You are looking at $\Gamma = \mathbb{Z}$ $\quad F = \{-1, 0, 1\}$.

~~First step is to go from an~~ You want to construct ~~the~~ a Morita equiv. between certain algebras. The first alg is $B = \mathcal{E}_{\Sigma_F} \rtimes \Gamma$, the alg. version; firm modules ~~...~~ are ~~the same as~~ given by $H, \Gamma, h_0^{1/2}$ $\left| \begin{array}{l} h_0^{1/2} u^n h_0^{1/2} = 0 \quad |n| \geq 2 \\ \sum u^n h_0 u^{-n} = id \end{array} \right.$

$$H \xrightarrow{\alpha} \mathbb{C}[\mathbb{Z}] \otimes V \xrightarrow{\beta} H \qquad V = h_0^{1/2} H$$

$$\mathbb{C}[\mathbb{Z}] \otimes V = \{f : \mathbb{Z} \to V\}_{\text{fin. supp}} \qquad \beta f = \sum_n u^n h_0^{1/2} f(n)$$

$$\xi \longmapsto (\alpha \xi)(n) = h_0^{1/2} u^{-n} \xi$$

Is it clear that $\alpha\xi$ has finite support? The point is that $\xi = \sum u^n h_0 u^{-n} \xi$ is a finite sum. for any $\xi \in H$. ~~So you use an~~ This gives β surjective namely $\xi = \beta(\alpha\xi)$.

You want to formulate the sum condition as

$\forall \xi$ $\quad h_0^{1/2} u^{-n} \xi$ is $= 0$ for almost all n and $\sum u^n h_0 u^{-n} \xi = \xi$.

$$(\alpha\beta f)(n) = \sum_m \left(h_0^{1/2} u^{-n+m} h_0^{1/2}\right) f(m)$$

Use $\mathbb{C}[\mathbb{Z}] \otimes V = \bigoplus_{n \in \mathbb{Z}} V z^n$. ~~What are~~

You are trying to set an algebraic Morita equivalence which should be close to the Hilbert space picture. Ultimately you get a projection, idempotent op on $\mathbb{C}[\mathbb{Z}] \otimes V$ as $\mathbb{C}[\mathbb{Z}]$-module. Think!

Here's what you should ~~...~~ do. Let A be Cuntz's P_F: generators p_n relations $p_n = \sum_i p_i p_{n-i}$, $p_n = 0$ for $|n| > 1$. This ring A is idempotent. Why? ~~...~~

In general $A = \sum \mathbb{C} p_n + \sum A p_n = \sum \underbrace{p_n \tilde{A}}_{\subset A^2} \subset A^2$

Let V be an A-module. Then on $\mathbb{C}[\Gamma]\otimes V$ you have a canonical ~~proj~~ idemp. op., a convolution op.

$$(pf)(n) = \sum_m p_{n-m} f(m)$$

~~While~~ Use notation $\mathbb{C}[\mathbb{Z}] = \mathbb{C}[z,z^{-1}]$

$\hat{p}(z) = \sum p_n z^n$, $\hat{f}(z) = \sum z^m f(m)$. You put $H = p\, V[z,z^{-1}]$. Now you have a functor $V \longmapsto p(V\otimes\mathbb{C}[\Gamma])$. (Note. When you identify elements of $V\otimes\mathbb{C}[\Gamma]$ with fns. $f:\Gamma\to V$ of finite support, it's irrelevant whether you put $\mathbb{C}[\Gamma]$ on the left or right. You have to choose ~~the tran~~ which translation ops to use:
$(L_t f)(s) = f(t^{-1}s)$ $\qquad (R_t f)(s) = f(st)$)

So you have a functor $V \longmapsto p(V\otimes\mathbb{C}[\Gamma])$.

$$(pf)(s) = \sum_t p_{t^{-1}s} f(t) \qquad \text{for a } L\text{-inv. op.}$$

$$= \sum_u p_u f(su^{-1})$$

$u = t^{-1}s$
$tu = s$

Put $E(V) = p(V\otimes\mathbb{C}[\Gamma])$. This is an right cont. exact functor of $V \in \mathrm{Mod}(\tilde{A})$. If $AV=0$ i.e. all $p_s = 0$, then $p=0$ so $E(V)=0$. Therefore

$$E(V) = E(\tilde{A})\otimes_A V = E(A)\otimes_A V$$

So now you have $E(A) = p(A\otimes\mathbb{C}[\Gamma])$.

Fascinating. $A = \begin{cases} \text{gen} & p_s \\ \text{relns.} & p_s = 0 \quad s \notin F, \quad p_t = \sum_s p_s p_{s^{-1}t} \end{cases}$

What is your problem? What do you have? For every $A = P_F$ module V you have a ~~canon~~ canon. projection p_V on $V \otimes \mathbb{C}[\Gamma]$ as left Γ-module

| office for Vanguard.

$$(pf)(s) = \sum_t p(t^{-1}s) f(t)$$

$$(L_u(pf))(s) = (pf)(u^{-1}s) = \sum_t p(t^{-1}u^{-1}s) f(t)$$

$$(p(L_u f))(s) = \sum_t p(t^{-1}s)(L_u f)(t) = \sum_t p(t^{-1}s) f(u^{-1}t)$$

$$= \sum_{ut} p((ut)^{-1}s) f(\cancel{u^{-1}u}\,t) = \sum_t p(t^{-1}u^{-1}s) f(t)$$

Aim to find a natural ring of operators on $E(V) = p(V \otimes \mathbb{C}[\Gamma])$ as Γ-module, i.e. L_u operators.

Recap. A gen. p_s, $s \in \Gamma$ | $p_s = \sum_t p_t p_{t^{-1}s}$, $p_s = 0 \quad s \notin F$

p on $V \otimes \mathbb{C}[\Gamma]$ is

$$(pf)(s) = \sum_t p(t^{-1}s) f(t) \qquad (L_u pf)(s) = (pf)(u^{-1}s)$$

$$(pL_u f)(s) = \sum_t p(t^{-1}s) f(u^{-1}t) \qquad = \sum_t p(\underbrace{t^{-1}u^{-1}}_{(ut)^{-1}}s) f(t)$$

Given an A-module V you get the Γ-module $E(V) = p(V \otimes \mathbb{C}[\Gamma])$ which is exact, right cont, and kills $V \ni AV = 0$.

~~[illegible]~~ It might help to examine the case: $\Gamma = \mathbb{Z}$, $F = \{-1, 0, 1\}$. Then A has three generators p_{-1}, p_0, p_1 subject to the 5 relations $\hat{p}(z)^2 = \hat{p}(z)$, where

$\hat{p}(z) = z^{-1}p_{-1} + p_0 + zp_1$. For any A-module V you have the $\mathbb{C}[\Gamma] = \mathbb{C}[u, u^{-1}]$-module

$$E(V) = p(\mathbb{C}[\mathbb{Z}] \otimes V).$$

Is it true that A and A^{op} are canonically isomorphic? Relations in A:

$$p(n) = \sum_k p(k)\,p(n-k) = \sum_{k+l=n} p(k)p(l)$$

become

$$p(n) = \sum_k p(n-k)\,p(k) \quad \text{in } A^{op}$$

$$= \sum_{k+l=n} p(l)\,p(k)$$

So the relations are preserved which means that you have an isom. $A \xrightarrow{\sim} A^{op}$ sending $p(n)$ to $p(n)$.

Next you would like to find a nice algebra of operators acting naturally on $E(V)$ for any V. ~~[illegible]~~ Two examples $\mathbb{C}[\mathbb{Z}]$

POINT: Because $\Gamma = \mathbb{Z}$ is commutative the left & right actions, $(L_t f)(s) = f(t^{-1}s)$ and $(R_t f)(s) = f(st)$ coincide up to -1, i.e. $L_t = R_{t^{-1}}$.

So what operators do you have on $E(V)$?

$E(V) = p(\mathbb{C}[\mathbb{Z}]\otimes V)$, so you have Γ equivariant maps

$$E(V) \xhookrightarrow{\alpha} \mathbb{C}[\mathbb{Z}]\otimes V \xrightarrow{\beta} E(V)$$

Think of $\mathbb{C}[\mathbb{Z}]$ as the ring of Laurent polys.

$p(z) = \sum_{|n|\le 1} z^n p_n$ acts on $\mathbb{C}[\mathbb{Z}]\otimes V = V\otimes \mathbb{C}[u,u^{-1}]$

β is determined by $\beta_0 : V \longrightarrow E(V)$

$$\beta f = \sum_n u^n \beta_0 f(n)$$

α is det. by $\alpha_0 : E(V) \longrightarrow V$: $(\alpha\xi)(n) = \alpha_0 u^{-n}\xi$

$$\beta\alpha\xi = \sum_n u^n \overset{h_0}{\beta_0\alpha_0} u^{-n}\xi \qquad \beta_0\alpha_0 = h_0 .$$

So what operators do you get on $E(V)$? You have $\mathbb{Z}$ autos and this h_0.

Recap: Given $\hat{p}(z) = z^{-1}p_{-1} + z^0 p_0 + z^1 p_1$ on V, $\hat{p}(z)^2 = \hat{p}(z)$ you construct $E(V) = p(\mathbb{C}[\mathbb{Z}]\otimes V)$, on which you have the mult. group $\{z^n\}$ acting and the operator $h_0 = \beta_0\alpha_0$. You then have on $E(V)$ a module structure over $\mathcal{E}_{\Sigma_F} \rtimes \Gamma = B$, B is a locally unitary alg and $E(V)$ is a F locally unitary B-module. In the converse direction you suppose given E with Γ action and h_0 satisfying $h_0 u^n h_0 = 0$ for $|n|\ge 2$ and $\sum u^n h_0 u^{-n} = 1$. I think you have to make a nuclearity assumption for $h_0 : E_0 \longrightarrow E_0$

Recap. V an A-module i.e. V is a v.s. equipped with $\hat{p}(z) = z^{-1}p_{-1} + z^0 p_0 + z p_1$ a Laurent poly family of projections. $E = \hat{p}(\mathbb{C}[z,z^{-1}]\otimes V)$. Then E is a Laurent poly module, equipped with $\alpha_0 : E \longrightarrow V$, $\beta_0 : V \longrightarrow E$ linear maps whence an operator $h_0 = \beta_0\alpha_0$ on E, and this satisfies $\sum_{n\in\mathbb{Z}} u^n h_0 u^{-n} = id$ on E. ~~OK life goes~~

So E is a B-module where $B = \mathcal{E}_{\Sigma_F} \rtimes \Gamma$ is an algebra with local left + right units. You have ~~Converse? Given a B module E~~ forgotten the support condition involving F.

$$E \xrightarrow{\alpha} \mathbb{C}[\Gamma]\otimes V \xrightarrow{\beta} E$$

$$\Big(\underset{\text{fin. supp}}{f : \Gamma \to V}\Big) \longmapsto \sum_n u^n \beta_0 f(n)$$

$$\xi \longmapsto (\alpha\xi)(n) = \alpha_0 u^{-n}\xi$$

$$\beta\alpha\xi = \sum_n u^n \underbrace{\beta_0\alpha_0}_{h_0} u^{-n}\xi = \xi$$

$$(\alpha\beta f)(n) = \sum_m \underbrace{\left(\alpha_0 u^{-n+m}\beta_0\right)}_{p_{-n+m}} f(m)$$

p_{-n+m} which is 0 for $|-n+m| \geq 2$.

So now what about the converse. Given E with Γ action and h_0

$\Lambda = \mathbb{C}[z,z^{-1}]$

Situation. V an A-module, means $\mathbb{C}[\Gamma]\otimes V$ has projection $\hat{p} = z^{-1}p_{-1} + p_0 + zp_1$. The L poly module $\Lambda\otimes V$ gen. by V has a certain kind of splitting.

$$E = E(V) = \hat{p}(\Lambda\otimes V) \qquad E \overset{\alpha}{\hookrightarrow} \Lambda\otimes V \overset{\beta}{\twoheadrightarrow} E$$

so $p = \alpha\beta$ $\qquad (\hat{p}f)(n) = \sum_m p(n-m) f(m)$ $\quad$ signs?

You want $\quad (\alpha\xi)(n) = \alpha_0 u^{-n}\xi$, $\quad \beta f = \sum_m u^m \beta_0 f(m)$

then $\quad \beta\alpha\xi = \sum_n u^n \overset{h_0}{\beta_0\alpha_0} u^{-n}\xi = \xi$.

$$(\alpha\beta f)(n) = \sum_m \underbrace{\alpha_0 u^{-n} u^m \beta_0}_{p(n-m)} f(m)$$

$$(\hat{p} f)(n) = \sum_m p(n-m) f(m)$$

i.e. $$\sum_n (\hat{p} f)(n) z^n = \sum_n \sum_m p(n-m) z^{n-m} f(m) z^m$$

$$\widehat{pf}(z) = \hat{p}(z)\hat{f}(z) \qquad \text{YES.}$$

What is going on? What's important

Words. E is a B module $\exists\, E = BE$, $B = \mathcal{E}_{\Sigma_F} \rtimes \Gamma$

this means (ultimately because B has local (right?) units) that there is a natural Γ action on E, also a partition of unity $\sum h_n = 1$ Γ-equivariant $h_n = u^n h_0 u^{-n}$ satisfying ~~support~~ independence condition $h_n h_m = 0 \quad |n-m| \geq 2$.

Your problem here is to factor h_0 into $\beta_0\alpha_0$

Is it possible to take $V = H$ somehow. The problem is to get $\boxed{\alpha \text{ defined}}$

$H \xrightarrow[\text{linear}]{} V$ $\quad$ when does it extend to $\quad H \longrightarrow \Lambda \otimes V$ Γ-linear

$\alpha_0 : H \longrightarrow V$ $\qquad$ $\alpha : H \longrightarrow \{f : \Gamma \to V\}$

$(\alpha\xi)(s) = \alpha_0 s^{-1}\xi$

$(\alpha t\xi)(s) = \alpha_0 s^{-1}t\xi$

$\boxed{(L_t(\alpha\xi))(s) = (\alpha\xi)(t^{-1}s) = \alpha_0 s^{-1}t\xi}$

When does $\alpha\xi$ have finite support? Need $\forall \xi \in H$ that α_0 sees ~~only~~ finitely many $s^{-1}\xi$. But H is generated by $h_0 H$, h_0. Maybe this was what lead to $kh_0 = h_0$. So you have $h_0 = \sum_{|n|\le 1} h_n h_0$ because $h_n h_0 = 0$ for $|n| \geq 2$. This gives then $h_0 = kh_0$ with $k = \sum_{|n|\le 1} h_n$. And so you factor

$$h_0 = kh_0 : E \xrightarrow{h_0} E \xrightarrow{k} E$$

So you take $h_0 = \beta_0\alpha_0$, $\alpha_0 = h_0$, $\beta_0 = k$

Let E be a B-module such that $E = BE$. We use that B has local left units. This has to be written out at some point. So where do we begin?

We start with E, Γ action, h_0 such that $\sum u^n h_0 u^{-n} = 1$.

$h_0 u^n h_0$

Try doing the independence first namely $h_0 u^n h_0 = 0$ for $|n| \geq 2$. The completeness condition amounts to

$\forall i$ $\sum_{n\in\mathbb{Z}} h_n h_i = h_i$ because $E = \underline{BE}$ implies that $E = \sum h_i E$.

$B = \sum h_i B$

So it seems the completeness condition + indep. amounts to $kh_0 = h_0$, the rest should follow from group ~~invariance~~ action.

So what actually happens. You have ~~[illegible]~~ the vector space E with Γ-action, operator h_0 satisfying
$$\begin{cases} h_0 u^n h_0 = 0 & |n| \geq 2 \\ \sum_{|n|\leq 1} h_n h_0 = h_0 \end{cases}$$

So now take $\overbrace{V = E}$, $\alpha_0 = h_0$, $\beta_0 = \sum_{|n|\leq 1} h_n$. Then define

$$E \xrightarrow{\alpha} \Lambda \otimes E \xrightarrow{\beta} E$$
$$(f : \Gamma \to E) \longmapsto \beta f = \sum_n u^n \beta_0 f(n)$$
$$\xi \longmapsto (\alpha\xi)(n) = \alpha_0 u^{-n}\xi$$

β is well-defined because f has finite support.

β is onto $\qquad k_0 = \sum_{|n|\leq 1} h_n$

Given E, factor $h_0 : E \xrightarrow{\alpha_0 = h_0} E \xrightarrow{\beta_0 = k_0} E$

$\beta_0 \alpha_0 = k_0 h_0 = h_0$ then you get

$$E \xrightarrow{\alpha} \Lambda \otimes E \xrightarrow{\beta} E$$
$$\{f : \Gamma \to E\} \longmapsto \beta f = \sum_n u^n \beta_0 f(n)$$
$$\xi \longmapsto (\alpha\xi)(n) = \alpha_0 u^{-n}\xi \qquad \beta\alpha\xi = \sum_n u^n \overbrace{\beta_0\alpha_0}^{h_0} u^{-n}\xi = \xi$$

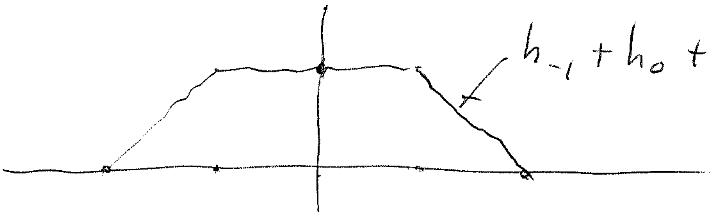
Check $\alpha\xi$ has finite support, You assume

$$E = BE \Rightarrow E = \sum_n h_n BE = \sum_n h_n E$$

$$(\alpha h_m \xi)(n) = \alpha_0 u^{-n} h_m \xi = \underbrace{h_0 u^{-n} u^m h_0 u^{-m}}_{\text{zero for } |n-m| \geq 2} \xi$$

Thus $(\alpha\beta f)(n) = \sum_m (\alpha_0 u^{-n} u^m \beta_0) f(m)$

$$p(n) = \alpha_0 u^{-n} \beta_0 = h_0 u^{-n} k_0$$
$$= h_0 u^{-n}(h_{-1} + h_0 + h_1)$$



graph of $h_{-1} + h_0 + h_1$

This projection doesn't have the same support.

Let's see if there is a way to get an equivalence of module categories. It seems that you want to take the inductive limit with respect to F. The first module category should consist of vector spaces V equipped with a Laurent polynomial projection

$\hat{p}(z) = \sum_n z^n p(n)$ $\qquad p(n) \in \mathcal{L}(V)$ $\qquad$ nilmd

$\hat{p}(z)^2 = \hat{p}(z).$ $\qquad$ nilmeds

The second module category should consist of Γ-modules E equipped with an operator h_0 such that $\exists\, F \overset{\text{finite}}{\subset} \Gamma$ such that $h_0 u^n h_0 = 0$ for $n \notin F$. and such that $\xi = \sum_{n \in \mathbb{Z}} u^n h_0 u^{-n} \xi \quad \forall \xi \in E$.

Program: To set up an equivalence of module categories. The first consists of vector spaces V equipped with a Laurent polyn. projection, bdd degree

$$\hat{p}(z) = \sum_{n\in F} z^n p(n) \qquad p(n) \in \mathrm{End}(V)$$

$$\hat{p}(z)^2 = \hat{p}(z), \qquad p(n) = \sum_m p(m)\, p(n-m)$$

Get a projection p on $\Lambda \otimes V$ $\qquad \Lambda = \mathbb{C}[\mathbb{Z}] = \mathbb{C}[u,u^{-1}]$

$$(pf)(n) = \sum_m p(n-m) f(m) \quad \Big| \text{ i.e. } (pf)^\wedge = p^\wedge f^\wedge$$

$$E(V) = p(\Lambda \otimes V)$$

Second kind of modules consists of vector spaces E with $\Gamma = \mathbb{Z}$ action equipped with an $h_0 \in \mathrm{End}(E)$ satisfying (i) $h_0 u^n h_0 = 0$ for $n \notin F$, F some finite subset of Γ. (ii) $\sum_n h_n \xi = \xi$ $\forall \xi \in E$, where $h_n = u^n h_0 u^{-n}$; this condition means $\{n \mid h_n\xi \neq 0\}$ is finite $\forall \xi$. Condition (ii) $\Longrightarrow$ $E = \sum u^n h_0 E$.

~~Define maps.~~

Perhaps you write the conditions differently, namely

~~$\forall \xi \in E$, you want $\{n \mid h_0 u^{-n}\xi \neq 0\}$ to be finite~~

~~and~~ (a) $E = \sum u^n h_0 E$

(b) $\{n \mid h_0 u^n h_0 \neq 0\}$ is finite F

this implies $\quad h_n u^m h_0 \xi = u^n h_0 u^{-n+m} h_0 \xi = 0$ for $|n-m| \notin F$

$\Longrightarrow \sum h_n u^m h_0 \xi$ is a finite sum. $\forall \xi$

ⓐ $E = \sum_n u^n h_0 E$ E gen. by $h_0 E$ under Γ

ⓑ $h_0 u^n h_0 = 0$ $n \notin F$ finite

ⓐ, ⓑ $\Longrightarrow$ $\sum' h_n \xi$ finite sum $\forall \xi \in E$.

can assume $\xi = h_0 \xi'$ $h_n h_0 = u^n \underbrace{h_0 u^{-n} h_0}$

ⓒ $\sum h_n \xi = \xi$ $\forall \xi \in E$

~~can say~~ alt, $\sum_n h_n h_0 = h_0$

then get $k_0 = \sum'_{n \in F} h_n$ $k_0 h_0 = h_0$

2nd module type: vs E with $\Gamma = \mathbb{Z}$-action (operators u^n, $n \in \mathbb{Z}$) and $h_0 \in End(E)$ satisfying

ⓐ $E = \sum_n u^n h_0 E$

ⓑ $F = \{n \mid h_0 u^{-n} h_0 \neq 0\}$ is finite

ⓐ + ⓑ $\Longrightarrow$ $h_n h_0 = u^n h_0 u^{-n} h_0 = 0$ for $n \notin F$

ⓒ $\sum'_{n \in F} h_n h_0 = h_0$

ⓐ + ⓑ + ⓒ $\Longrightarrow$ $\forall \xi \in E$ $\sum h_n \xi$ is defd $+ = \xi$.

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functors. Given V, $\hat{p}(z)$ have $E(V) = p(\Lambda \otimes V)$

$$E \xhookrightarrow{\alpha} \Lambda \otimes V \xrightarrow{\beta} E \qquad \begin{aligned} \beta\alpha &= id_E \\ \alpha\beta &= p \end{aligned}$$

$(\alpha \xi)(n) = \alpha_0 u^{-n} \xi$ $(\beta f)(n) = \sum u^n \beta_0 f(n)$

$h_0 = \alpha_0 \beta_0$ $(\beta \alpha f)(n) = \sum_m \alpha_0 u^{-n+m} \beta_0 f(n) = \sum_m p(n-m) f(m)$

Details of Morita equivalence:

1st kind of module is v.s. V with Laurent poly projection

$$\hat{p}(z) = \sum_n z^n p(n) \qquad p_n \in \mathrm{End}(V) \qquad p_n = 0 \quad n \notin F.$$

$\hat{p}^2 = \hat{p}$ equiv. $\quad p_n = \sum_m p_{n-m} p_m$

$\hat{p}$ corresp. to a projection p on the Laurent poly module $\Lambda \underset{\mathbb{C}}{\otimes} V$, where $\Lambda = \mathbb{C}[u, u^{-1}] = \mathbb{C}[\mathbb{Z}]$.

$$\begin{pmatrix} f : \Gamma \to V \\ \text{fin supp} \end{pmatrix} \qquad (pf)(n) = \sum_m p(n-m) f(m) \quad \Big| \quad \widehat{pf} = \hat{p}\hat{f}$$

If F is fixed, then these modules are the same as modules over the idemp. ring $A = P_F$

2nd kind of module is a v.s. E with Γ action and a operator h_0 on E + following hold. Put $h_n = u^n h_0 u^{-n}$. You want $h_n h_0 = 0 \quad n \notin F$ (F given)

$$\|$$

also $h_0 h_n = h_0 u^n h_0 u^{-n}$

Thus $h_n h_0 = 0$, $h_0 h_{-n} = 0$, $h_0 u^{-n} h_0 = 0$ equivalent

Want $\sum h_n \xi = \xi$ for all $\xi \in E$. This means that $h_n \xi \neq 0$ for only fin. many n.

$\therefore \sum h_n E = \sum u^n h_0 E = E$. Condition equivalent to $\displaystyle\sum_{n \in F} h_n h_0 = h_0$

<u>Example</u>: $E(V) = p(\Lambda \otimes V)$

$$\begin{tikzcd} E \arrow[r, hook, "\alpha"] \arrow[dr, bend right, "\alpha_0"'] & \Lambda \otimes V \arrow[r, "\beta"] \arrow[d, shift left] & E \\ & V \arrow[u, shift left] \arrow[ur, "\beta_0"'] & \end{tikzcd}$$

$h_0 = \beta_0 \alpha_0$

$h_n = u^n \beta_0 \alpha_0 u^{-n}$

$(\alpha\xi)(n) = \alpha_0 u^{-n} \xi$

$(\beta f) = \sum_n u^n \beta_0 f(n)$

$$(\beta\alpha)\xi = \sum_n \overbrace{u^n \beta_0 \alpha_0 u^{-n}}^{h_n} \xi =$$

$$(\alpha\beta f)(n) = \sum_m \underbrace{\alpha_0 u^{-n} u^m \beta_0}_{p(n-m)} f(m)$$

$$h_n h_0 = u^n \beta_0 \underbrace{(\alpha_0 u^{-n} \beta_0)}_{p_n} \alpha_0$$

$$h_0 h_n = \beta_0 \underbrace{(\alpha_0\ u^n \beta_0)}_{p(-n)} \alpha_0\ u^{-n}$$

~~disjoint~~

How to organize? Left right problem. Suppose you start with ~~V, p. Now $E(V)$ is~~ a $P_F = A$ module V, $\hat{p} = \sum_{n\in F} z^n p(n)$. You know that $E(V) = p(\Lambda\otimes V)$ is an exact rt cnt functor of V, so that $0 \longrightarrow AV \longrightarrow V \longrightarrow V/AV \longrightarrow 0$ leads to $0 \longrightarrow E(AV) \longrightarrow E(V) \longrightarrow \underset{0}{\overset{\shortparallel}{E(V/AV)}} \longrightarrow 0$. Also

$$E(V) \leftleftarrows E(\tilde{A}) \otimes_A V \qquad \text{by rt cnt.}$$

$E(\tilde{A})$ is clearly a flat firm A^{op} module.

$$E(A) = p(\Lambda \otimes A) \qquad ?$$

Given V you put $E(V) = p(\Lambda\otimes V)$ and let

$\alpha : E(V) \hookrightarrow \Lambda\otimes V$ be the inclusion.

$\beta : \Lambda\otimes V \twoheadrightarrow E(V)$ be induced by p

$$\Lambda\otimes V \xrightarrow{\beta} E(V) \xrightarrow{\alpha} \Lambda\otimes V$$

Is there something else here?

Something funny here! Do I believe this?

$$\begin{tikzcd} E \arrow[dr, "\alpha_0"] & & E' \\ & V \arrow[ur, "\beta_0"] & \end{tikzcd}$$

$$E \xrightarrow{\alpha} \operatorname{Hom}(\mathbb{C}[\Gamma], V) \hookleftarrow \mathbb{C}[\Gamma] \otimes V \xrightarrow{\beta} E'$$

Things learned yesterday. When defining $E = p(\Lambda \otimes V)$ you get

$$E \overset{\alpha}{\hookrightarrow} \Lambda \otimes V \overset{\beta}{\twoheadrightarrow} E \qquad \beta\alpha = id_E$$

so any $\xi \in E$ has the form $\xi = \beta f$ with $f : \Gamma \to V$ of finite support. $\therefore$ $E = \sum u^n \beta_0 V$

In fact $\xi = \beta(\alpha\xi)$, so $\alpha\xi$ is a minimal choice for f such that $\xi = \beta f$.

Review of GNS ~~construction~~ $\Gamma(\rho : A \to B)$. Here use unital setting, so that ρ is linear $\rho(1) = 1$. You assoc to $\rho : A \to B$ the category of (M, N, i, j) where M is an A-module, N a B-module, $i : N \to M$, $j : M \to N$ are linear satisf $j\,a\,i(n) = \rho(a)n$. In particular $ji = id_N$, so that N is determined by (M, i, j). I recall that the object (M, N, i, j) is equivalent to M with its natural module structure over $\Gamma(\rho : A \to B) = A \oplus A \otimes B \otimes A$ semi-direct product where $a' \otimes b \otimes a'' \longmapsto a'\,i\,b\,j\,a''$ on M, and $A \otimes B \otimes A$

has mult. $(a_1'\otimes b_1\otimes a_1'')(a_2'\otimes b_2\otimes a_2'') = a_1'\otimes b_1\rho(a_1 a_2')b_2\otimes a_2''$.

Given a B-module N, the different ways of "dilating" N to a Γ-module M are equivalent to factoring:

$$m \longmapsto (a' \mapsto \rho(a'm))$$

$$A\otimes N \longrightarrow M \longrightarrow \mathrm{Hom}(A,N)$$

$$a\otimes n \longmapsto a\,n$$

the canonical map $a\otimes n \longmapsto (a' \mapsto \rho(a'a)n)$ from $A\otimes N$ to $\mathrm{Hom}(A,N)$. $\exists$ minimal choice for M, namely, the image of this canonical map.

So back to $\hat{p}(z) = \sum_{n\in F} z^n p(n) \in \Lambda\otimes V \;\Big|\; \hat{p}^2 = \hat{p}$

p = corresp. projection in $\Lambda\otimes V$

$E = p(\Lambda\otimes V)$.

$$\begin{tikzcd} E \arrow[r, hook, "\alpha", "\text{inclusion}"'] \arrow[dr, "\alpha_0"'] & \Lambda\otimes V \arrow[r, "\beta=p"] \arrow[d] & E \\ & V \arrow[ur, "\beta_0"'] & \end{tikzcd}$$

You know that

$(\beta f) = \sum_n u^n \beta_0 f(n)$, $(\alpha\beta f)(n) = \sum_m \underbrace{(\alpha_0 u^{-m+n}\beta_0)}_{p(m-n)} f(m)$

$\beta\alpha = \sum_n u^n \beta_0 \alpha_0 u^{-n} = id_E$

New idea: We know that $E(V) = p(\Lambda\otimes V)$ is an exact functor from $A = P_F$-modules (i.e. vectors spaces V equipped with Laurent poly projection $\hat{p}(z)$, support in F) to $\Gamma = \mathbb{Z}$-modules with <u>appropriate equivariant</u> partition of 1.

Moreover $E(V) = 0$ when $AV = 0$ i.e. all $p_n = 0$.

Now A is idempotent, so ~~module~~ there is a unique nil and conil free A-module nil isom. to V namely

$$\mathrm{Im}\{AV \longrightarrow V/_A V\} = \mathrm{Im}\{A\otimes_A V \longrightarrow \mathrm{Hom}_A(A,V)\}$$

$$0 \to A \to \tilde{A} \to \mathbb{C} \to 0$$

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}_A(\mathbb{C}, V) \arrow[r] \arrow[d, equal] & \mathrm{Hom}_A(\tilde{A}, V) \arrow[r] & \mathrm{Hom}_A(A,V) \\
0 \arrow[r] & {}_AV \arrow[r] & V \arrow[r] & \mathrm{Hom}_A(A,V) \\
& {}_AV \cap AV \arrow[r] & AV \arrow[u, hook] &
\end{tikzcd}$$

$$A \otimes V \to V \to \mathrm{Hom}_A(A,V)$$

Look at

$$\begin{tikzcd}
E \arrow[rr, "\alpha_0"] \arrow[dr, two heads] & & V \arrow[rr, "\beta_0"] \arrow[dr, two heads] & & E \\
& \alpha_0 E \arrow[ur, hook] \arrow[dr, two heads] & & \beta_0 V \arrow[ur, hook] & \\
& & \overset{h_0}{\beta_0\alpha_0 E} \arrow[ur, hook] & &
\end{tikzcd}$$

This says that if we factor $E \xrightarrow{h_0} E$ through a vector space V, then V will be larger than $h_0 E$.

Here's what you should do? You want to start from the E side – a Γ-module with h_0 leading to an equiv. partition $\sum h_n = 1$. You need to

Still trying for Morita equivalence.

Given V with projection $p \in \Lambda \otimes \operatorname{End}(V)$, $p = \sum_{n \in F} u^n \otimes p_n$

get $E = p(\Lambda \otimes V)$ Γ-module with $h_0 = \beta_0 \alpha_0$

$$\begin{tikzcd} E \arrow[r, hook, "\alpha"] \arrow[dr, bend right, "\alpha_0"'] \arrow[rr, bend left, phantom, "1"] & \Lambda \otimes V \arrow[r, "\beta = p"] \arrow[d, shift left] & E \\ & {} \arrow[u, shift left] \arrow[ur, bend right, "\beta_0"'] & \end{tikzcd}$$

You know that $E(V)$ depends only on V up to nil isom, so

<u>V can be replaced by</u> $\operatorname{Im}\{AV \longrightarrow V/_A V\}$

~~You can make conjectures. If you have some sort of a Morita between $\mathcal{E}_{\Sigma_F} \rtimes \Gamma$ and P_F~~

Start with V, $p \in \Lambda \otimes \operatorname{End}(V)$ $p^2 = p$

$E = E(V) = p(\Lambda \otimes V)$ E is Γ-module with

$h_0 = \beta_0 \alpha_0 : E \longrightarrow E$ such that (i) $h_0 u^{-n} h_0 = 0$ for $n \notin F$

(ii) $\sum h_n \xi = \xi$ $\forall \xi \in E.$ $\xi = \beta\alpha\xi = \sum_n \underbrace{u^n \beta_0 \alpha_0 u^{-n}}_{h_n} \xi$

(Review: $(\alpha\xi)(n) = \alpha_0 u^{-n} \xi$

$\beta f = \sum_n u^n \beta_0 f(n)$ $\alpha\beta = p$

$$(\alpha\beta f)(n) = \alpha_0 u^{-n} \sum_m u^m \beta_0 f(m) = \sum_m \underbrace{(\alpha_0 u^{-n+m} \beta_0)}_{p(n-m)} f(m)$$

Since $p(n) = \alpha_0 u^{-n} \beta_0$, if $F = \{n \mid p(n) \neq 0\}$, then

you have $\alpha_0 u^{-n} \beta_0 = 0$ for $n \notin F = \operatorname{Supp}(p)$

hence $h_0 u^{-n} h_0 = 0$

Now, still using (V, p) let us ~~analyze~~ try to understand what happens as you replace V by smaller things. ~~You fix E first of all~~

E is fixed, but you start with $E \xrightarrow{\alpha_0} V \xrightarrow{\beta_0} E$

You have

$$\begin{tikzcd}
E \arrow[rr, "\alpha_0"] \arrow[dr] & & V \arrow[rr, "\beta_0"] \arrow[dr] & & E \\
& \alpha_0 E \arrow[ur] \arrow[dr] & & \beta_0 V \arrow[ur] & \\
& & \beta_0\alpha_0 E \arrow[ur] \arrow[d, phantom, "\parallel"] & & \\
& & h_0 E & &
\end{tikzcd}$$

Question: Is $\alpha_0 E$ an A-subm. of V? What do you know?

$$h_0 u^{-n} h_0 = 0 \qquad n \notin F.$$

$$\alpha_0 u^{-n} \beta_0 = 0 \qquad n \notin F$$

$$\sum_n \alpha_0 u^{-n} \beta_0 = \sum_n{}' p(n)$$

$$\sum_{n \in F} u^n h_0 u^{-n} h_0 = \sum_{n \in F}{}' h_n h_0 = h_0$$

$$\beta_0 = \sum_{n \in F} u^n \beta_0 \alpha_0 u^{-n} \beta_0 = \sum_{n \in F} h_n \beta_0 = \left(\sum_{n \in F} h_n \right) \beta_0$$

$$\alpha_0 = \sum_n \alpha_0 u^{-n} \beta_0 \alpha_0 u^n = \sum_{n \in F}{}' \alpha_0 h_{-n} = \alpha_0 \left(\sum_{n \in F}{}' h_{-n} \right)$$

Here is the point. Let $E = E(V)$ where V is an $A = P_F$-module i.e. a vector space equipped with idempotent $p \in \Lambda \otimes End(V)$, $p = \sum_{n \in F} u^n \otimes p(n)$.

You have

$$\begin{tikzcd} E \arrow[r, "\alpha = inc."] \arrow[dr] & \Lambda \otimes V \arrow[r, "\beta = p"] \arrow[d, shift right] & E \\ & V \arrow[u, shift right, "\alpha_0"] \arrow[ur, "\beta_0"'] & \end{tikzcd}$$

$\beta\alpha = id$ $\qquad (\alpha\xi)(n) = \alpha_0 u^{-n}\xi$

$((\alpha\beta)f)(n) = \sum_m (\alpha_0 u^{-n+m}\beta_0) f(m)$ $\qquad \beta f = \sum_n u^n \beta_0 f(n)$

On E you have Γ action and operator $h_0 = \beta_0\alpha_0$ properties: $h_0 u^{-n} h_0 = \beta_0 \underbrace{(\alpha_0 u^{-n} \beta_0)}_{p(n)} \alpha_0 = 0$ for $n \notin F$.

~~[illegible]~~? $\qquad \sum_n u^n h_0 u^{-n} = 1$ on E

So now what to expect. It's clear that any factorization $E \longrightarrow W \longrightarrow E$ of h_0 should lead to a direct embedding of E into $\Lambda \otimes W$. ~~You need to derive~~ Your idea is to canonically factor $h_0 = \beta_0\alpha_0$ as

$$\begin{tikzcd} E \arrow[r, "\alpha_0"] \arrow[d, two heads] & V \arrow[r, "\beta_0"] \arrow[d, two heads] & E \\ \alpha_0 E \arrow[ur, hook] \arrow[d, two heads] & \beta_0 V \arrow[ur, hook] & \\ h_0 E \arrow[ur, hook] & & \end{tikzcd}$$

$$\alpha_0 E = \alpha_0 \beta(\Lambda \otimes V) = \Big\{ \underbrace{\alpha_0 \sum_n u^{-n} \beta_0 f(n)}_{\sum_{n \in F} p(n) f(n)} \Big| f \in \Lambda \otimes V \Big\}.$$

so $\alpha_0 E = AV = \sum_{n \in F} p(n) V$. Similarly

$\beta_0 V = V / Ker(\beta_0 : V \to E)$ $\qquad$ ~~$Ker(\beta_0) = \{v \in V \mid \beta_0 \}$~~

β_0 is $V \hookrightarrow \Lambda \otimes V \xrightarrow{p} E$

$$Ker\, \beta_0 = Ker(\alpha\beta_0) = \{v \mid \alpha\beta_0 v = 0\} = \{v \mid \alpha_0 u^{-n}\beta_0 v = 0, \forall n\} = \bigcap Ker\, p(n)$$

Review the formulas. Let E be a Γ module equipped with an operator h_0 satisfying

$$\begin{cases} h_0 u^{-n} h_0 = 0 & \text{for } n \notin F \\ \sum h_n = 1 \end{cases}$$

Let $V = h_0 E$, ~~α_0~~ let $\overbrace{\alpha_0}^{=h_0} : E \twoheadrightarrow V$

$\beta_0 : V \hookrightarrow E$ inclusion, so that $h_0 = \alpha_0 \beta_0$

~~what to do~~ let ~~$p(n) = \beta_0 u^{-n} \alpha_0$~~

$V \xrightarrow{\beta_0} E \xrightarrow{\alpha_0} V$ $\qquad p(n) = \alpha_0 u^{-n} \beta_0$?

$h_0 E \hookrightarrow E \xrightarrow{h_0} h_0 E$?

$h_0 : E \longrightarrow E$ $\qquad \sum h_n = 1$

$h_0 u^{-n} h_0 = 0 \qquad n \notin F$

$$E \xrightarrow{\alpha_0 = h_0} h_0 E \xleftarrow{\beta_0 = \text{inc.}} E \xrightarrow{\alpha_0} h_0 E \qquad p(n) = \alpha_0 u^{-n} \beta_0$$

$h_0 E = V$, inj., $h_0 E = V$

$h_0 = \beta_0 \alpha_0$

$p(0) = \alpha_0 \beta_0$

$$h_0 u^{-n} h_0 = \beta_0 (\alpha_0 u^{-n} \beta_0) \alpha_0 \quad \text{surj}$$

So you find $p(n) = 0 \iff h_0 u^{-n} h_0 = 0$.

~~$h_0 E$ next you do what ever next?~~

$$\sum_m p(n-m) p(m) = \sum_m \alpha_0 u^{-n+m} \beta_0 \alpha_0 u^{-m} \beta_0$$

$$= \alpha_0 u^{-n} \underbrace{\sum_m u^m \beta_0 \alpha_0 u^{-m}}_{1} \beta_0 = p(n).$$

$p(-n)$

W_s

~~and now~~ The other point is that V is "round." ncnf. ~~I have to pursue this~~

$$V = h_0 E = \alpha_0 \beta (\Lambda \otimes V) = \alpha_0 \sum_n u^n \beta_0 V$$

$$V_0 \xhookrightarrow{\beta_0} E \xhookrightarrow{\alpha} \Lambda \otimes V$$

$$(\alpha\beta_0 v)(u) = (\alpha_0 u^{-n}\beta_0) v = p(u) v$$

$$\therefore \{ v \in V \mid p(u)v = 0 \quad \forall u \} = 0.$$

Now I know everything ~~works~~ should work.

~~[illegible]~~ Question to be explored is whether your old GNS picture in the unital ring context generalizes to the partition of 1 situation. ~~So what can you do next?~~

You should finish the Morita equivalence.

V a P_F module, i.e. a ~~[illegible]~~ vector space equipped with a projection p on $\Lambda \otimes V$, End get $E = E(V) = p(\Lambda \otimes V)$ exact functor of V which kills nil P_F modules.

$$E \xhookrightarrow{\alpha = in} \Lambda \otimes V \xrightarrow{\beta = p} E \qquad \left[\begin{array}{l} \beta\alpha = id_E \\ \alpha\beta = p \end{array} \right.$$

$$\Lambda \otimes V = \{ f : \Gamma \to V \text{ fin supp} \} \xmapsto{\beta} \sum_n u^n \beta_0 f(u)$$

~~So the next thing that comes~~

Consider general Γ, $\Lambda = \mathbb{C}[\Gamma]$, V vs. have free $\Lambda \otimes V = \{ f : \Gamma \to V \text{ fin supp} \}$ probably you want to use the rt. mult action $(R_t f)(s) = f(st)$

Want $p \in \Lambda \otimes End(V)$ acts on $\Lambda \otimes V$ commutes with rt mult.

~~$\Lambda \otimes V$ you want~~ You want $\Lambda \otimes \operatorname{End}(V) = \operatorname{End}(V) \otimes A$ to act on $\Lambda \otimes V$. Look at operators on $\Lambda \otimes V$ commuting with R_Γ.

Try $V \otimes \Lambda$ with operators $\operatorname{End}(V) \otimes A$

get operators on $V \otimes \Lambda$ comm. with R_s

Interesting $R_{t^{-1}}(v \otimes s) = v \otimes st^{-1}$

Puzzle $V \otimes \Lambda = \{ f : \Gamma \to V \mid f \text{ fin support} \}$

$$\sum_{s \in \Gamma} f(s) \otimes s \longleftarrow f$$

$$L_t \sum_{s\in\Gamma} f(s) \otimes s = \sum_{s \in \Gamma} f(s) \otimes ts = \sum_{s\in\Gamma} \underbrace{f(t^{-1}s)}_{(L_t f)(s)} \otimes s$$

$$R_t \sum_{s\in\Gamma} f(s) \otimes s = \sum_{s\in\Gamma} f(s) \otimes st^{-1} = \sum_{s\in\Gamma} \underbrace{f(st)}_{(R_t f)(s)} \otimes s$$

Consider $\Lambda \otimes V$ as Γ-module via left mult.

$t(s \otimes v) = ts \otimes v$ $\qquad t \sum_s s \otimes f(s) = \sum_s ts \otimes f(s) = \sum_s s \otimes \underbrace{f(t^{-1}s)}_{(L_t f)(s)}$

so if you identify $\Lambda \otimes V$ with $\{f : \Gamma \to V, \text{ fin supp}\}$ then $t \cdot f$ is $L_t f$. Next you want a proj. p on $\Lambda \otimes V$ commuting with the Γ-action. This should be given by a left invariant kernel:

$$(pf)(s) = \sum_t p(t^{-1}s)f(t) \qquad \text{which can write}$$

$$= \sum_t p((st)^{-1}s)f(st) = \sum_t p(t^{-1})\underbrace{f(st)}_{(R_t f)(s)}$$

$$\therefore \quad p = \sum_t p(t^{-1})R_t \in End(V)\otimes\Lambda$$

~~So what happens at this point?~~ You assume $p(t^{-1})$ supported in F and $p^2=p$, define $E(V) = p(\Lambda\otimes V)$

exact functor of V, kills V such that $p(t)=0$ for all t.

$$\begin{tikzcd} E \arrow[r, "\alpha"] \arrow[dr, "\alpha_1"'] & \Lambda\otimes V \arrow[r, "\beta"] \arrow[d, shift right] & E \\ & V \arrow[u, shift right] \arrow[ur, "\beta_1"'] & \end{tikzcd}$$

$$(\alpha\xi)(s) = \alpha_1 s^{-1}\xi \qquad \beta f = \sum_s s\beta_1 f(s)$$

~~Because $(\alpha(t\xi))(s) = (L_t\alpha\xi)(s) = (\alpha\xi)(t^{-1}s)$~~

~~take $s=1$ $\alpha_1(t\xi) = (\alpha\xi)(t^{-1})$~~

$$\alpha(t\xi) = L_t(\alpha\xi) \qquad (\alpha(t\xi))(s) = (\alpha\xi)(t^{-1}s)$$

$$\alpha(s^{-1}\xi)(1) = (\alpha\xi)(s)$$

Get

$$\beta\alpha\xi = \sum_s s\overbrace{\beta_1\alpha_1}^{h_1}s^{-1}\xi \qquad \therefore \sum h_s = 1.$$

$$(\alpha\beta f)(s) = (pf)(s) = \sum_t (\alpha_1 s^{-1}t\beta_1)f(t)$$

$$f(s) = \begin{cases} v & s = 1 \\ 0 & s \neq 1 \end{cases} \qquad h_1 s^{-1} h_1 = \beta_1 \underbrace{(\alpha_1 s^{-1} \beta_1)}_{p(s)} \alpha_1$$

So there's no problem

$$(pf)(s) = \sum_t p(t^{-1}s) f(t)$$

$$(\alpha\beta f)(s) = \sum_t \underbrace{(\alpha_1 s^{-1} t \beta_1)}_{p(t^{-1}s)} f(t)$$

Review. $\Lambda = \mathbb{C}[\Gamma]$, given (V, p)

V vector space, p is a Γ-left invariant idemp. operator on $\Lambda \otimes V$:

$$(pf)(s) = \sum_t p(t^{-1}s) f(t)$$

where $p(s) \in \mathrm{End}(V)$ and $p(s) = 0 \quad s \notin F$.

$$E = p(\Lambda \otimes V), \qquad E \xrightarrow{\alpha = \mathrm{inc}} \Lambda \otimes V \xrightarrow{\beta = p} E$$

$\beta f = \sum_s s \beta_1 f(s)$ because β is a Γ-mod. map.
(let $\iota_1 : V \to \Lambda \otimes V$, $\iota_1 v = 1 \otimes v$), $\beta_1 = \beta \iota_1$

Given $\xi \in E$ let $\alpha\xi = \sum s \otimes (\alpha\xi)(s)$. let $j_1 : \Lambda \otimes V \longrightarrow V$ be $j_1 f = f(1)$, and let $j_1 \alpha = \alpha_1 : E \longrightarrow \Lambda \otimes V \longrightarrow V$. Then $(\alpha\xi)(s) = j_1 s^{-1} \alpha \xi = j_1 \alpha s^{-1} \xi = \alpha_1 s^{-1} \xi$. $\boxed{(\alpha\xi)(s) = \alpha_1 s^{-1} \xi}$

$$pf = ((\alpha\beta) f)(s) = \sum_t \underbrace{(\alpha_1 s^{-1} t \beta_1)}_{p(t^{-1}s)} f(t) \qquad p(s) = \alpha_1 s^{-1} \beta_1 = 0 \quad s \notin F$$

$$\xi = \beta\alpha\xi = \sum_s s \beta_1 \alpha_1 s^{-1} \xi = \sum_s h_s \xi$$

so what happens? $h_1 s^{-1} h_1 = \beta_1 (\alpha_1 s^{-1} \beta_1) \alpha_1$

$\rho(s) = 0 \qquad s \notin F$

What to do? All kinds of things. You need the Morita equivalence, you want the Morita ~~context~~ in particular the dual pair over $A = P_F$. There is some duality game to be made explicit.

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things to review. Γ graded vector spaces

$V = \bigoplus_{s \in \Gamma} V_s$. Q: What is a homogeneous operator T? Logical procedure: Use $\otimes$

$$(V \otimes W)_s = \bigoplus_{tu = s} V_t \otimes W_u .$$

Idea: Preferred direction - operators on left or right. If you want ~~...~~ a left module

Then take $V_t = \begin{cases} \mathbb{C} & ? \\ & \end{cases}$

Have $\otimes$ for Γ-graded modules

$$(V \otimes W)_s = \bigoplus_{s = tu} V_t \otimes W_u$$

Now take $V_t = \begin{cases} \mathbb{C} & t = t_0 \\ 0 & t \text{ otherwise} \end{cases}$

Then $(V \otimes W)_s = V_{t_0} \otimes W_{t_0^{-1} s} = W_{t_0^{-1} s}$

left operator of degree a on W such be $T : W_s \longrightarrow W_{as}$

Needs more clarification. Basic idea should be a left Γ-graded module N over a Γ graded ring B: $B_s \otimes N_t \longrightarrow N_{st}$

Return now to $\mathbb{C}[\Gamma] \otimes V$.

Go back to E a Γ-module with $h_1 \ni \cdots$

simplest case $\underbrace{\quad}_{h_1 = \beta_1 \alpha_1}$ $E \xrightarrow{\alpha_1} V \xleftarrow{\beta_1} E$ with

$$\alpha_1 s \beta_1 = \begin{cases} 0 & s \neq 1 \\ id_V & s = 1 \end{cases} \qquad \text{and also} \quad \sum s \beta_1 \alpha_1 s^{-1} = id_E$$

~~Then types should get~~ β_1 extends to β, and α_1 coextends to α.

Somehow you should turn the above ~~situation~~ into ~~the contour base~~ simple intrinsic formulas, aim to

at the moment you have Γ left acting on E and $h_1 : E \to E$ ~~with~~ such that $h_1 s h_1 = \begin{cases} 0 & s \neq 1 \\ h_1 & s = 1. \end{cases}$

and $\sum_{s \in \Gamma} s h_1 s^{-1} = 1$. $\quad h_1 : E \xrightarrow{\alpha_1} V \xleftarrow{\beta_1} E$

$$\beta_1 (\alpha_1 s \beta_1) \alpha_1 = \begin{cases} 0 & s \neq 1 \\ \beta_1 \alpha_1 & s = 1 \end{cases} \qquad \alpha_1 s \beta_1 = \begin{cases} 0 & s \neq 1 \\ id & s = 1. \end{cases}$$

So what happens? ~~By habit you want~~

$$E \xrightarrow{\alpha_1} V \quad \rightsquigarrow \quad E \xrightarrow{\alpha} \mathrm{Hom}(\mathbb{C}[\Gamma], V)$$

$$\xi \longmapsto (s \mapsto \alpha_1 s \xi)$$

$$\mathrm{Hom}_\Gamma(E, \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}[\Gamma], V)) = \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}[\Gamma] \otimes_\Gamma E, V)$$

Start with a Γ-module E together with linear maps $E \xrightarrow{\alpha_1} V \xrightarrow{\beta_1} E$ where V is just a vector space

β_1 extends uniquely to a Γ-map $\beta : \Lambda \otimes V \to E$

α_1 coextends ——— Γ-map $\alpha : E \to \mathrm{Hom}(\Lambda, V)$

$$\beta(s \otimes v) = s\beta_1 v \qquad (\alpha\xi)(s) = \alpha_1 s\xi \qquad \text{why}$$

$$\mathrm{Hom}_\Gamma(\Lambda \otimes V, E) = \mathrm{Hom}(V, E)$$

$$\mathrm{Hom}_\Gamma(E, \mathrm{Hom}(\Lambda, V)) = \mathrm{Hom}(\Lambda \otimes_\Gamma E, V) = \mathrm{Hom}(E, V)$$

$$\big(\xi \mapsto (s \mapsto \alpha_1 s\xi)\big) \leftarrow\!\shortmid \big(s \otimes \xi \mapsto \alpha_1 s\xi\big) \leftarrow\!\shortmid \alpha_1$$

next point.

$$\begin{tikzcd}
 & V \arrow[d, "\text{dried } 1\otimes -"'] \arrow[dr, "\beta_1"] & \\
 & \Lambda \otimes V \arrow[r, "\beta"] \arrow[d, "\text{dried product}"] & E \\
E \arrow[r, "\alpha"] \arrow[dr, "\alpha_1"'] & \mathrm{Hom}(\Lambda, V) \arrow[d, "\text{evaluation at } 1"] & \\
 & V &
\end{tikzcd}$$

~~You want a canonical map~~

You want a canonical map in

$$\mathrm{Hom}_\Gamma(\Lambda \otimes V, \mathrm{Hom}(\Lambda, V))$$

$$\mathrm{Hom}(V, \mathrm{Hom}(\Lambda, V)) \;=\; \mathrm{Hom}_\Gamma(\Lambda \otimes V, \mathrm{Hom}(\Lambda, V)) \;=\; \mathrm{Hom}(\Lambda \otimes V, V)$$

$$v \mapsto (s \mapsto \delta_1(s)v) \qquad\qquad s \otimes v \mapsto \delta_1(s)v$$

$$\mathrm{Hom}(V, \mathrm{Hom}(\Lambda, V)) \simeq \mathrm{Hom}_\Gamma(\Lambda\otimes V, \mathrm{Hom}(\Lambda, V)) = \mathrm{Hom}(\Lambda\otimes V, V)$$

$$v \longmapsto (s \longmapsto \delta_1(s)v) \qquad (t\otimes v \mapsto (s \mapsto \delta_1(st)v)) \longleftarrow\!\!| (s\otimes v \mapsto \delta_1(s)v)$$

$$\hookrightarrow \quad (t\otimes v \mapsto \underbrace{t \text{ acting on } \delta_1 v}_{s \mapsto \delta_1(st)v})$$

Check $\chi(t\otimes v)(s) = \delta_1(st)v$

$$\mathrm{Hom}_A(A\otimes V, \mathrm{Hom}(A, W))$$

$$\mathrm{Hom}(V, \mathrm{Hom}(A, W)) \qquad\qquad \mathrm{Hom}(A\otimes V, W)$$

You reach a situation ~~[illegible]~~ familiar from GNS.

$A = \mathbb{C}[\Gamma]$, $B = \mathrm{End}(V)$, $M = E$, $N = V$

$$E \xrightarrow{\alpha_1} V \xrightarrow{\beta_1} E$$

$$M \xrightarrow{j} N \xrightarrow{i} M$$

So in the GNS you have $\rho : A \longrightarrow B$ linear

$\ni$ $\boxed{jai(n) = \rho(a)n}$. Recall that a

B module N, the possible M correspond to A module factorizations $A\otimes N \longrightarrow M \longrightarrow \mathrm{Hom}(A, N)$

$$a\otimes n \longmapsto ain \qquad m \longmapsto (a' \mapsto ja'n)$$

$$ain \longmapsto (a' \mapsto \underbrace{ja'ai\,n}_{\rho(a'a)})$$

of the canonical map

$$A\otimes N \longrightarrow \mathrm{Hom}(A, N)$$

$$a\otimes n \longmapsto (a' \mapsto \rho(a'a)n)$$

OK so back to E, Γ ~~what is~~ 920

$$E \xrightarrow{\alpha_1} V \xrightarrow{\beta_1} E$$

$$\Lambda \otimes V \xrightarrow{\beta} E$$

$$E \xrightarrow{\alpha} \mathrm{Hom}(\Lambda, V)$$

In GNS you have
$$\Lambda \otimes V \xrightarrow{\beta} E \xrightarrow{\alpha} \mathrm{Hom}(\Lambda, V)$$
$$a \otimes v \longmapsto a\beta_1 v \longmapsto (a' \mapsto (\alpha_1 a' a \beta_1) v)$$
$$\xi \longmapsto (a' \mapsto \alpha_1 a' \xi)$$

$$\rho(a) = \alpha_1 a \beta_1$$

~~want to understand~~

Take simplest case $\rho(s) = \delta_1(s) = \begin{cases} 0 & s \neq 1 \\ 1 & s = 1. \end{cases}$

$$s \otimes v \longmapsto s\beta_1 v \longmapsto (s' \longmapsto \underbrace{\delta_1(s's)}\, v)$$
$$= \begin{cases} 0 & s' \neq s^{-1} \\ v & s' = s^{-1}. \end{cases}$$

$$t \otimes v \longmapsto \delta_{t^{-1}} v$$

$$\sum_t t \otimes v_t \longmapsto \Big(\sum_t \delta_{t^{-1}} v_t\Big)(s) = \begin{cases} 0 & t \neq s^{-1} \\ v_{s^{-1}} & t = s^{-1} \end{cases}$$

---

things should become clearer using the GNS formalism

$$E \xrightarrow{\alpha_1} V \xrightarrow{\beta_1} E \qquad E \text{ an } A\text{-module}$$

β_1 extends to an A-map $\beta : A \otimes V \longrightarrow E$ $\quad \beta(a \otimes v) = a\beta_1 v$

α_1 coextends $\quad \alpha : E \longrightarrow \mathrm{Hom}(A, V)$ $\quad (\alpha\xi)(a') = \alpha_1 a' \xi$

Comp. $\alpha\beta : A \otimes V \longrightarrow E \longrightarrow \mathrm{Hom}(A, V)$

$$\alpha\beta(a \otimes v) = (a' \longmapsto \underbrace{(\alpha_1 a' a \beta_1)}_{\rho(a'a)} v)$$

So E appears in factoring this canon. map. Moreover there's a minimal E namely the image, ~~further you have a minimal~~ where β is surjective and α is injective. ~~...~~

Notice for this E, that elements of V give you generators for E, NO while elements of V^* give you A-module maps $E \longrightarrow A$. NO

~~You are close to something unclear, ... at least you have interesting elements of A, A^*, finite rank operators ... what to look at.~~

$$\begin{tikzcd} A\otimes V \arrow[rr, "\text{``}\rho\text{''}"] \arrow[dr, "\beta"'] & & \mathrm{Hom}(A,V) \\ & E \arrow[ur, "\alpha"'] & \end{tikzcd}$$

In the group case $A = \mathbb{C}[\Gamma]$, the ρ map is injective and the support hypothesis say the image of α is contained in the image of ρ. Thus α lifts ~~back to a~~ uniquely back thru "ρ":

$$\begin{tikzcd} A\otimes V \arrow[d, shift right, "\beta"'] \\ E \arrow[u, shift right, "\tilde{\alpha}"'] \end{tikzcd} \qquad \begin{aligned} \beta\tilde{\alpha} &= \text{identity} \\ \tilde{\alpha}\beta &= \text{idempotent} \end{aligned} \qquad ??$$

Not very clear.

Instead look at $A = \mathbb{C}[\Gamma]$ case. The ρ map arises from $\rho : A \xrightarrow[\text{linear}]{} \mathrm{End}(V)$
$\rho(a) = \alpha_1 a \beta_1$. There's an obvious choice when $A = \mathbb{C}[\Gamma]$

namely $\rho(s) = \delta_1(s) = \begin{cases} 0 & s \neq 1 \\ 1 & s = 1. \end{cases}$.

Why this map? $\rho(s) = \alpha_1 s \beta_1$ so $\rho(s) = \delta_1(s)$

corresponds to orthogonality. What is

$$\mathbb{C}[\Gamma] \otimes V \xrightarrow{\text{“}\rho\text{''}} \mathrm{Hom}(\mathbb{C}[\Gamma], V)$$
$$(s \otimes v) \longmapsto \underbrace{(s' \longmapsto \delta_1(s's)\, v)}_{\delta_{s^{-1}}\, v}$$

Uses this as standard

$$\sum_s s \otimes f(s) \longmapsto \left(\sum_s \delta_{s^{-1}} \otimes f(s) \right)$$

. this is the function taking t to $f(t^{-1})$ $\qquad \delta_{s^{-1}}(t) = 1$ for $t = s^{-1}$

What do you learn?

Review: You are using GNS ideas to find nice formulas. GNS idea: Let E be an A-module, V a vector space, linear maps

$$E \xrightarrow{\alpha_1} V \xrightarrow{\beta_1} E$$
$$\xi \longmapsto (a' \mapsto \alpha_1 a' \xi)$$
$$A \otimes V \xrightarrow{\beta} E \xrightarrow{\alpha} \mathrm{Hom}(A, V)$$
$$a \otimes v \longmapsto a\beta_1 v \longmapsto (a' \mapsto \underbrace{\alpha_1 a' a \beta_1 v}_{\rho(a'a)})$$

ρ linear map $A \longrightarrow \mathrm{End}(V)$

there's a minimal E corresp. to $\rho : A \to \mathrm{End}(V)$

namely E = Image of $A \otimes V \xrightarrow{\alpha\beta} \mathrm{Hom}(A, V)$

~~somehow you~~ so far you look at

$$V \xrightarrow{\beta_1} E \xrightarrow{\alpha_1} V \qquad \rho(a) = \alpha_1 a \beta_1$$

next look at $E \longrightarrow V \longrightarrow E$

$$\begin{array}{ccc} & A\otimes V & \xrightarrow{\beta} E \\ & \downarrow \# & \\ E \xrightarrow{\alpha} & \mathrm{Hom}(A,V) & \end{array}$$

You need $\#$, so restrict to $A = \mathbb{C}[\Gamma]$ where there should be canonical map $\#$ ~~from the~~, namely arising from $\rho = \delta_1 = \left(s \mapsto \begin{cases} 0 & s \neq 1 \\ 1 & s = 1 \end{cases}\right)$

$$t \otimes v \xmapsto{\#} \left(s \mapsto \delta_1(st)v\right) = \delta_{t^{-1}} v$$

So you get for $\#$

$$\mathbb{C}[\Gamma]\otimes V \longrightarrow \mathrm{Hom}(\mathbb{C}[\Gamma], V)$$

$$t \otimes v \longmapsto \delta_{t^{-1}} v$$

$$\sum_t t \otimes f(t) \longmapsto \sum_t \delta_{t^{-1}} f(t)$$

$$\#\left(\sum_t t\otimes f(t)\right)(s) = \sum_t \delta_{t^{-1}}(s)\, f(t) = f(s^{-1})$$

Let $\Phi = \# : \mathbb{C}[\Gamma]\otimes V \longrightarrow \mathrm{Hom}(\mathbb{C}[\Gamma], V) = \mathrm{Map}(\Gamma, V)$

$$\Phi(t\otimes v)(s) = \delta_1(st)v$$

$$\Phi\left(\sum_t t\otimes f(t)\right)(s) = \sum_t \delta_1(st) f(t) = f(s^{-1})$$

$$\Phi\left(u \sum t\otimes f(t)\right)(s) = \Phi\left(\sum t \otimes f(u^{-1}t)\right)(s) = f(u^{-1}s^{-1}) \quad ?$$

Again.

$$\begin{tikzcd} A\otimes V \arrow[rr] \arrow[dr, "\beta"'] & & \mathrm{Hom}(A,V) \\ & E \arrow[ur, "\alpha"'] & \end{tikzcd}$$

$$a\otimes v \overset{\beta}{\longmapsto} \langle a\beta, v\rangle, \quad \xi \longmapsto (a' \mapsto \langle \alpha, a'\xi\rangle)$$

$$\alpha\beta : a\otimes v \longmapsto (a' \mapsto \underbrace{\langle \alpha, a'a\beta\rangle}_{\rho(a'a)} v)$$

$$\mathbb{C}[\Gamma]\otimes V \longrightarrow \mathrm{Hom}(\mathbb{C}[\Gamma], V) = \mathrm{Map}(\Gamma, V)$$

$$t\otimes v \longmapsto \underbrace{(s \longmapsto \delta_1(st)\, v)}_{(s\mapsto \delta_{t^{-1}}(s)\, v)}$$

$$t^{-1}\otimes v \longmapsto \delta_t\, v$$

$$\sum_t t^{-1}\otimes v_t \longmapsto \sum_t \delta_t v_t = (s \longmapsto v_s)$$

So it seems that the good embedding of $\mathbb{C}[\Gamma]$ into $\mathrm{Map}(\Gamma,\mathbb{C})$, better the good identification between $\mathbb{C}[\Gamma]$ and $C_c(\Gamma)$ is

$$\sum_s s^{-1} f(s) \longleftrightarrow f$$

$$\| $$

$$\sum_s f(s)\, s^{-1}$$

Look at t action

$$t\sum_s s^{-1} f(s) = \sum s^{-1} f(st)$$

Thus left mult by t on $\mathbb{C}[\Gamma]$ corresp to R_t on $C_c(\Gamma)$

Let us now see ~~[illegible]~~ how α, β look.
You want to ~~compute~~ lift $\alpha : E \to \mathrm{Hom}(A, V)$ ~~back~~ into $A \otimes V$.

$$\begin{array}{ccc} \mathbb{C}[\Gamma] \otimes V & \overset{\beta}{\twoheadrightarrow} & E \\ \downarrow & & \\ V^\Gamma & \hookleftarrow^{\alpha} & E \end{array} \qquad \sum_s s \otimes \alpha_1 s^{-1} \xi \overset{\beta}{\longmapsto} \sum_s s \beta_1 \alpha_1 s^{-1} \xi$$

$$\xi \longmapsto (s \mapsto \alpha_1 s \xi)$$

If you start with V and the operators $\alpha_1 \overset{\rho(s)}{s} \beta_1 \in \mathrm{End}(V)$, then you have the canon. map

$$\mathbb{C}[\Gamma] \otimes V \xrightarrow{\beta} \xrightarrow{\alpha} V^\Gamma$$

$$t \otimes v \qquad\qquad (s \mapsto \rho(st) v)$$

and you take E to be the image, so that β surj and α injective. ~~Then~~ You want $\alpha E = \alpha\beta(\mathbb{C}[\Gamma] \otimes V)$ ~~to~~ consist of fin. support fns. need $\alpha_1 s t \beta_1$ fin. supp $\forall t$.

So you get

$$E \overset{\alpha}{\hookrightarrow} \mathbb{C}[\Gamma] \otimes V \xrightarrow{\beta} E$$

$$\xi \longmapsto \sum_s s \otimes \alpha_1 s^{-1} \xi \longmapsto \sum_s s \beta_1 \alpha_1 s^{-1} \xi$$

So you ~~[illegible]~~ find that your formulas for α and β are correct.

The interesting point seems to be ~~that~~ the embedding

$$\mathbb{C}[\Gamma]\otimes V \hookrightarrow V^\Gamma$$

$$\begin{tikzcd}
\sum_{s\in\Gamma} s\otimes f(s^{-1}) \arrow[r, mapsto] \arrow[d, mapsto, "t\cdot"'] & (f : s \mapsto f(s)) \arrow[d, mapsto] \\
\sum_{s\in\Gamma} ts\otimes f(s^{-1}) & (R_t f : s \mapsto f(st))
\end{tikzcd}$$

$$\| $$

$$\sum_s s\otimes f(s^{-1}t) \qquad (R_t f)(s^{-1})$$

Question: From $\mathbb{C}[\Gamma]\otimes V \overset{\beta}{\twoheadrightarrow} E \overset{\alpha}{\hookrightarrow} \cancel{\mathrm{Hom}(\mathbb{C}[\Gamma], V)}$ $\mathbb{C}[\Gamma]\otimes V$

~~[illegible]~~ you get generators for E from elements of V

and you get Γ-maps $E \to \mathbb{C}[\Gamma]$ from linear functionals on V. So there should be an obvious class of finite rank operators on E. e.g.

You ~~[illegible]~~ still have to ~~[illegible]~~ get the Morita equivalence. At the moment you have E with Γ action and $h_1 : E \circlearrowleft$ such that

$h_1 s h_1 = 0$ for $s\notin F$

$\sum_{s\in\Gamma} s h_1 s^{-1} = 1$ on E

given this you factor $h_1 = \beta_1\alpha_1 : E \overset{\alpha_1}{\twoheadrightarrow} V \overset{\beta_1}{\hookrightarrow} E$

whence $0 = h_1 s h_1 = \underset{\text{inj}}{\beta_1}(\alpha_1 s\beta_1)\underset{\text{surj}}{\alpha_1} \Longrightarrow \alpha_1 s\beta_1 = 0 \quad s\notin F$

get
$$E \xrightarrow{\alpha} \Lambda\otimes V \xrightarrow{\beta} E$$
$$\xi \longmapsto \sum_s s\otimes\alpha_1 s^{-1}\xi \longmapsto \sum_s s\beta_1\alpha_1 s^{-1}\xi = \xi$$

Again. The formulas.

$$\mathbb{C}[\Gamma]\otimes V \longrightarrow \mathrm{Hom}(\mathbb{C}[\Gamma], V)$$
$$t\otimes v \longmapsto (s \longmapsto \delta_1(st)v)$$

$$E \xrightarrow{\alpha_1} V \xrightarrow{\beta_1} E$$

$$\sum_{t\in\Gamma} t\otimes \underbrace{v(t)}_{\text{fin supp}} \longmapsto \sum_t \delta_1(st)v(t) = v(s^{-1})$$

$$\sum_t t\otimes \alpha_1 t^{-1}\xi \longleftrightarrow (\alpha\xi)(s) = \alpha_1 s\xi$$

So $\alpha : E \longrightarrow \mathbb{C}[\Gamma]\otimes V$ is

$$\alpha\xi = \sum_t t\otimes \alpha_1 t^{-1}\xi$$

and $\beta : \mathbb{C}[\Gamma]\otimes V \longrightarrow E$ is $\beta\sum_t t\otimes v(t) = \sum_{t\in\Gamma} t\beta_1 v(t)$

so $\beta\alpha\xi = \beta\sum t\otimes\alpha_1 t^{-1}\xi = \sum_{t\in\Gamma} t\underbrace{\beta_1\alpha_1}_{h_1} t^{-1}\xi$

So $\alpha\beta\sum_t t\otimes v(t) = \alpha\sum_t t\beta_1 v(t)$

$$= \sum_s s\otimes \alpha_1 s^{-1}\sum_t t\beta_1 v(t)$$

$$= \sum_s s\otimes \sum_t (\alpha_1 s^{-1}t\beta_1)v(t)$$

Is there a way to do this using ~~???~~ distribution ideas?

Morita equivalence

$$E \xrightarrow{\alpha_1} V \xrightarrow{\beta_1} E$$

A-mod v.s.

β_1 extends (uniquely?)

$$\mathrm{Hom}_A(\tilde{A}\otimes V, E) = \mathrm{Hom}(V, \overbrace{\mathrm{Hom}_A(\tilde{A},E)}^{E})$$

α_1 coextends

$$\mathrm{Hom}_A(E, \mathrm{Hom}(\tilde{A},V)) = \mathrm{Hom}(\overbrace{\tilde{A}\otimes_A E}^{E}, V)$$

~~What is happening with the group ring.~~

The group ring is unital

$$\begin{array}{ccc} a\otimes v & \longmapsto & a\beta_1 v \\ A\otimes V & \xrightarrow{\beta} & E \\ \downarrow \hat{\rho} & & \\ \end{array}$$

$$E \xrightarrow{\alpha} \mathrm{Hom}(A,V) \qquad \rho : A\otimes_A A \to \mathrm{End}(V)$$

$$\xi \longmapsto (a' \mapsto \alpha a' \xi)$$

You have to decide on a notation. Two induced modules. $\mathbb{C}[\Gamma]\otimes V$ and $\mathrm{Hom}(\mathbb{C}[\Gamma], V) = V^\Gamma$

$V^\Gamma = \{f : \Gamma \to V\}$ with $(tf)(s) = f(st)$ R_t action

What about $\mathbb{C}[\Gamma]\otimes V$. You have two choices.

$$\sum_{s\in\Gamma} s f(s) \quad \text{or} \quad \sum_{s\in\Gamma} s g(s^{-1}) = \sum_{s\in\Gamma} s^{-1} g(s)$$

f, g finite suppose

$$t\sum_s s f(s) = \sum_s ts f(s) = \sum_s s f(t^{-1}s) \qquad L_t \text{ action}$$

$$t\sum_s s^{-1} g(s) = \sum_s ts^{-1} g(s) = \sum_s (st^{-1})^{-1} g(s) = \sum_s s^{-1} g(st)$$

R_t action

~~So you have the action on functions~~

~~Summary: You have~~

There is a canonical map $\mathbb{C}[\Gamma]\otimes V \longrightarrow V^{\Gamma}$ arising from the embedding $V \hookrightarrow V^{\Gamma}$ $\quad v \mapsto \delta_1 v$

~~More prec~~ You have

$$\mathbb{C}[\Gamma]\otimes V \xrightarrow{\rho_1} V \xrightarrow{\iota_1} V^{\Gamma}$$

$\iota_1(v)$ is the function $\delta_1(s)v = \begin{cases} v & \text{if } s=1 \\ 0 & \text{otherwise} \end{cases}$

$$t\otimes v \overset{\hat{\rho}}{\longmapsto} (s \longmapsto \delta_1(st)v)$$

$$\cancel{\sum_{t\in\Gamma} t^{-1}f(t) \longrightarrow \Big(\sum_t t^{-1}f(t)\Big)(s)}$$

$$\hat{\rho}(tv)(s) = \rho(st)v = \delta_1(st)v$$

$$\hat{\rho}\Big(\sum_t t^{-1}f(t)\Big) = \sum_t \delta_1(st^{-1})f(t) = f(s)$$

Idea now: Use $\mathbb{C}[\Gamma]\otimes V \longrightarrow V^{\Gamma}$

$$\sum_s s^{-1}f(s) \longmapsto f$$

$$t\sum_s s^{-1}f(s) = \sum_s (st^{-1})^{-1}f(s) \qquad \therefore \text{ action given by } f \mapsto R_t f$$

$$= \sum_s s^{-1}f(st)$$

Recall $\mathbb{C}[\Gamma]\otimes V \longrightarrow V^{\Gamma}$

$$t\otimes v \longmapsto (s \mapsto \delta_1(st)v)$$

$$\sum t\otimes v_t \longmapsto \Big(s \mapsto \sum_t \delta_1(st)v_t\Big) = (s \mapsto v_{s^{-1}})$$

Assume $\left(\sum t v_t\right)(s) = f(s)$, then

$$\sum_t \delta_1(st)\, v_t = v_{s^{-1}} \Rightarrow \sum_t t\, v_{t^{-1}} \quad ?$$

$$\sum t v_t \longmapsto (s \longmapsto v_{s^{-1}})$$

$f(s) = v_{s^{-1}}$
$f(t^{-1}) = v_t$

$$\underset{\in \mathbb{C}[\Gamma]\otimes V}{\sum_t t f(t^{-1})} \longmapsto \underset{\in V^\Gamma}{f}$$

$$\boxed{\sum s^{-1} f(s) \rightsquigarrow f}$$

$$\downarrow$$

$$\sum_s t s^{-1} f(s) = \sum_s \underbrace{t(st)^{-1}}_{s^{-1}} f(st) \rightsquigarrow R_t f$$

~~So this~~ What this means is that I should write $\sum_s s^{-1} f(s)$ for an elt of $\mathbb{C}[\Gamma]\otimes V$ then f is the corresp. elt of V^Γ. So next what?

$$E \xrightarrow{\alpha} V \xrightarrow{\beta} E$$

$$E \longrightarrow V^\Gamma$$

$$\xi \longmapsto (s \mapsto \alpha_1 s \xi)$$ corresp. elt of $\mathbb{C}[\Gamma]\otimes V$ is

$$\sum_s s^{-1} \alpha_1 s \xi \quad \text{or} \quad \sum_s s\, \alpha_1 s^{-1} \xi$$

Similarly given $\sum s^{-1} f(s) \in \mathbb{C}[\Gamma]\otimes V$ corresp elt of E

$$E \xrightarrow{\alpha_1} V \xrightarrow{\beta_1} E$$

$$\alpha\xi = \sum_s s\alpha_1 s^{-1}\xi \qquad \beta \sum s f(s \quad ?$$

E Γ-module, $E \xrightarrow{\alpha_1} V \xrightarrow{\beta_1} E$

$$\begin{array}{ccccccc} E & \xrightarrow{\alpha} & \mathrm{Hom}(\mathbb{C}[\Gamma], V) & \longleftarrow & \mathbb{C}[\Gamma]\otimes V & \xrightarrow{\beta} & E \\ \xi & \longmapsto & (s \mapsto \alpha_1 s\xi) & & s\otimes v & \longmapsto & s\beta_1 v \end{array}$$

$$(s \mapsto \alpha_1 s\xi) \longmapsto \sum_s s\otimes\alpha_1 s^{-1}\xi \mapsto \sum s\beta_1\alpha_1 s^{-1}\xi$$

At the moment you have

$$\begin{array}{ccccc} E & \xrightarrow{\alpha} & \mathbb{C}[\Gamma]\otimes V & \xrightarrow{\beta} & E \\ \xi & \longmapsto & \sum_s s\otimes\alpha_1 s^{-1}\xi & \longmapsto & \sum_s s\beta_1\alpha_1 s^{-1}\xi \end{array}$$

$$\alpha\beta(t\otimes v) = \sum_s s\otimes\alpha_1 s^{-1}t\beta_1 v$$

$$\alpha\beta\sum_t t\otimes f(t^{-1}) = \sum_s\sum_t s\otimes\alpha_1 s^{-1}t\beta f(t^{-1})$$

Continue. Let's agree to write an elt of $\mathbb{C}[\Gamma]\otimes V$ in the form $\sum_s s^{-1}\otimes f(s)$ where $f: \Gamma \to V$ finite support

Then
$$\beta\sum_s s^{-1}\otimes f(s) = \sum_s s^{-1}\beta_1 f(s)$$

$$\alpha\Big(\sum_s s^{-1}\beta_1 f(s)\Big) = \sum_t t^{-1}\otimes\alpha_1 t\sum_s s^{-1}\beta_1 f(s)$$

$$\therefore\ \alpha\beta\sum s^{-1}\otimes f(s) = \sum_t t^{-1}\otimes\sum_s(\alpha_1 t s^{-1}\beta_1) f(s)$$

Old formulas.

$$E \xrightarrow{\alpha} \Lambda \otimes V \xrightarrow{\beta} E$$

$$\sum_s s \otimes f(s) \longmapsto \sum_s s\beta_1 f(s)$$

$$\xi \overset{\alpha}{\longmapsto} (\alpha\xi)(s) = \alpha_1 s^{-1}\xi$$

So identify $\Lambda \otimes V$ with fin. supp $f : \Gamma \to V$ via $f \mapsto \sum_s s \otimes f(s)$. Then $(\alpha\xi)(s) = \alpha_1 s^{-1}\xi$

$$\boxed{\beta\alpha\xi = \sum_s s\beta_1\alpha_1 s^{-1}\xi = \xi}$$

$$(\alpha\beta(f))(s) = \alpha_1 s^{-1}\beta f = \sum_t \alpha_1 s^{-1}t\beta_1 f(t)$$

$$\boxed{(\alpha\beta f)(s) = \sum_t (\alpha_1 s^{-1}t\beta_1) f(t)}$$

$$(\alpha\beta f)(s^{-1}) = \sum_t (\alpha_1 st\beta_1) f(t)$$

$$= \sum_t (\alpha_1 st^{-1}\beta_1) f(t^{-1})$$

---

forget notation, go back to Morita equiv.

~~Standard formulas~~

$$\Lambda \otimes V = \bigoplus_s s \otimes V \ni \sum_s s^{-1} \otimes f(s) \qquad f \text{ fin. support.}$$

$$\beta\left(\sum_s s^{-1} \otimes f(s)\right) = \sum_s s^{-1}\beta_1 f(s) \qquad \sum_s t \otimes \alpha_1 s\xi$$

$$\alpha\xi = \sum_s s^{-1} \otimes \alpha_1 s\xi \qquad \beta\alpha f = \sum_s s^{-1}\beta_1\alpha_1 s\xi$$

$$\alpha\beta \sum s^{-1} \otimes f(s) = \alpha \sum_t t^{-1}\beta_1 f(t) = \sum_s s^{-1} \otimes \sum_t \alpha_1 st^{-1}\beta_1 f(t)$$

$$(\alpha\beta f)(s) = \sum_t (\alpha_1 st^{-1}\beta_1) f(t)$$

formulas shouldn't matter too much.

~~Given a Γ module~~

Γ group, E Γ-module equipped with ~~ad~~ linear operator h_1 satisfying (i) $h_1 s h_1 = 0$ $\quad s \notin F$.

(ii) $\sum h_s = 1$ where $h_s = s h_1 s^{-1}$.

$$(ii) \Rightarrow E = \sum_s s V \quad \text{where} \quad V = h_1 E$$

Thus $\mathbb{C}[\Gamma] \otimes V \xrightarrow{\beta} E$ $\qquad$ β is Γ-linear ext. of β_1 = the inclusion of $V \subset E$

$$s \otimes v \longmapsto s\beta_1 v$$

Next let $\alpha_1 = h_1 : E \to h_1 E = V$, let

$$\alpha : E \longrightarrow \mathbb{C}[\Gamma] \otimes V$$

$$\xi \longmapsto \sum s \otimes \alpha_1 s^{-1} \xi$$

well defined because can suppose $\xi = t\beta_1 v$

$$\underset{\text{inj.}}{\beta_1} \alpha_1 s^{-1} t \beta_1 \underset{\text{surj.}}{\alpha_1} = h_1 s^{-1} t h_1 = 0 \qquad s^{-1}t \notin F.$$

$$E \xhookrightarrow{\alpha} \mathbb{C}[\Gamma] \otimes V \xrightarrow{\beta} E$$

$$\beta\alpha\xi = \sum_s \underbrace{s\beta_1\alpha_1 s^{-1}}_{h_s}\xi = \xi.$$

May better notation for elements of $\mathbb{C}[\Gamma] \otimes V$ is $\sum s\iota_1 f(s)$. Then $\beta \sum s\iota_1 f(s) = \sum_s s\beta_1 f(s)$

$$\begin{tikzcd} \mathbb{C}[\Gamma] \otimes V \arrow[r, "\rho = \delta_1"] & \mathrm{Hom}(\mathbb{C}[\Gamma], V) \arrow[dl, "J1"] \\ V \arrow[u, "\iota_1"] & \end{tikzcd}$$

It seems there is some idea hidden here

$$\rho(s) = J1\, s\, \iota_1 = \delta_1(s)$$

You had a good viewpoint this morning. namely special cases $\Gamma = 1$ or $F = \{1\}$.

For $\Gamma = 1$, ~~you have $E, h_1 : E \to E$~~ ask for a Γ-graded proj on V. Answer is a p_1 such that $p_1^2 = p_1$. Corresp E should be the image of p_1. ~~In general too~~ Given Γ group consider a Γ graded projection $p(s)$ on V supported on $F = \{1\}$. Get simply a

---

Free case. When is a Γ-module E free? When there exists a $\mathbb{C}$-linear operator h_1 on E, ~~such that~~ the projection onto the generator subspace V, which satisfies disjointness $h_1 s h_1 = \begin{cases} 0 & s \neq 1 \\ h_1 & s = 1 \end{cases}$

and completeness: $\sum s h_1 s^{-1} = 1$. Try to weaken a bit. 2nd cond. $\Longrightarrow$ $\sum_s s h_1 E = E$, so any elt ξ of E is a linear comb. of $sh\xi$. Put $V = h_1 E$. $\therefore \sum_{s \in \Gamma} sV = E$.

Next. $s h_1 s^{-1} t h_1 E$ is zero for $s \neq t$, and

$$t h_1 \xi = \sum_s s h_1 s^{-1} t h_1 \xi = t h_1^2 \xi \qquad \therefore h_1^2 = h_1.$$

~~Look from E end. Let $F = \{1\}$ so that $h_1 s h_1 = 0$ $s \neq 1$.~~

Look from Γ graded projection viewpoint.

$$p(s) = \sum_{tu = s} p(t)\, p(u) \qquad p(s) = 0 \quad s \neq 1.$$

$$p(1) = p(1)^2$$ Philosophy: Think of E as just a Γ-module with h_1 satisfying the two conditions $h_1 s h_1 = 0$

E Γ module, h_1 $\mathbb{C}$-lin. op on E

such that $\{s \mid h_1 s h_1 \neq 0\}$ is finite

$\forall \xi \in E \quad \{s \mid h_1 s^{-1}\xi \neq 0\}$ is finite

and $\sum_s s h_1 s^{-1} \xi = \xi$.

First case $\{s \mid h_1 s h_1 \neq 0\} \subset \{1\}$.

that is $h_1 s h_1 = 0$ for $s \neq 1$.

E Γ module, h_1 $\mathbb{C}$-linear operator on E

s.t. ① $\{s \mid h_1 s h_1 \neq 0\} = F$ is finite

② $\forall \xi \in E \quad \{s \mid h_1 s^{-1}\xi \neq 0\}$ is finite and

$$\sum s h_1 s^{-1}\xi = \xi$$

First case to consider is $\{s \mid h_1 s h_1 \neq 0\} = \{1\}$.

i.e. $h_1 s h_1 = 0$ for $s \neq 1$. ~~Note ② ⟹ any $\xi \in E$ is a finite sum $\xi = \sum_t t h_1 \xi_t$, or $E = \sum t h_1 E$~~

~~Where do you want to go?~~

~~Better might be to note that $s h_1 s^{-1} t h_1 t^{-1} = 0 \quad s \neq t$~~

~~and~~ Then

$$t h_1 t^{-1}\xi = \sum_s \underbrace{t h_1 t^{-1} s h_1}_{0 \text{ for } t^{-1}s \neq 1 \text{ i.e. } s \neq t} s^{-1}\xi$$

$$t h_1 t^{-1}\xi = (t h_1 t^{-1})^2 \xi$$

Thus the $h_s = s h_1 s^{-1}$ are mutually annihilating projectors, and their sum $= \mathrm{id}_E$.

Put $V = h_1 E$. Then $\sum h_s$.

Simplify the argument. $s h_1 s^{-1} h_1 = 0 \quad s \neq 1$

$$\sum_s s h_1 s^{-1} h_1 = h_1$$

wait. $\sum_s h_s h_t \xi = \sum_s s \underbrace{h_1 s^{-1} t h_1}_{0 \text{ for } s\neq t} t^{-1} \xi = h_t^2 \xi$

$\quad \overset{\shortparallel}{h_t \xi}$

so you have ~~all~~ anoihilating projectors h_s, whose sum is the identity. $h_s E = s h_1 E$.

$E = \bigoplus_s V_s \qquad V_s = h_s E = s V_1$

~~Now you are in a good situation~~

Conclusion: Given a Γ-module E and a $\mathbb{C}$-linear op h_1 on $E \ni \quad h_1 s h_1 = 0$ for $s \neq 1$ and $\forall \xi \quad \{s \mid h_1 s^{-1} \xi \neq 0\}$ finite and $\xi = \sum_{s \in \Gamma} s h_1 s^{-1} \xi$, then $E = \bigoplus_{s \in \Gamma} V_s$ where $V_s = s h_1 E = s V_1$.

$h_s = s h_1 s^{-1}$ is the projector killing $V_t \quad t \neq s$ etc.

Next to consider the general case. E Γ-module h_1 $\mathbb{C}$-linear $\ni \{s \mid h_1 s^{-1} h_1 \neq 0\}$ is finite; call this set F. and $\ni \forall \xi \in E \quad \sum_{s\in\Gamma} h_s \xi = \xi$ (sum assumed finite)

~~The basic idea should be to form the~~

Again put $V_s = s V_1$, $V_1 = h_1 E$

Not true that $h_s h_t = 0$ for $s \neq t$

But $h_s h_t = s h_1 s^{-1} t h_1 t^{-1} \neq 0 \iff t^{-1}s \in F$

Now ~~what~~ there's a problem about the symmetry.
To some extent you can replace F by $F \cup F^{-1}$
Let's organize this ~~all~~

$$h_1 s^{-1} h_1 = 0 \quad \text{for } s \notin F.$$

$$V = h_1 E \qquad E \xrightarrow{\alpha_1 = h_1} V \xrightarrow{\beta_1 = \mathrm{in}} E \qquad h_1 = \beta_1 \alpha_1$$

$$0 = h_1 s^{-1} t h_1 = \underset{\mathrm{inj}}{\beta_1} (\alpha_1 s^{-1} t \beta_1) \underset{\mathrm{surj}}{\alpha_1} \iff \alpha_1 s^{-1} t \beta_1 = 0$$

Do you get a Γ-graded projection on V?

$$\underline{p(s) = \alpha_1 s^{-1} \beta_1} \qquad p(t) p(t^{-1} s) = \alpha_1 t^{-1} \beta_1 \alpha_1 \quad ?$$

β_1 = inclusion of $h_1 E$ into E

β_s = ——— $s h_1 E = h_s E$ into E

$$\bigoplus_{s \in \Gamma} V_s \xrightarrow{\beta} E$$

~~So what am I going to do next?~~

E a Γ-module, $h_1 \in \mathrm{End}_{\mathbb{C}}(V)$

$F = \{s \mid h_1 s h_1 \neq 0\}$ is finite

$\sum_s s h_1 s^{-1} \xi = \xi \qquad \forall \xi \in E$. Put $V_1 = h_1 E$ then

$h_1 = \beta_1 \alpha_1 : E \xrightarrow{\alpha_1} V_1 \xrightarrow{\beta_1} E$ where $\alpha_1 = h_1$, $\beta_1 = \mathrm{inc}$.

$$h_1 s h_1 = \underset{\mathrm{surj}}{\beta_1} (\alpha_1 s \beta_1) \underset{\mathrm{inj}}{\alpha_1} \implies F = \{s \mid \alpha_1 s \beta_1 \neq 0\}.$$

$$V_s = sV_1 = sh_1s^{-1}E = sh_1E \subset E$$

$sh_1s^{-1} = s\beta_1\alpha_1s^{-1}$. ~~See what's going on~~

~~So much to~~ $\sum V_s = E$

$$E \xrightarrow{\alpha} \bigoplus V_s \xrightarrow{\beta} E \xrightarrow{\alpha} \bigoplus V_s$$

$\beta\iota_s$ = inclusion β_s of V_s in E

$p_s\alpha = h_s : E \twoheadrightarrow h_sE = V_s$

So what is

---

new notation. E is a Γ-module, $h_1 \in \mathrm{End}_{\mathbb{C}}(E)$,

$V_s = h_sE$, $h_s = \beta_s\alpha_s : E \xrightarrow{\alpha_s = h_s} V_s \xrightarrow[\text{incl.}]{\beta_s} E$

You now want to go over notation. Start with Γ-mod E equipped with operator $h_1 \in \mathrm{End}_{\mathbb{C}}(E)$

sat: $\{s \mid h_1sh_1 \neq 0\}$ finite

$\sum h_s = id$ where $h_s = sh_1s^{-1}$

$$\begin{tikzcd} E \arrow[rr, "h_1"] \arrow[dr, two heads, "\alpha_1 = h_1"'] & & E \\ & V_1 \arrow[ur, hook, "\beta_1 = \text{incl. of } h_1E"'] & \end{tikzcd} \qquad h_1 = \beta_1\alpha_1$$

Move things around by $s \in \Gamma$, so

$$\begin{tikzcd} E \arrow[rr, "h_s"] \arrow[dr, two heads, "\alpha_s = h_s"'] & & E \\ & V_s = sV_1 \arrow[ur, hook, "\beta_s = \text{inc}"'] & \end{tikzcd} \qquad \begin{array}{l} h_s = sh_1s^{-1} = s\beta_1\alpha_1s^{-1} \\ \beta_s = s\beta_1, \quad \alpha_s = \alpha_1s^{-1} \end{array}$$

So we have subspaces V_s $\sum_{s \in \Gamma} V_s = E$ permuted by Γ such that, which are not independent. Independent means ~~and~~ $\bigoplus_s V_s = E$ and that $\alpha_s : E \twoheadrightarrow V_s$, $\beta_s : V_s \hookrightarrow E$ are the p_s, ι_s rel. to this $\oplus$

In this situation ~~do you have a simple~~ You have the maps

$$\begin{tikzcd} E \arrow[r, "\alpha"] \arrow[dr, "\alpha_s"'] & \bigoplus V_s \arrow[r, "\beta"] \arrow[d, shift right, "j_s"'] & E \\ & V_s \arrow[u, shift right, "i_s"'] \arrow[ur, "\beta_s"'] & \end{tikzcd}$$

$\alpha\xi = (\alpha_s \xi)_{s\in S}$ $\qquad$ $\beta(\eta_s) = \sum_s \beta \iota_s \eta_s = \sum_s \beta_s \eta_s$

here (η_s) is a finite supp. map from Γ to E such that $\eta_s \in V_s = sV_1$.

Now the above situation is Γ-equivariant. Basically it amounts to the family of canonical factorizations

$h_s : E \overset{\alpha_s}{\twoheadrightarrow} V_s \overset{\beta_s}{\hookrightarrow} E \qquad h_s = \beta_s \alpha_s = s\beta_1\alpha_1 s^{-1}$

of h_s for each s and the fact that these maps produce

$$\begin{tikzcd} E \arrow[r, hook, "\alpha"] \arrow[rr, bend right, ""] & \bigoplus V_s \arrow[r, two heads, "\beta"] & E \end{tikzcd}$$

$$\sum h_s = id_E .$$

Your viewpoint here ~~uses the fact that does involves a~~ is supposed to ignore the group, to somehow describe a partition of unity in an operator sense. All that matters is the ~~maps~~ family (h_s), ~~and~~ ~~the~~ local finiteness, sum $= \underline{1}$.

$\{t \mid h_s h_t \neq 0\}$ etc.

What about order? ~~~~ How much can you prove from these assumptions. | local finiteness completeness.

So now you are looking at Cuntz's ~~[illegible]~~ noncommutative simplicial complexes really.

A first question might be to ~~[illegible]~~ see what happens with a finite index set.

$$h_1 + h_2 = 1 \qquad h_1 h_2 = 0 \iff h_2 h_1 = 0$$

~~[illegible]~~ What do you want?? ~~[illegible]~~

Consider $\mathbb{C}\langle h_0, \dots, h_n \rangle$ relation $\sum_j h_j = 1$.

You want a non unital alg.

$$(h_0 + h_1) h_j = h_j$$

Let E be a vector space equipped with operators x, y such that $(x+y)x = x$ and $(x+y)y = y$ on E. Also $E = xE + yE$. Thus $x+y = 1$ on E.

$$(x+y)(x+y) = \underset{\text{central}}{x+y}$$

~~[illegible]~~ Let $R =$ ring of operators on E gen. by x, y. Then $x+y = 1$ in R, so R is commutative.

so $y = 1-x$ on E

Let $A = T\langle x,y\rangle / \big((x+y)x = x,\ (x+y)y = y\big)$ then $(x+y)^2 = (x+y)$ in A; call $x+y = e$. Then $eA = A$ so e is a left identity for A.

$$A = T\langle x,y\rangle \Big/ \left((x+y) \begin{cases} x \\ y \end{cases} = \begin{matrix} x \\ y \end{matrix} \right) \qquad e = x+y \qquad ea = a$$

$e^2 = e$ and $eA = A$. ~~$A = eA + (1-e)A$~~

can write $A = Ae \oplus A(1-e)$

So what happens? ~~ex~~

Let M be a left A-module. Then $AM = M$ $\iff eM = M$, and in this case M should be firm. Check: ~~$eA = A \implies eM$~~ $eM = M \implies AM \supset M$ hence $AM = M$. Conv. $AM = M \implies M = eAM = eM$. Check for Morita equivalence

$$\begin{pmatrix} A & Ae \\ eA & eAe \end{pmatrix}$$

~~$\mathcal{M}(A) \longrightarrow \mathcal{M}(Ae)$~~

need $AeA = AA = A < eA < e^2A < AeA$

Note $Ae = eAe$ is unital. So it seems that

$$A = Ae \oplus A(1-e) = A \oplus N \quad \text{where } NA = 0$$

N unitary over $A = Ae$ left module

Let W be a right A-module.

Better to take $W = A$. split it into $Ae \oplus A(1-e)$

Is $A(1-e)$ nonzero? ~~xe~~ ?

What happens is $Ae = A(x+y)$

$(x+y)(x+y) = (x+y)$

Look at A^{op}. $\quad x(x+y) = x$ $\quad$ W an A^{op}-mod
$\quad y(x+y) = y$

Look at A having a left unit e: $ea = a$.

Then $A = \underset{\text{unital}}{Ae} \oplus \underset{\text{left unitary right nil}}{A(1-e)}$

$$e = \sum_{i=0}^{n} x_i \qquad e = x + y$$

Can you get this to work? A has two generators x, y and the relation says the linear comb. $x+y$ should be a left ident. Will get same ring posing any ${}^{\text{2 diml}}$ vector space and nonzero element.

Consider now a vector space X and nonzero element k, A is quotient of nonunital $T(X) = X \oplus X^{\otimes 2} \oplus \cdots$ by relations $kx = x$ $\forall x \in X$. How is this related to JC's R construction $RA = T_{\text{unital}}(A) \big/ 1_R = k$ a module

Keep it simple. Let M be a vector space with two operators k, h satisfying the relations $kh = h$ and $kk = k$. Let A be the nonunital ring corresp to these generators and relations. Better to have e, h satisfying $e^2 = e$ and $eh = h$. (This should give the same ring as the one with generators x, y subject to relations $(x+y)x = x$ and $(x+y)y = y$. Check $ex = x$ and $e(e-x) = e-x$)

M has operators $e, h \ni e^2 = e$, $eh = e$

You can split: $M = eM \oplus (1-e)M$. Then h can be any map from M to eM.

Look at ring $\left[\begin{array}{ll}\text{gens} & e, h \\ \text{relns} & e^2 = e,\ eh = h\end{array}\right.$

words in e, h what are the possibilities?

this yields a basis for the ring.

$h, h^2, h^3, \ldots\ldots$

$e, he, h^2e, h^3e, \ldots\ldots$

$$\boxed{\begin{aligned} A &= h\mathbb{C}[h] + \mathbb{C}[h]e \\ Ae &= \mathbb{C}[h]he + \mathbb{C}[h]e = \mathbb{C}[h]e \end{aligned}}$$

$$A = \mathbb{C}[h]h + \mathbb{C}[h]e$$

$$Ae = \mathbb{C}[h]he + \mathbb{C}[h]e^2 = \mathbb{C}[h]e$$

$$\mathbb{C}[h]h(1-e)$$

Review: You are trying to understand non-comm. simplices e.g. in dim 1, ring w. gens. x, y subj to relns. $(x+y)x = x$, $(x+y)y = y$. In particular you want to see that the relations $x(x+y) = x$, $y(x+y) = y$ do not necessarily hold

Change notation: $e = x+y$, $h = x$

The relations become $eh = h$ and $e(e-h) = e-h$

equiv. $eh = h$ and $e^2 = e$. Let A be the universal ring defd by these gen + relns. ~~...~~ for a basis closed under $h\cdot$ and $e\cdot$.

$h, h^2, h^3, \ldots$ $\qquad A = \mathbb{C}[h]h + \mathbb{C}[h]e$

$e, he, h^2e, \ldots$ $\qquad Ae = \mathbb{C}[h]he + \mathbb{C}[h]e = \mathbb{C}[h]e.$

Then $h - he$ seems to be $\neq 0$. To be more accurate we should represent A on $\mathbb{C}[h]^{\oplus 2}$

$$a = fh + ge \qquad \cancel{a = hf + eg}$$

$$\boxed{\begin{aligned} ha &= (hf)h + (hg)e \\ ea &= fh + g \end{aligned}}$$

$$h \rightsquigarrow \begin{pmatrix} h & 0 \\ 0 & h \end{pmatrix}$$

$$e \rightsquigarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

not convincing. Work a bit more.

A nonunital alg gen: e, h relns: $eh = h$, $e^2 = e$

~~[illegible]~~ Let A act on $\begin{pmatrix} \mathbb{C}[h] \\ \mathbb{C}[h] \end{pmatrix}$ $\quad \mathbb{C}[h] + \mathbb{C}[h]e$

$$h\begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} hf \\ hg \end{pmatrix} \qquad e\begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} f - f(0) \\ g + f(0) \end{pmatrix}$$

$$\tilde{A} = \frac{\langle e, h \rangle}{(e^2 = e,\ eh = h)}$$

$\tilde{A}$-modules are v.s. W equipped with a splitting $W = eW \oplus (1-e)W$ and an op. $h: W \longrightarrow eW$

you want to find ~~[illegible]~~ a W such that $h - he \neq 0$.

$$\begin{pmatrix} eW \\ \oplus \\ e^\perp W \end{pmatrix} \longleftarrow \begin{pmatrix} eW \\ \oplus \\ e^\perp W \end{pmatrix} \qquad h = \begin{pmatrix} * & * \\ 0 & \cancel{*} \end{pmatrix} \qquad e = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

clear.

~~[illegible] next consider~~ Repeat A gen. x_i $\quad i = 0, \dots, n$

relns. $\left(\sum_0^n x_i\right) x_j = x_j$. Set $e = \sum_0^n x_i$, $h_j = x_j$, $j \geq 1$.

$$e h_j = h_j \quad j \geq 1. \qquad e\big(\overbrace{e - \sum_{j \geq 1} h_j}^{x_0}\big) = \overbrace{e - \sum_{j \geq 1} h_j}^{x_0}$$

rep. of A is a v.s. W eq. w. $e, h_1, \dots, h_n$ such that $e^2 = e$, $eh_j = h_j$ ~~[illegible]~~ $1 \leq j \leq n$. $\quad W = eW + (1-e)W$

So you learn that both $\sum_s h_s h_t = h_t$, $\sum_s h_t h_s = h_t$ ~~have to be considered~~ should be assumed to hold.

Repeat the program: You are studying noncomm. partitions of unity, i.e. a vector space E equipped with collection of operators h_s satisfying local finiteness $\forall t$ $\{s \mid h_s h_t \neq 0\}$ is finite and in the other order and completeness $\sum_s^? h_s h_t = h_t = \sum_s h_t h_s$. Looked at case of 2 ops.

$e^2 = e$, $eh = h$.

You take ops e, h on E sat $e^2 = e$, $eh = h$ then require $E = eE + hE \implies e = 1$ whence $he = h$. So suppose finitely many h_s ~~on E whence~~ on E such that $\sum_s h_s h_t = h_t$ $\forall t$. Then assuming $E = AE$ you get $E = \sum_t h_t E$ $\implies \sum_s h_s = 1$ on $h_t E$ $\forall t$ hence on E $\implies h_t \sum_s h_s = h_t$ $\forall t$.

Repeat. Given v.s. E with h_s $s \in$ finite Γ $\ni \left(\sum_s h_s\right) h_t = h_t$ $\forall t$. Let B defined by gen h_s, $s \in \Gamma$ relns. $\left(\sum h_s\right) h_t = h_t$ $\forall t$. Then $B = B^2$ so can replace E by $BE = \sum h_t E$. Then get $\sum_s h_s = 1$ on $h_t E$ $\forall t$ $\sum h_s = 1$ on E whence $h_t \sum_s h_s = h_t$ $\forall t$.

<u>OKAY</u>. Now you want $h_s h_t = 0$ symm?

Again consider a finite index set Γ, factor

$$h_s = E \xrightarrow[h_s''\ \alpha_s]{} V_s \xrightarrow[\text{incl.}]{\beta_s\ \overset{\Vert}{f}} E \qquad 0 = h_s h_t = \beta_s \alpha_s \beta_t \alpha_t \implies \alpha_s \beta_t = 0$$

$$E \xrightarrow{\alpha} \bigoplus_s V_s \xrightarrow{\beta} E$$

$$(\eta_s)_{s\in\Gamma} \longmapsto \sum \beta_s \eta_s$$

$$\xi \longmapsto (\alpha_s \xi)_{s\in\Gamma} \longmapsto \sum \beta_s \alpha_s \xi = \sum h_s \xi = \xi$$

so you get a projector on $\bigoplus_s V_s$

$$\bigoplus V_s \xrightarrow{\beta} E \xrightarrow{\alpha} \bigoplus V_s$$

$$(\eta_s) \longmapsto \sum_t \beta_t \eta_t \longmapsto \left(\sum_t \alpha_s \beta_t \eta_t\right)$$

so on $\bigoplus V_s$ we have the operator with kernel $\alpha_s\beta_t$ i.e.

$\boxed{\text{OKAY}}$

$$\alpha\beta(\eta_s) = \sum_t (\alpha_s\beta_t)\eta_t$$

$\alpha\beta$ is a projector. Does this help?

What do you want!

~~In the finite case~~ Review the finite case

E, h_s $s\in\Gamma$ finite, $\sum_s h_s h_t = h_t \quad \forall t$

nonunital ring A with generators h_s $s\in\Gamma$ finite
relns $\sum_s h_s h_t = h_t \quad \forall t$

Put $\sum_{s\in\Gamma} h_s = e$, relns become $eh_t = h_t \quad \forall t$

e left unit for A $\quad\mathbb{Z}$ projective right A-mod

$$0 \to A \to \tilde{A} \to \mathbb{Z} \to 0$$

$$0 \to A \otimes_A M \to M \to M/AM \to 0$$

M firm $\iff AM = M$
$\iff e = 1$ on M.

So $\mathcal{M}(A)$ seems to be $\mathrm{Mod}(Ae)$. So finish the picture of A-modules. ~~Suppose~~ First the general theory. ~~$\mathcal{M}(A) \cong \mathcal{M}(Ae)$~~

$$\begin{pmatrix} A & Ae \\ eA & eAe \end{pmatrix} \qquad \text{need } AeA = A \qquad \begin{array}{l} A = Ae \oplus A(1-e) \\ \text{as left } A\text{-module} \\ A = eA \quad \text{as right} \\ A\text{-module} \end{array}$$

$$M \mapsto eM$$

so $P = eA = A$
$Q = Ae$

$$\mathcal{M}(A) \underset{Ae\otimes}{\overset{eA\otimes_A -}{\rightleftarrows}} \mathcal{M}(Ae) \ ?$$

$$\mathcal{M}(A) \rightleftarrows \mathrm{Mod}(Ae) \qquad \begin{array}{l} M \mapsto eM = M \\ N \mapsto Ae \otimes_{Ae} N = N \end{array}$$

~~So sticking to firm~~

Point: firm A-modules are the same as unitary Ae modules.

$$A: \begin{cases} \text{gen.} & h_s \quad s \in \Gamma \\ \text{relns.} & eh_s = h_s \quad \forall s \text{ where } e = \sum_{s\in\Gamma} h_s \end{cases}$$

~~An A-module is a v.s. M with~~

Choose ~~an~~ a basepoint of Γ, call it 1.

$$A: \begin{cases} \text{gen} & x_s,\ s \neq 1,\ e \\ \text{relations} & \begin{cases} ex_s = x_s & \forall s \neq 1 \\ e^2 = e \end{cases} \end{cases} \qquad \begin{array}{l} h_s = x_s \quad s \neq 1 \\ h_1 = e - \sum_{s\neq 1} x_s \end{array}$$

$$eh_1 = e^2 - \sum_{s\neq 1} ex_s = e - \sum_{s\neq 1} x_s = h_1$$

Then an A-module is a v.s. M equipped with idemp- $M = eM \oplus (1-e)M$ and operator $x_s : M \to eM$ for $s \neq 1$.

$e^2 = e$
$eh = h$

$$M = eM \oplus (1-e)M$$

Ask if $x_s e = x_s$

$$A \left| \begin{array}{l} \text{gen } h_s \quad s \in \Gamma \\ \text{rel } e h_s = h_s \quad \forall s \end{array} \right. \qquad e = \sum_s h_s \quad \Rightarrow e^2 = e$$

$$M = eM \oplus (1-e)M \quad \text{and} \quad h_s : M \longrightarrow eM$$
arb. with sum e

question $h_s e = h_s$? NO

~~Let M be coherent.~~

$$A \left| \begin{array}{l} \text{gen } h_s, \ s \in \Gamma \\ \text{rel } h_s e = h_s \quad \forall s \end{array} \right. \quad \Rightarrow \quad e^2 = e$$

$$M = eM \oplus (1-e)M \qquad h_s : M/(1-e)M \longrightarrow M$$
arb. $\sum_s h_s = e$

There should be nothing more

Next examine embedding of E into $\oplus V_s$.

$V_s = h_s E \qquad \alpha_s = h_s : E \twoheadrightarrow h_s E$, $\beta_s : h_s E \hookrightarrow E$

$$h_s = \beta_s \alpha_s$$

$$E \xrightarrow{\alpha = (\alpha_s)} \oplus V_s \xrightarrow{\beta = (\beta_s)} E \qquad \beta\alpha = \sum \beta_s \alpha_s = \sum h_s = 1.$$

$\alpha\beta$ is a projector $\qquad \alpha_s \beta_t : V_t \longrightarrow V_s$

$$(\alpha\beta)\overbrace{(v_t)_t}^{\eta} = \Big(\sum_t \alpha_s \beta_t \eta_t\Big)_s$$

Review. E vector space with partition of unity h_s, $s \in \Gamma$, Γ finite. Means $\sum_{s \in \Gamma} h_s = 1$. Weaker condition: let $e = \sum_{s \in \Gamma} h_s$, then $e h_s = h_s \quad \forall s \in \Gamma$.

Left identity property. If $E = \sum_{s \in \Gamma} h_s E$, then $e = 1$ on E, so E is unitary over non comm. simplex.