You need to get some insight ~~of~~ your problem. ~~Com~~ Formulate a problem.
You have an operation on entire functions of $s = \varphi$

$$f(s) \longmapsto \frac{f(s)-f(a)}{s-a} \longmapsto \frac{\frac{f(s)-f(a)}{s-a} - f'(a)}{s-a}$$

$$\longmapsto \frac{\frac{f(s)-f(a)-f'(a)(s-a)}{(s-a)^2} - f''(a)}{s-a} = \frac{f(s)-f'(a)(s-a) - \frac{1}{2!}f''(a)(s-a)^2}{(s-a)^3}$$

$$= \frac{f(s)}{(s-a)^3} - \frac{f(a)}{(s-a)^3} - \frac{f'(a)}{(s-a)^2} - \frac{f''(a)}{2!\,(s-a)}$$

~~Now~~ Suppose $f(s)$ has the form $\int \varphi(-x)\, e^{xs}\, dx$

Is $\frac{f(s)-f(a)}{s-a}$ in the same form.

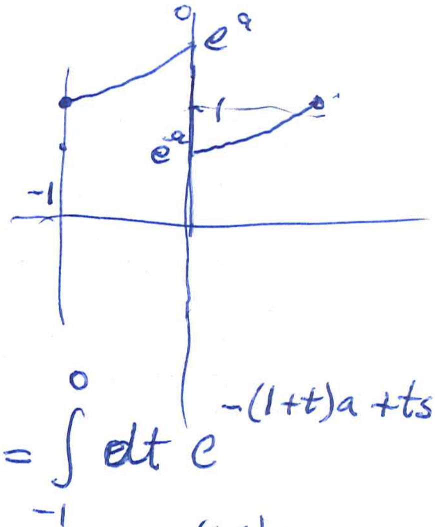
$$\frac{e^{xs}-e^{xa}}{s-a} = \int_0^1 x\,dt\; e^{\overbrace{xa+tx(s-a)}^{(1-t)xa+t(xs)}}$$

First suppose $\cancel{x}=1$ ~~with a~~ ~~Better use~~

$$\frac{e^s-e^a}{s-a} = \int_0^1 dt\; e^{(1-t)a+ts} = \int_0^1 dt\; \varphi(-t)\, e^{ts}$$

$$\varphi(-t) = e^{(1-t)a}\chi_{[0,1]}(t)$$

$$\varphi(t) = e^{(1+t)a}\chi_{[-1,0]}(t)$$



Next take $x=-1$.

$$\frac{e^{-s}-e^{-a}}{s-a} = \int_0^1 (-dt)\, e^{-(1-t)a-ts} = \int_{-1}^0 dt\; e^{-(1+t)a+ts}$$

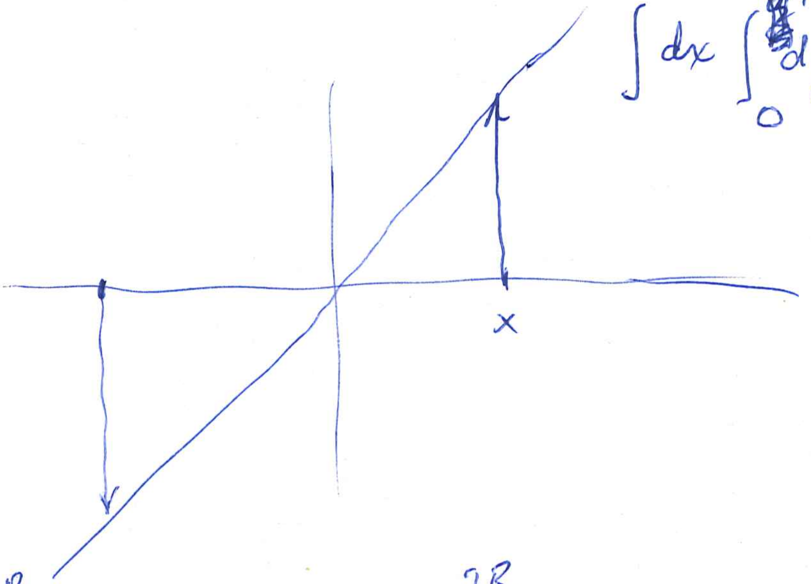
$$\varphi(-t) = e^{-(1+t)a}\chi_{[-1,0]}(t) \qquad \varphi(t) = e^{-(1-t)a}\chi_{[0,1]}(t)$$

Basic operator $f(s) \longmapsto \frac{f(s)-f(a)}{s-a}$ on entire functions.

$$e^{xs} \longmapsto \frac{e^{xs}-e^{xa}}{s-a} = \int_0^1 dt\, x\, e^{(1-t)xa+txs}$$

$$\int dx\, \varphi(-x)\, e^{xs} \longmapsto \int dx \int_0^1 dt\, x\varphi(-x)\, e^{(1-t)xa+txs}$$

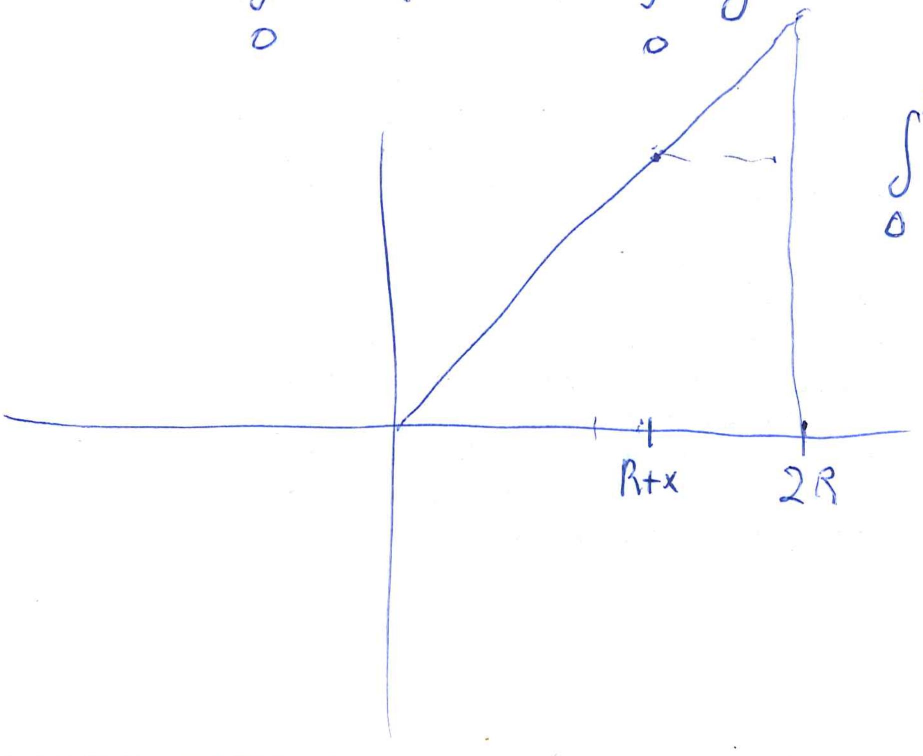
$$\int dx \int_0^x dy\, \varphi(-x)\, e^{(x-y)a+ys}$$



$$\int_{-R}^{R} dx\, \varphi(-x)\, e^{xs} = \int_0^{2R} dx\, \varphi(-x-R)\, e^{(R+x)s}$$

$$\longmapsto \int_0^{2R} dx\, \varphi(-R-x) \int_0^{R+x} dy\, e^{(R+x-y)a+ys}$$

$$= \int_0^{2R} dy\, e^{-ya} e^{ys} \int_{R-x}^{2R} dx\, \varphi(-R-x)\, e^{(R+x)a}$$



back to $\quad \dfrac{e^{xs}-e^{xa}}{s-a} = \displaystyle\int_0^x dy\; e^{(x-y)a+ys}$

$= $ convolution of $H(x)e^{xa}$ and $H(x)e^{xs}$

This should imply:

$$\frac{f(s)-f(a)}{s-a} = \int_0^\infty dx \underbrace{\cancel{e^{xs}}}_{\frac{e^{xs}-e^{xa}}{s-a}} \varphi(-x) = \int_0^\infty dx \int_0^x dy\; \underbrace{e^{(x-y)a+ys}\varphi(-x)}_{\varphi(-x)e^{(x-y)a}e^{ys}}$$

$$= \int_0^\infty dy \int_y^\infty dx\; \varphi(-x)\, e^{(x-y)a} e^{ys} = \int_0^\infty dy \int_0^\infty dx\; \varphi(-x) H(x-y) e^{(x-y)a} e^{ys}$$

this seems to be $\underbrace{\left(\varphi(-x) * (H(x)e^{xa}) * (H(x)e^{xs})\right)(0)}$

this is the function of u given by

$$\int \varphi(-t_0)\, e^{t_1 x} e^{t_2 s}$$

$t_0+t_1+t_2 = u \quad$ where $\quad t_i \geq 0$.

Perhaps the first thing to do is to set things up with the L.T.

$$\int_0^1 dw(-t)\, e^{-at+w(-st+at)}$$

$$\hat\varphi(s) = \int_0^\infty e^{-st}\varphi(t)\,dt$$

$$\int_0^1 dv(t)\, e^{-st+v(-at+st)}$$

$$\frac{\hat\varphi(s)-\hat\varphi(a)}{s-a} = \int_0^\infty dt\,\varphi(t) \frac{e^{-st}-e^{-at}}{s-a}$$

$$e^{-((1-v)s+va)t}$$

$$= \int_0^\infty dt\,\varphi(t)\int_0^1 dv\,(-t)\, e^{(1-v)(-st)+v(-at)}$$

$$= \int_0^\infty dt\,\varphi(t)\int_0^1 dw(-t)\, e^{-((1-w)a+ws)t}$$

$$\hat\varphi(s) = \int_0^\infty e^{-st}\varphi(t)\,dt \qquad \cancel{\frac{\hat\varphi(s)-\hat\varphi(a)}{s-a}}$$

$$\frac{e^{-st}-e^{-at}}{s-a} = \int_0^1 dw\,(-t)\,e^{-(a+w(s-a))t} = \int_0^1 dw(-t)\,e^{-[(1-w)a+ws]t}$$

$$\frac{\hat\varphi(s)-\hat\varphi(a)}{s-a} = \int_0^\infty dt\,\varphi(t)\int_0^1 dw\,(-t)\,e^{-[(1-w)a+ws]t} \qquad y = wt$$

$$= \int_0^\infty dt\,\varphi(t)\int_0^t dy\,(-1)\,e^{-[ta+y(s-a)]}$$

$$? \quad \int_0^t dy\,(-1)\left(e^{-ta-y(s-a)}\right) = \left[\frac{e^{-ta-y(s-a)}}{+(s-a)}\right]_0^t = \frac{e^{-ts}-e^{-ta}}{+(s-a)}$$

$$\frac{\hat\varphi(s)-\hat\varphi(a)}{s-a} = \int_0^\infty dt\int_0^t dy\;\varphi(t)(-1)\,e^{-ta+ya-ys}$$

$$e^{(-a)(t-y)}\,e^{(-s)y}$$

$$= \int_0^\infty dy\,\underbrace{\left(\int_y^\infty dt\;\varphi(t)(-1)\,e^{(-a)(t-y)}\right)}\,e^{-sy}$$

$$\psi(y) = \int_0^\infty du\;\varphi(y+u)(-1)\,e^{(-a)u}$$

Check: $\hat\varphi(s) = \int_0^\infty e^{-st}\varphi(t)\,dt$

$$\frac{\hat\varphi(s)-\hat\varphi(a)}{s-a} = \int_0^\infty dt\,\varphi(t)\,\frac{e^{-st}-e^{-at}}{s-a} = \int_0^1 dw\,(-t)\,e^{-[a+w(s-a)]t}$$

$$= \int_0^t dy\,(-1)\,e^{-at-y(s-a)}$$

$$= \int_0^\infty dt\int_0^t dy\;\varphi(t)(-1)\,e^{-a(t-y)}\,e^{-ys}$$

$\hat{\varphi}(s) = \int_0^\infty dt\, \varphi(t)\, e^{-st}$, $\quad \frac{\hat{\varphi}(s) - \hat{\varphi}(a)}{s-a} = \int_0^\infty dt\, \varphi(+t)\, \frac{e^{-st} - e^{-at}}{s-a}$

$$= \int_0^\infty dt\, \varphi(+t) \int_0^1 dw(-t)\, e^{-[a+w(s-a)]t} = \int_0^\infty dt\, \varphi(+t) \int_0^t dy\,(-1)\; e^{-at-y(s-a)}$$

$$= \int_0^\infty dy\, e^{-sy} \int_y^\infty dt \;\; \varphi(+t)(-1)\; e^{-at+ay}$$

$y+u=t$.

$$\psi(y) = \int_0^\infty \overset{du}{\cancel{dt}}\; \varphi(\overset{y+u}{t})(-1)\, e^{-a(\overbrace{t-y}^{u})} = -\int_0^\infty du\; e^{-au}\, \varphi(y+u)$$

$$\frac{d}{dy}\psi(y) = -\int_0^\infty du\; e^{-au}\, \varphi'(y+u)$$

$$= \left[-e^{-au}\varphi(y+u)\right]_0^\infty + \underbrace{\int_0^\infty du\; a\, e^{-au}\varphi(y+u)}_{a\psi(y)}$$

$$\left.\begin{aligned} (\partial_y - a)\psi(y) &= \varphi(y) \\ \psi(+\infty) &= 0 . \end{aligned}\right]$$

$$\psi(t) = e^{at}\int_t^\infty e^{-at'}\varphi(t')\,dt'$$

$\hat{\varphi}(s) = \int_0^\infty dt\; \varphi(t)\, e^{-st}$ $\quad\Big|\quad$ $\frac{\hat{\varphi}(s) - \hat{\varphi}(a)}{s-a} = \int_0^\infty dt\; \psi(t)\, e^{-st}$

Then $\quad \hat{\varphi}(s) - \hat{\varphi}(a) = \int_0^\infty dt\;\; \psi(t)\, (-\partial_t + a)\, e^{-st}$

$$= \left[-\psi(t)e^{-st}\right]_0^\infty + \int_0^\infty dt\; \left[(\partial_t - a)\psi(t)\right] e^{-st}\, dt$$

$$\hat{\varphi}(s) - \hat{\varphi}(a) = \psi(0) + \left[(\partial_t - a)\psi(t)\right]^\wedge$$

so $\quad \varphi(t) = (\partial_t - a)\psi(t) \quad$ modulo $\delta(t)$

$$\boxed{\hat{\varphi}(t) = \int_0^\infty dt\, \varphi(t) e^{-st}} \qquad \frac{\hat{\varphi}(s) - \hat{\varphi}(a)}{s-a} \simeq \int_0^\infty dt\, \varphi(t) \frac{e^{-st} - e^{-at}}{s-a}$$

$$\psi(t) = e^{at} \int_{+\infty}^{t} e^{-at'} \varphi(t')\, dt' \qquad \begin{cases} \psi(t) = 0 & t \gg 0 \\ \boxed{(\partial_t - a)\psi(t) = \varphi(t)} \end{cases}$$

$$\partial_t \psi = a\psi(t) + \cancel{e^{at}}\cancel{e^{-at}} \varphi(t)$$

$$(\mathcal{L}\psi')(s) = \int_0^\infty e^{-st} \psi'(t)\, dt$$

$$= \underbrace{\left[e^{-st}\psi(t)\right]_0^\infty}_{-\psi(0)} + \int_0^\infty s e^{-st} \psi(t)\, dt$$

$$-\psi(0) = -\int_{+\infty}^{0} e^{-at'} \varphi(t')\, dt' = \hat{\varphi}(a)$$

$\therefore$ ~~$\hat{\varphi}(s) - \hat{\varphi}(a) + s\hat{\varphi}$~~

$$\hat{\varphi}(s) = \mathcal{L}\varphi = \left(\partial_t \psi - a\psi\right)^{\wedge} = \underbrace{-\psi(0)}_{\hat{\varphi}(a)} + (s-a)\hat{\psi}$$

$$\therefore \boxed{\hat{\psi}(s) = \frac{\hat{\varphi}(s) - \hat{\varphi}(a)}{s-a}}$$

2nd derivation:

$$\frac{\hat{\varphi}(s) - \hat{\varphi}(a)}{s-a} = \int_0^\infty dt\, \varphi(t) \frac{e^{-st} - e^{-at}}{s-a}$$

$$\int_0^1 du\, e^{-[a+u(s-a)]t} (-t)$$

$$\parallel$$

$$\int_0^t (-1)\, dv\, e^{-at} e^{-(s-a)v}$$

$$= \int_0^\infty dt \int_0^t dv\, (-1) e^{-at+av-sv} \varphi(t) = \int_0^\infty (-1)\, dv \int_v^\infty dt\, e^{a\overbrace{(v-t)}^{-t'}} e^{-sv} \varphi(t)$$

$$v - t = -t'$$

$$= (-1)\int_0^\infty dv\, e^{-sv} \int_0^\infty dt'\, e^{-at'} \varphi \qquad ?$$

Try again for stubbornness.

$$\frac{\hat\varphi(s)-\hat\varphi(a)}{s-a} = \int_0^\infty dt\,\varphi(t)\,\frac{e^{-st}-e^{-at}}{s-a}$$

~~$e^{-st}-e^{-at}$~~

$$\frac{d}{du}\left(\frac{e^{-(at+(s-a)u)}}{s-a}\right) = e^{-(at+(s-a)u)}\cancel{(s-a)}(-1)$$

$$(-1)\int_0^t du\,\left(e^{-(at+(s-a)u)}\right) = \left[\frac{e^{-(at+(s-a)u)}}{s-a}\right]_{u=0}^{u=t} = \frac{e^{-st}-e^{-at}}{s-a}$$

$$\frac{\hat\varphi(s)-\hat\varphi(a)}{s-a} = -\int_0^\infty dt\int_0^t du\;\varphi(t)\,e^{-at}\,e^{-(s-a)u}$$

$$= -\int_0^\infty du\int_u^\infty dt'\;\varphi(t')\,e^{-at'}\,e^{-su}\,e^{au}$$

$$= -\int_0^\infty dt\,e^{-st}\left(e^{at}\int_t^\infty dt'\,\varphi(t')\,e^{-at'}\right)$$

$$= \int_0^\infty dt\,e^{-st}\,\psi(t) \qquad \psi(t) = e^{at}\int_\infty^t dt'\,e^{-at'}\varphi(t')$$

So this time suppose $\varphi(t)$ compact support, and see what happens.

$$\hat\varphi(s) = \int_{-\infty}^{\infty} dt\,\varphi(t)\,e^{-st} \qquad \text{entire fn of } s$$

$$\frac{\hat\varphi(s)-\hat\varphi(a)}{s-a}$$

So next see what happens when ~~$\varphi(t)$~~ ~~is not restricted~~ you try to handle $\varphi(t)$ of compact support but not contained in $\mathbb{R}_{\geq 0}$.

$$\hat{\varphi}(s) = \int_{-R}^{R} dt\, \varphi(t)\, e^{-st} \qquad \hat{\varphi}(i\xi) \text{ is the F.T. of } \varphi(t)$$

What can you say about $\dfrac{\hat{\varphi}(s) - \hat{\varphi}(a)}{s-a}$? Suppose this is $\hat{\psi}(s)$. Then ψ probably satisfies $(\partial_t - a)\psi = \varphi$.

$$(s-a)\hat{\psi}(s) = (s-a)\int_{-\infty}^{\infty} dt\, \psi(t)\, e^{-st}$$

$$= \int_{-\infty}^{\infty} dt\, \psi(t)\, -(\partial_t + a)e^{-st}$$

$$= \int_{-\infty}^{\infty} dt \left((\partial_t - a)\psi(t)\right) e^{-st}$$

Assume $\dfrac{\hat{\varphi}(s) - \hat{\varphi}(a)}{s-a} = \hat{\psi}(s) \implies \hat{\varphi}(s) - \hat{\varphi}(a) = (s-a)\hat{\psi}(s)$

$= \left((\partial_t - a)\psi\right)^{\wedge}(s)$.

Thus $\left(\varphi - (\partial_t - a)\psi\right)^{\wedge}(s) =$ the constant $\hat{\varphi}(a)$

$\varphi(t)$ comp supp. $\hat{\varphi}(s) = \int e^{-st}\varphi(t)\,dt$

entire function of s $\quad \dfrac{\hat{\varphi}(s) - \hat{\varphi}(a)}{s-a} = \int e^{-st}\psi(t)\,dt$

$$\hat{\varphi}(s) - \hat{\varphi}(a) = (s-a)\int e^{-st}\psi(t)\,dt = \int e^{-st}(\partial_t - a)\psi(t)\,dt$$

$$\int e^{-st}\left\{-\varphi(t) + (\partial_t - a)\psi(t)\right\} dt = -\hat{\varphi}(a) \quad \text{constant}$$

$$\therefore (\partial_t - a)\psi - \varphi = \begin{cases} 0 & t \neq 0 \\ \text{jumps by } -\hat{\varphi}(a) & \text{as } t \text{ crosses } 0. \end{cases}$$

$$\partial_t \psi - a\psi = \varphi$$

$$\psi(t) = e^{at}\int_{-\infty}^{t} e^{-at'}\varphi(t')\,dt' \qquad t<0$$

$$= e^{at}\int_{\infty}^{t} e^{-at'}\varphi(t')\,dt' \qquad t>0$$

$$\int_{\infty}^{t} - \int_{t}^{-\infty} = -\int_{-\infty}^{\infty} e^{-at'}\varphi(t')\,dt' = -\hat\varphi(a)$$

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derive $\quad (\partial_t - a)\psi^1(t,n) = b\psi^2(t,n) \qquad a = \frac{1}{2}|b|^2$

$\qquad \psi^1(t,n+1) - \psi^1(t,n) = \bar b\,\psi^2(t,n)$

exp solns. $\quad \psi(t,n) = e^{ts}\left(\frac{s+a}{s-a}\right)^n \begin{pmatrix} \frac{b}{s-a} \\ 1 \end{pmatrix} \cdot \text{const}$

$$\psi(t) = \begin{cases} \int_{+\infty}^{t} e^{a(t-t')}\varphi(t')\,dt' = \psi_+(t) \\ \int_{-\infty}^{t} e^{a(t-t')}\varphi(t')\,dt' = \psi_-(t) \end{cases} \qquad \psi_-(t) - \psi_+(t) = e^{at}\overbrace{\int_{-\infty}^{\infty} e^{-at'}\varphi(t')\,dt'}^{\hat\varphi(a)}$$

~~[illegible]~~ $\quad \hat\psi_+(s) = \frac{\hat\varphi(s)}{s-a} \qquad \hat\psi_-(s)$

Basic ~~[illegible]~~ phenomenon ~~[illegible]~~ to handle: ~~[illegible]~~ the F.T. of $\psi(t)$ with exponential asymptotics as $t \to \pm\infty$. You have to store this up with the rest of your ignorance

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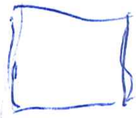
And so what do you know about grid space.

E should contain $\quad \frac{e^{xs}}{(s-a)^k}, \quad \frac{e^{xs}}{(s+a)^k} \quad$ for all t.

$$e^{xs} = \int dt\, e^{-st}\,\delta(t+x).$$ suppose you

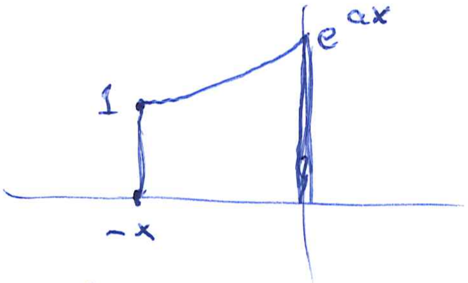
So let's see. Fix $x>0$ ask for what you get with horizontal translation $\mathbb{Z}$ also vertical translation

So you have elements $e^{ns}\frac{1}{s-a}$



$$\frac{e^{xs}-e^{xa}}{s-a} = \int e^{-st}\psi(t)\,dt \qquad \psi(t) = \int_{-\infty}^{t} e^{a(t-t')}\delta(t'+x)\,dt'$$

$$\varphi(t) = \delta(t+x) \qquad = \begin{cases} e^{a(t+x)} & -x<t\le 0 \\ 0 & t<-x \end{cases}$$



$$= -\int_{t}^{\infty} e^{a(t-t')}\delta(t'+x)\,dt' = 0 \qquad t>0$$

So what do you find?

Is there something to organize?

Review: If $\hat\varphi(s) = \int e^{-st}\varphi(t)\,dt$, then

$$\frac{\hat\varphi(s)-\hat\varphi(a)}{s-a} = \int e^{-st}\psi(t)\,dt, \text{ where } \psi(t) = \begin{cases} -\int_{t}^{\infty} e^{a(t-t')}\varphi(t')\,dt' & \text{for } t>0 \\ \int_{-\infty}^{t} e^{a(t-t')}\varphi(t')\,dt' & \text{for } t<0 \end{cases}$$

Suppose $\varphi(t) = \delta(t-x)$, $x>0$

$$\hat\psi(s) = \frac{\hat\varphi(s)-\hat\varphi(a)}{s-a} = \frac{e^{-xs}-e^{-as}}{s-a} \qquad \psi(t) = \begin{cases} 0 & t<0 \\ -e^{a(t-x)} & 0<t<x \\ 0 & x<t \end{cases}$$

$$\hat\varphi(s) = \int_0^\infty e^{-st}\varphi(t)\,dt \qquad (\partial_t - a)\psi(t) = \varphi(t)$$

$\psi(0)$ given const

$$(s-a)\hat\psi(s) = +\psi(0) + \hat\varphi(s) \Rightarrow -\psi(0) = \hat\varphi(a)$$

$$\hat\psi(s) = \frac{\hat\varphi(s)-\hat\varphi(a)}{s-a}$$

$$\psi(t) = e^{at}\int_0^t e^{-at'}\varphi(t')\,dt' - \hat\varphi(a)\,e^{at}$$

Use LT.

$$\hat{\varphi}_+(s) = \int_0^\infty e^{-st}\varphi(t)\,dt$$

$$\left(\begin{array}{ll} & (\partial_t - a)\,\psi_+(t) = \varphi(t) \\ t \geq 0 & \psi_+(+\infty) = 0 \end{array}\right) \qquad \psi_+(t) = -\int_t^\infty e^{a(t-t')}\varphi(t')\,dt' \quad t \geq 0.$$

$$-\psi_+(0) = \hat{\varphi}_+(a)$$

$$(s-a)\,\hat{\psi}_+(s) - \psi_+(0) = \hat{\varphi}_+(s)$$

$$\hat{\psi}_+(s) = \frac{\hat{\varphi}_+(s) - \hat{\varphi}_+(a)}{s-a}$$

$$\hat{\varphi}_-(s) = \int_{-\infty}^0 e^{-st}\varphi(t)\,dt$$

$$\left.\begin{array}{ll} (\partial_t - a)\,\psi_-(t) = \varphi(t) & t \leq 0 \\ \psi_-(-\infty) = 0 & \end{array}\right) \qquad \psi_-(t) = \int_{-\infty}^t e^{a(t-t')}\varphi(t')\,dt' \quad t \leq 0$$

$$\int_{-\infty}^0 e^{-st}(\partial_t - a)\psi_-(t)\,dt = \hat{\varphi}_-(s) \qquad \psi_-(0) = \hat{\varphi}_-(a)$$

$$\left[e^{-st}\psi_-(t)\right]_{-\infty}^0 + \int_{-\infty}^0 (s-a)\,e^{-st}\psi_-(t)\,dt$$

$$\hat{\varphi}_-(s) = \psi_-(0) + (s-a)\,\hat{\psi}_-(s)$$

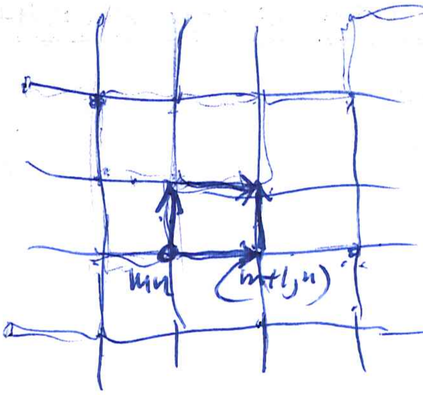
$$\hat{\psi}_-(s) = \frac{\hat{\varphi}_-(s) - \hat{\varphi}_-(a)}{s-a}$$

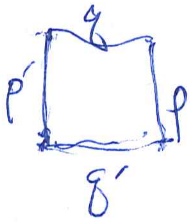
Adding these two you get

$$\hat{\varphi}(s) = \int_{-\infty}^\infty e^{-st}\varphi(t)\,dt \qquad \psi(t) = \begin{cases} \psi_+(t) & t \geq 0 \\ \psi_-(t) & t \leq 0. \end{cases}$$

$$\frac{\hat{\varphi}(s) - \hat{\varphi}(a)}{s-a} = \int_{-\infty}^\infty e^{-st}\psi(t)\,dt \quad \text{where}$$

In practical terms.





$$\underbrace{\begin{pmatrix} p \\ q \end{pmatrix} = \frac{1}{k}\begin{pmatrix} 1 & h \\ \bar h & 1 \end{pmatrix}\begin{pmatrix} p' \\ q' \end{pmatrix}}_{}\quad SU(1,1)$$

$|h| < 1 \qquad k = \sqrt{1-|h|^2}$

$$\begin{pmatrix} p \\ q' \end{pmatrix} = \begin{pmatrix} k & h \\ -\bar h & k \end{pmatrix}\begin{pmatrix} p' \\ q \end{pmatrix}$$

$\cap$

$U(2)$

$$\begin{pmatrix} \psi^1(m+1,n) \\ \psi^2(m,n+1) \end{pmatrix} = \frac{1}{k}\begin{pmatrix} 1 & h \\ \bar h & 1 \end{pmatrix}\begin{pmatrix} \psi^1(m,n) \\ \psi^2(m,n) \end{pmatrix}$$

$$\psi(m,n) = z^m w^n \begin{pmatrix} v^1 \\ v^2 \end{pmatrix} \qquad w = \left(\frac{k-z}{kz-1}\right)$$

$$\psi(m,n) = z^m \left(\frac{k-z}{kz-1}\right)^n \begin{pmatrix} \frac{h}{kz-1} \\ 1 \end{pmatrix} \text{const} \qquad z \in \mathbb{C} - \{0, k, k^{-1}\}$$

Set up analogy.

$$\begin{cases} (\partial_t - a)\psi^1(t,n) = b\,\psi^2(t,n) \\ \psi^2(t,n+1) - \psi^2(t,n) = \bar b\,\psi^1(t,n) \end{cases} \qquad a = \tfrac{1}{2}|b|^2$$

$s = i\xi$

exp solns $\quad \psi(t,n) = e^{st}\left(\frac{s+a}{s-a}\right)^n \begin{pmatrix} \frac{b}{s-a} \\ 1 \end{pmatrix} \times \text{const}$

$s \in \mathbb{C} - \{\pm a\}$

$$E = E_{hor} \oplus E_{vert}$$

E_{hor} = entire fns. of the form

E_{vert} = span of $\dfrac{1}{(s-a)^n}, \dfrac{1}{(s+a)^n}$, $n \geq 1$ = Span of $\left(\dfrac{s+a}{s-a}\right)^n \dfrac{b}{s-a}$ $\quad n \in \mathbb{Z}$

$$\hat\varphi(s) = \int_{-\infty}^{\infty} e^{-st}\varphi(t)\,dt$$

$\varphi(t)$ piecewise continuous of compact support.

Basically E_{hol} needs to consist of $\hat{\varphi}(s)$ for φ of compact support, ~~and clos~~ closed under $\hat{\varphi}(s) \longmapsto \dfrac{\hat{\varphi}(s)-\hat{\varphi}(a)}{s-a}$

$$\hat{\varphi}(s) = \underbrace{\int_0^\infty e^{-st}\varphi_+(t)\,dt}_{\hat{\varphi}_+} + \underbrace{\int_{-\infty}^0 e^{-st}\varphi_-(t)\,dt}_{\hat{\varphi}_-}$$

$$\begin{cases}(\partial_t - a)\psi_+(t) = \varphi_+(t) & t \geq 0 \\ \psi_+(+\infty) = 0 & \end{cases}$$

$$\psi_+(t) = -\int_t^\infty e^{a(t-t')}\varphi_+(t)\,dt \quad t \geq 0 \qquad \psi_+(0) = -\hat{\varphi}_+(a)$$

$$(s-a)\hat{\psi}_+ - \psi_+(0) = \hat{\varphi}_+$$

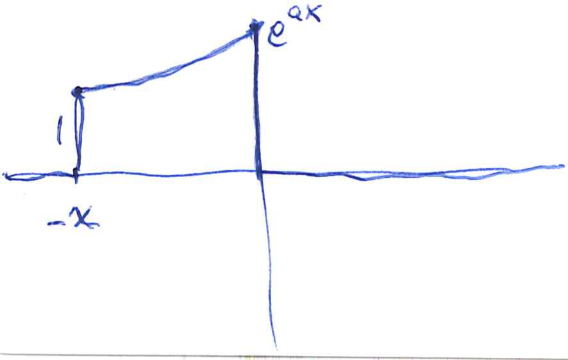
$$\therefore \quad \hat{\psi}_+(s) = \frac{\hat{\varphi}_+(s) - \hat{\varphi}_+(a)}{s-a}$$

This seems to be clear.

Actually you start with $\varphi(t) = \delta(t+x)$

$$\hat{\varphi}(s) = \int_{-\infty}^{\infty} e^{-st}\delta(t+x)\,dt = e^{xs} \qquad x > 0$$

Then $\dfrac{\hat{\varphi}(s)-\hat{\varphi}(a)}{s-a} \longleftrightarrow \begin{cases}\displaystyle\int_{-\infty}^{t} e^{a(t-t')}\delta(t'+x)\,dt' = e^{a(t+x)} & -x<t<0 \\ 0 \quad \text{for } t<-x,\ t>0. & \end{cases}$



If you apply operator n-times

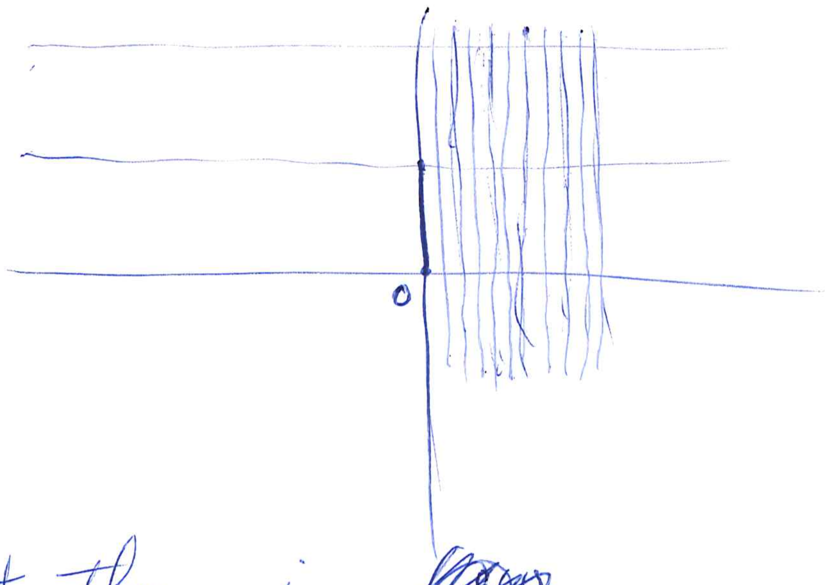
$$\frac{\hat{\varphi}(s) - \hat{\varphi}(a) - (s-a)\,\hat{\varphi}'(a) - \cdots - \frac{(s-a)^n}{n!}\hat{\varphi}^{(n)}(a)}{(s-a)^{n+1}}$$

Point $$\psi(t) = -\int_t^{\infty} e^{-a(t-t')}\varphi(t')\,dt'$$

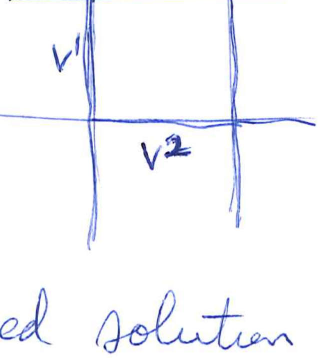
So now you should have the decomposition

$$E = E_{hor} \oplus E_{ver} \quad \mathbb{C}[\mu, \mu^{-1}]\frac{1}{s-a}$$

so can you find $IH(V', -)$.



First thing is .



$$E \xrightarrow{\sim} \mathbb{C}[z, z^{-1}, (z-k)^{-1}, (kz-1)^{-1}] \subset L^2(S^1, \tfrac{d\theta}{2\pi})$$

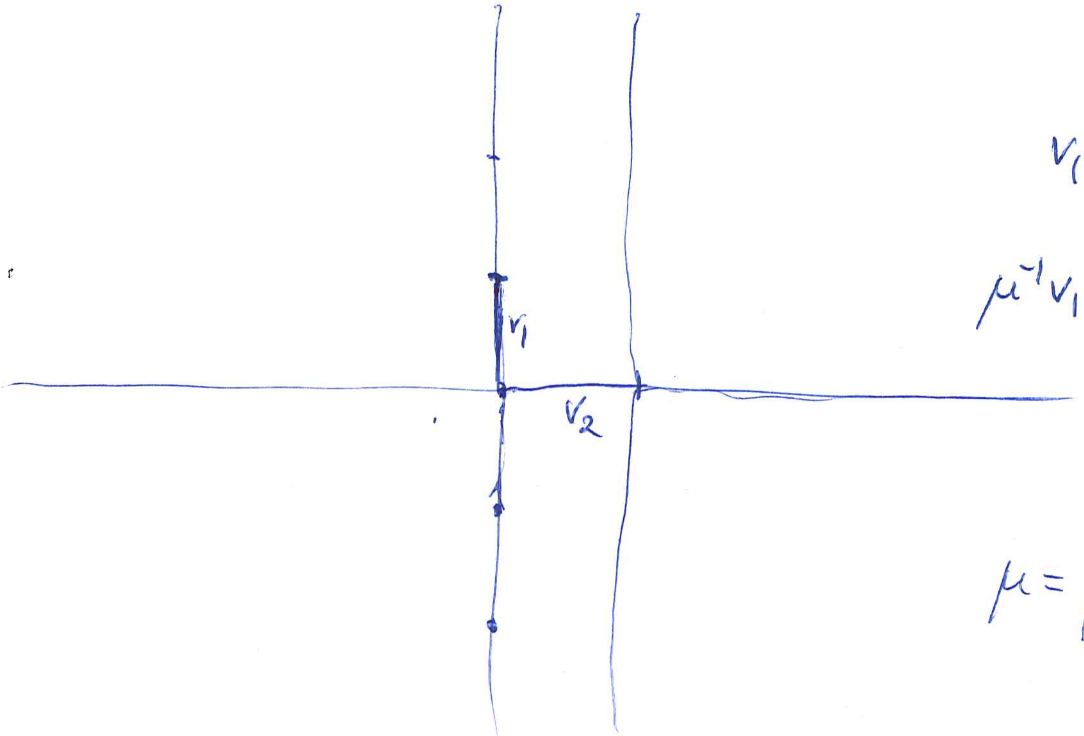
λ z.

μ $\dfrac{z-k}{kz-1}$

$V_2 \longmapsto 1$

$V_1 \longmapsto \dfrac{h}{kz-1}$

need solution of grid equation



$$v_1 = \frac{h}{kz-1}$$

$$\mu^{-1} v_1 = \frac{kz-1}{z-k}\,\frac{h}{kz-1} = \frac{h}{z-k}$$

$$\mu = \frac{z-k}{kz-1}$$

$$\left(v_2 \mid f(z) v_2\right) = \frac{1}{2\pi i}\int f(z)\,\frac{dz}{2\pi i z}$$

$$\mu^{-n}\frac{h}{kz-1} = \qquad n \geq 0.$$

$n \geq 1$ $$= \left(\frac{z-k}{kz-1}\right)^{-n}\frac{h}{kz-1} = \frac{(kz-1)^{n-1}}{(z-k)^n}$$

$\left(V_2 \middle| \hat{\varphi}(s)\, V_2\right)$ It should be a matter of organization. Suppose you have E defined as $E_{hor} + E_{ver}$. Can still show that $\overline{E_{hor}} = L^2\left(i\mathbb{R}, \frac{ds}{2\pi i}\right)$. ~~Shift~~

$\int e^{-st}\varphi(t)\,dt$ anyway

$\hat{\varphi}(s)$

grid (half cont.)

vectors $\lambda^t \mu^n \begin{pmatrix} v^1 \\ v^2 \end{pmatrix}$ $\quad (t,n) \in \mathbb{R}\times\mathbb{Z}$, ~~to realize them~~

$\lambda^t = e^{tD}$

$$\begin{cases} (D-a)v^1 = b v^2 \\ (\mu-1) v^2 = \bar{b} v^1 \end{cases} \qquad 2a = |b|^2$$

Realization as mero functions of s, $\quad D = $ mult by s $\quad v^1 = \frac{b}{s-a}$

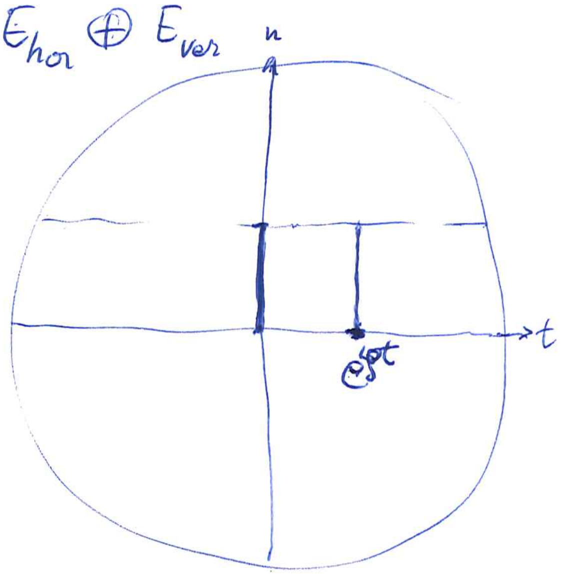
$\mu = $ mult by $\frac{s+a}{s-a}$ $\quad v^2 = 1$

$$\psi(t,n) = e^{st}\left(\frac{s+a}{s-a}\right)^n \begin{pmatrix} \frac{b}{s-a} \\ 1 \end{pmatrix} \qquad s \in \mathbb{C} - \{\pm a\}.$$

$$(f v^2 | g v^2) = \int_{-i\infty}^{i\infty} \overline{f(s)}\, g(s)\, \frac{ds}{2\pi i}$$

viewpoint. $\quad E = $ merom. functions of s.

$E_{hor} \oplus E_{ver}$



$s = i\rho$

Consider $f(s) \longmapsto \int_{-\infty}^{\infty} f(i\rho)\,\frac{d\rho}{2\pi}$

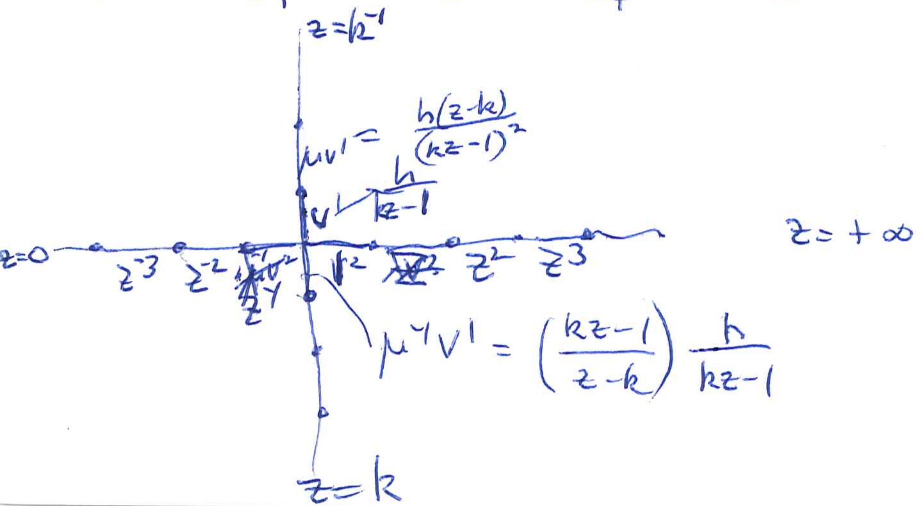
$f = \hat{\varphi}(s) = \int e^{-st}\varphi(t)\,dt$

should get $\varphi(0)$

$\hat{\varphi}(i\rho) = \int e^{-i\rho t}\varphi(t)\,dt$

$\varphi(t) = \int_{-\infty}^{\infty} e^{i\rho t}\hat{\varphi}(i\rho)\,\frac{d\rho}{2\pi}$

$$E = E^h_+ \oplus E^h_- \oplus E^v_+ \oplus E^v_-$$



Fix $E_{hor} = E^- \oplus E^+$

E^+ "spanned" by e^{xs}. $x>0$.

$\lambda^t v^2 = e^{tD} 1 = e^{ts}$. $t>0$.

$$E^+ = \left\{ \int_0^\infty e^{st} \varphi_+(t)\,dt \right\}$$

$$E^- = \left\{ \int_{-\infty}^0 e^{st} \varphi_-(t)\,dt \right.$$

$$\left[\begin{array}{l} (+\partial_t - a)\,\psi_+(t) = -\varphi_+(t) \qquad t \geq 0 \\ \psi_+(+\infty) = 0 \end{array}\right.$$

$$\psi_+(t) = e^{at}\int_\infty^t e^{-at'}\varphi_+(t')\,dt' \qquad \psi_+(0) =$$

$$e^{sx}\int e^{-st}\varphi(t)\,dt = \int e^{-s(t-x)}\varphi(t)\,dt$$

$$= \int e^{-st}\varphi(t+x)\,dt$$

$$\therefore \quad s\,\hat{\varphi}(s) = \widehat{\varphi'}(s)$$

$$\int e^{-st}\,\delta(t+x)\,dt = e^{sx}$$

E^+ appropriate lin comb of e^{ts} $t>0$.

Signs still are very confused.

$$\lambda^m \mu^n \begin{pmatrix} v^1 \\ v^2 \end{pmatrix} = z^m \left(\frac{z-k}{kz-1}\right)^n \begin{pmatrix} \frac{h}{kz-1} \\ 1 \end{pmatrix}$$

$$e^{tD} \mu^n \begin{pmatrix} v^1 \\ v^2 \end{pmatrix} = e^{ts} \left(\frac{s+a}{s-a}\right)^n \begin{pmatrix} \frac{b}{s-a} \\ 1 \end{pmatrix}$$

$$z^x = (z^\varepsilon)^{\frac{x}{\varepsilon}} = e^{(\log z)\varepsilon \frac{x}{\varepsilon}} = e^{(\log z)x}$$

$x = m\varepsilon$

$\therefore \quad e^{\log z} = e^s \qquad z \Rightarrow e^s$

basic idea is that ?

$z^x = e^{i\theta x}$

$m = \frac{x}{\varepsilon}$

$$(z^\varepsilon)^m \left(\frac{z^\varepsilon - k_\varepsilon}{k_\varepsilon z^\varepsilon - 1}\right)^n$$

so if $z^\varepsilon = 1 + \varepsilon s + O(\varepsilon^2)$

$$z^x = (z^\varepsilon)^{\frac{x}{\varepsilon}} = e^{\log(1+\varepsilon s + O(\varepsilon^2))\frac{x}{\varepsilon}}$$

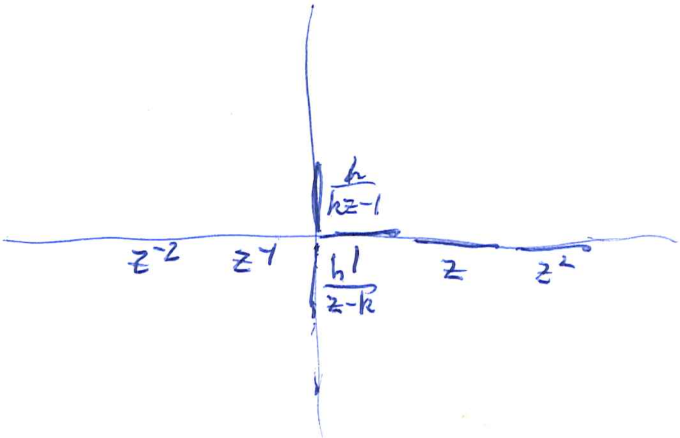
$$= e^{sx}$$

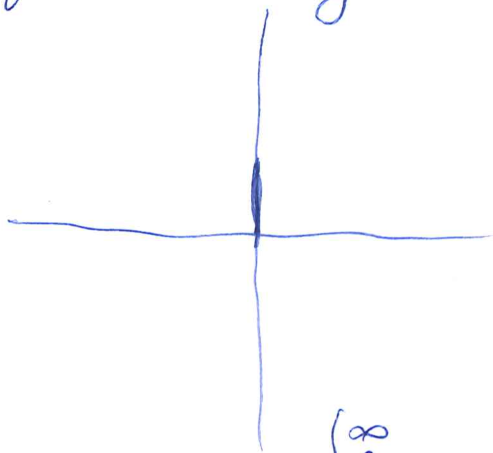
$k_\varepsilon = 1 - a\varepsilon \qquad a = \frac{1}{2}|b|$

$$\frac{z^\varepsilon - k_\varepsilon}{k_\varepsilon z^\varepsilon - 1} = \frac{1+\varepsilon s - (1-a\varepsilon)}{(1-a\varepsilon)(1+\varepsilon s) - 1} = \frac{s+a}{s-a}$$

So what happens is that z^m becomes $(z^\varepsilon)^{x/\varepsilon}$
$= (1+\varepsilon s + \text{higher})^{x/\varepsilon} \longrightarrow e^{sx}$ $\qquad s = i\xi$ etc.

Go back to — find analog





$$E_{hor,+} = \left\{ \sum_{n\geq 0} a_n z^n \right\} \qquad\qquad E_{hor,+} = \left\{ \int_0^\infty \varphi(t)\, e^{st}\, dt \right\}$$

$$z^m \left(\frac{z-k}{kz-1}\right)^n \begin{pmatrix} \frac{h}{kz-1} \\ 1 \end{pmatrix} \qquad\qquad e^{xs}\left(\frac{s+a}{s-a}\right)^n \begin{pmatrix} \frac{b}{s-a} \\ 1 \end{pmatrix}$$

$k = \sqrt{1-|h|^2}$ $\qquad\qquad$ $2a = |b|^2$

key point understand operation $\varphi(t) \longmapsto \psi(t)$ corresp. to $\hat{\varphi}(s) \longmapsto \frac{\hat{\varphi}(s) - \hat{\varphi}(a)}{s-a}$

$$\frac{\hat{\varphi}(s) - \hat{\varphi}(a)}{s-a} = \int_0^\infty e^{st}\psi(t)\,dt$$

$$\hat{\varphi}(s) - \hat{\varphi}(a) = \int_0^\infty (s-a)e^{st}\psi(t)\,dt = \int_0^\infty \left((\partial_t - a)e^{st}\right)\psi\,dt$$

$$= \left[e^{st}\psi(t)\right]_0^\infty - \int_0^\infty e^{st}(\partial_t + a)\psi(t)\,dt$$

$$= -\psi(0) - \widehat{(\partial_t + a)\psi}(s) = \hat{\varphi}(s) - \hat{\varphi}(a)$$

~~Let~~ Define $\psi(t)$ $\quad t \geq 0$ $\quad$ by

$$\left.\begin{aligned} +(\partial_t + a)\psi(t) &= -\varphi(t) \\ \psi(+\infty) &= 0 \end{aligned}\right| \qquad \psi(t) = e^{-at}\int_t^\infty e^{at'}\varphi(t')\,dt'$$

Arg. Given $\varphi(t)$ $t \geq 0$ piecewise cont. Comp. support

define $\psi(t)$ $t \geq 0$ by $\begin{cases} (\partial_t + a)\psi(t) = -\varphi(t) \\ \psi(+\infty) = 0 . \end{cases}$

i.e. $$\boxed{\psi(t) = e^{-at}\int_t^{\infty} e^{at'}\varphi(t')\,dt' \qquad \psi(0) = \hat{\varphi}(a)}$$

Then $$(s-a)\hat{\psi}(s) = \int_0^{\infty} \underbrace{(s-a)e^{st}}_{\partial_t - a}\psi(t)\,dt$$

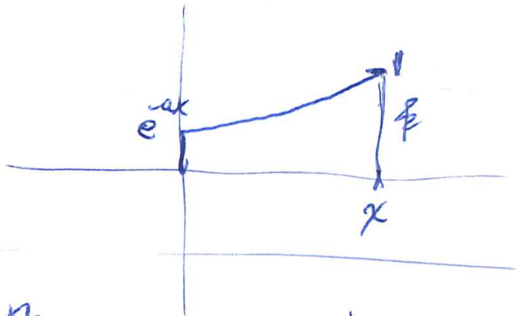
$$= \left[e^{st}\psi(t)\right]_0^{\infty} - \int_0^{\infty} e^{st}(\partial_t + a)\psi(t)\,dt$$

$$= -\psi(0) + \hat{\varphi}(s) \qquad \therefore \hat{\psi}(s) = \frac{\hat{\varphi}(s) - \hat{\varphi}(a)}{s-a}$$

$$e^{xs}\frac{1}{s-a} = \underbrace{\frac{e^{xs} - e^{xa}}{s-a}}_{\hat{\psi}(s)} + \frac{e^{xa}}{s-a}$$

$$\psi(t) = e^{at}\int_t^{\infty} e^{at'}\delta(t'-x)\,dt'$$

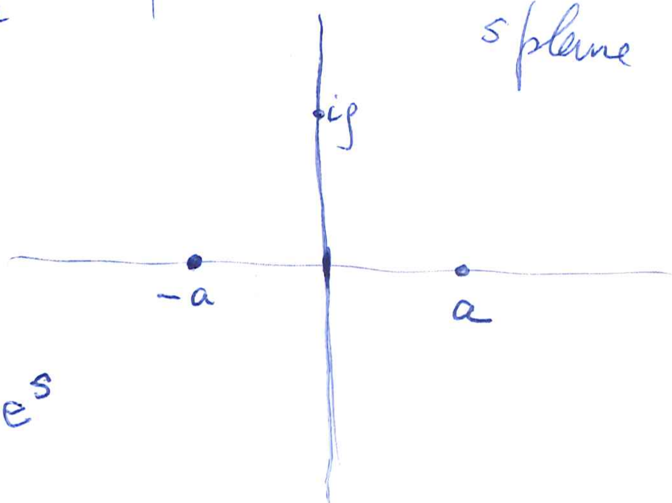
$$= \begin{cases} e^{a(t-x)} & 0 < t < x \\ 0 & x < t . \end{cases}$$



$$z^n\left(\frac{z-k}{kz-1}\right)^n \qquad e^{ts}\left(\frac{s+a}{s-a}\right)^n$$

$$z = k \longleftrightarrow s = -a$$

$$z = h^{-1} \qquad s = a$$



$$z \mapsto (z^{\varepsilon})^{1/\varepsilon} = (1+\varepsilon s)^{1/\varepsilon} \longrightarrow e^{s}$$

$$\hat{\varphi}_0(s) = \int_{-\infty}^{0} e^{st}\varphi(t)\,dt$$

$$(s-a)\hat{\psi}(s) = \int_{-\infty}^{0}(s-a)e^{st}\psi(t)\,dt$$

$$= \int_{-\infty}^{0}(\partial_t - a)e^{st}\psi(t)\,dt$$

$$= \left[e^{st}\psi\right]_{-\infty}^{0} - \int_{-\infty}^{0} e^{st}(\partial_t + a)\psi(t)\,dt$$

$$= \psi(0) - \left[(\partial_t + a)\psi\right]^{\wedge}(s)$$

so let ψ_- be

$$(\partial_t + a)\psi_- = -\varphi \qquad t \le 0$$

$$\psi(-\infty) = 0$$

$$\psi_-(t) = -e^{-at}\int_{-\infty}^{t} e^{at'}\varphi(t')\,dt' \qquad t \le 0$$

Then $\psi(0) = -\hat{\varphi}(a)$ and $\dfrac{\hat{\varphi}(s) - \hat{\varphi}(a)}{s-a} = \hat{\psi}(s)$

General formula is

$$\psi_+(t) = e^{-at}\int_{t}^{\infty} e^{at'}\varphi(t')\,dt' \qquad t \ge 0$$

$$\psi_-(t) = -e^{-at}\int_{-\infty}^{t} e^{at'}\varphi(t')\,dt' \qquad t \le 0.$$

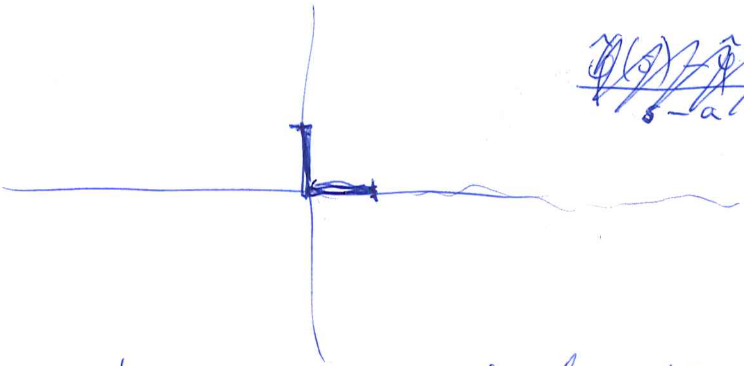
you solve $(\partial_t + a)\psi = -\varphi$

~~[illegible]~~ So now you understand E

Now you want to move on to the hermitian products.

Review the analogy

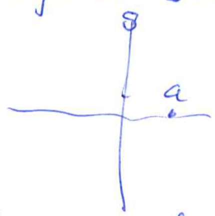
$$\lambda^m \mu^n \begin{pmatrix} v^1 \\ v^2 \end{pmatrix} \rightsquigarrow z^m \left(\frac{z-k}{kz-1}\right)^n \begin{pmatrix} \frac{h}{kz-1} \\ 1 \end{pmatrix}$$



$\cancel{\hat\varphi(s) - \hat\varphi(a)} / \cancel{s-a}$

$\mu = \frac{s+a}{s-a}$

introduce a sing. at $s=a$ like k^{-1}



back to ~~a~~ case both directions continuous and work out the cross

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change notation to x, y $\qquad e^{x\partial_x} e^{y\partial_y}$

$-\partial_x \psi^1 = i\psi^2$

$\partial_y \psi^2 = i\psi^1$

$$\psi = \begin{pmatrix} \psi^1(x,y) \\ \psi^2(x,y) \end{pmatrix} = \lambda^x \mu^y \begin{pmatrix} v^1 \\ v^2 \end{pmatrix} \rightsquigarrow e^{xw} e^{yw^{-1}} \Big($$

$-\overset{w}{D_x} v^1 = i v^2$

$\overset{w^{-1}}{D_y} v^2 = i v^1$

$D^2_{xy} = 1$

$w^{-1} v^2 = i v^1$

$-w v^1 = i v^2$

so can easily get to $\boxed{\begin{aligned} \partial_x \psi^1 &= \psi^2 \\ \partial_y \psi^2 &= \psi^1 \end{aligned}}$ $\quad \psi = e^{xw} e^{yw^{-1}} \begin{pmatrix} w^{-1} \\ 1 \end{pmatrix}$ const

So now your horizontal space consists of $\hat\varphi(w) = \int_{-\infty}^{\infty} e^{xw} \varphi(x)\, dx$

and you need to ~~[illegible]~~ remove the singularity of $e^{yw^{-1}} \hat\varphi(w)$ at $w=0$. Split $\hat\varphi(w) = \hat\varphi_+(w) + \hat\varphi_-(w)$

$$\hat\varphi(w) = \int_0^\infty e^{xw} \varphi(x)\, dx$$

need something in vertical space. ~~[illegible]~~ probably the upper half.

$$e^{w^{-1}} \hat\varphi(w) = \underset{\text{horiz.}}{\hat\Psi(w)} + \int_{0}^{\infty} e^{yw^{-1}} \psi(y)\, dy \quad \text{vert.}$$

$$e^{yw^{-1}}\hat{\varphi}(w) = \hat{\varphi}(w) + \int_0^y e^{y'w^{-1}} \varphi_2(y')\,dy$$

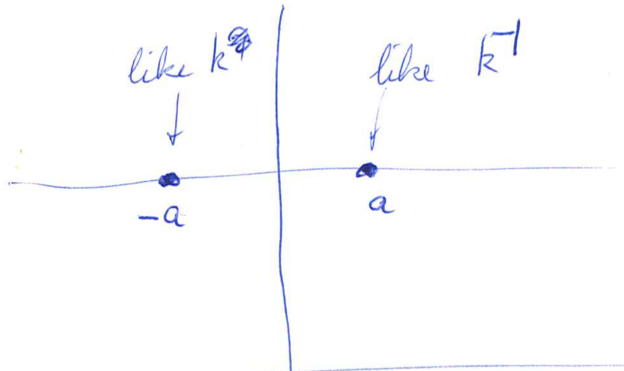
---

repeat: $\begin{cases} \partial_x \psi^1 = \psi^2 \\ \partial_y \psi^2 = \psi^1 \end{cases}$ $\quad \psi(x,y) = \lambda^x \mu^y \begin{pmatrix} v^1 \\ v^2 \end{pmatrix} \quad \lambda^x = e^{x\partial_x}, \ \mu^y = e^{y\partial_y}$

exp. solns. $\quad e^{xs + ys^{-1}} \begin{pmatrix} v^1 \\ v^2 \end{pmatrix}, \ s v^1 = v^2$.

You try to construct grid as a direct sum of horizontal + vertical subspaces, actually four subspaces

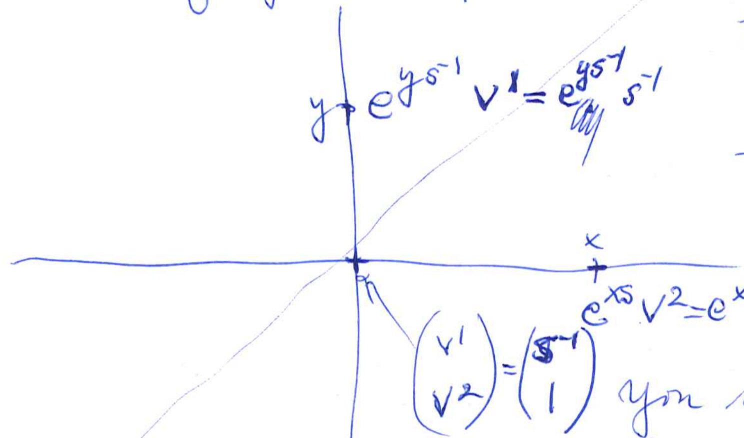
You need the analog of $e^{xs}\dfrac{1}{s-a}$. Apparently what happens is that your picture $\quad \mu = \dfrac{z-k}{\quad}$



$e^s \sim z \qquad e^{tD}\mu^n = e^{ts}\left(\dfrac{s+a}{s-a}\right)^n$

~~Anyway let us consider field~~.

Picture of grid space



Take $\hat{\varphi}$ in the positive horizontal space $\hat{\varphi}(s) = \int_0^\infty e^{sx}\varphi(x)\,dx$

then translate vertically

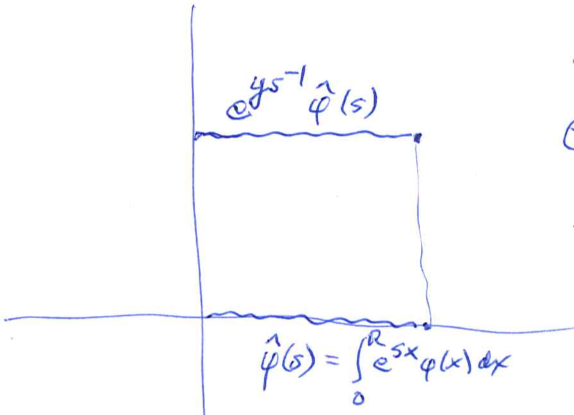
$$e^{ys^{-1}} \int_0^\infty e^{sx} \varphi(x)\,dx \qquad y > 0.$$

you want to represent this, split this into positive horiz + vert. compnents.

$$\psi(x,y) = e^{xs + ys^{-1}} \begin{pmatrix} s^{-1} \\ 1 \end{pmatrix} \overset{?}{=} \int e^{x's} \varphi_{h,+}(x')\,dx'$$

$x, y > 0$

$$+ \int e^{y's^{-1}} \varphi_{v,+}(y')\,dy'$$



Should be able to write $e^{ys^{-1}}\hat{\varphi}(s)$ as sum of vertical $\int_0^y e^{y's^{-1}}\psi_1(y')\,dy'$ and horizontal $\int_0^R e^{xs}\psi_2(x)\,dx$

$$\underbrace{e^{ys^{-1}}\int_0^R e^{sx}\varphi(x)\,dx}_{\text{Laurent series}} - \underbrace{\int_0^y e^{y's^{-1}}\psi_1(y')\,dy'}_{\text{power series in } s^{-1}} = \underbrace{\int_0^R e^{xs}\psi(x)\,dx}_{\substack{\text{entire} \\ \text{power series}}}$$

Try to do this for $\varphi(x) = \delta(x-R)$

$$\boxed{e^{ys^{-1}+sx} = \int_0^y \frac{e^{y's^{-1}}}{s^{-1}}\psi_1(y')\,dy' + \int_0^x e^{x's}\psi(x')\,dx'}$$

$$e^{ts}\left(\frac{s+a}{s-a}\right)^n = 1 \qquad \partial_x\partial_y\, e^{ys^{-1}+sx} = e^{ys^{-1}+sx}$$

Does this involve curvature? $\quad A^1 dx + A^2 dy$

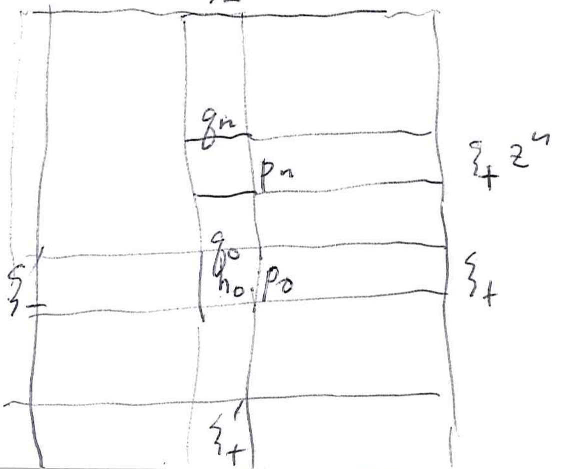
$\partial_x A - \partial_y A$

So things are quite confused, I wonder what is need to straighten things out.

Digress on inverse scattering to see if anything has become cleaner.

$$\begin{pmatrix}\xi_+\\ \xi_-\end{pmatrix} = \begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}\xi'_-\\ \xi'_+\end{pmatrix} \qquad \begin{pmatrix}\xi'_-\\ \xi'_+\end{pmatrix} = \begin{pmatrix}d & -b\\ -c & a\end{pmatrix}\begin{pmatrix}\xi_+\\ \xi_-\end{pmatrix}$$

$$\begin{pmatrix}\xi_+\\ \xi'_+\end{pmatrix} = \begin{pmatrix}\frac{1}{d} & \frac{b}{d}\\ -\frac{c}{d} & \frac{1}{d}\end{pmatrix}\begin{pmatrix}\xi'_-\\ \xi_-\end{pmatrix} \qquad \begin{pmatrix}\xi'_-\\ \xi_-\end{pmatrix} = \begin{pmatrix}\frac{1}{a} & -\frac{b}{a}\\ +\frac{c}{a} & \frac{1}{a}\end{pmatrix}\begin{pmatrix}\xi_+\\ \xi'_+\end{pmatrix}$$



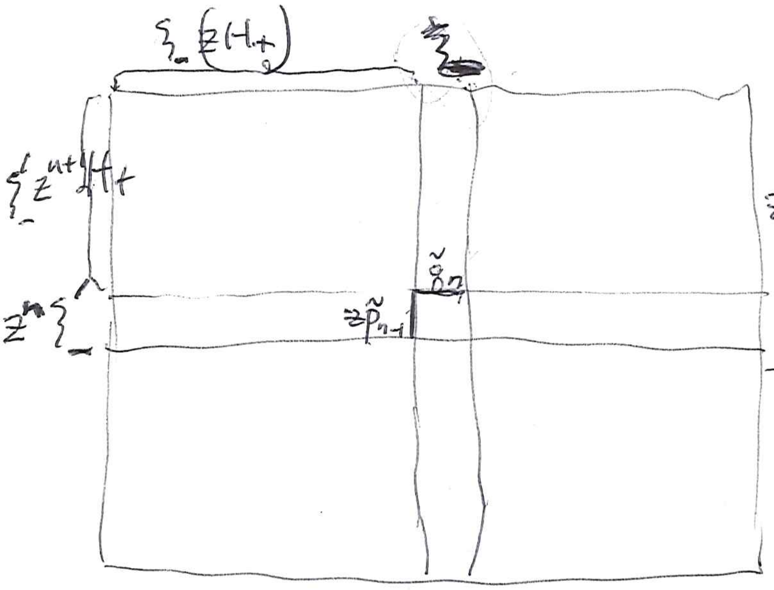
formula for PH, IH | start with IH,

$$IH(\xi_+ f + \xi_- g, \;``\;) = \|f\|^2 - \|g\|^2.$$

need $\xi_-' = \xi_+ d + \xi_-(-b)$

$$IH(\xi_-' f + \xi_- g, \;``\;) = IH(\xi_+ df + \xi_-(-bf+g))$$

$$= \|df\|^2 - \|-bf+g\|^2 = \int \begin{pmatrix} f \\ g \end{pmatrix}^* \begin{pmatrix} |d|^2-|b|^2 & +\bar b \\ +b & -1 \end{pmatrix} \begin{pmatrix} f \\ g \end{pmatrix} = \int \begin{pmatrix} f \\ g \end{pmatrix}^* \begin{pmatrix} 1 & \bar b \\ b & -1 \end{pmatrix} \begin{pmatrix} f \\ g \end{pmatrix}$$



$$z\tilde p_{n-1} \in \xi_- z^n + (\xi_-' z^n zH_+ + \xi_- zH_+)^\perp$$

$$z\tilde p_{n-1} = \xi_-' z^n (1-f_n) + \xi_-(-g_n)$$

$$\tilde g_n = \xi_- z^n(-\tilde f_n) + \xi_-(1-\varphi_n)$$

$$\int \begin{pmatrix} z^n zH_+ \\ zH_+ \end{pmatrix}^* \begin{pmatrix} 1 & \bar b \\ b & -1 \end{pmatrix} \begin{pmatrix} z^n(1-f_n) \\ -g_n \end{pmatrix} = 0$$

$$\int \begin{pmatrix} zH_+ \\ zH_+ \end{pmatrix}^* \begin{pmatrix} 1 & z^{-n}\bar b \\ bz^n & -1 \end{pmatrix} \begin{pmatrix} 1-f_n \\ -g_n \end{pmatrix} = 0$$

$\pi_+ =$ orth proj onto $zH_+ \subset L^2$

$$\boxed{\begin{aligned} \pi_+(1-f_n - z^{-n}\bar b g_n) &= 0 \\ \pi_+(bz^n(1-f_n) + g_n) &= 0 \end{aligned}}$$

$$f_n = -\pi_+(z^{-n}\bar b g_n)$$

$$\pi_+(bz^n) = \pi_+(bz^n f_n) - g_n$$

$T = \pi_+ \cdot bz^n \cdot \varepsilon_+ : zH_+ \to zH_+$

$T^* = \pi_+ z^{-n}\bar b \varepsilon_+$:

$$f = -T^* g$$

$$\pi_+(bz^n) = Tf - g$$

$$\pi_+ \begin{pmatrix} 1 & z^{-n}\bar b \\ bz^n & -1 \end{pmatrix} \varepsilon_+ \begin{pmatrix} f_n \\ g_n \end{pmatrix} = \begin{pmatrix} 0 \\ \pi_+(bz^n) \end{pmatrix}$$

$$\begin{pmatrix} 1 & T^* \\ T & -1 \end{pmatrix}^2 = \begin{pmatrix} 1+T^*T & 0 \\ 0 & 1+TT^* \end{pmatrix}$$

$$\boxed{\begin{pmatrix} 1 & T_n^* \\ T_n & -1 \end{pmatrix} \begin{pmatrix} f_n \\ g_n \end{pmatrix} = \begin{pmatrix} 0 \\ \pi_+(bz^n) \end{pmatrix}}$$

Consider $(-\partial_x^2 + u)\phi = k^2\phi$, assuming if necessary that $-\partial_x^2 + u = (\partial_x + v)(-\partial_x + v)$

i.e. $u = v' + v^2$, in which case you have the Dirac type eqn. $ik\psi = \begin{pmatrix} \partial_x & -v \\ v & -\partial_x \end{pmatrix}\psi$

$$\partial_x \psi^1 - v\psi^2 = ik\psi^1 \qquad (\partial_x - ik)\psi^1 = v\psi^2$$
$$-\partial_x \psi^2 + v\psi^1 = ik\psi^2 \qquad (\partial_x + ik)\psi^2 = v\psi^1$$

$$(\partial_x^2 - k^2)\psi^1 = (\partial_x + k)(v\psi^2) = v'\psi^2 + v(\partial_x + k)\psi^2 \ ?$$

So this point has to be understood better

$$\partial_x \psi^1 - v\psi^2 = k\psi^1$$
$$-\partial_x \psi^2 + v\psi^1 = +k\psi^2$$

$$(-\partial_x + v)(\psi^1 + \psi^2) = k(+\psi^1 - \psi^2)$$
$$(\partial_x + v)(\psi^1 - \psi^2) = k(\psi^1 + \psi^2)$$

then $(\partial_x + v)(-\partial_x + v)(\psi^1 + \psi^2) = k^2(\psi^1 + \psi^2)$

So let's consider a Dirac system

$$\partial_t \psi = \begin{pmatrix} \partial_x & -v \\ v & -\partial_x \end{pmatrix}\psi \qquad v \text{ real.}$$

then the above shows ψ^1

Start again

$$\partial_t \psi = \begin{pmatrix} \partial_x & iv \\ iv & -\partial_x \end{pmatrix}\psi \qquad k\psi = \begin{pmatrix} \frac{1}{i}\partial_x & v \\ v & +i\partial_x \end{pmatrix}\psi$$

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} -i\partial_x & v \\ v & i\partial_x \end{pmatrix}\begin{pmatrix} +1 & +1 \\ +1 & -1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\cancel{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}}$$

$$\frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$k\tilde{\psi} = \begin{pmatrix} v & \frac{1}{i}\partial_x \\ \frac{1}{i}\partial_x & -v \end{pmatrix}\tilde{\psi}$$

$$\partial_t \psi = \begin{pmatrix} \partial_x & -v \\ v & -\partial_x \end{pmatrix}\psi \qquad \begin{aligned} \lambda\psi^1 &= \partial_x\psi^1 - v\psi^2 \\ \lambda\psi^2 &= -\partial_x\psi^2 + v\psi^1 \end{aligned}$$

subtract
$$\lambda(\psi^1 - \psi^2) = \partial_x(\psi^1 + \psi^2) + v(-\psi^2 - \psi^1) = (\partial_x - v)(\psi^1 + \psi^2)$$

add
$$\lambda(\psi^1 + \psi^2) = \partial_x(\psi^1 - \psi^2) + v(\psi^1 - \psi^2) = (\partial_x + v)(\psi^1 - \psi^2)$$

$$\lambda\begin{pmatrix}\psi^1-\psi^2\\ \psi^1+\psi^2\end{pmatrix} = \begin{pmatrix}0 & \partial_x - v\\ \partial_x + v & 0\end{pmatrix}\begin{pmatrix}\psi^1-\psi^2\\ \psi^1+\psi^2\end{pmatrix}$$

$$\frac{1}{2}\begin{pmatrix}1 & -1\\ 1 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}1 & 1\\ -1 & 1\end{pmatrix} = \frac{1}{2}\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix} = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$$

$$\frac{1}{2}\begin{pmatrix}1 & -1\\ 1 & 1\end{pmatrix}\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\begin{pmatrix}1 & 1\\ -1 & 1\end{pmatrix} = \frac{1}{2}\begin{pmatrix}1 & -1\\ 1 & 1\end{pmatrix}\begin{pmatrix}1 & -1\\ 1 & 1\end{pmatrix} = \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}$$

$$\underbrace{(\partial_x + v)(\partial_x - v)}_{\partial_x^2 - (v' + v^2)}(\psi^1+\psi^2) = (\partial_x + v)\lambda(\psi^1 - \psi^2) = \lambda^2(\psi^1+\psi^2)$$

---

discuss the scattering, if you can. The point ~~[illegible]~~ maybe is that you ought now to be able to handle the factorizable Schroedinger equations.

$$\phi = \frac{\psi^1+\psi^2}{2}$$

$$\left(-\partial_x^2 + (v'+v^2)\right)\phi = -\lambda^2\phi = k^2\phi^2$$

So now ~~[illegible]~~ you want to discuss scattering assuming v decays,

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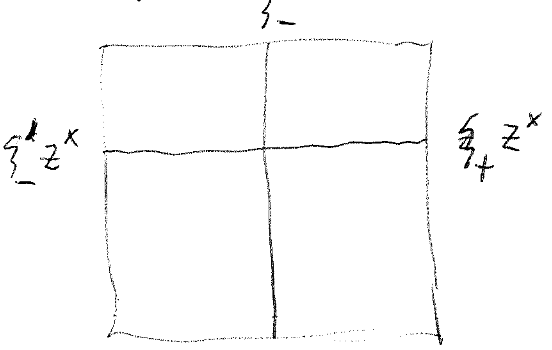
The thing you are aiming for

Set up continuous version, the principle should be that everything is described in terms of functions of k, ultimately pairs of functions in Hardy space

Your notation $\xi_\pm L^2, \xi'_\pm L^2$



$$p_x = \xi'_- z^x(1-f_x) + \xi_-(-g_x)$$

$$q_x = \xi'_- z^x(-\phi_x) + \xi_-(1-\psi_x)$$

There's nonunital ring stuff involved here. Why?

$$\int \begin{pmatrix} z^x H_+ \\ H_+ \end{pmatrix}^* \begin{pmatrix} 1 & \bar b \\ b & -1 \end{pmatrix} \begin{pmatrix} z^x(1-f) \\ -g \end{pmatrix} = 0$$

$$\pi_+ \begin{pmatrix} 1 & z^{-x}\bar b \\ b z^x & -1 \end{pmatrix} \begin{pmatrix} 1-f_x \\ -g_x \end{pmatrix} = 0$$

$$\underbrace{\pi_+ \begin{pmatrix} 1 & z^{-x}\bar b \\ b z^x & -1 \end{pmatrix} \varepsilon_+}\begin{pmatrix} f_x \\ g_x \end{pmatrix} = \pi_+ \begin{pmatrix} 1 \\ b z^x \end{pmatrix}$$

$$\begin{pmatrix} 1 & T_x^* \\ T_x & -1 \end{pmatrix} \begin{pmatrix} f_x \\ g_x \end{pmatrix} = \begin{pmatrix} 0 \\ \pi_+(b z^x) \end{pmatrix}$$

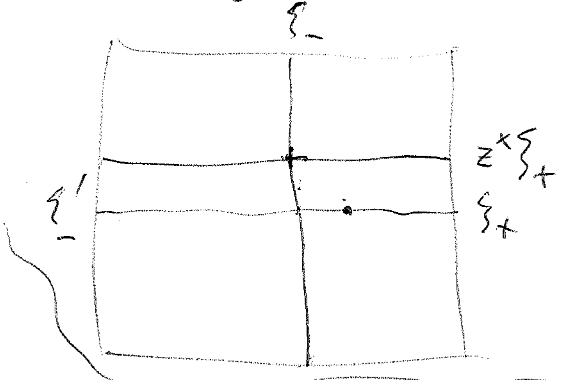
You have lots to do - interpret as Birkhoff factorization of S-matrix. There's lots of ~~intuition~~, insight needed. List problems.

getting h_x

$$\begin{pmatrix} & \end{pmatrix}$$

basic factorization

$$\begin{pmatrix} \xi_+ \\ \xi_- \end{pmatrix} = \begin{pmatrix} a_x^k & b_x^k \\ c_x^k & d_x^k \end{pmatrix} \begin{pmatrix} z^{-x} p_x \\ q_x \end{pmatrix} \qquad \begin{pmatrix} z^{-x} p_x \\ q_x \end{pmatrix} = \begin{pmatrix} a_x^l & b_x^l \\ c_x^l & d_x^l \end{pmatrix} \begin{pmatrix} \xi'_- \\ \xi'_+ \end{pmatrix}$$



$$\begin{pmatrix} p_0 \\ q_0 \end{pmatrix} = \begin{pmatrix} a^l & b^l \\ c^l & d^l \end{pmatrix} \begin{pmatrix} \xi'_- \\ \xi'_+ \end{pmatrix} = \begin{pmatrix} d^r & -b^r \\ -c^r & a^r \end{pmatrix} \begin{pmatrix} \xi_+ \\ \xi_- \end{pmatrix}$$ transfer picture

move to scattering picture

$$\begin{pmatrix} \xi_+ \\ \xi_- \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \xi'_- \\ \xi'_+ \end{pmatrix} \qquad \begin{pmatrix} \xi'_- \\ \xi'_+ \end{pmatrix} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \begin{pmatrix} \xi_+ \\ \xi_- \end{pmatrix}$$

$$\begin{pmatrix} \xi_+ \\ \xi'_+ \end{pmatrix} = \begin{pmatrix} \frac{1}{d} & \frac{b}{d} \\ -\frac{c}{d} & \frac{1}{d} \end{pmatrix} \begin{pmatrix} \xi'_- \\ \xi_- \end{pmatrix} \qquad \begin{pmatrix} \xi'_- \\ \xi_- \end{pmatrix} = \begin{pmatrix} \frac{1}{a} & -\frac{b}{a} \\ \frac{c}{a} & \frac{1}{a} \end{pmatrix} \begin{pmatrix} \xi_+ \\ \xi'_+ \end{pmatrix}$$

$$\begin{pmatrix} p_0 \\ q_0 \end{pmatrix} = \begin{pmatrix} a^l & b^l \\ c^l & d^l \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\frac{c}{d} & \frac{1}{d} \end{pmatrix} \begin{pmatrix} \xi'_- \\ \xi_- \end{pmatrix} = \begin{pmatrix} \frac{d^r}{d} & \frac{b^l}{d} \\ -\frac{c^r}{d} & \frac{d^l}{d} \end{pmatrix} \begin{pmatrix} \xi'_- \\ \xi_- \end{pmatrix}$$

$$\begin{pmatrix} a^r & b^r \\ c^r & d^r \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} d^l & -b^l \\ -c^l & a^l \end{pmatrix}$$

$$\frac{c^r}{d} = \frac{c d^l}{d} - c^l$$

$$\frac{d^r}{d} = a^l - b^l \frac{c}{d}$$

$$\begin{pmatrix} p_0 \\ q_0 \end{pmatrix} = \begin{pmatrix} d^r & -b^r \\ -c^r & a^r \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \frac{c}{a} & \frac{1}{a} \end{pmatrix} \begin{pmatrix} \xi_+ \\ \xi'_+ \end{pmatrix} = \frac{1}{a} \begin{pmatrix} -b^r & d^l \\ a^r & c^l \end{pmatrix} \begin{pmatrix} \xi_+ \\ \xi'_+ \end{pmatrix}$$

$$\begin{pmatrix} p_0 \\ q_0 \end{pmatrix} = \begin{pmatrix} a^l & b^l \\ c^l & d^l \end{pmatrix}\begin{pmatrix} \xi'_- \\ \xi'_+ \end{pmatrix} = \begin{pmatrix} d^r & -b^r \\ -c^r & a^r \end{pmatrix}\begin{pmatrix} \xi_+ \\ \xi_- \end{pmatrix}$$

$$= \frac{1}{d}\begin{pmatrix} d^r & b^l \\ -c^r & d^l \end{pmatrix}\begin{pmatrix} \xi'_- \\ \xi_- \end{pmatrix} = \frac{1}{a}\begin{pmatrix} a^l & -b^r \\ c^l & a^r \end{pmatrix}\begin{pmatrix} \xi_+ \\ \xi'_+ \end{pmatrix}$$

scattering matrix factors

$$\frac{1}{d}\begin{pmatrix} 1 & b \\ -c & 1 \end{pmatrix} = \begin{pmatrix} a^r & b^r \\ -c^l & a^l \end{pmatrix}\frac{1}{d}\begin{pmatrix} d^r & b^l \\ -c^r & d^l \end{pmatrix}$$

You want to explore the idea of splitting off the singularity using the Laurent series. You saw this nicely in the case of the half continuous grid

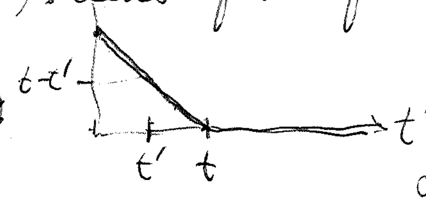
$$f(s) \longmapsto \frac{f(s)}{(s-a)^2} = \frac{f(s) - f(a) - f'(a)(s-a)}{(s-a)^2} + \cdots$$

$$f(s) = \hat{\varphi}(s) = \int_{-\infty}^{\infty} e^{sx}\varphi(x)\,dx = \hat{\varphi}_-(s) + \hat{\varphi}_+(s)$$

solve $\quad (\partial_t - a)^2 \psi_+(t) = \varphi(t) \qquad t \geq 0$

$$\psi_+(+\infty) = 0$$

Green's fn. for ∂_t^2 is



so $\psi(t) = -\int_t^{\infty} (t-t')\,\varphi(t')\,dt'$

$$\partial_t \psi(t) = -\int_t^{\infty} \varphi(t')\,dt'$$

$$\partial_t^2 \psi(t) = \varphi(t).$$

For $(\partial_t - a)^2$ you want

$$\psi(t) = -\int_t^\infty e^{a(t-t')}(t-t')\varphi(t')\,dt'$$

$$(\partial_t - a)\psi(t) = -\int_t^\infty e^{a(t-t')}\varphi(t')\,dt'$$

$$(\partial_t - a)^2\psi(t) = \varphi(t).$$ etc.

But now you want to combine the preceding with Birkhoff decomposition. Somehow the key idea here is the process of splitting off the singular part. You ~~[illegible]~~ start with $f(s)$ holom. multiply by singular thing e.g. $\frac{1}{s-a}$ and split off the singular part

continuous grid. $\psi(x,y) = e^{x\omega}e^{y\omega^{-1}}\begin{pmatrix} v^1 \\ v^2 \end{pmatrix}$

Motivation: $-\partial_x \psi^1 = i\psi^2$ $\quad \psi = e^{i(x\xi + y\eta)}\begin{pmatrix} v^1 \\ v^2 \end{pmatrix}$

$\partial_y \psi^2 = i\psi^1$

$\omega^{-1} = -i\xi^{-1}$

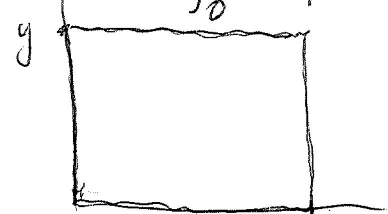
$-i\omega^{-1} = -\xi^{-1}$

$-\xi v^1 = v^2$

$\eta v^2 = v^1 \qquad \therefore\ \eta = -\xi^{-1}$

$$\psi = e^{i(x\xi - y\xi^{-1})}\begin{pmatrix} -\xi^{-1} \\ 1 \end{pmatrix} = e^{x\omega + y\omega^{-1}}\begin{pmatrix} -i\omega^{-1} \\ 1 \end{pmatrix} \qquad \omega = i\xi$$

$$e^{y\omega^{-1}}\int_0^\infty e^{x\omega}\varphi(x)\,dx = \int_0^y \frac{e^{y'\omega^{-1}}}{\omega}\overset{\varphi(y')}{\alpha(y,y')}\,dy' + \int_0^x e^{x'\omega}\beta(x,x')\,dx'$$



$$e^{y\omega^{-1}}e^{x\omega} = \quad ?$$

$\int_0^\infty e^{x\omega}\varphi(x)\,dx$

φ

$$e^{yw^{-1}} e^{xw} = \int_0^y \frac{e^{y'w^{-1}}}{w} \alpha(x,y,y')\,dy' + \int^x e^{x'w} \beta(\quad)\,dx'$$

ways to proceed. Laurent series.

$$e^{yw^{-1}+xw} = \sum \frac{y^k}{k!} \frac{x^l}{l!} w^{l-k}$$

Observation. In the discrete case the Hilbert space completions of $\mathbb{C}[\lambda]v^2$ and $\mathbb{C}[\mu]v^1$ are the same. Why?

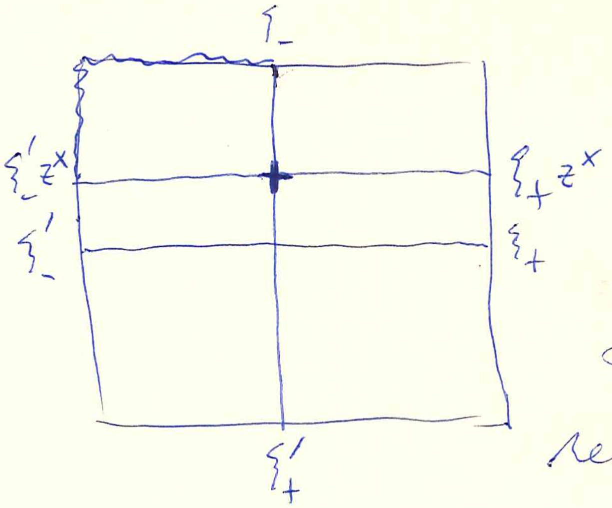
$$-\mu = \frac{-\lambda+k}{k\lambda-1} = \begin{pmatrix} -1 & k \\ k & -1 \end{pmatrix}(\lambda)$$

so $\lambda = \begin{pmatrix} 1 & k \\ k & 1 \end{pmatrix}(-\mu)$, ~~are related by~~

$\therefore$ λ, μ are ~~[illegible]~~ related by holomorphic autos of the closed unit disk. also $v^1 = \frac{h}{k\lambda-1} v^2$ are related by an invertible holom. fn. of λ + sin. of μ.

Is there an analog for the continuous case?

Return to the scattering ~~situation~~. ~~What~~ You want to relate Birkhoff factorization of S matrix to this idea of splitting off the singularities. Especially the picture (Mumford) leading to KdV type motion. So what to try? First, ~~[illegible]~~ can you recover the potential? Can you link your projection picture to the Schroedinger DE.



$$p_x \equiv \xi_-' z^x(1-f_x) + \xi_-(-g_x)$$

$$q_x \equiv \xi_-' z^x(-\phi_x) + \xi_-(1-\psi_x)$$

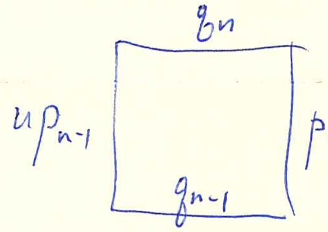
You then have the orth relations

$$\int \begin{pmatrix} z^x H_+ \\ H_+ \end{pmatrix}^* \begin{pmatrix} 1 & \bar b \\ b & -1 \end{pmatrix} \begin{pmatrix} z^x(1-f_x) \\ -g_x \end{pmatrix} = 0 \qquad \Bigg| \qquad \int \begin{pmatrix} H_+ \\ H_+ \end{pmatrix}^* \begin{pmatrix} 1 & \bar b z^{-x} \\ b z^x & -1 \end{pmatrix} \begin{pmatrix} 1-f_x \\ -g_x \end{pmatrix} = 0$$

$$\int \begin{pmatrix} z^x H_+ \end{pmatrix}^* \begin{pmatrix} \quad \end{pmatrix} \begin{pmatrix} -z^x\phi_x \\ 1-\psi_x \end{pmatrix} = 0 \qquad \Bigg| \qquad \int \begin{pmatrix} H_+ \\ H_+ \end{pmatrix}^* \begin{pmatrix} \quad \end{pmatrix} \begin{pmatrix} -\phi_x \\ 1-\psi_x \end{pmatrix} = 0$$

How to ~~you link these~~ go from these orth relns. to a DE? What do you want to get?

$$\partial_x \begin{pmatrix} p_x \\ q_x \end{pmatrix}$$



$$\begin{pmatrix} p_n \\ q_n \end{pmatrix} = \frac{1}{k_n} \begin{pmatrix} 1 & h_n \\ \bar h_n & 1 \end{pmatrix} \begin{pmatrix} u p_{n-1} \\ q_{n-1} \end{pmatrix}$$

$$\begin{pmatrix} p_x \\ q_x \end{pmatrix} = \begin{pmatrix} 1 & h_x \varepsilon \\ \bar h_x \varepsilon & 1 \end{pmatrix} \begin{pmatrix} u^\varepsilon p_{x-\varepsilon} \\ q_{x-\varepsilon} \end{pmatrix}$$

$$(1-u^\varepsilon)p_x + u^\varepsilon(p_x - p_{x-\varepsilon})$$

$$\frac{p_x - u^\varepsilon p_{x-\varepsilon}}{\varepsilon} = h_x q_{x-\varepsilon}$$

$$\frac{q_x - q_{x-\varepsilon}}{\varepsilon} = \bar h_x u^\varepsilon p_{x-\varepsilon}$$

$$\boxed{\begin{aligned} \partial_x p_x - ikp_x &= h_x q_x \\ \partial_x q_x &= \bar h_x p_x \end{aligned}} \qquad \boxed{\begin{aligned} (\partial_x - ik)p_x &= h_x q_x \\ \partial_x q_x &= \bar h_x p_x \end{aligned}}$$

Suppose that you fix an x, say $x=0$, whence you have the Birkhoff splitting. Then consider a variation in x. I think you want to allow powers of k in this business.

Suppose then you fix $x=0$, i.e. you have b given. Return to cont. grid situation for insight. You work with functions of s, horizontal space spanned by e^{xs}, $x\in\mathbb{R}$, vertical space by $e^{ys^{-1}}\frac{1}{s}$ $y\in\mathbb{R}$. grid space to consist of functions analytic for $s\in\mathbb{C}-\{0\}$. Generators are $v^2=1$ and $v^1=\frac{1}{s}$, translation operator $\lambda^x=$ mult by e^{xs} and $\mu^y=$ mult by $e^{ys^{-1}}$. $s=i\rho$. Expect Hilbert space to be $\int_{-i\infty}^{i\infty} f^* g \frac{ds}{2\pi i}$.

Problem: To take an elt of hor. space, e.g. $\int e^{xs}\varphi(x)dx$ φ compact supp, ~~translate~~ it to $\int e^{xs+ys^{-1}}\varphi(x)dx$ and split into horizontal and vertical components.

Look to first order.

$$\int e^{xs}s^{-1}\varphi(x)\,dx = \frac{\hat\varphi(s)}{s}$$

and you want to split it into $\dfrac{\hat\varphi(s)-\hat\varphi(0)}{s}+\dfrac{\hat\varphi(0)}{s}$

Don't forget to split into $\pm$ $x>0$ and $x<0$

Look for $\psi_+(t)$ such that $\hat\psi_+=\dfrac{\hat\varphi(s)-\hat\varphi(0)}{s}$

$$s\hat\psi_+(s)=\int_0^\infty \underbrace{se^{sx}}_{\partial_x(e^{sx})}\psi_+(x)\,dx = \left[e^{sx}\psi_+(x)\right]_0^\infty-\int_0^\infty e^{sx}\partial_x\psi_+(x)dx$$
$$= -\psi_+(0)-\widehat{(\partial_x\psi_+)}(s)$$

Choose $\psi_+(x)$ $x\geq 0$ such that $\partial_x\psi_+(x)=-\varphi(x)$, $\psi_+(+\infty)=0$

$$\psi_+(x)=\int_x^\infty \varphi(x')\,dx' \qquad \psi_+(0)=\hat\varphi(0)$$

$$s\hat\psi_+=\hat\varphi(s)-\hat\varphi(0).$$

to 2nd order. Keep $x>0$ but you need the Green's function for ∂_x^2

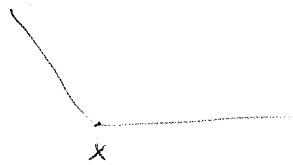
$$\int_0^\infty s^{-2} e^{xs}\varphi(x)\,dx = \underbrace{\frac{\hat\varphi(s)-\hat\varphi(0)-s\hat\varphi'(0)}{s^2}}_{\hat\psi(s)} + \frac{\hat\varphi(0)-s\hat\varphi'(0)}{s^2}$$

$$s^2\hat\psi(s) = \int_0^\infty s^2 e^{xs}\,\psi(x)\,dx = \int_0^\infty \partial_x^2(e^{xs})\,\psi(x)\,dx$$

$$= \int_0^\infty \left[\partial_x\left(\partial_x(e^{xs})\psi - e^{xs}\partial_x\psi\right) + e^{xs}\partial_x^2\psi\right]dx$$

$$= \underbrace{\left[se^{xs}\psi(x) - e^{xs}(\partial_x\psi)(x)\right]_0^\infty}_{=-\psi(0)s+(\partial_x\psi)(0)} + \underbrace{\widehat{\partial_x^2\psi}(s)}_{\hat\varphi(s)}$$

$$\partial_x^2\psi(x) = \varphi(x) \qquad x\geq 0$$
$$\psi(+\infty) = 0.$$



$$\psi(x) = -\int_x^\infty (x-x')\,\varphi(x')\,dx'$$

So what happens is simple

$$\psi_n(x) = (-\partial_x)^{-n}\varphi(x)$$
$$\psi_n(+\infty) = 0$$

$$\psi_0(x) = \varphi(x)$$
$$\psi_1(x) = -\int_\infty^x \varphi(x')\,dx'$$
$$\psi_2(x) = \int_\infty^x (x-x')\,\varphi(x')\,dx'$$
$$\psi_n(x) = (-1)^n\int_\infty^x \frac{(x-x')^{n-1}}{(n-1)!}\,\varphi(x')\,dx'$$

$$s\hat\psi_n(s) = \int_0^\infty \partial_x(e^{sx})\,\psi_n(x)\,dx$$
$$= \left[e^{sx}\psi_n(x)\right]_0^\infty + \int e^{sx}(-\partial_x)\psi_n(x)$$
$$s\hat\psi_n(s) = -\psi_n(0) + \hat\psi_{n-1}(s)$$

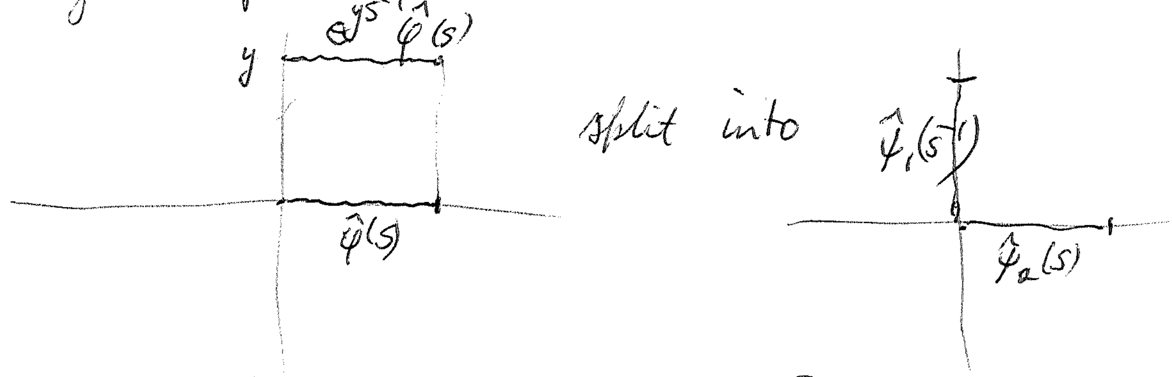
Repeat yesterday's calculations

Problem: Given $\hat{\varphi}(s) = \int_0^\infty e^{xs}\varphi(x)\,dx$, $\varphi(x)=0$ $x>>0$

to split $e^{ys^{-1}}\hat{\varphi}(s)$, the Laurent series, into

~~power series~~ $\int e^{xs}\psi_2(x)\,dx + \int_0^y \frac{e^{y's^{-1}}}{s}\psi_1(y')\,dy'$. Picture

in grid space



enough to do for $\varphi(x) = \delta$ function at some x'.
Thus you want to split $e^{ys^{-1}+xs}$ into
$\pm$ parts. $e^{xs+ys^{-1}} = \sum_{k,l\geq 0} \frac{x^k y^l}{k!\,l!} s^{k-l} =$

$n = k-l$

$l = k-n$

take $n < 0$

$$\sum_{k-l=n}{}' \frac{x^k y^l}{k!\,l!} = \sum_k{}' \frac{x^k y^{k-n}}{k!\,(k-n)!}$$

$$= \sum_{k\geq 0}{}' \frac{x^k y^{k+m}}{k!\,(k+m)!}$$

$$= \underbrace{\left(\sum_{k\geq 0} \frac{(xy)^k}{k!\,(k+m)!}\right)}_{\text{some Bessel function } J_m(xy)} y^m$$

Refresh memory. Bessel fns. arise from the Laplacean in polar coords.

$r\,d\theta \;,\; dr$

$\frac{1}{r}\partial_\theta \;,\; \partial_r$

$$\nabla u = (\partial_r u)\,\hat{e}_r + \left(\frac{1}{r}\partial_\theta u\right)\hat{e}_\theta$$

$$\iint \left(\left(\frac{\partial u}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial u}{\partial \theta}\right)^2 \right) r\,dr\,d\theta$$

$$\iint \frac{\partial u}{\partial r}\frac{\partial u}{\partial r}\, r\,dr\,d\theta = -\int u \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) r\,dr\,d\theta$$

$$\therefore\ \Delta u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}u\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = \frac{\partial^2 u}{\partial t^2}$$

$$\left(r\frac{\partial}{\partial r}\right)^2 u - m^2 u = \underbrace{-\omega^2}_{\lambda} r^2 u$$

$$u = e^{i\omega t} e^{im\theta} u(r).$$

$$\boxed{\Delta(r^m e^{im\theta}) = \frac{1}{r^2}\left(m^2 + (im)^2\right) = 0.}$$

$$u(r) = \sum a_n r^n$$

$$n^2 a_n r^n - m^2 a_n r^n = \lambda a_{n-2} r^n$$

$$(n^2 - m^2)\,a_n = \lambda a_{n-2}$$

$$a_n = \frac{\lambda}{n^2 - m^2} a_{n-2}$$

$$\frac{\lambda}{4\,1\cdot(m+1)} \qquad \frac{\lambda^2}{4^2\, 2!\,(m+1)(m+2)}$$

$$a_m = 1.$$

$$a_{m+2} = \frac{\lambda}{(m+2)^2 - m^2} = \frac{\lambda}{(2m+2)2}$$

$$a_{m+4} = \frac{\lambda}{(m+4)^2 - m^2} a_{m+2} = \boxed{\frac{\lambda}{(2m+4)(4)}\frac{\lambda}{(2m+2)2}}$$

e.g. if $m=0$, then you get.

$$u(r) = \sum a_n r^{2n} \qquad 4n^2 a_n = \lambda a_{n-1}$$

$$a_n = \frac{\lambda/4}{n^2} a_{n-1}. \qquad a_n = \frac{(\lambda/4)^n}{(n!)^2}$$

What else to know? $\quad \frac{1}{2\pi i}\oint e^{xs+ys^{-1}} \frac{ds}{s^m}$

gives the $\int e^{\cos\theta}$ version.

Puzzle - why should the wave equation $\partial_t^2 u = \Delta u$ in the Euclidean plane, which yields integral in Bessel functions be linked to the massive Dirac or Klein-Gordon in 2diml Minkowski space. ?

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back to yesterday's approach.

Idea here $\quad e^{ys^{-1}}\hat{\varphi}(s) = \hat{\varphi}(s) + y\frac{\hat{\varphi}(s)}{s} + \frac{y^2}{2!}\frac{\hat{\varphi}(s)}{s^2} + \cdots$

the holom. part at $s=0$ should be

$$\hat{\varphi}_0(s) + y\hat{\varphi}_1(s) + \frac{y^2}{2!}\hat{\varphi}_2(s) + \cdots$$

$$\hat{\varphi}_0 = \hat{\varphi} \qquad \hat{\varphi}_1 = \frac{\hat{\varphi}(s)-\hat{\varphi}(0)}{s} \qquad \hat{\varphi}_2 = \frac{\hat{\varphi}(s)-\hat{\varphi}(0)-\hat{\varphi}'(0)s}{s^2}$$

$$s\hat{\varphi}_n(s) = \int_0^\infty \partial_x(e^{xs})\varphi_n(x)\,dx = -\varphi_n(0) + \int_0^\infty e^{xs}\underbrace{(-\partial_x)\varphi_n(x)}_{\varphi_{n-1}(x)}dx$$

So $\quad s\hat{\varphi}_n(s) = -\varphi_n(0) + \widehat{\varphi_{n-1}}(s)$

$$\begin{cases} -\partial_x\varphi_n(x) = \varphi_{n-1}(x) \\ \varphi_n(x) = 0 \quad x \gg 0 \end{cases} \qquad \varphi_n(x) = \int_x^\infty \varphi_{n-1}(x')\,dx'$$

Let connect the preceding with Bessel.

$$e^{ys^{-1}} e^{s x_0} = e^{ys^{-1}} \hat{\delta}_{x_0}(s) = e^{ys^{-1}} \int_0^\infty e^{x' s} \underbrace{\delta(x'-x_0)}_{\varphi(x')} \, dx'$$

$$\psi_1(x) = \int_x^\infty \delta(x'-x_0)\, dx' = H(x_0 - x)$$

$$\psi_2(x) = \int_x^{x_0} H(x_0 - x')\, dx' = \qquad \text{too hard.}$$

$$e^{ys^{-1}} \hat{\varphi}(s) = e^{ys^{-1}} \int_0^\infty e^{xs} \varphi(x)\, dx \qquad H(x'-x) \text{ for } n=1$$

$$\psi_n(x) = (-\partial_x)^{-n} \varphi(x) = \int_x^\infty \frac{(x'-x)^{n-1}}{(n-1)!} \varphi(x')\, dx'$$

So the holom. part should be the series

$$\sum \frac{y^n}{n!} \hat{\psi}_n(s)$$

Take $\varphi(x') = \delta(x'-b)$

$$\psi_n(x) = \int_x^\infty \frac{(x'-x)^{n-1}}{(n-1)!} \delta(x'-b)\, dx \qquad b>0$$

$$= \frac{(b-x)^{n-1}}{(n-1)!} H(b-x)$$

$$e^{bs} + \sum_{n \geq 1} \frac{y^n}{n!} \int_0^b e^{xs} \frac{(b-x)^{n-1}}{(n-1)!} H(b-x)\, dx$$

there's something here which interferes

Try something formal

$$e^{ys^{-1}} \hat{\varphi}(s) = \sum \frac{y^n}{n!} \widehat{(-\partial_x)^{-n} \varphi}$$

non comm. residue in dim 1. | on the circle or line, say the circle. You have $f(x)$ combined with ξ. cross product algebra, you want some sort of trace. ~~The value is also~~ The functions have basis of exponentials, So you need a trace on the algebras of $g(\xi)$ which is ~~preserved~~ invariant under translation.

What are the formulas.

$$e^{ys^{-1}}\hat{\varphi}(s) = \hat{\varphi}(s) + y\frac{\hat{\varphi}(s)-\hat{\varphi}(0)}{s} + \frac{y^2}{2!}\frac{\hat{\varphi}(s)-\hat{\varphi}(0)-\hat{\varphi}'(0)s}{s^2} + \cdots$$

$$\hat{\psi}_0 + y\hat{\psi}_1 + \frac{y^2}{2!}\hat{\psi}_2 + \cdots$$

$$\hat{\varphi}(s) = \int_0^\infty e^{sx}\varphi(x)\,dx = \sum \frac{s^n}{n!}\int_0^\infty x^n\varphi(x)\,dx$$

sing. part is

$$\frac{y\,\hat{\varphi}(0)}{s} + \frac{y^2}{2!}\left(\frac{\hat{\varphi}(0)+\hat{\varphi}'(0)s}{s^2}\right) + \frac{y^3}{3!}\left(\frac{\hat{\varphi}(0)+\hat{\varphi}'(0)s+\frac{1}{2!}\hat{\varphi}''(0)}{s^3}\right.$$

$$\hat{\varphi}(0) = e^{bs}$$

$$e^{bs}\left(e^{ys^{-1}}-1\right) + be^{bs}s\left(e^{ys^{-1}}-1-\frac{y}{s}\right)$$

$$+\frac{1}{2!}b^2e^{bs}s^2\left(e^{ys^{-1}}-1-\frac{y}{s}-\frac{1}{}\right.$$

$$y\,\frac{\hat\varphi(0)}{s} + \frac{y^2}{2!}\,\frac{\hat\varphi(0)+\hat\varphi'(0)s}{s^2} + \frac{y^3}{3!}\,\frac{\hat\varphi(0)+\hat\varphi'(0)s+\frac{1}{2!}\hat\varphi''(0)s^2}{s^3}+\cdots$$

$$= \hat\varphi(0)\left(e^{\frac{y}{s}}-1\right) + \hat\varphi'(0)s\left(e^{\frac{y}{s}}-1-\frac{y}{s}\right) + \frac{1}{2!}\hat\varphi''(0)s^2\left(e^{\frac{y}{s}}-1-\frac{y}{s}-\frac{1}{2!}\frac{y^2}{s^2}\right)$$

$$= \boxed{e^{xs}\left(e^{\frac{y}{s}}-1\right) + e^{xs}xs\left(e^{\frac{y}{s}}-1-\frac{y}{s}\right) + \frac{1}{2!}e^{xs}x^2s^2\left(e^{\frac{y}{s}}-1-\frac{y}{s}-\frac{1}{2}\frac{y^2}{s^2}\right)}$$

$$\parallel$$

$$\int_0^y e^{y's^{-1}}\left(\text{function of } x, y'\right)dy'$$

Repeat. This time try to split $e^{ys^{-1}+sx}$ into $\sum_{n\le -1} a_n(x,y)s^n + \sum_{n\ge 0} a_n(x,y)s^n$. This you can do using Bessel functions as well as the

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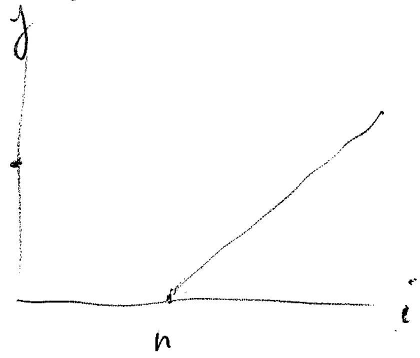
Go back to $e^{xs+ys^{-1}} = \sum_{n\in\mathbb{Z}} a_n(x,y)s^n$

$$= \sum_{i,j\ge 0}\frac{x^i}{i!}\frac{y^j}{j!}s^{i-j} \qquad \therefore\ a_n(x,y) = \sum_{i-j=n}\frac{x^i y^j}{i!\,j!}$$



$n\ge 0$ then $a_n(x,y)$

$$= \sum_{j\ge 0}\frac{x^{n+j}y^j}{(n+j)!\,j!} = x^n J_n(xy)$$

$$a_n(x,y) = \sum_{j=i-n,\ i\ge 0}\frac{x^i y^{i-n}}{i!\,(i-n)!}$$

$$a_n(x,y) = x^n J_n(xy) \qquad n\ge 0$$

$$= y^{-n} J_{-n}(xy) \qquad n\le 0.$$

Go back to

$$e^{xs+ys^{-1}} = \sum_{n\geq 0} x^n J_n(xy)\, s^n + \sum_{n<0} J_{-n}(xy)\, y^{-n} s^{+n}$$

Repeat formulas

you've been assuming $x, y \geq 0$

$$e^{ys^{-1}}\,\hat{\varphi}(s) = e^{ys^{-1}} \int_0^{\infty} e^{xs}\,\varphi(x)\,dx$$

$$= \int \sum_{n\geq 0} x^n J_n(xy)$$

$$= \sum_{n\geq 0} s^n \int x^n J_n(xy)\,\varphi(x)\,dx + \sum_{n\leq -1} y^{-n} s^n \int J_{-n}(xy)\,\varphi(x)\,dx$$

$$= \sum_{n\geq 0} s^n \int x^n J_n(xy)\,\varphi(x)\,dx + \sum_{n\geq 1} \frac{y^n}{s^n} \int J_n(xy)\,\varphi(x)\,dx$$

$$e^{xs+ys^{-1}} = \sum_{n\geq 0} x^n J_n(xy)\, s^n + \sum_{n\geq 1} y^n J_n(xy)\, s^{-n}$$

Start again with $\hat{\varphi}(s) = \int e^{xs}\varphi(x)\,dx$
φ compact support so that $\hat{\varphi}(s)$ is entire.

Consider $e^{ys^{-1}}\hat{\varphi}(s)$, try to understand this formally at least. What is $\frac{1}{s^n}\hat{\varphi}(s)$? You want to split this into the regular + sing. parts

$$\frac{1}{s}\hat{\varphi}(s) = \frac{\hat{\varphi}(s)-\hat{\varphi}(0)}{s} + \frac{\hat{\varphi}(0)}{s}$$

$$s\hat{\varphi}(s) = \int \partial_x(e^{xs})\,\varphi(x)\,dx = \cancel{[e^{xs}\varphi(x)]} + \int e^{xs}(-\partial_x)\varphi(x)\,dx$$

Things are confused.

$$e^{xs+ys^{-1}} = \sum_{i,j\geq 0}\frac{x^i}{i!}\frac{y^j}{j!}s^{i-j} = \sum_{n\in\mathbb{Z}} s^n\left(\sum_{i-j=n}\frac{x^i y^j}{i!\,j!}\right)$$

$i-j=-n$
$i+n=j$

$$= \sum_{n\geq 0} s^n \sum_{j\geq 0}\frac{x^{j+n}y^j}{(j+n)!\,j!} + \sum_{n\geq 1} s^{-n}\sum_{i\geq 0}\frac{x^i y^{i+n}}{i!\,(i+n)!}$$

$$= \sum_{n\geq 0} s^n x^n J_n(xy) + \sum_{n\geq 1} s^{-n}y^n J_n(xy)$$

$$e^{ys^{-1}}\int e^{xs}\varphi(x)\,dx = \sum_{n\geq 0} s^n\int x^n J_n(xy)\,\varphi(x)\,dx + \sum_{n\geq 1} s^{-n}y^n\int J_n(xy)\,\varphi(x)\,dx$$

what is first order term in y.

$$s^{-1}y\int\varphi(x)\,dx$$

$$J_n(xy) = \frac{1}{n!} + \frac{xy}{(1+n)!}$$

$$\sum_{n\geq 0} s^n\int\frac{x^n}{e^{sx}}\left(\frac{1}{n!}+\frac{xy}{(1+n)!}\right)\varphi(x)\,dx \qquad \left(\frac{e^{sx}-1}{s}\right)y$$

$$\frac{\hat\varphi(0)}{s} + \frac{\hat\varphi(s)-\hat\varphi(0)}{s}$$

So apparently the Bessel expansion will take

$\hat\varphi(s) = \int e^{sx}\varphi(x)\,dx$ to $\dfrac{\hat\varphi(0)}{s} + \dfrac{\hat\varphi(s)-\hat\varphi(0)}{s}$

but will not write $\dfrac{\hat\varphi(s)-\hat\varphi(0)}{s}$ in the form $\hat\psi(s)$, with $\psi(x)$ of compact support.

Repeat. $e^{ys^{-1}} f(s) = f + y\dfrac{f}{s} + \dfrac{y^2}{2!}\dfrac{f}{s^2} + \cdots$

regular part is

$$f + y\,\frac{f-f(0)}{s} + \frac{y^2}{2!}\,\frac{f-f(0)-f'(0)s}{s^2} + \cdots$$

singular part is

$$y\,\frac{f(0)}{s} + \frac{y^2}{2!}\,\frac{f(0)+f'(0)s}{s^2} + \cdots$$

$$J_m(z) = \sum_{n\geq 0} \frac{z^n}{n!\,(n+m)!} \qquad ds^2 = dr^2 + r^2 d\theta^2$$

Lap. in polar coords. $dr,\ r\,d\theta$ $\quad \nabla f$ has components $\dfrac{\partial f}{\partial r},\ \dfrac{1}{r}\dfrac{\partial f}{\partial \theta}$

$$\|\nabla f\|^2 = \iint\left(\left(\frac{\partial f}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial f}{\partial \theta}\right)^2\right) r\,dr\,d\theta$$

$$\int \frac{\partial f}{\partial r}\frac{\partial f}{\partial r}\, r\,dr = -\int f\,\partial_r(r\partial_r f)$$

$$\int \frac{\partial f}{\partial r}\, r\frac{\partial f}{\partial r}\,dr = -\int f\,\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) r\,dr$$

$$\Delta u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = -\omega^2 u$$

$$\Delta u = -\omega^2 u \qquad \left(r\frac{\partial}{\partial r}\right)^2 u + \left(\frac{\partial}{\partial \theta}\right)^2 u = -\omega^2 r^2 u$$

$e^{ys^{-1}} \int e^{xs}\varphi(x)\,dx$ | discuss philosophy, you have this operation dividing by s, indefinite integral on the other side.

$$\hat{\varphi}(s) = \int e^{xs}\varphi(x)\,dx \qquad s\hat{\varphi}(s) = \widehat{(-\partial_x\varphi)}(s)$$

something cohomological. So there's an indeterminacy on the x side of a constant, and on the s side there is this process of removing singularities

It is not precise enough to proceed.
What do you mean?

Precise question. Take $\int e^{xs}\varphi(x)dx$ in E_{hor} apply vertical translation $e^{ys^{-1}}$, can you explicitly decribe it's splitting into hor + ver components. Use your inner products. But

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Let's begin with the calculation which should show that if $f(s)$ is in E_{hor}, then $e^{ys^{-1}}f(s)$ ~~splits~~ ~~into~~ lies in $E_{hor} + E_{ver}$

I think you've learned that the Bessel formula for the translation operator $e^{xs+ys^{-1}} = \sum_{n\geq 0}' s^n x^n J_n(xy) + \cdots$ is inadequate. You do get the splitting ~~but it's not~~ into entire functions of s and s^{-1}, but it is not clear that they have the required form. In fact it seems unlikely, because splitting ~~of $\varphi(x)$ into~~ $x>0$ and $x<0$ doesn't occur. To first order in y you split $\frac{1}{s}f(s)$ into $\frac{f(s)-f(0)}{s} + \frac{f(0)}{s}$ so the question is why there is a $\varphi(x)$ of comp. supp.