next look at $\Gamma$ infinite $E$, v.s. with locally finite partition of unity $(h_s)$ indexed by $\Gamma$. means $\forall \xi \in E$ $\{s \mid h_s \xi \neq 0\}$ finite and $\sum_s h_s \xi = \xi$

Notice that in this case $E = \sum_{t \in \Gamma} h_t E$ so you have $\{s \mid h_s h_t \xi \neq 0\}$ is finite and $\sum_s h_s h_t \xi = \xi$

Strengthen to have $\{s \mid h_s h_t \neq 0\}$ finite $\forall t$ and $\sum_s h_s h_t = h_t \quad \forall t$.

---

next ingredient?? $\sum h_s = 1$ understood in finite case. Condition $h_s h_t = 0 = \overset{\text{inj}}{\beta_s} \alpha_s \beta_t \overset{\text{surj}}{\alpha_t} \Rightarrow \alpha_s \beta_t = 0$

$$\alpha_s \beta_t : V_t \xhookrightarrow{\beta_t} E \xrightarrow{\alpha_s} V_s$$ (some nilpotence?)

these are the components of $p = \alpha\beta$

where to??? K-theory ~~comp~~ questions

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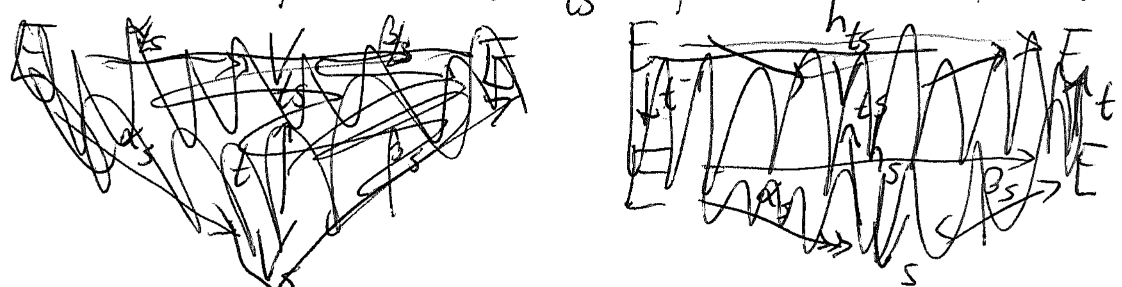
Suppose $\Gamma$ acts on $E$ $(h_s)_{s \in \Gamma}$ $\quad t h_s t^{-1} = h_{ts}$

$\Gamma$ situation: Suppose $E$ has partition of 1 indexed by $\Gamma$. Suppose group $\Gamma$ acts on $E$, that $(h_s)_{s\in\Gamma}$ is a partition of 1 indexed by $\Gamma$, that $t h_s t^{-1} = h_{ts}$

Then $t h_s t^{-1} = t \beta_s \alpha_s t^{-1}$, $h_{ts} = \beta_{ts} \alpha_{ts} \Longrightarrow \beta_{ts} = t \beta_s t^{-1}$, $\alpha_{ts} = t \alpha_s t^{-1}$



$$\begin{tikzcd} E \arrow[r, two heads, "\alpha_s = h_s"] \arrow[d, "t"'] & h_s E \arrow[r, hook, "\beta_s"] \arrow[d, "t"] & E \arrow[d, "t"] \\ E \arrow[r, two heads, "\alpha_{ts} = h_{ts}"] & h_{ts} E \arrow[r, hook, "\beta_{ts}"] & E \end{tikzcd}$$

$$\begin{tikzcd} E \arrow[r, "\alpha_s"] \arrow[d, "t"'] & V_s \arrow[r, hook] & E \\ E \end{tikzcd}$$

$$\begin{tikzcd} E \arrow[r, two heads, "\alpha_s"] \arrow[d, "t"'] & V_s \arrow[r, "\beta_s"] \arrow[d, "t"] & E \arrow[d, "t"] \\ E \arrow[r, two heads, "\alpha_{ts}"] & V_{ts} \arrow[r, "\beta_{ts}"] & E \end{tikzcd}$$

$V_s = \operatorname{Im} sh_1$

$\beta_{ts} = t\beta_s t^{-1}$

$\alpha_{ts} = t\alpha_s t^{-1}$

You set up.

$$E \xrightarrow{\alpha} \bigoplus_{s\in\Gamma} V_s \xrightarrow{\beta} E \xrightarrow{\alpha} \bigoplus_{s\in\Gamma} V_s$$

$$\alpha\xi = (\alpha_s\xi)_s \qquad \beta(\eta_s) = \sum_s \beta_s\eta_s$$

$$\beta\alpha\xi = \sum_s \beta_s\alpha_s\xi = \sum_s h_s\xi = \xi.$$

$$\alpha\beta(\eta_t) = \sum_t \alpha_s\beta_t\eta_t$$

Now what happens? ~~Can you proceed? Again.~~
Look at the $\Gamma$ module $\bigoplus_{s\in\Gamma} V_s$; its graded wrt $\Gamma$ compatibly. $tV_s = V_{ts}$. Choose a basepoint
The index set for the partition of 1 is a $\Gamma$ torsor.
~~Say you pick~~ use 1 as basepoint

$$\begin{tikzcd} V_1 \arrow[r, "s"] \arrow[dr, "\beta_1"'] & V_s \arrow[d, "\beta_s"] \\ & E \end{tikzcd}$$

There's a coord. change. A family

$(\eta_s)_{s\in\Gamma}$ $\eta_s \in V_s$ can be written $\eta_s = sf(s)$
with $f(s) \in V_1$ $\forall s$. Then $\beta\eta = \sum_s \beta_s\eta_s$

$$= \sum_s s\beta_1 s^{-1} s f(s) = \sum_s s\beta_1 f(s), \quad \text{and}$$

$$\alpha_s\xi = s\alpha_1 s^{-1}\xi = sg(s) \quad \text{where } g(s) = \alpha_1 s^{-1}\xi. \text{ Then}$$

$$\alpha\beta(\eta) = \alpha\sum_t t\beta_1 f(t) = s\sum_t \alpha_1 s^{-1}t\beta_1 f(t)$$

$$(\alpha\beta f)(s) = \sum_t \underbrace{(\alpha_1 s^{-1}t\beta_1)}_{\text{a left invariant kernel}} f(t)$$

$$\cancel{(pf)(s) = \sum_t (\alpha_1 s^{-1} t \beta_1) f(t)}$$

$$\|\qquad\qquad\qquad\|$$

$$\cancel{(\tilde{p}f)(s^{-1}) \qquad \sum (\alpha_1 s^{-1} t \beta_1) \tilde{f}(t} \qquad ?$$

Change variable $\quad f(t) = g(t^{-1})$

$$(pf)(s) = \sum_t (\alpha_1 s^{-1} t\,[\cdot]\,\beta_1)\, g(t[\cdot]^{-1}) \qquad ?$$

---

$$\eta = (\eta_s)_{s\in\Gamma} = (s g(s^{-1}))_{s\in\Gamma} \qquad g(s^{-1}) \in V_1$$

$$\beta\eta = \sum \beta_s \eta_s = \sum_s s\beta_1 \cancel{s^{-1}s}\, g(s^{-1})$$

$$\beta\eta = \sum_{t\in\Gamma} t\beta_1 g(t^{-1})$$

$$(\alpha\beta\eta)(s) = s \underbrace{\alpha_1 s^{-1} \sum_t t\beta_1 g(t^{-1})}_{(pg)(s^{-1})}$$

If so $\quad (pg)(s) = \sum_t \alpha_1 s t \beta_1 g(t^{-1})$

---

Review: Looked a partition of 1 on $E$ $\qquad \sum h_s = id$

$$h_s = \beta_s\alpha_s : E \twoheadrightarrow V_s = h_s E \hookrightarrow E$$

$$0 = h_s h_t = \underline{\beta_s}\alpha_s\beta_t\underline{\alpha_t} \iff \alpha_s\beta_t = 0.$$

$$E \xrightarrow{\alpha = (\alpha_s)_s} \bigoplus V_s \xrightarrow{\beta f = \sum_s \beta_s f(s)} E \xrightarrow{\alpha} \bigoplus V_s$$

$$(\alpha\beta f)(s) = \sum_t \alpha_s \beta_t f(t)$$

Next $\Gamma$ gp acts on $E$, partition indexed by $\Gamma$

$t h_s t^{-1} = h_{ts}$ equivalent $tV_s = V_{ts}$ $\qquad V_s = sV_1$

so $f = (f(s) \in V_s) = (s g(s))$ where $g : s \mapsto g(s) \in V_1$

$$\begin{array}{ccccc} E & \xrightarrow{\alpha_s} & V_s & \xhookrightarrow{\beta_s} & E \\ t\downarrow & & t\downarrow & & t\downarrow \\ E & \xrightarrow{\alpha_{ts}} & V_{ts} & \xrightarrow{\beta_{ts}} & E \end{array}$$

$(\alpha_s \xi) = (s\alpha_1 s^{-1}\xi)$ corresp to f. $\alpha_1 s^{-1}\xi \in V_1$

$$(\alpha\beta f)(s) = \sum_t \alpha_s \beta_t f(t)$$

$$= \sum_t s\alpha_1 s^{-1} t \beta_1 \cancel{t^{-1} t}\, g(t) \qquad \Big| \qquad (\alpha\beta g)(s) = \sum_t (\alpha_1 s^{-1} t \beta_1)\, g(t)$$

---

formulas:

$$E \xrightarrow{\alpha} \bigoplus_s V_1 \xrightarrow{\beta} E$$

$$\xi \overset{\alpha}{\longmapsto} (\alpha_1 s^{-1}\xi)_{s\in\Gamma} \qquad (s \longmapsto \alpha_1 s^{-1}\xi$$

$$(g_s)_{s\in\Gamma} \overset{\beta}{\longmapsto} \sum s\beta_1 g_s$$

$$\beta\alpha\xi = \sum_s s\beta_1\alpha_1 s^{-1}\xi = \xi$$

$$\alpha\beta(g_s)_s = \sum_t (\alpha_1 s^{-1} t \beta_1)\, g_t$$

$$\alpha\beta(g_{u^{-1}})_u = \sum_t (\alpha_1 u t \beta_1)\, g_t = \sum_t (\alpha_1 u t^{-1} \beta_1)\, g$$

$$\alpha\beta(s \mapsto g_s) = \Big(s \longmapsto \sum_t (\alpha_1 s^{-1} t \beta_1)\, g_t\Big)$$

---

$$(\alpha\beta f)_s = \sum_t (\alpha_1 s^{-1} t \beta_1)\, f_t$$

$$E \xrightarrow{\alpha} \bigoplus_{s\in\Gamma} V_1 \xrightarrow{\beta} E \xrightarrow{\alpha} \bigoplus_{s\in\Gamma} V_1$$

$$\xi \overset{\alpha}{\longmapsto} (\alpha\xi)_s = \alpha_1 s^{-1}\xi$$

$$(f : s \mapsto f_s) \overset{\beta}{\longmapsto} \beta f = \sum_{s\in\Gamma} s\beta_1 f_s$$

$$\xi \longmapsto \beta\alpha\xi = \sum_{s\in\Gamma} s\beta_1\alpha_1 s^{-1}\xi = \xi$$

$$f \longmapsto \sum_{t\in\Gamma} t\beta_1 f_t \longmapsto (\alpha\beta f)_s = \sum_{t\in\Gamma} (\alpha_1 s^{-1} t \beta_1) f_t$$

Conversely given $V_1$ and a projection of $p$ on $\bigoplus_{s\in\Gamma} V_1$ commuting with $(L_t f)_s = f_{t^{-1}s}$. Then

$$(pf)_s = \sum_t k_{s,t} f_t$$

$$(pL_u f)_s = \sum_t k_{s,t} f_{u^{-1}t} = \sum_t k_{s,ut} f_t$$

$$(L_u pf)_s = \sum_t k_{u^{-1}s,t} f_t \qquad \therefore \quad k_{u^{-1}s,t} = k_{s,ut}$$

which means that $k_{s,t} = k_{us,ut}$ for all $u$

$k_{s,t} = k_{1,s^{-1}t}$. and then for $p$ to carry $\oplus V_1$ into itself you need

$k_{1,s^{-1}t} = k(s^{-1}t)$ to have finite support in $s$ for each $t$, i.e. $k(\ )$ finite support.

So this viewpoint is that you have $V_1$ a left invariant operator on $\bigoplus_\Gamma V_1$, fin. supp.

projection. $(pf)_s = \sum_t k_{s^{-1}t} f_t$

$$(p(pf))_s = \sum_t k_{s^{-1}t}(pf)_t = \sum_t k_{s^{-1}t} \sum_u k_{t^{-1}u} f_u$$

$$(p^2 f)_s = \sum_u \left( \sum_t k_{s^{-1}t} k_{t^{-1}u} \right) f_u$$

$p^2 = p$ amounts to the convolution property $k * k = k$. for the function $k_s$. Example: $\alpha_1 s^{-1}t \beta_1 = k_{s^{-1}t}$

$$\sum_{tu = s \text{ fixed}} k_t k_u = \sum_{s = tu} \alpha_1 t \overbrace{\beta_1 \alpha_1}^{h_1} u \beta_1 = \sum_t \alpha_1 h_t \overbrace{tu}^{s} \beta_1$$
$$= \alpha_1 s \beta_1 = k_s$$

the formulas are now clear, you need now the Morita equivalence. So you have to sort out the rings.

What do you know at this point?

Given $E$ $\Gamma$ module, $h_s$ equiv partition of 1 on $E$ finite overlaps, you know such an $E$ is equivalent to a $\Gamma$-graded projector $p_s$ on a vector space $V_1$, finite support, $V_1 = \sum p_s V_1$ $0 = \bigcap_s \operatorname{Ker}(p_s \text{ in } V_1)$. This equivalence of module categories should translate to a Morita equivalence between finite rings. You know this should work, but you are missing details. ~~consider next the~~

So what to do? Set it up carefully. ~~Let~~ $V$ ~~be~~ a $P_F$ module, $V$ a vector space equipped with a left $\Gamma$ invariant projector on $\bigoplus_\Gamma V$ with supp in $F$.

You know any ~~[illegible]~~ linear map $\bigoplus_\Gamma V \xrightarrow{K} \prod_\Gamma V$ is given by a kernel: $(Kf)_s = \sum_{t\in\Gamma} K_{s,t} f(t)$, where $K_{s,t} \in \mathrm{End}(V)$ can be arbitrary. Next ~~impose~~ left $\Gamma$-invariance $\iff$ $K_{s,t} = k(s^{-1}t)$. Ask when $K$ maps $\bigoplus_\Gamma V$ into $\bigoplus_\Gamma V$? Enough ~~[illegible]~~ to look at $f$ ~~[illegible]~~ supported at a point $t_0$. You have ~~[illegible]~~ $k(s^{-1}t_0)f(t_0) = 0$ a.a. $s$. This can happen without $k(s^{-1}t_0)$ ~~[illegible]~~ $=0$ a.a. $s$. You can have a family of $\varphi_n \in \mathrm{End}(V)$ such that $\varphi_n \neq 0$ $\forall n$ but ~~[illegible]~~ $\forall v \in V$ $\varphi_n(v) = 0$ for a.a. $n$. e.g. $V = \mathbb{C}^{(\infty)}$ $\varphi_n = pr$ on $n$-th comp. Put another ~~way Hom(V,~~

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Impose condition $\alpha_1 s \beta_1 \neq 0 \implies s \notin F$ fixed finite set

$$(pf)_s = \cancel{\phantom{x}} (\alpha\beta f)_s = \alpha_1 s^{-1}\beta f = \alpha_1 s^{-1}\sum_t t\beta_1 f_t$$

$$= \sum_t \alpha_1 s^{-1}t\beta_1 f(t).$$

$$\alpha_1\left(\sum t\beta\alpha_1 t^{-1}\right)s\beta_1$$
$$\|$$

$$\boxed{p_s = \alpha_1 s\beta_1} \qquad \sum_t p_t p_{t^{-1}s} = \sum_t \alpha_1 t\beta_1\alpha_1 t^{-1}s\beta_1 = p_s$$

$A = p_F$, (non unital) ring gen $p_s$ $s\in\Gamma$

relns. $\left(\begin{array}{l} p_s = 0 \quad s\notin F \\ p_s = \sum_t p_t p_{t^{-1}s} \end{array}\right.$ $\implies A = \sum_{t\in F} p_t A$

$A = A^2$

If ~~[illegible]~~ $V$ is an $A$-module, then define $E(V) = p(\Lambda\otimes V)$

details of Mor. equiv. Just exactly what should be done? At the moment you have two ~~categories of~~ firm left module categories. You have two idempotent rings defined by generators and relations and an equivalence between their ~~left~~ firm left module categories. Review:

$A = P_F$, $F$ finite subset of the group $\Gamma$.

gen: $p_s$, $s \in \Gamma$, relns $\left| \begin{array}{ll} p_s = 0 & s \notin F \\ p_s = \sum_t p_t p_{t^{-1}s} & \end{array} \right.$

$C$ gens: $h_s$, $s \in \Gamma$ relns: $h_s h_t = 0$ for $s^{-1}t \notin F$

you've left out the crossproduct with $\Gamma$.

$$\forall t \quad h_t = \sum_{s \in tF^{-1}} h_s h_t = \sum_{s \in tF} h_t h_s$$

Note that $C$ has local left + right units, i.e. $\forall c \ \exists F_1, F_2$ finite $\subset \Gamma$ such that $\sum_{s \in F_1} h_s c = c$ and $c \sum_{s \in F_2} h_s = c$.

~~[illegible]~~

$$B = C \rtimes \Gamma = \bigoplus_s Cs \qquad \Gamma\text{-graded alg.}$$

$B$ should have local left + right units.

$C$ is the algebra of functions w. comp. supp on $E_\Gamma$. This ring $C$ contains the partition of unity $h_s$ you need. Let $E$ be a $C$-module, i.e. a vector space equipped with ~~[illegible]~~ a partition of unity ~~you want~~ $h_s$, $s \in \Gamma$. Define $V_s = h_s E$, assume $\sum_s V_s = E$ whence you should have $\sum h_s = 1$ on $E$, using the local left units. ~~And then you put~~

Focus. Consider a ring $A$ with local left units, equivalently, such that $\mathbb{Z}$ is a flat $A^{op}$-module

$A$ has a left unit: $ea = a \quad \forall a \in A$

$\Leftrightarrow$ $\mathbb{Z}$ is projective as right $A$-module

$$0 \longrightarrow A \longrightarrow \tilde{A} \longrightarrow \mathbb{Z} \longrightarrow 0$$

$\mathbb{Z}$ proj $\Leftrightarrow$ splits $\Leftrightarrow$ $\exists\ 1-e \in \tilde{A}$ such that $(1-e)A = 0$

$A$ has local left units $\overset{\text{def}}{\Longleftrightarrow}$ $\forall a \ \exists e \ni (1-e)a = 0$

Given $a_1, \ldots, a_n$ $\exists e \ni (1-e)a_i = 0$

$(1-e')a_i = 0 \qquad i = 1, \ldots, n-1$

Pick $e''$ $\underbrace{(1-e'')(1-e')}_{1-e} a_n$

$$\tilde{A} \xrightarrow{\ a_\cdot\ } \tilde{A}^n \longrightarrow \tilde{A}P$$

$\downarrow m_\cdot$

$M$

$\sum_{i=1}^n m_i a_i = 0$

$S = 1 - A$

$$A \otimes_A M \longrightarrow M$$

$\sum a_i \otimes m_i \longmapsto \sum a_i m_i = 0$

$\exists e \quad (1-e)a_i = 0 \quad \forall i$

$k = \sum a_i \otimes m_i \longmapsto \sum a_i m_i = 0$

$e a_i = a_i \quad \exists e.$

$ek = \sum e a_i \otimes m_i = \sum a_i \otimes m_i = k$

$= \sum_i e \otimes a_i m_i = e \otimes 0 = 0.$

$$(e)\sum_i a_i \otimes m_i = \sum_i e a_i \otimes m_i = \sum a_i \otimes m_i$$

$$= e \otimes \sum a_i m_i$$

Question: Suppose $A$ has local left units. Is $A$ Morita equalent to a ring with both local left and local right units?

$e$ idemp. in $A$ get $\begin{pmatrix} A & Ae \\ eA & eAe \end{pmatrix}$ which gives a M.eq. when $A = AeA$. If $ea = a$ $\forall a$ then ~~$eAe\ Ae$~~ $e^2 = e$ and $eA = A$, $eAe = Ae$. A left $A$-mod is firm iff $eM = M$, in which case $M$ is a unitary $Ae$ module $\therefore \mathcal{M}(A) = \mathcal{M}(Ae)$. If $N$ an $A^{op}$ module, then $N = Ne \oplus N(1-e)$, where $Ne$ is unitary $(Ae)^{op}$ module and $N(1-e)$ is killed by $A$. $N$ an $A^{op}$-module

Then $NA = NeA$ ? $\qquad NA = N$ ?

Consider $\quad A = eA = \underbrace{Ae}_{\text{unitaly ring}} \oplus \underbrace{A(1-e)}_{\text{left ideal right ann. by } A}$

$A = Ae \oplus A(1-e)$ semi-direct product of the unital ring $Ae$ with bimodule $A(1-e)$ over $Ae$ unitary on left nil on right.

$$N \otimes_A A = N \otimes_A Ae \oplus N \otimes_A A(1-e)$$
$$N = Ne \oplus N(1-e)$$

---

$A$, $e$ $\quad ea = a \quad \forall a \quad e^2 = e$

$e^2 = e$, $eA = A \qquad A = Ae \oplus A(1-e)$

$A = B \oplus K$

In general for $I$ ideal in $A$ $\ni IA = 0$

$$I \hookrightarrow A \twoheadrightarrow B \qquad \boxed{IA = 0, \quad AI = I}$$

$$\cancel{I \otimes_A A} \longrightarrow A \otimes_A A \xrightarrow{\sim} B \otimes_B A \longrightarrow 0$$

st $A = A$

A is a firm left B-mod when A is a firm A-mod

$$\begin{pmatrix} A & B = A/I \\ A & B \end{pmatrix}$$

$$A \otimes_A M \xrightarrow{\sim} M$$

$$A \otimes_A AM = B \otimes_{A/I} M$$

$\therefore \mathcal{M}(A) = \mathcal{M}(A/I)$

$$A \otimes_A A/I = \overset{A}{A \otimes_A A} \Big/ \overset{I}{A \otimes_A I}$$

---

$$\begin{pmatrix} 0 & I \\ 0 & I \end{pmatrix} \subset \begin{pmatrix} A & A \\ A & A \end{pmatrix} \twoheadrightarrow \begin{pmatrix} A & B \\ A & B \end{pmatrix}$$

---

Look at group situation for ~~[illegible]~~ insight.

Let $A \doteq$ gen $h_s$, $s \in \Gamma$ — relns. $h_s h_t = 0$ $s^{-1}t \notin F$

A has local left units. — $\sum_{s} h_s h_t = h_t$, $s \in tF^{-1}$

Let $M$ be an A-module such that $AM = M$ so that $M = \sum_t h_t M$ and elts of $M$ have local left units (in $A$). ~~[illegible]~~ In fact we can be more precise $e_K = \sum_{s \in K} h_s$, ~~is not~~ as $K$ ranges over all finite subsets of $\Gamma$, is ~~a not~~ an approx. left identity on $M$. So for any $\overset{t \in \Gamma}{m \in M}$ one has $e_K h_t m = h_t m$ for $K$ large enough. ?

Take $M = A$. so that $e_K h_t = h_t$ $\quad \forall t \, \exists K$

$\cancel{e_K h \, e_L}$ $\qquad h_1 e_K e_L$

$A$ : gen $h_s$ $s \in \Gamma$

relns $\begin{cases} h_s h_t = 0 & s^{-1}t \notin F \\ \sum_s h_s h_t = h_t \end{cases}$ $\quad$ ~~$s^{-1}$~~ $\notin$ ~~$s$~~$Ft^{-1}$ $\quad s \notin$ ~~$t$~~$F^{-1}$

each gen $h_t$ has local left unit $\overbrace{\sum_K h_s}^{e_K}$ $\quad K \supset tF^{-1}$

This means that $A$ has the approximate left unit $(e_K)$, $K$ ranges over all finite subsets of $\Gamma$. This will also ~~be~~ true for any $A$-module $M$ such that $M = AM$ equiv. $M = \sum_t h_t M$. Clearly then we have

$\sum h_s = 1$ on $M$. What does this mean?

It means for any $m \in M$ that $\sum_s h_s m$ is a finite sum equal to $m$. Therefore also

$\sum_s h_t h_s m$ is a finite sum equal to $h_t m$

But $\sum_s \underbrace{h_t h_s}_{\neq 0} m = \left(\sum_{s \in tF} h_t h_s\right) m$ $\quad \therefore$ you have

$\Downarrow$

$t^{-1}s \in F \Rightarrow s \in tF$ $\qquad \sum_{s \in tF} h_t h_s = h_t$

on any firm module.

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$A$ gen $h_s$ $s \in \Gamma$ relns. $\quad h_s h_t = 0 \quad s^{-1}t \notin F$

$\qquad t^{-1}s \notin F^{-1}$

$\sum_{s \in K = tF^{-1}} h_s h_t = h_t \qquad s \notin tF^{-1}$

Also $A = \sum h_t A$

$$\sum_{s \in tF^{-1} \cap uF} h_u h_s h_t = h_u h_t \qquad \text{~~now sum over a large~~}$$

~~set of $u$~~ $\quad \parallel \qquad \left(\sum_{s \in uF} h_u h_s - h_u\right) h_t = 0$

$$\sum_{s \in uF} h_u h_s h_t$$

What you've learned: Suppose $\Gamma$ finite, let $A$ have gen. $h_s$, $s \in \Gamma$ subject to relation $\sum_{s \in \Gamma} h_s h_t = h_t$. Put $e = \sum_{s \in \Gamma} h_s$, so that $e h_t = h_t$ $\forall t$, and $e^2 = e$. Then $h_u e h_t = h_u h_t$ or $(h_u e - h_u) h_t = 0$ $\forall t$. You would like $h_u e = h_u$ for all $u$, but you've seen that this isn't true. $A(1-e) \neq 0$ and one has $A(1-e)A = 0$; the ring $A$ has a right nil submodule namely $A(1-e)$. The idea is that an $A$-module $M$ ~~is the same as~~ is a vector space with ~~idempotent e~~ operators $h_s$, $s \in \Gamma$ whose sum ~~is an idempotent e~~ $e = \sum_{s \in \Gamma} h_s$ sat. $e h_s = h_s$ $\forall s$.

Then $M = eM + (1-e)M$ where $h_s : M \to eM$ can be arb. with sum $e$. Is $h_u(1-e) = 0$ $\forall u$ means $h_u : M/e^{\perp}M \longrightarrow eM$. means the $h_u : M \to eM$ factor: $M/(1-e)M \longrightarrow eM$ & have sum $e$. not

---

So the finite partition of $1$ case is clear.

Infinite case. gen. $h_s$, $s \in \Gamma$ relns. $h_s h_t = 0$ true for $s$ outside a finite set dep. on $t$, true for $t$ outside a finite set dep. on $s$. Want

group case

You want

$$\sum_{s \in tF^{-1}} h_s h_t = h_t, \quad \sum_{s \in uF} h_u h_s = h_u$$

$$\sum_s h_u h_s h_t = h_u h_t$$

Anyway you need both relations it seems. What should be noted is that the left ~~identity~~ $e$ on $A$ is not usually a right ~~identity~~: you have $A = Ae \oplus A(1-e)$ with $A(1-e) \neq 0$ in general, but $A(1-e)A = 0$ so you have an ideal $A(1-e)$ killed by $A$ on the right.

Proceed to group case. $C$ gen. $h_s, s \in \Gamma$ relns.

$$h_s h_t = 0 \text{ for } s^{-1}t \notin F, \quad \sum_{s \in tF^{-1}} h_s h_t = h_t, \quad \sum_{s \in tF} h_t h_s = h_t$$

the relation says that $C$ has local left units, and the other relations says $C$ has local right units, in fact $e_K = \sum_{s \in K} h_s$ is a local identity.

Let $E$ be a left $C$ module such that $E = CE$ equiv $E = \sum_{t \in \Gamma} h_t E$ whence $\sum_s h_s = id$ on $E$. Usual business. $V_t = h_t E = t h_1 E = t V_1$

$$\begin{tikzcd}
E \arrow[r, "\alpha_s = h_s"] \arrow[d, "t"] & V_s \arrow[r, hook, "\beta_s = inc"] \arrow[d, "t"] & E \arrow[d, "t"] \\
E \arrow[r, "\alpha_{ts}"] & V_{ts} \arrow[r, hook, "\beta_{ts}"] & E
\end{tikzcd}$$

$$E \xrightarrow{\alpha} \bigoplus_{s \in \Gamma} V_s \xrightarrow{\beta} E$$

$$\xi \longmapsto (\alpha_s \xi)_s = (s\alpha_1 s^{-1}\xi)_s$$

$$(f_s)_s \longmapsto \sum_s \beta_s f_s = \sum_s s\beta_1 s^{-1} f_s$$

$$\xi \longmapsto (s\alpha_1 s^{-1}\xi)_s \longmapsto \sum_s \underbrace{s\beta_1 s^{-1} s\alpha_1 s^{-1}}_{h_s} \xi = \xi$$

$$\alpha\beta(g_s) = \alpha \sum_t t\beta_1 g_t$$

$$\alpha\beta(g_s) = \Big(\alpha_1 s^{-1} \sum_t t\beta_1 g_t\Big)_s$$

$$= \Big(\sum_t \underbrace{(\alpha_1 s^{-1} t \beta_1)}_{p(s^{-1}t)} g_t\Big)$$

---

$$E \xrightarrow{\alpha} \bigoplus_s V_1 \xrightarrow{\beta} E$$

$$\xi \longmapsto (\alpha_1 s^{-1}\xi)_s \quad \Big| \quad (g_s)_s \longmapsto \sum_s s\beta_1 g_s$$

$f_s = s g_s \qquad \beta\alpha = id \qquad \alpha\beta(g_s)$

$\Gamma$ graded projection.

What you prob. need now is to go back to showing your $V_1$ is ~~minimal~~ nonf over $A = P_F$. ~~for what~~

$$E \xrightarrow{\alpha_1} V_1 \xhookrightarrow{\beta_1} E \xrightarrow{\alpha_1} V_1$$

$$p(s) = \alpha_1 s \beta_1 \qquad \sum_s p(s) V_1 = \sum_s \alpha_1 s \beta_1 V$$
$$= \alpha_1 \underline{\sum s\beta_1 V} = \alpha_1 E = V_1$$

$$\bigcap_s \operatorname{Ker} p(s) = \bigcap_s \operatorname{Ker}\{\alpha_1 s \beta_1 : V_1 \to V_1\}$$

$$= \bigcap_s \operatorname{Ker}$$

$$\bigcap_s \operatorname{Ker} p(s) = \bigcap_s \{\alpha_1 s\beta_1 : V_1 \to V_1\} = \{x \in V_1 \mid \forall s\ \alpha_1 s \beta_1 x = 0\}$$
$$= \{x \in V_1 \mid \forall s,\ s\beta_1 x \in \operatorname{Ker} \alpha_1$$

---

$$p(s)v = \alpha_1 s \beta_1 v : V_1 \longrightarrow E \xrightarrow{\alpha_1} V_1$$

$p(s)v = 0 \quad \forall s \quad$ means $\quad s\beta_1 v \in \operatorname{Ker} \alpha_1$ for all $s$

$$\begin{tikzcd} \mathbb{C}[\Gamma]\beta_1 V \arrow[r] \arrow[rr, bend right, "0"'] & E \arrow[r, hook, "\alpha"] & \bigoplus_\Gamma V_1 \end{tikzcd}$$

see p 912

---

$$E \xrightarrow{\alpha} \bigoplus_\Gamma V_1 \xrightarrow{\beta} E \xrightarrow{\alpha} \bigoplus_\Gamma V_1$$

$$(f : \Gamma \to V_1) \overset{\beta}{\longmapsto} \sum_t t\beta_1 f_t \overset{\alpha}{\longmapsto} \Big(\sum_t \alpha_1 s^{-1} t \beta_1 f_t\Big)_s$$

~~point of the calculation~~

$$p(s) = \alpha_1 s \beta_1 : V_1 \longrightarrow V_1$$

$$\sum_s p(s) V_1 = \alpha_1 \sum_s s\beta_1 V_1 = \alpha_1 \beta \bigoplus_\Gamma V = \alpha_1 E = V_1$$

$$p(s^{-1}) = \alpha_1 s^{-1} \beta_1$$

$v \in \operatorname{Ker} \alpha_1 s^{-1}\beta_1 \iff \forall s$
$\beta_1 v \in \operatorname{Ker} \alpha_1 s^{-1}\beta_1 \iff$
$\alpha\beta_1 v = 0. \iff \beta_1 v = 0$
$\iff v_1 = 0$

~~what to do~~

$$\alpha : E \hookrightarrow \bigoplus V_1$$
$$\xi \longmapsto (\alpha_1 s^{-1}\xi)_s$$

You wanted ~~[illegible]~~ to review this for some reason. To review the position

Given $\Gamma$, $F$ get $P_F$

gens $p_s$ $s \in \Gamma$
rels $p_s = \sum_t p_t p_{t^{-1}s}$ $p_s = 0$ $s \notin F$.

---

Review. $B = C \rtimes \Gamma$, $C$ gen $h_s$ $s \in \Gamma$
rel $h_s h_t = 0$ $s^{-1}t \notin F$

$$\sum_{s \in tF^{-1}} h_s h_t = \sum_{s \in tF} h_t h_s = h_t$$

$C$ has left and right local units, same should be true for $B$, $B = \bigoplus_s Cs = \bigoplus_s \sum_t h_t Cs = \sum_t h_t B$.

~~[illegible]~~ a $B$-module $E$ should be firm if $E = \sum h_t E$ whence $\sum h_s = 1$ on $E$, etc.

What you have now: You have an explicit equivalence between firm $B$-modules and ~~firm~~ $A = P_F$-modules, rather nonf modules over $P_F$.

Perhaps some insight arises from looking at any $A = P_F$-module, i.e. vector space $V$ equipped with operators $p_s$ etc. ~~What [illegible]~~

Still to construct the dual pair ~~corresponding~~ yield the Morita equivalence you have.

NOTE: Let $V$ be an $A = P_F$ module, and $E(V)$ the associated $B = C \rtimes \Gamma$-module. ~~The~~ Recall $E(V) = \underline{p(\mathbb{C}[\Gamma] \otimes V)}$ = the image of the operator on $\mathbb{C}[\Gamma] \otimes V = \bigoplus_{s \in \Gamma} V$ given by

$$(pf)_s = \sum_t p_{s^{-1}t} f_t.$$

~~What the~~

$$\begin{tikzcd}
E(V) \arrow[r, "\alpha"] \arrow[dr, "\alpha_1"'] & \mathbb{C}[\Gamma] \otimes V \arrow[r, "\beta"] \arrow[d, shift right, "\partial_1"'] & E(V) \\
& V \arrow[u, shift right, "\iota_1"'] \arrow[ur, "\beta_1"'] &
\end{tikzcd}$$

$\beta_1 \alpha_1 = h_1$

$$\begin{tikzcd}
E \arrow[rr, "\alpha_1"] \arrow[dr, two heads] & & V \arrow[rr, "\beta_1"] \arrow[dr, two heads] & & E \\
& \alpha_1 E \arrow[ur, hook] \arrow[dr, two heads] & & \beta_1 V \arrow[ur, hook] & \\
& & h_1 E \arrow[ur, hook] & &
\end{tikzcd}$$

Analyze the situation a bit abstractly. You have idempotent rings $A, B$ and an equivalence $\mathcal{M}(A) \simeq \mathcal{M}(B)$. Abstractly you know that functors in two directions are given by $P \otimes_A -$, $Q \otimes_B -$.

$P \otimes_A V = E(V)$     $P$ should be $E(A)$. So look at the other functor taking a firm $B$-module $E$ to the image of $h_1$.

Key idea involves choosing any factorization of $h_1 : E \xrightarrow{\alpha_1} V \xrightarrow{\beta_1} E$.

Proceed carefully. Start with $E$ a $B = C \rtimes \Gamma$-module which is firm, so that $E = \sum h_s E$ and $\Gamma$ acts on $E$. If you ~~sta~~ start with any $B$ module $N$ then $CN = \sum h_s N$ should be firm. (There's much to prove here). Now you want something like

---

~~[illegible] You are trying to make the M. eq~~

You have ~~[illegible]~~ constructed the explicit M. eq. between $A = P_F$ and $B = C_F \rtimes \Gamma$. Now you want the dual pair assoc. to this M. eq. ~~One of [illegible]~~

$$\begin{array}{r} \cancel{36.16} \\ \cancel{11.84} \\ \hline \cancel{48.00} \end{array} \qquad \begin{array}{r} \cancel{49.718} \\ \cancel{17.80} \\ \hline 31.918 \end{array}$$

---

Start again: $A = P_F$ $\left(\begin{array}{l} \text{gens } p_s,\ s \in \Gamma \\ \text{rels } p_s = \sum_t p_t p_{t^{-1}s},\ p_t = 0 \text{ for } t \notin F. \end{array}\right.$

$C_F$: gens $h_s, s \in \Gamma$
rels $h_s h_t = 0$ for $s^{-1}t \notin F$, $\sum_{s \in tF^{-1}} h_s h_t = \sum_{s \in tF} h_t h_s = h_t$

$C_F$ has local left units and local right units.

$B = C_F \rtimes \Gamma$. A firm $B$-module should be a v.s. with $\Gamma$-action and family of ~~[illegible]~~ $\mathbb{C}$-linear ops $h_s, s \in \Gamma$ s.t. $t h_s t^{-1} = h_{ts}$ $\sum h_s = 1$. ~~[illegible]~~

$\mathcal{M}(B) \longrightarrow \mathcal{M}(A)$

$E \longmapsto \operatorname{Im}(h_1 \text{ on } E)$ but you want a firm version maybe. What does this mean?

~~What does this mean???~~ It should be ~~some~~ factorization of $h_1 : E \xrightarrow{\alpha_1} V \xrightarrow{\beta_1} E$ where $\beta_1$

might not be injective, in fact you expect to define by relations in $E$ involving $\beta_1$.

$$\begin{tikzcd}[column sep=small]
E \arrow[rr, "\alpha_1"] \arrow[dr, two heads] & & V \arrow[rr, "\beta_1"] \arrow[dr, two heads] & & E \\
& \alpha_1 E \arrow[ur, hook] \arrow[dr, two heads] & & \beta_1 V \arrow[ur, hook] & \\
& & h_1 E \arrow[ur, hook] & &
\end{tikzcd}$$

this is the general situation i.e. general fact.

For $V$ to be firm over $P_F$ I think you want $\alpha_1$ to be surjective, so the above becomes.

$$\begin{tikzcd}
E \arrow[r, two heads, "\alpha_1"] & V \arrow[rr, "\beta_1"] \arrow[dr, two heads] & & E \\
& & \beta_1 V \arrow[ur, hook] &
\end{tikzcd}$$

Some how you want a natural quotient of $E$. Here's an idea. Take a general factorization $h_1 = \beta_1 \alpha_1 : E \longrightarrow V \longrightarrow E$. ~~to replace $V$~~

~~by $\sum_{s\in F} p_s V = \sum_{s\in F} \alpha_1 s \beta_1 V = \alpha_1\left(\sum_s s\beta_1\alpha_1 E\right) = \alpha_1\left(\sum_s s\beta_1\alpha_1 s^{-1} E\right) = \alpha_1 E$~~

$$\sum_{s\in F} p_s V = \sum_s \alpha_1 s \beta_1 V = \alpha_1\left(\sum_s s\beta_1 V\right) = \alpha_1 E$$

$$\bigcap_{s\in F} \mathrm{Ker}(p_s : V \to V) = \{v \mid \forall s \quad \alpha_1 s \beta_1 v = 0\} = \{v \mid \beta_1 v \in \mathrm{Ker}(\alpha)\}$$

$$(\alpha \xi)_s = \alpha_1 s^{-1} \xi \qquad = \mathrm{Ker}(\beta_1)$$

$$\sum_{s\in F} p_s V = \sum_{s\in\Gamma} \alpha_1 s\beta_1 V = \alpha_1\Big(\sum_s s\beta_1 V\Big) = \alpha_1\big(\beta(\mathbb{C}[\Gamma]V)\big)$$
$$= \alpha_1 E$$

$$\bigcap_{s\in F} \mathrm{Ker}(p_s : V\to V) = \{v \mid \forall s\ \ \alpha_1 s\beta_1 v = 0\} = \{v \mid \forall s\ (\alpha\beta_1 v)_s = 0\}$$
$$= \{v \mid \alpha\beta_1 v = 0\} = \mathrm{Ker}(\beta_1 : V \to E).$$

~~What you need is a coherent~~

Repeat: Given $E$ firm $B$-module, you want the corresponding firm $A$-module $V$. This should ~~be~~ occur in a factorization $E \xrightarrow{\alpha_1} V \xrightarrow{\beta_1} E$ where $\alpha$ is surjective, because $V$ firm implies $V = AV$ and $AV = \sum p_s V = \alpha_1 E$ as above.

Now you want to define $V$ as a cokernel of operators on $E$. What relations? $h_t h_s = 0$ for $t \neq s \in F$

$$E \xrightarrow[s\in ?]{\sum h_s} E \xrightarrow{h_1} h_1 E \qquad \sum_{s\in F} h_1 h_s = h_1$$

So define $V_1 = \mathrm{Coker}\Big(E \xrightarrow[s\in F]{\sum h_s} E\Big)$ and let ~~$\alpha_1$ obvious~~

$\alpha_1 : E \longrightarrow V$ ? Inside $B$ you have

$$h_1 \sum_{s\in F} h_s = h_1$$

$$\begin{tikzcd} E \arrow[r, "\sum h_s"] & E \arrow[r, two heads, "\alpha_1"] \arrow[d, "h_1"] & V \arrow[d, "\beta_1"] \arrow[dll, dashed] \\ & h_1 E \arrow[r, hook, "\beta_1"'] & E \end{tikzcd}$$

This is confused but it looks OK.

$$\begin{tikzcd} E \arrow[r, "k_1 = \sum_{s\in F} h_s"] & E \arrow[r, two heads, "\alpha_1"] \arrow[d] \arrow[dr, "h_1"] & \mathrm{Coker}(k_1) \arrow[d, "\beta_1"] \\ & h_1 E \arrow[r, hook] & E \end{tikzcd}$$

Is $V_1 = E \Big/ \Big(\sum_{s\in F} h_s\Big)E$ independent of enlarging $F$?

$$\begin{tikzcd}
E \arrow[r, "\sum_{s\in F} h_s"] & E \arrow[r, two heads, "\alpha_1"] \arrow[d, shift right, "h_1"'] & V_1 \arrow[dl, "\beta_1" near start] \\
E \arrow[r, "\sum_{s\in F'} h_s"] & E \arrow[r, two heads, "\alpha_1'"'] \arrow[u, shift right, "h_1"'] & V_1' \arrow[ul, "\beta_1'"' near start]
\end{tikzcd}$$

$$\begin{tikzcd}
E \arrow[r, "\sum_{s\in F} h_s"] & E \arrow[r, two heads, "\alpha_1"] \arrow[dr, "h_1"'] & V \arrow[d, "\beta_1"] \\
 & & E
\end{tikzcd}$$

?

---

Try again. Given $E$ the smallest $V$ is $V \cong h_1 E$ and it corresp to the fact. $E \xrightarrow{\alpha_1} V \xrightarrow{\beta_1} E$ $\alpha_1$ surj, $\beta_1$ injective. Instead you want a quotient of $E$ defined by the operators $h_s$ on $E$.

$$E \xrightarrow{\sum_{s\in F} h_s} E \xrightarrow{h_1} E \xrightarrow{\sum_{s\in F^{-1}} h_s} E$$

If $F = F^{-1}$ things might be simpler.

Examine the relations. $kh = h = hk$

To consider the nonunital ring gen $h, k$
rel $hk = h = kh$

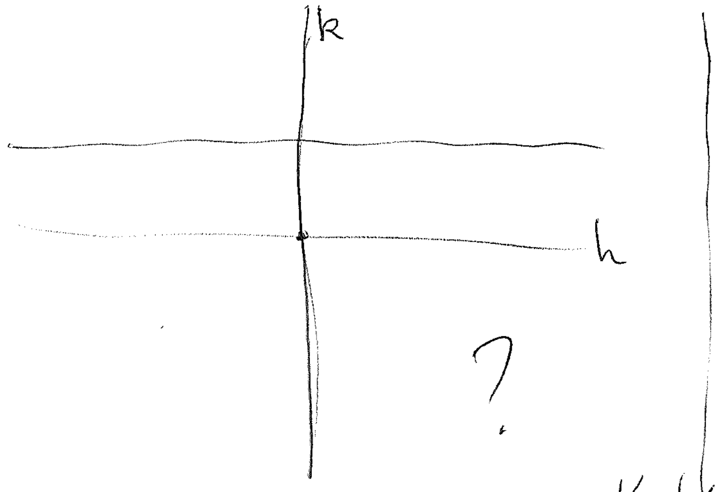
ring commutative. Look at rep. ~~[illegible]~~ in a field.

$h = 0$, $k$ arb. $\qquad hk^2 = hk = h \qquad h(k^2-k) = 0$

$h \neq 0$, $k = 1$.

$\mathbb{C}[h,k] \Big/ \big(h(k-1) = 0\big)$ plane curve union of $h=0$, $k=1$

want the ideal ~~where [illegible]~~ vanishing at origin




What seems to be relevant is

$$\begin{tikzcd}
 & E \arrow[r, "k"] \arrow[d, "h"'] & E \arrow[r, two heads] \arrow[d, "h"] \arrow[dl, "h"'] & E/kE \\
\operatorname{Ker} k' \arrow[r, hook] & E \arrow[r, "k'"'] & E &
\end{tikzcd}$$

~~So what are the relevant points. What is relevant.~~

Look at $kh = h = hk$.

$$\begin{tikzcd}
{}_kE \arrow[r] \arrow[d, "0"'] & E \arrow[r, "k"] \arrow[d, "h"'] & E \arrow[r] \arrow[d, "h"] \arrow[dl, "h"'] & E/kE \arrow[d, "0"] \\
{}_{k'}E \arrow[r] & E \arrow[r, "k'"'] & E \arrow[r] & E/k'E
\end{tikzcd}$$

it looks the ~~map~~ of complexes is homotopic to 0

~~[illegible]~~ Try again:

You have a functor from $A = P_F$ modules to firm $B = C_F \rtimes \Gamma$ - modules which ~~kills~~ is exact and kills nil modules, and a functor in the opposite direction ~~with~~ yielding nonf modules. But to describe the Morita equivalence as a dual pair you normally use firm bimodules.

~~What you don't~~ Where to begin? find bimodules suitable. Given an $A = P_F$ module $V$ you ~~take~~ form $E(V) = p(\mathbb{C}[\Gamma] \otimes V)$. Bimodule is $p(\mathbb{C}[\Gamma] \otimes A)$ same for $A, \tilde{A}, \tilde{A}/A$

~~So what is at stake?~~ Where to start?

Simplest case $F = \{1\}$. $A = \mathbb{C}p_1$ $p_1^2 = p_1$ $p_s = 0$ $s \neq 1$.

$V$ an $A$-module $V = p_1 V \oplus (1-p_1)V$

$$\bigoplus_t V \xrightarrow{\beta} E \xrightarrow{\alpha} \bigoplus_s V$$

$$f \longmapsto \sum_t t\beta_1 f_t \longmapsto \sum_t \overbrace{(\alpha_1 s^{-1} t \beta_1)}^{p(s^{-1}t)} f_t$$

when $F=1$ you have $f \longmapsto (pf)_s = \sum_t p(s^{-1}t) f_t$

$= p_1 f_s$

---

Repeat formulas. $A = P_\Phi$ : gens $p_s$, $s \in \Gamma$

rels $p_s = 0$, $s \notin \Phi$ | $p_s = \sum_t p_t p_{t^{-1}s}$

$V$ be an $A$-module

$$A = \sum_s p_s A = \sum_s A p_s$$

$$AV = \sum p_s V$$

$${}_A V = \bigcap_s \mathrm{Ker}\{p_s \text{ on } V\}$$

Example $\Gamma = \mathbb{Z}$, $\Phi = \{-1, 0, 1\}$

$A$ has 3 gens $p_{-1}, p_0, p_1$

5 relns. $(p_{-1} + p_0 + p_1)(p_{-1} + p_0 + p_1) = 0$ components have homog. degrees $-2, -1, 0, 1, 2$

What is $E(A)$? What next?

$E(A) \subset \mathbb{C}[\Gamma] \otimes A$ $\Gamma$ graded ring

$\Gamma$-graded module.

Problem: to find firm bimodules for the M-equiv.

$$E(A) = p(\mathbb{C}[\Gamma]\otimes A) \qquad \mathbb{C}[\Gamma]\otimes V = \bigoplus_{s\in\Gamma} V$$

$$E(V) = p(\mathbb{C}[\Gamma]\otimes \cancel{V}) \qquad = \{ f:\Gamma\to V \mid \mathrm{supp}(f) \text{ finite}\}$$

$$\mathbb{C}[\Gamma]\otimes V \xrightarrow{\beta} E(V) \xrightarrow{\alpha} \mathbb{C}[\Gamma]\otimes V \xrightarrow{\beta} E(V)$$

$$(f_s) \longmapsto \sum_s s\beta_1 f_s$$

$$\xi \longmapsto (\alpha\xi)_s = \alpha_1 s^{-1}\xi$$

$$f=(f_s)_{s\in\Gamma} \overset{\beta}{\longmapsto} \sum_t t\beta_1 f_t \overset{\alpha}{\longmapsto} \Big(\sum_t \underbrace{(\alpha_1 s^{-1}t\beta_1)}_{p_{s^{-1}t}} f_t\Big)_{s\in\Gamma}$$

~~[illegible]~~ Does $p$ belong to $\mathbb{C}[\Gamma]\otimes A$ in some sense? $\mathbb{C}[\Gamma]\otimes A = \bigoplus_{s\in\Gamma} A = \{ f:\Gamma\to A,\ s\mapsto f_s \mid \mathrm{supp}\, f \text{ finite}\}$

~~[illegible]~~ give $\mathbb{C}[\Gamma]\otimes A$ the tensor product ring structure

There are many issues you still do not understand.

Start with $B = C_\Phi \rtimes \Gamma$, $C_\Phi$: gens $h_s$, $s\in\Gamma$ rels. $h_s h_t = 0$ for $s^{-1}t\notin\Phi$, $\sum_{s\in t\Phi} h_s h_t = \sum_{s\in t\Phi} h_t h_s = h_t$

~~[illegible]~~ You have to understand how $B$ acts on $E(V)$ for any $A=P_\Phi$-module $V$.

Questions: ring structure on $\mathbb{C}[\Gamma]\otimes A$?

~~does~~ Is $p$ an idempotent in $\mathbb{C}[\Gamma]\otimes A$?

Let's ~~[illegible]~~ get this clear

$$\mathbb{C}[\Gamma]\otimes A = \bigoplus_s A$$

~~So what goes on?~~ In $A$ you have generator ~~elements~~ $p_s,\ s\in\Gamma$ rels $p_s=0$ $s\notin\Phi$, $\sum_s p_s p_{s^{-1}t}=p_t$.

What is the universal property of $A$? Notion of a $\Gamma$-graded alg. $B=\bigoplus_{s\in\Gamma} B_s$ where $B_sB_t\subset B_{st}$. $A=P_F$ is a $\Gamma$-graded algebra $A=\bigoplus A_s$ where $p_s\in A_s$.

Anything to check.

---

more about $\Gamma$ graded ~~algebras~~. v.s.

same as $V \xrightarrow{\Delta_V} \Lambda\otimes V \underset{1_V\otimes\Delta}{\overset{\Delta_\Lambda\otimes 1_V}{\rightrightarrows}} \Lambda\otimes\Lambda\otimes V$

Why? $\Delta_V(v)=\sum_s s\otimes e_s(v) \longrightarrow \sum_s s\otimes s\otimes e_s(v)$

$$\longrightarrow \sum_s s\otimes\sum_t t\otimes e_te_s(v)$$

$$\sum_{t,s} s\otimes t\otimes e_te_s = \sum s\otimes s\otimes e_s$$

$e_te_s=\delta_{s,t}e_s$ annihilating proj

non unitaly comodule

Counit $\eta:\Lambda\longrightarrow\mathbb{C}$ $\eta(s)=s$ $\forall s$.

$$\begin{tikzcd} V \arrow[r] \arrow[dr, bend right, "1"'] & \Lambda\otimes V \arrow[d, "\eta\otimes 1"] \\ & V \end{tikzcd}$$

$\sum e_s v = v$.

So now you have some experience with $\Gamma$ graded v.s. equiv. $\hat\Gamma$-modules. So go back to Morita equiv.

~~Anyway you have this variant~~ When $\Gamma$ is a group $\hat\Gamma$-modules form a $\otimes$ cat.

$$V=\bigoplus_s V_s \qquad W=\bigoplus_t W_t$$

$$V\otimes W=\bigoplus_{s,t} V_s\otimes W_t = \bigoplus_u \underbrace{\Big(\bigoplus_{u=st} V_s\otimes W_t\Big)}_{(V\otimes W)_u}$$

$\Gamma$-Graded alg. $\quad A = \oplus A_s \qquad A_s A_t \subset A_{st}$

Given $\Gamma$, $\Phi$ finite $\subset \Gamma$, you have $P_\Phi$ gens $p_s$, $s \in \Gamma$, rels $p_s = 0$, $s \notin \Phi$, $p_s = \sum_t p_t p_{t^{-1}s}$. To see that $A = P_\Phi$ is naturally $\Gamma$-graded. To construct the splitting $A = \oplus A_s$ using the universal prop of $A = P_\Phi$

Question: If $A$ is a $\Gamma$-graded alg, can you relate this to the coaction $A \longrightarrow \mathbb{C}[\Gamma] \otimes A$. What does this mean? Do for $V, W$

$$V \longrightarrow \Lambda \otimes V \qquad V \otimes W \longrightarrow$$
$$W \longrightarrow \Lambda \otimes W$$

$$\Delta v = \sum_s s \otimes e_s v \qquad \Delta w = \sum_t t \otimes e_t w$$

$e_u$ on $V \otimes W \qquad e_u = \sum_{u=st} e_s \otimes e_t$

$$\Delta(v \otimes w) = \sum_u u \otimes \sum_{u=st} e_s u \otimes e_t w$$

$$V \otimes W \xrightarrow{\Delta_V \otimes \Delta_W} \Lambda \otimes V \otimes \Lambda \otimes W = \Lambda \otimes \Lambda \otimes V \otimes W$$

$$v \otimes w \longmapsto \sum_{s,t} (s \otimes e_s v) \otimes (t \otimes e_t w) \qquad \downarrow \mu \otimes 1 \otimes 1$$

$$\Lambda \otimes (V \otimes W)$$

$$a' \quad a'' \qquad \sum_{s,t} s \otimes t \otimes e_s a' \otimes e_t a'' \qquad \sum_u u \otimes \sum_{u=st} e_s a' \otimes e_t a''$$

so $A \otimes A \longrightarrow \Lambda \otimes \Lambda \otimes A \otimes A \longrightarrow \Lambda \otimes A \otimes A$

The question is whether the map $A \otimes A \xrightarrow{\Delta_{A \otimes A}} \Lambda \otimes A \otimes A$ corresp. to the $\Gamma$ grading of $A \otimes A$ is an alg map

$$\begin{array}{ccc} A \otimes A & \xrightarrow{\Delta_{A\otimes A}} & \Lambda \otimes A \otimes A \\ \downarrow \mu & & \downarrow 1 \otimes \mu \\ A & \xrightarrow{\Delta_A} & \Lambda \otimes A \end{array}$$

$$a' \otimes a'' \longmapsto \sum_s s \otimes e_s(a' \otimes a'') = \sum_u u \otimes \sum_{u=st} e_s a' \otimes e_t a'' \to$$

$$\sum_u \sum_{u=st} e_s a' \otimes e_t a'' = e_u(a' \otimes a'')$$

$$a' \otimes a'' \xrightarrow{\Delta \otimes \Delta} \sum_s s \otimes a'_s \otimes \sum_t t \otimes a''_t$$

$$\Big\downarrow \text{mult in } \Lambda \otimes A$$

$$\sum_{s,t} st \otimes a'_s a''_t \, .$$

$$\|$$

$$\sum_u u \otimes (a'a'')_u$$

So you learn that if $A$ is a $\Gamma$-graded alg then the $\Gamma$-action $A \xrightarrow{\Delta} \Lambda \otimes A$, $a \mapsto \sum_s s \otimes a_s$ is an alg. hom., in fact a $\Gamma$-graded alg homom. ($a = a_1$) (where the second $A$ has trivial grading). ~~With this~~

Now look at $P_\Phi$. should be obvious

For any alg $A$, $B = \Lambda \otimes A$ is a $\Gamma$-graded alg. where $B_s = s \otimes A$. But $A = P_\Phi$ has canon. elts $p_s$, so you get a ! homom. $A \to \Lambda \otimes A$ sending $p_s$ to $s \otimes p_s$. check relns. $s \otimes p_s = 0$ $s \notin \Phi$

$$s \otimes p_s \overset{?}{=} \underbrace{\sum_t (t \otimes p_t)(t^{-1}s \otimes p_{t^{-1}s})}_{\sum_t s \otimes p_t p_{t^{-1}s} \;=\; s \otimes p_s}$$

$$\begin{tikzcd} A \arrow[r, "\Delta_A"] \arrow[d, "\Delta_A"'] & \Lambda \otimes A \arrow[d, "\Delta_\Lambda \otimes A"] \\ \Lambda \otimes A \arrow[r, "1 \otimes \Delta_A"] & \Lambda \otimes \Lambda \otimes A \end{tikzcd}$$

$$\Delta_A(p_s) = s \otimes p_s$$

$$(\Delta_\Lambda \otimes 1)\Delta_A p_s = s \otimes s \otimes p_s$$

$$(1 \otimes \Delta_A)\Delta_A(p_s) = (1 \otimes \Delta_A)(s \otimes p_s)$$

$$= s \otimes s \otimes p_s$$

$$A \xrightarrow{\Delta_A} \Lambda \otimes A \xrightarrow{\eta \otimes 1} A$$

$$p_s \longmapsto s \otimes p_s \longmapsto p_s$$

But $\Delta_A$ and $\eta$ are alg. homos.

mystified. So start again, discuss the problems.

$P_\Phi$ is a ~~universal~~ $\Gamma$-graded algebra equipped with a projection $p = \sum_s p_s$ $p^2 = p$, $\sum_{u=st} p_s p_t = p_u$ with support in $\Phi$: $p_s = 0$ $s \notin \Phi$. universal with these properties. You have a canonical morphism ~~map~~ of $\Gamma$-graded algebras $P_\Phi \longrightarrow \mathbb{C}[\Gamma] \otimes P_\Phi$, injective in fact there's a retraction morphism. So ask about $E(\mathbb{C}[\Gamma] \otimes P_\Phi)$?

Another point. Is there ~~an~~ analogue of $X \times_Y X = X \times \Gamma$ in this graded ~~situation~~ ~~case~~?

---

$P_\Phi$ is a $\Gamma$-graded alg, so there is an algebra ~~hom~~ morphism $P_\Phi \longrightarrow \mathbb{C}[\Gamma] \otimes P_\Phi$, to simplify notation

$$\begin{array}{ccc} A & \longrightarrow & \mathbb{C}[\Gamma] \otimes A = \bigoplus_s A \\ \cup & & \cup \\ A_s & \longrightarrow & \delta_s \otimes A_s \end{array}$$

~~We~~ Let's not get overly concerned with notation, yet

What is the point? You want something

Put $A = P_\Phi$. Canonical idempotent in $\mathbb{C}[\Gamma] \otimes A$ namely $p = \sum \delta_s \otimes p_s \longleftrightarrow (p_s) \in \bigoplus_s A$

~~Look again. You~~ must decide what you want.

$B = C_\Phi \rtimes \Gamma$ $\qquad C_\Phi$: gens: $h_s$, $s \in \Gamma$
rels: $h_s h_t = 0$ for $s^{-1}t \notin \Phi$.

~~What is~~ $B = \bigoplus B_s$

$$\sum_{s \in t\Phi^{-1}} h_s h_t = h_t = \sum_{s \in t\Phi} h_t h_s$$

$B_s = C_\Phi s$, $B$ is $\Gamma$-graded

$B$ has

canonical proj in $C^*$-case. $h_1 = h_1^{1/2} h_1^{1/2} = \beta_1 \alpha_1$

$$p_s = h_1^{1/2} s h_1^{1/2} = h_1^{1/2} h_s^{1/2} s$$

$$\sum_t p_t p_{t^{-1}s} = \sum_t h_1^{1/2} t h_1 t^{-1} s h_1^{1/2} = h_1^{1/2} s h_1^{1/2} = p_s .$$

$(p_s)$ is a "$\Gamma$-graded projection" meaning? an idempotent in $\bigoplus_s A$.

---

Begin with $B = C_\Phi \rtimes \Gamma$

a firm $B$-module $E$ is a $\Gamma$-module equipped with operator $h_1$ $\ni \cdots$

$$\begin{cases} h_1 s h_1 = 0 & s \notin \Phi \\ \sum_s s h_1 s^{-1} = id \end{cases}$$

Then you have this ~~analysis~~ equivalence with ncaf $P_F$-modules. $V = h_1 E$

$$E \xrightarrow{\alpha} \bigoplus_s V \xrightarrow{\beta} E \xrightarrow{\alpha} \bigoplus_s V$$

$\xi$ $\quad \alpha\xi = (\alpha_1 s^{-1} \xi)_t$ $\quad \beta f = \sum_t t \beta_1 f_t$ $\quad \beta\alpha = id_E$

$$\alpha\beta f = \Big( \sum_t \underbrace{(\alpha_1 s^{-1} t \beta_1)}_{p_{s^{-1}t}} f_t \Big)_{s \in \Gamma}$$

~~How to make this clearer? How to make~~

Can you say something about $C_\Phi$ as $B$ module?

No feeling yet. Can you say something about $\sum p_s \in A = P_\Phi$. It seems to be ~~a projection~~ idempotent

Look at $\Gamma = \mathbb{Z}$ $\quad \Phi = \{-1, 0, 1\}$.

Concerned with idempotent ~~operators~~ Laurent poly operators.

$$p_{-1} z^{-1} + p_0 + p_1 z$$

So you learn that $(p_s) \longmapsto \sum_{s\in\Gamma} p_s$ means evaluating a $\Gamma$-graded projection at the identity character.

Now $(p_s) \in \bigoplus_s A = \mathbb{C}[\Gamma]\otimes A$ tensor product where factors comm.

so ~~an~~ alg morph. $\mathbb{C}[\Gamma] \longrightarrow D$ yields an idempotent in $D\otimes A$.

Not much today.

It might help to go over $(p_s) \in \mathbb{C}[\Gamma]\otimes A$

---

Go back over grading. $\Gamma$ set, graded vector space $V = \bigoplus_{s\in\Gamma} V_s$ wrt $\Gamma$. coalg $\mathbb{C}[\Gamma]$, $\Delta_\Gamma s = s\otimes 1 + 1\otimes s$

$V$ is a comod over $\mathbb{C}[\Gamma]$ means given

$$V \xrightarrow{\Delta_V} \mathbb{C}[\Gamma]\otimes V \underset{1\otimes\Delta_V}{\overset{\Delta_\Gamma\otimes 1}{\rightrightarrows}} \mathbb{C}[\Gamma]\otimes\mathbb{C}[\Gamma]\otimes V$$

$$\Delta_V v = \sum_{s\in\Gamma} s\otimes e_s(v) \xmapsto{\Delta_\Gamma\otimes 1} \sum_s s\otimes s\otimes e_s(v)$$

$$\xmapsto[1\otimes\Delta_V]{} \sum_s s\otimes \sum_t t\otimes e_t e_s(v)$$

$$e_t e_s(v) = \begin{cases} 0 & t\neq s \\ e_s(v) & t = s.\end{cases}$$

$\therefore$ $e_s$ $s\in\Gamma$ mutually annihilating projections (idempotent ops) on $V$.

A comodule over $\mathbb{C}[\Gamma]$ is thus a family of annihilating idempotents indexed by $\Gamma$. ~~Not~~

But $\mathbb{C}[\Gamma]$ is counital $\eta: \mathbb{C}[\Gamma] \longrightarrow \mathbb{C}$ $\eta(s) = 1$ $\forall s$.

so if you ~~impose~~ require $V$ to be counitary

$$V \xrightarrow{\Delta_V} \mathbb{C}[\Gamma]\otimes V \xrightarrow{\eta\cdot 1} V$$

$$v \longmapsto \sum_{s\in\Gamma} s\otimes e_s v \longmapsto \sum e_s v$$

$\Gamma$ group. Then have $\otimes$ operation on comodules for $\mathbb{C}[\Gamma]$

$$V\otimes W \xrightarrow{\Delta_V\otimes\Delta_W} \mathbb{C}[\Gamma]\otimes V\otimes\mathbb{C}[\Gamma]\otimes W$$

$$\| $$

$$\mathbb{C}[\Gamma]\otimes\mathbb{C}[\Gamma]\otimes V\otimes W \xrightarrow{\mu\otimes1\otimes1} \mathbb{C}[\Gamma]\otimes V\otimes W$$

$$(\Delta_V\otimes\Delta_W)(v\otimes w) = \sum_s s\otimes e_s(v)\otimes\sum_t t\otimes e_t(w)$$

$$\longrightarrow \sum_{u} u\otimes\sum_{u=st} e_s(v)\otimes e_t(w)$$

$\therefore\ e_u$ on $\cancel{\Delta_V\otimes\Delta_W}$ $V\otimes W$

---

Actually the coalg $\mathbb{C}[\Gamma]$ is not really worth mentioning

$\Gamma$ set, $V=\bigoplus_s V_s$ $\Gamma$-graded vector space

~~tensor product~~

$\Gamma$ gp $\quad V\otimes W = \bigoplus_u\Big(\bigoplus_{u=st} V_s\otimes W_t\Big)$

$$= \bigoplus_u\Big(\bigoplus_t V_{ut^{-1}}\otimes W_t\Big)$$

---

what's interesting to us is a splitting of $\underline{\mathbb{C}[\Gamma]\otimes V}$ as $\mathbb{C}[\Gamma]$-module. $\quad \mathbb{C}[\Gamma]\otimes V = \bigoplus_s V = \{f:\Gamma\to V \mid \text{supp} f \text{ finite}\}$

$$\sum_s s\otimes f_s \longleftarrow f$$

$$\Big\downarrow t\cdot$$

$$\sum_s ts\otimes f_s = \sum_s s\otimes f_{t^{-1}s} \longleftarrow (L_t f)_s = f_{t^{-1}s}$$

$T: \mathbb{C}[\Gamma]\otimes V \longrightarrow \mathbb{C}[\Gamma]\otimes W$

$\cancel{(Tf)_s = \sum_t T_{s,t} f_t}$ $\qquad (kf)_s = \sum_t k_{s^{-1}t}\, f_t$

Why is $P_\Phi$ a $\Gamma$-graded algebra?

~~Any other question Wh do what do you do?~~

homogeneous generators and relations.

your proof was to use the universal property to construct a morphism of algs $\varphi : P_\Phi \longrightarrow \Lambda \otimes P_\Phi$ sending $p_s$ to $s \otimes p_s$, check $(\Delta_\Gamma \otimes 1)\varphi = (1 \otimes \varphi)\varphi$

$$(\Delta_\Gamma \otimes 1)\varphi p_s = (\Delta_\Gamma \otimes 1)(s \otimes p_s) = s \otimes s \otimes p_s$$

$$(1 \otimes \varphi)\varphi p_s = (1 \otimes \varphi)(s \otimes p_s) = s \otimes s \otimes p_s$$

Also check $(\eta \otimes 1)\varphi p_s = (\eta \otimes 1)(s \otimes p_s) = p_s$.

Do there some other meaning to $A \longrightarrow \Lambda \otimes A$

Adjoint functors?

Compare $\Gamma$ graded and ungraded vector spaces

$F$: $V$ $\Gamma$-graded $\xrightarrow{\text{forget}}$ $V \in \mathcal{V}$

$G$: $X \in \mathcal{V} \longrightarrow \mathbb{C}[\Gamma] \otimes X$ is $\Gamma$ graded.

canon

$$V \xrightarrow{\beta} \mathbb{C}[\Gamma] \otimes V = GFV$$

$$FGX = \mathbb{C}[\Gamma] \otimes X \xrightarrow{\alpha = \eta} X$$

$$\alpha : FG \longrightarrow 1$$

$$\alpha.F : FGF \to F$$

$$\beta : 1 \longrightarrow GF$$

$$F.\beta : F \longrightarrow FGF$$

$$\mathrm{Hom}(FX, Y) = \mathrm{Hom}(X, GY)$$

$$\alpha : FGY \longrightarrow Y \qquad \beta : X \longrightarrow GFX$$

$$\begin{tikzcd}
\mathrm{Hom}(FX,Y) \arrow[r, "G"] & \mathrm{Hom}(GFX, GY) \arrow[r, "\beta^*"] & \mathrm{Hom}(X, GY) \arrow[dl, "F."] \\
& \mathrm{Hom}(FX, FGY) \arrow[ul, "\alpha_*"] &
\end{tikzcd}$$

$$FX \xrightarrow{F.\beta} FGFX \xrightarrow{\alpha . F} FX$$

$$V = \bigoplus_s V_s \longrightarrow \mathbb{C}[\Gamma] \otimes V$$

$$V_s \longrightarrow s \otimes V_s$$

$$\mathcal{V} \underset{\text{forget grading}}{\overset{1\otimes -}{\rightleftarrows}} \mathcal{V}_\Gamma \qquad V \overset{G}{\longmapsto} \bigoplus_s V = \mathbb{C}[\Gamma] \otimes V$$

$$(V_s)_s \overset{F}{\longmapsto} \bigoplus_s V_s$$

---

You need to check this.

$$\mathcal{A} \rightleftarrows \mathcal{A}_\Gamma$$

$$\bigoplus_s M_s \longleftarrow (M_s)_{s\in\Gamma}$$

$$M \longmapsto (M)_{s\in\Gamma}$$

an obj of $\mathcal{A}_\Gamma$ is a family $(M_s)_{s\in\Gamma}$ obvious maps of families.

---

Given a graded module $(M_s)_{s\in\Gamma}$ you send it to $\bigoplus_{s\in\Gamma} M_s$

~~module~~ $N$ you send it to $(N)_{s\in\Gamma}$

It might be clearer if you took $\mathcal{A}$ modules, $\mathcal{A}_\Gamma$ modules with $\Gamma$-grading.

$$M \longmapsto \mathbb{C}[\Gamma]\otimes M$$

$$\mathcal{A} \rightleftarrows \mathcal{A}_\Gamma$$

$$N \longleftarrow N = \bigoplus_{s\in S} N_s \text{ with grading}$$

$$\text{adj:} = \underbrace{\eta}_{\text{counit}} : \mathbb{C}[\Gamma]\otimes M \longrightarrow M.$$

$$\alpha : FGM \longrightarrow M$$

$$N = \bigoplus_s N_s \overset{\beta}{\longrightarrow} \underset{GFN}{\mathbb{C}[\Gamma]\otimes N}$$

$$(f_s) \qquad \sum s\otimes f_s$$

Repeat: $\bigoplus_{s\in\Gamma} V_s$ graded $\overset{F}{\longmapsto} \bigoplus_{s\in\Gamma} V_s$ ungraded

$$\mathbb{C}[\Gamma]\otimes W = \bigoplus_{s\in\Gamma} W \overset{G}{\longleftarrow} W.$$

$$\mathrm{Hom}_{\Gamma\text{-gr}}\Big(\bigoplus_{s\in\Gamma} V_s, \underbrace{\bigoplus_s W}_{GW}\Big) = \prod_s \mathrm{Hom}(V_s, W)$$

$$= \mathrm{Hom}_{\mathbb{C}}\Big(\underbrace{\bigoplus_s V_s}_{F(V_s)}, W\Big)$$

$$\mathrm{Hom}_{\hat{\Gamma}}\big((V_s)_s, (W)_s\big) = \prod_s \mathrm{Hom}_{\mathbb{C}}(V_s, W) = \mathrm{Hom}_{\mathbb{C}}\Big(\bigoplus_s V_s, W\Big)$$

$G$ $F$

$$\mathrm{Hom}(V, GW) = \mathrm{Hom}(FV, W)$$

$\alpha : FGW \longrightarrow W$

$$\bigoplus_s W \longrightarrow W$$
$$\mathbb{C}[\Gamma] \otimes W \xrightarrow{\eta \otimes 1} W$$

$\beta : V \longrightarrow GFV$

$$(V_s)_s \longmapsto \Big(\bigoplus_t V_t\Big)_s$$

$\forall s \quad V_s \longrightarrow \bigoplus_t V_t$

is the inclusion corresp to $t = s$.

---

Review: Claim you have adjoint functors

$$\mathcal{V} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{V}^\Gamma \qquad F(V_s)_{s\in\Gamma} = \bigoplus_{s\in\Gamma} V_s$$
$$GW = (W)_{s\in\Gamma}$$

$$\mathrm{Hom}_{\mathcal{V}}\Big(\bigoplus_{s\in\Gamma} V_s, W\Big) = \prod_{s\in\Gamma} \mathrm{Hom}(V_s, W) = \mathrm{Hom}_{\mathcal{V}^\Gamma}\big((V_s)_{s\in\Gamma}, (W)_{s\in\Gamma}\big)$$

$\alpha : FG \longrightarrow \mathrm{id}$

$$\mathrm{Hom}(FX, Y) = \mathrm{Hom}(X, GY)$$

$$\bigoplus_s W \longrightarrow W$$

$\beta : \mathrm{id} \longrightarrow GF$

$$(V_s)_{s\in\Gamma} \longmapsto \Big(\bigoplus_{t\in\Gamma} V_t\Big)_{s\in\Gamma}$$
$$\underbrace{\bigoplus V_s}_{V} \longrightarrow \mathbb{C}[\Gamma] \otimes \bigoplus_{t\in\Gamma} V_t$$
$$f \longmapsto \sum_s s \otimes f_s$$

I'm sure these are the correct maps, but the notation might be cleaner.

Instead of $\mathcal{V}^\Gamma$ = families $(V_s)_{s\in\Gamma}$, try $\mathcal{V}^\Gamma$ = $V$ equ. with

$V = \bigoplus_{s\in\Gamma} V_s$

$F$ = forget the grading

$$GW = \mathbb{C}[\Gamma] \otimes W = \bigoplus_{s\in\Gamma} s \otimes W$$

$$\mathrm{Hom}_{\mathcal{V}^\Gamma}\Big(\bigoplus_s V_s, \overset{GW}{\bigoplus_s s\otimes W}\Big) = \prod_s \mathrm{Hom}(V_s, W) = \mathrm{Hom}_{\mathcal{V}}\Big(\overset{F(\oplus V_s)}{\bigoplus_s V_s}, W\Big)$$

$$\left(\bigoplus_s W \xrightarrow{id} \bigoplus_s W\right) \longmapsto \left(W \xrightarrow{\iota_s} \bigoplus_\Gamma W\right)_s$$

$$\longmapsto \left(\bigoplus_{s\in\Gamma} W \xrightarrow{\Sigma} W\right)$$

$$\mathrm{Hom}_{\mathcal{V}}\left(\bigoplus_\Gamma V_s, \bigoplus_\Gamma V_s\right).$$

$$\left(\bigoplus_\Gamma V_s \xrightarrow{id} \bigoplus_\Gamma V_t\right) \longleftrightarrow \left(V_s \xrightarrow{\iota_s} \bigoplus_{t\in\Gamma} V_t\right)_s$$

$$\longleftrightarrow \bigoplus_s V_s \xrightarrow{\oplus \iota_s} \underbrace{\bigoplus_s \left(\bigoplus_t V_t\right)}_{\bigoplus_s s\otimes \bigoplus_t V_t}$$

$$\bigoplus_s V_s \cup V_s$$

---

~~What maps V and~~ You need a good notation.

$$V = \bigoplus_{s\in\Gamma} V_s \qquad \left( \begin{aligned} j_s \iota_t &= \delta_{st} \\ \sum \iota_s j_s &= id. \end{aligned} \right.$$

maybe better would be $h_s h_t = 0 \quad s\neq t$

$\sum h_s = id.$ $\quad \left(\Longrightarrow h_s h_t = \begin{cases} 0 & s\neq t \\ h_t & s=t \end{cases}\right)$

---

To see if this helps.

$V$ $\Gamma$-graded i.e. have $h_s$ $s\in\Gamma$ satisf- above. Go back to adjunction. Let

Given $W$ v.s. get $\mathbb{C}[\Gamma]\otimes W = \bigoplus_{s\in\Gamma} s\otimes W$

$$\begin{array}{ccc} \mathrm{Hom}_{\hat\Gamma}(V, \mathbb{C}[\Gamma]\otimes W) & = & \mathrm{Hom}(V, W) \\ \| & & \| \\ \prod_s \mathrm{Hom}(V_s, W) & = & \mathrm{Hom}(\bigoplus_s V_s, W) \end{array}$$

---

try new notation. $\hat\Gamma$ ring gens $e_s$ $s\in\Gamma$ relations $e_s e_t = \begin{cases} 0 & s\neq t \\ e_t & s=t \end{cases}$. Firmness condition $W = \sum e_s W$. OKAY. Look at adjoint fns.

$$\mathrm{Hom}_{\hat\Gamma}\Big(V, \underbrace{\overset{G(W)}{\mathbb{C}[\Gamma]\otimes W}}_{\bigoplus_s s\otimes W}\Big) \qquad e_t(s\otimes w) = \begin{cases} 0 & t\neq s \\ s\otimes w & t=s. \end{cases}$$

$$= \prod_s \mathrm{Hom}(V_s, W) = \mathrm{Hom}_{\mathbb{C}}\Big(\overset{F(V)}{\bigoplus_{s\in\Gamma} V_s}, \overset{W}{W}\Big)$$

Canonical $\alpha : \underbrace{FG(W)}_{} \longrightarrow W$

$$\bigoplus_s s\otimes W \longrightarrow W$$

$$V = \bigoplus_s s\otimes W \xrightarrow{\ id\ } \bigoplus_s s\otimes W$$

~~set of inc.~~ $\cancel{s\otimes W \hookrightarrow \bigoplus_t t\otimes W}$

$\cancel{\bigoplus_s s\otimes W \quad \bigoplus_t}$

corresp to family $s\otimes W \longrightarrow W$

corresp to $\bigoplus_s s\otimes W \longrightarrow W$

$$\mathrm{Hom}_{\hat{\Gamma}}(V, GW)$$

Take $V = GW$

$(GW)_s = W \quad \forall s$

$$\parallel$$

$$\prod_s \mathrm{Hom}(V_s, W)$$

get familys $V_s \xrightarrow{f_s} W$

$$\parallel$$

$$\mathrm{Hom}(\underbrace{\bigoplus_s V_s}_{FV}, W)$$

get $\bigoplus$

---

$$\mathrm{Hom}_{\hat{\Gamma}}(GW, GW)$$

$$\parallel$$

$$\prod_s \mathrm{Hom}(W, W)$$

$$\parallel$$

$$\mathrm{Hom}(\underbrace{\bigoplus_s W}_{FGW}, W)$$

get $\alpha = \sum pr_s$

---

$$\mathrm{Hom}(\bigoplus_s V_s, \bigoplus_s V_s)$$

id

$\updownarrow$

$$\parallel$$

$$\prod_s \mathrm{Hom}(V_s, \bigoplus_t V_t)$$

$\iota_s : V_s \hookrightarrow \bigoplus_t V_t$

$$\parallel$$

$$\mathrm{Hom}_{\hat{\Gamma}}(V, G \bigoplus_t V_t)$$

$$\bigoplus_s V_s \xrightarrow{\oplus \iota_s} \overbrace{\bigoplus_s \bigoplus_t V_t}^{\mathbb{C}[\Gamma] \otimes V}$$

$$V_s \longrightarrow s \otimes V_s \subset \mathbb{C}[\Gamma] \otimes V$$

So now consider a $\hat{\Gamma}$-algebra

$$A = \bigoplus_{s\in\Gamma} A_s \qquad A_s A_t \subset A_{st}$$

$$\mathrm{Hom}_{\hat{\Gamma}\text{-alg}}(A, GB)$$
$$\cap$$
$$\prod_s \mathrm{Hom}(A_s, B)$$

$$GB = \mathbb{C}[\Gamma]\otimes B = \bigoplus_s B$$
$$(GB)_s = s\otimes B \quad \forall s$$
$$(s\otimes b)(t\otimes b') = st\otimes bb'$$

Q: Let $u_s : A_s \longrightarrow B$ be linear maps for $s\in\Gamma$

~~Then~~ When is $\bigoplus_{s\in\Gamma} A_s \xrightarrow{(s\otimes u_s)} \bigoplus_{s\in\Gamma} s\otimes B$

an alg morphism.

$$\begin{tikzcd}
A_s \otimes A_t \arrow[r, "s\otimes u_s \otimes t\otimes u_t"] \arrow[d] & (s\otimes B)\otimes(t\otimes B) \arrow[d] \\
A_{st} \arrow[r, "st\otimes u_{st}"] & st\otimes B
\end{tikzcd}$$

$$u_{st}(a_s b_t) \overset{?}{=} (s\otimes u_s(a_s))(t\otimes u_t(a_t)) = st\otimes u_s(a_s)u_t(a_t)$$

---

So what's up. Functors $F : \hat{\Gamma}\text{-space} \to V\text{-space}$

$$(V_s)_{s\in\Gamma} \overset{F}{\longmapsto} \bigoplus_{s\in\Gamma} V_s$$

$$W \overset{G}{\longmapsto} (GW)_s = W_s$$

what's important is that that all the components are can isom.

$$G(W) = \text{"constant" } \Gamma\text{-graded vector space}$$
$$G(W) \simeq \mathbb{C}[\Gamma]\otimes W$$

In general $FV = \bigoplus_{s\in\Gamma} V_s$ has proj. $e_s$

$GW$ has $(GW)_s$ — $W$

$$\mathrm{Hom}_{\mathbb{C}}(F(V_s), W) = \mathrm{Hom}_{\hat{\Gamma}}((V_s), GW)$$

$$\mathrm{Hom}_{\mathbb{C}}\left(\bigoplus_{s\in\Gamma} V_s, W\right) = \mathrm{Hom}_{\hat{\Gamma}}\left(\bigoplus_s V_s, \overbrace{\bigoplus_{s\in\Gamma} W}^{GW}\right)$$

operators on $GW$: $W \underset{i_s}{\overset{f_s}{\rightleftarrows}} \bigoplus_s W$ $\qquad e_s = i_s f_s$

OKAY this is not completely clear yet.

$$\bigoplus_{s\in\Gamma} V_s = \left\{ f \in \prod_{s\in\Gamma} V_s \text{ fin. supp} \right\}.$$

---

Check adjointness between algebras and $\hat{\Gamma}$-algebras

~~Hom~~ Let $B = \bigoplus_{s\in\Gamma} B_s$ be a $\hat{\Gamma}$-alg, $A$ alg.

From $B$ get an alg $\bigoplus_{s\in\Gamma} B_s$ by forget grading

From $A$ ——— $\hat{\Gamma}$ ——— $\bigoplus_{s\in\Gamma} A = \mathbb{C}[\Gamma]\otimes A$

functions of finite supp. under convolution

$$\sum_s s f_s \sum_t t g_t = \sum_u u\left(\sum_{u=st} f_s g_t\right)$$

$$(f*g)_u = \sum_{u=st} f_s g_t. \qquad \text{Life is tricky!!!!}$$

~~So what goes on??~~ Let $\theta : \bigoplus_{s\in\Gamma} B_s \longrightarrow \bigoplus_{s\in\Gamma} A$ be a $\hat{\Gamma}$ alg ~~hom.~~ morph. i.e. $\forall s$ have $\theta_s : B_s \longrightarrow A$ $\ni$

$$\begin{array}{ccc} B_s \times B_t & \xrightarrow{\theta_s \times \theta_t} & A \times A \\ \downarrow & & \downarrow \\ B_{st} & \xrightarrow{\theta_{st}} & A \end{array}$$

clearly same as $\theta : \bigoplus_{s\in\Gamma} B_s \longrightarrow A$ alg morph.

So now you have straightened out the adjoint functors between algebras and $\hat{\Gamma}$ algebras.

Go back to Morita equivalence. Where to start?

$P_\Phi$ alg. gens $p_s$ $\overset{s\in\Gamma}{\checkmark}$ rels $\begin{cases} p_s = 0 & s \notin \Phi \\ p_s = \sum_t p_t p_{t^{-1}s} \end{cases}$

$P_\Phi$ is naturally $\Gamma$-graded. Why? unique hom.

$$P_\Phi \longrightarrow \mathbb{C}[\Gamma]\otimes P_\Phi \xrightarrow{\eta} P_\Phi$$

$$p_s \longmapsto s\otimes p_s$$

$$\begin{array}{ccccc} & & P_\Phi & & \\ & & \uparrow{\scriptstyle \eta\otimes 1} & & \\ P_\Phi & \longrightarrow & \mathbb{C}[\Gamma]\otimes P_\Phi & \underset{1\otimes\Delta_{P_\Phi}}{\overset{\Delta_\Gamma\otimes 1}{\rightrightarrows}} & \mathbb{C}[\Gamma]\otimes\mathbb{C}[\Gamma]\otimes P_\Phi \end{array}$$

$$\begin{array}{ccccc} p_s & & s\otimes p_s & \longmapsto & s\otimes s\otimes p_s \\ & & & \longmapsto & s\otimes s\otimes p_s \end{array}$$

---

Now $\exists$ $p = \sum s\otimes p_s \in \mathbb{C}[\Gamma]\otimes P_\Phi$ such that $p^2 = p$. Can form $p(\mathbb{C}[\Gamma]\otimes P_\Phi)$.

~~You have to check~~ You have trouble going further.

$$p(\mathbb{C}[\Gamma]\otimes P_\Phi)\underset{P_\Phi}{\otimes} V$$

Is there something special about $p(\mathbb{C}[\Gamma]\otimes P_\Phi)$

---

Go back to the operators. Pick ~~a ring~~ an alg $A$ form assoc. $\hat{\Gamma}$ alg $\mathbb{C}[\Gamma]\otimes A = \bigoplus_{s\in\Gamma} A$

$$\left(\sum s\otimes f_s\right)\left(\sum t\otimes g_t\right) = \sum_{s,t} st\otimes f_s g_t$$

$$= \sum_{s,u} s(s^{-1}u)\otimes f_s g_{s^{-1}u} = \sum_u u\otimes \sum_s f_s g_{s^{-1}u}$$

try to make some progress.

review: basic adjunction

$$\mathrm{Hom}_{\mathbb{C}}\Big(\bigoplus_{s\in\Gamma} V_s, W\Big) = \prod_s \mathrm{Hom}(V_s, W) = \mathrm{Hom}_{\hat{\Gamma}\text{-mod}}\Big(\bigoplus_{s\in\Gamma} V_s, \bigoplus_{s\in\Gamma} W\Big)$$

$$\mathrm{Hom}_{\text{algs}}\Big(\bigoplus_s B_s, A\Big) = \mathrm{Hom}_{\hat{\Gamma}\text{-algs}}\Big(\bigoplus_s B_s, \bigoplus_s A\Big)$$

equivalence between $\theta : \bigoplus_{s\in\Gamma} B_s \longrightarrow A$ and

$\left(\theta_s : B_s \to A\right)_{s\in\Gamma}$ such that

$$\begin{array}{ccc} B_s \times B_t & \longrightarrow & B_{st} \\ \downarrow \theta_s\times\theta_t & & \downarrow \theta_{st} \\ A\times A & \longrightarrow & A \end{array}$$

adjunction maps

$$\bigoplus_{s\in\Gamma} A \longrightarrow A \qquad\qquad B_s \xrightarrow{\iota_s} \bigoplus_{t\in\Gamma} B_t = B$$

$$\mathbb{C}[\Gamma]\otimes \bigoplus_t B_t$$

---

Begin with $\boxed{C_\Phi \rtimes \Gamma}$ You know what the firm modules are. Let $B = C_\Phi \rtimes \Gamma$ ~~where~~

$m(u)$ $E$ $\Gamma$-module, $th_st^{-1} = h_{ts}$ $\sum h_s = 1$

$h_s h_t = s h_1 s^{-1} t h_1 t^{-1} = 0$ for $s^{-1}t \notin \Phi$. ~~You have a~~

~~$\mathbb{C}[\Gamma]\otimes$~~

You want to ~~[illegible] do~~ find a firm dual pair over $B$

Q. $A \longrightarrow \mathbb{C}[\Gamma]\otimes A$ $\hat{\Gamma}$ alg map.

$$E(A) \longrightarrow \underbrace{E(\mathbb{C}[\Gamma]\otimes A)}_{p(\mathbb{C}[\Gamma]\otimes\mathbb{C}[\Gamma]\otimes A)}$$

Begin with the alg $C_\Phi \rtimes \Gamma$ whose fin modules $E$ ~~should be~~ you understand in terms of ncnf modules over $P_\Phi$.

Simplest case for $\Phi$: $\Phi = \{1\}$. Then $C_\Phi$:

gens $h_s$, $s \in \Gamma$ rels $h_s h_t = \begin{cases} 0 & s \neq t \\ h_t & s = t \end{cases}$. Then

$C_\Phi = \bigoplus_{s\in\Gamma} \mathbb{C} h_s$ where the $h_s$ are mut. orth. projs

$P_\Phi$ gen $p_1$ rels $p_1^2 = p_1$.

$\underbrace{p_s, \; s \in \Phi \qquad \boxed{p_s = \sum_t p_t p_{t^{-1}s}}}$ $\qquad p_s = 0 \quad s \neq 1$.

$$E \xrightarrow{\alpha_1} V \xrightarrow{\beta_1} E \qquad \beta_1\alpha_1 = h_1$$

$$E \xrightarrow{\alpha} \bigoplus_{s\in\Gamma} V_f \xrightarrow{\beta} E \xrightarrow{\alpha} \bigoplus$$

$$f \longmapsto \sum_t t\beta_1 f_t \longmapsto \underbrace{\sum_t \alpha_1 s^{-1} t \beta_1 f_t}_{(pf)_s}$$

$$\xi \longmapsto (\alpha\xi)_s = \alpha_1 s^{-1}\xi$$

$$\sum_{s=tu} p_t p_u = \sum_{s=tu} \alpha_1 t\beta_1 \alpha_1 u \beta_1 = \sum_{s=tu} \alpha_1 t h_1 t^{-1} tu\beta_1 = \alpha_1 s\beta_1 = p_s$$

$\Gamma$

---

~~You want to find~~ proj

$A = P_F$ use $A \hookrightarrow \mathbb{C}[\Gamma] \otimes A$

$A_s \hookrightarrow s \otimes A$

Can you get this to work?

---

You really need some insight into the basic Constructions.

Let's start again with the basic gadget

$$\mathcal{M}(\underbrace{C_\Phi \rtimes \Gamma}_{B}) \underset{Q\otimes_B -}{\overset{P\otimes_A -}{\rightleftarrows}} \mathcal{M}(\underbrace{P_\Phi}_{A})$$

$$\begin{pmatrix} A & Q \\ P & B \end{pmatrix}$$

$\otimes$

$$P = p(\mathbb{C}[\Gamma]\otimes A)$$

Given $N$ a firm B-module i.e. $\Gamma$-action, $h_1$

$\ni$ $h_1 \, t h_1 = 0$ $t \notin \Phi$ $t$'s $\in \Phi$

$$\sum_{s \in t\Phi^{-1}} h_s h_t = h_t = \sum_{s\in t\Phi} h_t h_s$$

So take $N = B$ and look at ~~coherent~~ $h_1$

Here's the ~~problem~~ point you missed. You should look at the obvious firm B-modules and find the corresponding $A = P_F$-modules. The obvious B-module is the algebra $C_\Phi$ with gens. $h_s$ relns. $h_s h_t = 0$ $s^{-1}t \notin \Phi$

$C_\Phi$ gen. $h_s, s\in\Gamma$ rels $h_s h_t = 0$, $s^{-1}t \notin \Phi$

$$\sum_{s\in t\Phi^{-1}} h_s h_t = h_t = \sum_{s\in t\Phi} h_t h_s$$

and $C_\Phi$ is a B-module. So look at the operator $h_1$

Put $E = C_\Phi$ $\quad E \xrightarrow{\alpha_1} h_1 E \xrightarrow{\beta_1} E$

~~Somehow – non~~ Is there some significance to $\left(\sum_{s\in\Phi^{-1}} h_s\right) h_1 = h_1$

You had the idea of replacing $h_1E$ by a cokernel

$$\xrightarrow{h_1} E \xrightarrow{1-\sum_{s\in Y} h_s} E \xrightarrow{h_1} h_1E \longrightarrow 0$$

$$E \xrightarrow{1-\sum_{s\in K} h_s} E.$$

---

You have the basic relation

$$(1-k)h_1 = h_1(1-k) = 0.$$

so you have a complex

$$\xrightarrow{h_1} E \xrightarrow{1-k} E \xrightarrow{h_1} E \longrightarrow$$

a supercomplex $\quad E \underset{1-k}{\overset{h_1}{\rightleftarrows}} E$

---

$$\begin{tikzcd} D \arrow[rr, "i"] & & D \arrow[dl, "j"] \\ & E \arrow[ul, "k"] & \end{tikzcd}$$

exact couple

$$(jk)^2 = j\overbrace{kj}^{0}k = 0$$

$$Z = \mathrm{Ker}(jk) \xrightarrow{k} \mathrm{Ker}(j) = i(D)$$

$$jky = 0 \iff ky \in iD \iff y \in k^{-1}iD$$

$B = jkE \qquad \cancel{iy \iff iy \in kE}$

Given $x \in iD$ $\ ix = 0$, then $\exists y \in E$ with $ky = x$

and $jky = jx$ ? Given $x = ix'$ and $ix = iix' = 0$

then $\exists y$ $\ ky = ix'$ $\qquad D' = \mathrm{Ker}\, j = iD$

$$\begin{tikzcd} & \mathrm{Ker}\, j \arrow[r, "i"] & \mathrm{Ker}\, j \\ \mathrm{Ker}\, jk \arrow[ur] & & \end{tikzcd}$$

$jx = 0 \qquad ix = 0 \iff \overset{\exists y}{ix = ky} \Rightarrow jky = 0$

$$\begin{tikzcd} D \arrow[rr, "i"] & & D \arrow[dl, "j"] \\ & E \arrow[ul, "k"] & \end{tikzcd} \qquad \begin{tikzcd} iD \arrow[r, "i'"] & iD \arrow[d, "j' = ji^{-1}"] \\ Z/B \arrow[u, "\mathrm{res\,of}\,k = k'"] & Z/B \end{tikzcd}$$

Let $x \in Z = \mathrm{Ker}(jk)$. $k'x = kx$ $\quad jkx = 0 \iff kx \in iD \iff x \in k^{-1} iD$

also $k\,\mathrm{Im}(jk) = kjkE = 0$. ~~Also if~~

$jky$

---

Given exact couple
$$\begin{tikzcd} D \arrow[rr, "i"] & & D \arrow[dl, "j"] \\ & E \arrow[ul, "k"] & \end{tikzcd}$$
put $D' = iD$, $E' = \dfrac{\mathrm{Ker}(jk)}{\mathrm{Im}(jk)}$

define maps

$$\begin{tikzcd} D' \arrow[rr, "i'"] & & D' \arrow[dl, "j'"] \\ & E' \arrow[ul, "k'"] & \end{tikzcd}$$

$i'(ix) = iix$

$j'(ix) = jx \mod \overbrace{\mathrm{Im}(jk)}^{B}$

$k'(y+B) = ky \qquad y \in Z$

check well defined: clear for $i'$, for $j'$: $jkjx = 0$, $\therefore jx \in Z$

$ix = 0 \Rightarrow \exists y \in E \;\; x = ky \Rightarrow jx = jky \in B$

for $k'$: $y \in Z \Rightarrow jky = 0 \Rightarrow ky \in D'$

$y \in B \Rightarrow y = jky' \Rightarrow k'y = kjky' = 0$.

compositions $= 0$. $\quad j'i'ix = j'iix = jix = 0$

$k'j'ix = k'(jx + B) = 0$, $\quad i'k'(y+B) = iky = 0$.

~~exactness: $k'(y+B) = 0 \Rightarrow ky = 0$~~
~~$y \in Z \Rightarrow jky = 0$~~

exactness ass $i'(ix) = 0$, then $i(ix) = 0 \Rightarrow$

$\exists y \in E \quad ky = ix \Rightarrow jky = jix = 0 \Rightarrow y \in Z$

so $ix \in k'(Z/B)$. Also $\overset{j'x}{j'(ix)} = 0 \Rightarrow jx \in B \Rightarrow$

$\exists y \quad jx = jky \Rightarrow x - ky \in iD \Rightarrow x' = ix \in iD'$.

Ass $k'(y+B)=0$ where $\overset{y\in Z}{jky=0}$. Thus $ky=0$

so $\exists x \in D$, $y = jx$: then $ix \in D'$ and $j'(ix) = jx + B \;(= y+B)$

---

Let's see if progress can be made. How? Main direction from $P_\Phi$ to $C_\Phi \rtimes \Gamma$. Let $V$ be a $P_\Phi$-module i.e. a vector space equ. w. $\overset{\in\Gamma}{s} \longmapsto p_s \in End(V)$ s.t.

$$\left| \begin{array}{ll} p_s = 0 & \text{for } s \notin \Phi \\ p_s = \sum_t p_t p_{t^{-1}s} & \end{array} \right.$$

$$p : \Gamma \longrightarrow End(V), \quad s \longmapsto p_s$$

Then get $p : \mathbb{C}[\Gamma] \otimes V \circlearrowleft$

$$\mathbb{C}[\Gamma] \otimes V = \bigoplus_{s\in\Gamma} V = \{ f : \Gamma \to V,\ s \mapsto f_s \mid f \text{ fin. supp} \}.$$

$$(pf)_s = \sum_{t\in\Gamma} p_{s^{-1}t} f_t$$

$$(pL_u f)_s = \sum_t p_{s^{-1}t} (L_u f)_t = \sum_t p_{s^{-1}t} f_{u^{-1}t}$$

$$(L_u p f)_s = (pf)_{u^{-1}s} = \sum_t p_{(u^{-1}s)^{-1}t} f_t \qquad \sum_t p_{s^{-1}ut} f_t$$

---

$\Gamma$-modules, $\hat{\Gamma}$-modules

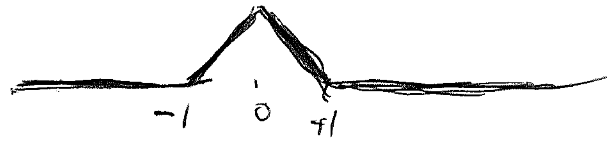
$$\mathrm{Hom}_\Gamma(\mathbb{C}[\Gamma] \otimes V, W) = \mathrm{Hom}_{\mathbb{C}}(V, W)$$

$$\mathrm{Hom}_{\hat{\Gamma}}(\underbrace{\underset{\bigoplus_{s\in\Gamma} V_s}{V}, \underset{\bigoplus_{s\in\Gamma} W}{\mathbb{C}[\Gamma]\otimes W}}_{\prod_{s\in\Gamma} \mathrm{Hom}(V_s, W)}) = \mathrm{Hom}_{\mathbb{C}}(V, W)$$

Why do you care about $\hat{\Gamma}$ modules? 

~~The answer is clear!!!!!!!~~

Example $\Gamma = \mathbb{Z}$ $\quad \Phi = \{u^{-1}, u^0, u^1\}$. OKAY

$h_0$  or

$C_c(\mathbb{R}) = E$ $\quad$ $\Gamma$-module with $h_0$ such that $\sum h_n = 1$ $\quad h_0 u^n h_0 = 0$ $\quad |n| \geq 2$.

According to your theory, $C_c(\mathbb{R})$ ~~is Mor~~ corresponds to a ncuf $P_\Phi$-module $V$, where $V = h_0 C_c(\mathbb{R})$

When is $f \in C_c(\mathbb{R})$ divisible by $h_0$, answer ~~when~~ should be when $\lim\limits_{x \downarrow -1} \frac{f(x)}{x+1}$ $\exists$ $\quad \lim\limits_{x \uparrow 1} \frac{f(x)}{x-1}$ $\exists$

If $f = h_0 g$, then $g = \frac{f}{h_0}$ cont. on $(-1, 1)$. ~~Now how do you see that~~

Can you see that $h_0 C_c(\mathbb{R}) = V$ is a $P_\Phi$ module?

~~It should be~~ Is it true that $h_0 C_c(\mathbb{R}) = h_0 C([-1,1])$?

Yes $\quad C_c(\mathbb{R}) \xrightarrow[\text{by Tietze}]{\alpha_0 = \text{res}} C([-1,1]) \xrightarrow{\beta_0 = h_0} C_c(\mathbb{R})$

Lecture. $A = A^2$ an $A$-module $M$ firm when

$\mu_M : A \otimes_A M \xrightarrow{\sim} M$ $\qquad a \otimes m \longmapsto am$

$e$ left unit for $A$ $\qquad ea = a \quad \forall a$

$A \otimes_A M$

$ea = a \quad \forall a$ then $M$ is firm iff ~~...~~

$M$ is firm
$\Downarrow$
$AM = M$
$\Downarrow$
$em = m \quad \forall m.$

$$\begin{tikzcd} M \arrow[r, two heads] \arrow[rr, bend left, "id"] & A\otimes_A M \arrow[r, "\mu"] & M \end{tikzcd}$$
$$m \longmapsto e\otimes m \longmapsto m$$

$$a\otimes m = ea\otimes m = e\otimes am$$

---

A has local left units when $\forall a\in A \ \exists a' \ni a'a = a$

---

forgot A has a left unit $e$ $\Leftrightarrow$ $\mathbb{Z}$ is a right proj A mod

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] & \tilde{A} \arrow[r] \arrow[l, bend right, "e\cdot"'] & \mathbb{Z} \arrow[r] \arrow[l, bend right] & 0 \end{tikzcd}$$

$$1-e \longmapsto 1$$

$$(1-e)A = 0.$$

---

<u>Def.</u> A has <u>local left units</u> when

(i) $\forall a_1 \in A \quad \exists a\in A$ s.t. $(1-a)a_1 = 0.$

(ii) $\forall a_1,\dots,a_n \in A \quad \exists a\in A$ s.t. $(1-a)a_i = 0 \quad i = 1,\dots,n$

(i) $\Rightarrow$ (ii) by induction on $n$.

$\exists a$ st $(1-a)a_i = 0 \qquad i = 1,\dots,n-1$

$\exists a'$ st. $(1-a')(1-a)a_n = 0$

$\therefore \quad \underbrace{(1-a')(1-a)}_{1-(a'+a-a'a)}$ kills $a_1,\dots,a_n$

<u>Prop:</u> A has local left units $\Leftrightarrow$ $\mathbb{Z}$ is a flat right A-module

If so $M$ is firm $\Leftrightarrow$ $M = AM$.

$$0 \longrightarrow A \longrightarrow \tilde{A} \longrightarrow \mathbb{Z} \longrightarrow 0 \qquad \text{rt mod.}$$

$$\mathrm{Tor}_1^{\tilde{A}}(\mathbb{Z}, M) \to A\otimes_A M \to \underset{\substack{\parallel \\ M}}{\tilde{A}\otimes_A M} \to M/AM \to 0$$

$$\begin{tikzcd} \tilde{A}^s \arrow[r, "a_{ij}"] \arrow[dr, "\sum m_i a_{ij} = 0"'] & \tilde{A}^n \arrow[r, "(a'_{ki})"] \arrow[d, "m_i"] & \tilde{A}^g \arrow[dl, "(m'_k)"] \\ & M & \end{tikzcd}$$

$$\sum_i m_i a_{ij} = 0 \quad \forall j \implies \exists\ m_i = \sum_k m'_k a'_{ki}, \ \sum_i a'_{ki} a_{ij} = 0$$

---

A has local left units $\forall a_1\ \exists a \quad (1-a)a_1 = 0.$

$S$ = mult. system $1-a$

$S^{-1}A \qquad \cancel{s_1^{-1}a_1 \sim s_2^{-1}a_2 \iff \exists s \quad s}$

$$(s_1, a_1) \longrightarrow (ss_1, sa_1)$$

~~Form cat~~ $\left\{ \begin{array}{l} \cancel{Ob = S} \\ \cancel{Hom(s_1, s_2)} = \{ s \mid ss_1 = s_2 \}. \end{array} \right.$

$s_1$

$s_2$

$Ob = pt$

$Hom(\cdot, \cdot) = S.$

$$\cdot \underset{1-a_2}{\overset{1-a_1}{\rightrightarrows}} \cdot \xrightarrow{1-a} \qquad (1-a)(a_1 - a_2) = 0$$

---

$(s_1, a_1) \sim (s_2, a_2) \iff \exists s \qquad ss_1 = ss_2$ and $sa_1 = a_2$

$$\begin{tikzcd} s_1 \arrow[r] & ss_1 = ss_2 \\ s_2 \arrow[u] & \end{tikzcd} \qquad s_1 \underset{s'}{\overset{s}{\rightrightarrows}} s_2 \qquad ss_1 = s_2 = s's_1$$

---

$$J_a = \{ x \in A \mid (1-a)x = 0 \}.$$

$S = \cancel{\ker(1-a)} \quad 1-A$

$R$ unital, $M$ unitary

$M$ is $R$-flat iff any linear relation in $M$:

$$\sum_{i \in I} r_i m_i = 0$$

is a consequence of linear relations in $R$:

$$\exists \quad r'_{ij}, \ m'_j \quad \text{s.t.} \quad m_i = \sum r'_{ij} m'_j, \quad \sum r_i r'_{ij} = 0$$

$$\begin{tikzcd}
R \arrow[r, "\cdot(r_i)"] \arrow[dr, "0"'] & R^I \arrow[r, "\cdot(r'_{ij})"] \arrow[d, "\cdot(m_i)"] & R^J \arrow[dl, "\cdot(m'_j)"] \\
& M &
\end{tikzcd}$$

---

$M = AM$

$$\begin{tikzcd}
\tilde{A} \arrow[r, "(r_i)"] \arrow[dr, "0"'] & \tilde{A}^I \arrow[r, "(a_{ij})"] \arrow[d, "(m_i)"] & A^J \arrow[d, "(m'_j)"] \\
& M \arrow[r, equal] & M
\end{tikzcd}$$

$$0 = \sum r_i m_i = \sum (r_i a_{ij}) m'_j$$

$$\begin{tikzcd}
\tilde{A} \arrow[r, "(a_i)"] & \tilde{A}^I \arrow[d, "m_i"] \\
& M
\end{tikzcd}$$

review the situation; maybe fill in details.

$\Gamma$ group, $\Phi$ finite subset, $C_\Phi$ alg gen by $h_s$ $s \in \Gamma$ subj to rels

$$h_s h_t = 0 \qquad s^{-1}t \notin \Phi$$

$$\sum_{t\Phi^{-1}} h_s h_t = h_t = \sum_{t\Phi} h_t h_s$$

$C_\Phi$ has local left and local right units. in fact an approx identity.

$$\underbrace{C_\Phi \rtimes \Gamma}_{B} \longrightarrow \underbrace{\tilde{C}_\Phi \rtimes \Gamma}_{R} \longrightarrow \mathbb{C}[\Gamma]$$

~~B has 1 approximate approximate identities~~ ~~Let M~~

You claim that a $B$-module $M$ is firm iff $M = BM$

---

Assume $R/A$ flat.

$$\begin{tikzcd} R \arrow[r, "a_1"] \arrow[dr, "0"'] & R \arrow[r, "b_i"] \arrow[d, "1"] & R^I \arrow[dl, "(x_i+A)"] \\ & R/A & \end{tikzcd}$$

$b_i a_1 = 0$

$1 - \sum x_i b_i \in A$

$1 - \sum x_i b_i = a$

$\sum x_i b_i = 1 - a$

$0 = \sum x_i b_i a_1 = (1-a)a_1$

---

Conversely. ~~R/A~~

$R/A$ flat over $R^{op}$

$$\begin{tikzcd} R \arrow[r, "a_1 \cdot"] \arrow[dr, "0"'] & R \arrow[r, "(b_i \cdot)"] \arrow[d, "1+A"] & R^I \arrow[dl, "(x_i+A)"] \\ & (R/A & \end{tikzcd}$$

$b_i a_1 = 0 \qquad i \in I$

$\sum_i x_i b_i \equiv 1 \pmod{A}$

$\sum_i x_i b_i = 1 - a \qquad a \in A$

$\therefore (1-a)a_1 = 0$

$A$ has local left units $\overset{\Longleftarrow}{\underset{\Longrightarrow}{\Leftrightarrow}}$ $R/A$ right flat

first step to show $\forall (a_i, i \in I)$ $\exists\, a \ni (1-a)a_i = 0$ all $i$.

$$0 \to A \to R \to R/A \to 0$$

$R/A$ is $R^{op}$-flat $\iff \mathrm{Tor}_1^R(R/A, M) = 0$ $\forall$ $R$-mods $M$.

$\iff A \otimes_R M \to M$ inj. $\forall$ — .

$k = \sum_i a_i \otimes m_i \in \mathrm{Ker}(\text{———})$ i.e. $\sum a_i m_i = 0$

choose $a \ni (1-a)a_i = 0.\ \forall i$

$$ak = k \qquad ak = \sum_i a a_i \otimes m_i = \sum_i a \otimes a_i m_i$$

$$= a \otimes \left(\sum_i a_i m_i\right) = 0.$$

For a ring $A$ with local left units ~~the~~ ~~firm modules are etc~~ one has

(i) $M$ firm

(ii) $M = AM$

(iii) $\forall m \in M \ \exists a$ st $(1-a)m = 0$