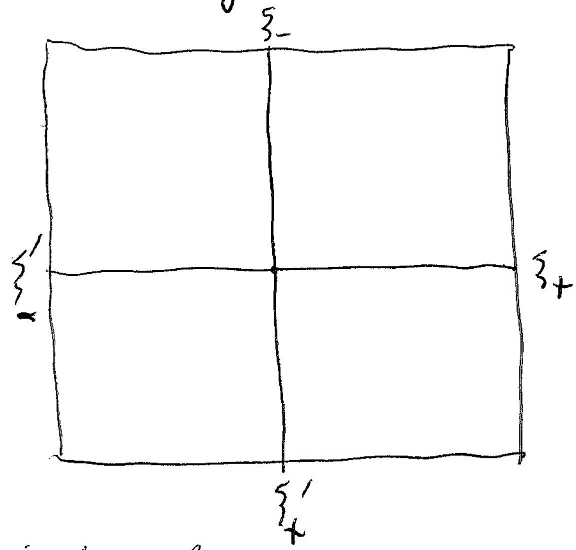
$$\begin{pmatrix} \xi'_- \\ \xi'_+ \end{pmatrix} \longleftarrow \begin{pmatrix} u_x \\ v_x \end{pmatrix} = \begin{pmatrix} z^{-x} p_x \\ q_x \end{pmatrix} \longrightarrow \begin{pmatrix} \xi_+ \\ \xi_- \end{pmatrix}$$

here the systems stand for vectors in E essentially or for a ~~specific~~ generic solution of the DE. The Green's function should be some operator on E?

$$\begin{pmatrix} u_0 \\ v_0 \end{pmatrix} = \begin{pmatrix} a^l & b^l \\ c^l & d^l \end{pmatrix} \begin{pmatrix} \xi'_- \\ \xi'_+ \end{pmatrix}$$

Wait: It should be possible to interpret elements of E as a pair of functions of λ, this is clear – there are many natural coordinates, so maybe you want view E as sections of a vector bundle of rank 2 over the λ plane.

So worry about how to recover h from the scattering data.



$$\begin{pmatrix} \rho_0 \\ \xi_0 \end{pmatrix} = \begin{array}{c} \begin{array}{cc} \tilde{H}_+ & H_- \end{array} \\ \begin{pmatrix} d^r & -b^r \\ -c^r & a^r \end{pmatrix} \\ \begin{array}{cc} H_+ & \tilde{H}_- \end{array} \end{array} \begin{pmatrix} \xi_+ \\ \xi_- \end{pmatrix}$$

$$\begin{pmatrix} d^r & -b^r \\ -c^r & a^r \end{pmatrix} \begin{pmatrix} 1 & \beta \\ \bar{\beta} & 1 \end{pmatrix} = \begin{array}{c} \begin{array}{cc} \tilde{H}_- & H_+ \end{array} \\ \begin{pmatrix} \frac{a^l}{a} & \frac{b^l}{d} \\ \frac{c^l}{a} & \frac{d^l}{d} \end{pmatrix} \\ \begin{array}{cc} \hat{H}_- & \tilde{H}_+ \end{array} \end{array}$$

integral equations are
$$\begin{cases} d^r - b^r\bar{\beta} \in \tilde{H}_- \\ d^r\beta - b^r \in H_+ \end{cases}$$

In more detail? What is d^r?

$$d^r(\lambda) = 1 + \int_0^\infty e^{\lambda x} d(x)\, dx \quad \in 1 + H_+$$

$$b^r(\lambda) = \int_{-\infty}^{0} e^{\lambda y} b(y)\, dy \quad \in 1 + H_-$$

$$\beta(\lambda) = \int_{-\infty}^{\infty} e^{\lambda z} \beta(z)\, dz$$

$$d^r(\lambda)\beta(\lambda) = \beta(\lambda) + \underbrace{\int_0^\infty e^{\lambda x} d(x)\, dx \int_{-\infty}^{\infty} e^{\lambda z} \hat{\beta}(z)\, dz}$$

$$\int_0^\infty dx \int_{-\infty}^{\infty} dz\, e^{\lambda \underbrace{(x+z)}_{y}} d(x) \hat{\beta}(\underbrace{z}_{y-x}) = \int_{-\infty}^{\infty} dy\, e^{\lambda y} \int_0^\infty d(x)\hat{\beta}(y-x)\, dx$$

So
$$d^r(\lambda)\beta(\lambda) = \int_{-\infty}^{\infty} dy\, e^{\lambda y} \left\{ \hat{\beta}(y) + \int_0^\infty d(x)\hat{\beta}(y-x)\, dx \right\}$$

$$b^r(\lambda) = \int_{-\infty}^{0} dy\, e^{\lambda y}\, b(y)$$

$d^\wedge(\lambda)\beta(\lambda) - b^\wedge(\lambda) \in H_+$ seems to mean

$$\boxed{\hat\beta(y) + \int_0^\infty d(x)\hat\beta(y-x)\,dx = b(y) \quad \text{for } y < 0.}$$

$$\overline{\beta(\lambda)} = \int_{-\infty}^{\infty} dz\, e^{-\lambda z}\, \overline{\hat\beta(z)} = \int_{-\infty}^{\infty} dz\, e^{\lambda z}\, \overline{\hat\beta(-z)}$$

"$\lambda \in i\mathbb{R}$"

$$b^\wedge(\lambda)\overline{\beta(\lambda)} = \int_{-\infty}^{\infty} dy\, e^{\lambda y} \int_{-\infty}^{\infty} dx\, b(x)\, \overline{\hat\beta(x-y)}$$

$$d^\wedge(\lambda) = 1 + \int_0^\infty dx\, e^{\lambda x} d(x)$$

$d^\wedge(\lambda) - b^\wedge(\lambda)\overline{\beta(\lambda)} \in 1 + H_+$ seems to mean

$$\boxed{d(y) = \int_{-\infty}^{0} dx\, b(x)\, \overline{\hat\beta(x-y)} \quad \text{for } y > 0}$$

$$\tilde p_0 = \xi_+ + \sum_{j>0} d_j u^j \xi_+ - \sum_{k<0} b_k u^k \xi_-$$

$$\overline{(u^{k-j}\xi_- | \xi_+)} = \beta_{k-j}$$

$$0 \overset{j>0}{=} (u^j \xi_+ | \tilde p_0) = d_j - \sum_{k<0} b_k \underbrace{(u^j \xi_+ | u^k \xi_-)}_{\overline{\beta_{k-j}}} \quad \text{for } j > 0$$

$$0 \overset{k<0}{=} (u^k \xi_- | \tilde p_0) = \beta_k + \sum_{j>0} d_j \beta_{k-j} - b_k \quad \text{for } k < 0$$

You should set this up with ~~varying~~ varying position so that a^k a^l etc. depend on x.

$$\partial_x \begin{pmatrix} z^{-x} p_x \\ q_x \end{pmatrix} = \begin{pmatrix} 0 & h e^{-\lambda x} \\ \bar{h} e^{\lambda x} & 0 \end{pmatrix} \begin{pmatrix} z^{-x} p_x \\ q_x \end{pmatrix}$$

Assuming h_x decays fast enough i.e. $|h e^{-\lambda x}| = |h| e^{-Re(\lambda)x}$, $|\bar{h} e^{\lambda x}| = |h| e^{Re(\lambda)x}$ bounded then we should have convergence as $x \to \infty$ or $-\infty$. Here think of $z^{-x} p_x$ and q_x as fns. of λ, say analytic in the strip $|Re(\lambda)| < \varepsilon$. If

$$\begin{pmatrix} z^{-x} p_x \\ q_x \end{pmatrix} \xrightarrow[\text{as } x \to +\infty]{} \begin{pmatrix} \xi_+ \\ \xi_- \end{pmatrix}$$

then $\begin{matrix} p_x \sim z^x \xi_+, \\ q_x \sim \xi_- \end{matrix}$ so if $Re(\lambda) > 0$, then p_x blows up unless $\xi_+ = 0$. Thus $\xi_+ = 0$ describes the decaying soln as $x \to +\infty$, when $Re(\lambda) > 0$. If $Re(\lambda) < 0$, then $\xi_- = 0$ should describe the decaying solution as $x \to +\infty$. This is ~~pretty~~ unclear.

Let us calculate the Green's function $G_\lambda(x, y)$ defined by $\left(\partial_x - V_\lambda(x)\right) G_\lambda(x, y) = \delta(x-y)$ id matrix

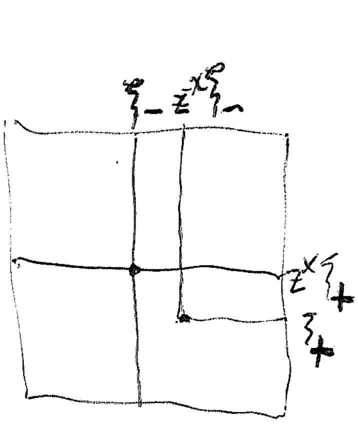
$$\lim_{x \to +\infty} G_\lambda(x, y) \text{ proportional to } \quad ?$$

$$G_\lambda(x, y) = \begin{pmatrix} d_x^r & -b_x^r \\ -c_x^r & +a_x^r \end{pmatrix} \begin{pmatrix} a_y^r & b_y^r \\ c_y^r & d_y^r \end{pmatrix}$$

Take $y = 0$, and find $G_\lambda(x, 0)$

$$\begin{pmatrix} z^{-x} p_x \\ q_x \end{pmatrix} = \begin{pmatrix} \overset{\tilde{H}_+}{d_x^r} & \overset{z^{-x} H_-}{-b_x^r} \\ \underset{z^x H_+}{-c_x^r} & \underset{\tilde{H}_-}{+a_x^r} \end{pmatrix} \begin{pmatrix} \xi_+ \\ \xi_- \end{pmatrix}$$

?



Problem – understand $G_\lambda(x,y)$, defined by

$$\left(\partial_x - V_\lambda(x)\right) G_\lambda(x,y) = 0 \quad \text{for} \quad x \neq y$$

and $G_\lambda(y^+,y) - G_\lambda(y^-,y) = I$. Also for $Re(\lambda)<0$

$$\xleftarrow{x\to-\infty} \quad G_\lambda(x,y) \quad \xrightarrow{x\to+\infty} \quad \begin{pmatrix} \quad \end{pmatrix}$$

$$\begin{pmatrix} z^{-x} p_x \\ q_x \end{pmatrix} \qquad G$$

$$\shortparallel$$

Consider $\psi_1(x) = \begin{pmatrix} a^\ell_x & b^\ell_x \\ c^\ell_x & d^\ell_x \end{pmatrix} \begin{pmatrix} \xi'_- \\ \xi'_+ \end{pmatrix}$

This has asymptotics $\lim\limits_{x\to-\infty} \psi_1(x) = \begin{pmatrix} \xi'_- \\ \xi'_+ \end{pmatrix}$

Consider also

$$\psi_2(x) = \begin{pmatrix} d^r_x & -b^r_x \\ -c^r_x & a^r_x \end{pmatrix} \begin{pmatrix} \xi_+ \\ \xi_- \end{pmatrix}$$

This has asymptotics $\lim\limits_{x\to+\infty} \psi_2(x) = \begin{pmatrix} \xi_+ \\ \xi_- \end{pmatrix}$

$$\begin{pmatrix} d_\ell & -b_\ell \\ -c_\ell & a_\ell \end{pmatrix} \begin{pmatrix} d_r & -b_r \\ -c_r & a_r \end{pmatrix}$$

Is it possible that $\psi_1(x) = \psi_2(x)$. This happens iff $\begin{pmatrix} \xi_+ \\ \xi_- \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \xi'_- \\ \xi'_+ \end{pmatrix}$

What ~~b~~ conditions are nice at $x=-\infty$.

Review. $\partial_x \begin{pmatrix} z^{-x} p_x \\ q_x \end{pmatrix} = \begin{pmatrix} 0 & h e^{-\lambda x} \\ \bar{h} e^{\lambda x} & 0 \end{pmatrix} \begin{pmatrix} z^{-x} p_x \\ q_x \end{pmatrix}$

Assuming h decays enough as $x \to \pm\infty$ ~~all~~ any solution has limits

$$\begin{pmatrix} \xi'_- \\ \xi'_+ \end{pmatrix} \longleftarrow \begin{pmatrix} z^{-x} p_x \\ q_x \end{pmatrix} \longrightarrow \begin{pmatrix} \xi_+ \\ \xi_- \end{pmatrix}$$