What next? Go back to

grid eqn. $(\partial_x - a)\psi^1 = b\psi^2$ $\qquad a = \frac{1}{2}|b|^2$

$(\mu - 1)\psi^2 = \bar{b}\psi^1$

~~exp.~~ solns. $\psi = e^{i x \rho}\mu^n \begin{pmatrix} v^1 \\ v^2 \end{pmatrix}$ $\qquad (i\rho - a)v^1 = bv^2$

$(\mu - 1)v^2 = \bar{b}v^1$

$$\mu = 1 + \frac{|b|^2}{i\rho - a} = \frac{i\rho + a}{i\rho - a} \qquad v^1 = \frac{b}{i\rho - a}v^2$$

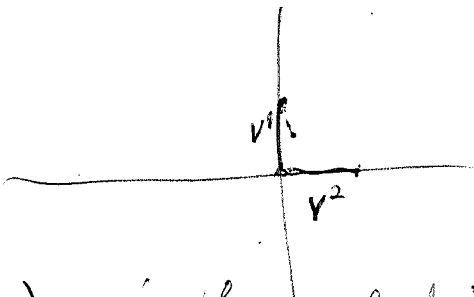
Spectrum – you need

$$\{(\rho, \mu) \mid \rho \neq \infty,\ \mu \neq 0, \infty\} = \{\rho \mid \rho \neq \infty, \pm ia\}.$$

Go back to $\quad -\partial_x \psi^1 = i\psi^2$

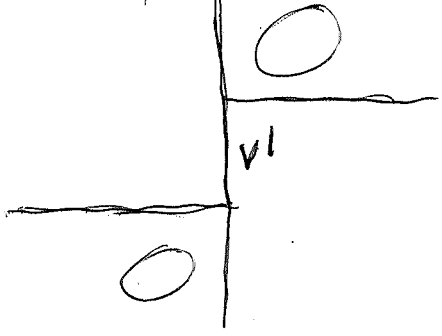
$\partial_s \psi^2 = i\psi^1$.

First consider $\quad \begin{pmatrix} (k\lambda - 1)\psi^1 = h\psi^2 \\ (k\mu - 1)\psi^2 = \bar{h}\psi^1 \end{pmatrix}$ $\quad$ disc. grid eqns.



The point to understand: Green's functions.

$(v^1, -)$ is the solution = zero on the space-cones and = 1 on v^1.



~~understand~~ The question is how to get this solution.

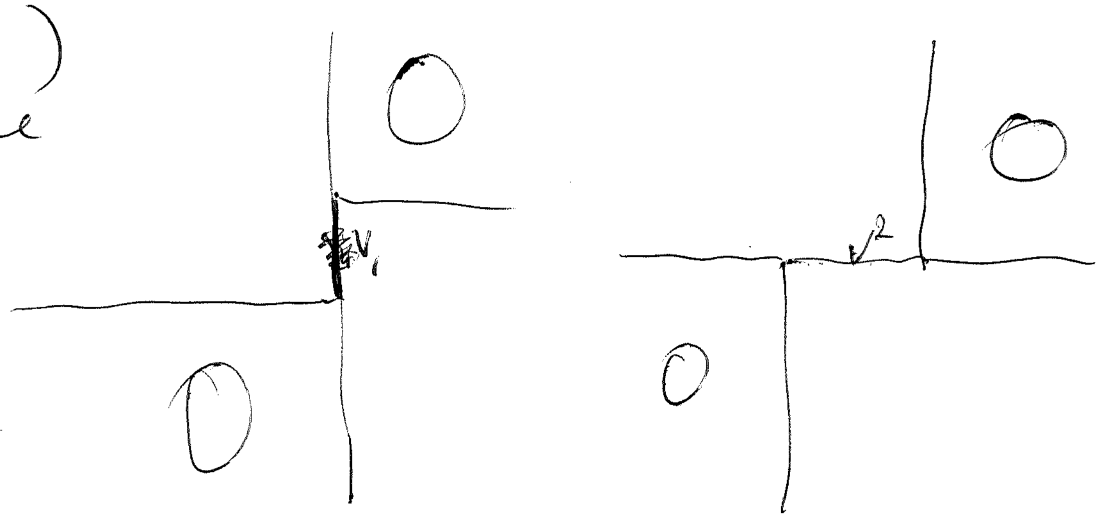
solution = linear fnl on grid space. But grid space = rational fns. of z ~~with poles~~ reg outside of $0, \infty, k, k^{-1}$

$$-\partial_r \psi^1 = i\psi^2 \qquad -\rho v^1 = v^2$$
$$\partial_s \psi^2 = +i\psi^1 \qquad \sigma v^2 = v^1$$

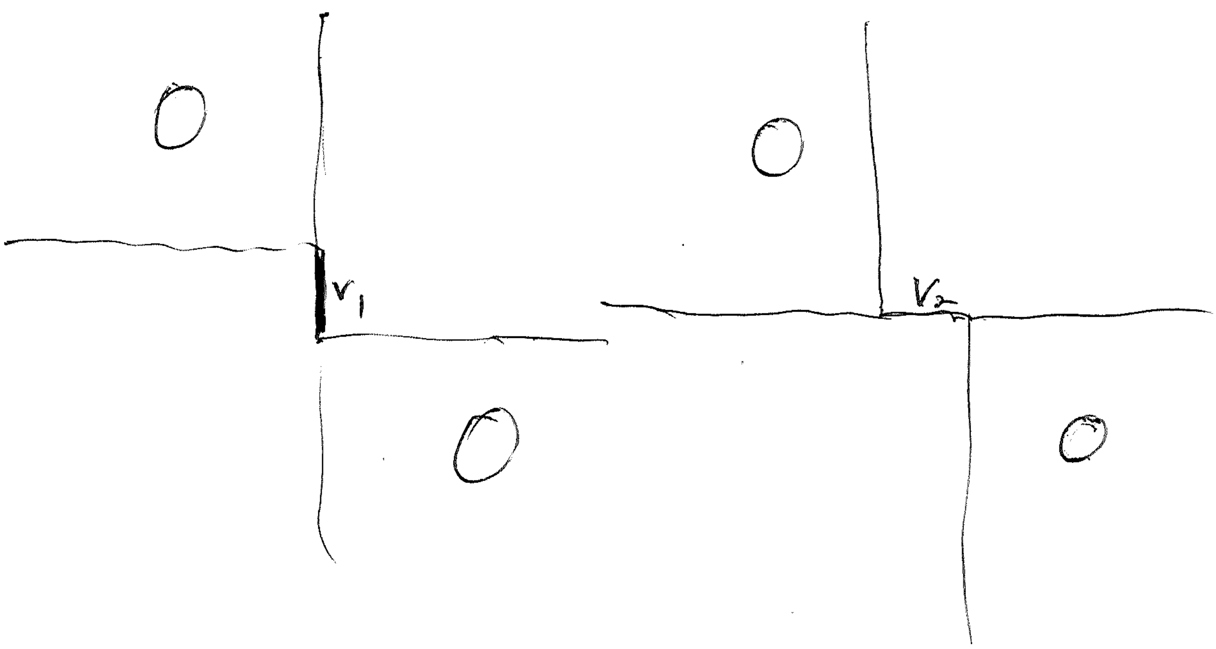
Your aim is to construct solutions of the grid equations which should be given by [illegible] ? A linear fnl f on grid space E determines $\psi^1(r,s) = f(\lambda^r \mu^s v^1)$, $\psi^2(r,s) = f(\lambda^r \mu^s v^2)$. Therefore from the elts $v^1, v^2 \in E$ you get 4 solutions, namely $f = (v^i|-)$ and $f = IH(v^i|-)$ $i=1,2$.

In the discrete case

(1)
case



IH
case



Next do cont. case when v^1, v^2 are δ-fn. type "vectors". So what do you do? You solve the Cauchy problem. The point is that the solutions you seek which are supported in opposite quadrants, have δ functions for Cauchy data along the space axis $t=0$, or the time axis $x=0$. You've studied the Cauchy problem and should know the kernels:

$$\psi(x,t) = e^{t\begin{pmatrix}\partial_x & i\\ i & -\partial_x\end{pmatrix}}\psi(x,0) \qquad \begin{pmatrix}k & 1\\ 1 & -k\end{pmatrix}$$

$$\psi(x,t) = e^{x\begin{pmatrix}\partial_t & -i\\ i & -\partial_t\end{pmatrix}}\psi(0,t) \qquad \begin{pmatrix}\omega & -1\\ 1 & -\omega\end{pmatrix}$$

$$(+\partial_t - \partial_x)\psi^1 = i\psi^2 \qquad \partial_x\psi^1 = \partial_t\psi^1 - i\psi^2$$

$$(\partial_t + \partial_x)\psi^2 = i\psi^1 \qquad \partial_x\psi^2 = i\psi^1 - \partial_t\psi^2$$

Consider first

$$\psi(x,t) = e^{t\begin{pmatrix}\partial_x & i\\ i & -\partial_x\end{pmatrix}}\psi(x,0)$$

$$= e^{t\begin{pmatrix}\partial_x & i\\ i & -\partial_x\end{pmatrix}}\int e^{ikx}\,\hat{\psi}_0(k)\,\frac{dk}{2\pi}$$

$$= \int e^{ikx}\, e^{it\underbrace{\begin{pmatrix}k & 1\\ 1 & -k\end{pmatrix}}_{A_k}}\,\hat{\psi}_0(k)\,\frac{dk}{2\pi} \qquad A_k^2 = \overbrace{\begin{pmatrix}k & 1\\ 1 & -k\end{pmatrix}\begin{pmatrix}k & 1\\ 1 & -k\end{pmatrix}}^{\begin{pmatrix}\omega^2 & 0\\ 0 & \omega^2\end{pmatrix}}$$

$$e^{itA_k} = \cos(\omega t) + i\,\frac{\sin\omega t}{\omega}A$$

$$\|$$

$$\sum_{n\geq 0}\frac{(-1)^n t^{2n}\omega^{2n}}{2n!} + \sum_{n\geq 0}(-1)^n\frac{(itA)^{2n+1}}{(2n+1)!}$$

$$(-1)^n(t\omega)^{2n}\, itA$$

$$\psi(x,t) = \cos(\omega t) I + \frac{i \sin \omega t}{\omega} A \qquad \frac{e^{i\omega t} - e^{-i\omega t}}{2\omega} A$$

$$= \frac{e^{i\omega t}}{2\omega}\begin{pmatrix} \omega + k & 1 \\ 1 & \omega - k \end{pmatrix} + \frac{e^{-i\omega t}}{2\omega}\begin{pmatrix} \omega - k & -1 \\ -1 & \omega + k \end{pmatrix}$$

What do you want? You seek a (global) solution of the grid equations with certain Cauchy data

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$$A^2 = \omega^2 \qquad 1 = \frac{\omega + A}{2\omega} + \frac{\omega - A}{2\omega}$$ ~~proj. op~~

~~Actually what you want is a kers~~

Thoughts this morning. a characteristic Cauchy problem might be handled, or require, Mellin transform – Melrose theory.

You want a solution of the grid equation

~~[illegible]~~ viz.

$$-\partial_r \psi^1 = i\psi^2$$
$$\partial_s \psi^2 = i\psi^1$$

Cindy check

To find solutions which are zero in the space like quadrants (1st + 3rd). Obvious procedure (using non-charact. $t=0$) is

$$\psi(x,t) = e^{t\begin{pmatrix}\partial_x & i \\ i & -\partial_x\end{pmatrix}} \psi_0(x) = \int \frac{dk}{2\pi} e^{ikx} e^{it\begin{pmatrix}k & 1 \\ 1 & -k\end{pmatrix}} \hat{\psi}_0(k) \qquad A$$

Take $\psi_0(x) = \delta(x)\begin{pmatrix}1 & 0 \\ 0 & 1\end{pmatrix}$, get $\psi(x,t) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ikx} e^{it(\;)}$

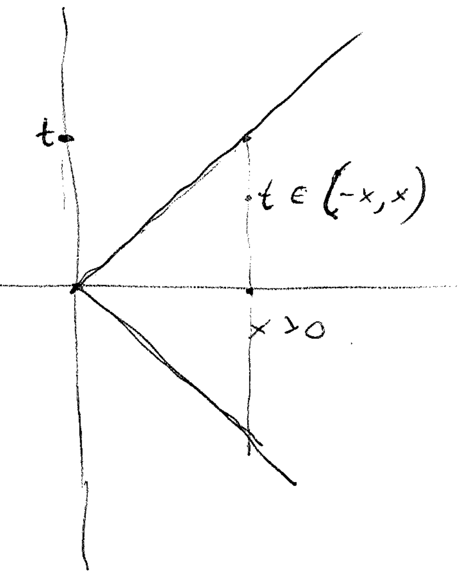
$$\psi(x,t) = \int \frac{dk}{2\pi} e^{ikx}\left(\cos(\omega t)\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} + i\frac{\sin(\omega t)}{\omega}A\right)$$

$$= \int \frac{dk}{2\pi}\frac{e^{ikx}}{2\omega}\left\{ e^{i\omega t}\begin{pmatrix}\omega+k & 1\\ 1 & \omega-k\end{pmatrix} + e^{-i\omega t}\begin{pmatrix}\omega-k & -1\\ -1 & \omega+k\end{pmatrix}\right\}$$

You want to ~~set~~ check that $\psi(x,t)$ is zero for $x > |t|$, $x < -|t|$.

The point is analyticity.
The integrand is an entire function of k, so you can ~~[illegible]~~ move ~~all~~ the contour

$\cancel{\int \frac{dk}{2\pi} e^{ikx} \cos}$



Look at $e^{i(kx+\omega t)}$ $\qquad \omega = \pm\sqrt{k^2+1} \sim \pm k$?

for k

$$\int_{-\infty}^{\infty} e^{ikx}\left(\cos\omega t\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} + i\frac{\sin\omega t}{\omega}A\right)\frac{dk}{2\pi}$$

Suppose $x > 0$, $-x < t < x$

One point is that ω is ~~well~~ nicely defined as an analytic fn of k outside the cut from $k=-1$ to 1.

$$\omega = \sqrt{k^2+1} = k\left(1+k^{-2}\right)^{+1/2} = k\left(1+\tfrac{1}{2}k^{-2}+\cdots\right)$$

So you can use the 2nd formula.

$$A = \begin{pmatrix} k & 1 \\ 1 & -k \end{pmatrix} \qquad A^2 = \omega^2 I. \qquad \omega^2 = k^2 + 1$$

$$e^{itA} = e^{it\omega}\frac{\omega + A}{2\omega} + e^{-it\omega}\frac{\omega - A}{2\omega}$$

$$\int e^{ikx} e^{itA} \frac{dk}{2\pi} = \int \left(e^{i(kx+t\omega)} \frac{\omega + A}{2\omega} + e^{i(kx-t\omega)} \frac{\omega - A}{2\omega} \right) \frac{dk}{2\pi}$$

$$e^{itA} = \cos(\omega t) I + i \frac{\sin \omega t}{\omega} A$$

What's the problem? You want $(v^1|-)$, $(v^2|-)$

$(v^1|-)$ should be the solution (of grid eqn.) reducing to $\begin{pmatrix} \delta(x) \\ 0 \end{pmatrix}$ on $t=0$. What you want to do is to take ?

Try for the g picture. First point: Choose ω branch.

Properties of $\displaystyle\int_{-\infty}^{\infty} e^{ikx} e^{itA} \frac{dk}{2\pi}$ matrix function of (x,t) whose columns should be the solutions $(v^1|-)$, $(v^2|-)$ of the grid eqns.

$$e^{x\partial_x + t\partial_t} \ ?$$

You have spectral representation for solutions of grid equations, ~~nearly~~ no spectral rep for the desired grid space. Try again.

Try to get duality straight.

$$-\partial_r \phi^1 = i\phi^2$$
$$\partial_s \phi^2 = i\phi^1$$

You want ~~to~~ to make sense of "the universal solution, the general solution", grid space should be a representation of the translation group $\mathbb{R}^2$. You should have generators $e^{(r\partial_r + s\partial_s)}\begin{pmatrix} v^1 \\ v^2 \end{pmatrix}$.

Exp. solutions $e^{i(r\rho + s\sigma)}\begin{pmatrix} v^1 \\ v^2 \end{pmatrix}$ $\quad -\rho v^1 = v^2$, $\sigma v^2 = v^1$

All this is too confusing. Let's try to translate between (x,t) and (r,s) coords. Stick to solutions.

Repeat. $(\partial_t - \partial_x)\psi^1 = i\psi^2 \quad (\partial_t + \partial_x)\psi^2 = i\psi^1 \quad \partial_t\psi = \begin{pmatrix} \partial_x & i \\ i & -\partial_x \end{pmatrix}\psi$

$(\partial_x - \partial_t)\psi^1 = -i\psi^2 \quad (\partial_x + \partial_t)\psi^2 = i\psi^1 \quad \partial_x\psi = \begin{pmatrix} \partial_t & -i \\ i & -\partial_t \end{pmatrix}\psi$

$$\psi(x,t) = e^{t\begin{pmatrix} \partial_x & i \\ i & -\partial_x \end{pmatrix}}\psi_0(x) = \int \frac{dk}{2\pi} e^{ikx} e^{itA_k}\hat{\psi}_0(k) \qquad A_k = \begin{pmatrix} k & 1 \\ 1 & -k \end{pmatrix}$$

$$\psi(x,t) = e^{x\begin{pmatrix} \partial_t & -i \\ i & -\partial_t \end{pmatrix}}\psi_0(t) = \int \frac{d\omega}{2\pi} e^{i\omega t} e^{ixB_\omega}\hat{\psi}_0(\omega) \qquad B_\omega = \begin{pmatrix} \omega & -1 \\ 1 & -\omega \end{pmatrix}$$

So what happens? If you take $\psi_0(x) = \delta(x)I$, so that $\hat{\psi}_0(k) = I \quad \forall k$, then you get solution

$$\psi(x,t) = \int \frac{dk}{2\pi} e^{ikx + itA_k}$$

and similarly for $\psi_0(t) = \delta(t)I$ you get soln.

$$\psi(x,t) = \int \frac{d\omega}{2\pi} e^{i\omega t + ixB_\omega}$$

[redacted] You should check the supports of these solutions.

$$e^{i(kx + A_k t)} = e^{ikx}\left(\cos(\omega t) + i\frac{\sin(\omega t)}{\omega}A_k\right)$$ even fn. of ω

$$= e^{ikx}\left(e^{i\omega t}\frac{\omega + A}{2\omega} + e^{-i\omega t}\frac{\omega - A}{2\omega}\right)$$

~~OK [illegible]~~

$$B_\omega^2 = \begin{pmatrix} \omega & -1 \\ 1 & -\omega \end{pmatrix}\begin{pmatrix} \omega & -1 \\ 1 & -\omega \end{pmatrix} = (\omega^2-1)I$$

$$e^{i(\omega t + B_\omega x)} = e^{i\omega t}\left\{ e^{ikx}\frac{k+B_\omega}{2k} + e^{-ikx}\frac{k-B_\omega}{2k} \right\}$$

$$= e^{i\omega t}\left(\cos(kx) + i\,\frac{\sin(kx)}{k} B_\omega \right)$$

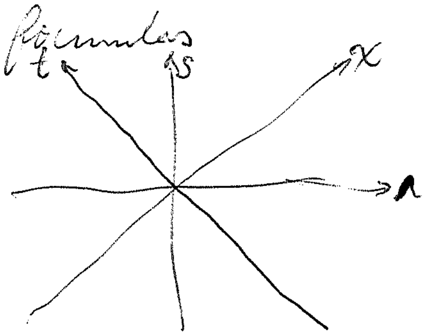
what is interesting here is the fact that you need all $\omega \in \mathbb{R}$ to get $\delta(t)$, but for $|\omega|<1$ $k^2=\omega^2-1<0$. So it seems that ~~we get a clear answer~~ the cycle (contour) for integration involves imaginary k.

~~[illegible]~~

~~[illegible]~~

Maybe what you have to do is to write these solutions as integrals over ρ, these should give different cycles, which then might be used ~~[illegible] to~~ in the formulas for $(\ )$ and $IH(\ ,\ )$. This looks OK. e.g. in the first case you have an integral over $k \in \mathbb{R}$, which probably means $\rho \in \mathbb{R}$, the second is over $\omega \in \mathbb{R}$ which means maybe $\rho \in \mathbb{R}$ and $\rho \in S^1$.

Formulas



$$\partial_r = -\partial_t + \partial_x \qquad \partial_r f = \partial_t f(-1) + \partial_x f(1)$$

$$\partial_s = \partial_t + \partial_x \qquad \partial_s f = \partial_t f(1) + \partial_x f(1)$$

$$t = -r + s \qquad \partial_t \psi = \begin{pmatrix} \partial_x & i \\ i & -\partial_x \end{pmatrix} \psi$$

$$x = r + s$$

$$(\partial_t - \partial_x)\psi^1 = i\psi^2$$
$$(\partial_t + \partial_x)\psi^2 = i\psi^1$$

$$(\partial_x - \partial_t)\psi^1 = -i\psi^2$$
$$(\partial_x + \partial_t)\psi^2 = i\psi^1$$

$$\partial_x \psi = \begin{pmatrix} \partial_t & -i \\ i & -\partial_t \end{pmatrix} \psi$$

$$A_k = \begin{pmatrix} k & 1 \\ 1 & -k \end{pmatrix} \qquad A_k^2 = (k^2+1)I$$

Cauchy problem. $t=0$.

$$\psi(x,t) = \exp t\begin{pmatrix} \partial_x & i \\ i & -\partial_x \end{pmatrix} \int \frac{dk}{2\pi} e^{ikx} \hat{\psi}_0(k) = \int \frac{dk}{2\pi} e^{i(kx + A_k t)} \hat{\psi}_0(k)$$

$$e^{iA_k t} = e^{i\omega t} \frac{\omega + A_k}{2\omega} + e^{-i\omega t} \frac{\omega - A_k}{2\omega} \qquad \omega^2 = k^2 + 1$$

$$= \cos(\omega t) I + i \frac{\sin(\omega t)}{\omega} A_k$$

You want the $\psi(x,t) = \int \frac{dk}{2\pi} e^{i(kx + A_k t)}$ corresp to

$\psi_0(x) = \delta(x) I$

$$\rho = \omega + k \qquad k = \frac{\rho - \rho^{-1}}{2} \qquad kx \mp \omega t = \frac{\rho - \rho^{-1}}{2} x \mp \frac{\rho + \rho^{-1}}{2} t$$

$$\rho^{-1} = \omega - k \qquad \omega = \frac{\rho + \rho^{-1}}{2} \qquad = \rho\left(\frac{x \mp t}{2}\right) - \rho^{-1}\left(\frac{x \pm t}{2}\right)$$

$$= \rho r - \rho^{-1} s$$

$$\psi(r,s) = \int \frac{dk}{2\pi} \left\{ \frac{e^{i(kx - \omega t)}}{2\omega} \begin{pmatrix} \omega - k & -1 \\ -1 & \omega + k \end{pmatrix} + \frac{e^{i(kx + \omega t)}}{-2\omega} \begin{pmatrix} -\omega - k & -1 \\ -1 & -\omega + k \end{pmatrix} \right\}$$

$$= \int \frac{dk}{2\pi} \left\{ \frac{e^{i(\rho r - \rho^{-1} s)}}{\rho + \rho^{-1}} \begin{pmatrix} \rho^{-1} & -1 \\ -1 & \rho \end{pmatrix} + \frac{e^{i(\rho s - \rho^{-1} r)}}{-(\rho + \rho^{-1})} \begin{pmatrix} -\rho & -1 \\ -1 & -\rho^{-1} \end{pmatrix} \right\}$$

Note: If $r = s$ i.e. $t = 0$, this becomes

$$\int \frac{dk}{2\pi} e^{ikx} = \delta(x) I.$$

observe that $\rho \mapsto k = \frac{\rho - \rho^{-1}}{2}$ maps $\mathbb{R}^\times \xrightarrow{\sim} \mathbb{R}$

$\rho, -\rho^{-1}$ yield same k. Also

$$dk = \frac{1+\rho^{-2}}{2}\, d\rho = \frac{\rho+\rho^{-1}}{2} \frac{d\rho}{\rho} \quad \frac{dk}{2\pi(\rho+\rho^{-1})} = \frac{d\rho}{4\pi\rho}$$

$$\int_{-\infty}^{\infty} \frac{dk}{2\pi} \frac{e^{i(\rho r - \rho^{-1}s)}}{\rho+\rho^{-1}} \begin{pmatrix} \rho^{-1} & -1 \\ -1 & \rho \end{pmatrix} = \int_0^\infty \frac{d\rho}{4\pi\rho}\, e^{i(\rho r - \rho^{-1}s)} \begin{pmatrix} \rho^{-1} & -1 \\ 1 & \rho \end{pmatrix}$$

$$\int_{-\infty}^{0-} \frac{d\rho}{4\pi\rho}\, e^{i(\rho r - \rho^{-1}s)} \begin{pmatrix} \rho^{-1} & -1 \\ -1 & \rho \end{pmatrix}$$

$$= \int_0^{+\infty} \frac{d\rho}{4\pi\rho}(-1)\, e^{i(\rho s - \rho^{-1}r)} \begin{pmatrix} -\rho & -1 \\ -1 & -\rho^{-1} \end{pmatrix} = \int_0^\infty \frac{d\rho}{4\pi\rho}\, e^{i(\rho s - \rho^{-1}r)} \begin{pmatrix} \rho & 1 \\ 1 & \rho^{-1} \end{pmatrix}$$

So

$$\boxed{\psi(r,s) = \int_{-\infty}^{\infty} \frac{d\rho}{4\pi\rho}\, e^{i(\rho r - \rho^{-1}s)} \begin{pmatrix} \rho^{-1} & -1 \\ -1 & \rho \end{pmatrix}}$$

Next ~~example~~ case $\qquad B_\omega^2 = (\omega^2 - 1)I$

$$\psi(x,t) = e^{x\begin{pmatrix} \partial_t & -i \\ i & -\partial_t \end{pmatrix}} \psi(0,t)$$

$$= \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\, e^{i(\omega t + B_\omega x)}\, \hat{\psi}_0(\omega) \qquad B_\omega = \begin{pmatrix} \omega & -1 \\ 1 & -\omega \end{pmatrix}$$

Again you want $\hat{\psi}_0(\omega) = I$ corresp. to $\psi(0,t) = \delta(t) I$

$$= \int \frac{d\omega}{2\pi} e^{i\omega t} \left\{ e^{ikx} \frac{k + B_\omega}{2k} + e^{-ikx} \frac{-k + B_\omega}{-2k} \right\}$$

$$= \int \frac{d\omega}{2\pi} \left\{ \frac{e^{i(kx+\omega t)}}{2k} \begin{pmatrix} k+\omega & -1 \\ 1 & k-\omega \end{pmatrix} + \frac{e^{i(-kx+\omega t)}}{-2k} \begin{pmatrix} -k+\omega & -1 \\ 1 & -k-\omega \end{pmatrix} \right\}$$

$$kx+\omega t = \frac{\rho-\rho^{-1}}{2}x + \frac{\rho+\rho^{-1}}{2}t = \rho\left(\frac{x+t}{2}\right) - \rho^{-1}\left(\frac{x-t}{2}\right)$$

$$(\partial_t-\partial_x)\psi^1 = i\psi^2 \qquad \partial_x\psi^1 = \partial_t\psi^1 - i\psi^2$$
$$(\partial_t+\partial_x)\psi^2 = i\psi^1 \qquad \partial_x\psi^2 = -\partial_t\psi^2 + i\psi^1 \qquad \partial_x\psi = \begin{pmatrix}\partial_t & -i\\ i & -\partial_t\end{pmatrix}\psi$$

$$\psi(x,t) = e^{x\begin{pmatrix}\partial_t & -i\\ i & -\partial_t\end{pmatrix}}\psi_0(t) = \int_{-\infty}^{\infty} e^{i\omega t} e^{ixB_\omega}\hat{\psi}_0(\omega)\frac{d\omega}{2\pi}$$

You want case $\hat{\psi}_0(\omega) = I$. $\qquad B_\omega = \begin{pmatrix}\omega & -1\\ 1 & -\omega\end{pmatrix}$, $B_\omega^2 = (\omega^2-1)I$

$$\psi(x,t) = \int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\, e^{i\omega t}\left\{e^{ikx}\left(\frac{k+B_\omega}{2k}\right) + e^{-ikx}\left(\frac{-k+B_\omega}{-2k}\right)\right\}$$

$$= \int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\left\{\frac{e^{i(\omega t+kx)}}{2k}\begin{pmatrix}k+\omega & -1\\ 1 & k-\omega\end{pmatrix} + \frac{e^{i(\omega t-kx)}}{-2k}\begin{pmatrix}-k+\omega & -1\\ 1 & -k-\omega\end{pmatrix}\right\}$$

check: Ass $x=0$, then get $\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}e^{i\omega t} = I.$

$t=-r+s$
$x = r+s$

$$\omega t+kx = \omega(-r+s)+k(r+s) = (-\omega+k)r+(\omega+k)s$$
$$= \rho s - \rho^{-1}r$$
$$\omega t-kx = \omega(-r+s) - k(r+s) = -(\omega+k)r+(\omega-k)s$$
$$= -\rho r+\rho^{-1}s$$

$$\psi(r,s) = \int_{-\infty}^{\infty}\frac{d\omega}{4\pi}\left\{\frac{e^{i(\rho s-\rho^{-1}r)}}{\frac{\rho-\rho^{-1}}{2}}\begin{pmatrix}\rho & -1\\ 1 & -\rho^{-1}\end{pmatrix} + \frac{e^{i(\rho^{-1}s-\rho r)}}{\frac{-\rho+\rho^{-1}}{2}}\begin{pmatrix}\rho^{-1} & -1\\ 1 & -\rho\end{pmatrix}\right\}$$

$$d\omega = \frac{1}{2}(1-\rho^{-2})d\rho = \frac{\rho-\rho^{-1}}{2}\frac{d\rho}{\rho} \qquad \rho \longmapsto \frac{\rho+\rho^{-1}}{2} = \omega$$

for each $1<\omega<\infty$ you have two ρ's >0, mutually inverse so that

$$\int_1^\infty\frac{d\omega}{2\pi}\left\{\frac{e^{i(\rho s-\rho^{-1}r)}}{\rho-\rho^{-1}}(\;)+\cdots\right\} = \int_0^\infty\frac{d\rho}{4\pi\rho}\frac{e^{i(\rho s-\rho^{-1}r)}}{\rho-\rho^{-1}}\begin{pmatrix}\rho & -1\\ 1 & -\rho^{-1}\end{pmatrix}$$

Similarly $\int_{-\infty}^{-1}\frac{d\omega}{2\pi} = \int_{-\infty}^{0}\frac{d\rho}{4\pi\rho} \cdots$

lastly $\int_{-1}^{1} \frac{d\omega}{2\pi} = \int_{|\rho|=1} \frac{d\rho}{4\pi\rho} \cdots$

So you seem to have the answer. The cycle giving IH appears to be $R \cup S^1$. ~~what does~~ You need to check this by a direct method, starting with

$-\partial_r \psi^1 = i\psi^2$ exp. solns $e^{i(r\rho + s\sigma)}$

$\partial_s \psi^2 = i\psi^1$ $-\rho v^1 = v^2$

$\sigma v^2 = v^1$ $e^{i(r\rho - s\rho^{-1})}\begin{pmatrix}1\\-\rho\end{pmatrix}$

~~This seems to be a mistake as if you have~~

typical exp. solution $\quad e^{i(r\rho - s\rho^{-1})}\begin{pmatrix}1\\-\rho\end{pmatrix}$

change $\rho \mapsto -\rho^{-1}$ $\quad e^{i(-r\rho^{-1} + s\rho)}\begin{pmatrix}1\\+\rho^{-1}\end{pmatrix}$

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semi-discrete situation.

$$(\partial_r - a)\psi^1 = b\psi^2 \qquad \psi^1 = \frac{b}{\partial_r - a}\psi^2$$

$$(\mu - 1)\psi^2 = \bar{b}\psi^1$$

$$= \frac{|b|^2}{\partial_r - a}\psi^2$$

$$\mu\psi^2 = \left(1 + \frac{2a}{\partial_r - a}\right)\psi^2 = \frac{\partial_r + a}{\partial_r - a}\psi^2 \qquad \text{YES.}$$

exponential solutions $\quad (i\rho - a)v^1 = bv^2$

$$(\mu - 1)v^2 = \bar{b}v^1$$

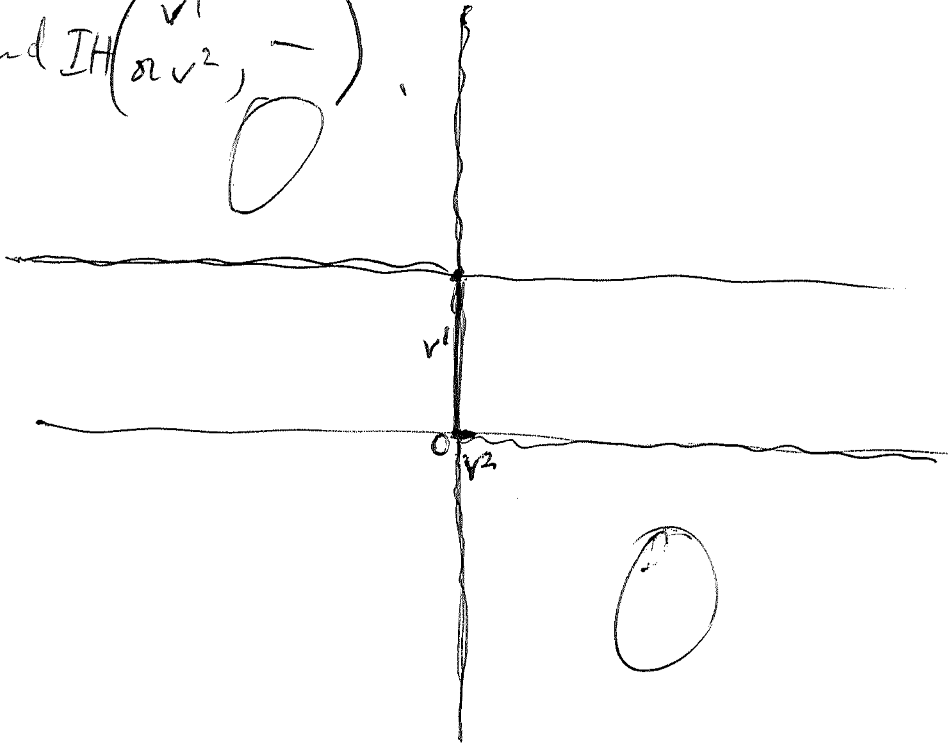
$$\mu = 1 + \frac{\overset{2a}{\cancel{|b|^2}}}{i\rho - a} = \frac{i\rho + a}{i\rho - a} \qquad v^1 = \frac{b}{i\rho - a}v^2$$

So what happens is that you have exp. solutions of the form

$$\psi(r,n) = e^{i\lambda\rho}\left(\frac{i\rho+a}{i\rho-a}\right)^{n}\begin{pmatrix}\frac{b}{i\rho-a}\\ 1\end{pmatrix}$$

Here $\rho \in \mathbb{C} - \{\pm ia\}$

Your problem is to ~~[illegible]~~ construct the solutions $\left(\begin{smallmatrix}v^1\\ \text{or } v^2\end{smallmatrix}\Big| -\right)$ and $\mathbb{H}\left(\begin{smallmatrix}v^1\\ \text{or } v^2\end{smallmatrix}, -\right)$.



$$\psi_0^1(r,n) = ($$

OK go back to

$$-\partial_r \psi^1 = i\psi^2$$
$$\partial_s \psi^2 = i\psi^1$$

can you solve the Cauchy problem on $s=0$.

Now what does this mean? Pass to F.T.

$$-\rho\sigma = 1$$

$$\left.\begin{aligned} -\rho\hat\psi^1 &= \hat\psi^2 \\ \sigma\hat\psi^2 &= \hat\psi^1 \end{aligned}\right) \rightsquigarrow \hat\psi = \begin{pmatrix} 1 \\ -\rho \end{pmatrix} \hat\psi^1$$

What should be true is that the Cauchy data is ψ^1. It looks like you can prescribe ψ^1 then $\psi^2 = -\frac{1}{i}\partial_r \psi^1$. ~~[illegible]~~

~~have~~ $\cancel{\psi(r,s) = \int e^{i(r\rho - s\rho^{-1})} \begin{pmatrix} 1 \\ -\rho \end{pmatrix} f(\rho) \frac{d\rho}{4\pi\rho}}$

~~for general solution. Assuming a soln of this form gives~~ $\cancel{\psi(r,0) = \int e^{ir\rho} \begin{pmatrix} 1 \\ -\rho \end{pmatrix} f(\rho) \frac{d\rho}{4\pi\rho}}$

Let

$$\psi(r,s) = \cancel{\int e^{i(r\rho - s\rho^{-1})} f(\rho) \frac{d\rho}{2\pi}}$$

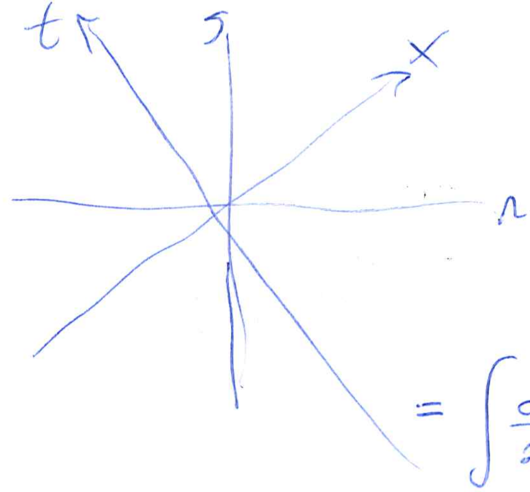
$$= \int \frac{d\rho}{2\pi} e^{i(r\rho - s\rho^{-1})} \begin{pmatrix} -\rho^{-1} \\ 1 \end{pmatrix} f(\rho)$$

This should be a solution of the DE such that

$$\psi^2(r,0) = \int \frac{d\rho}{2\pi} e^{ir\rho} f(\rho).$$

But you learned yesterday that you want to integrate over $\mathbb{R} \cup S^1$.

Go over the formulas.



$$t = -r+s \qquad \partial_r = -\partial_t + \partial_x$$
$$x = r+s \qquad \partial_s = \partial_t + \partial_x$$

$$\psi(x,t) = e^{t\begin{pmatrix}\partial_x & i \\ i & -\partial_x\end{pmatrix}} \underbrace{\psi(x,0)}_{\delta(x) I_2}$$

$$= \int \frac{dk}{2\pi} e^{i(kx + A_k t)} \hat{\psi}_0(k) \qquad A_k = \begin{pmatrix} k & 1 \\ 1 & -k \end{pmatrix}$$

$$= \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ikx} \left\{ e^{i\omega t} \frac{\omega + A_k}{2\omega} + e^{-i\omega t} \frac{+\omega - A_k}{+2\omega} \right\} \cancel{\hat{\psi}_0(k)}$$

$$= \int_{-\infty}^{\infty} \frac{dk}{2\pi} \left[\frac{e^{i(r\rho - s\rho^{-1})}}{(\rho+\rho^{-1})} \begin{pmatrix} \rho^{-1} & -1 \\ -1 & \rho \end{pmatrix} + \frac{e^{i(r\rho^{-1} + s\rho)}}{+(\rho+\rho^{-1})} \begin{pmatrix} +\rho & +1 \\ +1 & +\rho^{-1} \end{pmatrix} \right\}$$

$$k = \frac{\rho - \rho^{-1}}{2} \qquad dk = \frac{1+\rho^{-2}}{2} d\rho = \frac{\rho + \rho^{-1}}{2} \frac{d\rho}{\rho}$$

$$\boxed{\psi(r,s) = \int_{-\infty}^{\infty} \frac{d\rho}{4\pi\rho} e^{i(r\rho - s\rho^{-1})} \begin{pmatrix} \rho^{-1} & -1 \\ -1 & \rho \end{pmatrix}}$$

Two solns of grid eqn whose restriction to $r = s$ is needs clarification

Other direction

$$\psi(x,t) = e^{x\begin{pmatrix}\partial_t & -i \\ i & -\partial_t\end{pmatrix}} \overbrace{\psi(0,t)}^{\delta(t) I}$$

$$\partial_t \psi^1 - i\psi^2 = \partial_x \psi^1$$
$$\partial_t \psi^2 + \partial_x \psi^2 = i\psi^1$$

$$= \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{i(\omega t + B_\omega x)} \qquad B_\omega = \begin{pmatrix} \omega & -1 \\ 1 & -\omega \end{pmatrix} \qquad B_\omega^2 = (\omega^2 - 1) I$$

$$= \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{i\omega t} \left\{ e^{ikx} \frac{k + B_\omega}{2k} + e^{-ikx} \frac{k - B_\omega}{2k} \right\}$$

$$-\omega t + kx = r\rho - s\rho^{-1} \qquad \text{change } \omega \mapsto -\omega \qquad \begin{matrix}\rho \to -\rho^{-1} \\ k \to k\end{matrix}$$

$$\boxed{\omega t + kx = -r\rho^{-1} + s\rho} \qquad \boxed{\omega t - kx = -r\rho + s\rho^{-1}} \qquad \text{related by } \rho \leftrightarrow \rho^{-1}$$

$$\psi(r,s) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \left\{ \frac{e^{i(-r\rho^{-1}+s\rho)}}{\rho-\rho^{-1}} \begin{pmatrix} \rho & -1 \\ 1 & -\rho^{-1} \end{pmatrix} + \frac{e^{i(-r\rho+s\rho^{-1})}}{\rho-\rho^{-1}} \begin{pmatrix} -\rho^{-1} & 1 \\ -1 & +\rho \end{pmatrix} \right\}$$

double covering (ramified) $\rho \longmapsto \omega = \frac{\rho+\rho^{-1}}{2}$ $\quad d\omega = \frac{\rho - \rho^{-1}}{2}\frac{d\rho}{\rho}$

$$\psi(r,s) = \int \frac{d\rho}{4\pi\rho} e^{-i(r\rho - s\rho^{-1})} \begin{pmatrix} -\rho^{-1} & 1 \\ -1 & \rho \end{pmatrix}$$

Simplest thing to do is to change the sign of ρ.

$$\boxed{\psi(r,s) = \int_C \frac{d\rho}{4\pi\rho} e^{i(r\rho - s\rho^{-1})} \begin{pmatrix} \rho^{-1} & 1 \\ -1 & -\rho \end{pmatrix}}$$

$\int_C$ C is the 1-cycle in the ρ plane mapped to $\int_{-\infty}^{\infty}$ in the ω plane.

Properties: $\psi(r,s)$ satisfies $\begin{matrix} -\partial_r \psi^1 = i\psi^2 \\ \partial_s \psi^2 = i\psi^1 \end{matrix}$

Thus $\int_C$ is $\int_{-\infty}^{0} + \oint_{|\rho|=1} + \int_0^{\infty}$

Suppose restrict to $x = r+s = 0$.

$r\rho - s\rho^{-1} = r(\rho+\rho^{-1}) = (r-s)\,\omega = t\omega$

$$\psi(r,-r) = \int_C \frac{d\rho}{4\pi\rho} e^{-it\omega} \begin{pmatrix} \rho^{-1} & 1 \\ -1 & -\rho \end{pmatrix}$$

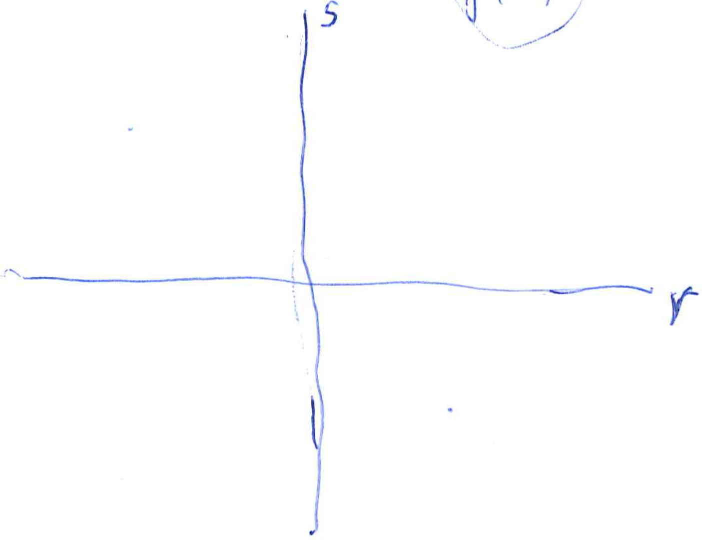
not clear.

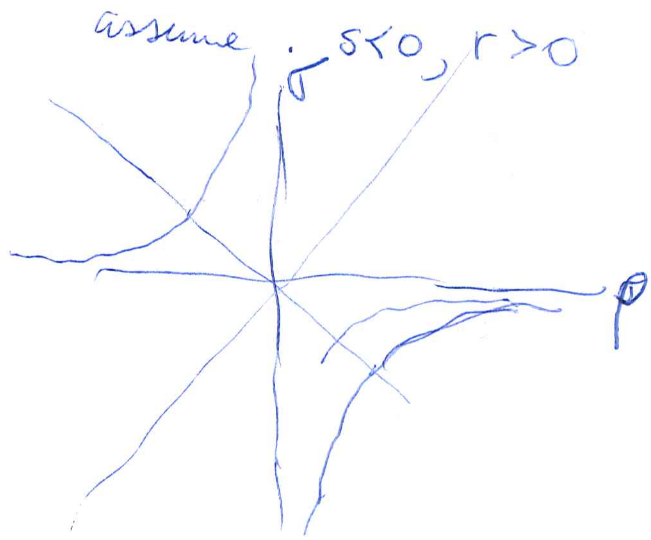
Consider $\psi(r,s) = \int_C \frac{d\rho}{4\pi\rho} e^{i(r\rho - s\rho^{-1})} \begin{pmatrix} \rho^{-1} & 1 \\ -1 & -\rho \end{pmatrix}$

$\psi(r,s)$ should be a solution of $\left(\begin{array}{l} -\partial_r \psi^1 = i\psi^2 \\ \partial_s \psi^2 = i\psi^1 . \end{array} \right.$

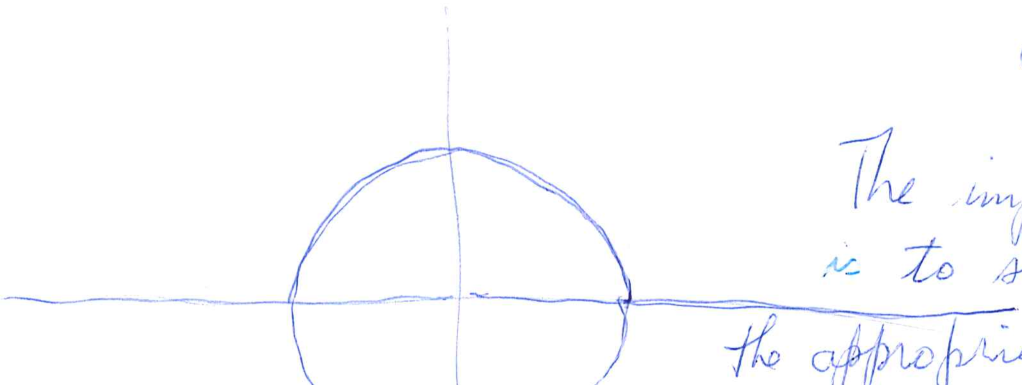
for any current C. Suppose now that

$\int_C = \int_{-\infty}^{0} + \int_{0}^{\infty} + \oint_{|\rho|=1}$ appropriate version





You want the complex ρ-plane.



The important thing is to show vanishing in the appropriate quadrant by deforming the contours. So you first look at $e^{i(r\rho - s\rho^{-1})}$ for $r>0$, $s>0$.

and understand why just $\int_{-\infty}^{\infty}$ can be deformed

$r, s > 0$. ρ is real initially, $\rho = a + ib$

$b > 0$. ~~That~~ Point is that $\rho \in UHP \implies \rho, \bar{\rho}^{-1} \in UHP$

$$-\frac{1}{x+iy} = \frac{-x+iy}{x^2+y^2} \qquad \therefore \qquad r, s > 0$$

It seems fairly clear that you can push

Go back to

$$-\partial_r \psi^1 = i\psi^2 \qquad -\rho V^1 = V^2$$
$$+\partial_s \psi^2 = i\psi^1 \qquad \sigma V^2 = V^1$$

exp. solns.

$$e^{i(r\rho - s\rho^{-1})}\begin{pmatrix} \rho^{-1} & 1 \\ -1 & -\rho \end{pmatrix}$$

~~Consider the exponential factor~~

Solutions.

$$\int_C e^{i(r\rho - s\rho^{-1})}\begin{pmatrix} \rho^{-1} \\ -1 \end{pmatrix}$$

C 1-current in complex ρ-plane.

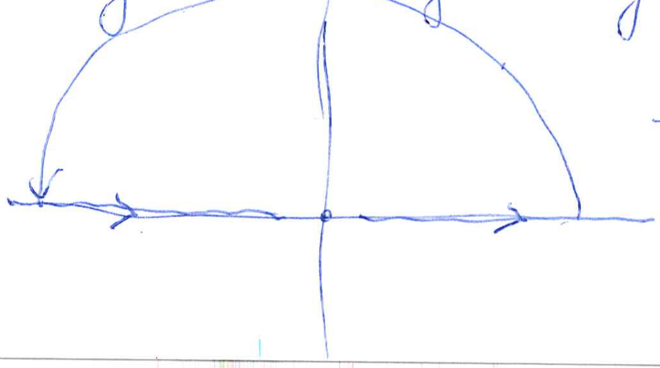
But you want solutions with vanishing properties. First ask for vanishing in 1st 3rd quadrants.

1st $r, s > 0$. ~~$r\rho \pm s\rho^{-1}$~~

Point is ~~that~~ to deform $\rho \in \mathbb{R}$ into UHP

$$\operatorname{Im}(r\rho - s\rho^{-1}) = r(\operatorname{Im}\rho) + s(\operatorname{Im}(-\rho^{-1}))$$

Only the singularity at $\rho = 0$ should contribute



Try next to understand 2nd + 4th quad.

$$e^{i(r\rho - s\rho^{-1})} \qquad r>0,\ s<0.$$

To simplify suppose $r=\frac{1}{2}$ $s=-\frac{1}{2}$

$$e^{i\left(\frac{\rho+\rho^{-1}}{2}\right)} \qquad \omega = \frac{\rho+\rho^{-1}}{2} \qquad d\omega = \frac{1-\rho^{-2}}{2}\,d\rho = \frac{\rho-\rho^{-1}}{2}\,\frac{d\rho}{\rho}$$

What you need to do is to find suitable contours,

$e^{ir\rho}$, $r>0$

decays as $\mathrm{Im}(\rho) \nearrow +\infty$.

$e^{-is\rho^{-1}}$ $s<0$ decays as $\rho \to$ from the LHP. So one contour you can use is

$$e^{-is(-i\varepsilon)^{-1}} = e^{s\varepsilon^{-1}}$$ decays for $s<0$, $\varepsilon \downarrow 0+$

$$\int_0^\infty = \int$$

$$\int_{-\infty}^0 = \int$$

$-\partial_r \psi^1 = i\psi^2$ exp. solns. $e^{i(r\rho - s\rho^{-1})}\begin{pmatrix}\rho\\ -1\end{pmatrix}$ or $\begin{pmatrix}1\\ -\rho\end{pmatrix}$

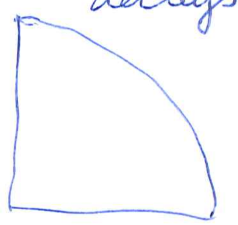
$\partial_s \psi^2 = i\psi^1$

~~Suppose~~ for r real $\neq 0$ $e^{ir\rho}$ oscillatory

at $\rho \to \pm\infty$ for s real $\neq 0$ $e^{-is\rho^{-1}}$ ————

as $\rho \to 0^{\pm}$ Assume this means convergences for these limits suppose $r > 0$, then $e^{i(r\rho)}$ decays as $\rho \to +i\infty$, so

$$\int_a^{+\infty} = \int_a^{+i\infty}$$



suppose $s > 0$ then $e^{-is\rho^{-1}}$ decays as $\rho \to +i0+$

So it looks like

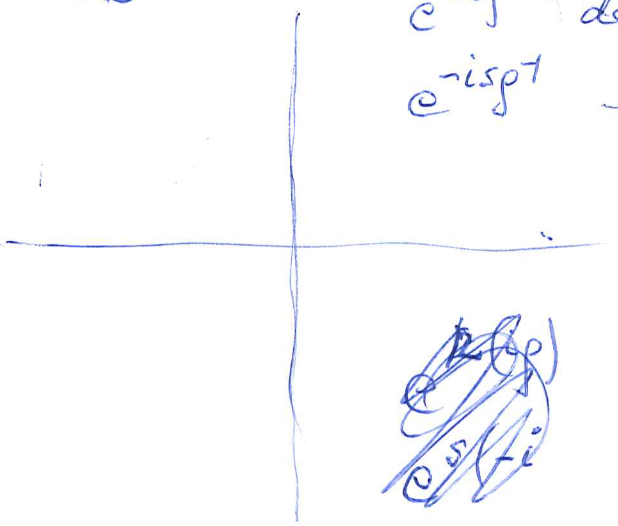
$$\int_{0+}^{+\infty} = \int_{i0+}^{+i\infty}$$

$-is\frac{1}{+i0^+} = -\frac{s}{0^+}$

$-is(i\varepsilon)^{-1} = -\frac{s}{\varepsilon}$

$\rho \to -i\infty$ $e^{ir\rho}$ $ir(-i\infty) = r\underset{+\infty}{\overset{>0}{\infty}}$

$\rho \to -\infty$



$e^{ir\rho}$ decaying in UHP for $r > 0$

$e^{-is\rho^{-1}}$ ———— RHS for $s > 0$?

Right hand sector of 0

$e^{ir\rho}$ decays $\operatorname{Im}\rho \to +\infty$

$e^{is(-\rho^{-1})}$ decays $\operatorname{Im}(-\rho^{-1}) \to +\infty$

So contour $\rho \in \mathbb{R}$ can be shoved upward to get zero.

$e^{ir\rho}$ $\quad r>0 \quad$ decays for $\quad \rho \to +i\infty$

$e^{is(-\rho^{-1})}$ $\quad s<0 \quad$ decays for $\quad \rho \to -i0_+$

So the contour for $r>0,\ s<0$ is should go from $i(0_-)$ to $i(+\infty)$

$$\partial_t \psi = \begin{pmatrix} \partial_x & i \\ i & -\partial_x \end{pmatrix} \psi \qquad \begin{aligned} \partial_x \psi^1 &= \partial_t \psi^1 - i\psi^2 \\ \partial_x \psi^2 &= i\psi^1 - \partial_t \psi^2 \end{aligned} \qquad \partial_x \psi = \begin{pmatrix} \partial_t & -i \\ i & -\partial_t \end{pmatrix} \psi$$

$$\psi(x,t) = e^{x\begin{pmatrix} \partial_t & -i \\ i & -\partial_t \end{pmatrix}} \underbrace{\psi(0,t)}^{\delta(t) I} = \int \frac{d\omega}{2\pi} e^{i\omega t + iB_\omega x} \qquad B_\omega = \begin{pmatrix} \omega & -1 \\ 1 & -\omega \end{pmatrix}$$

$$= \int \frac{d\omega}{2\pi} e^{i\omega t} \left\{ e^{ikx} \frac{k + B_\omega}{2k} + e^{-ikx} \frac{k - B_\omega}{2k} \right\} \qquad \begin{aligned} t &= -r+s \\ x &= r+s \end{aligned}$$

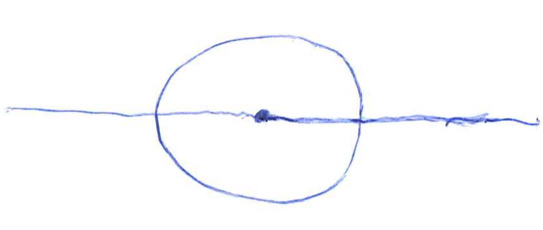
$$\omega t + kx = \frac{\rho+\rho^{-1}}{2} t + \frac{\rho-\rho^{-1}}{2} x = \rho\left(\frac{t+x}{2}\right) + \rho^{-1}\left(\frac{t-x}{2}\right) = \rho s - \rho^{-1} r$$

$$\omega t - kx = \rho\left(\frac{t-x}{2}\right) + \rho^{-1}\left(\frac{t+x}{2}\right) = -\rho r + \rho^{-1} s$$

$$\psi(r,s) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \left\{ \frac{e^{i(s\rho - r\rho^{-1})}}{-\rho+\rho^{-1}} \begin{pmatrix} -\rho & +1 \\ -1 & +\rho^{-1} \end{pmatrix} + \frac{e^{i(-r\rho + s\rho^{-1})}}{\rho - \rho^{-1}} \begin{pmatrix} -\rho^{-1} & +1 \\ -1 & +\rho \end{pmatrix} \right\}$$

$$\omega = \frac{1}{2}(\rho + \rho^{-1}) \qquad d\omega = \frac{\rho - \rho^{-1}}{2} \frac{d\rho}{\rho}$$

$$\psi(r,s) = \int \frac{d\rho}{4\pi\rho} e^{i(-r\rho + s\rho^{-1})} \begin{pmatrix} -\rho^{-1} & 1 \\ -1 & \rho \end{pmatrix}$$



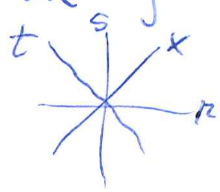
Repeat.

$$\partial_t\psi = \begin{pmatrix}\partial_x & i\\ i & -\partial_x\end{pmatrix}\psi \qquad \begin{aligned}\partial_x\psi^1 &= \partial_t\psi^1 - i\psi^2\\ \partial_x\psi^2 &= i\psi^1 - \partial_t\psi^2\end{aligned} \qquad \partial_x\psi = \begin{pmatrix}\partial_t & -i\\ i & -\partial_t\end{pmatrix}\psi$$

$$\psi(x,t) = e^{x\begin{pmatrix}\partial_t & -i\\ i & -\partial_t\end{pmatrix}}(\delta(t)I) \simeq \int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\, e^{i\omega t + iB_\omega x} \qquad B_\omega = \begin{pmatrix}\omega & -1\\ 1 & -\omega\end{pmatrix}$$

$$= \int \frac{d\omega}{2\pi}\, e^{i\omega t}\left\{e^{ikx}\,\frac{k+B_\omega}{2k} + e^{-ikx}\,\frac{k-B_\omega}{2k}\right\} \qquad B_\omega^2 = (\omega^2-1)I$$

$$\begin{aligned}t &= -r+s\\ x &= r+s\end{aligned} \qquad r = \frac{x-t}{2} \qquad s = \frac{x+t}{2}$$



$$\omega t + kx = \frac{\xi+\xi^{-1}}{2}t + \frac{\xi-\xi^{-1}}{2}x = \xi\left(\frac{t+x}{2}\right) + \xi^{-1}\left(\frac{t-x}{2}\right) = \xi s - \xi^{-1}r$$

$$\omega t - kx = \frac{\xi+\xi^{-1}}{2}t - \frac{\xi-\xi^{-1}}{2}x = \xi\left(\frac{t-x}{2}\right) + \xi^{-1}\left(\frac{t+x}{2}\right) = -\xi r + \xi^{-1}s$$

$$\psi(x,t) = \int\frac{d\omega}{2\pi}\left\{\frac{e^{i(\xi s-\xi^{-1}r)}}{\xi-\xi^{-1}}\begin{pmatrix}\xi & -1\\ 1 & -\xi^{-1}\end{pmatrix} + \frac{e^{i(-\xi r+\xi^{-1}s)}}{\xi-\xi^{-1}}\begin{pmatrix}-\xi^{-1} & 1\\ -1 & \xi\end{pmatrix}\right\}$$

$$\omega = \frac{\xi+\xi^{-1}}{2} \qquad d\omega = \frac{1-\xi^{-2}}{2}\,d\xi = \frac{\xi-\xi^{-1}}{2}\cdot\frac{d\xi}{\xi}$$

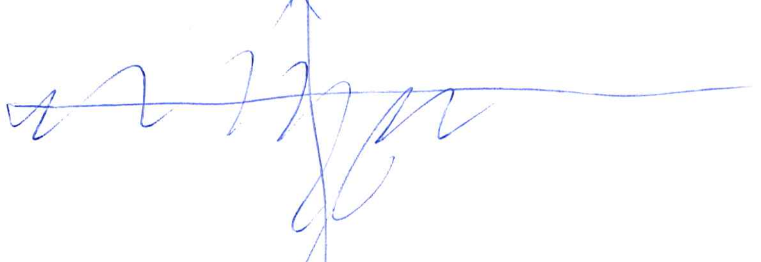
$$\psi(x,t) = \int_C \frac{d\xi}{4\pi\xi}\, e^{i(-\xi r+\xi^{-1}s)}\begin{pmatrix}-\xi^{-1} & 1\\ -1 & \xi\end{pmatrix}$$

to get the form you want put $\rho = -\xi$.

where is the ~~contour~~ path in the complex ξ plane which maps under $\xi \mapsto \frac{\xi+\xi^{-1}}{2}$ to the path $-\infty<\omega<\infty$

Check this carefully. $\xi \mapsto \omega$ is double covering.

Take $\omega > 1$ i.e. $\xi > 0$.



Split ω axis, line into $\omega<-1$, $-1<\omega<1$, $\omega>1$.

$$\cancel{\int_1^\infty \frac{d\omega}{2\pi(\xi-\xi^{-1})}(\cdot) = \int_1 \frac{d\xi}{4\pi\xi}\, e^{i(-\xi r+\xi^{-1}s)}\begin{pmatrix}-\xi^{-1} & 1\\ -1 & \xi\end{pmatrix}}$$

$$\cancel{\int_0^1 \frac{d\omega}{2\pi(\xi-\xi^{-1})}(\quad) = \int \frac{d\xi}{4\pi\xi}}$$

$$F(\xi) = e^{i(-\xi r+\xi^{-1}s)}\begin{pmatrix}-\xi^{-1} & 1\\ -1 & \xi\end{pmatrix}$$

$$F(\xi^{-1}) = e^{i(\xi s-\xi^{-1}r)}\begin{pmatrix}-\xi & 1\\ -1 & \xi^{-1}\end{pmatrix}$$

$$\int_1^\infty \frac{d\omega}{2\pi}\left(\frac{F(\xi)}{\xi-\xi^{-1}}+\frac{F(\xi^{-1})}{\xi^{-1}-\xi}\right) = \int_1^\infty \frac{d\omega}{2\pi}\,\frac{F(\xi)-F(\xi^{-1})}{\xi-\xi^{-1}}$$

$$= \int_1^\infty \frac{d\xi}{4\pi\xi}\left(F(\xi)-F(\xi^{-1})\right) = \int_0^\infty$$

$$\cancel{\int_0^1 \frac{d\xi}{4\pi\xi}F(\xi) = \int_{+\infty}^1 -\frac{d\xi}{4\pi\xi}F(\xi^{-1}) \neq}$$

$$\int_0^1 \frac{d\xi}{4\pi\xi}F(\xi) = \int_\infty^1 \frac{d\eta}{4\pi\eta}(-1)F(\eta^{-1}) = \int_1^\infty \frac{d\eta}{4\pi\eta}F(\eta^{-1})$$

$$\int_{\xi=0}^{\xi=1} \frac{d\xi}{4\pi\xi}F(\xi) = \int_{t=\infty}^{t=0} \underbrace{\frac{d(t^{-1})}{4\pi(t^{-1})}}_{\frac{-t^{-2}}{4\pi t^{-1}}dt \,=\, -\frac{dt}{4\pi t}} F(t^{-1}) = \int_{t=\infty}^{t=1} \left(-\frac{dt}{4\pi t}\right)F(t^{-1})$$

$$\infty$$

$$= \int_1 \frac{dt}{4\pi t}F(t^{-1})$$

$$F(\xi) = e^{i(-\xi r + \xi^{-1} s)} \begin{pmatrix} -\xi^{-1} & 1 \\ -1 & \xi \end{pmatrix}$$

$$\psi(x,t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{F(\xi) - F(\xi^{-1})}{\xi - \xi^{-1}}$$

$$= \int \frac{d\xi}{4\pi\xi}\left(F(\xi) - F(\xi^{-1})\right)$$

$$\omega = \frac{\xi + \xi^{-1}}{2}$$

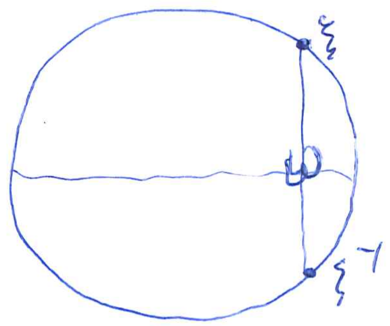
$$d\omega = \frac{1-\xi^{-2}}{2} d\xi$$

$$= \frac{\xi - \xi^{-1}}{2} \frac{d\xi}{\xi}$$

$$\frac{d\omega}{2\pi(\xi - \xi^{-1})} = \frac{d\xi}{4\pi\xi}$$

Check $\quad \xi^2 - 2\omega\xi + 1 = 0$

$$\xi = \omega \pm \sqrt{\omega^2 - 1}$$



$$\int \frac{d\xi}{4\pi\xi}\left(F(\xi) - F(\xi^{-1})\right) \overset{?}{=}$$

$$\int_{-1}^{1} \frac{d\omega}{2\pi}\left(\frac{F(e^{i\theta}) - F(e^{-i\theta})}{e^{i\theta} - e^{-i\theta}}\right) = \int_{+\pi}^{0} \frac{\overbrace{d(\cos\theta)}^{-\sin\theta\, d\theta}}{2\pi} \frac{F(e^{i\theta}) - F(e^{-i\theta})}{2i\sin\theta}$$

$$= \int_{\pi}^{0} \frac{i\, d\theta}{4\pi}\left(F(e^{i\theta}) - F(e^{-i\theta})\right)$$

$$= \frac{1}{4\pi i}\int_{0}^{\pi} d\theta \left(F(e^{i\theta}) - F(e^{-i\theta})\right)$$

$$\int_0^{\pi} F(e^{-i\theta})\, d\theta = \int_0^{-\pi} F(e^{i\theta})(-d\theta) = \int_{-\pi}^{0} F(e^{i\theta})\, d\theta$$

$$\int_{-1}^{1} \frac{d\omega}{2\pi}\left(\frac{F(e^{i\theta})-F(e^{-i\theta})}{e^{i\theta}-e^{-i\theta}}\right) = \frac{1}{4\pi i}\left(\int_0^{\pi} F(e^{i\theta})\,d\theta - \int_{-\pi}^{0} F(e^{i\theta})\,d\theta\right)$$

$F(z) = z$.

$$\frac{1}{\pi} = \frac{1}{4\pi i}\underbrace{\left(\left[\frac{e^{i\theta}}{i}\right]_0^{\pi} - \left[\frac{e^{i\theta}}{i}\right]_{-\pi}^{0}\right)}$$

$$\left(\frac{-1}{i} - \frac{1}{i}\right) - \left(\frac{1}{i} - \frac{-1}{i}\right)$$

$$= \frac{-2}{i} - \frac{2}{i} = 4i$$

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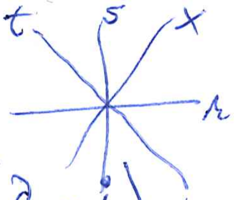
where are we? with

$$\psi(r,s) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\,\frac{F(\xi)-F(\xi^{-1})}{\xi-\xi^{-1}} = \underline{\int \frac{d\xi}{4\pi\xi}\left(F(\xi)-F(\xi^{-1})\right)}$$

where $F(\xi) = e^{i(-\xi r + \xi^{-1}s)}\begin{pmatrix} -\xi^{-1} & 1 \\ -1 & \xi \end{pmatrix}$

$$\int_{\gamma} \frac{d\xi}{4\pi\xi} F(\xi) = \int_{\gamma'} \left(-\frac{d\xi}{4\pi\xi}\right) F(\xi^{-1})$$

Repeat $\quad t = -r+s \quad x = r+s$



$$\partial_r = -\partial_t + \partial_x \qquad \partial_s = \partial_t + \partial_x$$

$$\partial_t \psi = \begin{pmatrix} \partial_x & i \\ i & -\partial_x \end{pmatrix} \psi \qquad \partial_x \psi = \begin{pmatrix} \partial_t & -i \\ i & -\partial_t \end{pmatrix} \psi \qquad B_\omega = \begin{pmatrix} \omega & -1 \\ 1 & -\omega \end{pmatrix}$$

$$\psi(x,t) = e^{x\begin{pmatrix} \partial_t & -i \\ i & -\partial_t \end{pmatrix}} \delta(t) I = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{i\omega t} e^{iB_\omega x}$$

$$= \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{i\omega t} \left\{ e^{ikx} \frac{k+B_\omega}{2k} + e^{-ikx} \frac{k-B_\omega}{2k} \right\} \qquad \frac{t+x}{2} = s \quad \frac{t-x}{2} = -r$$

$$\omega t \pm kx = \frac{\xi+\xi^{-1}}{2} t \pm \frac{\xi-\xi^{-1}}{2} x = \xi\left(\frac{t\pm x}{2}\right) + \xi^{-1}\left(\frac{t\mp x}{2}\right)$$

$$\omega t + kx = \xi s - \xi^{-1} r \qquad \omega = \frac{\xi+\xi^{-1}}{2} \qquad \frac{d\omega}{\xi-\xi^{-1}} = \frac{d\xi}{2\xi}$$

$$\omega t - kx = -\xi r + \xi^{-1} s$$

$$\psi(r,s) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi(\xi-\xi^{-1})} \left\{ \underbrace{e^{i(-\xi^{-1}r+\xi s)} \begin{pmatrix} \xi & -1 \\ 1 & -\xi^{-1} \end{pmatrix}}_{-F(\xi^{-1})} + \underbrace{e^{i(-\xi r+\xi^{-1}s)} \begin{pmatrix} -\xi^{-1} & 1 \\ -1 & +\xi \end{pmatrix}}_{F(\xi)} \right\}$$

$$\therefore \quad \psi(r,s) = \int_C \frac{d\xi}{4\pi\xi} \left(F(\xi) - F(\xi^{-1})\right)$$

where C has to be understood, it should be some sort of 1-chain in the complex ξ plane. It looks as if:

$$\int_C \frac{d\xi}{4\pi\xi} F(\xi) = \int_{C^{-1}} \frac{d\xi}{4\pi\xi} \left(-F(\xi^{-1})\right)$$

better

$$\int_C \frac{d\xi}{4\pi\xi} \left(-F(\xi^{-1})\right) = \int_{C^{-1}} \frac{d\xi}{4\pi\xi} F(\xi)$$

so that $\psi(r,s) = \int_{C+C^{-1}} \frac{d\xi}{4\pi\xi} F(\xi)$.

Work this out, ~~[illegible]~~ find C.

Start with the 1-chain $-\infty < \omega < \infty$ in the complex ω-plane, this splits into three pieces

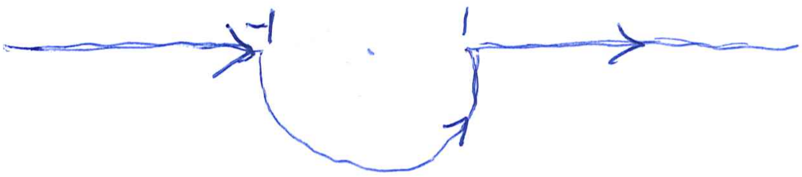
$-\infty < \omega < -1$ here use $\xi = \omega + \sqrt{\omega^2-1}$, $-\infty < \xi < -1$

$-1 < \omega < 1$ here use $\xi = e^{i\phi}$, $-\pi < \phi < 0$

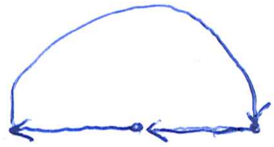
$1 < \omega < \infty$ here let $\xi = \omega + \sqrt{\omega^2-1}$

so over $1 < \omega < \infty$ you use $1 < \xi = \omega + \sqrt{\omega^2-1} < \infty$

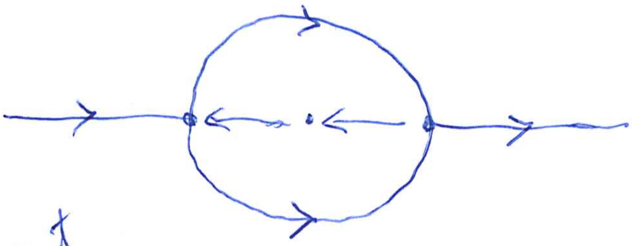
C



C^{-1}



$C + C^{-1}$

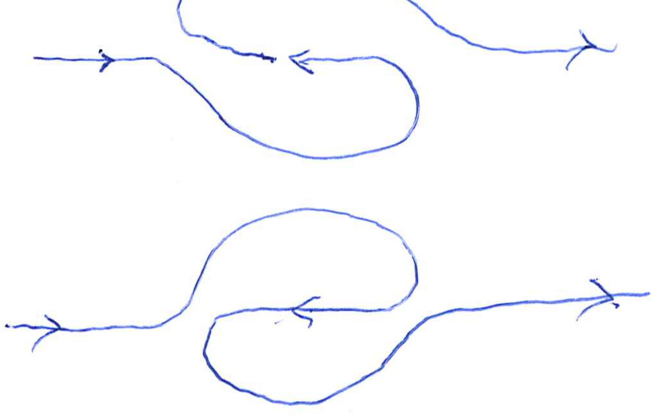


$$F(\xi) = e^{i(-\xi r + \xi^{-1} s)} \begin{pmatrix} -\xi^{-1} & 1 \\ -1 & \xi \end{pmatrix}$$

$e^{-i\xi r}$ $\quad r > 0$

$e^{i\xi^{-1} s}$ $\quad s < 0$

not relevant



$e^{i(-\xi^{-1})(-s)}$

$$\psi(x,t) = e^{t\begin{pmatrix}\partial_x & i\\ i & -\partial_x\end{pmatrix}}\delta(x)I = \int_{-\infty}^{\infty}\frac{dk}{2\pi}e^{ikx}e^{iA_k t} \qquad A_k = \begin{pmatrix}k & 1\\ 1 & -k\end{pmatrix}$$

$\cancel{t = -r+s}$

$x = r+s$

$$= \int_{-\infty}^{\infty}\frac{dk}{2\pi}e^{ikx}\left\{e^{i\omega t}\frac{\omega + A_k}{2\omega} + e^{-i\omega t}\frac{\omega - A_k}{2\omega}\right\}$$

$$kx - \omega t = \frac{\rho-\rho^{-1}}{2}x - \frac{\rho+\rho^{-1}}{2}t = \rho\overbrace{\left(\frac{x-t}{2}\right)}^{r} - \rho^{-1}\overbrace{\left(\frac{x+t}{2}\right)}^{s}$$

$$kx + \omega t = \quad = \rho\underbrace{\left(\frac{x+t}{2}\right)}_{s} + \rho^{-1}\underbrace{\left(\frac{-x+t}{2}\right)}_{-r}$$

$$kx - \omega t = \rho r - \rho^{-1}s \qquad dk = \frac{1+\rho^{-2}}{2}d\rho = \frac{\rho+\rho^{-1}}{2}\frac{d\rho}{\rho} \qquad \frac{dk}{\rho+\rho^{-1}} = \frac{d\rho}{2\rho}$$

$$kx + \omega t = \rho s - \rho^{-1}r$$

$$\psi(r,s) = \int_{-\infty}^{\infty}\frac{dk}{2\pi(\rho+\rho^{-1})}\left\{e^{i(\rho s - \rho^{-1}r)}\begin{pmatrix}\rho & 1\\ 1 & \rho^{-1}\end{pmatrix} + e^{i(\rho r - \rho^{-1}s)}\begin{pmatrix}\rho^{-1} & -1\\ -1 & \rho\end{pmatrix}\right\}$$

$$= \int_C \frac{d\rho}{4\pi\rho}\left(F(\rho) - F(-\rho^{-1})\right) = \int \frac{d\rho}{4\pi\rho}F(\rho)$$

C + image of C under $\rho \mapsto -\rho^{-1}$

$k = \frac{\rho-\rho^{-1}}{2} \qquad -\infty < k < \infty$ same as path $0 < \rho < \infty$.

goes under $\rho \longmapsto -\rho^{-1}$

into $-\infty \xrightarrow{-\rho^{-1}} 0$

---

prepare to write up.

$t = -r+s \qquad \frac{x+t}{2} = s \quad \Big| \quad \frac{x-t}{2} = r$

$x = r+s$

$$\partial_t\psi = \begin{pmatrix}\partial_x & i\\ i & -\partial_x\end{pmatrix}\psi \qquad \partial_r = -\partial_t + \partial_x \qquad \partial_s = \partial_t + \partial_x$$

$$-\partial_r\psi^1 = i\psi^2 \qquad \partial_s\psi^2 = i\psi^1$$

$$e^{i(\rho r + \sigma s)}\begin{pmatrix}v^1\\ v^2\end{pmatrix} \qquad \boxed{\begin{matrix}-\rho v^1 = v^2\\ \sigma v^2 = v^1\end{matrix}} \qquad \text{O.K.}$$

$$e^{i(\rho r - \rho^{-1}s)}\begin{pmatrix}1\\ -\rho\end{pmatrix}$$

$$\psi(x,t) = e^{t\begin{pmatrix}\partial_x & i \\ i & -\partial_x\end{pmatrix}}\delta(x) = \int_{-\infty}^{\infty}\frac{dk}{2\pi}e^{ikx}\, e^{iA_k t} \qquad A_k = \begin{pmatrix} k & 1 \\ 1 & -k\end{pmatrix}$$

$$A_k^2 = \underbrace{(k^2+1)}_{\omega^2} I \qquad \omega = \pm\sqrt{k^2+1}$$

$$= \int_{-\infty}^{\infty}\frac{dk}{2\pi}\, e^{ikx}\left\{ e^{i\omega t}\frac{\omega + A_k}{2\omega} + e^{-i\omega t}\frac{\omega - A_k}{2\omega}\right\}$$

$$kx + \omega t = k(r+s) + \omega(-r+s) = (-\omega+k)r + (\omega+k)s = -\rho^{-1}r + \rho s$$
$$kx - \omega t = k(r+s) - \omega(-r+s) = (\omega+k)r + (-\omega+k)s = \rho r - \rho^{-1}s$$

$$\psi(r,s) = \int_{-\infty}^{\infty}\frac{dk}{2\pi(\rho+\rho^{-1})}\left\{ e^{i(\rho r - \rho^{-1}s)}\begin{pmatrix}\rho^{-1} & -1 \\ -1 & \rho\end{pmatrix} + e^{i(-\rho^{-1}r+\rho s)}\begin{pmatrix}\rho & 1 \\ 1 & \rho^{-1}\end{pmatrix}\right\}$$

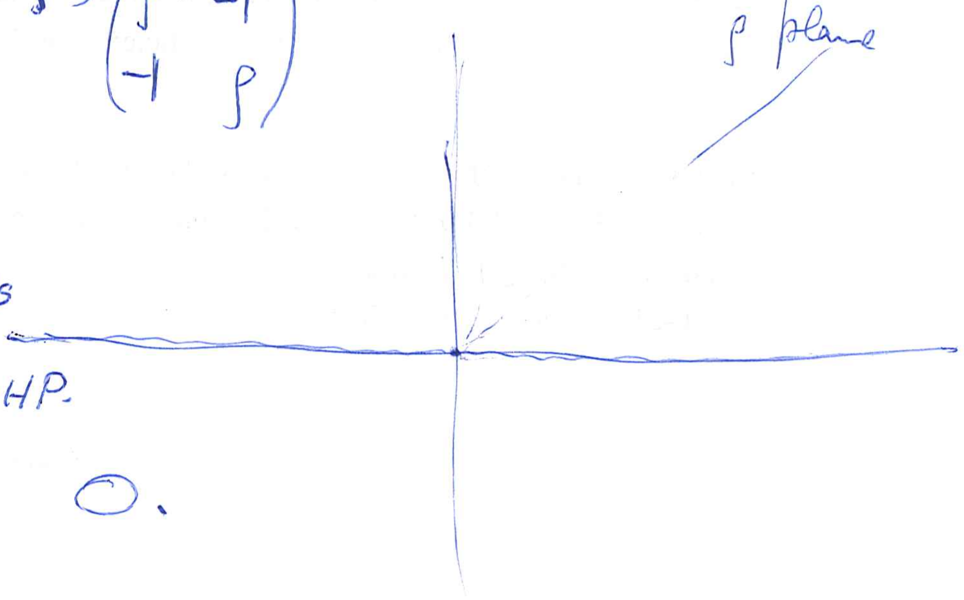
$$= \int_{\omega=-\infty}^{\omega=\infty}\underbrace{\frac{dk}{2\pi(\rho+\rho^{-1})}}_{\frac{d\rho}{4\pi\rho}}\left(F(\rho) - F(-\rho^{-1})\right)$$

$$k = \frac{\rho - \rho^{-1}}{2} \qquad \frac{dk}{\rho+\rho^{-1}} = \frac{d\rho}{2\rho}$$

$$= \int_0^{\infty}\frac{d\rho}{4\pi\rho}\left(F(\rho) - F(-\rho^{-1})\right)$$

C = the path $0<\rho<\infty$

$$= \int_{-\infty}^{\infty}\frac{d\rho}{4\pi\rho}\, e^{i(\rho r - \rho^{-1}s)}\begin{pmatrix}\rho^{-1} & -1 \\ -1 & \rho\end{pmatrix}$$



suppose $r, s > 0$.

then $e^{i\rho r}$, $e^{i(-\rho^{-1})s}$

decaying for $\rho \in UHP$.

It seems you get 0.

Indefinite case.

$$\psi(x,t) = e^{x\begin{pmatrix}\partial_t & -i\\ i & -\partial_t\end{pmatrix}}\delta(t)I = \int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\, e^{i\omega t}\, e^{iB_\omega x}$$

$B_\omega = \begin{pmatrix}\omega & -1\\ 1 & -\omega\end{pmatrix}$

$B_\omega^2 = (\omega^2-1)I$

$k = \pm\sqrt{\omega^2-1}$

$$= \int \frac{d\omega}{2\pi}\, e^{i\omega t}\left\{ e^{ikx}\,\frac{k+B_\omega}{2k} + e^{-ikx}\,\frac{k-B_\omega}{2k}\right\}$$

$$\omega t + kx = \omega(-r+s) + k(r+s) = (-\omega+k)r + (\omega+k)s = -\xi^{-1}r + \xi s$$
$$\omega t - kx = \omega(-r+s) - k(r+s) = (-\omega-k)r + (\omega-k)s = -\xi r + \xi^{-1}s$$

$$\psi(r,s) = \int_{-\infty}^{\infty}\frac{d\omega}{2\pi(\xi-\xi^{-1})}\left\{ e^{i(-\xi r+\xi^{-1}s)}\begin{pmatrix}k-\omega & 1\\ -1 & k+\omega\end{pmatrix} + e^{i(-\xi^{-1}r+\xi s)}\begin{pmatrix}k+\omega & -1\\ 1 & k-\omega\end{pmatrix}\right\}$$

$\omega = \frac{\xi+\xi^{-1}}{2}$

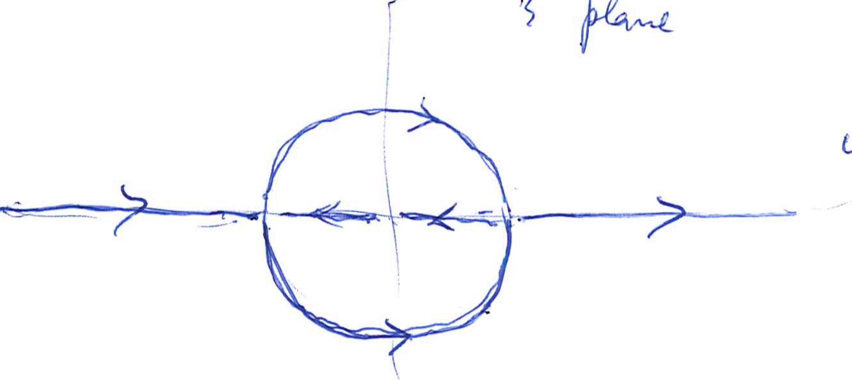
$\frac{d\omega}{\xi-\xi^{-1}} = \frac{d\xi}{2\xi}$

$$= \int_C \frac{d\xi}{4\pi\xi}\left\{ \underbrace{e^{i(-\xi r+\xi^{-1}s)}\begin{pmatrix}-\xi^{-1} & 1\\ -1 & \xi\end{pmatrix}}_{F(\xi)} + \underbrace{e^{i(-\xi^{-1}r+\xi s)}\begin{pmatrix}\xi & -1\\ 1 & -\xi^{-1}\end{pmatrix}}_{-F(\xi^{-1})}\right\}$$

lifts $-\infty \xrightarrow{\omega} \infty$

$$= \int_C \frac{d\xi}{4\pi\xi}\left(F(\xi) - F(\xi^{-1})\right) = \int_{C+(C)^{-1}} \frac{d\xi}{4\pi\xi}\, F(\xi).$$

ξ plane



$\omega = \frac{\xi+\xi^{-1}}{2}$

LHP

$-i(\xi)\, r^{>0}$

$$\frac{F(\xi)}{\xi} = e^{i(-\xi r+\xi^{-1}s)}\,\frac{1}{\xi}\begin{pmatrix}-\xi^{-1} & 1\\ -1 & \xi\end{pmatrix}$$

$s<0$.

UHP

$e^{-i(-\xi^{-1})s^{<0}}$

A direct proof. Consider

$$F_1(\rho) = e^{i(\rho r - \rho^{-1}s)}\begin{pmatrix} \rho^{-1} & 1 \\ -1 & -\rho \end{pmatrix}$$

$$F_1(\rho^{-1}) = e^{i(\rho^{-1}r - \rho s)}\begin{pmatrix} \rho & 1 \\ -1 & -\rho^{-1} \end{pmatrix}$$

$$\omega = \frac{\rho+\rho^{-1}}{2} \qquad d\omega = \frac{\rho-\rho^{-1}}{2}\frac{d\rho}{\rho} \qquad \frac{d\omega}{\rho-\rho^{-1}} = \frac{d\rho}{2\rho}$$

$$\int \frac{d\rho}{4\pi\rho}\left(F_1(\rho) - F_1(\rho^{-1})\right) = \int \frac{d\omega}{2\pi}\,\frac{F_1(\rho)-F_1(\rho^{-1})}{\rho-\rho^{-1}}$$

If $r+s = x = 0$, then $\rho r - \rho^{-1}s = \left(\frac{\rho+\rho^{-1}}{2}\right)(r-s) = -\omega t$

$-r+s = t$ $\qquad \rho^{-1}r - \rho s = \left(\frac{\rho^{-1}+\rho}{2}\right)(r-s) = -\omega t.$

signs messy again

$$\frac{F_1(\rho)-F_1(\rho^{-1})}{\rho-\rho^{-1}} = e^{-i\omega t}(-1)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

too hard.

Go to semi-discrete situation

$$\begin{pmatrix} \lambda^\varepsilon v^1 \\ \mu v^2_{\sqrt{\varepsilon}} \end{pmatrix} = \frac{1}{k_\varepsilon}\begin{pmatrix} 1 & b\sqrt{\varepsilon} \\ \bar{b}\sqrt{\varepsilon} & 1 \end{pmatrix}\begin{pmatrix} v^1 \\ v^2\sqrt{\varepsilon} \end{pmatrix}$$

$$\frac{k_\varepsilon \lambda^\varepsilon v^1 - v^1}{\varepsilon} = b v^2 \qquad k_\varepsilon = \sqrt{1-|b|^2\varepsilon} = 1 - \underbrace{\tfrac{1}{2}|b|^2}_{a}\varepsilon$$

$$k_\varepsilon \mu v^2 - v^2 = \bar{b} v^1$$

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$$(-a+i\rho)\, v^1 = b\, v^2 \qquad \lambda^\varepsilon = e^{i\rho\varepsilon}$$

$$(\mu-1)\, v^2 = \bar b\, v^1 \qquad \mu v^2 = \left(1+\frac{|b|^2}{-a+i\rho}\right) v^2 = \left(\frac{a+i\rho}{-a+i\rho}\right) v^2$$

So what's going on? OKAY.

$$\begin{pmatrix} \psi^1(r+\varepsilon, n) \\ \psi^2(r, n+1)\sqrt{\varepsilon} \end{pmatrix} = \frac{1}{k_\varepsilon}\begin{pmatrix} 1 & b\sqrt{\varepsilon} \\ \bar b\sqrt{\varepsilon} & 1 \end{pmatrix}\begin{pmatrix} \psi^1(r,n) \\ \psi^2(r,n)\sqrt{\varepsilon} \end{pmatrix}$$

$$\begin{pmatrix} \psi^1(r+\varepsilon, n) \\ \psi^2(r, n+1) \end{pmatrix} = \frac{1}{k_\varepsilon}\begin{pmatrix} 1 & b\varepsilon \\ \bar b & 1 \end{pmatrix}\begin{pmatrix} \psi^1(r,n) \\ \psi^2(r,n) \end{pmatrix}$$

$$\boxed{\begin{aligned} (-a+\partial_r)\,\psi^1(r,n) &= b\,\psi^2(r,n) \\ \psi^2(r,n+1)-\psi^2(r,n) &= \bar b\,\psi^1(r,n) \end{aligned}}$$

$$\boxed{\psi^1(r,n) = \frac{b}{\partial_r - a}\,\psi^2(r,n)}$$

$$\psi^2(r,n+1) = \left(1+\frac{|b|^2}{\partial_r - a}\right)\psi^2(r,n) = \left(\frac{\partial_r + a}{\partial_r - a}\right)\psi^2(r,n)$$

$$\boxed{\psi^2(r,n) = \left(\frac{\partial_r + a}{\partial_r - a}\right)^n \psi^2(r,0)}$$

These equations describe continuous linear functionals on the hypothetical grid space.

So what are you going to do?

Spectral repn.

$\lambda^n \mu^n \begin{pmatrix} v^1 \\ v^2 \end{pmatrix}$ "universal solution"

$\lambda^n = e^{i\rho n}$ $\quad \frac{1}{i}\partial_n \lambda^n = \rho \lambda^n$

isom $\quad E \longrightarrow$ analytic functions of $\rho \in \mathbb{C} - \{\pm ia\}$

$\lambda^n \quad e^{in\rho}$.

$\mu \quad \frac{i\rho + a}{i\rho - a}$.

$v^1 \quad \frac{b}{i\rho - a} v_2$

$v^2 \quad 1$.

$L^2\left(\mathbb{R}, \frac{d\rho}{2\pi}\right)$

$L^2(\mathbb{R}$

$$(v^2|v^1) = \int_{-\infty}^{\infty} \frac{b}{i\rho - a}\frac{d\rho}{2\pi} = \int_{-\infty}^{\infty} \frac{b}{\rho + ia}\frac{d\rho}{2\pi i}$$

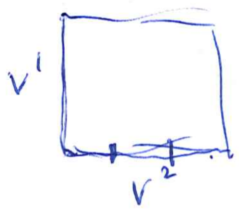
$E \simeq \mathbb{C}[z, z^{-1}, (z-k)^{-1}, (kz-1)^{-1}]$

$(k\lambda - 1)v^1 = h v^2$
$(k\mu - 1)v^2 = \bar{h} v^1$

$\lambda \quad z$.

$\mu \quad \frac{z-k}{kz-1}$.

E module over $\mathbb{C}[\lambda, \lambda^{-1}, \mu, \mu^{-1}]$
modulo $(k\lambda - 1)(k\mu - 1) = 1 - k^2$



$\frac{1}{\varepsilon}$

$\{\lambda^n v^2\}$ orth set

$v^1 = \frac{h}{k\lambda - 1} v^2$

form $\bar{E} = L^2$ completion
$E \hookrightarrow \bar{E}$

v^1

$v^2_\varepsilon \approx v^2\sqrt{\varepsilon}$

$$\begin{pmatrix} \lambda^\varepsilon v^1 \\ \mu\, v^2_\varepsilon \end{pmatrix} = \frac{1}{k_\varepsilon}\begin{pmatrix} 1 & b\sqrt{\varepsilon} \\ \bar{b}\sqrt{\varepsilon} & 1 \end{pmatrix}\begin{pmatrix} v^1 \\ v^2_\varepsilon \end{pmatrix}$$

$$(k_\varepsilon \lambda^\varepsilon - 1)\, v^1 = b\sqrt{\varepsilon}\, v^2_\varepsilon$$
$$(k_\varepsilon \mu - 1)\, v^2_\varepsilon = \bar{b}\sqrt{\varepsilon}\, v^1$$

$$\frac{k_\varepsilon \lambda^\varepsilon_\varepsilon - 1}{\varepsilon}\, v^1 = b\,\frac{v^2_\varepsilon}{\sqrt{\varepsilon}}$$
$$(k_\varepsilon \mu - 1)\,\frac{v^2_\varepsilon}{\sqrt{\varepsilon}} = \bar{b}\, v^1$$

$$(-a + \partial_r)\, v^1 = b v^2 \qquad 2a = |b|^2$$
$$(\mu - 1)\, v^2 = \bar{b}\, v^1$$

$$E \xrightarrow{\sim} \mathbb{C}[z, z^{-1}, (z-k)^{-1}, (z-k^{-1})^{-1}] \subset L^2(S^1, \tfrac{d\theta}{2\pi})$$
$$\lambda, \mu, v^2, v^1 \qquad z, \frac{z-k}{kz-1}, 1, \frac{h}{kz-1}$$

$$E_\varepsilon \qquad\qquad L^2(\mathbb{R}, \tfrac{d\rho}{2\pi})$$

$$\lambda^r \rightarrow e^{i\rho r}$$
$$v^2 \rightarrow 1 \qquad \mu \mapsto 1 + \frac{\overbrace{|b|^2}^{2a}}{-a+i\rho} = \frac{a+i\rho}{-a+i\rho}$$
$$v^1 \rightarrow \frac{b}{-a+i\rho}$$

~~seems to be the~~

$$E^? \longrightarrow L^2(\mathbb{R}, \tfrac{d\rho}{2\pi})$$
$$\lambda^r, \mu^n, v^2, v^1 \longmapsto e^{i\rho r}, \left(\frac{i\rho+a}{i\rho-a}\right)^n, 1, \frac{b}{i\rho-a}$$

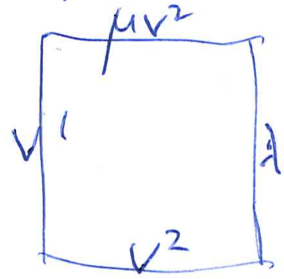
horizontal space spanned by $\lambda^r v^2 = e^{i\rho r}$ for $r \in \mathbb{R}$

vertical space spanned by $\mu^m v^1$ = rational functions vanishing at ∞ poles at $\pm ia$.

Continuous grid equations.

$$\begin{cases} (-a + \partial_r)\, \psi^1(r,n) = b\,\psi^2(r,n) \\ \psi^2(r,n+1) - \psi^2(r,n) = \bar{b}\,\psi^1(r,n) \end{cases}$$

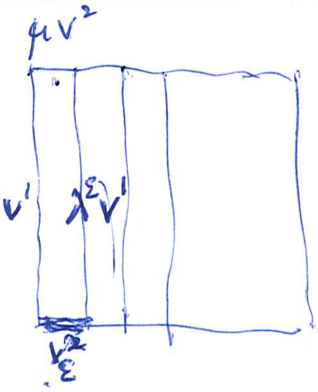
$$(k\lambda-1)v^1 = h v^2$$
$$(k\mu-1)v^2 = \bar{h} v^1$$

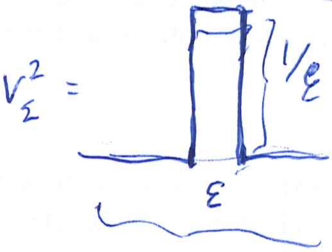


$$E \hookrightarrow \mathbb{C}[z, z^{-1}, (z-k)^{-1}, (z-k^{-1})^{-1}]$$
$$\lambda^m, \mu^n, v^1, v^2 \qquad z^m, \left(\frac{z-k}{kz-1}\right)^n, \frac{h}{kz-1}, 1$$
$$\bar{E} \hookrightarrow L^2\left(S^1, \frac{d\theta}{2\pi}\right)$$

$$(fv^2|gv^2) = \int_{|z|=1} f^* g \frac{dz}{2\pi i z}$$

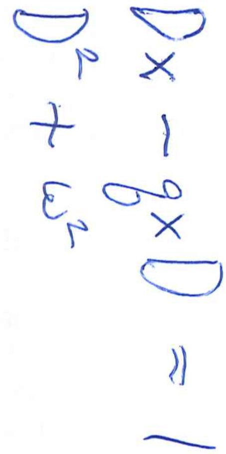
IH =





$\longrightarrow \delta$

$$\| \quad \| = \frac{1}{\sqrt{\varepsilon}}$$

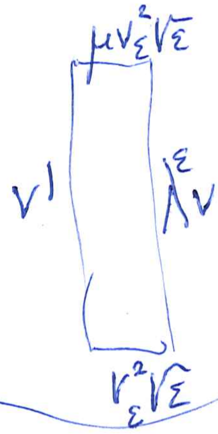


$$v^2_\varepsilon = \frac{1}{\varepsilon}\chi_{[0,\varepsilon]} \longrightarrow \delta(r) \qquad \|v^2_\varepsilon\|^2 = \frac{1}{\varepsilon}$$

if $\qquad \therefore \|v^2_\varepsilon \sqrt{\varepsilon}\| = 1.$

$$\frac{v^2_\varepsilon}{\|v^2_\varepsilon\|} = v^2_\varepsilon\sqrt{\varepsilon}$$

$$h_\varepsilon = b\sqrt{\varepsilon}$$



$$\underbrace{(k_\varepsilon\lambda^\varepsilon - 1)}_{\varepsilon} v^1 = h_\varepsilon v^2_\varepsilon\sqrt{\varepsilon} = b v^2_\varepsilon \cancel{\varepsilon}$$
$$(k_\varepsilon\mu - 1)v^2_\varepsilon\cancel{\sqrt{\varepsilon}} = \bar{h}_\varepsilon v^1 = \bar{b}\cancel{\sqrt{\varepsilon}}\, v^2_\varepsilon$$

$E^?$

$$\bar{E}^? = L^2\left(\mathbb{R}, \frac{dp}{2\pi}\right) \qquad \lambda^n, \mu^n, v^1, v^2$$
$$e^{ipr}, (\quad)^*, \frac{b}{p-a}, 1.$$

Constant coeff. descrete grid equations.

$$\begin{cases} k\psi^1(m+1,n) - \psi^1(m,n) = h\psi^2(m,n) \\ k\psi^2(m,n+1) - \psi^2(m,n) = \bar{h}\psi^1(m,n) \end{cases}$$

$$k\lambda^{m+1}\mu^n v^1 - \lambda^m\mu^n v^1 = h\lambda^m\mu^n v^2$$

$$k\lambda^m\mu^{n+1}v^2 - \lambda^m\mu^n v^2 = \bar{h}\lambda^m\mu^n v^1$$

$$\begin{pmatrix}\psi^1(m,n)\\ \psi^2(m,n)\end{pmatrix} = \psi\left(\lambda^m\mu^n\begin{pmatrix}v^1\\ v^2\end{pmatrix}\right)$$

$$k_\varepsilon\psi^1(r+\varepsilon,n) - \psi^1(r,n) = \overset{b\sqrt{\varepsilon}}{h_\varepsilon}\psi^2(r,n)\sqrt{\varepsilon}$$

$$k_\varepsilon\psi^2(r,n+1)\sqrt{\varepsilon} - \psi^2(r,n)\sqrt{\varepsilon} = \underset{b\sqrt{\varepsilon}}{\bar{h}_\varepsilon}\psi^1(r,n)$$

$$\begin{pmatrix}\psi^1(r,n)\\ \psi^2(r,m)\end{pmatrix} = \psi\begin{pmatrix}\lambda^r\mu^n v^1\\ \lambda^r\mu^n v^2_\varepsilon\sqrt{\varepsilon}\end{pmatrix} \qquad v^2 = \lim v^2_\varepsilon$$

$$\boxed{\begin{aligned}(\partial_r - a)\psi^1(r,n) &= b\psi^2(r,n)\\ \psi^2(r,n+1)-\psi^2(r,n) &= \bar{b}\psi^1(r,n)\end{aligned}} \qquad a = \tfrac{1}{2}|b|^2$$

$$\psi^2(r,n+1) = \psi^2(r,n) + \frac{\bar{b}b}{\partial_r - a}\psi^2(r,n) = \frac{\partial_r + a}{\partial_r - a}\psi^2(r,n)$$

$$\therefore\ \psi^2(r,n) = \left(\frac{\partial_r + a}{\partial_r - a}\right)^n\psi^2(r,0)$$

$$\psi^1(r,n) = \left(\frac{\partial_r + a}{\partial_r - a}\right)^n\frac{b}{\partial_r - a}\psi^2(r,0)$$

~~const. coeff.~~ $(\partial_r - a)\psi^1(r,n) = b\,\psi^2(r,n)$

$\psi^2(r,n+1) - \psi^2(r,n) = \bar{b}\,\psi^1(r,n)$ $\quad 2a = |b|^2$

grid space is the universal solution, being const. coeff. look for exp. solutions.

$$\psi(r,n) = e^{i\rho r}\mu^n\begin{pmatrix} v^1 \\ v^2 \end{pmatrix}$$

where $\quad (i\rho - a)v^1 = bv^2 \qquad \mu = 1 + \frac{|b|^2}{i\rho - a} = \frac{i\rho + a}{i\rho - a}$

$(\mu - 1)v^2 = \bar{b}v^1$

i.e. $\quad \psi(r,n) = e^{i\rho r}\left(\frac{i\rho+a}{i\rho-a}\right)^n \begin{pmatrix} \frac{b}{i\rho - a} \\ 1 \end{pmatrix} \times \text{const.}$

Here $\rho \in \mathbb{C} - \{\pm ia\}$.

Logic: You have grid equations, you want to ~~first study~~ produce a suitable universal solution. First you can study $\mathbb{C}$-valued solutions. What properties should $\psi(r,n)$ have? Need ∂_r to be defined, maybe $\partial_r - a$ to be invertible.

In any case

Repeat. Start with the grid equations

$$(\partial_r - a)\psi^1(r,n) = b\psi^2(r,n) \qquad |b|^2 = 2a$$

$$\Delta\psi^2(r,n) = \bar{b}\psi^1(r,n)$$

~~which this makes sense for~~ this makes sense for, where $\psi = \begin{pmatrix} \psi^1(r,n) \\ \psi^2(r,n) \end{pmatrix}$ is a diff. function from $\mathbb{R}\times\mathbb{Z}$ to a TVS. V. For any V get a vector space of solutions $Z(V)$, covariant functor of V, ask to be representable const coeff $\Rightarrow$ action of translation group $\mathbb{R} \times \mathbb{Z}$ on $Z(V)$, ~~don't go ahead characters~~ ask about decomposing $Z(V)$ into irred reps i.e. characters, irred subrepresentations, of $Z(V)$, should be of form

$$\psi(r,n) = e^{i\rho r} \left(\frac{i\rho + a}{i\rho - a}\right)^n \begin{pmatrix} \frac{b}{i\rho - a} \\ 1 \end{pmatrix} v^2$$

where $v^2 = \psi^2(0,0)$. $\quad \rho \in \mathbb{C} - \{\pm ia\}$

What is the picture? In the discrete case there's no topology needed, you have ~~a module~~ finitely generated over the group ring $\mathbb{C}[\mathbb{Z} \times \mathbb{Z}]$.

You want to understand representing of the grid space as functions. The group ring is, can be identified with a ring of functions, namely functions on the dual. ~~Dual corresp~~.

$\mathbb{C}[\mathbb{Z} \times \mathbb{Z}] = \mathbb{C}[\lambda, \lambda^{-1}] \otimes \mathbb{C}[\mu, \mu^{-1}]$

are alg functions, We are dealing with a subvariety: $\mu = \frac{\lambda - k}{k\lambda - 1}$

You need some version of the group ring for $\mathbb{R}$, it should be a subring of functions on the dual.

Formulate idea: Even though ~~you~~ don't know exactly what the grid space E should be, ~~you~~ can attempt to understand E by studing specific examples of grid solutions. For example if M is a finitely generated A-module, A f.gen. alg over $\mathbb{C}$ then for each max ideal $\mathfrak{m}$ of A

You say this terribly. You can try to understand the A-module M in terms of the ~~family of~~ pairs $(A/\mathfrak{m},\ M/\mathfrak{m}M)$ $\quad \mathfrak{m} \in \mathrm{Max}(A)$

Start with grid eqns.

$$(\partial_r - a)\psi^1 = b\psi^2$$
$$(\mu - 1)\psi^2 = \bar{b}\psi^1 \qquad a = \tfrac{1}{2}|b|^2$$

find all exponential solution.

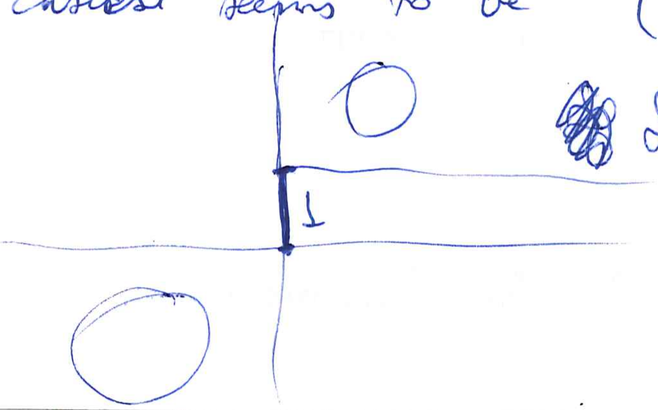
$$\psi = e^{i\rho r}\mu^n \begin{pmatrix} \frac{b}{i\rho - a} \\ 1 \end{pmatrix} c \qquad \rho \in \mathbb{C} - \{\pm ia\}$$

This gives a trivial line bundle L over the alg. variety $X = \mathbb{C} - \{\pm ia\}$, so the grid space you seek should appear as sections. Let v^1 = universal $\psi^1(0,0)$, v^2 = universal $\psi^2(0,0)$, then you have

$$\text{"}E\text{"} \longrightarrow \Gamma(X, L)$$
$$\lambda^r, \mu^n, v^1, v^2 \longmapsto e^{i\rho r}, \left(\frac{i\rho + a}{i\rho - a}\right)^n, \frac{b}{i\rho - a}, 1$$

What are your aims? On the level of the grid eqns, you want the grid solutions $(v\delta|-)$ and $IH(v\delta|-)$

Easiest seems to be $(v^1|-)$. Question:



Is $\left(v^1\middle|\ \lambda^r\mu^n\begin{pmatrix} v^1 \\ v^2 \end{pmatrix}\right)$

Look at $(v^2|-)$. Note that $\lambda^r v^2$ $r \in \mathbb{R}$ is a δ fn. orth basis, so you get

$$L^2\left(\mathbb{R}, \frac{d\rho}{2\pi}\right) \longrightarrow \bar{E} \qquad\Big|\qquad L^2(\mathbb{R}, dr) \longleftrightarrow \bar{E}$$

$$f(\rho) \qquad\qquad f(r) \longmapsto \left(\int f(r)\lambda^r dr\right) v^2$$

---

idea: $\{\lambda^r v^2, r \in \mathbb{R}\}$ δ fn. orth. set, so

$$\int dr\, \phi(r)\, \lambda^r v^2 \longmapsto \int dr\, \phi(r) e^{i\rho r} = \hat{\phi}(-\rho)$$

If this is unitary then

$$\int \frac{d\rho}{2\pi} \left(\int dr\, \phi(r) e^{i\rho r}\right)^* \left(\int dr'\, \phi(r') e^{i\rho r'}\right)$$

$$= \int dr\, dr'\, \overline{\phi(r)}\, \delta(r-r')\, \phi(r') = \int |\phi(r)|^2 dr$$

Next you look at $v^1 = \frac{b}{i\rho - a} = -\int_0^\infty e^{(i\rho - a)r} dr$

$v^1 = -\int_0^\infty dr\, e^{-ar} e^{i\rho r}$. So what do you <u>learn</u>??

$$E \longrightarrow L^2\left(S^1, \frac{d\theta}{2\pi}\right)$$

$$\lambda^m, \mu^n, v^1, v^2 \longmapsto z^m, \left(\frac{z-k}{kz-1}\right)^n, \frac{h}{kz-1}, 1$$

Anyway

Your program? to construct a candidate for the grid space

Focus on finding a candidate for the grid space and the residue formulas.

<u>Question</u>: The Hilbert space picture amounts some kind of equivalence between unitary + self adjoint operators given by C.T. The point is maybe that the horizontal and vertical spaces when completed become the same. How are the horizontal + vertical spaces related for IH? One is >0, the other is <0, but these spaces are not orthogonal. But the grid space <u>should be their direct sum</u>.

Start with vertical space - spanned by

$\frac{1}{(\rho-ia)^n}$, $\frac{1}{(\rho+ia)^n}$ $\quad n\geq 0$. This is the space of rational functions of ρ vanishing at ∞ regular off $\{\pm ia)$. You know that

$$\mu^n v' = \left(\frac{i\rho+a}{i\rho-a}\right)^n \frac{b}{i\rho-a} = \left(\frac{\rho-ia}{\rho+ia}\right)^n \frac{-ib}{\rho+ia}$$

is an orth basis ~~for~~ (1), also for IH w opp. sign.

You should ~~get E~~ generate E using the vertical space V + horizontal translation. Algebraically you take. $e^{i\rho x} V$

Good idea on the white board, for understanding the situation. $e^{i\rho x}\frac{1}{(i\rho-a)^n}$?

First idea is that ~~[illegible]~~ merom. fns. ~~[illegible]~~ on $\mathbb{C}$ with poles at most at $\{\pm ia\}$ reg. outside $\pm ia$ split into entire function + these rational functions. Is this Hadamard's finite part idea?

e.g. $$e^{i\rho r}\frac{1}{\rho - ia} = \frac{e^{-ar}}{\rho - ia} + \frac{e^{i\rho r} - e^{-ar}}{\rho - ia}$$

$$\cancel{\int_0^1 e^{(a+t(i\rho - a))r}\,dt}$$

$$\int_0^1 e^{i(ia+t(\rho - ia))r}\, ir\,dt$$

$$= \int_0^1 e^{-ar + t(i\rho + a)r}\, ir\,dt$$

$$= \left[\frac{e^{(-a+t(i\rho+a))r}}{(i\rho+a)r}\, ir\right]_0^1 = \frac{e^{i\rho r} - e^{-ar}}{\underbrace{-i(i\rho + a)}_{\rho - ia}}$$

You get an entire fn. of ρ!! made up of $e^{i(tr)\rho} = e^{i\rho(tr)}$

$$\cancel{\int_0^1 e^{i\rho(ia+t\ldots)}}$$

$$\psi^1(r,n) = \frac{b}{\partial_r - a}\,\psi^2(r,n)$$

$$\psi^2(r,n+1) = \left(1 + \frac{|b|^2}{\partial_r - a}\right)\psi^2(r,n) = \left(\frac{\partial_r + a}{\partial_r - a}\right)\psi^2(r,n)$$

Look at Cauchy problem on line $n=0$.

$$\psi^2(r,n) = \left(\frac{\partial_r + a}{\partial_r - a}\right)^n \psi^2(r,0) \qquad \Bigg| \qquad \psi^2(r,0) = \int_{-\infty}^{\infty} \frac{d\rho}{2\pi}\, e^{i\rho r}\, \hat{\psi}^2_0(\rho)$$

$$= \int_{-\infty}^{\infty} \frac{d\rho}{2\pi}\, e^{i\rho r} \left(\frac{i\rho + a}{i\rho - a}\right)^n \hat{\psi}^2_0(\rho)$$

$$\psi^1(r,n) = \int_{-\infty}^{\infty} \frac{d\rho}{2\pi}\, e^{i\rho r} \left(\frac{i\rho + a}{i\rho - a}\right)^n \frac{b}{i\rho - a}\, \hat{\psi}^2_0(\rho)$$

Apparently the Cauchy data along $n=0$ is just the function $\psi^2(r,0)$. I do not know the meaning of inverting $\partial_r \pm a$. Maybe this is important.

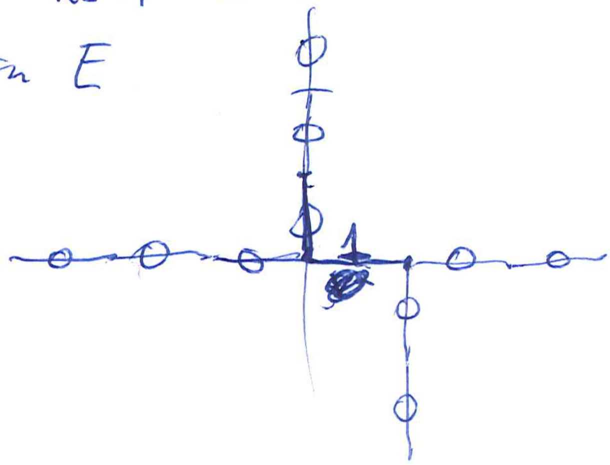
---

ideas. Consider $\quad E = \mathbb{C}[z, z^{-1}, (z-k)^{-1}, (kz-1)^{-1}]$

$\lambda, \mu, v^1, v^2 \qquad z,\ \frac{z-k}{kz-1},\ \frac{h}{kz-1},\ 1$

Look at the linear functional on E

$$f \longmapsto \oint_{|z|=1} f \,\frac{dz}{2\pi i z}$$



Solution with ψ

$n \geq 0 \qquad \dfrac{(z-k)^n h}{(kz-1)^{n+1}}$

$n \leq -1 \qquad \dfrac{(kz-1)^{-n-1} h}{(z-k)^{-n}} \qquad \dfrac{(kz-1)^{a-1} h}{(z-k)^{a}} \qquad a \geq 1.$

Go back to ~~...~~ $(\partial_r - a)\psi^1(r,n) = b\,\psi^2(r,n)$

Consider the discrete case again — look at arbitrary solutions of the grid equations, same as linear functionals on $E = \mathbb{C}[\lambda,\lambda^{-1}]v^2 \oplus \mathbb{C}[\mu,\mu^{-1}]v^1$

Today you want to ~~...~~ find a candidate for grid space, this should be a class of merom. fns. of ρ regular off $\pm ia$. ~~...~~

~~...~~ List ideas. Take vertical space $E^\vee$ having the basis $\mu^n v^1 = \frac{(i\rho+a)^n}{(i\rho-a)^{n+1}}\, b$.

Can you define λ^r = mult by $e^{i\rho r}$

$$e^{i\rho r}\frac{1}{i\rho - a} = e^{sr}\frac{1}{s-1} = \frac{e^{sr}-e^{r}}{s-1} + \frac{e^{r}}{s-1}$$

$$\frac{d}{dr}\left(\frac{e^{sr}-e^{r}}{s-1}\right) = \frac{se^{sr}-e^{r}}{s-1} = e^{s}$$

$$\frac{e^{sr}-e^{r}}{s-1} = e^{r}\left(\frac{e^{(s-1)r}-1}{s-1}\right) = e^{r}\int_0^r e^{(s-1)r'}\,dr'$$

$$= \int_0^r e^{r+(s-1)r'}\,dr' = \int_0^r e^{\underbrace{r-r'+sr'}_{\text{path from } r \text{ to } sr}}\,dr'$$

$$e^{sr}\frac{1}{s-a} = \frac{e^{ar}}{s-a} + \frac{e^{sr}-e^{ar}}{s-a}$$

$$e^{ar}\,\frac{e^{(s-a)r}-1}{s-a} = e^{ar}\int_0^r e^{(s-a)x}\,dx$$

$$e^{sr}\frac{1}{s-a} = \frac{e^{ar}}{s-a} + \underbrace{\frac{e^{sr}-e^{ar}}{s-a}}$$

$$e^{ar}\frac{e^{(s-a)r}-1}{s-a} = e^{ar}\int_0^r e^{(s-a)x}\,dx$$

$$= \int_0^r e^{sx+a(r-x)}\,dx$$

first

$$\frac{1}{s-a}e^{s}$$

Suppose you ~~start with try~~ start with $C_c(\mathbb{R})$ convolution algebra, group alg of $\mathbb{R}$.

$$\varphi\in C_c(\mathbb{R}),\ \varphi \longmapsto \hat\varphi(\rho) = \int dr\,\varphi(r)e^{-i\rho r}$$

$$\widehat{\varphi'}(\rho) = \int dr\ \varphi'(r)e^{-i\rho r}$$

$$= \int dr\ \varphi(r)\overbrace{(-\partial_r)(e^{-i\rho r})}^{i\rho e^{-i\rho r}} = i\rho\,\hat\varphi(\rho).$$

$$\int_0^r e^{br_1}\,e^{a(r-r_1)}\,dr_1 \qquad \text{Convolution of } H(r)e^{br} \text{ and } H(r)e^{ar}$$

? Somehow organize this.

$$e^{sr}\frac{1}{s-a} = e^{i\rho r}\frac{1}{i\rho-a} \rightsquigarrow \overbrace{\int\frac{d\rho}{2\pi}e^{i\rho x}e^{i\rho r}\frac{1}{i\rho-a}}^{\phi(x)}$$

$$\phi(x) = \int\frac{d\rho}{2\pi i}e^{i\rho(x+r)}\frac{1}{\rho+ia}$$

$\cdot -ia$

$$\phi = \int_{-\infty}^{\infty} \frac{d\rho}{2\pi i}\, e^{i\rho(x+r)} \frac{1}{\rho + ia} = \begin{cases} -e^{a(x+r)} & x+r < 0 \\ 0 & x+r > 0 \end{cases}$$

111

Begin again: $s = i\rho$

$\mu = \frac{i\rho + a}{i\rho - a} = \frac{\rho - ia}{\rho + ia}$

$$e^{sr}\frac{1}{s-a} = \frac{e^{ar}}{s-a} + \underbrace{\frac{e^{sr} - e^{ar}}{s-a}}$$

mult by μ removes sing at ia
add ——— $-ia$

$$e^{ar}\left(\frac{e^{(s-a)r} - 1}{s-a}\right) = e^{ar}\int_0^r e^{(s-a)r'}\, dr'$$

$$= \int_0^r e^{sr'}\left(e^{a(r-r')}\right) dr'$$

~~Fourier transf of~~

linear combination of exponentials $e^{i\rho x}$ $0 \leq x \leq r$

You like the UHP for $\rho = is$ $\therefore$ $\underline{Re(s) > 0}$.

~~[illegible]~~ / ~~Op~~erator

Start again to find a candidate for the grid space E consisting of meromorphic functions of $s = i\rho$ regular for $s \neq \pm a$. You have a splitting $E = E_{hor} \oplus E_{vert}$; better to say E contains all rational functions regular for $s \neq \pm a$ and vanishing at ∞, call this space E_{vert}, it has basis $\frac{1}{(s-a)^n}$ and $\frac{1}{(s+a)^n}$ for $n \geq 1$. Then you have the above splitting with E_{hor} the subspace of entire functions in E.

You assume E is closed under multiplication by e^{xs} $\forall x \in \mathbb{R}$ and by $\left(\frac{s+a}{s-a}\right)^n$ $\forall n \in \mathbb{Z}$. ~~It follows~~ Since $e^{xs}\cdot$ preserves entire functions, it follows that E_{hor} is closed under mult by e^{xs} for all s.

There are two ways to proceed. First note that E_{hor} is stable under the group $e^{\mathbb{R}s}$, E_{vert} is stable under $\mu^{\mathbb{Z}}$, where $\mu = \frac{s+a}{s-a}$. We can try to find e^{xs} on ~~$E_{hor} \oplus$~~ E_{vert}, or μ^n on E_{hor}. This might mean solving Cauchy problems with initial data on $x = 0$, resp. $n = 0$.

(Another question is whether E should consist of differentials rather than functions. This seems better in view of the ~~fact that~~ residue formulas

So [redacted] where are you?

E merom. functions of s regular off $\{\pm a\}$.

E closed under $\{e^{xs}, x \in \mathbb{R}\}$ $\mu^{\mathbb{Z}}$ $\mu = \frac{s+a}{s-a}$

and contains $v' = \frac{b}{s-a}$ hence all $\frac{1}{(s-a)^n}, \frac{1}{(s+a)^n}$

$n \geq 1$. Then $E = E_{hor} \oplus E_{vert}$.

Two ways to proceed, define

define $e^{xs} f(s)$ for $f \in E_{vert}$.

define $\mu^n f$ for $f \in E_{hor}$. 2nd looks easier

$$\mu f = \left(\frac{s+a}{s-a}\right) f(s)$$

$$= \left(\frac{s+a}{s-a}\right) f(a) + \underbrace{\frac{(s+a)}{(s-a)}}_{1 + \frac{2a}{s-a}} \big(f(s) - f(a)\big)$$

Simplest point is maybe

$$\frac{1}{s-a} f(s) = \frac{f(a)}{s-a} + \underbrace{\frac{f(s)-f(a)}{s-a}}$$

Intuitively you expect the cont. bases $e^{\mathbb{R}s}$ for E_{horiz} and $\mu^{\mathbb{Z}} v'$ for E_{vert}. Another point is that if we complete then $s - a = ip - a$ becomes invertible, and $\overline{E_{hor}} = \overline{E_{vert}}$

$$e^{xs}\frac{1}{s-a} = \frac{e^{xa}}{s-a} + \frac{e^{xs}-e^{xa}}{s-a}$$

$$e^{xs}\frac{1}{(s-a)^n}$$

In general $\quad f(s) = f(a) + f'(a)(s-a) + \cdots + f^{(n)}(a)\frac{(s-a)^n}{n!} + R_n$

$$\frac{f(s)}{(s-a)^n} = \frac{f(a)}{(s-a)^n} + \frac{f'(a)}{(s-a)^{n-1}} + \cdots + \frac{f^{(n)}(a)}{(s-a)^0} + \frac{R_n}{(s-a)^n}$$

What's the usual formula for R_n.

$$f(a+t(s-a))$$

$$f(s) - f(a) = \int_0^1 \overbrace{\frac{d}{dt} f(a+t(s-a))}^{f'(x_t)(s-a)}\, d(t-1)$$

$$\cancel{\int \frac{d}{dt} f(a+t(s-a))(t-1)}$$

$$= \int_0^1 \left[\frac{d}{dt}\left(\frac{d}{dt}(f(a+t(s-a))\cdot(t-1))\right) - \left(\frac{d}{dt}\right)^2\left(f(a+t(s-a))\right)(t-1)\right]dt$$

$$= \left[(f'(x_t)(s-a))\cdot(t-1)\right]_0^1 - \int f''(x_t)(s-a)^2(1-t)\,dt$$

$$= f'(a)(s-a) - \int_0^1 f''(x_t)(s-a)^2(1-t)\,dt$$

$$\cancel{f(s) = f(a) + \int_0^1 f'(x_t)(s-a)\,dt}$$

$$f(1) = f(0) + \int_0^1 f'(t)\,dt$$

$$\int_0^1 \left(\partial_t\left[f'(t)(t-1)\right] - f''(t)(t-1)\right) dt$$

$$= f(0) + f'(0)(1) - \underbrace{\int_0^1 f''(t)\,(t-1)\,dt}$$

$$\int_0^1 \left(\partial_t\left[f''(t)\frac{(t-1)^2}{2}\right] - f^{(3)}(t)\frac{(t-1)^2}{2}\right) dt$$

$$= f(0) + f'(0)1 + \frac{f''(0)}{2}(1)^2 - \int_0^1 f^{(3)}(t)\frac{(t-1)^2}{2!}\,dt$$

Apply to $f(x_t)$ $\quad x_t = a + t(s-a)$

$$f(s) = f(a) + f'(a)(s-a) + \frac{f''(a)}{2!}(s-a)^2 - \int_0^1 f^{(3)}(x_t)(s-a)^3\frac{(t-1)^2}{2!}\,dt$$

$$e^{xs} = e^{xa} + xe^{xa}(s-a) + x^2e^{xa}\frac{(s-a)^2}{2!} - \int_0^1 e^{x(a+t(s-a))}x^3\frac{(s-a)^3}{2!}(t-1)^3\,dt$$

$$e^{xa}\left(e^{x(s-a)}\right) = e^{xs}$$

Basically you have this $f(s) =$ F.T. of $\varphi(x)$

say e^{xs} ~~which is the F.T. of~~

and then $\frac{1}{(s-a)^n} f(s) =$ F.T. of $\frac{1}{(\partial_x - a)^n}\varphi(x)$

Repeat – today you understand this stuff.

First do Taylor with remainder.

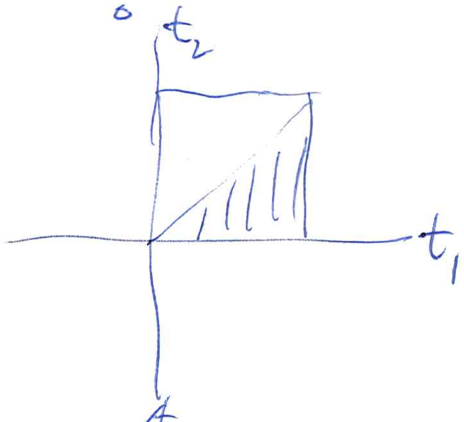
$D^n\left(f(t)\frac{(t-1)^n}{n!}\right)$?

$$D\big(f(t)(t-1)\big) = f'(t)(t-1) + f(t)$$

$$e^D - 1 - D - \dots - \frac{D^n}{n!} \qquad \int e^{tD} D\,dt$$

$$f(x) = f(0) + \int_0^x f'(t)\,dt = f(0) + \int_0^1 f'(tx)\,x\,dt$$

$$e^D = 1 + \int_0^1 e^{tD} D\,dt$$



$$e^{tD} = 1 + \int_0^t dt_1 D e^{t_1 D}$$

$$= 1 + \int_0^t dt_1 D + \int_0^t dt_1 \int_0^{t_1} dt_2\, D^2 e^{t_2 D}$$

$$= 1 + tD + \int_0^1 dt_2\, D^2 e^{t_2 D} \int_{t_2}^1 dt_1 \; (1-t_2)$$

$$= 1 + tD + \int_0^1 dt_2 (1-t_2) D^2 e^{t_2 D}$$

you need a summary of Taylor's formula.

basic function $\frac{(1-t)^n}{n!}\chi_{[0,1]}$

$$D\left(\frac{(1-t)^n}{n!}\chi_{[0,1]}\right) = -\frac{(1-t)^{n-1}}{(n-1)!}\chi[0,1] + \frac{(1-t)^n}{n!}\left(\delta(t) - \overset{\substack{0 \text{ for } n\geq 1 \\ \|}}{\delta(t-1)}\right)$$

$$D \quad f(t)\frac{(1-t)^n}{n!}\chi_{[0,1]}$$

$$\varphi_n(t) = \frac{(1-t)^n}{n!}\chi_{[0,1]}$$

$$D\varphi_n = -\varphi_{n-1} + \frac{1}{n!}\delta(t)$$

~~Actually you have a convolution.~~

$$\cancel{f * p_n \qquad p_n(t) = \frac{t^n}{n!}\chi_{(0,1)}}$$

$$\cancel{Dp_n = p_{n-1} - \frac{1}{n!}\delta(t-1)}$$

$$\cancel{\int_0^1 e^{-st}\frac{t^n}{n!}dt = \int_0^\infty e^{-st}\frac{t^n}{n!}dt - \int_1^\infty e^{-st}\frac{t^n}{n!}dt}$$

Taylor formula

$$\int (Df)\chi_{(0,1)}\,dt = -\int f\left(D\chi_{(0,1)}\right)dt$$

$$= -\int f\left(\delta(t) - \delta(t-1)\right)dt = -f(0) + f(1)$$

$$\int (D^2 f)\left((1-t)\chi_{[0,1]}\right) + \int 2Df\, D\left((1-t)\chi\right) + \int f\, D^2\left((1-t)\chi\right) = 0$$

NO

$$\int (D^2 f)\underbrace{\left((1-t)\chi_{(0,1)}\right)}_{\varphi} = -\int Df\, D\varphi = -\int Df\left(-\chi_{(0,1)} + \delta(t)\right)$$

$$= -(Df)(0) + \left(-f(0) + f(1)\right)$$

At the moment you have two approaches

$$e^{tD} = 1 + \int_0^t D e^{t_1 D}\, dt_1$$

$$= 1 + \int_0^t D\, dt_1 + \int_0^t dt_1 \int_0^{t_1} dt_2\, D^2 e^{t_2 D}$$

$$= 1 + tD + \frac{t^2}{2} D^2 + \underbrace{\int_0^t dt_1 \int_0^{t_1} dt_2 \int_0^{t_2} dt_3}_{\int_0^t dt_3 \frac{(t-t_3)^2}{2}} D^3 e^{t_3 D}$$

$$e^{tD} = 1 + \int_0^t dt_1\, D e^{t_1 D}$$

$$= 1 + \int_0^t dt_1\, D \left\{ 1 + \int_0^{t_1} dt_2\, D e^{t_2 D} \right\}$$

$$= 1 + tD + \iint_{0 \le t_2 \le t_1 \le t} dt_1\, dt_2\, D^2 e^{t_2 D}$$

$$= 1 + tD + \int_0^t dt_2 (t-t_2) D^2 e^{t_2 D}$$

$$= 1 + tD + \int_0^t dt_2 (t-t_2) D^2 + \underbrace{\int_0^t dt_2 \int_0^{t_2} dt_3}_{\int_0^t dt_3 \int_{t_3}^t dt_2 (t-t_2) = \frac{(t-t_3)^2}{2}} (t-t_2) D^3 e^{t_3 D}$$

not transp.

$$\frac{1}{s-D} = \frac{1}{s} + \frac{1}{s} D \frac{1}{s-D}$$

$$= \frac{1}{s} + \frac{1}{s} D \frac{1}{s} + \frac{1}{s} D \frac{1}{s} D \frac{1}{s-D}$$

$$e^{tD} = 1 + tD + \int_0^t (t-t') D^2 e^{t'D} dt'$$

$$e^{tD} = \sum_{j=0}^{n} \frac{t^j D^j}{j!} + \int_0^t \frac{(t-t')^n}{n!} D^{n+1} e^{t'D} dt'$$

$$f(t) = \sum_{j=0}^{n} \frac{t^j}{j!} f^{(j)}(0) + \int_0^t \frac{(t-t')^n}{n!} f^{(n+1)}(t') dt'$$

Let's go back to our candidate for grid space. First recall your initial ~~and~~ candidate for E_{hor} was ~~a free module of rank~~ the group for $\mathbb{R}$ consisting of smooth functions with compact support, $\varphi(r)$, under convolution.

$$\varphi(r) = \int \frac{d\rho}{2\pi} e^{i\rho r} \hat{\varphi}(\rho) \qquad \hat{\varphi}(\rho) = \int dr\, e^{-i\rho r} f(r)$$

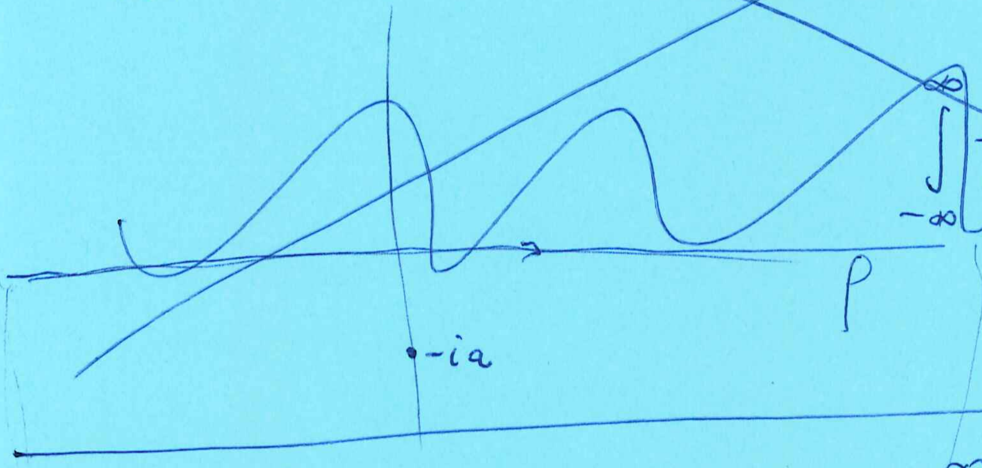
$$\partial_r \varphi \longmapsto i\rho \hat{\varphi}(\rho)$$

smooth function with comp. support means.

$$|\hat{\varphi}(\rho)| \leq C e^{|\mathrm{Im}\,\rho|} (1+|\rho|)^{-N}$$

But I think the $\varphi(r)$ occurring are not always smooth.

$$\varphi(r) = \int_{-\infty}^{\infty} \frac{e^{i\rho r} - e^{ar}}{i\rho - a}\, e^{-i\rho r}\,\frac{d\rho}{2\pi} = \int_{-\infty}^{\infty} \frac{1 - e^{ar}e^{-i\rho r}}{i\rho - a}\,\frac{d\rho}{2\pi}$$



$$\int_{-\infty}^{\infty} \frac{1}{i\rho - a}\,\frac{d\rho}{2\pi}$$

$r > 0$

$$\varphi(r) = \int_{-\infty}^{\infty} \frac{e^{i\rho} - e^{a}}{i\rho - a}\, e^{-i\rho r}\,\frac{d\rho}{2\pi} = \int_{-\infty}^{\infty}\left(\frac{e^{i\rho(1-r)}}{i\rho - a} - \frac{e^{a}e^{-i\rho r}}{i\rho - a}\right)\frac{d\rho}{2\pi}$$

You have to understand the operation

$$f(\rho) \longmapsto \frac{f(\rho) - f(-ia)}{i\rho - a}$$

on F.T. of distribution with compact support.

Basic example. $\quad e^{rs} \longmapsto \dfrac{e^{rs} - e^{ra}}{s - a}$

$$\frac{e^{rs} - e^{ra}}{s - a} = \int_0^1 e^{r(a + t(s-a))}\, dt \ ?$$

$$r(s-a) = \partial_t\{r(a + t(s-a))\}$$

$$\partial_t\, e^{(1-t)a + ts} = e^{(1-t)a + ts}(-a + s)$$

$$\left[e^{(1-t)a+ts}\right]_0^1 = \left(\int_0^1 e^{(1-t)a+ts}\,dt\right)(s-a)$$

Somehow you are puzzled by s.

operation $\quad f(s) \longmapsto \dfrac{f(s)-f(a)}{s-a}$

Assume $\quad f(s) = \int e^{sx}\,\varphi(x)\,dx$. Then

$$\frac{f(s)-f(a)}{s-a} = \int \frac{e^{sx}-e^{ax}}{s-a}\,\varphi(x)\,dx$$

The important thing to do is to write $\dfrac{e^{sx}-e^{ax}}{s-a}$ in terms of e^{sy} with $y \in [0, x]$

~~[illegible]~~ You want a path ~~[illegible]~~ starting from $e^{ax}e^{s0}$, ending with $e^{a0}e^{sx}$

$\cancel{a + t(s-a)}$ $\qquad e^{((1-t)a+ts)x} = e^{(1-t)a}\,e^{tsx}$

$$e^{sx}-e^{ax} = \int_0^1 \partial_t\left\{e^{[a+t(s-a)]x}\right\}dt$$

$$\boxed{\frac{e^{sx}-e^{ax}}{s-a} = \int_0^1 e^{[a+t(s-a)]x}\,x\,dt}$$

$$= \int_0^x e^{a(x-y)+sy}\,dy$$

$$=$$

$$f(s) = \cancel{\int dx\, \varphi(x)\, e^{xs}} = \int dx\, \varphi(-x)\, e^{xs}$$

$$\frac{f(s)-f(a)}{s-a} = \int dx\, \varphi(-x) \frac{e^{xs}-e^{xa}}{s-a}$$

to expand using e^{ys} $y \in [0,x]$.

$$\partial_y \left(e^{xa+y(s-a)} \right) = e^{xa+y(s-a)} \, s-a$$

$$e^{xs} - e^{xa} = \int_0^x dy\, \partial_y \left(e^{xa+y(s-a)} \right) = \int_0^x dy\, e^{xa+y(s-a)} (s-a)$$

$$\therefore \frac{f(s)-f(a)}{s-a} = \int dx\, \varphi(-x) \frac{e^{xs}-e^{xa}}{s-a}$$

$$= \int dx\, \varphi(-x) \int_0^x dy\, e^{xa+y(s-a)}$$

$$= \int_0^\infty dy\, e^{ys} \underline{\int_y^\infty dx\, \varphi(-x)\, e^{(x-y)a}}$$

$$\psi(-y) = \int_y^\infty dx\, \varphi(-x)\, e^{(x-y)a}$$

$$\psi(u) = \int_{-u}^\infty dx\, \varphi(-x)\, e^{(x+u)a} = \int_{-\infty}^{u} dx\, \varphi(+x)\, e^{(u-x)a}$$

$$\psi(x) = \int_{-\infty}^{x} dy\, \varphi(y)\, e^{(x-y)a} = \int_{-\infty}^{\infty} dy\, \varphi(y)\, H(x-y)\, e^{(x-y)a}$$

$$\psi = \varphi * \left(H(x)e^{xa}\right)$$

$$(\partial_x - a)\left(H(x)e^{xa}\right) = e^{xa}(\partial_x)H(x) = \delta$$

Repeat everything.

$$f(s) = \int e^{-st}\varphi(t)\,dt$$

$$\frac{f(s)-f(a)}{s-a} = \int \frac{e^{-st}-e^{-at}}{s-a}\varphi(t)\,dt$$

$$\partial_\lambda \frac{e^{-(a+\lambda(s-a))t}}{+(s-a)} = e^{+(a+\lambda(s-a))(-t)}(-\lambda t) \quad ?$$

$0 \le \lambda \le 1$

$$f(s) = \int e^{st}\varphi(-t)\,dt \qquad at + \lambda(s-a)t$$

$$\frac{f(s)-f(a)}{s-a} = \int \underbrace{\frac{e^{st}-e^{at}}{s-a}}\varphi(-t)\,dt$$

$$\int_0^1 \underbrace{e^{at+\lambda(s-a)t}\,t\,d\lambda}_{\partial_\lambda\left(\frac{e^{at+\lambda(s-a)t}}{s-a}\right)} = \int e^{\overbrace{at(1-\lambda)}^{a(t-t\lambda)}+\lambda st}\,d(t\lambda)$$

$$\frac{e^{xs}-e^{xa}}{s-a}$$

path $xa+y(s-a)$

$(x-y)a+ys.$ $\quad 0\le y\le x$

$$\boxed{\frac{e^{xs}-e^{xa}}{s-a} = \int_0^x dy\; e^{(x-y)a+ys}}$$

because $\quad \partial_y\, e^{(x-y)a+ys} = e^{(x-y)a+ys}(s-a)$

$$e^{xs}-e^{xa} = \Big[e^{(x-y)a+ys}\Big]_0^x = \int_0^x dy\, e^{(x-y)a+ys}(s-a)$$

$$f(s) = \int dx\, e^{sx}\varphi(-x) = \int dx\, e^{-sx}\varphi(x)$$

$$\varphi(x) = \int \frac{ds}{2\pi i}\, e^{sx} f(s).$$

$$\frac{f(s)-f(a)}{s-a} = \int dx\, \frac{e^{sx}-e^{ax}}{s-a}\,\varphi(-x)$$

$$= \int_{-\infty}^{\infty} dx \left(\int_0^x dy\; e^{(x-y)a+ys}\right)\varphi(-x)$$

$$= \int_0^{\infty} dy\; e^{ys}\overbrace{\left[\int_y^{\infty} dx\, e^{(x-y)a}\,\varphi(-x)\right]}^{\psi(-y)}$$

set $\quad \psi(+y) = \displaystyle\int_{-y}^{\infty} dx\; e^{(x+y)a}\,\varphi(-x) = \int_{-\infty}^{\overset{\infty}{+y}} dx\, \overbrace{e^{(y-x)a}}^{H(y-x)}\,\varphi(x)$

Convolution of $H(x)e^{xa}$ and $\varphi(x)$

So $\psi = He^{xa} * \varphi$

$$\int_0^\infty dx\, \cancel{H(x)}\, e^{xa} e^{-sx} = \frac{1}{s-a} \qquad \mathrm{Re}(s) > a.$$

Point: $f(s) \longleftrightarrow \varphi$

$$\Longrightarrow \quad \frac{f(s)-f(a)}{s-a} \longleftrightarrow \underbrace{\overbrace{\boxed{H(x)e^{xa} * \varphi}}^{\psi}}_{\text{should have compact support.}}$$

$$\psi(x) = \int_{-\infty}^{x} dy\, \cancel{H(x-y)}\, e^{(x-y)a}\varphi(y)$$

$$\psi(x) \neq 0 \Longrightarrow \exists\, y \leq x \ni \varphi(y) \neq 0.$$

Suppose $\varphi(y) = \delta(y)$.

Q

K

support φ

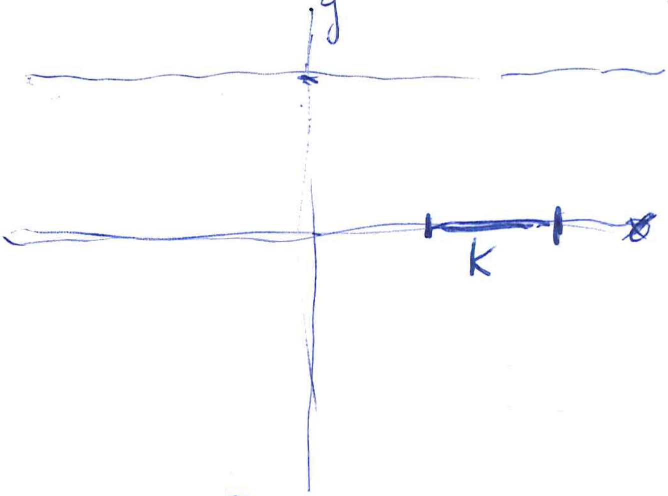
Then $\psi(x) = \begin{cases} e^{xa} & \text{if } x>0 \\ 0 & \text{if } x<0 \end{cases}$

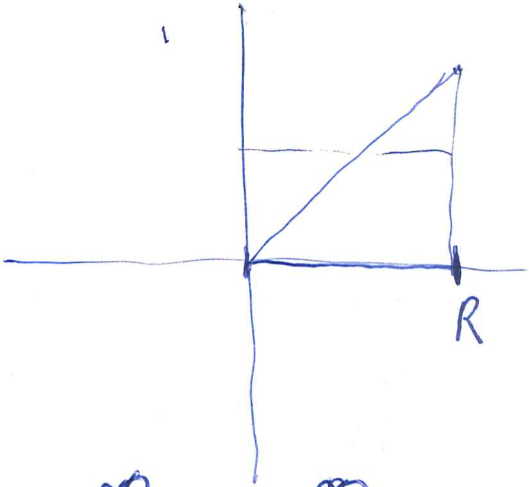
There was a puzzle yesterday, review:

$$f(s) = \int dx\, \varphi(-x)\, e^{xs} = \int dx\, \varphi(x)\, e^{-xs}$$

$$\frac{f(s)-f(a)}{s-a} = \int dx\, \varphi(-x)\, \frac{e^{xs}-e^{xa}}{s-a} = \int dx\, \varphi(-x) \int_0^x dy\, e^{xa+y(s-a)}$$

$$=$$





$$\int_0^\infty dx\,\varphi(-x)\int_0^x dy\; e^{xa+y(s-a)} = \int_0^\infty dy\; e^{ys}\int_y^\infty dx\,\varphi(-x)\,e^{(x-y)a}$$

$$\int_y^\infty dx\,\varphi(-x)\,e^{(x-y)a} = \int_0^\infty du\,\varphi(-y-u)\,e^{ua} = \psi(-y)$$

$x-y=u$

$x=y+u$

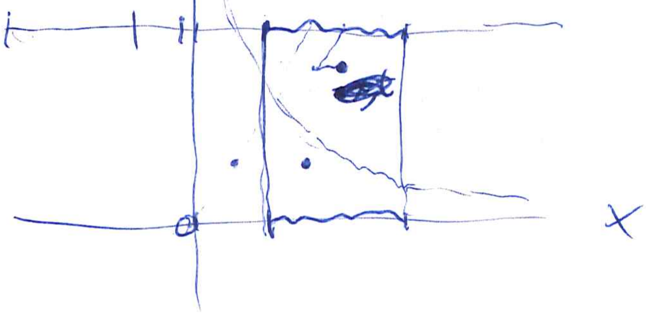
where $\psi(y)=\int_0^\infty du\,\varphi(y-u)\,e^{ua}$

Start again

$$f(s) = \int dx\,\varphi(x)\,e^{-xs} = \int dx\,\varphi(-x)\,e^{xs} \qquad (1-t)xa + txs$$

$$\frac{f(s)-f(a)}{s-a} = \int dx\,\varphi(-x)\,\frac{e^{xs}-e^{xa}}{s-a} = \int dx\,\varphi(-x)\,x\int_0^1 dt\; e^{xa+t(xs-xa)}$$

$$= \int dx\int_0^1 dt\; e^{(tx)s}\,\varphi(-x)\,x\,e^{(1-t)xa}$$



If you start with a set K = support $\varphi(-x)$ then you get all exp. $e^{(tx)s}$ $x\in K$, $t\in[0,1]$

$$\frac{e^{xs}-e^{xa}}{s-a} = \int_0^1 dt\, x\, e^{xa+t(xs-xa)} \qquad y = tx$$

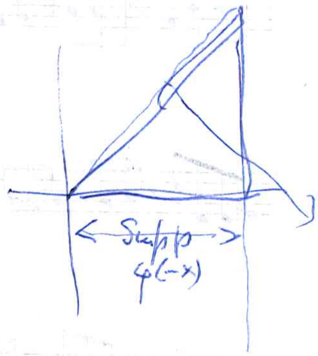
$$= \int_0^x dy\, e^{xa+y(s-a)}$$

$$f(s) = \int_{-\infty}^{\infty} dx\, \varphi(+x)\, e^{-xs} = \int_{-\infty}^{\infty} dx\, \varphi(-x)\, e^{xs} \qquad \varphi \text{ comp. supp.}$$

$$\frac{f(s)-f(a)}{s-a} = \int_0^{\infty} dx\, \varphi(-x) \int_0^x dy\, e^{xa+ys-ya}$$

supposing $\varphi(-x)$ supp in $\mathbb{R}_{\geq 0}$.

$$= \int_0^{\infty} dx \int_0^{\infty} dy\, e^{ys} H(x-y)\, e^{(x-y)a}\, \varphi(-x)$$



$$= \int_0^{\infty} dy\, e^{ys}\, \psi(-y)$$

when $$\psi(-y) = \int_0^{\infty} dx\, H(x-y)\, e^{(x-y)a}\, \varphi(-x)$$

$$\psi(y) = \int_0^{\infty} dx\, H(x+y)\, e^{(x+y)a}\, \varphi(-x)$$

$$= \int_{-\infty}^{0} dx\, H(y-x)\, e^{(y-x)a}\, \varphi(x) =$$

the convolution of $\varphi(x)$ supported on $\mathbb{R}_{\leq 0}$ and $H(x)e^{xa}$ (fundamental soln to $\partial_x - a$) supported in $\mathbb{R}_{\geq 0}$. So where is $\psi(y)$ supported?

$$\psi(x) \qquad H(x_1)e^{x_1 a}\, \varphi(x_2) \qquad x = x_1 + x_2$$

$$x_1 > 0 \qquad x_2 < 0$$

$$\varphi(-x) = \delta(x-1)$$

$$\psi(-y) = \int_0^\infty dx\, H(x-y)\, e^{(x-y)a}\, \delta(x-1)$$

Confusion reigns about convolution

$$\frac{e^{xs} - e^{xa}}{s-a} = \int_0^x dy\, e^{(x-y)a + ys}$$

observe this is the convolution of $H(x)e^{xa}$ and $H(x)e^{xs}$

as a check note that $(\partial_x - a) * (H(x)e^{xa}) = \delta(x)$

so $(\partial_x - a) * H(x)e^{xa} * H(x)e^{xs} = \delta * H(x)e^{xs} = H(x)e^{xs}$

should be true, which is OKAY since

$$(\partial_x - a) * \frac{e^{xs} - e^{xa}}{s-a} = \frac{(s-a)e^{xs}}{s-a} = e^{xs}$$

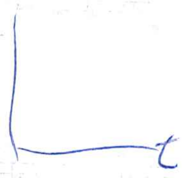
So it ~~is~~ should be no surprise that

$$\int_0^\infty dx\, \varphi(-x)\, \frac{e^{xs} - e^{xa}}{s-a} \quad \text{is a convolution}$$

~~also~~ for suitable $\varphi(-x)$. The setting - functions of $x \in \mathbb{R}_{>0}$, but you maybe should think of x as being t as in L.T.

$$\int_0^\infty e^{-st} (f*g)(t)\, dt = \int_0^\infty dt\, e^{-st} \int_0^t f(t-t')g(t')\, dt'$$

$$= \int_0^\infty dt \int_0^t dt'\, e^{-s(t-t')} f(t-t')\, e^{-st'} g(t')$$

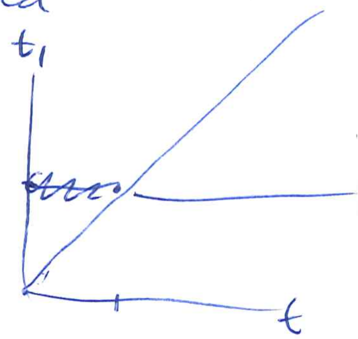


$$= \int_0^\infty dt' \int_{u=0}^\infty d(t'+u)\, e^{-su} f(u)\, e^{-st'} g(t') = \hat{f}(s)\, \hat{g}(s).$$

$$\int_0^\infty dx\, \varphi(p-x)\,\frac{e^{xs}-e^{xa}}{s-a}$$

$$= \left(\varphi * H(x)e^{xa} * H(x)e^{xs}\right)(p) \qquad \text{O.K.}$$

Repeat: First get convolution formalism straight. Choice between $\mathbb{R}$ and $\mathbb{R}_{\geq 0}$, cont. version of Laurent series versus power series. Convolution is related to composition of operators. ~~Conv~~ You want to link the pairing $\int fg$ and convolution $f*g$



$$\hat\varphi(s) = \int_0^\infty e^{-st}\varphi(t)\,dt \qquad t_2 = t - t_1$$

$$(\hat\varphi\hat\psi)(s) = \int_0^\infty dt_1 \int_0^\infty dt_2\; e^{-s\overbrace{(t_1+t_2)}^{t}}\varphi(t_1)\,\psi(t_2)$$

$$= \underbrace{\int_0^\infty dt_1 \int_{t_1}^\infty dt}_{\int_0^\infty dt \int_0^t dt_1}\; e^{-st}\,\varphi(t_1)\,\psi(t-t_1)$$

~~pairing~~ Look at $L^2(S^1)$ first, $(f|g) = \int f^* g \,\frac{d\theta}{2\pi}$

$= (1|f^*g)$ $\qquad (ab)_n = \sum_{i+j=n} a_i b_j \qquad (a|b) = \sum \bar a_n b_n$

$$(a^*b)_n = \sum_{i+j=n} \bar a_{-i}\, b_j \qquad (1|a^*b) = \sum \bar a_{-n} b_n$$