Go back over things. First begin with an $S(z) = \sum_{n\in\mathbb{Z}} S_n z^{-n}$ where $1-|S(z)|^2 \geq \varepsilon$, $S(z)$ continuous on S^1. Then get Hilbert space ~~[illegible]~~ with

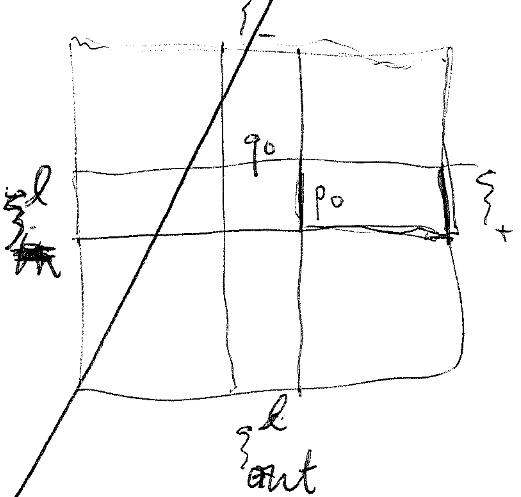
$$\|f\xi_+ + g\xi_-\|^2 = \int \begin{pmatrix} f \\ g\end{pmatrix}^* \begin{pmatrix} 1 & S^* \\ S & 1\end{pmatrix}\begin{pmatrix} f \\ g\end{pmatrix} \frac{d\theta}{2\pi}$$

Thus

$$(g\xi_- \mid f\xi_+) = \int \bar{g}\, S f \,\frac{d\theta}{2\pi}$$

$$(u^{-k}\xi_- \mid u^j\xi_+) = \int z^k S z^j \frac{d\theta}{2\pi} = S_{j+k}$$

Fill in the scattering picture!!! This means ~~first~~ write $1-|S(z)|^2 = |T(z)|^2$ with $T(z)$ analytic & invertible on D. How? ~~So what~~



~~ξ [illegible]~~

$$\xi_+ = \alpha\,\xi^l_{in} + \beta\,\xi_-$$

$$(\xi_- \mid u^n \xi_+) = (\xi_- \mid u^n \beta\, \xi_-)$$

$$= \int z^n \beta \,\frac{d\theta}{2\pi}$$

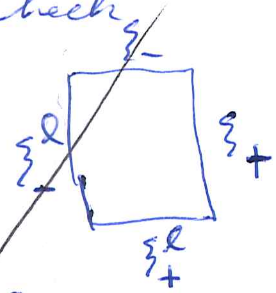
$$\begin{pmatrix} \xi^r_+ \\ \xi^l_+ \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}\begin{pmatrix} \xi^l_- \\ \xi^r_- \end{pmatrix}$$

function of z values in $U(2)$ with $\alpha = \delta \in H_+$

Begin with a contraction

$S: L^2(S^1)\xi_+ \longrightarrow L^2(S^1)\xi_-$ commuting with u. Check notation

$$\begin{pmatrix}\xi_+\\ \xi^\ell_+\end{pmatrix} = \begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\begin{pmatrix}\xi^\ell_-\\ \xi_-\end{pmatrix}$$



$$S_n = (\xi_- \mid u^n \xi_+) \qquad \xi_+ = \alpha\xi^\ell_- + \beta\xi_-$$

$$(\xi_- \mid u^n\xi_+) = (\xi_- \mid u^n\beta\xi_-) = \int z^n\beta\,\frac{d\theta}{2\pi} = (z^{-n}\mid\beta)$$

So given $\beta(z) = \sum z^{n}(z^{-n}\mid\beta)$ a good contraction, say smooth fn. of z and $|\beta| \le 1-\varepsilon$.

If you restrict β to $H_+\xi_+ \longrightarrow H_-\xi_-$

$$(u^{-k}\xi_- \mid u^j\xi_+) = \beta_{j+k} \qquad j\ge 0,\ k\ge 1,$$

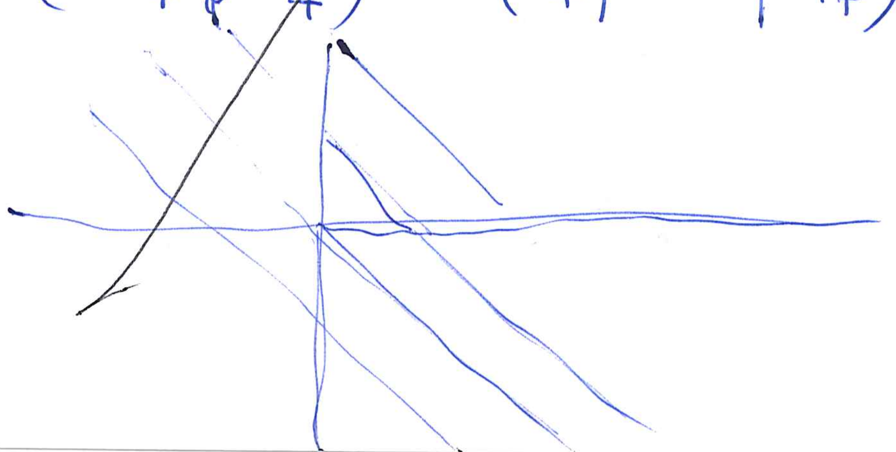
you get a contraction.

Similarity with Toeplitz operators on H_+

In this case the operator β of mult. by $\beta(z)$ is compressed to H_+.

$$H_+ \subset L^2 \xrightarrow{\beta} L^2 \longrightarrow H_+$$

$$(u^k\xi_+ \mid \beta u^j\xi_+) = (\xi_+ \mid u^{j-k}\beta\xi_+) = (z^{j-k}\mid\beta)$$

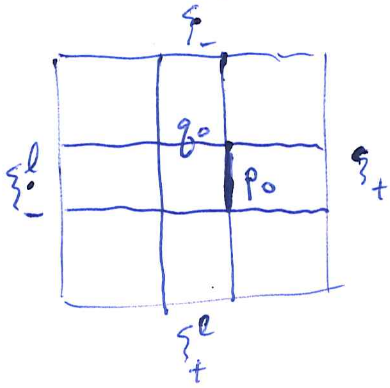


This operator $H_+ \subset L^2 \xrightarrow{\beta} L^2 \to H_-$
~~should~~ might be closely related to $[F,\beta]$.

But you know I think that given ~~the contraction $\beta : L^2(S^1)\xi_+ \to L^2(S^1)\xi_-$~~ $\beta(z)$ ~~[illegible]~~ smooth contraction a L^2 comm. with z you do get a sequence h_n which can be truncated

---

Start with a $\beta(z) = \sum \beta_n z^n$ $\qquad \beta_n = (\xi_- | u^n \xi_+)$
smooth fn. on S^1 s.t. $|\beta(z)| \le 1-\varepsilon$.



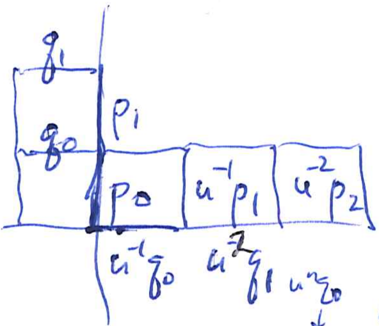
$$\begin{pmatrix}\xi_+\\ \xi_+^\ell\end{pmatrix} = \begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\begin{pmatrix}\xi_-^\ell\\ \xi_-\end{pmatrix}$$

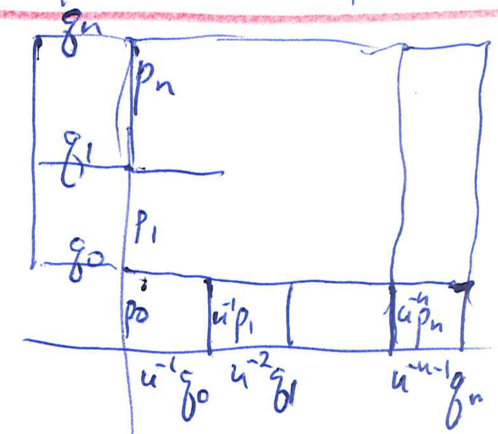
$$\xi_+ = \alpha\,\xi_-^\ell + \beta\,\xi_-$$

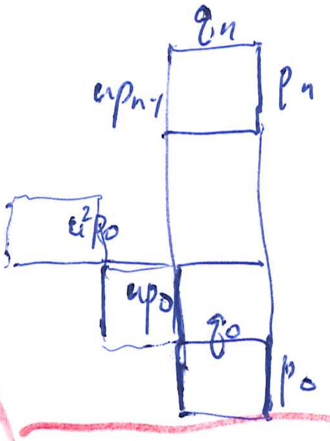
$$(\xi_- | u^n \xi_+) = (\xi_- | u^n \beta\, \xi_-) = \int z^n \beta = (z^{-n} | \beta)$$

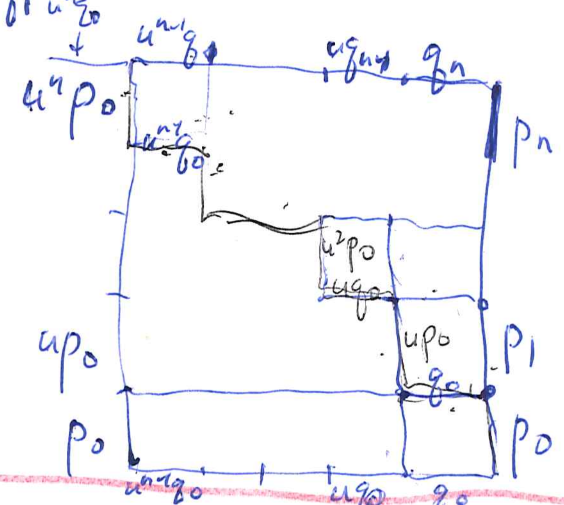
$$\beta = \sum \bar z^n (\bar z^n | \beta) = \sum z^{-n} \beta_n \qquad \beta_n = (\xi_- | u^n \xi_+)$$

From (h_n) to a d1dDE







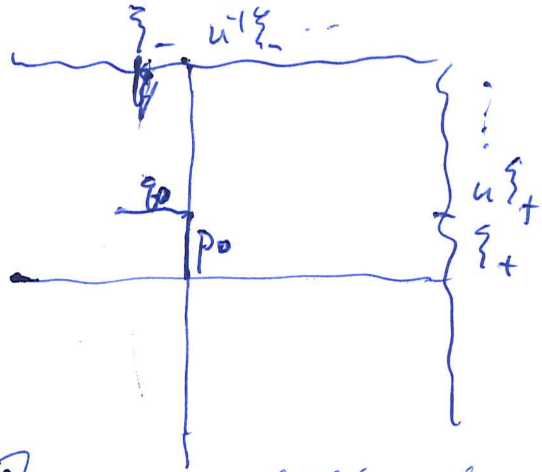


$p_n \in [u,\dots,u^n]p_0 + [1,\dots,u^{n-1}]q_0$

$q_n \in$ ______

$\begin{pmatrix}p_n\\ q_n\end{pmatrix} \in \Big(\qquad \Big)\begin{pmatrix}p_0\\ q_0\end{pmatrix}$

$$\begin{pmatrix}\xi_+\\ \xi_-\end{pmatrix} = \begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}p_0\\ q_0\end{pmatrix} \qquad \begin{pmatrix}p_0\\ q_0\end{pmatrix} = \begin{pmatrix}d & -b\\ -c & a\end{pmatrix}\begin{pmatrix}\xi_+\\ \xi_-\end{pmatrix}$$



$$p_0 \in H_+\xi_+ + H_-\xi_-$$

$$p_0 = \sum_{j\geq 0} d_j u^j \xi_+ + \sum_{k\geq 1} b_k u^{-k}\xi_-$$

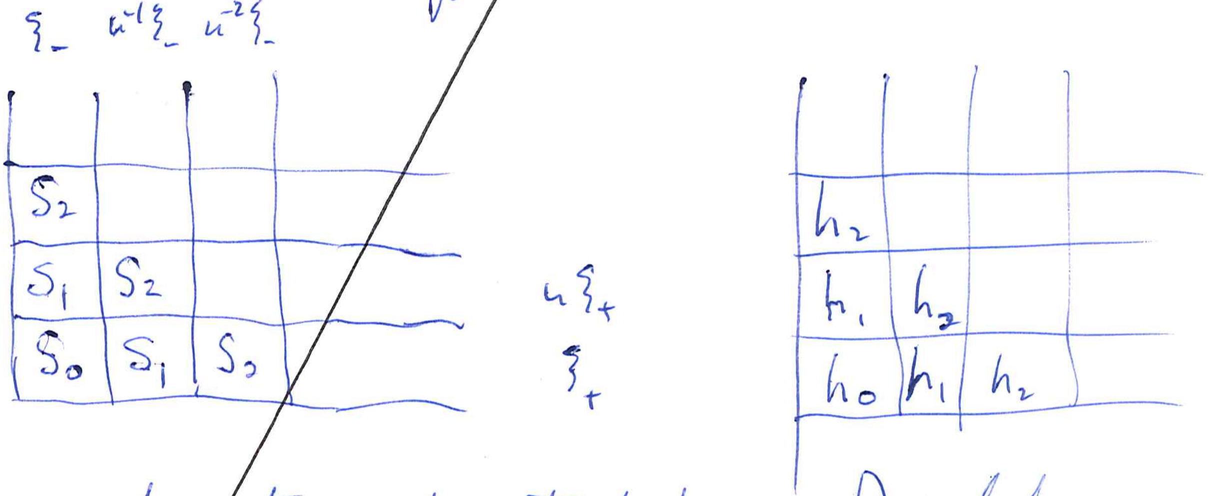
$$u^{-1}q_0 \in H_+\xi_+ + H_-\xi_-$$

There are lot's of questions about a_0, d_0

In the inverse scattering you start with

$$S(z) = \sum z^{-n} S_n \qquad S_n = (u^{-n}\xi_- | \xi_+)$$

Begin the inverse process.



You need to get started. Develop.

Try something

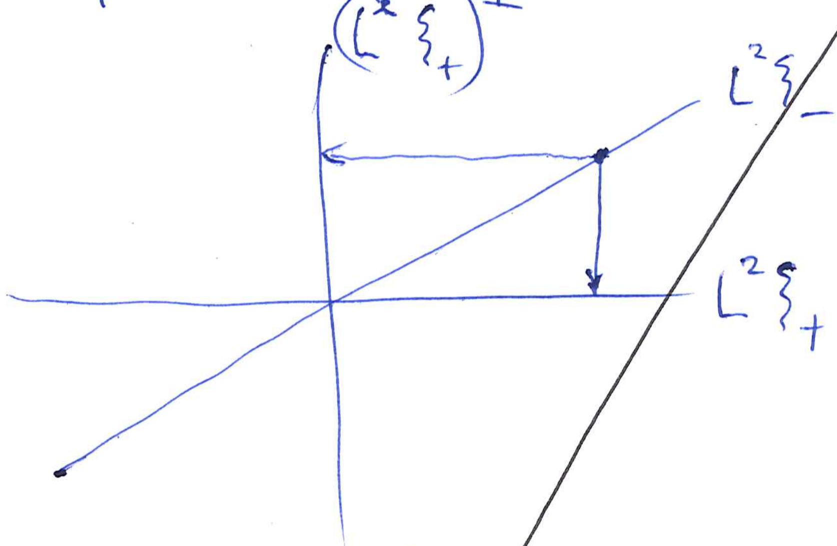
You have experienced problems finding $\xi_\pm'$

This is something to focus upon. Start with $E = L^2\xi_+ + L^2\xi_-$ $\qquad (\xi_- | u^n \xi_+) = S_n$

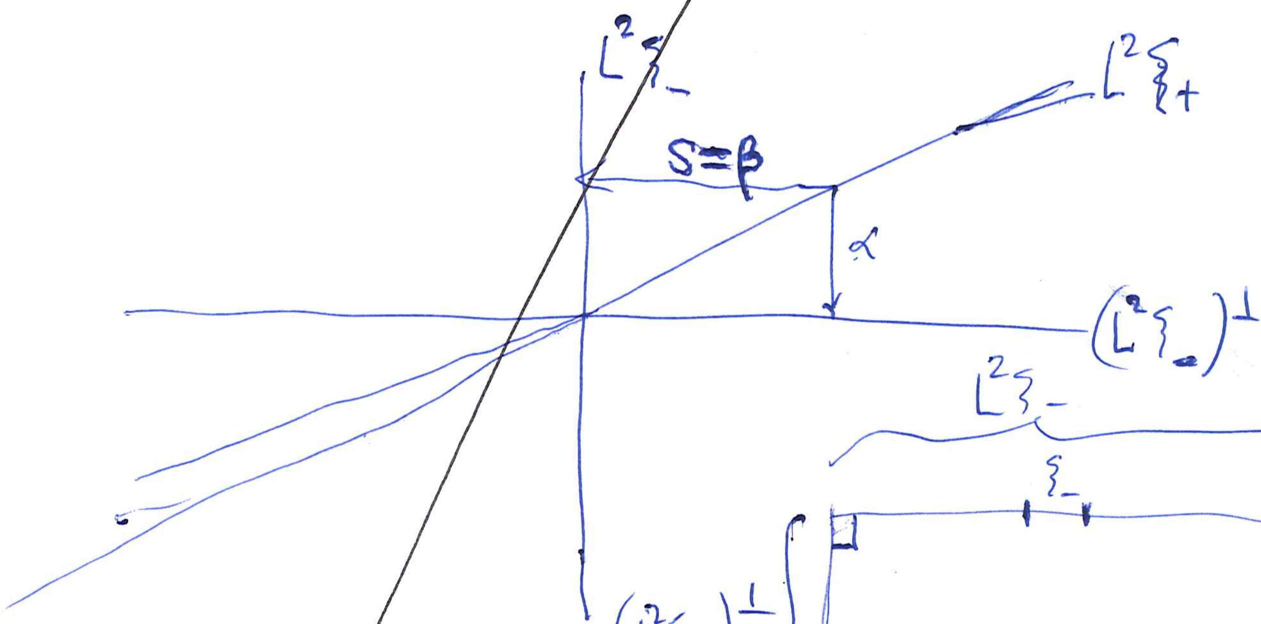
$$\|f_1\xi_+ + f_2\xi_-\|^2 = \int \begin{pmatrix}f_1\\ f_2\end{pmatrix}^* \begin{pmatrix}1 & S^*\\ S & 1\end{pmatrix}\begin{pmatrix}f_1\\ f_2\end{pmatrix} \frac{d\theta}{2\pi}$$

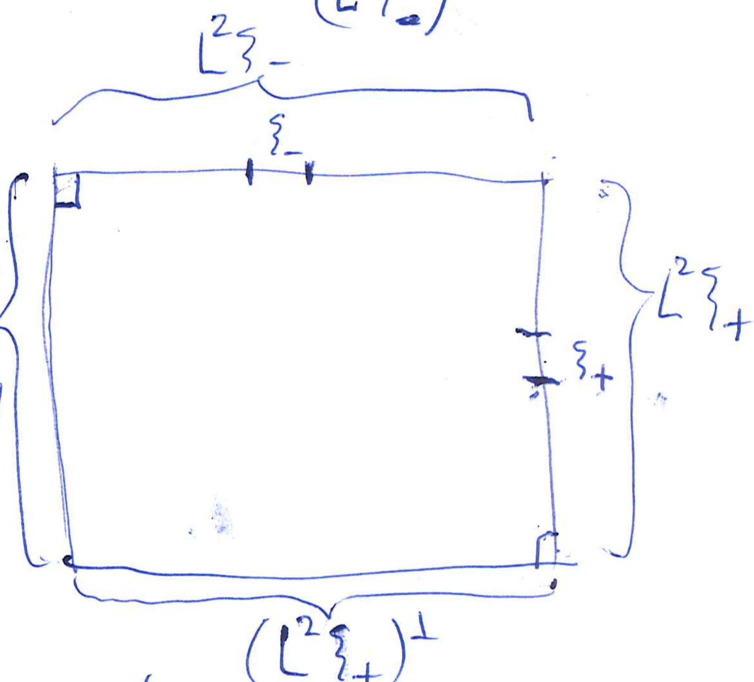
$$(f_2\xi_- \mid f_1\xi_+) = \int f_2^* S f_1 \frac{d\theta}{2\pi} .$$

Aim: $E = L^2\xi_+ \oplus (L^2\xi_+)^\perp$, so $L^2\xi_-$ and $(L^2\xi_+)^\perp$ being two complements of $L^2\xi_+$ are related somehow.



Consider instead.

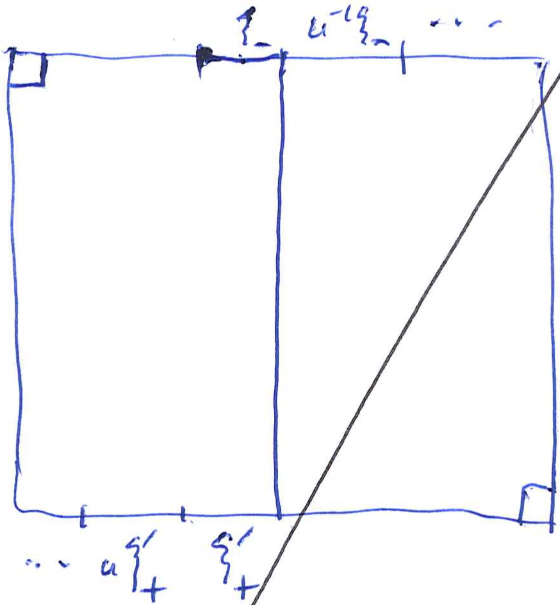




You start with E having two transversal subspaces $L^2\xi_+$, $L^2\xi_-$. You have canonical isomorphisms

$L^2\xi_+ \xrightarrow{\sim} E/L^2\xi_-$ $\qquad$ $L^2\xi_- \xrightarrow{\sim} E/L^2\xi_+$

The problem is how ~~...~~ ξ'_+ arises inside of $(L^2\xi_+)^\perp$. $\xi'_+ = \xi^l_{out}$. We have a canon. bij. $L^2\xi_- \xrightarrow{\sim} (L^2\xi_+)^\perp$, so the orth basis $u^{-k}\xi_-$ gives a basis for $(L^2\xi_+)^\perp$. The incoming ~~...~~ tog. subscript on $L^2\xi_-$ means $\exists$ natural filtration increasing under u, namely $H_-\xi_-$. You expect a decreasing filtration on the other — probably orthogonal



$$\xi_-^{0} + \sum_{k\geq 1}\lambda_k u^{-k}\xi_- + \sum_{j\in\mathbb{Z}}\mu_j u^j\xi_+ \quad \text{to satisfies}$$

$$\underline{(u^j\xi_+ \mid \xi_-^{0})} + \sum_{k\geq 1}\lambda_k (u^j\xi_+ \mid u^{-k}\xi_-) + \mu_j = 0 \qquad \forall j$$

$$\boxed{\bar{S}_j + \sum_{k\geq 1}\bar{S}_{j+k}\lambda_k = \mu_j \qquad \forall j\in\mathbb{Z}}$$

$$\boxed{\lambda_k + \sum_{j\in\mathbb{Z}}\mu_j S_{k+j} = 0 \qquad \forall k\geq 1}$$

$$\bar{s}_j = \mu_j - \underbrace{\sum_{k\geq 1} \bar{s}_{k+j} \sum_{j_1\in\mathbb{Z}} s_{k+j_1}\, \mu_{j_1}}_{\sum_{j_1}\left(\sum_{k\geq 1} \bar{s}_{k+j}\, s_{k+j_1}\right)\mu_{j_1}}$$

$$\tilde{\xi}'_+ = \xi_- + \tau \qquad \tau \in H_-\xi_- + L^2\xi_+$$

and $\tilde{\xi}'_+ \perp \tau$. So

$$\xi_- = \tilde{\xi}'_+ - \tau, \quad 1 = \|\xi_-\|^2 = \|\tilde{\xi}'_+\|^2 + \|\tau\|^2.$$

Existence of ξ'_+ demonstrated via transfer: $\xi'_+ \in R_0\xi_- + H_-\xi_- + L^2\xi_+$.
But it's easier to understand scattering ?

$$\xi'_+ \in L^2\xi_- ?$$

Review situation. Given $S(z) = \sum z^{-n} S_n$

$E = L^2\xi_+ + L^2\xi_-$ with

$$\|f_1\xi_+ + f_2\xi_-\|^2$$

$$= \left(f_1(u)\xi_+ + f_2(u)\xi_-\right)^*\left(f_1(u)\xi_+ + f_2(u)\xi_-\right)$$

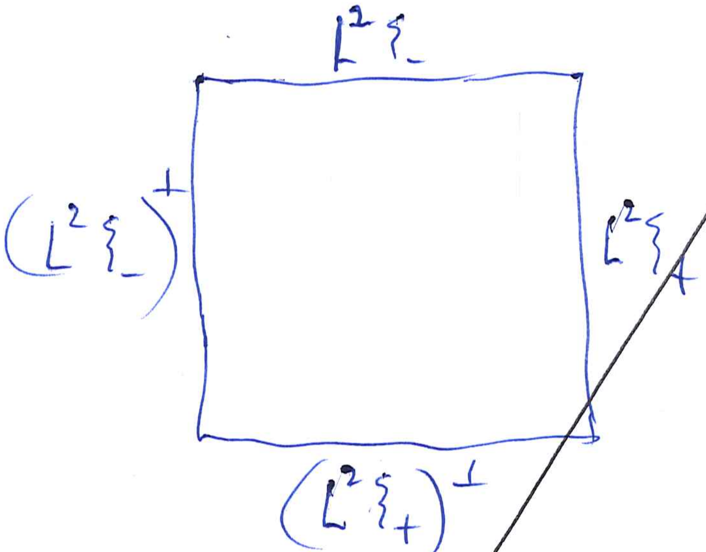
$$= \begin{pmatrix}\xi_+ \\ \xi_-\end{pmatrix}^* (f_1\ \ f_2)^* \,(f_1\ \ f_2)\begin{pmatrix}\xi_+ \\ \xi_-\end{pmatrix}$$

$$= \begin{pmatrix}\xi_+\\ \xi_-\end{pmatrix}^* \begin{pmatrix} f_1^*f_1 & f_1^*f_2 \\ f_2^*f_1 & f_2^*f_2\end{pmatrix}\begin{pmatrix}\xi_+\\ \xi_-\end{pmatrix}$$

So what comes next?

$$= \int \begin{pmatrix} f_1\\ f_2\end{pmatrix}^* \begin{pmatrix}1 & S^*\\ S & 1\end{pmatrix}\begin{pmatrix}f_1\\ f_2\end{pmatrix}\frac{d\theta}{2\pi}$$

So you have this ~~...~~ $E = L^2\xi_+ + L^2\xi_-$



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Get isoms $L^2\xi_- \xrightarrow{\sim} (L^2\xi_+)^\perp$ alg. isoms.

$L^2\xi_+ \xrightarrow{\sim} (L^2\xi_-)^\perp$ maybe not important

What's important seems to be ~~the~~ the subspaces $H_+\xi_+ \subset L^2\xi_+$, $H_-\xi_- \subset L^2\xi_-$ and the

What is $(L^2\xi_+)^\perp$? Ask when $f\xi_+ + g\xi_- \in (L^2\xi_+)^\perp$

$$0 = (f_1\xi_+ \mid f\xi_+ + g\xi_-) = (f_1\xi_+ \mid f\xi_+ + gS^*\xi_+)$$

$\forall f_1$. i.e. $f + gS^* = 0$ $\quad f = -gS^*$.

Thus $(L^2\xi_+)^\perp$ consists of $g(\xi_- - S^*\xi_+)$

Check. $f\xi_+ + g\xi_- = \sum a_j u^j \xi_+ + \sum b_k u^{-k}\xi_- \in L^2\xi_+^{\perp}$

$$0 = a_j + \sum b_k \underbrace{(u^j\xi_+ \mid u^{-k}\xi_-)}_{\bar{S}_{j+k}}$$

$$\iff f = -gS^*$$

$$-gS^*\xi_+ + g\xi_- = g(\xi_- - S^*\xi_+)$$

$$\xi_- - S^*\xi_+ = \xi_- - \sum \bar{S}_n u^n \xi_+ \in (L^2\xi_+)^{\perp}$$

$$(u^j\xi_+ \mid \xi_-) - \bar{S}_j = 0.$$

$(L^2\xi_+)^{\perp}$ has basis $u^n(\xi_- - S^*\xi_+)$

$$\left(\xi_- - S^*\xi_+ \mid u^n(\xi_- - S^*\xi_+)\right)$$

Repeat & get into better shape. There's a functional angle which is important.
Make E Hilbert space of $\begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$ with inner product

$$\int \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}^* \begin{pmatrix} 1 & S^* \\ S & 1 \end{pmatrix} \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \frac{d\theta}{2\pi}$$

$$= \|f_1 + S^*f_2\|^2 + \|f_2\|^2 - \|S^*f_2\|^2$$

What do you hope to accomplish? Enough familiarity, control to establish the scattering picture. Begin with $\beta(z) = \sum_{n\in\mathbb{Z}} z^{-n}\beta_n$ form

$$L^2 \underset{\varepsilon_-}{\overset{\varepsilon_+}{\rightrightarrows}} E \qquad \varepsilon_+^*\varepsilon_+ = 1 = \varepsilon_-^*\varepsilon_- \qquad \varepsilon_-^*\varepsilon_+ = \beta$$

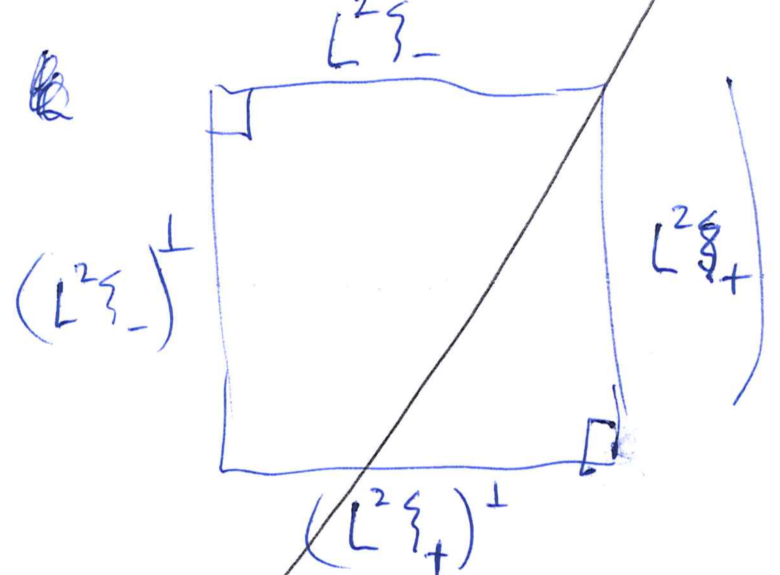
$$\|\varepsilon_+ f_+ + \varepsilon_- f_-\|^2 = \int \underbrace{\left(|f_+|^2 + f_-^*\beta f_+ + f_+^*\beta^* f_- + |f_-|^2\right)}_{\begin{pmatrix} f_+ \\ f_- \end{pmatrix}^* \begin{pmatrix} 1 & \beta^* \\ \beta & 1 \end{pmatrix} \begin{pmatrix} f_+ \\ f_- \end{pmatrix}} \frac{d\theta}{2\pi}$$

$$= \|f_+ + \beta^* f_-\|^2 + \left(f_-, (1-\beta\beta^*) f_-\right)$$

$$(u^{-k}\xi_- \mid u^j \xi_+) = S_{k+j}$$

$$S(z) = \sum z^{-n} S_n$$

$$S_n = (z^{-n} \mid S_u) = \int z^n S(z) \frac{d\theta}{2\pi}$$



contraction of $L^2\xi_+ \to L^2\xi_-$ given by orthog proj is

$$\xi_+ \longmapsto \sum_k u^{-k}\xi_- (u^{-k}\xi_- \mid \xi_+) = \sum u^{-k}\xi_- S_k = S(u)\xi_-$$

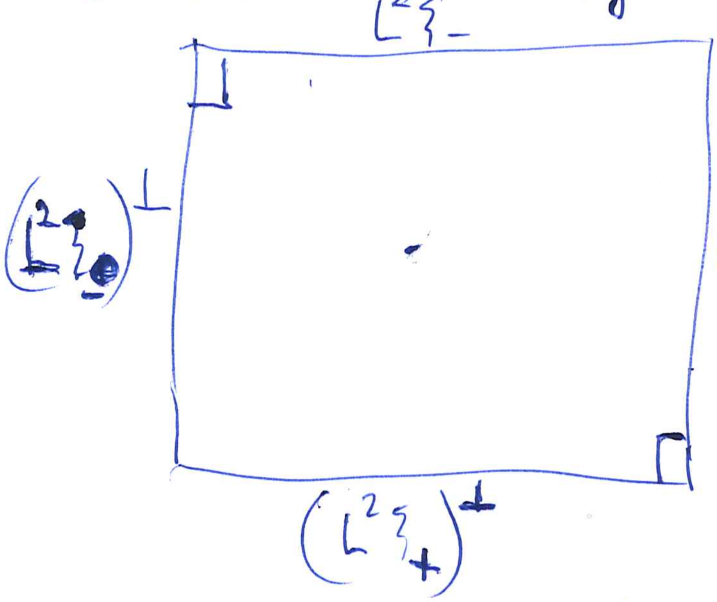
$(L^2\xi_+)^\perp$ ~~spanned by~~ consists of $\quad f\xi_+ - fS\xi_-$

get a basis $u^j(\xi_+ - S\xi_-)$. But you want an orth. basis.

Idea: You have a basis ~~and~~ which is ordered: two natural flags which can be orthonormalized

Review the steps. $E = L^2\xi_+ + L^2\xi_-$

$(u^{-k}\xi_- | u^j \xi_+) = \delta_{k+j}$



$L^2\xi_- \longrightarrow E/L^2\xi_+ \simeq (L^2\xi_+)^\perp$

$\cancel{f\xi_- \mapsto f(\xi_- - \delta\xi_+)}$

$L^2\xi_+ \to E/L^2\xi_- \simeq (L^2\xi_-)^\perp$

$f\xi_+ \mapsto f(\xi_+ - \delta\xi_-)$

So we have something

$$\xi_+ = \underbrace{(\xi_+ - \delta\xi_-)}_{\in (L^2\xi_-)^\perp} + \underbrace{\delta\xi_-}_{\in L^2\xi_-}$$

The next thing to do is to write

$\xi_+ - \underset{\overset{\shortparallel}{\beta}}{\delta}\xi_- = \alpha\xi_-'$. How do you find α?

Idea here: The condition $(\xi_-' | u^n \xi_-') = \delta_{n0}$ is ~~[illegible]~~ some sort of generalization of a unit vector in the Hilbert module setting. ~~[illegible]~~ E seems to be more than a Hilbert space with u

What is going on? E module over $C(S^1)$, E is also a Hilbert space. If I pick a

Look at $L^2(S^1)$. $\boxed{\text{Strong unit vector}}$ means a ξ such that $(\xi|u^n\xi) = \delta_n$, and this is equivalent to $|\xi(z)| = 1$ on S^1.

~~Given ... of $L^2(S^1)$ ... get measures ... You~~

~~...~~ life

Think of E as the space of L^2 ~~functions~~ sections of a vector bundle ~~over the circle equipped~~ with hermitian inner product over S^1, the inner product on E being obtained by integrating over S^1.

$$s = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \qquad \text{norm} \qquad |s|^2 = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}^* \begin{pmatrix} 1 & S^* \\ S & 1 \end{pmatrix} \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$$

Suppose given ~~a vector~~ a hermitian vector bundle V over S^1. If s is a section, then we get a measure on S^1 namely $|s|^2 \frac{d\theta}{2\pi}$ s.t.

$$(s|fs) = \int f|s|^2 \frac{d\theta}{2\pi}$$

so $\quad (\xi|z^n\xi) = \delta_n^0 \iff |\xi|^2 = 1$

In your ~~situation~~ $E = L^2\xi_+ + L^2\xi_-$

the orthogonal complement of $L^2\xi_+$?

$$\|f_1\xi_+ + f_2\xi_-\|^2 = \int \underbrace{\begin{pmatrix} f_1 \\ f_2 \end{pmatrix}^* \begin{pmatrix} 1 & S^* \\ S & 1 \end{pmatrix} \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}}_{\begin{matrix} f_1^*f_1 & f_1^*S^*f_2 \\ f_2^*Sf_1 & f_2^*f_2 \end{matrix}} \frac{d\theta}{2\pi}$$

~~$f\xi_+$~~

$$(f_1\xi_+ + f_2\xi_- | g\xi_+) = (f_1|g) + (f_2|gS)$$

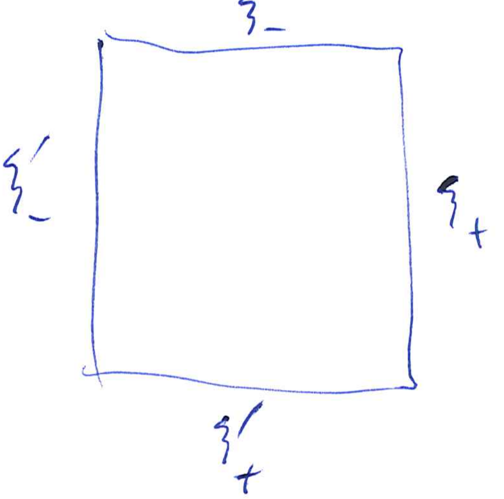
$$\therefore \left(L^2 \xi_+\right)^\perp = \left\{ f_1 \xi_+ + f_2 \xi_- \,\middle|\, f_1 + S^* f_2 = 0 \right\}$$

$$= \left\{ f_2 \left(\xi_- - S^* \xi_+ \right) \right\} = L^2\left(\xi_- - S^* \xi_+\right)$$

$$\left(L^2 \xi_-\right)^\perp = \left\{ f_1 \xi_+ + f_2 \xi_- \,\middle|\, S f_1 + f_2 = 0 \right\}$$

$$= \left\{ f_1 \left(\xi_+ - S \xi_- \right) \right\} = L^2\left(\xi_+ - S \xi_-\right)$$

You want ~~[illegible]~~ to understand how



to get $\xi'_\pm$. ~~Basic idea is that~~ Picture: trivial rank 2 vb over S^1 w herm. scalar product

$$\begin{pmatrix} 1 & S^* \\ S & 1 \end{pmatrix} \qquad \text{nice section}$$

$$\xi_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad \text{orth. comp. is } \begin{pmatrix} 1 \\ -S \end{pmatrix}$$

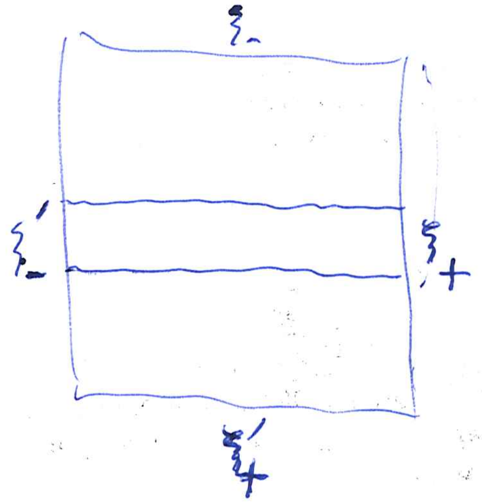
ξ'_+ will be a section of the form $f\begin{pmatrix} 1 \\ -S \end{pmatrix}$ having norm 1 everywhere:

$$\left(\begin{pmatrix} 1 \\ -S \end{pmatrix} f\right)^* \begin{pmatrix} 1 & S^* \\ S & 1 \end{pmatrix} \begin{pmatrix} 1 \\ -S \end{pmatrix} f = f^* \begin{pmatrix} 1 & -S^* \end{pmatrix} \begin{pmatrix} 1 & S^* \\ S & 1 \end{pmatrix} \begin{pmatrix} 1 \\ -S \end{pmatrix} f$$

$$= f^* \begin{pmatrix} 1 & -S^* \end{pmatrix} \begin{pmatrix} 1 - S^* S \\ 0 \end{pmatrix} f$$

$$= f^* \left(1 - S^* S\right) f \qquad \therefore |f|^2 \left(1 - S^* S\right) = 1$$

The problem is how to specify the phase of f. Somehow done by analyticity.



$$\xi'_- = f(\xi_+ - S\xi_-)$$

what's the criterion determining f? ~~twice~~

The condition is $\xi'_- \in (H_+\xi_+ + L^2\xi_-) \cap (uH_+\xi_+ + L^2\xi_-)^\perp$

$$\begin{pmatrix}\xi_+\\ \xi'_+\end{pmatrix} = \begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\begin{pmatrix}\xi'_-\\ \xi_-\end{pmatrix}$$

$$\begin{pmatrix}\xi_+\\ \xi_-\end{pmatrix} = \begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}\xi'_-\\ \xi'_+\end{pmatrix}$$

$$\begin{pmatrix}\xi'_-\\ \xi'_+\end{pmatrix} = \begin{pmatrix}d & -b\\ -c & a\end{pmatrix}\begin{pmatrix}\xi_+\\ \xi_-\end{pmatrix}$$

$$\xi_+ = \alpha\xi'_- + \beta\xi_- \qquad \xi'_- = \frac{1}{\alpha}\xi_+ - \frac{\beta}{\alpha}\xi_-$$

You are using things which are true from the transfer matrix situation, i.e. when (h_n) given

But now you want to start with $\beta = S$. ~~Make~~ ~~assumption~~ Assumption $|\beta(z)| \le 1-\varepsilon$.

$$\xi'_- = \sum_{j\ge 0} d_j u^j \xi_+ + \sum_{k\in\mathbb{Z}} b_k u^{-k}\xi_- - \sum_{j\ge 0}\sum_{k\in\mathbb{Z}} d_j S_{k+j} u^{-k}\xi_-$$

$$0 = \sum_{j\ge 0} d_j S_{k+j} - b_k$$

$$\xi'_- = \sum_{j\ge 0} d_j u^j\Big(\xi_+ - \overbrace{\sum_k S_{k+j} u^{-k-j}\xi_-}^{S\xi_-}\Big) \qquad \text{autom. } \perp L^2\xi_-$$

Anyway the other conditions are

$$(u^{j}\xi_+ \mid \xi_-') = 0 \qquad j \geq 1$$

$$\Big(u^{j'}\xi_+ \,\Big|\, \sum_{j\geq 0} d_j u^j(\xi_+ - \delta\xi_-)\Big) = d_{j'} - \sum_j d_j (u^{j'}\xi_+ \mid u^j \delta\xi_-)$$

$$= d_{j'} - \sum_j d_j \underbrace{\Big(\xi_+ \,\Big|\, u^{j-j'}\sum \delta_n u^{-n}\xi_-\Big)}_{(\xi_+ \mid \delta_n u^{-n+j-j'}\xi_-)}$$

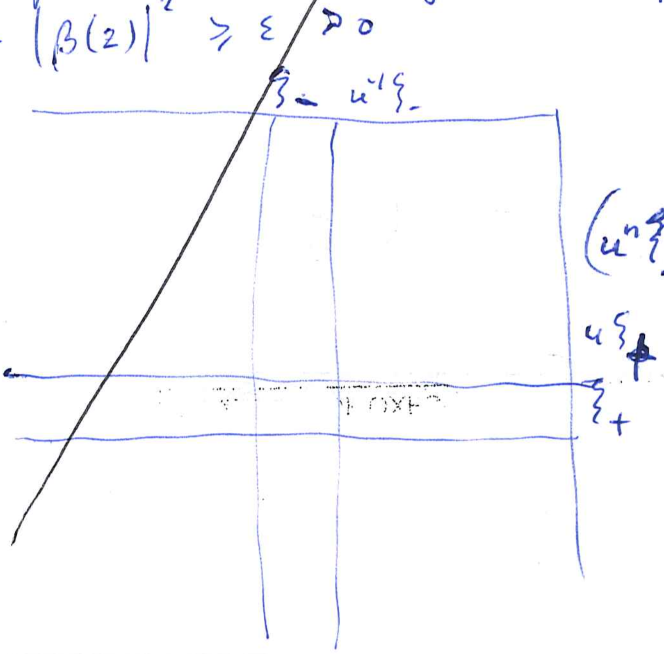
✓

$$(\xi_+ \mid \delta_n u^{-n+j-j'}\xi_-) = \sum_n \delta_n \overline{\delta}_{n-j+j'}$$

$$0 = d_{j'} - \sum_{j,n} d_j \delta_{n+j}\overline{\delta}_{n+j'} \qquad \begin{matrix} j \geq 0 \\ n \in \mathbb{Z} \end{matrix} \qquad j' \geq 1$$

Anyway what next? Why have I been stuck the past two days? Repeat. Given $\beta(z)$

$1 - |\beta(z)|^2 \geq \varepsilon > 0$

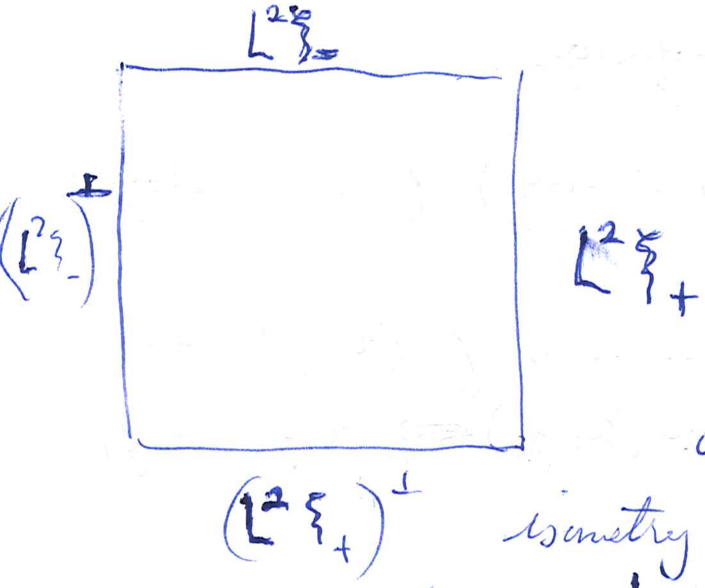


form $E = L^2\xi_+ + L^2\xi_-$ where

$$(\xi_- \mid u^j\xi_+) = \int z^j \beta(z)\frac{d\theta}{2\pi}$$

$$(u^n\xi_- \mid f(\xi_+ - \beta\xi_-)) = (\xi_- \mid u^n f(\xi_+ - \beta\xi_-))$$

$$= \int (z^n f\beta - z^n f\beta) = 0$$



~~[illegible]~~ basis

$u^n(\xi_+ - \beta\xi_-)$ for $(L^2\xi_-)^\perp$

$$(L^2\xi_-)^\perp \Leftarrow L^2\left(S^1, (1-|\beta|^2)\frac{d\theta}{2\pi}\right)$$

$$u^n(\xi_+ - \beta\xi_-) \longleftarrow z^n$$

isometry because

$$\left(\xi_+ - \beta\xi_- \mid f(\xi_+ - \beta\xi_-)\right) = \int f(1-\beta\bar\beta)\frac{d\theta}{2\pi}$$

~~By Szegö theory get~~ since $1-\beta\bar\beta > 0$ smooth

$\alpha(0) > 0$

$\exists\ \alpha(z)$ analytic invertible on $D \ni |\alpha|^2 = 1-|\beta|^2$ on S^1. Put $\xi_-' = \frac{1}{\alpha}(\xi_+ - \beta\xi_-)$. Then

$$(\xi_-' \mid f\,\xi_-') = \left(\frac{1}{\alpha}(\xi_+ - \beta\xi_-) \,\Big|\, f\,\frac{1}{\alpha}(\xi_+ - \beta\xi_-)\right)$$

$$= \int f \frac{1}{|\alpha|^2}(1-|\beta|^2)\frac{d\theta}{2\pi} = \int f \frac{d\theta}{2\pi}$$

Also find $\xi_+ = \alpha\xi_-' + \beta\xi_-$.

Another way to proceed is to treat $u^n\underbrace{(\xi_+ - \beta\xi_-)}_{\eta}$ as orth polys. Orthog conditions

~~[illegible]~~ You seek $\sum_{j\geq 0} d_j u^j \overbrace{(\xi_+ - \beta\xi_-)}^{\eta}$

such that $0 = \left(u^i\eta \,\Big|\, \sum_{j\geq 0} d_j u^j \eta\right) = \qquad \forall i > 0$

$$\sum_{j\geq 0} d_j(\eta \mid u^{j-i}\eta) = \sum_{j\geq 0} d_j \int z^{j-i}(1-|\beta|^2)\frac{d\theta}{2\pi}$$

It seems that your problems, difficulties arise ~~other~~ with the orthogonality relations. These force you to look at Fourier coeffs rather than functions on the circle.

ξ'_- is defined by

$$\xi'_- \in \left(H_+\xi_+ + L^2\xi_-\right) \cap \left(uH_+\xi_+ + L^2\xi_-\right)^\perp$$

(Also to get the notation straight you expect

$$\begin{pmatrix} \xi_+ \\ \xi_- \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \xi'_- \\ \xi'_+ \end{pmatrix}$$

$$\begin{pmatrix} \xi'_- \\ \xi'_+ \end{pmatrix} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \begin{pmatrix} \xi_+ \\ \xi_- \end{pmatrix} \qquad \xi'_- = d\xi_+ - b\xi_-$$

so

$$\xi'_- = \sum_{j\geq 0} d_j u^j \xi_+ - \sum_{n\in\mathbb{Z}} b_n u^n \xi_-$$

$$0 = \left(u^{-k}\xi_- \mid \xi'_-\right) = \sum_{j\geq 0} d_j \underbrace{\left(u^{-k}\xi_- \mid u^j \xi_+\right)}_{\delta_{k+j}} - b_k$$

$$b_k = \sum_{j\geq 0} \delta_{k+j} d_j$$

Other approach

$$0 = \left(\xi_- \mid f\xi'_-\right) = \left(\xi_- \mid f(d\xi_+ - b\xi_-)\right)$$

$$= \int \left(f d\beta - fb\right)\frac{d\theta}{2\pi} \implies b = d\beta \quad (\text{fns of } z)$$

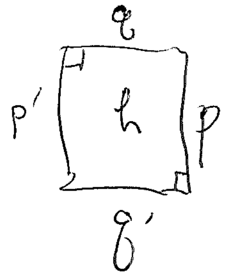
Future work

1) What happens if instead of

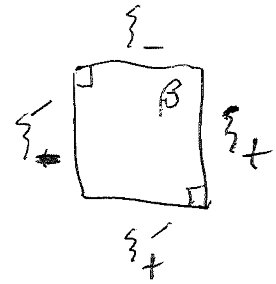
$$\alpha \xi'_- = \xi_+ - \beta \xi_-$$

you replace α by $\bar{\alpha}$?

2) generalizing



to



ξ strong unit vector when $(\xi | u^n \xi) = \delta^0_n$, herm. vector bundle over S^1.

3) Relations between (β_n) and (h_n)

4) Case $h_n = 0$ $n \le 0$ where

$p_0 = \xi'_-$ $\quad q_0 = \xi'_+$

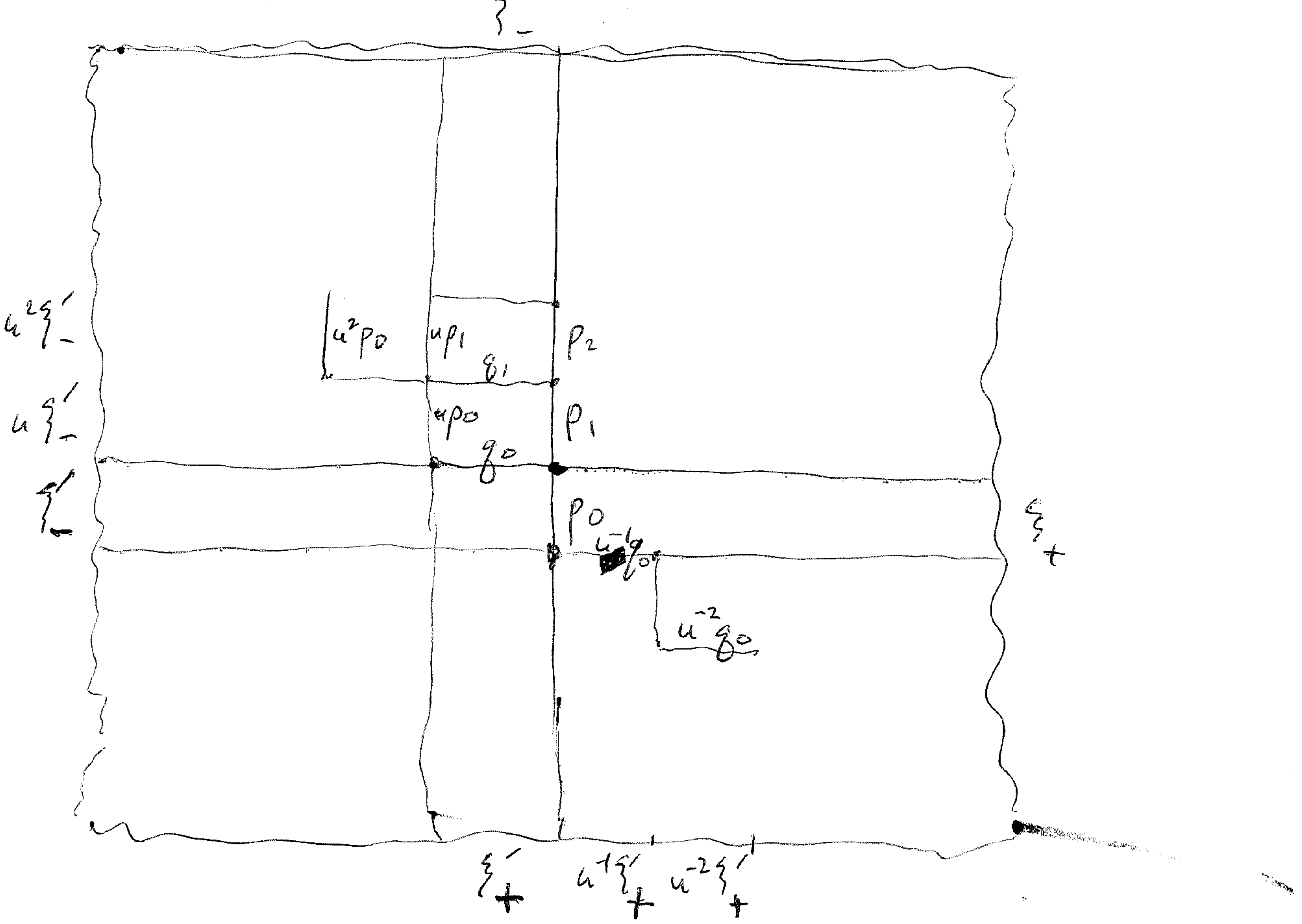
5) Link with $h_0 = 1$ or $\beta = \frac{\bar{q}}{q}$

6)

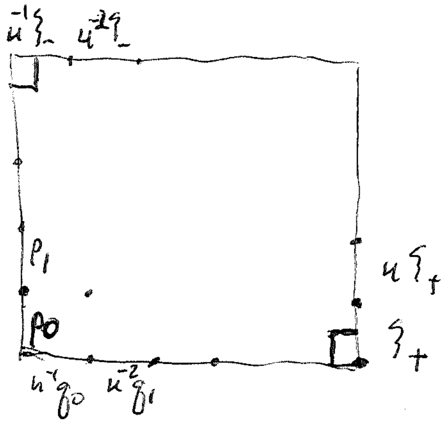
	ξ_-	$u^{-1}\xi_-$	$u^{-2}\xi_-$	
	S_2			$u^2\xi_+$
	S_1	S_2		$u\xi_+$
	S_0	S_1	S_2	ξ_+

h_2			
h_1	h_2		
h_0	h_1	h_3	

Review what happens to a d l d OE with
$h_n = 0$ for $n \leq 0$.



You are seeing?



The central problem seems how to ~~[illegible]~~ split β.
The idea here is that in constructing p_0, q_0 from ξ_+, ξ_- you use only β_n for $n<0$

IDEA two orthonormal bases

$p_0, p_1, \dots; u^{-1}\xi_-, u^{-2}\xi_-, \dots$

$u^{-1}q_0, u^{-1}q_1, \dots; \xi_+, u\xi_+, \dots$

Review formulas $p_0, u^{-1}q_0 \in H_+\xi_+ + H_-\xi_-$

of $\begin{pmatrix}\xi_+\\ \xi_-\end{pmatrix} = \begin{pmatrix} a & b\\ c & d\end{pmatrix}\begin{pmatrix} p_0\\ q_0\end{pmatrix}$ $\qquad \begin{pmatrix} p_0\\ q_0\end{pmatrix} = \begin{pmatrix} d & -b\\ -c & a\end{pmatrix}\begin{pmatrix}\xi_+\\ \xi_-\end{pmatrix}$

$c \in zH_+,\ d \in H_+ \qquad b \in (zH_+)^* = z^{-1}zH_- = H_-$

$$p_0 = \sum_{j\ (\geq 0)} d_j u^j \xi_+ - \sum_{k\ (<0)} b_k u^k \xi_-$$

$d_j = 0 \quad j<0$
$d_0 > 0.$
$b_k = 0 \quad k \geq 0$

$$\underset{k<0}{0 =} (u^k\xi_- | p_0) = \sum_j d_j \underbrace{(u^k\xi_- | u^j\xi_+)}_{\beta_{k-j}} - b_k \qquad \therefore\ b - d\beta \in H_+$$

$$0 \overset{j>0}{=} (u^j\xi_+ | p_0) = d_j - \sum_k b_k \underbrace{(u^j\xi_+ | u^k\xi_-)}_{\bar\beta_{k-j}} \qquad d - b\bar\beta \in zH_-$$

What do these equations mean?

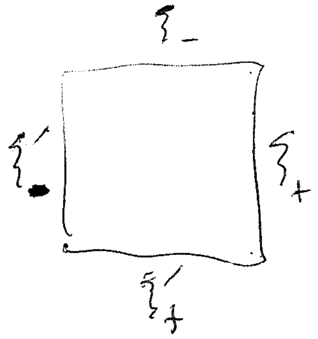
$$zH_- \ni \sum d_j z^j - \sum_k b_k z^k \sum_j \bar\beta_{k-j} z^{-k+j} = d - b\bar\beta$$

$$\boxed{\begin{array}{ll} b - d\beta \in H_+ & d \in H_+ \\ d - b\bar\beta \in zH_- & b \in H_- \end{array}}$$

Key idea. Given β_n for $n<0$ satisfying the appropriate positivity, i.e. $\beta_{k-j} = (u^k\xi_- | u^j\xi_+)$ for $j \geq 0,\ k<0$ is the matrix of a contraction

Go back to scattering matrix + factorization.

Given

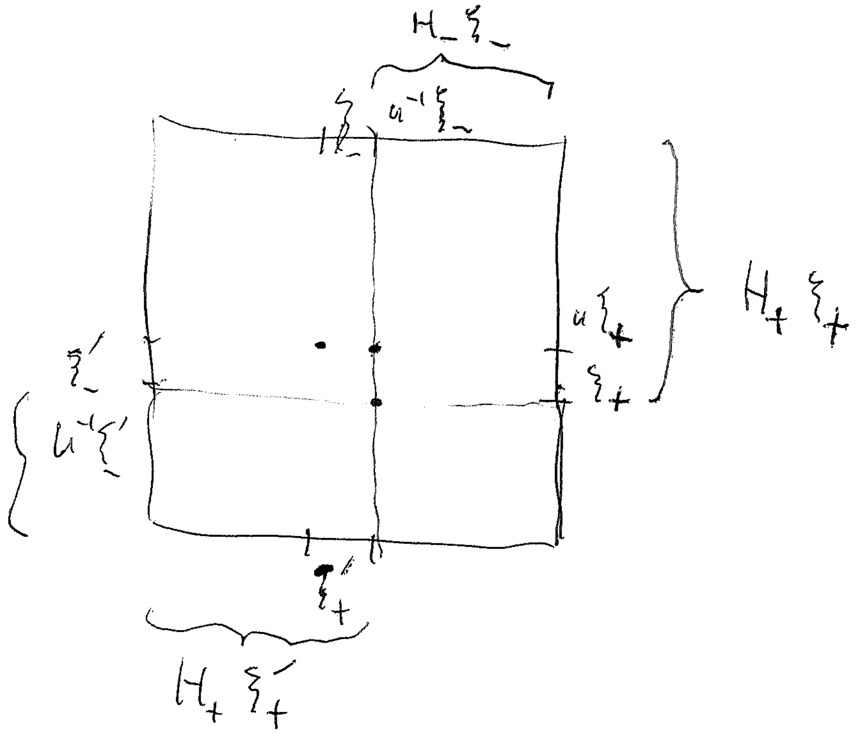


$$\begin{pmatrix} \xi_+ \\ \xi_- \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \xi'_- \\ \xi'_+ \end{pmatrix} \qquad \begin{pmatrix} \xi'_- \\ \xi'_+ \end{pmatrix} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \begin{pmatrix} \xi_+ \\ \xi_- \end{pmatrix}$$

$$\begin{pmatrix} \xi_+ \\ \xi'_+ \end{pmatrix} = \begin{pmatrix} \frac{1}{d} & +\frac{b}{d} \\ -\frac{c}{d} & \frac{1}{d} \end{pmatrix} \begin{pmatrix} \xi'_- \\ \xi_- \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \alpha \end{pmatrix} \begin{pmatrix} \xi'_- \\ \xi_- \end{pmatrix}$$

You have $S = \begin{pmatrix} \alpha & \beta \\ \gamma & \alpha \end{pmatrix}$ expressing $\begin{pmatrix} \xi_+ \\ \xi'_+ \end{pmatrix}$ in terms of $\begin{pmatrix} \xi'_- \\ \xi_- \end{pmatrix}$. We want the splitting of E corresp to a vector point of the grid

When $\beta = 0$, $\alpha = 1$



outgoing subspace $\quad H_+\xi_+ \oplus H_+\xi_+'$

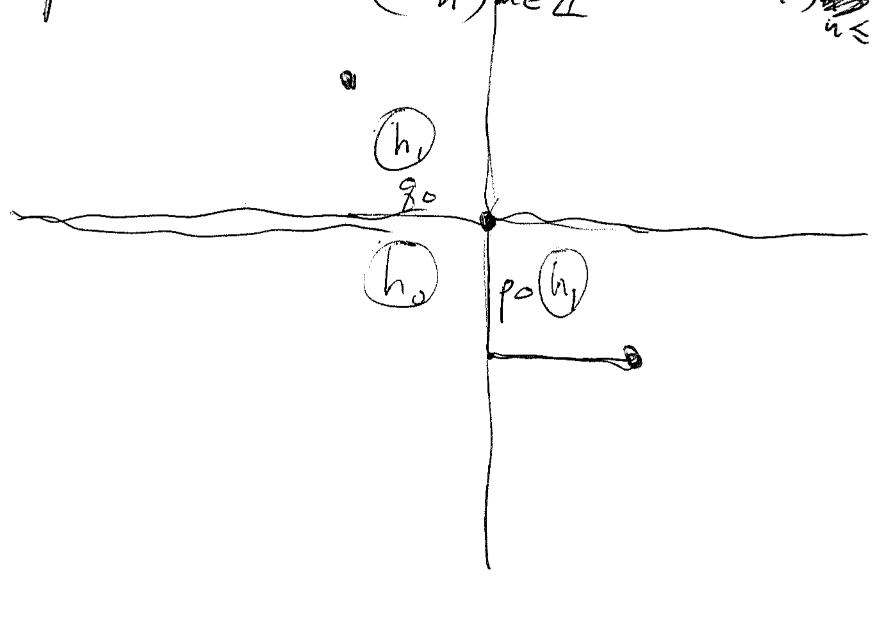
incoming " $\quad H_-\xi_- \oplus H_-\xi_-'$

~~For the picture these should be ortho-complementary~~

~~basic splitting~~ Bergman kernel idea

~~Idea~~ Problem. You want to construct the splitting $E = (H_+\xi_+ \overset{\perp}{\oplus} H_-\xi_-) \overset{\perp}{\oplus} (H_+\xi_+' + H_-\xi_-')$.

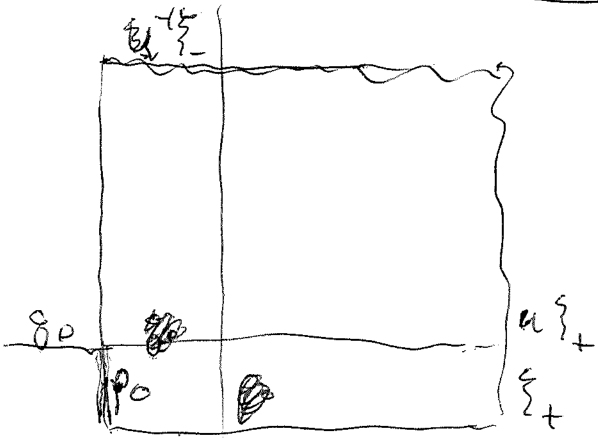
what ideas. ~~[illegible]~~ This splitting corresponds to $(h_n)_{n\in\mathbb{Z}} = (h_n)_{n\leq 0} + (h_n)_{n\geq 1}$



What kind of picture do you want eventually. 84

The Hilbert space of states is a $\mathfrak{a}$-module, ~~algebraically it is a rank 2~~ i.e. *representation of $\mathcal{L}_r(\mathbb{Z}) = C(S^1)$. Basic picture of L^2 sections of rank 2 hermitian v.b. over S^1.

Review:



$$p_0 = \sum_{j\,(\geq 0)} d_j u^j \xi_+ - \sum_{k\,(<0)} b_k u^k \xi_-$$

$$\underset{k<0}{0 =} (u^k\xi_- | p_0) = \sum_j d_j \underbrace{(u^k\xi_- | u^j \xi_+)}_{\beta_{k-j}} - b_k$$

$$\underset{j\geq 0}{0 =} (u^j\xi_+ | p_0) = d_j - \sum_k b_k \underbrace{(u^j\xi_+ | u^k \xi_-)}_{\bar\beta_{k-j}}$$

$$\sum d_j z^j - \sum_k b_k z^k \underbrace{\sum_j \bar\beta_{k-j} z^{j-k}}_{\text{ind of } k.} \in zH_-$$

$$d(z) - b(z)\bar\beta(z) \in zH_- \qquad b \in H_-$$

$$b(z) - d(z)\beta(z) \in H_+ \qquad d \in H_+$$

Is it true that up to a scalar factor there is a unique pair ~~d,b~~ $d \in H_+$ $b \in H_-$ such that $b - d\beta \in H_+$? equivalently $\frac{b}{d} - \beta \in H_+$

$$b - d\beta \in H_+ \qquad d \in H_+$$

$$d - b\bar\beta \in zH_- \qquad b \in zH_- = \overline{H_+}$$

$$d \in b\bar\beta + zH_- \qquad b \in \beta d + H_+$$

$$d \in \bar\beta(\beta d + H_+) + zH_- \qquad \begin{pmatrix} d & +b \\ +c & d \end{pmatrix} \qquad \begin{pmatrix} \frac{ab-cd}{d} & \frac{b}{d} \\ -\frac{c}{d} & \frac{1}{d} \end{pmatrix}$$

$$d \in |\beta|^2 d + \bar\beta H_+ + zH_-$$

$$(1-|\beta|^2)d \in \bar\beta H_+ + zH_-$$

~~The transfer matrix~~ You are trying to factor the transfer matrix.

$$T_{\infty,-\infty} = T_{\infty,0}\, T_{0,-\infty}$$

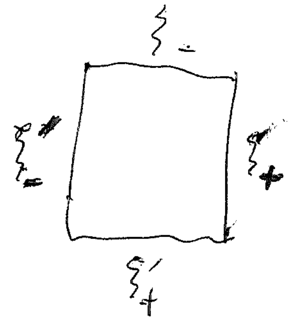
$n>m$ $$T_{\infty,0} = \lim_{n\to\infty} \frac{1}{k_n}\begin{pmatrix} 1 & h_n z^{-n} \\ \bar h_n z^{n} & 1 \end{pmatrix} \cdots \frac{1}{k_1}\begin{pmatrix} 1 & h_1 z^{-1} \\ \bar h_1 z^{1} & 1 \end{pmatrix}$$

$$\in \begin{pmatrix} zH_- & H_- \\ zH_+ & H_+ \end{pmatrix} = \begin{pmatrix} [1, z^{-1}, \dots] & [z^{-1}, z^{-2}, \dots] \\ [z, z^2, \dots] & [1, z, \dots] \end{pmatrix}$$

$$T_{0,-\infty} = \frac{1}{k_0}\begin{pmatrix} 1 & h_0 \\ \bar h_0 & 1 \end{pmatrix} \frac{1}{k_1}\begin{pmatrix} 1 & h_{-1} z \\ \bar h_{-1} z^{-1} & 1 \end{pmatrix} \cdots$$

$$\in \begin{pmatrix} [1, z^{-1}, \dots] & [1, z, \dots] \\ [1, z^{-1}, \dots] & [1, z, \dots] \end{pmatrix} = \begin{pmatrix} zH_- & H_+ \\ zH_- & H_+ \end{pmatrix}$$

Consider
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \overset{\begin{pmatrix}\xi_+ \\ \xi_-\end{pmatrix}}{=} \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \overset{\begin{pmatrix}p_0 \\ q_0\end{pmatrix}}{} \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} \begin{pmatrix}\xi'_- \\ \xi'_+\end{pmatrix}$$



$$\begin{pmatrix}\xi_+ \\ \xi_-\end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix}\xi'_- \\ \xi'_+\end{pmatrix}$$

$$\begin{pmatrix}\xi_+ \\ \xi'_+\end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}\begin{pmatrix}\xi'_- \\ \xi'_+\end{pmatrix}$$

$$\frac{1}{d}\left(\xi_- - c\,\xi'_-\right) = \xi'_+$$

$$\gamma = -\frac{c}{d} \qquad \delta = \frac{1}{d}$$

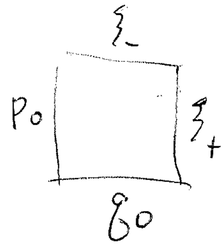
$$\xi_+ = \left(a - \frac{bc}{d}\right)\xi'_- + \frac{b}{d}$$

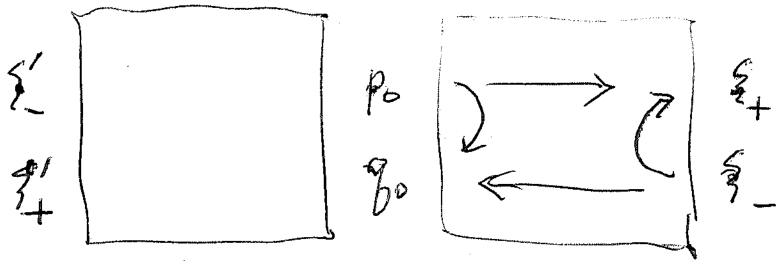
$$\beta = \frac{b}{d} \qquad \alpha = \frac{ad-bc}{d}$$

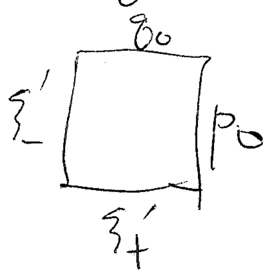
$$d = c_1 b_2 + d_1 d_2$$

$$\frac{d}{d_1 d_2} = 1 + \frac{c_1}{d_1}\frac{b_2}{d_2} = 1 - \gamma_1\beta_2$$

$$\alpha = \frac{d_1\alpha_2}{1-\gamma_1\beta_2}$$







6 variables related by 4 relations.

$$\xi_+ = \alpha_1 p_0 + \beta_1 \xi_- \qquad p_0 =$$
$$q_0 = \gamma_1 p_0 + \delta_1 \xi_- \qquad \xi'_- =$$

$$\frac{b}{d} = \frac{a_1 b_2 + b_1 d_2}{c_1 b_2 + d_1 d_2} = \frac{\frac{a_1 b_2}{d_1 d_2} + \frac{b_1}{d_1}}{\frac{c_1}{d_1}\frac{b_2}{d_2} + 1} = \frac{\frac{a_1}{d_1}\beta_2 + \beta_1}{-\gamma_1\beta_2 + 1}$$

$$-\frac{c_1}{d_1}\frac{b_1}{d_1}$$

$$\frac{b}{d} - \beta_1 = \frac{\frac{a_1}{d_1}\beta_2 + \cancel{\beta_1} + \gamma_1\beta_2\beta_1 - \cancel{\beta_1}}{-\gamma_1\beta_2 + 1} = \frac{\beta_2}{1-\gamma_1\beta_2}\left(\frac{a_1}{d_1} + \gamma_1\beta_1\right)$$

$$= \frac{\beta_2}{1-\gamma_1\beta_2}\,\frac{1}{d_1^2}\,(a_1d_1-b_1c_1)$$

$$\therefore\ \beta = \beta_1 + \alpha\,\frac{\beta_2}{1-\gamma_1\beta_2}\,\alpha_1$$

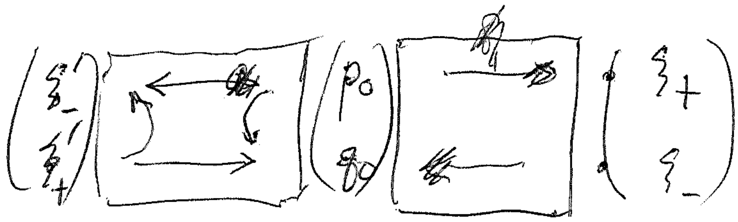
$$\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}\begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} = \begin{pmatrix} a_1b_2+b_1d_2 \\ c_1b_2+d_1d_2 \end{pmatrix}$$

$$\beta = \frac{a_1b_2+b_1d_2}{c_1b_2+d_1d_2} = \frac{\frac{a_1}{d_1}\beta_2 + \cancel{\beta_1} - (-\gamma_1\beta_2+\cancel{1})\beta_1}{-\gamma_1\beta_2+1} + \beta_1$$

$$= \beta_2\left(\frac{a_1}{d_1} + \gamma_1\beta_1\right) \qquad \frac{a_1}{d_1} - \frac{c_1}{d_1}\frac{b_1}{d_1} = \frac{a_1d_1-c_1b_1}{d_1^2}$$

$$\beta - \beta_1 = a_1b_2+b_1d_2$$

$$\beta = \beta_1 + \frac{\delta_1\beta_2\alpha_1}{1-\gamma_1\beta_2}$$



$$\left(\xi\right.$$

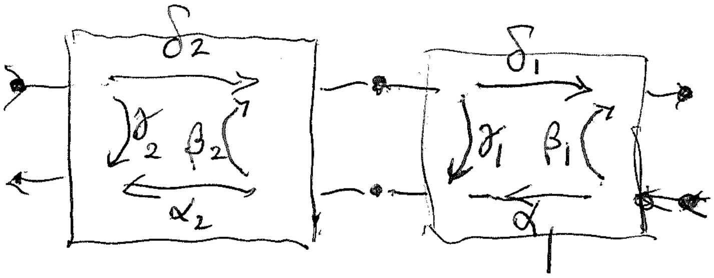
$$\begin{pmatrix} a & b \\ c & d\end{pmatrix} = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1\end{pmatrix}\begin{pmatrix} a_2 & b_2 \\ c_2 & d_2\end{pmatrix} = \begin{pmatrix} a_1a_2+b_1c_2 & a_1b_2+b_1d_2 \\ c_1a_2+d_1c_2 & c_1b_2+d_1d_2\end{pmatrix}$$

$$\beta = \frac{b}{d} = \frac{a_1b_2+b_1d_2}{c_1b_2+d_1d_2} = \frac{\frac{a_1}{d_1}\beta_2 + \beta_1}{-\gamma_1\beta_2+1}$$

$$\beta - \beta_1 = \frac{1}{1-\gamma_1\beta_2}\underbrace{\left(\frac{a_1}{d_1}\beta_2 + \cancel{\beta_1} - \beta_1(\cancel{1} - \gamma_1\beta_2)\right)}_{\left(\underbrace{\frac{a_1}{d_1}+\gamma_1\beta_1}\right)\beta_2}$$

$$\frac{a_1}{d_1} - \frac{c_1}{d_1}\frac{b_1}{d_1} = \frac{a_1d_1 - b_1c_1}{d_1d_1} = \delta_1\alpha_1$$

$$\therefore\quad \beta = \beta_1 + \delta_1\beta_2\frac{1}{1-\gamma_1\beta_2}\alpha_1$$



$$\delta = \frac{1}{d} = \frac{1}{d_1d_2+c_1b_2} = \frac{1}{d_1}\,\frac{1}{1+\frac{c_1}{d_1}\frac{b_2}{d_2}}\,\frac{1}{d_2} = \delta_1\frac{1}{1-\boxed{\gamma_1}\beta_2}\delta_2$$

$$\gamma = -\frac{c}{d} = -\frac{c_1a_2+d_1c_2}{d_1d_2+c_1b_2} = -\frac{+\frac{c_1}{d_1}\frac{a_2}{d_2}+\frac{c_2}{d_2}}{1+\frac{c_1}{d_1}\frac{b_2}{d_2}} = \frac{+\gamma_1\frac{a_2}{d_2}+\gamma_2}{1-\gamma_1\beta_2}$$

$$\gamma - \gamma_2 = \frac{\gamma_1 \frac{a_2}{d_2} + \cancel{\gamma_2} - \gamma_2(\cancel{1} - \gamma_1 \beta_2)}{1 - \gamma_1 \beta_2}$$

$$= \frac{\gamma_1}{1-\gamma_1\beta_2} \underbrace{\left(\frac{a_2}{d_2} + \left(-\frac{c_2}{d_2}\right)\left(\frac{b_2}{d_2}\right)\right)}_{\frac{1}{d_2}\left(\frac{a_2 d_2 - b_2 c_2}{d_2}\right) = \alpha_2 \delta_2}$$

$$\therefore \gamma = \gamma_2 + \alpha_2 \frac{1}{1-\gamma_1\beta_2} \gamma_1 \delta_2$$

Focus on

$$\beta = \beta_1 + \delta_1 \beta_2 \frac{1}{1-\gamma_1\beta_2} \delta_1$$

Idea: $\quad b - d\beta \in H_+$

$\Rightarrow \quad \frac{b}{d} - \beta \in H_+ \quad \Rightarrow \frac{b}{d}$ might be the H_- part of β

Program: Fit $T_{\infty,-\infty} = T_{\infty,0} T_{0,\infty}$ factorization into the framework.

$$T_{\infty,0} = \prod_{n \in (\infty, 1]} \frac{1}{k_n} \begin{pmatrix} 1 & h_n z^{-n} \\ \bar{h}_n z^n & 1 \end{pmatrix} = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}$$

$$T_{0,\infty} = \prod_{n \in [0, -\infty)} \quad " \quad = \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}$$

$$\begin{pmatrix} \xi_+ \\ \xi_- \end{pmatrix} = \overset{1}{T}_{\infty,0} \begin{pmatrix} p_0 \\ q_0 \end{pmatrix} \qquad \begin{pmatrix} p_0 \\ q_0 \end{pmatrix} = \overset{2}{T}_{0,\infty} \begin{pmatrix} \xi'_- \\ \xi'_+ \end{pmatrix}$$

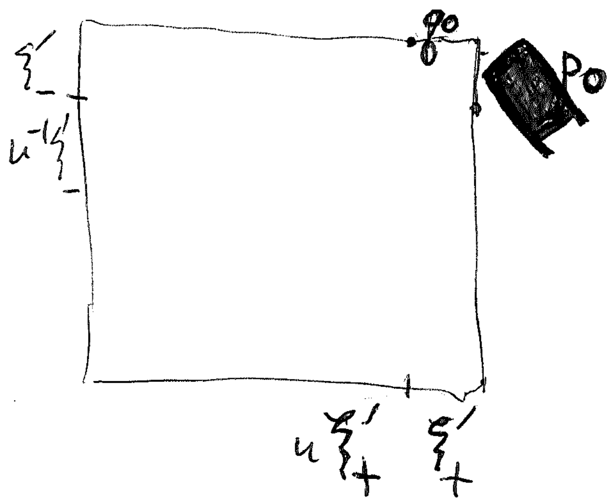
First property $T_{\infty,0}$ concerns $h_1, h_2, \ldots$

so we know $c_1 \in zH_+ \quad d_1 \in H_+$

$b_1 = \bar{c}_1 \in H_- \quad a_1 = \bar{d}_1 = zH_-$

$$\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} = T_{\infty,0} \in \begin{pmatrix} zH_- & H_- \\ zH_+ & H_+ \end{pmatrix}$$

$$T_{0,\infty} = \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} \in \begin{pmatrix} zH_- & H_+ \\ zH_- & H_+ \end{pmatrix}$$



$p_0 \in zH_- \xi'_- + H_+ \xi'_+$

$p_0 = a_2 \xi'_- + b_2 \xi'_+$

$q_0 \in zH_- \xi'_- + H_+ \xi'_+$

$$\beta_2 = \frac{b_2}{d_2} \in H_+ \qquad \gamma_1 = -\frac{c_1}{d_1} \in \frac{zH_+}{H_+} \in zH_+$$

$$\therefore \quad \frac{1}{1-\gamma_1\beta_2} \in 1 + zH_+$$

$$\delta_1 = \frac{1}{d_1} \in H_+ \qquad \delta_2 \cancel{=} \frac{1}{d_2} \in H_+$$

$$\alpha_1 = \frac{1}{d_1}$$

Start again

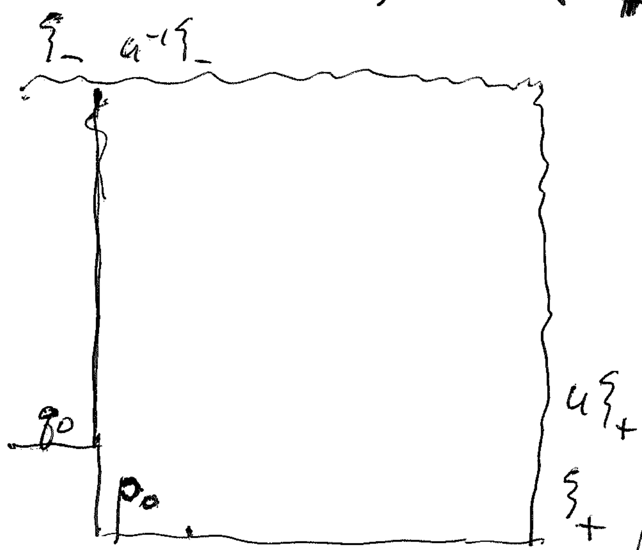
$$T_{\infty,-\infty} = T_{\infty,0}\, T_{0,-\infty} = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}\begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}$$

$$\prod_{n\in(\infty,1]} \frac{1}{k_n}\begin{pmatrix} 1 & h_n z^{-n} \\ \bar{h}_n z^n & 1 \end{pmatrix} \qquad \prod_{n\in[0,-\infty)} \frac{1}{k_n}\begin{pmatrix} & \\ & \end{pmatrix}$$

analyze $T_{\infty,0}$ as follows

$$\begin{pmatrix} \xi_+ \\ \xi_- \end{pmatrix} = \lim_{n\to\infty}\begin{pmatrix} u^{-n}p_n \\ q_n \end{pmatrix} = \lim_{n\to\infty} T_{n,0}\begin{pmatrix} p_0 \\ q_0 \end{pmatrix} = T_{\infty,0}\begin{pmatrix} p_0 \\ q_0 \end{pmatrix}$$

$$\begin{pmatrix} p_0 \\ q_0 \end{pmatrix} = T_{\infty,0}^{-1}\begin{pmatrix} \xi_+ \\ \xi_- \end{pmatrix} = \begin{pmatrix} d_1 & -b_1 \\ -c_1 & a_1 \end{pmatrix}\begin{pmatrix} \xi_+ \\ \xi_- \end{pmatrix}$$



$p_0 \in H_+\xi_+ + H_-\xi_-$

$\boxed{d_1 \in H_+,\ b_1 \in H_-}$

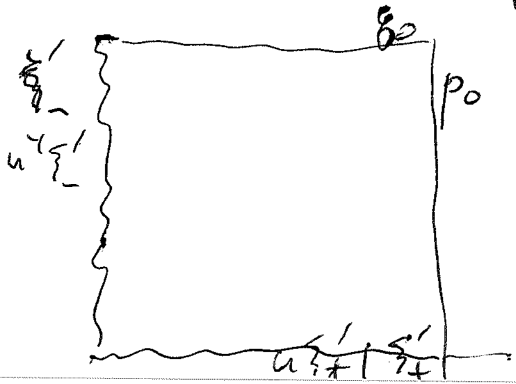
$u^{-1}q_0 \in H_+\xi_+ + H_-\xi_-$

$q_0 \in zH_+\xi_+ + zH_-\xi_-$

$\boxed{c_1 \in zH_+,\ a_1 \in zH_-}$

Consistent with $a_1 = \bar{d}_1 \quad c_1 = \bar{b}_1$

So look at $\begin{pmatrix} p_0 \\ q_0 \end{pmatrix} = T_{0,-\infty}\begin{pmatrix} \xi'_- \\ \xi'_+ \end{pmatrix} = \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}\begin{pmatrix} \xi'_- \\ \xi'_+ \end{pmatrix}$



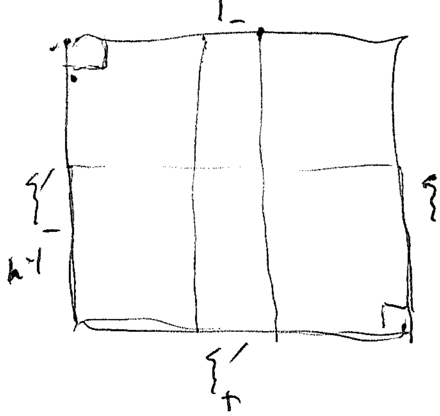
$p_0, q_0 \in zH_-\xi'_- + H_+\xi'_+$

$a_2, c_2 \in zH_-$	$b_2, d_2 \in H_+$

$$\begin{pmatrix}\xi_+\\ \xi_-\end{pmatrix} = \begin{pmatrix} a & b\\ c & d\end{pmatrix}\begin{pmatrix}\xi'_-\\ \xi'_+\end{pmatrix}$$

$$\xi_+ \in uH_- \xi'_- + L^2 \xi'_+$$

$$\boxed{a \in uH_- \;\Big|\; b \in L^2}$$



$$\xi_- \in \overset{d}{H_+}\,\xi'_+ + \overset{c}{L^2}\,\xi'_-$$

$$\boxed{d \in H_+ \;\Big|\; c \in L^2}$$

$$\begin{pmatrix} a & b\\ c & d\end{pmatrix} = \begin{pmatrix} a_1 & b_1\\ c_1 & d_1\end{pmatrix}\begin{pmatrix} a_2 & b_2\\ c_2 & d_2\end{pmatrix}$$

$$d = c_1 b_2 + d_1 d_2 \in zH_+ H_+ + H_+ H_+ = H_+$$

$$b = \underbrace{a_1}_{zH_-}\underbrace{b_2}_{H_+} + \underbrace{b_1}_{H_-}\underbrace{d_2}_{H_+}$$

If you express out in terms of in

$$\begin{pmatrix}\xi_+\\ \xi'_+\end{pmatrix} = \begin{pmatrix} \frac{1}{d} & \frac{b}{d}\\ -\frac{c}{d} & \frac{1}{d}\end{pmatrix}\begin{pmatrix}\xi'_-\\ \xi_-\end{pmatrix}$$

then transmission coeffs are in H_+

But if you express in in terms of out:

$$\begin{pmatrix}\xi'_-\\ \xi'_+\end{pmatrix} = \begin{pmatrix} d & -b\\ -c & a\end{pmatrix}\begin{pmatrix}\xi_+\\ \xi_-\end{pmatrix} \qquad \begin{pmatrix}\xi'_-\\ \xi_-\end{pmatrix} = \begin{pmatrix} \frac{1}{a} & -\frac{b}{a}\\ \frac{c}{a} & \frac{1}{a}\end{pmatrix}\begin{pmatrix}\xi_+\\ \xi'_+\end{pmatrix}$$

then trans. coeffs $\frac{1}{a} \in H_-$

go back to $\begin{pmatrix} \xi_+ \\ \xi'_+ \end{pmatrix} = \begin{pmatrix} \frac{1}{d} & \frac{b}{d} \\ -\frac{c}{d} & \frac{1}{d} \end{pmatrix} \begin{pmatrix} \xi'_- \\ \xi_- \end{pmatrix}$

$$\xi_+ = \frac{1}{d}\xi'_- + \frac{b}{d}\xi_-$$

so

$$(u^k \xi_- \mid u^j \xi_+) = \left(u^{k-j}\xi_- \,\middle|\, \frac{b}{d}\xi_-\right)$$

$$= \beta_{k-j}$$

$$\xi_- = \frac{c}{a}\xi_+ + \frac{1}{a}\xi'_+$$

$$(u^j \xi_+ \mid u^k \xi_-) = \left(u^j \xi_+ \,\middle|\, u^k \frac{c}{a}\xi_+\right) = \int z^{k-j}\frac{c}{a}$$

$$= \overline{\int z^{j-k}\frac{b}{d}}$$

$$= \bar{\beta}_{j-k}$$

$$\beta = \frac{a_1 b_2 + b_1 d_2}{c_1 b_2 + d_1 d_2} = \frac{\frac{a_1}{d_1}\beta_2 + \cancel{\beta_1} - (\cancel{1} - \gamma_1\beta_2)\beta_1}{1 - \gamma_1\beta_2} + \beta_1$$

$$= \frac{\beta_2}{1-\gamma_1\beta_2}\underbrace{\left(\frac{a_1}{d_1} + \cancel{\gamma_1\beta_1} - \frac{c_1 b_1}{d_1^2}\right)}_{\frac{1}{d_1}\frac{a_1 d_1 - b_1 c_1}{d_1} = \delta_1\alpha_1}$$

$$\beta = \alpha_1 \beta_2 \frac{1}{1-\gamma_1\beta_2}\delta_1$$

N.B. $\alpha_1 = \delta_1$ is our case

$$\boxed{\beta = \beta_1 + \alpha_1 \beta_2 \frac{1}{1-\gamma_1\beta_2}\alpha_1 \implies \beta - \beta_1 \in H_+}$$

$$\beta_2 = \frac{b_2}{d_2} \in H_+ \qquad \gamma_1 = -\frac{c_1}{d_1} \in \frac{zH_+}{H_+} = zH_+$$

$$\alpha_1 = \frac{1}{d_1} \in H_+ \qquad \beta_1 = \frac{b_1}{d_1} \; ?$$

$$\cancel{\delta = d^{-1} = (d_1 d_2 + c_1 b_2)^{-1} = d_2^{-1}(1 + d_1^{-1} c_1 b_2 d_2^{-1})^{-1} d_1^{-1}}$$

$$\cancel{= \delta_2 (1-\gamma_1\beta_2)^{-1}\delta_1 \qquad \gamma_1 = d_1^{-1} c_1 \qquad \beta_2 = b_2 d_2^{-1}}$$

$$\cancel{\beta = bd^{-1} = (a_1 b_2 + b_1 d_2)\, d_2^{-1} (1-\gamma_1\beta_2)^{-1}\delta_1}$$

$$\cancel{= (a_1\beta_2 + b_1)(1-\gamma_1\beta_2)^{-1}\delta_1}$$

$$\cancel{\beta_1 =}$$

$$\gamma = -\frac{c}{d} = -\frac{c_1 a_2 + d_1 c_2}{d_1 d_2 + c_1 b_2} = \frac{\gamma_1 \frac{a_2}{d_2} + \cancel{\gamma_2} - \gamma_2(1-\gamma_1\beta_2)}{1-\gamma_1\beta_2} + \gamma_2$$

$$= \gamma_2 + \gamma_1 \left(\frac{1}{1-\gamma_1\beta_2}\right)\left(\frac{a_2}{d_2} + \left(\frac{-c_2}{d_2}\right)\left(\frac{b_2}{d_2}\right)\right)$$

$$\boxed{\gamma = \gamma_2 + \frac{\gamma_1}{1-\gamma_1\beta_2}\alpha_2\delta_2 \implies \gamma - \gamma_2 \in zH_+}$$

$$\boxed{\begin{pmatrix} \alpha_1 & \beta_1 \\ \gamma_1 & \delta_1 \end{pmatrix} \in \begin{pmatrix} H_+ & \\ zH_+ & H_+ \end{pmatrix} \qquad \begin{pmatrix} \alpha_2 & \beta_2 \\ \gamma_2 & \delta_2 \end{pmatrix} \in \begin{pmatrix} H_+ & H_+ \\ & H_+ \end{pmatrix}}$$

I now need to connect the condition

$\beta - \beta_1 \in H_+$ i.e. $\beta - \frac{b_1}{d_1} \in H_+$

equiv $d_1\beta - b_1 \in H_+$

$\gamma - \gamma_2 \in zH_+$ $\gamma + \frac{c_2}{d_2} \in zH_+$

The other condition is $d_1 - b_1\bar{\beta} \in zH_-$

i.e. $\left. \begin{array}{l} a_1 - c_1\beta \in H_+ \\ b_1 - d_1\beta \in H_+ \end{array} \right)$ $\left(\begin{array}{l} d_1\beta - b_1 \in H_+ \\ -c_1\beta + a_1 \in H_+ \end{array} \right.$

$$\underbrace{\begin{pmatrix} d_1 & -b_1 \\ -c_1 & a_1 \end{pmatrix}}_{T_{\infty,0}^{-1}} \begin{pmatrix} \beta \\ 1 \end{pmatrix} \in \begin{pmatrix} H_+ \\ H_+ \end{pmatrix}.$$

$$\begin{pmatrix} p_0 \\ q_0 \end{pmatrix} = \begin{pmatrix} d_1 & -b_1 \\ -c_1 & a_1 \end{pmatrix} \begin{pmatrix} \xi_+ \\ \xi_- \end{pmatrix}$$

Let's check this. The orthogonality conditions determining $\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}$ are

$b_1 - d_1\beta \in H_+$ with $b_1 \in H_-$, $d_1 \in H_+$

$d_1 - b_1\bar{\beta} \in zH_-$

$\to a_1 - c_1\beta \in H_+$

In fact $\begin{pmatrix} \bar{d}_1 & \bar{c}_1 \\ c_1 & d_1 \end{pmatrix}$

You are confused.

$$\begin{pmatrix} \xi_+ \\ \xi_- \end{pmatrix} = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \begin{pmatrix} p_0 \\ q_0 \end{pmatrix} \qquad \begin{pmatrix} p_0 \\ q_0 \end{pmatrix} = \begin{pmatrix} d_1 & -b_1 \\ -c_1 & a_1 \end{pmatrix} \begin{pmatrix} \xi_+ \\ \xi_- \end{pmatrix}$$

$$\begin{pmatrix} p_0 \\ q_0 \end{pmatrix} = \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} \begin{pmatrix} \xi'_- \\ \xi'_+ \end{pmatrix}$$

Recap: Given β you construct E, u etc.
In fact you can first construct

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \frac{\alpha\delta-\beta\gamma}{\delta} & \frac{\beta}{\delta} \\ -\frac{\gamma}{\delta} & \frac{1}{\delta} \end{pmatrix}$$

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} \frac{ad-bc}{d} & \frac{b}{d} \\ -\frac{c}{d} & \frac{1}{d} \end{pmatrix}$$

Recall the method: You find $1-|\beta|^2 = |\delta|^2$ where $\delta \in H_+$ invertible $\delta = \frac{1}{d}$. Then

$$\cancel{\begin{pmatrix} \xi'_- \\ \xi'_+ \end{pmatrix} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \begin{pmatrix} \xi_+ \\ \xi_- \end{pmatrix}}$$

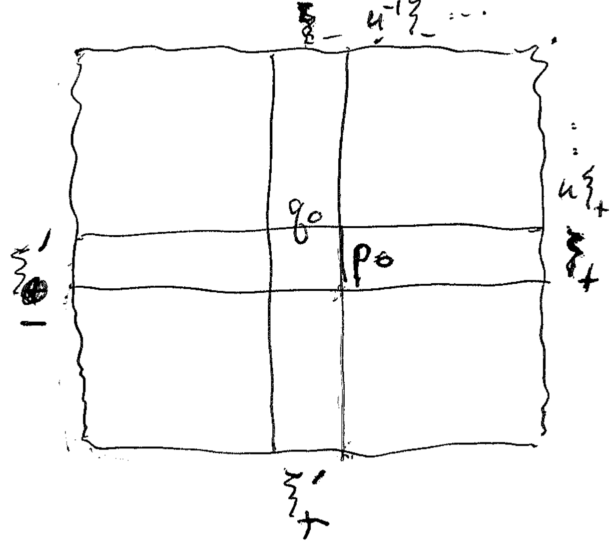
$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} \bar{\delta} & \beta \\ -\frac{\delta}{\bar{\delta}}\bar{\beta} & \delta \end{pmatrix}$$

so then $b = \frac{\beta}{\delta}$ $\quad d = \frac{1}{\delta}$

Given β you construct $E, u, \xi_\pm$ $\xi'_\mp$

$\delta = \frac{1}{d} \in H_+$ $\qquad \left|\frac{1}{d}\right|^2 = 1-|\beta|^2$

Repeat. The idea is to take $p_0 = d_1\xi_+ - b_1\xi_-$ and to project it into $L^2\xi_\pm$, $L^2\xi'_\mp$. Projection into $L^2\xi_-$ gives the condition



$$(d_1\beta - b_1)\xi_- = \text{proj of } p_0 \in H_+\xi_-$$

$$\therefore \boxed{d_1\beta - b_1 \in H_+}$$

similarly proj. into $L^2\xi_+$ yields $\boxed{d_1 - b_1\bar{\beta} \in zH_-}$

proj. into $L^2\xi'_- = (L^2\xi_-)^\perp$ yields since

$$\xi_+ = \alpha\xi'_- + \beta\xi_- = \frac{1}{d}\xi'_- + \frac{b}{d}\xi_-$$

that $p_0 \longmapsto d_1\frac{1}{d}\xi'_- \in H_+\xi'_-$ $\quad\boxed{\therefore\ d_1\frac{1}{d} \in H_+ \text{ equiv } d_1 \in H_+}$

Then

$$\begin{pmatrix}\xi'_- \\ \xi'_+\end{pmatrix} = \begin{pmatrix} d & -b \\ -c & a\end{pmatrix}\begin{pmatrix}\xi_+ \\ \xi_-\end{pmatrix}$$

$$\begin{pmatrix}\xi'_- \\ \xi_-\end{pmatrix} = \begin{pmatrix} \frac{1}{a} & -\frac{b}{a} \\ +\frac{c}{a} & \frac{1}{a}\end{pmatrix}\begin{pmatrix}\xi_+ \\ \xi'_+\end{pmatrix} \qquad \xi_- = \frac{c}{a}\xi_+ + \frac{1}{a}\xi'_+$$

proj of $p_0 = d_1\xi_+ - b_1\xi_-$ into $L^2\xi'_+$ is

$p_0 \longmapsto -b_1\frac{1}{a}\xi'_+ \in H_-\xi'_+$ $\quad\boxed{\therefore\ \frac{b_1}{a} \in H_- \text{ equiv. } b_1 \in H_-}$

Another idea. Assume $\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \Rightarrow \begin{array}{l} \bar{d}_1 = a_1 \\ \bar{c}_1 = b_1 \end{array}$ $\quad$ det $= 1$

Then conditions become

$$d_1 \beta - b_1 \in H_+$$
$$-c_1 \beta + a_1 \in H_+$$

which should be clear from

$$\begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} = \begin{pmatrix} d_1 & -b_1 \\ -c_1 & a_1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

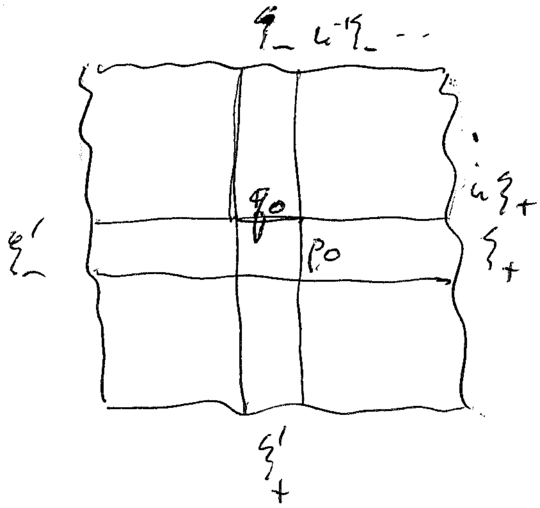
$$\Longrightarrow \quad \begin{pmatrix} b_2 \\ d_2 \end{pmatrix} = \begin{pmatrix} d_1 & -b_1 \\ -c_1 & a_1 \end{pmatrix} \begin{pmatrix} b \\ d \end{pmatrix}$$

$$\boxed{\begin{aligned} \frac{b_2}{d} &= d_1 \beta - b_1 \\ \frac{d_2}{d} &= -c_1 \beta + a_1 \end{aligned}}$$

$b_2 \in H_+$
$d_2, d \in H_+$

Can you do something with

$$\begin{pmatrix} 1 & -\beta \\ -\bar{\beta} & 1 \end{pmatrix} \begin{pmatrix} b_1 \\ d_1 \end{pmatrix} \in \begin{pmatrix} H_+^{\infty} \\ zH_-^{\infty} \end{pmatrix} \quad ?$$



$$\begin{pmatrix}\xi_+\\ \xi_-\end{pmatrix} = \begin{pmatrix}a_1 & b_1\\ c_1 & d_1\end{pmatrix}\begin{pmatrix}p_0\\ q_0\end{pmatrix}$$

$$\begin{pmatrix}p_0\\ q_0\end{pmatrix} = \begin{pmatrix}d_1 & -b_1\\ -c_1 & a_1\end{pmatrix}\begin{pmatrix}\xi_+\\ \xi_-\end{pmatrix}$$

$$p_0 = d_1\xi_+ - b_1\xi_- \in H_+\xi_+ + H_-\xi_-$$
$$q_0 = -c_1\xi_+ + a_1\xi_- \in zH_+\xi_+ + zH_-\xi_-$$

project p_0 into $L^2\xi_-$ $\qquad d_1\beta - b_1 \in H_+$

project p_0 into $L^2\xi_+$ $\qquad d_1 - b_1\bar\beta \in H_-$

At the moment you have reformulated the orthog conditions as saying
$$\begin{pmatrix}d_1 & -b_1\\ -c_1 & a_1\end{pmatrix}\begin{pmatrix}\beta\\ 1\end{pmatrix} \in \begin{pmatrix}H_+\\ H_+\end{pmatrix}$$
$$\begin{pmatrix}d_1 & -b_1\\ -c_1 & a_1\end{pmatrix}\begin{pmatrix}b\\ d\end{pmatrix} \in \begin{pmatrix}H_+\\ H_+\end{pmatrix}$$

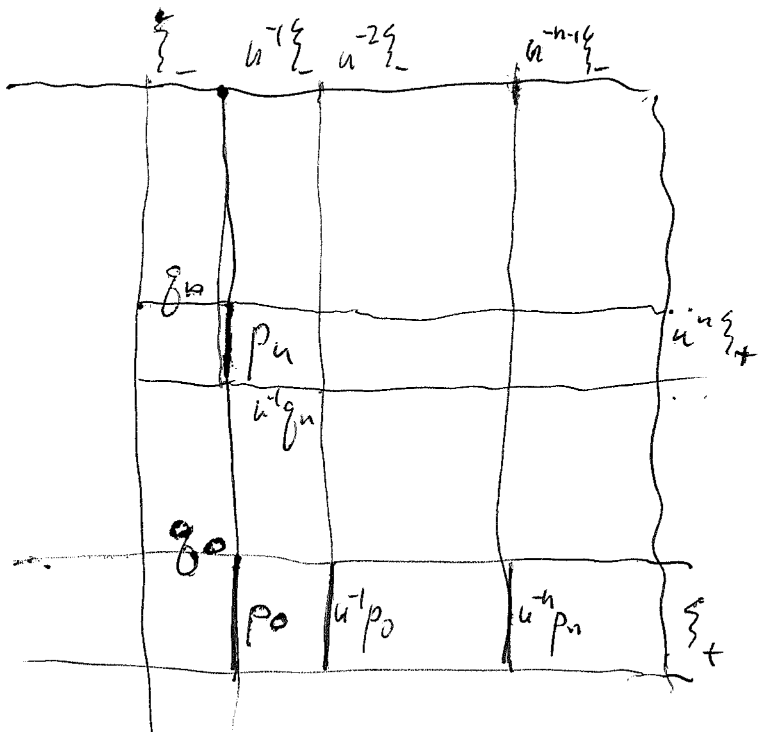
$$\begin{pmatrix}d_1 & -b_1\\ -c_1 & a_1\end{pmatrix}\begin{pmatrix}a & b\\ c & d\end{pmatrix} = \begin{pmatrix}a_2 & b_2\\ c_2 & d_2\end{pmatrix}$$

$$\therefore \begin{pmatrix}d_1 & -b_1\\ -c_1 & a_1\end{pmatrix}\begin{pmatrix}b\\ d\end{pmatrix} = \begin{pmatrix}b_2\\ d_2\end{pmatrix}$$

orthog relations are simply for $p_0 = d_1\xi_+ - b_1\xi_-$

$$d_1\beta - b_1 \in H_+ \qquad \beta_1\beta \in H_+$$
$$-\bar b_1\beta + \bar d_1 \in H_+ \qquad \bar d_1(1-\bar\beta_1\beta) \in H_+$$

Today you look at $\begin{pmatrix} u^{-n}p_n \\ q_n \end{pmatrix}$ 1079.03



$$u^{-n}p_n \in \underline{H_+\xi_+ + z^{-n}H_-\xi_-}$$

$$u^{-1}q_n \in$$

$$q_n \in z^{n+1}H_+\xi_+ + zH_-\xi_-$$

so if $\begin{pmatrix} u^{-n}p_n \\ q_n \end{pmatrix} = \underbrace{\begin{pmatrix} d_n & -b_n \\ -c_n & a_n \end{pmatrix}}_{\in \begin{pmatrix} H_+ & z^{-n}H_- \\ z^n z H_+ & zH_- \end{pmatrix}} \begin{pmatrix} \xi_+ \\ \xi_- \end{pmatrix}$

project into $L^2\xi_-$

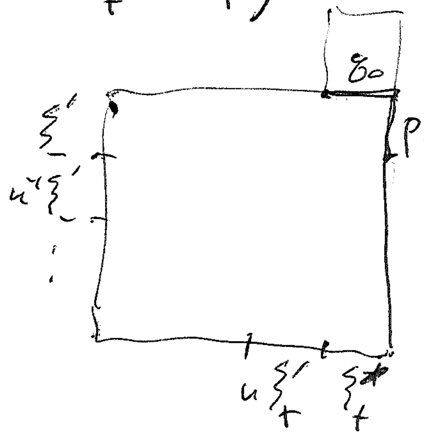
$$u^{-n}p_n = d_n\xi_+ - b_n\xi_-$$

$$\boxed{\begin{aligned} d_n\beta - b_n &\in z^{-n}H_+ \\ d_n - b_n\bar\beta &\in z^n z H_- \end{aligned}}$$

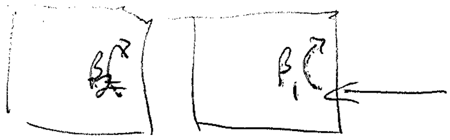
special case $h_n = 0$ for $n \leq 0$. Then can you describe those β occurring? In this case $p_0 = \xi'_-$ $\quad q_0 = \xi'_+$ so

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \in \begin{pmatrix} H_- & H_- \\ zH_+ & H_+ \end{pmatrix}$$

$$\boxed{\begin{pmatrix} p_0 \\ q_0 \end{pmatrix} = \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}\begin{pmatrix} \xi'_- \\ \xi'_+ \end{pmatrix} \qquad \text{gen case} \quad \begin{matrix} a_2 \in H_- & b_2 \in H_+ \\ c_2 \in H_- & d_2 \in H_+ \end{matrix}}$$



Note: If $h_n = 0$ for $n \geq 1$, then $\beta = \frac{b}{d} = \frac{b_2}{d_2} \in H_+$



Q. $\quad h_n = 0$ for $n \geq 1 \iff \beta = \frac{b}{d} \in H_+$?

$(\Longrightarrow)$ $h_n = 0$ for $n \geq 1$, iff $\quad q_0 = q_1 = \cdots = \xi_-$
$\quad p_0 = u^{-1}p_1 = \cdots = \xi_+$

in which case $T_{\infty,-\infty} = T_{0,\infty} = \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}$ whence $\beta = \beta_2 \in H_+$

Conversely if $\beta \in H_+$, then $\beta_{k-j} = (u^k\xi_- \mid u^j\xi_+)$ is zero for $k < j$

$$(u^k\xi_- \mid u^j\xi_+) = (z^{k-j} \mid \beta) = 0 \text{ if } k - j < 0$$

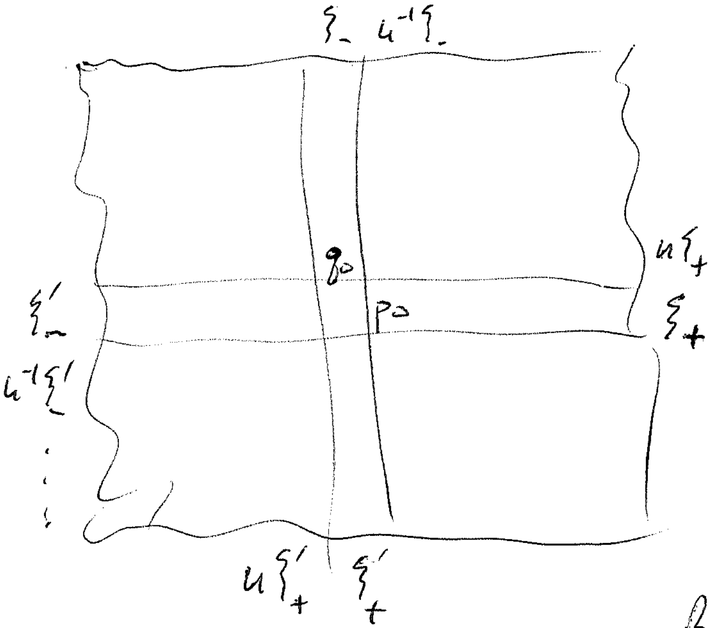
$\therefore \quad u^{<0}\xi_- \perp u^{\geq 0}\xi_+$

~~Important thing is [illegible]~~

Suppose $\beta = \sum_{n\geq 0} \beta_n z^n \in H_+$ $\qquad \beta = \frac{b}{d}$ d inv. hol.

$$\beta \in H_+ \iff b \in H_+$$

$$\begin{pmatrix}\xi_+\\ \xi_-\end{pmatrix} = \begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}\xi'_-\\ \xi'_+\end{pmatrix}$$



$$\xi_+ = a\xi'_- + b\xi'_+$$

$$(u^k\xi_- \mid u^j\xi_+) = (z^{k-j} \mid \beta)$$

$$= \beta_{k-j} = 0 \quad \text{for } k-j<0$$

find $p_0 = \xi_+$, $q_0 = \xi_-$

so $h_0 = (q_0 \mid p_0) = (\xi_- \mid \xi_+) = \beta_0$

$$\beta_0 = \int \beta(z)\,\frac{d\theta}{2\pi} \qquad |\beta_0| \leq \|\beta\|_\infty$$

The point somehow should be roughly that given β there is a sequence of approximations $\beta^{(n)}$ to β. Schur expansion

You want to express the idea of building from the left

$$\begin{pmatrix}a_n & b_n\\ c_n & d_n\end{pmatrix} = \frac{1}{k_n}\begin{pmatrix}1 & h_n z^n\\ \bar{h}_n z^n & 1\end{pmatrix}\begin{pmatrix}a_{n-1} & b_{n-1}\\ c_{n-1} & d_{n-1}\end{pmatrix}$$

all functions of z. What about $\begin{pmatrix}u^{-n}p_n\\ q_n\end{pmatrix}$

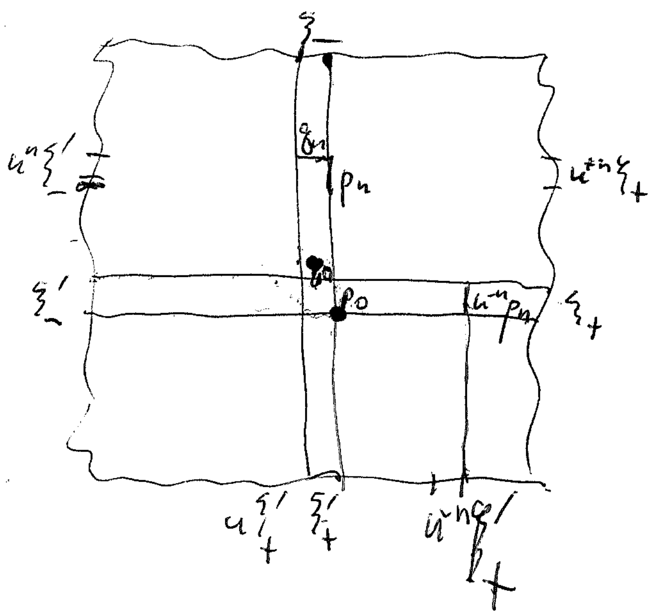
$$\begin{pmatrix}p_0\\ q_0\end{pmatrix} = \begin{pmatrix}a_0 & b_0\\ c_0 & d_0\end{pmatrix}\begin{pmatrix}\xi'_-\\ \xi'_+\end{pmatrix}$$

this is an eqn in E, but each elt of E is equivalent to two functions of z.

You have the rough idea that [redacted] ~~should~~ working from the left corresponds to polar behavior of β.

begin again, where? ~~We~~ Start with β ~~so what?~~ The problem is to start with $\beta(z)$ smooth and $|\ | < 1$, then to show that h_n goes to zero as $|n| \to \infty$. There was a new idea, namely, a picture of ~~E~~ elements of E as pairs of functions on the circle. Let

$$T_{n,-\infty} = \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix} \quad \text{so that} \quad \begin{pmatrix} \text{[redacted]}u^{-n}p_n \\ q_n \end{pmatrix} = \underbrace{\begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix}}_{\in \begin{pmatrix} zH_- & z^{-n}H_+ \\ z^{n+1}H_- & H_+ \end{pmatrix}} \begin{pmatrix} \xi'_- \\ \xi'_+ \end{pmatrix}$$



$$q_n \in H_+ \xi'_+ + z^{n+1} H_- \xi'_-$$

$$u^{-1} p_n \in z^{-n} H_+ \xi'_+ + z H_- \xi'_-$$

so $\quad \dfrac{b_n}{d_n} \in z^{-n} H_+$

New ~~idea~~ idea about fns. $\begin{pmatrix} b_n \\ d_n \end{pmatrix}$ is the solution of the Dirac equation ~~..~~ such that?

$$q_n = c_n \xi'_- + d_n \xi'_+ \longrightarrow \xi'_+ \quad \text{as } n \to -\infty$$

so we know $c_n \to 0$ and $d_n \to 1$ as $n \to -\infty$ in the L^2 sense

$$T_{n,-\infty} = \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix} \in \begin{pmatrix} zH_- & z^{-n}H_+ \\ z^{n+1}H_- & H_+ \end{pmatrix}$$

$$\begin{pmatrix} u^{-n}p_n \\ q_n \end{pmatrix} = \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix} \begin{pmatrix} \xi'_- \\ \xi'_+ \end{pmatrix} \quad \text{in } E.$$

If $h_n = 0$ all $n > n_0$.

Then

$$\begin{array}{ccc} & q_{n_0+1} & \\ up_{n_0} & \boxed{\;=\;} & p_{n_0+1} \\ & q_{n_0} & \end{array}$$

~~Reflection coefficient~~

$$T_{n,-\infty} = \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix}$$ parametrizes

solutions of the DE in terms of the asymptotic behavior to the left ~~ends~~.

---

idea yesterday: go back to the standard way of dealing with scattering ~ solutions of the DE $\psi(x,\lambda)$ with certain asymptotics. 2 component function of λ, sections of rank 2 vector bundle over $\mathbb{R}$ ~~depending~~ ~~asy then an~~ Stick to the discrete case

$$\begin{pmatrix} u^{-n}p_n \\ q_n \end{pmatrix} = T_{n,-\infty}\begin{pmatrix} \xi'_- \\ \xi'_+ \end{pmatrix} = \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix}\begin{pmatrix} \xi'_- \\ \xi'_+ \end{pmatrix} \quad \in \begin{pmatrix} zH_- & z^{-n}H_+ \\ z^{n+1}H_- & H_+ \end{pmatrix}$$

What kind of sollutions? Interesting solns.

Pose question of splitting E at space time point.

Splitting $\left(H_+\xi_+' + zH_-\xi_-'\right) \overset{\perp}{\oplus} \left(zH_+\xi_+ + H_-\xi_-\right)$

In general you want to construct the projection belonging to the splitting

$$\left(z^{m+1}H_-\xi_-' + z^nH_+\xi_+'\right) \overset{\perp}{\oplus} \left(z^mH_+\xi_+ + z^{n+1}H_-\xi_-\right)$$

$$\left(H_-\xi_-' + H_+\xi_+'\right) \overset{\perp}{\oplus} \left(H_+\xi_+ + H_-\xi_-\right)$$

$$\begin{pmatrix} f_- & f_+ \end{pmatrix}\begin{pmatrix} \xi_-' \\ \xi_+' \end{pmatrix} = \begin{pmatrix} f_- & f_+ \end{pmatrix}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}\begin{pmatrix} \xi_+ \\ \xi_- \end{pmatrix}$$

better maybe is

$$\begin{pmatrix} f_+ & f_- \end{pmatrix}\begin{pmatrix} \xi_+ \\ \xi_- \end{pmatrix} = \begin{pmatrix} f_+ & f_- \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} \xi_-' \\ \xi_+' \end{pmatrix}$$

$$= \begin{pmatrix} af_+ + cf_- & bf_+ + df_- \end{pmatrix}\begin{pmatrix} \xi_-' \\ \xi_+' \end{pmatrix}$$

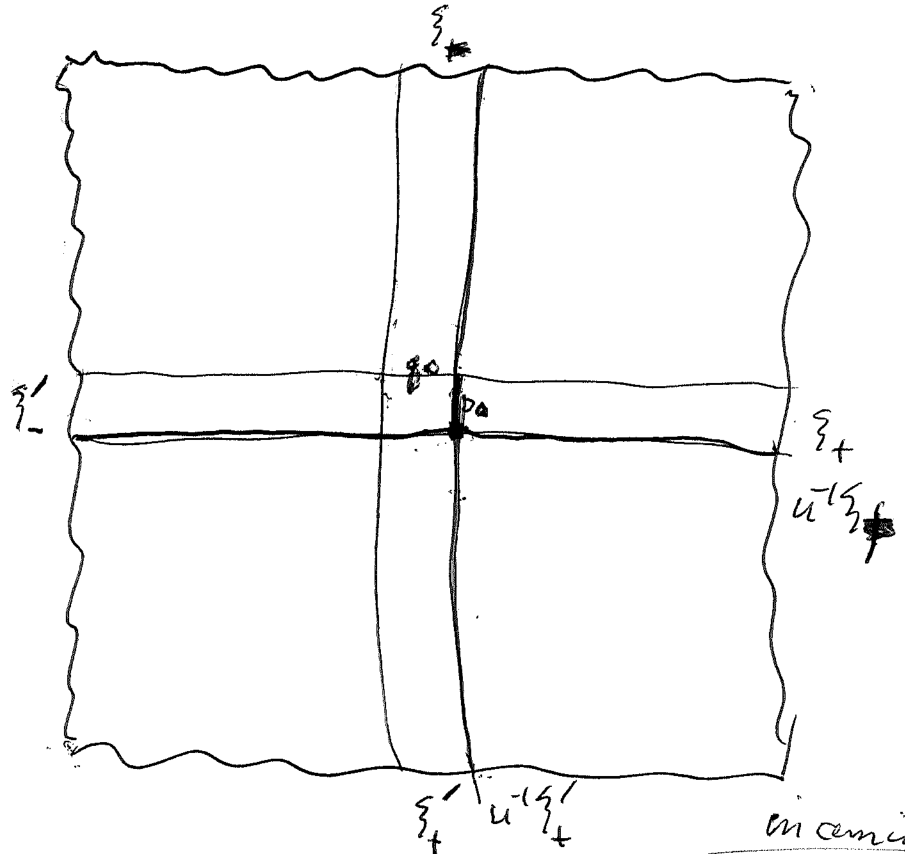
You're interested in the subspace $H_+\xi_+ + H_-\xi_-$

You have a non-orth basis for its orthogonal complement which you might adjust

ought to play with

$$\begin{pmatrix} 1 & 0 \\ -\frac{c}{d} & \frac{1}{d} \end{pmatrix}^{-1} = d\begin{pmatrix} \frac{1}{d} & 0 \\ \frac{c}{d} & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ c & d \end{pmatrix}$$

Old question



incoming in E.

If $\quad H_+p_0 + H_+q_0 = H_+\xi'_- + H_+\xi_-$

and $\quad H_-p_0 + H_-q_0 = H_-\xi'_+ + H_-\xi_+$

these should be complementary, so you seem to have a unitary S with incoming and outgoing subspaces

Let's check this out.

$$\begin{pmatrix} p_0 \\ q_0 \end{pmatrix} = \begin{pmatrix} a_0 & b_0 \\ c_0 & d_0 \end{pmatrix} \begin{pmatrix} \xi'_- \\ \xi'_+ \end{pmatrix} \qquad \begin{pmatrix} a_0 & b_0 \\ c_0 & d_0 \end{pmatrix} \in \begin{pmatrix} zH_- & H_+ \\ zH_- & H_+ \end{pmatrix}$$

$$\begin{pmatrix} \xi_+ \\ \xi_- \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \xi'_- \\ \xi'_+ \end{pmatrix} \qquad \begin{pmatrix} \xi'_- \\ \xi'_+ \end{pmatrix} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \begin{pmatrix} \xi_+ \\ \xi_- \end{pmatrix}$$

$$\begin{pmatrix} \xi_+ \\ \xi'_+ \end{pmatrix} = \begin{pmatrix} \frac{1}{d} & \frac{b}{d} \\ -\frac{c}{d} & \frac{1}{d} \end{pmatrix} \begin{pmatrix} \xi'_- \\ \xi_- \end{pmatrix} \qquad \begin{pmatrix} \xi'_- \\ \xi_- \end{pmatrix} = \begin{pmatrix} & \\ & \frac{1}{a} \end{pmatrix} \begin{pmatrix} \xi_+ \\ \xi'_+ \end{pmatrix}$$

$$\begin{pmatrix} p_0 \\ q_0 \end{pmatrix} = \begin{pmatrix} a_0 & b_0 \\ c_0 & d_0 \end{pmatrix} \begin{pmatrix} \xi'_- \\ -\frac{c}{d}\xi'_- + \frac{1}{d}\xi_- \end{pmatrix}$$

$$= \begin{pmatrix} \left(a_0 - b_0\frac{c}{d}\right)\xi'_- + \frac{b_0}{d}\xi_- \\ \left(c_0 - d_0\frac{c}{d}\right)\xi'_- + \frac{d_0}{d}\xi_- \end{pmatrix}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a_+ & b_+ \\ c_+ & d_+ \end{pmatrix} \begin{pmatrix} a_0 & b_0 \\ c_0 & d_0 \end{pmatrix}$$

$$\cancel{\begin{pmatrix} d_0 & -b_0 \\ -c_0 & a_0 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} d_0 a - b_0 c & d_0 b - b_0 d \\ -c_0 a + a_0 c & -c_0 b + a_0 d \end{pmatrix}}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} d_0 & -b_0 \\ -c_0 & a_0 \end{pmatrix} = \begin{pmatrix} \overset{a_+}{a d_0 - b c_0} & -a b_0 + b a_0 \\ \underset{c_+}{c d_0 - d c_0} & \underset{d_+}{-c b_0 + d a_0} \end{pmatrix}$$

$$\therefore \quad p_0 = \frac{\overset{d_+}{a_0 d - b_0 c}}{d}\xi'_- + \frac{b_0}{d}\xi_-$$

$$\begin{pmatrix} 1 & \bar{h}_2 z^{-2} \\ \bar{h}_2 z^2 & 1 \end{pmatrix} \begin{pmatrix} 1 & h_1 z^{-1} \\ \bar{h}_1 z & 1 \end{pmatrix} = \begin{pmatrix} 1 + \bar{h}_2 \bar{h}_1 z^{-1} \\ \bar{h}_2 z^2 + \bar{h}_1 z \end{pmatrix} \qquad \begin{pmatrix} H_+ & H_+ \\ zH_+ & H_+ \end{pmatrix}$$

U

$$\begin{pmatrix} p_0 \\ q_0 \end{pmatrix} = \begin{pmatrix} a_0 & b_0 \\ c_0 & d_0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\frac{c}{d} & \frac{1}{d} \end{pmatrix} \begin{pmatrix} \xi'_- \\ \xi_- \end{pmatrix}$$

$$= \begin{pmatrix} a_0 - b_0\frac{c}{d} & \frac{b_0}{d} \\ c_0 - d_0\frac{c}{d} & \frac{d_0}{d} \end{pmatrix} \begin{pmatrix} \xi'_- \\ \xi_- \end{pmatrix} \approx \begin{pmatrix} \frac{a_+}{d} & \frac{b_0}{d} \\ -\frac{c_+}{d} & \frac{d_0}{d} \end{pmatrix}$$

$$\begin{pmatrix} \xi'_- \\ \xi_- \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ c & d \end{pmatrix}\begin{pmatrix} d_0 & -b_0 \\ -c_0 & a_0 \end{pmatrix}\begin{pmatrix} p_0 \\ q_0 \end{pmatrix}$$

$$= \begin{pmatrix} d_0 & -b_0 \\ \underbrace{cd_0 - dc_0}_{c_+} & \underbrace{-cb_0 + da_0}_{d_+} \end{pmatrix}\begin{pmatrix} p_0 \\ q_0 \end{pmatrix}$$

So

$$\begin{pmatrix} p_0 \\ q_0 \end{pmatrix} = \frac{1}{d}\begin{pmatrix} d_+ & b_0 \\ -c_+ & d_0 \end{pmatrix}\begin{pmatrix} \xi'_- \\ \xi_- \end{pmatrix}$$

$$\begin{pmatrix} \xi'_- \\ \xi_- \end{pmatrix} = \begin{pmatrix} d_0 & -b_0 \\ c_+ & d_+ \end{pmatrix}\begin{pmatrix} p_0 \\ q_0 \end{pmatrix}$$

$$\begin{pmatrix} \xi_+ \\ \xi_- \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} \xi'_- \\ \xi'_+ \end{pmatrix} = \begin{pmatrix} a_> & b_> \\ c_> & d_> \end{pmatrix}\overbrace{\begin{pmatrix} a_0 & b_0 \\ c_0 & d_0 \end{pmatrix}\begin{pmatrix} \xi'_- \\ \xi'_+ \end{pmatrix}}^{\begin{pmatrix} p_0 \\ q_0 \end{pmatrix}}$$

$$\begin{pmatrix} \xi_+ \\ \xi'_+ \end{pmatrix} = \begin{pmatrix} \frac{1}{d} & \frac{b}{d} \\ -\frac{c}{d} & \frac{1}{d} \end{pmatrix}\begin{pmatrix} \xi'_- \\ \xi_- \end{pmatrix} \qquad \xi'_+ = \begin{pmatrix} -\frac{c}{d} & \frac{1}{d} \end{pmatrix}\begin{pmatrix} \xi'_- \\ \xi_- \end{pmatrix}$$

$$\begin{pmatrix} p_0 \\ q_0 \end{pmatrix} = \begin{pmatrix} a_0 & b_0 \\ c_0 & d_0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -\frac{c}{d} & \frac{1}{d} \end{pmatrix}\begin{pmatrix} \xi'_- \\ \xi_- \end{pmatrix}$$

$$= \begin{pmatrix} a_0 - b_0\frac{c}{d} & \frac{b_0}{d} \\ c_0 - d_0\frac{c}{d} & \frac{d_0}{d} \end{pmatrix}\begin{pmatrix} \xi'_- \\ \xi_- \end{pmatrix}$$

$$\begin{pmatrix} a_> & b_> \\ c_> & d_> \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} d_0 & -b_0 \\ -c_0 & a_0 \end{pmatrix} = \begin{pmatrix} ad_0 - bc_0 & -ab_0 + ba_0 \\ cd_0 - dc_0 & -cb_0 + da_0 \end{pmatrix}$$

$$\boxed{\begin{pmatrix} p_0 \\ q_0 \end{pmatrix} = \begin{pmatrix} \frac{d_>}{d} & \frac{b_0}{d} \\ -\frac{c_>}{d} & \frac{d_0}{d} \end{pmatrix}\begin{pmatrix} \xi'_- \\ \xi_- \end{pmatrix}}$$

$$\cancel{\begin{pmatrix} \xi'_- \\ \xi_- \end{pmatrix} = \begin{pmatrix} d_0 & -b_0 \\ -c_0 & a_0 \end{pmatrix}\begin{pmatrix} p_0 \\ q_0 \end{pmatrix}}$$

$$\begin{pmatrix} \xi'_- \\ \xi'_+ \end{pmatrix} = \begin{pmatrix} d_0 & -b_0 \\ -c_0 & a_0 \end{pmatrix}\begin{pmatrix} p_0 \\ q_0 \end{pmatrix}$$

$$\begin{pmatrix} \xi'_- \\ \xi_- \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -\frac{c}{d} & \frac{1}{d} \end{pmatrix}^{-1}\begin{pmatrix} d_0 & -b_0 \\ -c_0 & a_0 \end{pmatrix}\begin{pmatrix} p_0 \\ q_0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ c & d \end{pmatrix}\begin{pmatrix} d_0 & -b_0 \\ -c_0 & a_0 \end{pmatrix}\begin{pmatrix} p_0 \\ q_0 \end{pmatrix}$$

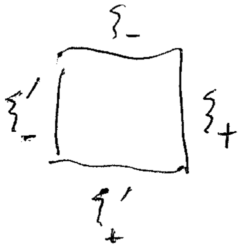
$$\begin{pmatrix} d_0 & -b_0 \\ cd_0 - dc_0 & da_0 - cb_0 \end{pmatrix}$$

$$\begin{array}{cc} c_> & d_> \end{array}$$

$$\boxed{\begin{pmatrix} \xi'_- \\ \xi_- \end{pmatrix} = \begin{pmatrix} d_0 & -b_0 \\ c_> & d_> \end{pmatrix}\begin{pmatrix} p_0 \\ q_0 \end{pmatrix}}$$

$$\begin{vmatrix} d_0 & -b_0 \\ c_> & d_> \end{vmatrix} = d$$

Next work out $\begin{pmatrix} p_0 \\ q_0 \end{pmatrix}$ in terms of $\begin{pmatrix} \xi_+ \\ \xi'_+ \end{pmatrix}$



$$\begin{pmatrix} \xi_+ \\ \xi_- \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \xi'_- \\ \xi'_+ \end{pmatrix} \qquad \begin{pmatrix} \xi'_- \\ \xi'_+ \end{pmatrix} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \begin{pmatrix} \xi_+ \\ \xi_- \end{pmatrix}$$

$$\begin{pmatrix} \xi_+ \\ \xi'_+ \end{pmatrix} = \begin{pmatrix} \frac{1}{d} & \frac{b}{d} \\ -\frac{c}{d} & \frac{1}{d} \end{pmatrix} \begin{pmatrix} \xi'_- \\ \xi_- \end{pmatrix} \qquad \begin{pmatrix} \xi'_- \\ \xi_- \end{pmatrix} = \begin{pmatrix} \frac{1}{a} & -\frac{b}{a} \\ +\frac{c}{a} & \frac{1}{a} \end{pmatrix} \begin{pmatrix} \xi_+ \\ \xi'_+ \end{pmatrix}$$

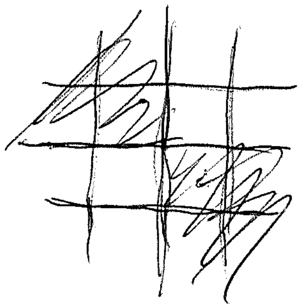
$$\begin{pmatrix} p_0 \\ q_0 \end{pmatrix} = \begin{pmatrix} d_> & -b_> \\ -c_> & a_> \end{pmatrix} \begin{pmatrix} \xi_+ \\ \xi_- \end{pmatrix} = \begin{pmatrix} d_> & -b_> \\ -c_> & a_> \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \frac{c}{a} & \frac{1}{a} \end{pmatrix} \begin{pmatrix} \xi_+ \\ \xi'_+ \end{pmatrix}$$

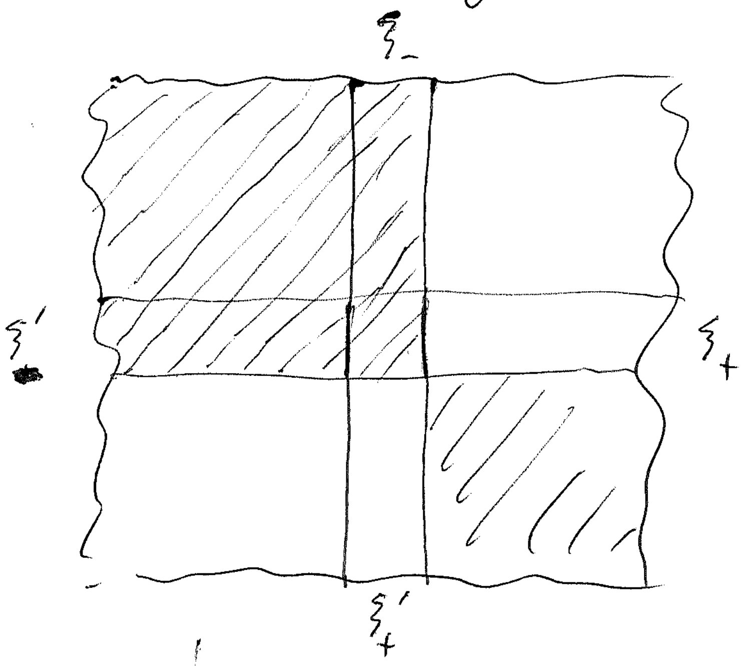
$$= \begin{pmatrix} d_> - b_> \frac{c}{a} & -b_> \frac{1}{a} \\ -c_> + a_> \frac{c}{a} & \frac{a_>}{a} \end{pmatrix} \begin{pmatrix} \xi_+ \\ \xi'_+ \end{pmatrix}$$

$$\begin{pmatrix} a_0 & b_0 \\ c_0 & d_0 \end{pmatrix} = \begin{pmatrix} d_> & -b_> \\ -c_> & a_> \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} d_> a - b_> c & d_> b - b_> d \\ -c_> a + a_> c & -c_> b + a_> d \end{pmatrix}$$

$$\begin{pmatrix} p_0 \\ q_0 \end{pmatrix} = \begin{pmatrix} \frac{a_0}{a} & -\frac{b_>}{a} \\ \frac{c_0}{a} & \frac{a_>}{a} \end{pmatrix} \begin{pmatrix} \xi_+ \\ \xi'_+ \end{pmatrix} = \frac{1}{a} \underbrace{\begin{pmatrix} a_0 & -b_> \\ c_0 & a_> \end{pmatrix}}_{\begin{pmatrix} zH_- & H_- \\ zH_- & zH_- \end{pmatrix}} \begin{pmatrix} \xi_+ \\ \xi'_+ \end{pmatrix}$$

What needs understanding. You have complementary subspaces, outgoing and incoming, depending on a grid vertex. think of these as light cones





outgoing $\quad H_+\xi'_- + H_+\xi_-$

incoming $\quad H_-\xi_+ + H_-\xi'_+$

these should be complementary.

because $\left(H_-\xi_+ \overset{\perp}{\oplus} H_-\xi'_+\right)^\perp = H_+\xi_+ \overset{\perp}{\oplus} H_+\xi'_+$

and I have shown these are both equal to $H_+p_0 + H_+q_0$. Then you have a 2 par. family

$$z^n H_+\xi'_- + z^m H_+\xi_-$$

$$z^n H_-\xi_+ + z^m H_-\xi'_+$$

half an hour to ~~make some~~ understand better 112
~~the~~ S, H_+, H_- situation. There are two cases to look at, ~~and~~ to unify, to find a common framework for, namely rank 1 and 2 - Main idea is the vector bundle ~~over~~ $\mathbb{P}^1$ with clutching function S. What you normally do is to ~~form~~ twist by the line bundle $\mathcal{O}(n)$ and ~~look at~~ form cohomology. ~~But~~ In ~~this scattering~~ the rank 2 situation it seems you twist by a line bundle of degree 0. Maybe this is why you need determinants.

What you need to do now is ~~to~~ to fix

$$S = \begin{pmatrix} \alpha & \beta \\ \gamma & \alpha \end{pmatrix}$$

~~unitary~~ unitary isom.

Intrinsically you have the Hilbert space E of finite energy states ~~and~~ for the Dirac eqn. and incoming and outgoing repns.

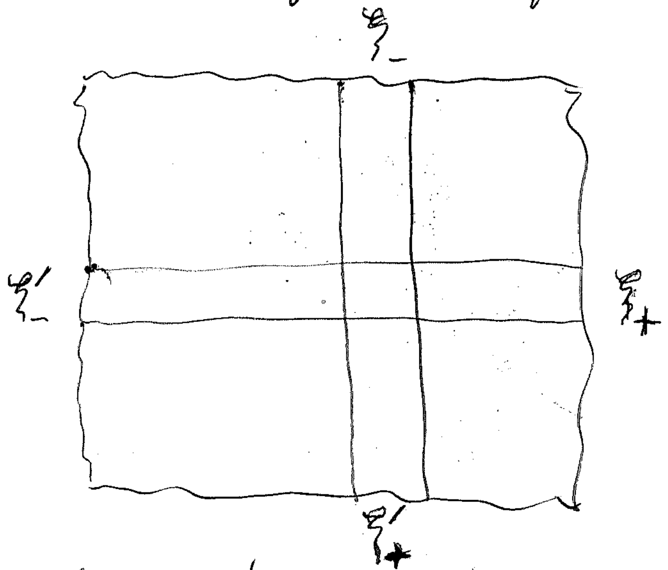
$$L^2(S^1)^{\oplus 2} \xrightarrow[\sim]{\begin{pmatrix} \xi'_- \\ \xi_- \end{pmatrix}} E \xleftarrow[\sim]{\begin{pmatrix} \xi_+ \\ \xi'_+ \end{pmatrix}} L^2(S^1)^{\oplus 2}$$

$$\begin{pmatrix} \xi_+ \\ \xi_- \end{pmatrix} = \begin{pmatrix} & \beta \\ & \end{pmatrix}\begin{pmatrix} \xi'_- \\ \xi_- \end{pmatrix} \qquad \xi_-^* \xi_+ = \beta$$

~~But~~ The problem is what to do with this situation. ~~~~ You have (E, u) and the incoming and outgoing representations. Something like the following should work. Namely you keep (ξ_-, ξ'_-) fixed but change (ξ_+, ξ'_+) to $(z^m \xi_+, z^n \xi'_+)$. Not quite correct.

~~Criteria~~ Properties desired - Action of $\mathbb{Z} \times \mathbb{Z}$ trivial for $\Delta\mathbb{Z}$.

Look for a formula for h_0.



$$\begin{pmatrix} \xi_+ \\ \xi_- \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \xi'_- \\ \xi'_+ \end{pmatrix}$$

$$\begin{pmatrix} \xi_+ \\ \xi'_+ \end{pmatrix} = \begin{pmatrix} \frac{1}{d} & \frac{b}{d} \\ -\frac{c}{d} & \frac{1}{d} \end{pmatrix} \begin{pmatrix} \xi'_- \\ \xi_- \end{pmatrix}$$

Go back to outgoing $\qquad H_+ \xi'_- + H_+ \xi_-$

and ~~[illegible]~~ complementary incoming $\qquad H_- \xi_+ + H_- \xi'_+$

Is there something you can say about S, H_+, H_- in the rank 1 case. Factorization? Assume that

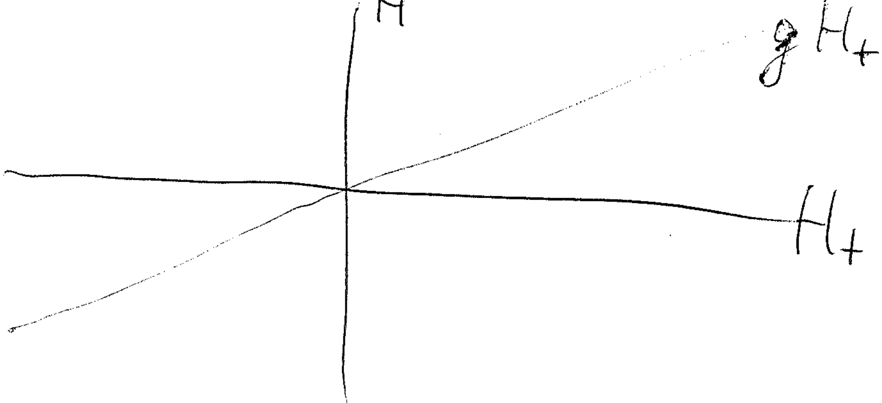
$H_+ \oplus S H_- \xrightarrow{\simeq} L^2$. Write

Given $g = \delta : S^1 \to U(1)$, look at Toeplitz op.

$$H_+ \subset H \xrightarrow{g} H \twoheadrightarrow H_+$$

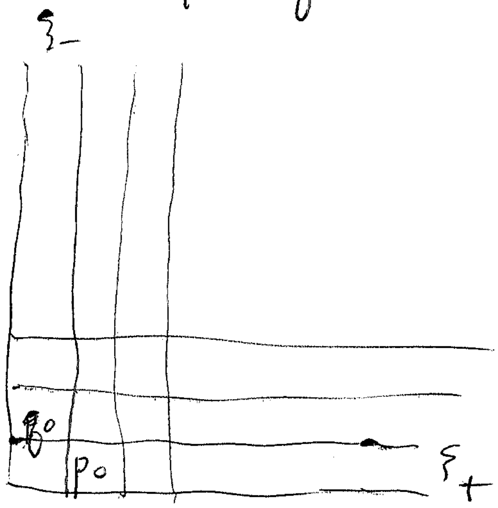
$$\text{Kernel} = H_+ \cap g^{-1} H_- \xrightarrow{\sim} gH_+ \cap H_-$$

$$\text{Img} = gH_+ + H_- / H_- \subset H/H_- = H_+$$



Look again at $L^2(S^1, d\mu)$ $\quad d\mu = \frac{1}{|q|^2}\frac{d\theta}{\text{norm}}$

$\xi_0 = 1.$ $\quad \xi_+ = \bar{q}$ $\quad \xi_- = q$



$(z^k \xi_- | \xi_+) =$

$\int z^{-k} \bar{q}\bar{q} \frac{1}{|q|^2} \frac{d\theta}{~~norm~~}$

$= \int z^{-k} \frac{\bar{q}}{q} \frac{d\theta}{2\pi}$

$\beta = \frac{\bar{q}}{q}$ You want to somehow work inside $L^2(S^1, d\mu)$ and attach incoming + outgoing subspaces.

Maybe the way to proceed goes as follows. You have this loop $S: S^1 \to u(1)$ interacting with the splitting $L^2 = H_+ \oplus H_-$. You want ~~the~~ asymptotics. ~~How~~

$L^2(S^1, d\mu)$ $\quad d\mu = \frac{1}{|q|^2}\frac{d\theta}{2\pi}$ $\quad$ q norm $\ni \int d\mu = 1.$

$\xi_0 = 1$ $\quad q$

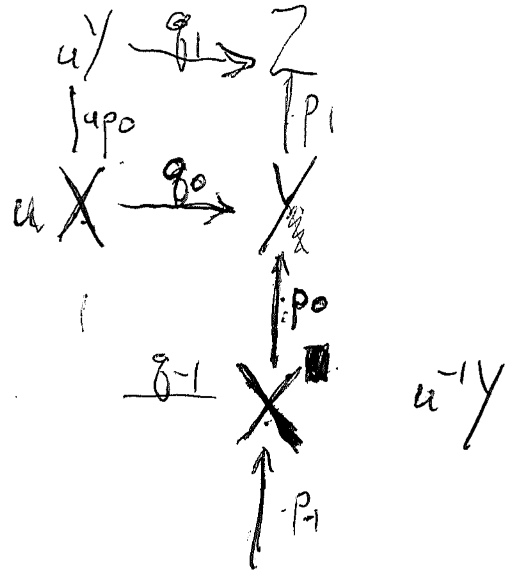
---

continue with the study of things not well understood. ~~Why not~~ Go back to partial unitary $Y = aX + V_+ = bX + V_-$

Picture is $\quad uY$

Y

Picture u^2Y



$$Y = X \oplus V_+ \quad = uX \oplus V_-$$

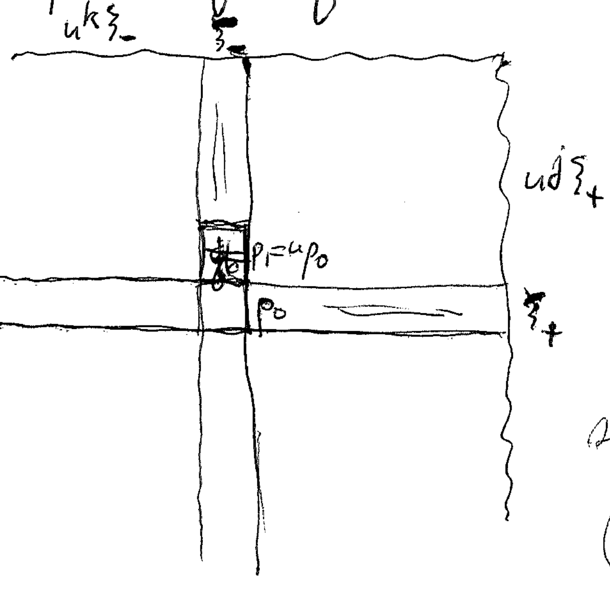
General remark is that you do not ~~make much~~ use eigenvectors very much. Can you use contractions?

Fix ~~notation~~ $\quad Y = X_0 \qquad X = X_{-1} \qquad Z = X_1$

Assume given $\quad X_{-1} \underset{b=ua}{\overset{a=\text{inc.}}{\rightrightarrows}} X_0 \qquad c = a^*b$

You propose to have $h_n = 0 \quad n \geq 1$, so that

$p_n = u^n p_0 \quad q_n = q_0 \quad n \geq 0$. What happens.



$$(u^k\xi_- \mid u^j\xi_+) = (u^{k-j}\xi_- \mid \xi_+)$$
$$= (z^{k-j} \mid \beta)$$

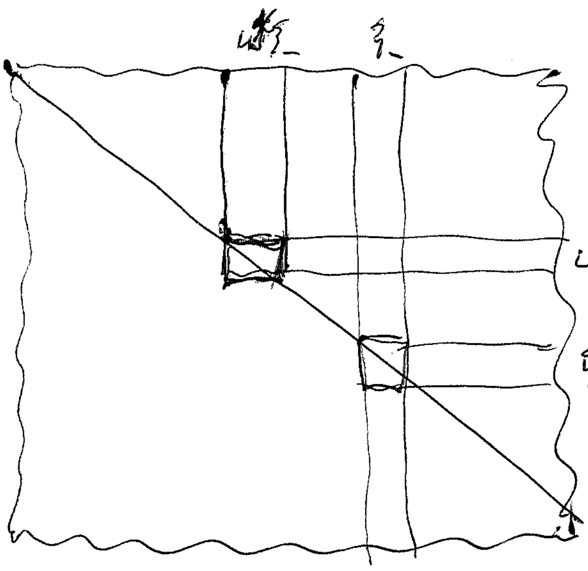
this $= 0$ if $k-j > 0$.

Here is a nice situation where $\beta \in zH_-$

$$\begin{pmatrix} \alpha & \beta \\ -\frac{\alpha}{\bar\alpha}\bar\beta & \alpha \end{pmatrix}$$

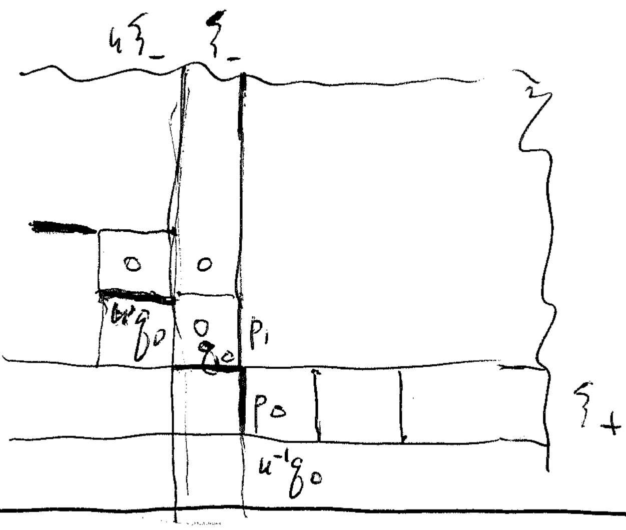
First case $\beta = \sum \beta_n z^n$ with $\beta \in H_+$

$(u^k \xi_- | u^j \xi_+) = \beta_{k-j} = 0$ for $k-j<0$



seems to be the case $h_n \equiv 0$ for $n>0$.

So $\beta(z)$ analytic seems to be the situation for a b-prod.



You really ought to look at eigenfunctions. For each $z \neq 0, \infty$ you have a 2 dim'l space of eigenfunctions

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Go over contractions.

$$Y = aX \oplus \mathbb{C}\xi_+ = bX \oplus \mathbb{C}\xi_-$$

$$\begin{tikzcd} uX \arrow[r, "b"] & Y \\ & X \arrow[u, "a"] \arrow[ul, "u"] \end{tikzcd}$$

At the moment you have located partial unitaries within scattering framework. In terms of the grid picture, you have $h_1 = h_2 = \dots = 0$

so that $p_n = u^n p_0$ for $n \geq 0$ $\qquad \xi_+ = p_0$

$q_n = q_0$ $\qquad \xi_- = q_0$

The partial unitary admits ~~certain~~ extensions to a contraction. $c_0 = a^* u a = a^* b$ NO you an operator on Y, namely $c_0 = u a a^* = b a^*$

So $c_h = ba^* + \xi_- h \xi_+^*$. $|h| \le 1$

parametrizes extensions of (a,b) to a contraction.
Resolvent of $c = c_h$. Deal with c, c^*. There
are two resolvents $\frac{1}{z-c}$ and $\frac{1}{1-zc^*}$ to consider

$$\frac{1}{z-c} = \frac{z^{-1}}{1-z^{-1}c} = \sum_{n\ge 0} z^{-n-1}c^n \quad \text{defined for } |z|>1 \text{ always}$$

$$\frac{1}{1-zc^*} = \sum_{n\ge 0} z^n c^{*n} \quad \text{defined for } |z|<1. \text{ —.}$$

~~Set~~ isometric embedding. Let $y \in Y$.

$$\xi_+^* \frac{1}{z-c_0} y = \sum_{n\ge 0} z^{-n-1} \xi_+^* c_0^n y \quad \in L^2(S^1)$$

$$1 - c_0^* c_0 = 1 - (ba^*)^*(ba^*) = 1 - aa^* = \xi_+ \xi_+^*$$

$$\left\| \xi_+^* \frac{1}{z-c_0} y \right\|^2 = \sum_{n\ge 0} \underbrace{\| \xi_+^* c_0^n y \|^2}_{(c_0^n y \mid (1-c_0^*c_0) c_0^n y)} = \|y\|^2 - \lim_{n\to\infty} \|c_0^n y\|^2$$

So you have $y \mapsto \xi_+^* \frac{1}{z-c_0} y$, $Y \longrightarrow L^2(S^1)$ is
an isometric embedding iff $c_0^n y \longrightarrow 0$ for all y.

Replace c_0 by

$$c_h = ba^* + \xi_- h \xi_+^*$$

$$c_h^* = ab^* + \xi_+ \bar{h} \xi_-^*$$

$$c_h^* c_h = aa^* + \xi_+ |h|^2 \xi_+^* = aa^* + (1-aa^*)|h|^2$$

$$1 - c_h^* c_h = 1 - aa^* - (1-aa^*)|h|^2 = (1-aa^*)(1-|h|^2) = (1-|h|^2)\xi_+\xi_+^*$$

To get an isometric embedding you want

$$\frac{1}{\sqrt{1-|h|^2}} \xi_+^* \frac{1}{z-c_h} \quad \text{not ~~anal~~ holom in } h.$$

Here's something to check, that this semi-embedding condition holds in a scattering situation. Work 3 hours.

You need to find a way to control things. Find something to say. New point is contraction, ~~all~~ almost unitary contraction. Resolvent of such. Algebra

$$\sum_{n\geq 0} z^{-n} C^n + \sum_{n\geq 0} z^n C^{*n}$$

$$= \frac{z^{-1}C}{1-z^{-1}C} + \frac{1}{1-zC^*}$$

$$= \frac{1}{1-z^{-1}C}\Big(\underbrace{z^{-1}C(1-zC^*) + 1 - z^{-1}C}_{1-CC^*}\Big)\frac{1}{1-zC^*}$$

$$= \frac{1}{1-zC^*}\Big(\underbrace{(1-zC^*)z^{-1}C + 1 - z^{-1}C}_{1-C^*C}\Big)\frac{1}{1-z^{-1}C}$$

$$\boxed{\begin{aligned}\sum_{n\geq 0} z^{-n}C^n + \sum_{n\geq 1} z^n C^{*n} &= \frac{1}{1-z^{-1}C}(1-CC^*)\frac{1}{1-zC^*}\\ &= \frac{1}{1-zC^*}(1-C^*C)\frac{1}{1-z^{-1}C}\end{aligned}}$$

Is there some way to exploit the ~~Rey~~ picture: functions on S^1, $C(S^1)$, a contraction β in this algebra, and the Hardy splitting $H = H_+ \oplus H_-$, Hilbert transform!

Abstract the game you are playing. E Hilb. space with u, subspace Y. General algebra involves free product $\mathbb{C}[u,u^{-1}] * \mathbb{C}[F]$, which is $QA \otimes \mathbb{C}[F]$, $A = \mathbb{C}[u,u^{-1}]$. General Y is complicated. You want simple stuff

---

return to $T_{0,-\infty} = \begin{pmatrix} a_0 & b_0 \\ c_0 & d_0 \end{pmatrix} = T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$$\cap$$

$$\begin{pmatrix} uH_- & H_+ \\ uH_- & H_+ \end{pmatrix}$$

$$\xi_+ = \frac{1}{d}\xi'_- + \beta_\cdot \xi_-$$

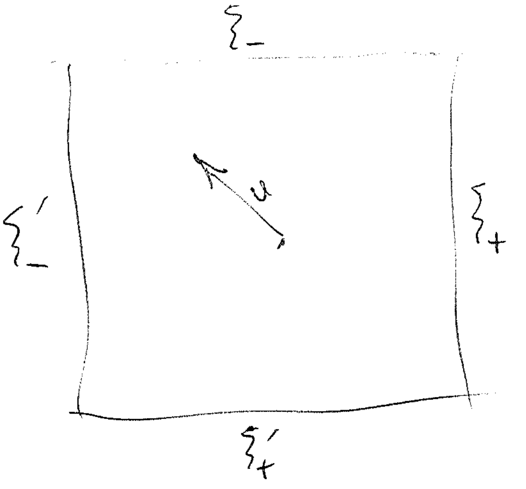
you have $\begin{pmatrix} p_0 \\ q_0 \end{pmatrix} = \begin{pmatrix} \xi_+ \\ \xi_- \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \xi'_- \\ \xi'_+ \end{pmatrix}$ $\qquad \begin{pmatrix} \xi_+ \\ \xi'_+ \end{pmatrix} = \begin{pmatrix} \frac{1}{d} & \frac{b}{d} \\ -\frac{c}{d} & \frac{1}{d} \end{pmatrix} \begin{pmatrix} \xi'_- \\ \xi_- \end{pmatrix}$

Digress.

$$\begin{pmatrix} \xi_+ \\ \xi_- \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \xi'_- \\ \xi'_+ \end{pmatrix} \qquad \begin{pmatrix} \xi'_- \\ \xi'_+ \end{pmatrix} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \begin{pmatrix} \xi_+ \\ \xi_- \end{pmatrix}$$

$$\begin{pmatrix} \xi_+ \\ \xi'_+ \end{pmatrix} = \begin{pmatrix} \frac{1}{d} & \frac{b}{d} \\ -\frac{c}{d} & \frac{1}{d} \end{pmatrix} \begin{pmatrix} \xi'_- \\ \xi_- \end{pmatrix} \qquad \begin{pmatrix} \xi'_- \\ \xi_- \end{pmatrix} = \begin{pmatrix} \frac{1}{a} & -\frac{b}{a} \\ \frac{c}{a} & \frac{1}{a} \end{pmatrix} \begin{pmatrix} \xi_+ \\ \xi'_+ \end{pmatrix}$$

indeed conjugate transpose



~~You want something clean.~~
perturbation stuff.

$$c_h = ba^* + \xi_- h \xi_+^*$$

$$Y = aX \oplus \mathbb{C}\xi_+ = bX \oplus \mathbb{C}\xi_-$$

$$a^*a = b^*b = 1_X, \quad ua = b.$$

$$c_h^* c_h = aa^* + \xi_+ |h|^2 \xi_+^*$$

$$1 = aa^* + \xi_+ \xi_+^*$$

$$1 - c_h^* c_h = \xi_+ (1 - |h|^2) \xi_+^*$$

$$\xi_+^* \frac{1}{z - c_h} = \xi_+^* \frac{1}{z - c_0} + \xi_+^* \frac{1}{z - c_0} \xi_- h \xi_+^* \frac{1}{z - c_0} + \cdots$$

$$+ \left(\xi_+^* \frac{1}{z - c_0} \xi_-\right) h \left(\xi_+^* \frac{1}{z - c_0} \xi_- h\right) \xi_+^* \frac{1}{z - c_0} +$$

$$\xi_+^* \frac{1}{z - c_h} = \frac{1}{1 - \underbrace{\xi_+^* \frac{1}{z - c_0} \xi_-}_{S_0(z^{-1})} h} \xi_+^* \frac{1}{z - c_0}$$

can this be β somehow

Answer is no because $\xi_+^* \frac{1}{z - c_0} \xi_-$ should be in H_- analytic in D_- vanish at ∞.

~~Mistake~~ Check $\quad c_h = c_0 + \delta$

$$\frac{1}{z - c_h} = \frac{1}{z}\left(\frac{1}{1 - z^{-1}c_0} + \frac{1}{1 - z^{-1}c_0} z^{-1}\delta \frac{1}{1 - z^{-1}c_0} + \cdot\right.$$

$$= \frac{1}{z - c_0} + \frac{1}{z - c_0} \delta \frac{1}{z - c_0} +$$

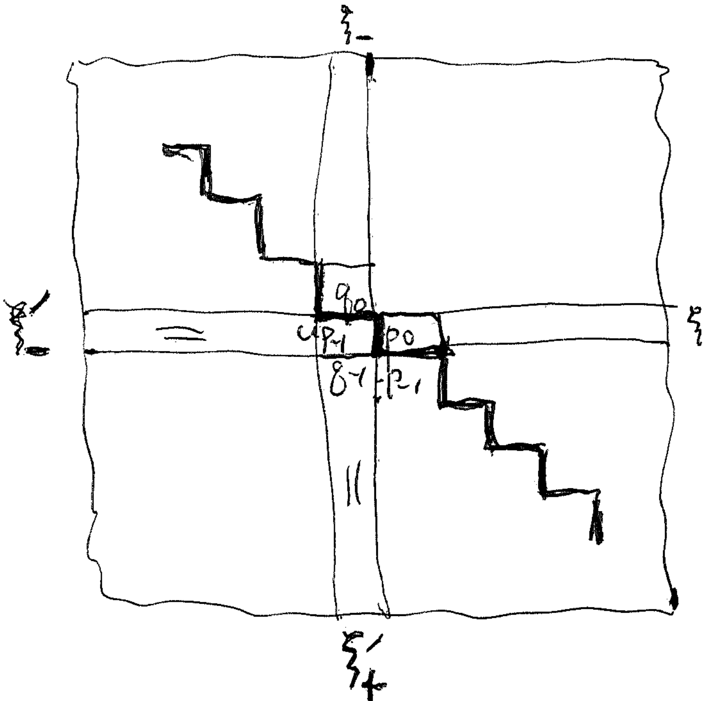
Put $\quad S_h(z^{-1}) = \xi_+^* \frac{1}{z - c_h} \xi_-$. Then

$$S_h(z^{-1}) = \frac{1}{1 - S_0(z^{-1}) h} S_0(z^{-1})$$

$$\boxed{1 + S_h(z^{-1}) h = \frac{1}{1 - S_0(z^{-1}) h}}$$

Relate these functions to the scattering data.

Connect $L^2(S^1, d\mu)$ with scattering. you want to take $L^2(S^1,d\mu)$ and the orth poly system $\begin{pmatrix} p_n \\ q_n \end{pmatrix}$ inside, ~~and~~ then understand the ~~scattering~~ associated partial unitary. My idea is ~~[illegible]~~ to glue $L^2(S^1, d\mu)$. Picture should be:



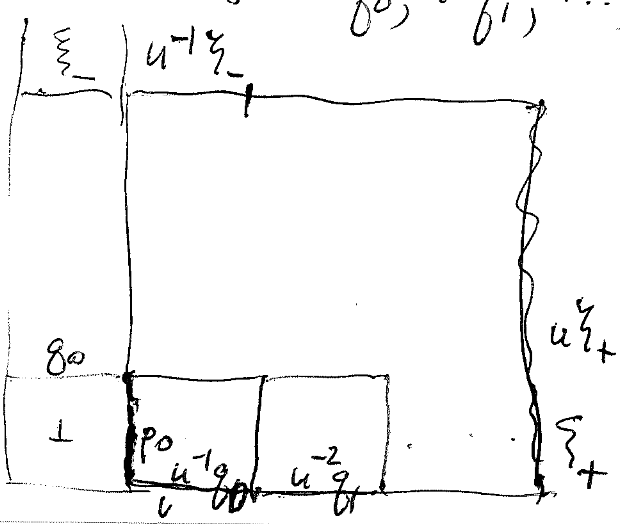
To constuct this picture you dilate the partial unitary. What is Y and X etc.

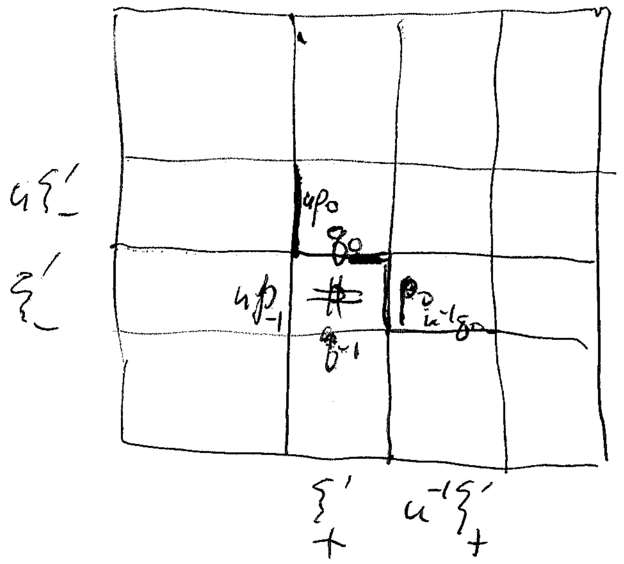
Inside $L^2(S^1, d\mu)$ $\qquad d\mu = \frac{1}{|g|^2}\frac{d\theta}{2\pi}$ $\qquad g$ norm so that $\int d\mu = 1$.

Subspaces $F_n = [z^0, \dots, z^n]$ $\qquad F_\infty = \langle p_0, p_1, \dots \rangle$

rest of orth basis is $u^{-1}\xi_-, u^{-2}\xi_-, \dots$

other orth basis is $u^{-1}q_0, u^{-2}q_1, \dots; \xi_+, u\xi_+, \dots$ So you have





One thing worth saying is that

$\langle p_0, p_1, \dots \rangle = \mathbb{C}[u]p_0$

OKAY whence $q_0 = h_0 p_0$

$h_0 \neq 0$.

Puzzle: If $p_0 = \xi'_-$, then u

Say given DE ~~with for~~ involving $\begin{pmatrix} p_n \\ q_n \end{pmatrix}$ for $n \geq 0$ ~~so~~ so $h_1, h_2, \dots$ given and maybe h_0 for a bdry condition. Then you ~~want~~ have a propagator whence $\begin{pmatrix} \xi_+ \\ \xi_- \end{pmatrix} = \begin{pmatrix} a_> & b_> \\ c_> & d_> \end{pmatrix} \begin{pmatrix} p_0 \\ q_0 \end{pmatrix}$ where

$\begin{pmatrix} a_> & b_> \\ c_> & d_> \end{pmatrix} \in \begin{pmatrix} zH_- & H_- \\ zH_+ & H_+ \end{pmatrix}$

$$\begin{pmatrix} p_0 \\ q_0 \end{pmatrix} = \begin{pmatrix} d_> & -b_> \\ -c_> & a_> \end{pmatrix} \begin{pmatrix} \xi_+ \\ \xi_- \end{pmatrix}$$

$$\begin{pmatrix} \xi_+ \\ \xi_- \end{pmatrix} = \underbrace{\begin{pmatrix} a_> & b_> \\ c_> & d_> \end{pmatrix} \frac{1}{h_0} \begin{pmatrix} 1 & h_0 \\ h_0 & 1 \end{pmatrix}} \underbrace{\begin{pmatrix} up_{-1} \\ q_{-1} \end{pmatrix}}_{\begin{pmatrix} \xi'_- \\ \xi'_+ \end{pmatrix}}$$

What to hope for?

$$\begin{pmatrix} zH_- & zH_- \\ H_+ & H_+ \end{pmatrix} \ni \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$\begin{pmatrix} \xi_+ \\ \xi'_+ \end{pmatrix} = \begin{pmatrix} \frac{1}{d} & \frac{b}{d} \\ -\frac{c}{d} & \frac{1}{d} \end{pmatrix} \begin{pmatrix} \xi'_- \\ \xi_- \end{pmatrix}$$

$\in H_+$

This is the ref. coeff. you build E from. Should coincide with response of the contraction.

Recap. ~~Ass~~ The problem: ~~Gt Case~~ Fit ~~past~~ work on partial unitaries, contractions into DE. Combine them. Start with ~~...~~ (h_n)

$h_n = 0$ for $n < 0$. ~~Treat h_0 as bdry condn~~

Vary h_0, you want to allow $h_0 \nearrow 1$, perturbation

Assume (h_n) summable.

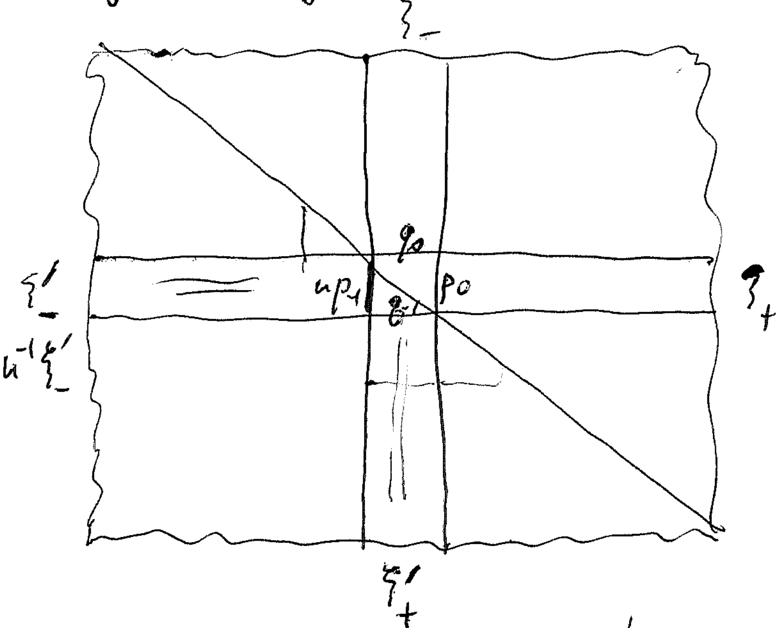
$$\begin{pmatrix} \xi_+ \\ \xi_- \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \xi'_- \\ \xi'_+ \end{pmatrix} = \begin{pmatrix} a_> & b_> \\ c_> & d_> \end{pmatrix} \frac{1}{k_0} \begin{pmatrix} 1 & h_0 \\ \bar h_0 & 1 \end{pmatrix} \begin{pmatrix} \xi'_- \\ \xi'_+ \end{pmatrix}$$

$$\begin{pmatrix} zH_- & zH_- \\ H_+ & H_+ \end{pmatrix} \qquad \begin{pmatrix} zH_- & H_- \\ zH_+ & H_+ \end{pmatrix} \qquad \begin{pmatrix} \xi_+ \\ \xi'_+ \end{pmatrix} = \begin{pmatrix} \frac{1}{d} & \frac{b}{d} \\ -\frac{c}{d} & \frac{1}{d} \end{pmatrix} \begin{pmatrix} \xi'_- \\ \xi_- \end{pmatrix}$$

$\gamma = -\frac{c}{d} \in H_+$ is the reflection coeff. $\xi'_+ = \gamma \xi'_- + \frac{1}{d}\xi_-$

so $(u^k \xi'_- \mid u^j \xi'_+) = (u^{k-j}\xi'_- \mid \gamma \xi'_-) = (z^{k-j} \mid \gamma)_{L^2}$

$= 0$ if $k - j < 0$.



all squares ~~below~~ are $\perp$ below the diagonal

Now ~~partial~~ find the contraction c_0, identify $\gamma(z)$ with the response of the partial unitary

Can you prove that h_n rapidly decaying $\Longrightarrow$ γ smooth or analytic.

Go back to $S(z)$ a smooth loop in $U(1)$ of degree 0, whence Birkhoff factorization $S(z) = \frac{\bar q}{q}$ with ~~....~~ q invertible on H_+.

Then how to proceed? Case $p_0 = q_0$

What happens is that $\xi'_- = \xi'_+$ i.e. $\gamma = 1$.

S-matrix becomes degenerate.

Take $p_0 = q_0 = 1$. $\xi_- = q_\infty = q$, $\xi_+ = \bar{q}$

so $(u^k \xi_- | u^j \xi_+) = (u^{k-j}\xi_- | \xi_+) = \int \overline{z^{k-j}}\, \bar{q}\, \bar{q}\, \frac{1}{|q|^2} \frac{d\theta}{2\pi}$

$= (z^{k-j} | S)$

$\xi_+ = S(u)\xi_-$

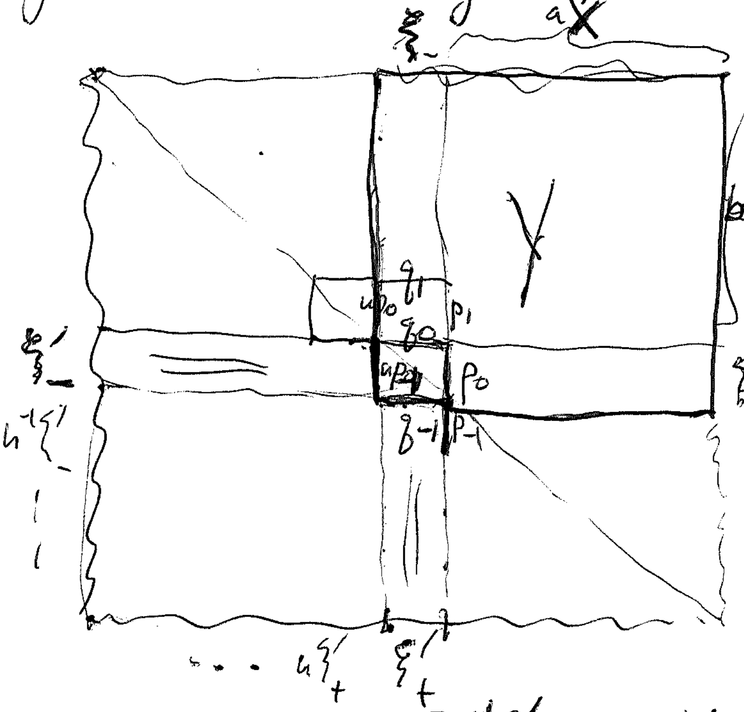
What is the bifiltration in this case?

$H_+ \xi_- \quad z^n H_- \xi_+$

want $\perp$: $H_- \xi_- + z^n H_+ \xi_+ \simeq H_- + z^n H_+ \frac{\bar{q}}{q}$

$$H_- + z^n H_+ \xrightarrow[\cdot \bar{q}]{\sim} H_- + z^n H_+ \frac{\bar{q}}{q}$$

First clean up relation between various extensions of a partial unitary $Y = aX \oplus \underbrace{\mathbb{C}\xi'_+}_{q_{-1}} = bX \oplus \underbrace{\mathbb{C}\xi'_-}_{up_{-1}}$



$$aX \begin{pmatrix} p_0 \\ q_0 \end{pmatrix} = \frac{1}{k_0} \begin{pmatrix} 1 & h_0 \\ \bar{h}_0 & 1 \end{pmatrix} \begin{pmatrix} up_{-1} = \xi'_- \\ q_{-1} = \xi'_+ \end{pmatrix}$$

Recall the picture of the dilation.

$\mathbb{C}u^{-1}\xi'_- \oplus aX \oplus \mathbb{C}\xi'_+ \oplus \mathbb{C}u\xi'_+ \oplus \cdots$

$\mathbb{C}\xi'_- \oplus bX$

$SL_2(\mathbb{Z})$ graph of unit vectors in a Hilbert space?

So conclude

$$Y = H_+\xi_+ + \cancel{z}H_-\xi_-$$

$$aX = H_+\xi_+ + H_-\xi_-$$

$$bX = zH_+\xi_+ + zH_-\xi_-$$

$$Y = aX \oplus \mathbb{C}\xi_+' = bX \oplus \mathbb{C}\xi_-' \qquad \text{good.}$$

Note that $\varphi_{-1} = \xi_+'$, $\xi_-' = u\varphi_{-1}$ are given by the orthogonality relations involving $(u^k\xi_- \mid \xi_+) = \beta_k$ for $k<0$. Maybe $k=0$ also if $h_0 \neq 0$.

On the other hand [redacted] we can build up E using the subspaces $H_+\xi_+' + z^n z H_-\xi_-'$ and the reflection $(u^k\xi_-' \mid u^j\xi_+') = \gamma_{k-j}$. Check the formulas

$$\begin{pmatrix}\xi_+\\ \xi_-\end{pmatrix} = \begin{pmatrix} a & b\\ c & d\end{pmatrix}\begin{pmatrix}\xi_-'\\ \xi_+'\end{pmatrix} \qquad \begin{pmatrix}\xi_+\\ \xi_+'\end{pmatrix} = \underbrace{\begin{pmatrix}\frac{1}{d} & \frac{b}{d}\\ -\frac{c}{d} & \frac{1}{d}\end{pmatrix}}_{\begin{pmatrix}\delta & \beta\\ \gamma & \delta\end{pmatrix}}\begin{pmatrix}\xi_-'\\ \xi_-\end{pmatrix}$$

$$\beta = -\frac{\delta}{\bar\delta}\bar\gamma$$

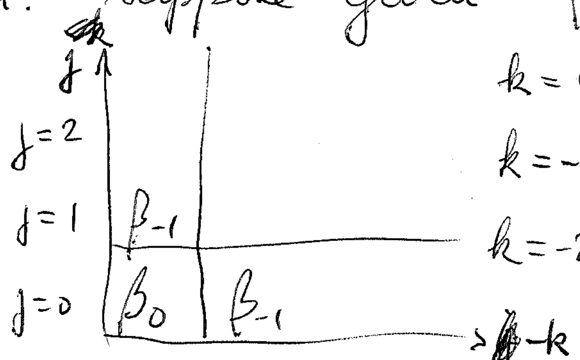
$$\xi_+' = \gamma\xi_-' + \delta\xi_-$$

$$(u^k\xi_-' \mid \xi_+') = (u^k\xi_-' \mid \gamma\xi_-') = \gamma_k.$$

One nice thing is that $\gamma_k = 0$ for $k<0$, even $\gamma_0 = 0$ if $h_0 = 0$.

$$\underset{k\le 0}{} \quad \underset{j\ge 0}{} \quad (u^k\xi_- \mid u^j\xi_+)$$

Question: Suppose given β matrix



$$\begin{array}{c|ccc} & j=0 & j=1 & j=2\\ \hline k=0 & \beta_0 & \beta_{-1} & \beta_{-2}\\ k=-1 & \beta_{-1} & \beta_{-2} & \beta_{-3}\\ k=-2 & \beta_{-2} & \beta_{-3} & \beta_{-4}\end{array}$$

You want this to be a contraction

There is a puzzle here - not as clear as I would like, namely it seems that if you are given a matrix of the form

$$\begin{pmatrix} \beta_0 & \beta_{-1} & \beta_{-2} & \cdots \\ \beta_{-1} & \beta_{-2} & & \\ \beta_{-2} & & & \\ \vdots & & & \end{pmatrix}$$

Hankel matrix

which is a contraction, then you can find coefficients $\beta_1, \beta_2, \ldots$ such that $\beta(z) = \sum \beta_n z^n$ is a contraction on $L^2(S^1)$.

Suppose given $\gamma(z) = \sum' \gamma_n z^n$ $\left(\begin{array}{l} n<0 \\ \gamma_n = 0 \end{array}\right.$

Assume $|\gamma(z)| \leq 1-\varepsilon$ $z \in S^1$. ⊚

Possibly this works - somehow the contraction version of a partial unitary.

Start again. Viewpoint. You want to consider partial unitaries, contractions, besides unitaries

Suppose given $(h_n)_{n \geq 1}$, then you get a partial unitary corresp to the half line DE without the bdry cond.

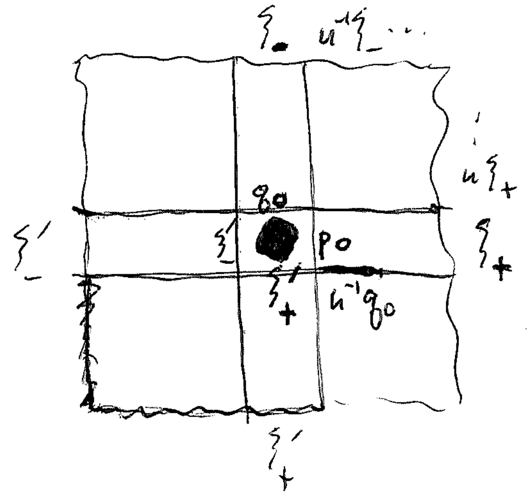
Suppose h_n summable, then get $\begin{pmatrix} a_> & b_> \\ c_> & d_> \end{pmatrix}$

You want to work out the scattering situation

Basically you have $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a_> & b_> \\ c_> & d_> \end{pmatrix} \frac{1}{k_0} \begin{pmatrix} 1 & h_0 \\ \bar{h}_0 & 1 \end{pmatrix}$

$$\begin{pmatrix} \xi_+ \\ \xi_- \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \xi_-' \\ \xi_+' \end{pmatrix} \qquad \begin{pmatrix} zH_- & zH_- \\ H_+ & H_+ \end{pmatrix} \qquad \begin{pmatrix} zH_- & H_- \\ zH_+ & H_+ \end{pmatrix}$$

$$\begin{pmatrix} \xi_+ \\ \xi_+' \end{pmatrix} = \begin{pmatrix} \frac{1}{d} & \frac{b}{d} \\ -\frac{c}{d} & \frac{1}{d} \end{pmatrix} \begin{pmatrix} \xi_-' \\ \xi_- \end{pmatrix}$$



$$Y = H_+\xi_+ + uH_-\xi_-$$
$$X = H_+\xi_+ + H_-\xi_-$$

The problem. To construct Y you need the inner products $(u^k\xi_- | u^j\xi_+)$ $\quad k \le 0$, $j \ge 0$

$\Rightarrow k-j \le 0.$

~~There must~~ There are ~~two~~ (four) constructions of E

E can be constructed as $L^2\xi'_- + L^2\xi'_+$, you inner prod. $\left(u^k\xi'_- \middle| u^j\xi'_+\right)$ $\forall k, j$, but these $= 0$ for $k-j<0$.

$$\left(u^k\xi_- \middle| \xi_+\right) = \left(u^k\xi_- \middle| \cancel{\tfrac{1}{d}\xi'_-} + \tfrac{b}{d}\xi_-\right)$$
$$= \left(z^k \middle| \tfrac{b}{d}\right)$$
$$\left(u^k\xi'_- \middle| \xi'_+\right) = \left(u^k\xi'_- \middle| -\tfrac{c}{d}\xi'_- + \cancel{\tfrac{1}{d}\xi_-}\right) = \left(z^k \middle| -\tfrac{c}{d}\right)$$

$L^2(S^1, d\mu)$ $\qquad d\mu = \frac{1}{|g|^2}\frac{d\theta}{2\pi}$ $\qquad g$ norm $g(0) > 0$ $\qquad \int d\mu = 1.$

$\quad\shortparallel$

$\quad Y$ $\qquad p_0 = q_0 = \xi_0$

Recall that $E = H_-\xi'_- \oplus X \oplus H_+\xi'_+$

First Orth basis for X is $1 = p_0, p_1, \ldots, z^{-1}g, z^{-2}g, \ldots$

If this is true, then $X = L^2(S^1, d\mu)$

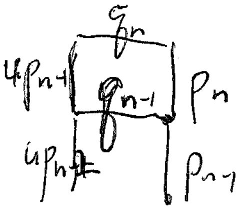
Begin with $L^2(S^1, d\mu)$ $\qquad \int d\mu = 1 \qquad d\mu = \frac{1}{|q|^2}\frac{d\theta}{2\pi}$

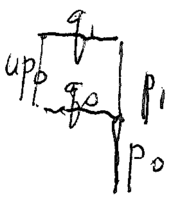
orthog polys $\begin{pmatrix} p_n \\ q_n \end{pmatrix}$ $n \geq 0$. Szegö thm

$q_\infty = q$. Check

$$\int \bar{q}\, z^n \frac{1}{|q|^2}\frac{d\theta}{2\pi}$$

$$= \int z^n \frac{1}{q}\frac{d\theta}{2\pi} = 0 \qquad n > 0.$$





$$L^2(S^1, d\mu) = \overbrace{\langle p_0, p_1, \dots \rangle}^{H_+\xi_0} \overset{\perp}{\oplus} H_-\xi_-$$

$$= \underbrace{\langle u^{-1}q_0, u^{-2}q_1, \dots \rangle}_{H_-\xi_0} \overset{\perp}{\oplus} H_+\xi_+$$

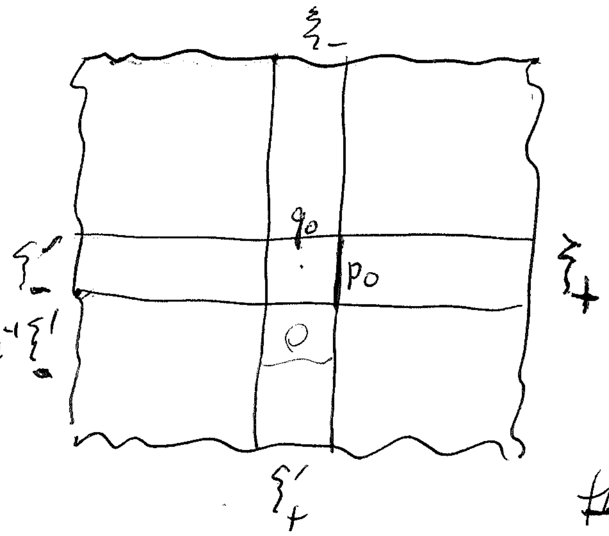
Consider $\begin{pmatrix}\xi_+ \\ \xi_-\end{pmatrix} = \underbrace{\begin{pmatrix} a & b \\ c & d \end{pmatrix}}_{\cap}\begin{pmatrix}\xi'_- \\ \xi'_+\end{pmatrix}$ for potentials (h_n) $\qquad h_n = 0 \quad n<0$

$$\begin{pmatrix} zH_- & zH_- \\ H_+ & H_+ \end{pmatrix} \qquad \begin{pmatrix}\xi_+ \\ \xi'_+\end{pmatrix} = \begin{pmatrix} \frac{1}{d} & \frac{b}{d} \\ -\frac{c}{d} & \frac{1}{d} \end{pmatrix}\begin{pmatrix}\xi'_- \\ \xi_-\end{pmatrix}$$

$\beta = \frac{b}{d}$ determines E

$$\left(u^k \xi_- \,\middle|\, \xi_+\right) = \left(z^k \,\middle|\, \frac{b}{d}\right)$$

$$\left(u^k \xi'_- \,\middle|\, \xi'_+\right) = \left(z^k \,\middle|\, -\frac{c}{d}\right)$$



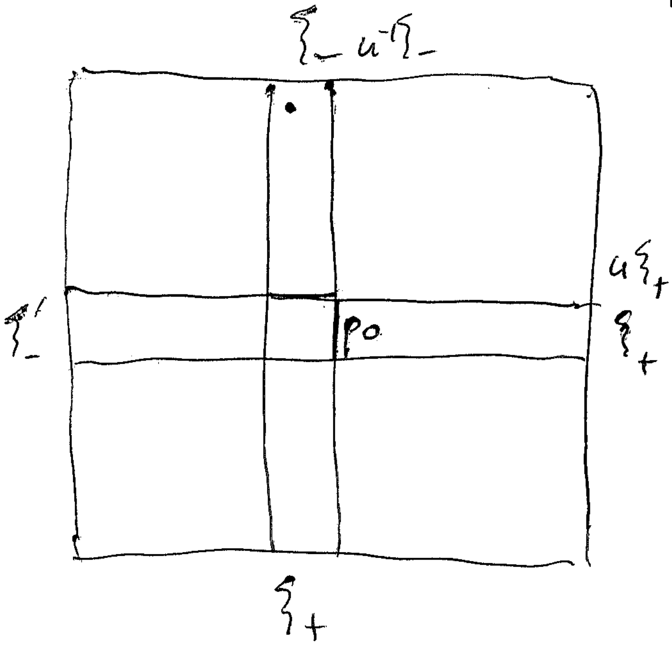
What is the puzzle? The coefficients $\beta_k = (z^k | \frac{b}{d})$ for $k \leq 0$ suffice to determine ~~the subspace $H_+\xi_+ + zH_-\xi_-$~~ E and all its structure. β is a contraction of some sort commuting with u. Toeplitz contraction? What seems to happen is that from $\beta \le 0$ you get a $\gamma_{(0)}$.

Let's look at the process in the other direction.

$$T_{+\infty,-\infty} = T_{\infty,0}\, T_{0,-\infty}$$

$$\begin{pmatrix}\xi_+\\ \xi_-\end{pmatrix} = \underbrace{\begin{pmatrix} a_> & b_> \\ c_> & d_> \end{pmatrix}}_{\cap \begin{pmatrix} zH_- & H_- \\ zH_+ & H_+\end{pmatrix}} \begin{pmatrix} p_0 \\ q_0 \end{pmatrix} \qquad \begin{pmatrix} p_0 \\ q_0\end{pmatrix} = \overbrace{\begin{pmatrix} a_0 & b_0 \\ c_0 & d_0\end{pmatrix}}^{\begin{pmatrix} zH_- & H_+ \\ zH_- & H_+\end{pmatrix}\,\cup} \begin{pmatrix} \xi'_- \\ \xi'_+\end{pmatrix}$$

explain orth relations for p_0



$$p_0 \in H_+ \xi_+ + H_- \xi_-$$
$$q_0 \in zH_+ \xi_+ + zH_- \xi_-$$
$$\begin{pmatrix} p_0 \\ q_0\end{pmatrix} \in \begin{pmatrix} d_> & -b_> \\ -c_> & a_> \end{pmatrix}\begin{pmatrix}\xi_+ \\ \xi_-\end{pmatrix}$$
$$p_0 \perp zH_+ \xi_+ + H_- \xi_-$$
$$q_0 \perp zH_+ \xi_+ + H_- \xi_-$$

$$p_0 = \sum_j d_j u^j \xi_+ - \sum_k b_k u^k \xi_-$$

$b_k = 0$ for $k \geq 0$
$d_j = 0$ for $j < 0$

$$\underset{k<0}{0 =} \left(u^k \xi_- \middle| p_0\right) = \sum_j d_j \beta_{k-j} - b_k = 0$$

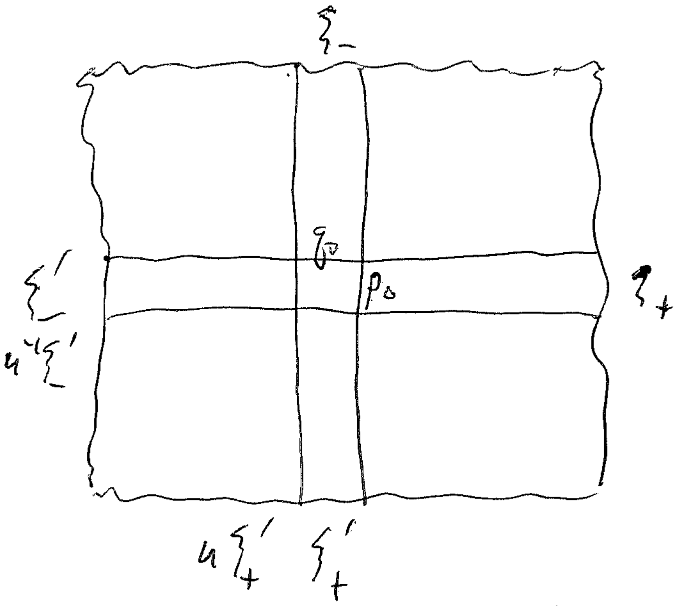
$$\underset{j>0}{0 =} \left(u^j \xi_+ \middle| p_0\right) = d_j - \sum_k b_k \bar\beta_{k-j}$$

$d_> \in H_+$
$b_> \in H_-$

$$d_> \beta - b_> \in H_+$$

$$d_> - b_> \bar{\beta} \in zH_-$$

$$\begin{pmatrix} zH_- & H_- \\ zH_+ & H_+ \end{pmatrix} \qquad \begin{pmatrix} zH_- & H_+ \\ zH_- & H_+ \end{pmatrix}$$



$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a_> & b_> \\ c_> & d_> \end{pmatrix}\begin{pmatrix} a_0 & b_0 \\ c_0 & d_0 \end{pmatrix} = \begin{pmatrix} a_>a_0 + b_>c_0 & a_>b_0 + b_>d_0 \\ c_>a_0 + d_>c_0 & c_>b_0 + d_>d_0 \end{pmatrix}$$

$$\begin{pmatrix} \xi_+ \\ \xi_- \end{pmatrix} = \begin{pmatrix} a_> & b_> \\ c_> & d_> \end{pmatrix}\begin{pmatrix} p_0 \\ q_0 \end{pmatrix} \qquad \begin{pmatrix} p_0 \\ q_0 \end{pmatrix} = \begin{pmatrix} d_> & -b_> \\ -c_> & a_> \end{pmatrix}\begin{pmatrix} \xi_+ \\ \xi_- \end{pmatrix}$$

$$\begin{pmatrix} \xi_+ \\ \xi_- \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} \xi'_- \\ \xi'_+ \end{pmatrix} \qquad \begin{pmatrix} \xi'_- \\ \xi'_+ \end{pmatrix} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}\begin{pmatrix} \xi_+ \\ \xi_- \end{pmatrix}$$

$$\begin{pmatrix} \xi_+ \\ \xi'_+ \end{pmatrix} = \begin{pmatrix} \frac{1}{d} & \frac{b}{d} \\ -\frac{c}{d} & \frac{1}{d} \end{pmatrix}\begin{pmatrix} \xi'_- \\ \xi_- \end{pmatrix} \qquad \begin{pmatrix} \xi'_- \\ \xi_- \end{pmatrix} = \begin{pmatrix} \frac{1}{a} & -\frac{b}{a} \\ \frac{c}{a} & \frac{1}{a} \end{pmatrix}\begin{pmatrix} \xi_+ \\ \xi'_+ \end{pmatrix}$$

Basic condition is that

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1}\begin{pmatrix} a_> & b_> \\ c_> & d_> \end{pmatrix} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}\begin{pmatrix} a_> & b_> \\ c_> & d_> \end{pmatrix} = \begin{pmatrix} da_> - bc_> & db_> - bd_> \\ -ca_> + ac_> & -cb_> + ad_> \end{pmatrix}$$

$$db_> - bd_> \in H_+$$

$$-cb_> + ad_> \in zH_-$$

$$= \begin{pmatrix} a_0 & b_0 \\ c_0 & d_0 \end{pmatrix}^{-1} \in \begin{pmatrix} H_+ & H_+ \\ zH_- & zH_- \end{pmatrix}$$

Again ·

$$\begin{pmatrix}\xi_+\\ \xi_-\end{pmatrix} = \begin{pmatrix}a_> & b_>\\ c_> & d_>\end{pmatrix}\overbrace{\begin{pmatrix}a_0 & b_0\\ c_0 & d_0\end{pmatrix}\begin{pmatrix}\xi'_-\\ \xi'_+\end{pmatrix}}^{\begin{pmatrix}p_0\\ q_0\end{pmatrix}}$$

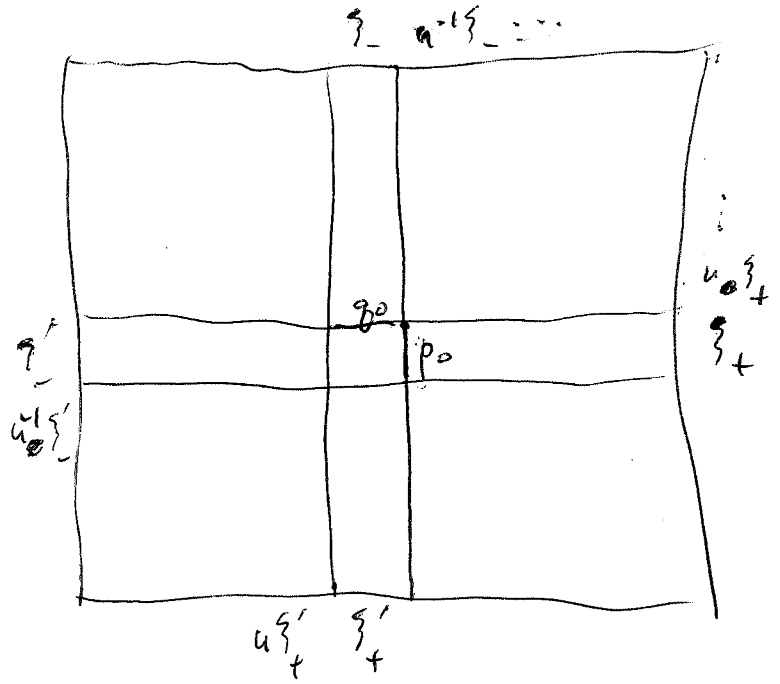
$$\begin{pmatrix}a_> & b_>\\ c_> & d_>\end{pmatrix} \in \begin{pmatrix}zH_- & H_-\\ zH_+ & H_+\end{pmatrix} \qquad \begin{pmatrix}a_0 & b_0\\ c_0 & d_0\end{pmatrix} \in \begin{pmatrix}zH_- & H_+\\ zH_- & H_+\end{pmatrix}$$

$q_0 \in zH_-\xi'_- + H_+\xi'_+$

$q_0 \in zH_+\xi_+ + zH_-\xi_-$

$p_0 \in H_+\xi_+ + H_-\xi_-$

$p_0 \in zH_-\xi'_- + H_+\xi'_+$?



$$\begin{pmatrix}p_0\\ q_0\end{pmatrix} = \begin{pmatrix}a_0 & b_0\\ c_0 & d_0\end{pmatrix}\begin{pmatrix}\xi'_-\\ \xi'_+\end{pmatrix}, \qquad \begin{pmatrix}a_0 & b_0\\ c_0 & d_0\end{pmatrix} \in \begin{pmatrix}zH_- & H_+\\ zH_- & H_+\end{pmatrix}$$

$$\begin{pmatrix}\xi_+\\ \xi_-\end{pmatrix} = \begin{pmatrix}a_> & b_>\\ c_> & d_>\end{pmatrix}\begin{pmatrix}p_0\\ q_0\end{pmatrix}, \qquad \begin{pmatrix}a_> & b_>\\ c_> & d_>\end{pmatrix} \in \begin{pmatrix}zH_- & H_-\\ zH_+ & H_+\end{pmatrix}$$

$$\begin{pmatrix}p_0\\ q_0\end{pmatrix} \in \begin{pmatrix}d_> & -b_>\\ -c_> & a_>\end{pmatrix}\begin{pmatrix}\xi_+\\ \xi_-\end{pmatrix}, \qquad \begin{pmatrix}d_> & -b_>\\ -c_> & a_>\end{pmatrix} \in \begin{pmatrix}H_+ & H_-\\ zH_+ & zH_-\end{pmatrix}$$

$$\cancel{\begin{pmatrix}d & -b\\ -c & a\end{pmatrix}\begin{pmatrix}a_> & b_>\\ c_> & d_>\end{pmatrix}} = \begin{pmatrix}a_0 & b_0\\ c_0 & d_0\end{pmatrix} \in \begin{pmatrix}zH_- & H_+\\ zH_- & H_+\end{pmatrix}$$

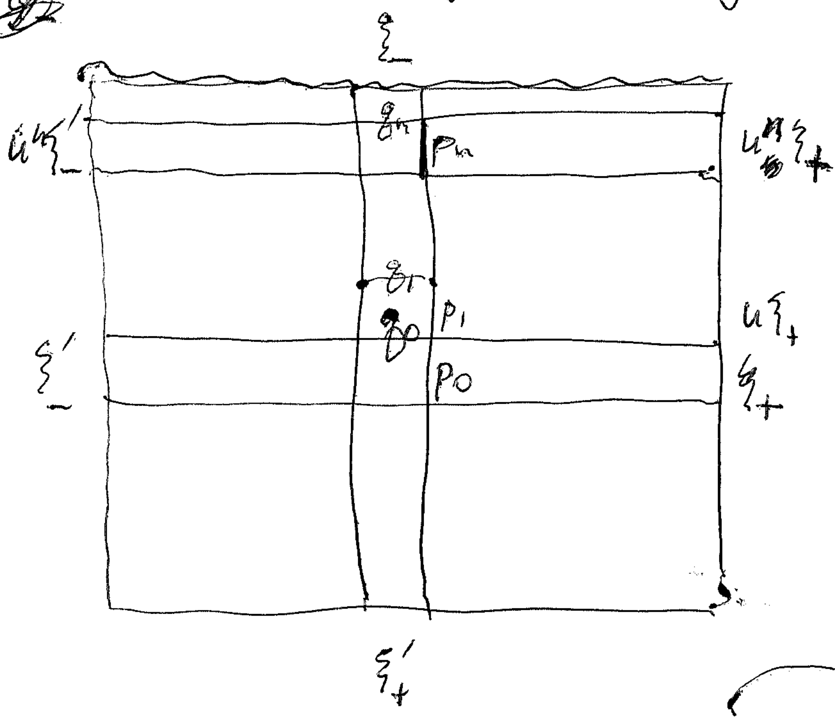
$$\begin{pmatrix}d_> & -b_>\\ -c_> & a_>\end{pmatrix}\begin{pmatrix}a & b\\ c & d\end{pmatrix} = \begin{pmatrix}d_>a - b_>c & d_>b - b_>d\\ -c_>a + a_>c & -c_>b + a_>d\end{pmatrix}$$

$$d_> b - b_> d \in H_+ \iff d_> \beta - b_> \in H_+$$

$$d_> a - b_> c \in zH_- \iff d_> - b_> \left(\frac{c}{a}\right) \in zH_-$$

$$\frac{c}{a} = \overline{\left(\frac{b}{d}\right)} = \bar\beta .$$

So now you bring n into the game.



$$q_n, p_n \in u^{n+1} H_- \xi'_- + H_+ \xi'_+$$

$$u^{-n} p_n \in H_- \xi'_- + z^{-n} H_+ \xi'_+$$

$$\begin{pmatrix} \xi^*_+ \\ \xi_- \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \xi'_- \\ \xi'_+ \end{pmatrix} = \begin{pmatrix} a_> & b_> \\ c_> & d_> \end{pmatrix} \overbrace{\begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix} \begin{pmatrix} \xi'_- \\ \xi'_+ \end{pmatrix}}^{\begin{pmatrix} u^{-n} p_n \\ q_n \end{pmatrix}}$$

$$\cap \qquad\qquad \cap$$

$$\begin{pmatrix} zH_- & z^{-n} H_- \\ z^{n+1} H_- & H_+ \end{pmatrix} \qquad \begin{pmatrix} zH_- & z^{-n} H_+ \\ z^n z H_- & H_+ \end{pmatrix}$$

$$\begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix} = \begin{pmatrix} d_> & -b_> \\ -c_> & a_> \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} d_> a - b_> c & d_> b - b_> d \\ -c_> a + a_> c & -c_> b + a_> d \end{pmatrix}$$

$$\boxed{\begin{array}{l} d_> \beta - b_> \in z^{-n} H_+ \\ d_> - b_> \bar\beta \in z H_- \end{array}}$$

$$\beta = \frac{b}{d}$$

$d_> \in H_+ \qquad b_> \in z^{-n} H_{\cancel{+}}$

$$\boxed{\begin{aligned} d_> \beta - b_> &\in z^{-n} H_+ \\ d_> - b_> \bar{\beta} &\in z H_- \end{aligned}}$$

determines $\beta + z^{-n} H_+$

$d_>(z^{-n} H_+) = z^{-n} H_+$

$b_>(z^n z H_-) \subset z^{-n} H_- \cdot z^{n+1} H_-$
$= H_-$

Do orthog.

$$p_0 = \sum d_j u^j \xi_+ \;\cancel{-}\; \sum b_k u^k \xi_- \in H_+ \xi_+ + H_- \xi_-$$

$$0 = (u^k \xi_- | p_0) = \sum d_j \beta_{k-j} - b_k \qquad : \; d_> \beta - b_> \in H_+$$
$k \leq -1.$

$$0 = (u^j \xi_+ | p_0) = d_j - \sum b_k \overline{\beta_{k-j}} \qquad d_> - b_> \bar{\beta} \in \cancel{z} H_-$$
$j \geq 1$

~~[illegible]~~ Assume these equations have a unique solution $\ni$ $b_> \in H_-$, $d_> \in H_+$.
Consider $\delta\beta$ not changing the solution.

$$d_> \delta\beta \in H_+ \iff \delta\beta \in H_+$$

$$b_> \overline{\delta\beta} \in z H_- \iff \delta\beta \in H_+$$

$H_+ \quad zH_-$

Look at other side

$$-c_> \cdot + a_> \left(\frac{c}{a}\right) \in z H_-$$
$\bar{\beta}$

$$-c_> \left(\frac{b}{d}\right) + a_> \cdot \in H_+$$
β

Assume $\exists$ unique soln with $c_> \in z H_+ \quad a_> \in H_-$

$$a_> \overline{\delta\beta} \in z H_- \iff \overline{\delta\beta} \in z H_-$$
$\delta\beta \in H_+$

$$c_> \delta\beta \in H_+ \Longleftarrow \delta\beta \in H_+$$

The moral is that to get p_0, q_0 you need only $\beta + H_+$ i.e. the $(z^n|\beta)$ with $n<0$.

$$\begin{pmatrix} u^{-n}p_n \\ q_n \end{pmatrix} = \begin{pmatrix} d_> & -b_> \\ -c_> & a_> \end{pmatrix}\begin{pmatrix} \xi_+ \\ \xi_- \end{pmatrix}$$

$$\cap$$

$$\begin{pmatrix} H_+ & z^{-n}H_- \\ z^n z H_+ & zH_- \end{pmatrix}$$

$\therefore\ p_n = z^n d_> \xi_+ - z^n b_> \xi_-$

$\in z^n H_+ \xi_+ + H_- \xi_-$

$q_n = -c_> \xi_+ + a_> \xi_-$

$\in z^{n+1} H_+ \xi_+ + H_- \xi_-$

But also

$$\begin{pmatrix} u^{-n}p_n \\ q_n \end{pmatrix} = \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix}\begin{pmatrix} \xi'_- \\ \xi'_+ \end{pmatrix}$$

$$\cap$$

$$\begin{pmatrix} zH_- & z^{-n}H_+ \\ z^n z H_- & H_+ \end{pmatrix}$$

$p_n = z^n a_n \xi'_- + z^n b_n \xi'_+$

$\in z^{n+1} H_- \xi'_- + H_+ \xi'_+$

$q_n = c_n \xi'_- + d_n \xi'_+$

$\in z^{n+1} H_- \xi'_- + H_+ \xi'_+$

$$\begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix} = \begin{pmatrix} d_> & -b_> \\ -c_> & a_> \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} d_>a - b_>c & d_>b - b_>d \\ -c_>a + a_>c & -c_>b + a_>d \end{pmatrix}$$

$d_> - b_> \frac{c}{a} \in zH_-$ $\qquad$ $\boxed{d_> - b_>\bar\beta \in zH_-}$

$d_> \frac{b}{d} - b_> \in z^{-n}H_+$ $\qquad$ $\boxed{d_>\beta - b_> \in z^{-n}H_+}$

$-c_> a + a_> c \in z^{n+1}H_-$ $\qquad$ $\boxed{-c_> + a_>\bar\beta \in z^{n+1}H_-}$

$-c_> b + a_> d \in H_+$ $\qquad$ $\boxed{-c_>\beta + a_> \in H_+}$

Next case

$$\begin{pmatrix} a_> & b_> \\ c_> & d_> \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} d_n & -b_n \\ -c_n & a_n \end{pmatrix} = \begin{pmatrix} ad_n - bc_n & -ab_n + ba_n \\ cd_n - dc_n & -cb_n + da_n \end{pmatrix}$$

$$\cap \qquad\qquad\qquad\qquad \cap$$

$$\begin{pmatrix} zH_- & z^{-n}H_- \\ z^n z H_+ & H_+ \end{pmatrix} \qquad\qquad \begin{pmatrix} zH_- & z^{-n}H_- \\ z^{n+1}H_+ & H_+ \end{pmatrix}$$

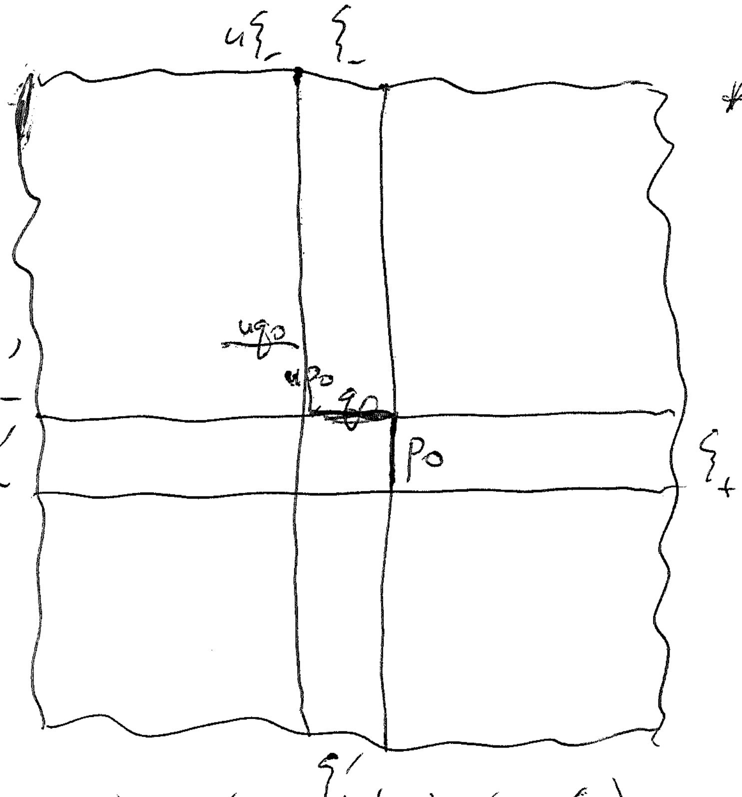
$$\begin{pmatrix} a_> & b_> \\ c_> & d_> \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} d_0 & -b_0 \\ & \end{pmatrix}$$

$ad_n - bc_n \in zH_-$ equiv. $d_n - \frac{b}{a} c_n \in zH_-$

$-ab_n + ba_n \in z^{-n}H_-$ equiv $-b_n + \frac{b}{a} a_n \in z^{-n}H_-$

$cd_n - dc_n \in z^{n+1}H_+$ " $\frac{c}{d} d_n - c_n \in z^{n+1}H_+$

$-cb_n + da_n \in H_+$ " $-\frac{c}{d} b_n + a_n \in H_+$



The subspace gen by $u^n p_0$, $u^n q_0$ $n \geq 0$ is the "light cone"

$$H_+\xi'_- + H_+\xi_-$$

$$\begin{pmatrix} p_0 \\ q_0 \end{pmatrix} = \begin{pmatrix} d_> & -b_> \\ -c_> & a_> \end{pmatrix}\begin{pmatrix} \xi_+ \\ \xi_- \end{pmatrix}$$

$$\begin{pmatrix} \xi_+ \\ \xi_- \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} \xi'_- \\ \xi'_+ \end{pmatrix}$$

$$\begin{pmatrix} \xi_+ \\ \xi'_+ \end{pmatrix} = \begin{pmatrix} \frac{1}{d} & \frac{b}{d} \\ -\frac{c}{d} & \frac{1}{d} \end{pmatrix}\begin{pmatrix} \xi'_- \\ \xi_- \end{pmatrix}$$

$$\begin{pmatrix} \xi_+ \\ \xi_- \end{pmatrix} = \begin{pmatrix} \frac{1}{d} & \frac{b}{d} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \xi'_- \\ \xi_- \end{pmatrix}$$

$$\begin{pmatrix} p_0 \\ q_0 \end{pmatrix} = \underbrace{\begin{pmatrix} d_> & -b_> \\ -c_> & a_> \end{pmatrix} \begin{pmatrix} \frac{1}{d} & \frac{b}{d} \\ 0 & 1 \end{pmatrix}} \begin{pmatrix} \xi'_- \\ \xi_- \end{pmatrix}$$

$$\begin{pmatrix} \frac{d_>}{d} & d_> \frac{b}{d} - b_> \\ -\frac{c_>}{d} & -c_> \frac{b}{d} + a_> \end{pmatrix} \begin{pmatrix} \xi'_- \\ \xi_- \end{pmatrix}$$

$$\begin{pmatrix} a_0 & b_0 \\ c_0 & d_0 \end{pmatrix} = \begin{pmatrix} d_> & -b_> \\ -c_> & a_> \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

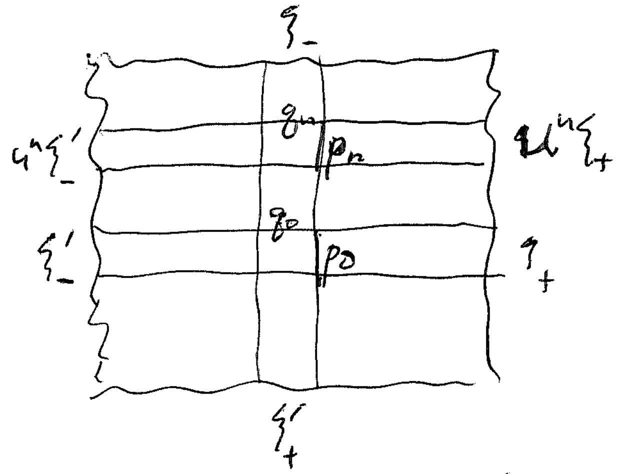
$$= \begin{pmatrix} & d_> b - b_> d \\ & -c_> b + a_> d \end{pmatrix}$$

$$\begin{pmatrix} p_0 \\ q_0 \end{pmatrix} = \frac{1}{d} \begin{pmatrix} d_> & b_0 \\ -c_> & d_0 \end{pmatrix} \begin{pmatrix} \xi'_- \\ \xi_- \end{pmatrix}$$

$$\begin{pmatrix} \xi'_- \\ \xi_- \end{pmatrix} = \begin{pmatrix} d & -b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a_> & b_> \\ c_> & d_> \end{pmatrix} \begin{pmatrix} p_0 \\ q_0 \end{pmatrix}$$

$$= \begin{pmatrix} \overset{d_0}{\overbrace{d a_> - b c_>}} & \overset{-b_0}{\overbrace{d b_> - b d_>}} \\ c_> & d_> \end{pmatrix} \begin{pmatrix} \\ \end{pmatrix}$$

$$\subset \begin{pmatrix} H_+ & H_+ \\ zH_+ & H_+ \end{pmatrix}$$



$$\begin{pmatrix} p_n \\ q_n \end{pmatrix} = \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix} \begin{pmatrix} \xi'_- \\ \xi'_+ \end{pmatrix}$$

$$\xi'_+ = -\frac{c}{d}\xi'_- + \frac{1}{d}\xi_-$$

$$\begin{pmatrix} p_n \\ q_n \end{pmatrix} = \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\frac{c}{d} & \frac{1}{d} \end{pmatrix} \begin{pmatrix} \xi'_- \\ \xi_- \end{pmatrix} = \begin{pmatrix} \overbrace{a_n - b_n\frac{c}{d}}^{a_>/d} & \frac{b_n}{d} \\ \underbrace{c_n - d_n\frac{c}{d}}_{-c_>/d} & \frac{d_n}{d} \end{pmatrix}$$

$$\cancel{\begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix} = \begin{pmatrix} d_> & -b_> \\ -c_> & a_> \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}}$$

$$\begin{pmatrix} a_> & b_> \\ c_> & d_> \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} d_n & -b_n \\ -c_n & a_n \end{pmatrix} = \begin{pmatrix} \cdot & \cdot \\ cd_n - dc_n & -cb_n + da_n \end{pmatrix}$$

$$\begin{pmatrix} p_n \\ q_n \end{pmatrix} = \frac{1}{d} \begin{pmatrix} a_> & b_n \\ -c_> & d_n \end{pmatrix} \begin{pmatrix} \xi'_- \\ \xi_- \end{pmatrix}$$

---

recap.

$$\begin{pmatrix} \xi_+ \\ \xi_- \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \xi'_- \\ \xi'_+ \end{pmatrix} \qquad \begin{pmatrix} \xi_+ \\ \xi'_+ \end{pmatrix} = \begin{pmatrix} \frac{1}{d} & \frac{b}{d} \\ -\frac{c}{d} & \frac{1}{d} \end{pmatrix} \begin{pmatrix} \xi'_- \\ \xi_- \end{pmatrix}$$

$$\begin{pmatrix} \xi'_- \\ \xi'_+ \end{pmatrix} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \begin{pmatrix} \xi_+ \\ \xi_- \end{pmatrix} \qquad \begin{pmatrix} \xi'_- \\ \xi_- \end{pmatrix} = \begin{pmatrix} \frac{1}{a} & -\frac{b}{a} \\ \frac{c}{a} & \frac{1}{a} \end{pmatrix} \begin{pmatrix} \xi_+ \\ \xi'_+ \end{pmatrix}$$

$$\begin{pmatrix} a_> & b_> \\ c_> & d_> \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} d_n & -b_n \\ -c_n & a_n \end{pmatrix} = \begin{pmatrix} ad_n - bc_n & -ab_n + ba_n \\ cd_n - dc_n & -cb_n + da_n \end{pmatrix}$$

$$\begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix} = \begin{pmatrix} d_> & -b_> \\ -c_> & a_> \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} d_> a - b_> c & d_> b - b_> d \\ -c_> a + a_> c & -c_> b + a_> d \end{pmatrix}$$

$$\begin{pmatrix} u^{-n} p_n \\ q_n \end{pmatrix} = \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix}\begin{pmatrix} \xi'_- \\ \xi'_+ \end{pmatrix} = \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -\frac{c}{d} & \frac{1}{d} \end{pmatrix}\begin{pmatrix} \xi'_- \\ \xi_- \end{pmatrix}$$

$$= \frac{1}{d}\begin{pmatrix} a_n d - b_n c & b_n \\ c_n d - d_n c & d_n \end{pmatrix}\begin{pmatrix} \xi'_- \\ \xi_- \end{pmatrix}$$

$$\boxed{\begin{pmatrix} u^{-n} p_n \\ q_n \end{pmatrix} = \frac{1}{d}\underbrace{\begin{pmatrix} d_> & b_n \\ -c_> & d_n \end{pmatrix}}_{\in}\begin{pmatrix} \xi'_- \\ \xi_- \end{pmatrix}}$$

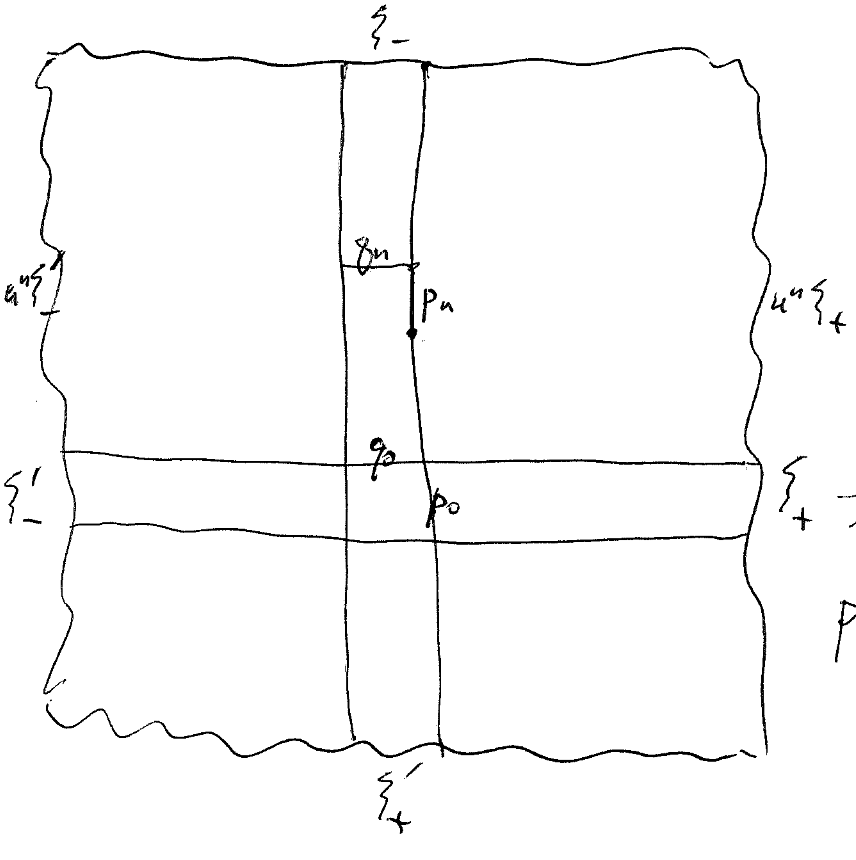
$$\begin{pmatrix} H_+ & z^{-n}H_+ \\ z^{n+1}H_+ & H_+ \end{pmatrix}$$

$$\begin{pmatrix} u^{-n} p_n \\ q_n \end{pmatrix} = \begin{pmatrix} d_> & -b_> \\ -c_> & a_> \end{pmatrix}\begin{pmatrix} \xi_+ \\ \xi_- \end{pmatrix} = \frac{1}{a}\begin{pmatrix} d_> & -b_> \\ -c_> & a_> \end{pmatrix}\begin{pmatrix} a & 0 \\ c & 1 \end{pmatrix}\begin{pmatrix} \xi_+ \\ \xi'_+ \end{pmatrix}$$

$$= \frac{1}{a}\begin{pmatrix} d_> a - b_> c & -b_> \\ -c_> a + a_> c & a_> \end{pmatrix}\begin{pmatrix} \xi_+ \\ \xi'_+ \end{pmatrix}$$

$$\boxed{\begin{pmatrix} u^{-n} p_n \\ q_n \end{pmatrix} = \frac{1}{a}\underbrace{\begin{pmatrix} a_n & -b_> \\ c_n & a_> \end{pmatrix}}_{\in}\begin{pmatrix} \xi_+ \\ \xi'_+ \end{pmatrix}}$$

$$\begin{pmatrix} zH_- & z^{-n}H_- \\ z^{n+1}H_- & zH_- \end{pmatrix}$$



you want to know p_n, q_n in scattering terms.

$$p_n \in z^n H_+ \xi'_- + H_+ \xi_-$$