orth. case: given $H : V \xrightarrow{\sim} V^*$ $\quad H^t = H$. 1

$$so(V,H) = \{X \in End(V) \mid X^t H + HX = 0\}.$$

$\Lambda^2 V$

$$so(V) \xrightarrow{\sim} \Lambda^2 V^*$$

$$X \longmapsto HX$$

$$H^{-1}\omega \longleftarrow \omega$$

$Cliff(V)$ $\quad \psi_v^2 = v^t H v$

$\cancel{[\psi_v^2, \psi_{v'}] =}$

$\cancel{[[\psi_{v_1}, \psi_{v_2}]_+, \psi_x]_- = [[\psi_{v_1}, \psi_x]_+, \psi_{v_2}]_- + [\psi_{v_1}, [\psi_{v_2}, \psi_x]]}$

$$\frac{1}{2}[\psi_{v_1}\psi_{v_2} - \psi_{v_2}\psi_{v_1}, \psi_x]$$

$$= (v_1^t H x)\psi_{v_2} + \psi_{v_1}(v_2 H x) - (v_2^t H x)\psi_{v_1} - \psi_{v_2}(v_1^t H x)$$

$$= \psi_{v_2} v_1^t H x - \psi_{v_1}(v_2^t H x) = \psi_{(v_2 v_1^t H - v_1 v_2^t H)x}$$

$$v_1 \wedge v_2 \longmapsto \frac{1}{2}(v_1 \otimes v_2 - v_2 \otimes v_1) \longmapsto v_1 v_2^t H - v_2 v_1^t H$$

$$\Lambda^2 V \xrightarrow{\sim} so(V) \xrightarrow{\sim} \Lambda^2 V^*$$

$$v_1 v_2 \longmapsto (v_1 v_2^t - v_2 v_1^t) H \longmapsto H(v_1 v_2^t - v_2 v_1^t) H$$

$\cancel{\psi_{v_1}\psi_{v_2}\psi_v - \psi_{v_2}\psi_{v_1}\psi_v}$ $\qquad \psi_v^2 = v^t H v$

$$\psi_{x+y}^2 - \psi_x^2 - \psi_y^2 = \psi_x\psi_y + \psi_y\psi_x$$
$$= x^tHy + y^tHx$$

$$[\psi_x\psi_y, \psi_v] = \psi_x(\psi_y\psi_v + \psi_v\psi_y) - (\psi_x\psi_v + \psi_v\psi_x)\psi_y$$
$$= \psi_x(y^tHv) - (x^tHv)\psi_y = \psi_{xy^tHv - yx^tHv}$$

$$[\psi_x\psi_y, \psi_v\psi_w] = [\psi_x\psi_y, \psi_v]\psi_w + \psi_v[\psi_x\psi_y, \psi_w]$$
$$= \psi_x\psi_w y^tHv - \psi_y\psi_w x^tHv + \psi_v\psi_x y^tHw - \psi_v\psi_y x^tHw$$

$$[XY, VW] = X(Y,V)W - Y(X,V)W$$
$$V(Y,W)X - V(X,W)Y$$

---

bosonic $[\phi_x\phi_y, \phi_v] = \phi_x y^t\omega v + \phi_y x^t\omega v$

$$= \phi_{x(y^t\omega v) + y(x^t\omega v)}$$

---

simple oscillator $\qquad \dim V = 2. \qquad V = \mathbb{R}^2$

$H = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \omega = \begin{pmatrix} 0 & -1 \\ +1 & 0 \end{pmatrix} \qquad \begin{matrix} \dot q = p \\ \dot p = -q \end{matrix}$

$$X = \omega^{-1}H = \begin{pmatrix} 0 & +1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

Let q, p basis for V.

$[\phi_p^2 + \phi_q^2,$

~~Let V have basis~~ q, p

$V = \mathbb{R}^2$ standard bases $e_1 = \begin{pmatrix}1\\0\end{pmatrix}$ $e_2 = \begin{pmatrix}0\\1\end{pmatrix}$

typical elt of V is $e_1 q + e_2 p = \begin{pmatrix}q\\p\end{pmatrix}$

$$\begin{pmatrix}q\\p\end{pmatrix}^t H \begin{pmatrix}q'\\p'\end{pmatrix} = qq' + pp'$$

$$\begin{pmatrix}q\\p\end{pmatrix}^t \omega \begin{pmatrix}q'\\p'\end{pmatrix} = (q\ \ p)\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\begin{pmatrix}q'\\p'\end{pmatrix} = pq' - qp'$$

$$X\begin{pmatrix}q\\p\end{pmatrix} = \begin{pmatrix}0 & +1\\ -1 & 0\end{pmatrix}\begin{pmatrix}q\\p\end{pmatrix} = \begin{pmatrix}+p\\-q\end{pmatrix}$$

$$S^2V \longrightarrow sp(V) \longrightarrow S^2V^*$$

$\boxed{H = \frac{p^2}{2} + \frac{q^2}{2}}$ Standard stuff.

W = gen. q, p subj to $[p, q] = \frac{1}{i}$

$H = \frac{p^2}{2} + \omega^2 \frac{q^2}{2}$ $\quad [iH, q] = i\left[\frac{p^2}{2}, q\right] = p$

$$[iH, p] = i\left[\omega^2 \frac{q^2}{2}, p\right] = -\omega^2 q$$

ad(iH) gives time flow

$$\begin{pmatrix}q(t)\\p(t)\end{pmatrix} = e^{itH}\begin{pmatrix}q\\p\end{pmatrix}e^{-itH} = \exp\left\{t\begin{pmatrix}0 & 1\\ -\omega^2 & 0\end{pmatrix}\right\}\begin{pmatrix}q\\p\end{pmatrix}$$

In the Weyl alg. you have $\phi(V)$, $\phi(S^2V)$ etc. but to quantize you need a polarization of V, this means a choice of maximal isotropic subspace V_0 wrt ω, then get irred cyclic repn of $\mathrm{Weyl}(V)/\mathrm{Weyl}(V)V_0 \underset{\text{add.}}{\simeq} S(V/V_0)$
You should be able to understand the frequency.

standard calculation is

$$[p,q] = \frac{\hbar}{i} \qquad H = \frac{1}{2}p^2 + \frac{1}{2}\omega_0^2 q^2$$

$$= \frac{1}{2}(\omega_0 q - ip)(\omega_0 q + ip)$$

$$\left[\frac{\omega_0 q + ip}{r}, \frac{\omega_0 q - ip}{s}\right] = \frac{2\omega_0\hbar}{rs} = 1$$

$$a = \frac{\omega_0 q + ip}{r} \qquad a^* = \frac{\omega_0 q - ip}{s}$$

$$H = \frac{1}{2} rs\, a^*a = \frac{1}{2}(\omega_0 q - ip)(\omega_0 q + ip)$$

$$\hbar\omega_0 a^*a = \frac{1}{2}\left(p^2 + \omega_0^2 q^2 - \hbar\omega_0\right)$$

$$H = \hbar\omega_0\left(a^*a + \frac{1}{2}\right) = \frac{1}{2}\left(p^2 + \omega_0^2 q^2\right).$$

In principle I understand. You have these two forms H, ω $(=\Omega?)$, you want polarization of V.

Fermionic. $V = \mathbb{R}^2$ $\quad H = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $\quad A = \begin{pmatrix} 0 & -\omega_0 \\ \omega_0 & 0 \end{pmatrix}$

Let's try another viewpoint. So far have treated V as a real v.s. with symp. & quad forms.

Now complexify. Recall viewpoint.

~~Review oscillator theory.~~ Continue towards super symmetry. First you need polarization which you ignored before. Involves complexification so that symp. form A and symm. form S become hyperbolic. You should be able to treat this in previous setting.

bosonic. V $\quad A : V \to V^*$ $\quad A^t = -A$.

Weyl(V): $\quad [\phi_x, \phi_y] = x^t A y$

$$[\phi_x \phi_y, \phi_v] = \phi_x y^t A v + \phi_y x^t A v$$

$$[xy, v] = x y^t A v + y x^t A v$$

$$S^2 V \longrightarrow sp(V) \longrightarrow S^2 V^*$$

$$xy \longmapsto xy^t A + yx^t A \longmapsto Axy^t A + Ayx^t A$$

Polarization. The notion of polarization is part of kinematics — something attached to V, A. Simply a max isot. subspace for A and any two are related by symp. transf.

What is your aim? to finish harm. osc. stuff. Situation V real, $A : V \xrightarrow{\sim} V^*$, $S : V \xrightarrow{\sim} V^*$, $A^t = -A$, $S^t = +S$. ~~Thus~~ Actually you also want $S > 0$. Wait.

Consider the bosonic operator ~~Fock~~ picture where V_c is a linear space of operators ϕ_v with conjugation = adjoint and $[\phi_x, \phi_y] = x^t A y$

Let's proceed carefully. You have V with skew form x^tAy. Better you have the vector space V with forms A, S, and you get

$X = A^{-1}S$
$$X^tA = (A^{-1}S)^tA = -SA^{-1}A = -AX$$
$$X^tS = -SA^{-1}S = -SX$$

~~But~~ preserving both A, S also

$Y = S^{-1}A$
$$Y^tS = -AS^{-1}S = -A = -SY$$
$$Y^tA = -AS^{-1}A = -AY.$$

preserving both A, S.

What do you want to understand? How the pair S, A determine polarizations. Look abstractly. You have the vector space V, the non-degenerate ~~skew~~-symm. A over some field. You want a splitting of V into complementary max isot. subspaces for A, arising somehow from a given non deg symm. form S.

Over the reals with S positive, you know
$X = A^{-1}S$ preserves S, i.e. is skew-symmetric so X^2 is diagonalizable and < 0. So $-X^2$ has a unique pos. square roots $|X|$, so the phase $\frac{X}{|X|} = J$ is $\ni$ $J^2 = -1$. This means a complex polarization of some sort.

V, $A, S : V \xrightarrow{\sim} V^*$ $A^t = -A$, $S^t = S$. Aim to construct polarizations of (V, A) (V, S) which are compatible.

Yesterday I looked again at picture of a harmonic oscillator as a complex Hilbert space V together with real structure, i.e. conjugation, no relations between them. Example $V = \mathbb{C}^2$ symm. gp. $U(2)$, symm. group $GL_2(\mathbb{R})$, look at

$U(2) \backslash GL(2,\mathbb{C}) / GL(2,\mathbb{R})$, better $GL(2,\mathbb{R}) \backslash GL(2,\mathbb{C}) / U(2)$

because $GL(2,\mathbb{C})/U(2)$ = space of pos. def. herm. inner products, pos. def. herm. matrices. 4 diml acted on by $GL(2,\mathbb{R})$
4 dim. $g^*(S + iA)g = g^tSg + i\, g^tAg$. The action is transitive on the S component, pick basepoint $S = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, stabilizer is $O(2)$, which acts as ± 1 on A-comp.
So orbit space is $\mathbb{R}_{\geq 0}$, > 0 if A nondeg.

Other approach - look at real 2 planes M in $\mathbb{C}^2$ which generate i.e. $\Lambda^2_{\mathbb{R}} M \longrightarrow \Lambda^2_{\mathbb{C}} V$ nonzero. So ~~you are in~~ you have real Grass. $G_2(\mathbb{R}^4 = \mathbb{C}^2)$ with $U(2)$ acting. ~~Precisely But you~~ You want M such that $iM \wedge M = 0$ since $i(iM \wedge M) = M \wedge iM = iM \wedge M$.

will be 0 or 1 complex dim. So remove $\mathbb{P}^1\mathbb{C}$ from $G_2(\mathbb{R}^4 = \mathbb{C}^2)$. Can you see a numerical invariant, an angle associated to a real 2 plane M in $\mathbb{C}^2$ $M \cap \mathbb{C}\begin{pmatrix}1\\0\end{pmatrix}$. This business might be quaternionic

What seems to work is to take an orth basis for M wrt the real part S of the herm. scalar product, then look at the imaginary part iA which should give a number.

$\boxed{\text{IDEA}}$, vague hope, that as one goes to inf. dimensions the commutation relations should be relaxed to relations modulo compacts.

Go back to polarization. ~~[illegible]~~ Suppose given V complex, A, S non. deg bil. forms A anti-sym, S symm. No.

Take a harm. osc. situation: V real vector space equipped with a pos. symm. form S, ~~non~~ symp. form A. Get $X = A^{-1}S$ (time flow) preserving A, S. Also have $X^{-1} = S^{-1}A$

The eigenvalues of X are [illegible] imaginary, so when V is complexified, [illegible] V_c splits $= W^+ \oplus W^-$ giving a polarization for both A, S. Here A, S are extended $\mathbb{C}$ linearly to V_c. ~~[illegible]~~ Whether you use X or X^{-1} shouldn't matter, except maybe for the sign.

dim $V = 2$. ~~What next?~~

$$V_{\mathbb{R}} = \mathbb{R}^2 \qquad S = \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix} \qquad A = \begin{pmatrix}0 & -1\\1 & 0\end{pmatrix}h$$

$$X = A^{-1}S = \begin{pmatrix}0 & h^{-1}\\-h^{-1} & 0\end{pmatrix}$$

$$\begin{pmatrix} 0 & \frac{1}{h} \\ -\frac{1}{h} & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}\begin{pmatrix} \frac{i}{h} & 0 \\ 0 & -\frac{i}{h} \end{pmatrix}$$

$$V_c = \mathbb{C}^2 = \underbrace{\mathbb{C}\begin{pmatrix} 1 \\ i \end{pmatrix}}_{W^+} \oplus \underbrace{\mathbb{C}\begin{pmatrix} 1 \\ -i \end{pmatrix}}_{W^-}$$

W^+, W^- isotropic for both A, S

Put $\quad c = \begin{pmatrix} 1 \\ i \end{pmatrix} \quad \bar{c} = \begin{pmatrix} 1 \\ -i \end{pmatrix}$

$$[\phi_c, \phi_{\bar{c}}] = c^t A \bar{c} = (1 \ \ i)\begin{pmatrix} 0 & -h \\ h & 0 \end{pmatrix}\begin{pmatrix} 1 \\ -i \end{pmatrix}$$

$$= (1 \ \ i)\begin{pmatrix} +hi \\ h \end{pmatrix} = hi + ih = 2ih$$

$$\frac{1}{2}(\psi_c \psi_{\bar{c}} + \psi_{\bar{c}} \psi_c) = c^t S \bar{c} = 1 + i(-i) = 2$$

~~Start again with~~

You need supersymmetry examples, meaning?

You want ~~same~~ simplest examples, meaning?

Perhaps you are wrong to take $V_r, A^{\text{symp}}, S > 0$ to form Weyl(V, A), Cliff(V, S) to use ~~polarization~~ the obvious polarization (assoc. to the phase of $A^{-1}S$ also $S^{-1}A$

don't understand supersymmetry yet.

basic example should be de Rham-Koszul complex $S(V) \otimes \Lambda V$.

The Hilbert space structure should be transparent, because it's a tensor product of commuting situations.

$[a, a^*] = 1$, ground state $a\xi = 0$. Then

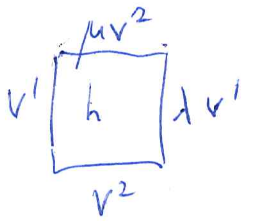
$$[a, a^{*n}\xi] = n a^{*n-1}\xi \qquad [a^*a, a^{*n}] = n a^{*n} \quad \text{etc.}$$

Review the problem. Consider a harmonic oscillator (V, A, S). Apparently (V, A) has a polarization determined by S and (V, S) has a polarization determined by A, so you should have irred repns. of $\mathrm{Weyl}(V, A)$ and $\mathrm{Cliff}(V, S)$ ~~which~~ $S(E)$, $\Lambda E)$. The supersymmetry should be some interesting operator (like d) on $S(E) \otimes \Lambda E)$. Example: holom. fms. repn. of CCR. Get mixture of deRham and Koszul complexes.

---

time for something new, go back to IH in the continuous case. Review last stuff examined.

$$\begin{pmatrix} \lambda v^1 \\ \mu v^2 \end{pmatrix} = \frac{1}{k}\begin{pmatrix} 1 & h \\ \bar h & 1 \end{pmatrix}\begin{pmatrix} v^1 \\ v^2 \end{pmatrix}$$



$$\psi^1_{mn} = \lambda^m \mu^n v^1 \qquad (k\lambda - 1)v^1 = h v^2$$

$$\psi^2_{mn} = \lambda^m \mu^n v^2 \qquad (k\mu - 1)v^2 = \bar h v^1$$

$$\mu = \frac{1}{k}\left(1 + \frac{1-k^2}{k\lambda - 1}\right) = \frac{k^{\cancel{2}}\lambda - \cancel{k}\,1 + 1 - k^2}{k(k\lambda - 1)} = \frac{\lambda - k}{k\lambda - 1}$$

$$k\psi^1_{m+1,n} = \psi^1_{m,n} + h\psi^2_{mn}$$

$$k_\varepsilon \psi^1_{x+\varepsilon,y} = \psi^1_{x,y} + b\varepsilon\psi^2_{x,y} \rightsquigarrow \partial_x\psi^1 = b\psi^2$$

$$k\psi^2_{m,n+1} = \psi^2_{m,n} + \bar h\psi^1_{mn}$$

$$k_\varepsilon \psi^2_{x,y+\varepsilon} = \psi^2_{xy} + \bar b\varepsilon\psi^1_{xy} \rightsquigarrow \partial_y\psi^2 = \bar b\psi^1$$

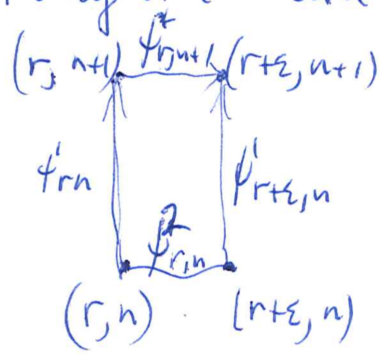
$\psi = e^{i(r\rho + s\sigma)}\begin{pmatrix} v^1 \\ v^2 \end{pmatrix}$ $\qquad$ $i\rho v^1 = h v^2$
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $i\sigma v^2 = \bar{h} v^1$

$-(\rho\sigma) = |h|^2$ $\qquad$ take $h = i$ get

$\rho v^1 = v^2$
$-\sigma v^2 = +v^1$ $\qquad$ $e^{i(r\rho - s\rho^{-1})}\begin{pmatrix} \frac{1}{\rho} \\ 1 \end{pmatrix} v^2$

---

horizontal cont. limits $\quad$ Idea is $\quad$ $b\varepsilon$



$$k_\varepsilon \psi^1_{r+\varepsilon,n} = \psi^1_{r,n} + h_\varepsilon \psi^2_{r,n}$$

$$k_\varepsilon \psi^2_{r,n+1} = \underbrace{\bar{h}_\varepsilon}_{b} \psi^1_{r,n} + \psi^2_{r,n}$$

$$k_\varepsilon = \sqrt{1 - |b|^2 \varepsilon} = 1 - a\varepsilon \qquad \boxed{a = \tfrac{1}{2}|b|^2}$$

$$(-a + \partial_r)\psi^1_{rn} = b\psi^2_{rn}$$

$$\psi^2_{r,n+1} - \psi^2_{r,n} = \bar{b}\psi^1_{r,n}$$

Not too clear. It's probably better to have $\bar{b}$ replaced by an independent b'.

Other viewpoint, from grid space, you want v^2 to approach a δ fn. ~~Thus v^2~~

In any case

$$(-a + i\rho)v^1 = bv^2$$

$$(\mu - 1)v^2 = \bar{b}v^1$$

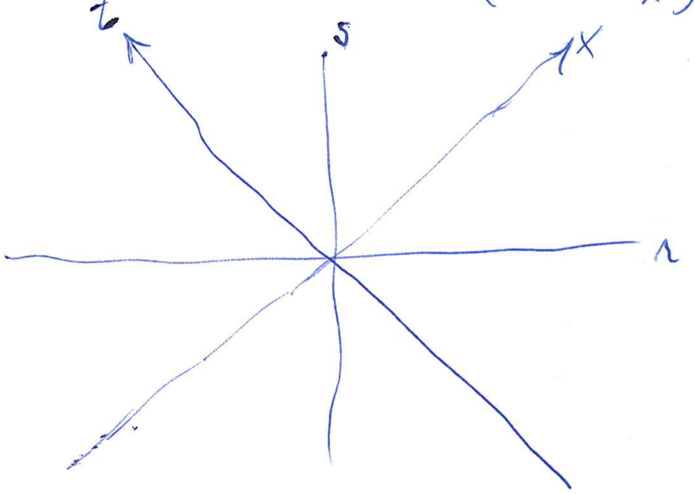
$$\mu = 1 + \frac{2a}{-a + i\rho} = \frac{a + i\rho}{-a + i\rho}$$

Analyze first the discrete grid

No let's do everything in continuous space time.

$$\partial_t \psi = \begin{pmatrix} \partial_x & i \\ i & -\partial_x \end{pmatrix} \psi$$

$$(\partial_t - \partial_x)\psi^1 = i\psi^2$$
$$(\partial_t + \partial_x)\psi^2 = i\psi^1$$



$$\partial_r f = \partial_x f \frac{\partial x}{\partial r} + \partial_t f \frac{\partial t}{\partial r}$$
$$\partial_s f = \partial_x f(1) + \partial_t f(1)$$
$$x = r + s$$
$$t = -r + s$$

$$\partial_r = \partial_x - \partial_t \qquad -\partial_r \psi^1 = i\psi^2 \qquad -\rho\hat{\psi}^1 = \hat{\psi}^2$$
$$\partial_s = \partial_x + \partial_t \qquad \partial_s \psi^2 = i\psi^1 \qquad \sigma\hat{\psi}^2 = \hat{\psi}^1$$

$$\sigma = -\rho^{-1}. \qquad \psi(r,s) = \int_{-\infty}^{\infty} e^{i(r\rho - s\rho^{-1})} \begin{pmatrix} 1 \\ -\rho \end{pmatrix} f(\rho)\, d\rho$$

I recall analyzing the Cauchy problem.

$$\partial_t \psi = \begin{pmatrix} \partial_x & i \\ i & -\partial_x \end{pmatrix} \qquad \psi(x,t) = e^{t\begin{pmatrix} \partial_x & i \\ i & -\partial_x \end{pmatrix}} \psi_0(x)$$

$$\psi(x,0) \text{ given} = \psi_0 \qquad = e^{t\begin{pmatrix} \partial_x & i \\ i & -\partial_x \end{pmatrix}} \int e^{ikx} \hat{\psi}_0(k) \frac{dk}{2\pi}$$

$$= \int e^{ikx} e^{it\overbrace{\begin{pmatrix} k & 1 \\ 1 & -k \end{pmatrix}}^{A}} \hat{\psi}_0(k) \frac{dk}{2\pi} \qquad A^2 = \begin{pmatrix} k & 1 \\ 1 & -k \end{pmatrix}\begin{pmatrix} k & 1 \\ 1 & -k \end{pmatrix} = \begin{pmatrix} \omega^2 & 0 \\ 0 & \omega^2 \end{pmatrix} = \omega^2 I$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} t^{2n} \omega^{2n} I + \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} t^{2n+1} \frac{\omega^{2n+1}}{\omega} iA$$

$$\cos(\omega t) I + i \frac{\sin(\omega t)}{\omega} A$$

Yesterday you reached the problem of quantizing a loop with values in $SU(1,1)$, make precise. Grid space assoc. to (h_n) is a free module of rank 2 over $\mathbb{C}[u,u^{-1}]$ with ~~different bases~~ $SU(1,1)$ structure.

Start with $SU(1,1) = SL(2,\mathbb{R})$. Get clear the notion of $SU(1,1)$ structure.

V, 2 diml over $\mathbb{C}$, conjugation σ, volume

V complex vector with hermitian form $H(v,v')$ and conjugation σ. Restrict to V^σ to get a $\mathbb{C}$ valued $\mathbb{R}$-bilinear form, Condition can write $H = S + iA$

$H(v',v) = \overline{H(v,v')}$ n

means S symm. A anti-symm.

What about non-degeneracy, positivity?

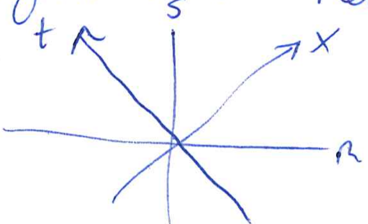
~~What about mistaken.~~

Your mistake?- to assume $H>0$ on V_c means $S>0$ on $V_{\mathbb{R}}$, Certainly $H>0 \Rightarrow S>0$

Check carefully $H = S + iA$ a herm. matrix $g \in GL_n(\mathbb{R})$ acts $g^t H g = g^t S g + i\, g^t A g$.

$$\begin{pmatrix} z_1' \\ z_2' \end{pmatrix}^* \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = i\left(\bar{z}_2' z_1 - \bar{z}_1' z_2\right)$$

In the next few days you need to find IH for the wave equation. Review carefully. Use char coords



$t = -r + s$

$x = r + s$

$\frac{\partial f}{\partial r} = \frac{\partial f}{\partial x} - \frac{\partial f}{\partial t}$ $\quad \partial_s = \partial_x + \partial_t$

$$\partial_t \psi = \begin{pmatrix} \partial_x & i \\ i & -\partial_x \end{pmatrix} \psi \qquad \boxed{\begin{matrix} -\partial_r \psi^1 = i\psi^2 \\ \partial_s \psi^2 = i\psi^1 \end{matrix}} \qquad \boxed{\begin{matrix} -\rho \hat{\psi}^1 = \psi^2 \\ \sigma \hat{\psi}^2 = \psi^1 \end{matrix}}$$

$$(1+\rho\sigma)\hat{\psi}^j = 0 \qquad \hat{\psi} = e^{i(r\rho - s\rho^{-1})}\begin{pmatrix} 1 \\ -\rho \end{pmatrix}\hat{\psi}^1$$

There are 4 Cauchy type problems to look at

$\boxed{t=0}$ $\quad \partial_t \psi = \begin{pmatrix} \partial_x & i \\ i & -\partial_x \end{pmatrix}\psi$, $\psi(x,0) =$ given $\psi_0(x)$

soln. $\quad \psi(x,t) = \exp\left(t\begin{pmatrix} \partial_x & i \\ i & -\partial_x \end{pmatrix}\right)\psi_0(x) \qquad A = \begin{pmatrix} k & 1 \\ 1 & -k \end{pmatrix} \quad A^2 = \omega^2 I$

$\omega^2 = k^2 + 1.$

$$= \int e^{ikx} \underbrace{\exp\left(it\begin{pmatrix} k & 1 \\ 1 & -k \end{pmatrix}\right)} \hat{\psi}_0(k)\,\frac{dk}{2\pi}$$

$$e^{itA} = \cos(\omega t)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + i\,\frac{\sin(\omega t)}{\omega}\begin{pmatrix} k & 1 \\ 1 & -k \end{pmatrix}$$

$$\boxed{e^{itA} = e^{i\omega t}\frac{1}{2\omega}\begin{pmatrix} \omega+k & 1 \\ 1 & \omega-k \end{pmatrix} + e^{-i\omega t}\frac{1}{2\omega}\begin{pmatrix} \omega-k & -1 \\ -1 & \omega+k \end{pmatrix}}$$

Energy $\quad \int \psi^* \psi \, dx = \int \left(e^{itA}\hat{\psi}_0\right)^*\left(e^{itA}\hat{\psi}_0\right)\frac{dk}{2\pi}$

$$= \int (\hat{\psi}_0)^*(\hat{\psi}_0)\,\frac{dk}{2\pi}$$

$\boxed{x=0}$ $\quad 0 = \begin{pmatrix} -\partial_t + \partial_x & i \\ -i & +\partial_t + \partial_x \end{pmatrix}\psi \qquad \partial_x \psi = \begin{pmatrix} \partial_t & -i \\ i & -\partial_t \end{pmatrix}\psi$

$\psi(0,t) = \psi_0(t).$

soln. $\quad \psi(x,t) = e^{x\begin{pmatrix} \partial_t & -i \\ i & -\partial_t \end{pmatrix}}\psi_0(t)$

$$= \int e^{\omega t} e^{ix\underbrace{\begin{pmatrix} \omega & -1 \\ 1 & -\omega \end{pmatrix}}_{B}} \hat{\psi}_0(\omega)\,\frac{d\omega}{2\pi}$$

$$B^2 = \begin{pmatrix} \omega & -1 \\ 1 & -\omega \end{pmatrix}\begin{pmatrix} \omega & -1 \\ 1 & -\omega \end{pmatrix} = \begin{pmatrix} \omega^2-1 & 0 \\ 0 & \omega^2-1 \end{pmatrix} = \overbrace{(\omega^2-1)}^{k^2} I .$$

$$e^{ixB} = \cos(kx)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + i\,\frac{\sin(kx)}{k}\begin{pmatrix} \omega & -1 \\ 1 & -\omega \end{pmatrix}$$

$$\boxed{e^{ixB} = e^{ikx}\frac{1}{2k}\begin{pmatrix} k+\omega & -1 \\ 1 & k-\omega \end{pmatrix} + e^{-ikx}\frac{1}{2k}\begin{pmatrix} k-\omega & 1 \\ -1 & k+\omega \end{pmatrix}}$$

Here $k = \sqrt{\omega^2-1}$, but $\cos(kx)$, $\frac{\sin(kx)}{k}$ are entire functions of $k^2 = \omega^2-1$.

There's a problem here that you are assuming $\psi_0(t)$ can be represented as $\int e^{i\omega t}\hat{\psi}_0(\omega)\frac{d\omega}{2\pi}$. This perhaps is not serious, because you only need the case where $\psi_0(t) = \delta(t)\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\delta(t)\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ whence $\hat{\psi}_0(\omega) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ for all $\omega \in \mathbb{R}$.
So you have the representation and a corresponding Green's function. a singularity $\delta(t)\delta(x)$ times $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. The difficulty ~~seems to be~~ is solutions involving $|\omega| < 1$ have $k = \pm\sqrt{\omega^2-1}$ imaginary, so the solution grows in the x direction. e.g. time dep. $\omega = 0$.

$k = i$, $\omega = 0$

$$e^{-x}\frac{1}{2i}\begin{pmatrix} i & -1 \\ 1 & i \end{pmatrix} + e^{x}\frac{1}{2i}\begin{pmatrix} i & 1 \\ -1 & +i \end{pmatrix}$$

$$= \begin{pmatrix} \cosh x & -i\sinh x \\ i\sinh x & \cosh x \end{pmatrix}$$

$$\int \left(\psi^* \varepsilon \psi\right)_{x=0} dt = \int \left(\psi_0^{1*}\psi_0^1 - \psi_0^{2*}\psi_0^2\right)(t)\, dt$$

$$= \int \left(\hat\psi_0^{1*}\hat\psi_0^1 - \hat\psi_0^{2*}\hat\psi_0^2\right) \frac{d\omega}{2\pi}$$

check ind of x.

$$\int \left(\psi^* \varepsilon \psi\right)(x,t)\, dt = \int \left(e^{ixB}\hat\psi_0(\omega)\right)^* \varepsilon \left(e^{ixB}\hat\psi_0(\omega)\right) \frac{d\omega}{2\pi}$$

$$= \int \hat\psi_0(\omega)^* \varepsilon \hat\psi_0(\omega) \frac{d\omega}{2\pi}$$

But $iB = \begin{pmatrix} i\omega & -i \\ i & -i\omega \end{pmatrix} \in \mathrm{Lie}\, SU(1,1)$

This is a formula for $IH(\psi)$

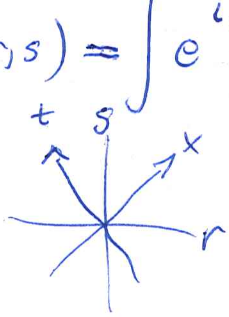
$\boxed{s=0}$

$$-\partial_r \psi^1 = i\psi^2 \qquad -\rho\hat\psi^1 = \hat\psi^2$$
$$\partial_s \psi^2 = i\psi^1 \qquad \sigma\hat\psi^2 = \hat\psi^1$$

exp. solutions $\quad \psi(r,s) = e^{i(r\rho - s\rho^{-1})}\begin{pmatrix} 1 \\ -\rho \end{pmatrix}$ const

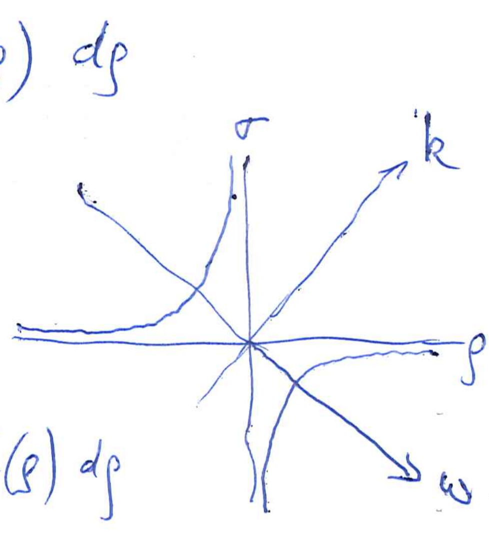
gen. soln. $\quad \psi(r,s) = \int e^{i(r\rho - s\rho^{-1})}\begin{pmatrix} 1 \\ -\rho \end{pmatrix} f(\rho)\, d\rho$

$$x = r+s \qquad t = -r+s$$



$$\partial_r = -\partial_t + \partial_x \qquad \partial_s = \partial_t + \partial_x$$

$$s = \frac{x+t}{2}, \quad r = \frac{x-t}{2}$$



$$\psi(x,t) = \int e^{i\left(x \underbrace{\frac{\rho-\rho^{-1}}{2}}_{k} - t \underbrace{\frac{\rho+\rho^{-1}}{2}}_{\omega}\right)} \begin{pmatrix} 1 \\ -\rho \end{pmatrix} f(\rho)\, d\rho$$

~~So what next?~~ continue with Cauchy problem

Want $-\partial_r \psi^1 = i\psi^2$ $\qquad \psi(r,0) = \begin{pmatrix}\psi_0^1(r)\\ \psi_0^2(r)\end{pmatrix}$

$\partial_s \psi^2 = i\psi^1$

$-\partial_r \psi^1(r,0) = i\psi^2(r,0) \qquad \therefore \quad \psi_0^2(r) = i\partial_r \psi_0^1(r)$

So the necessary Cauchy data consists of a function $\psi_0^1(r)$

$$\psi(r,s) = \int e^{i(r\rho - s\rho^{-1})}\begin{pmatrix}1\\ -\rho\end{pmatrix}\hat{\psi}_0^1(\rho)\, d\rho$$

So your solution method consists of taking $\psi_0^1(r)$, the Cauchy data, transf. to $\psi_0^1(r) = \int e^{ir\rho}\hat{\psi}_0^1(\rho)\frac{d\rho}{2\pi}$ and then

$$\psi(r,s) = \int e^{ir\rho}e^{-is\rho^{-1}}\begin{pmatrix}1\\ -\rho\end{pmatrix}\hat{\psi}_0^1(\rho)\frac{d\rho}{2\pi}$$

whence

$$\psi(x,t) = \int e^{i\left(x\frac{\rho-\rho^{-1}}{2} - t\frac{\rho+\rho^{-1}}{2}\right)}\begin{pmatrix}1\\ -\rho\end{pmatrix}\hat{\psi}_0^1(\rho)\frac{d\rho}{2\pi}$$

Energy $\quad \int \psi(x,0)^*\psi(x,0)\,dx =$

Wait $\qquad \psi^*\psi\, dx \qquad \psi^*\varepsilon\psi\, dt \qquad \begin{pmatrix}\partial_x & i\\ i & -\partial_x\end{pmatrix}$

$$\partial_t(\psi^*\psi) = (X\psi)^*\psi + \psi^* X\psi \qquad X = \varepsilon\partial_x + A$$

$$= (\varepsilon\partial_x\psi)^*\psi + \psi^*\varepsilon\partial_x\psi + \underbrace{(A\psi)^*}_{\psi^*(-A)}\psi + \psi^*(A\psi)$$

$$= \partial_x(\psi^*\varepsilon\psi)$$

$\psi^*\psi\,(dr+ds) + \psi^*\varepsilon\psi\,(-dr+ds)$ $\qquad \varepsilon = \begin{pmatrix}1&0\\0&-1\end{pmatrix}$

$= \psi^*(1-\varepsilon)\psi\,dr + \psi^*(1+\varepsilon)\psi\,ds$ $\qquad 1-\varepsilon = \begin{pmatrix}0&0\\0&2\end{pmatrix}$

$= 2\psi^{2*}\psi^2\,dr + 2\psi^{1*}\psi^1\,ds$ $\qquad 1+\varepsilon = \begin{pmatrix}2&0\\0&0\end{pmatrix}$

~~[illegible]~~ increasing staircase ~~[illegible]~~ $\qquad dr = ds = \frac{dx}{2}$

decreasing $\qquad -dr = +ds = \frac{dt}{2}$

---

set $t=0$. $\quad \psi(x,0) = \int e^{ix\frac{\rho-\rho^{-1}}{2}} \begin{pmatrix}1\\-\rho\end{pmatrix} f(\rho)\,\frac{d\rho}{2\pi}$

$$\psi(x,0)^* = \int e^{-ix\frac{\rho-\rho^{-1}}{2}} \begin{pmatrix}1\\-\rho\end{pmatrix}^* f(\rho)^*\,\frac{d\rho}{2\pi}$$

$$\int dx \int d\rho_1\,d\rho_2\; e^{-ix\left(\frac{\rho_1-\rho_1^{-1}}{2} - \frac{\rho_2-\rho_2^{-1}}{2}\right)} \begin{pmatrix}1\\-\rho_1\end{pmatrix}^* \begin{pmatrix}1\\\rho_2\end{pmatrix} f(\rho_1)^* f(\rho_2)$$

This looks to hard, but perhaps you can write things ~~[illegible]~~ in terms of k. Put $k = \frac{\rho-\rho^{-1}}{2}$ two ρ' values for each k.

ρ and $-\rho^{-1}$. Look at $\qquad A = \begin{pmatrix}k&1\\1&-k\end{pmatrix}$

$$\psi(x,t) = e^{t\begin{pmatrix}\partial_x & i\\ i & -\partial_x\end{pmatrix}}\psi_0(x) = \int e^{ikx} e^{itA}\hat{\psi}_0(k)\,\frac{dk}{2\pi}$$

$$= \int e^{ikx}\left\{ e^{i\omega t}\frac{1}{2\omega}\begin{pmatrix}\omega+k & 1\\ 1 & \omega-k\end{pmatrix} + e^{-i\omega t}\frac{1}{2\omega}\begin{pmatrix}\omega-k & -1\\ -1 & \omega+k\end{pmatrix}\right\}\hat{\psi}_0(k)$$

Your problem is to go between the repn.

$$\psi(x,t) = \int_{-\infty}^{\infty} e^{ikx}\, e^{it\begin{pmatrix} k & 1 \\ 1 & -k \end{pmatrix}} \hat{\psi}_0(k)\, \frac{dk}{2\pi}$$

and the repn.

$$\psi(x,t) = \int_{-\infty}^{\infty} e^{i\left(x\frac{\rho-\rho^{-1}}{2} - t\frac{\rho+\rho^{-1}}{2}\right)} \begin{pmatrix} 1 \\ -\rho \end{pmatrix} f(\rho)\, d\rho$$

Here $\hat{\psi}_0(k)$ consists of 2 functions of k. whereas $f(\rho)$ $\quad \frac{\rho-\rho^{-1}}{2} = k$ consists of 2 functions of k, namely for $0<\rho<\infty$ and $-\infty<\rho<0$. How can you get these things together

---

Review $\quad \partial_t\psi = \begin{pmatrix} \partial_x & i \\ i & -\partial_x \end{pmatrix}\psi \qquad \begin{aligned} -\partial_r\psi^1 &= i\psi^2 \\ \partial_s\psi^2 &= i\psi^1 \end{aligned}$

$$\psi(x,t) = e^{t\begin{pmatrix} \partial_x & i \\ i & -\partial_x \end{pmatrix}}\psi_0(x) = \int e^{ikx}\, e^{it\begin{pmatrix} k & 1 \\ 1 & -k \end{pmatrix}} \hat{\psi}_0(k)\, \frac{dk}{2\pi}$$

$$= \int e^{ikx}\, e^{it(\quad)} \int e^{-ikx'}\psi_0(x')\,dx'\, \frac{dk}{2\pi}$$

$$\cancel{\psi(x,t) = \int \frac{dk\,dx'}{2\pi}\, e^{ik(x-x')}\, e^{it\begin{pmatrix} k & 1 \\ 1 & -k \end{pmatrix}}}$$

$$\psi(x,t) = \int \underbrace{K(x-x', t-0)}\, \psi(x',0)\, dx'$$

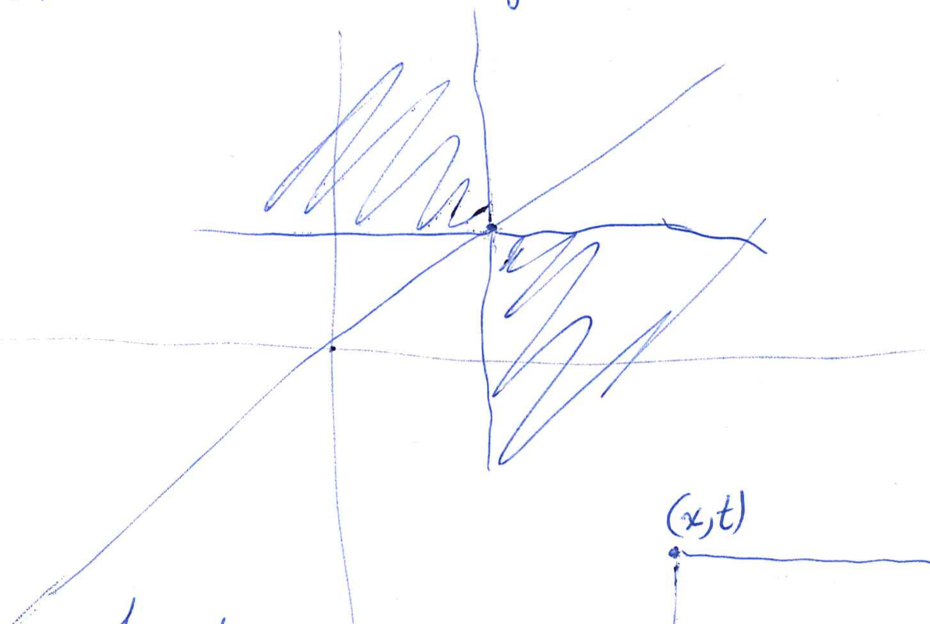
$$\int \frac{dk}{2\pi}\, e^{ik(x-x')}\, e^{it\begin{pmatrix} k & 1 \\ 1 & -k \end{pmatrix}}$$

What is your aim? To calculate $IH(\psi) = \int (\psi^* \varepsilon \psi)(x,t)\,dt$ for "any" global solution

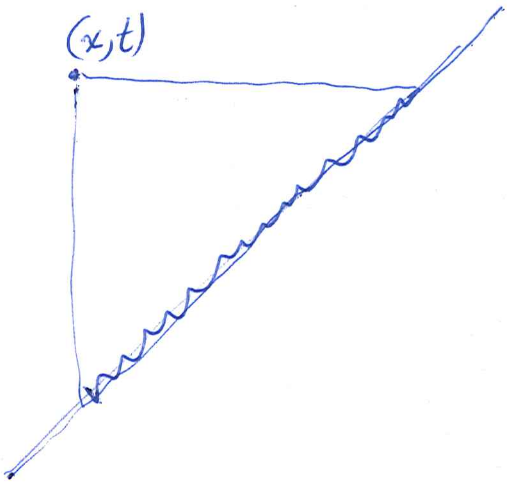
~~[illegible]~~ What do you know about global solutions? You should be able to prescribe $\psi(x,0)$ more or less arbitrarily, ~~in the regu~~ because the kernel of $e^{t\begin{pmatrix}\partial_x & i \\ i & -\partial_x\end{pmatrix}} : \psi_0(x) \longmapsto \psi_t(x)$

$$K(x,t;x',0) = \int \frac{dk}{2\pi}\, e^{ik(x-x')}\, e^{it(A)}$$

is supported in a light cone



better picture



Now I don't understand the class of solutions, but it's clear that the grid space consists of $\psi_0(x) \in C_c^\infty(\mathbb{R})$. ~~This is you first guess but it eliminates~~ In the mass zero case you want a group ring for $\mathbb{R}$.

~~[illegible]~~ So take $\varphi_0(x) \in C_c(\mathbb{R})$

look at $\quad \psi(x,t) = \left(e^{tD}\psi_0\right)(x) \qquad D = \begin{pmatrix} \partial_x & i \\ i & -\partial_x \end{pmatrix}$

can you calculate $\int \psi^* \varepsilon \psi \, dt$, is it defined

$$\psi(x,t) = \int e^{ikx} e^{itA} \hat{\psi}_0(k) \frac{dk}{2\pi}$$

~~[illegible]~~

$$\psi = e^{tD}\psi_0$$

$$\phi = e^{tD}\phi_0$$

$$\phi^* \varepsilon \psi$$

$$\psi(0,t) = \int \exp\left(it\begin{pmatrix} k & 1 \\ i & -k \end{pmatrix}\right) \hat{\psi}_0(k) \frac{dk}{2\pi}$$

$$\psi(0,t)^* = \int \frac{dk}{2\pi} \hat{\psi}_0(k)^* \exp\left(-it\begin{pmatrix} k & 1 \\ 1 & -k \end{pmatrix}\right)$$

$$\psi(0,t)^* \varepsilon \psi(0,t) = \int \frac{dk_1}{2\pi}\frac{dk_2}{2\pi} \hat{\psi}_0(k_1)^* e^{-it\overset{A(k_1)}{A}} \varepsilon e^{it\overset{A(k_2)}{A}} \hat{\psi}_0(k_2)$$

So what ~~[illegible]~~ is $\displaystyle\int_{-\infty}^{\infty} e^{-itA(k_1)} \varepsilon e^{itA(k_2)} dt$?

$$\int_{-\infty}^{\infty} \underbrace{e^{-it\begin{pmatrix} k_1 & 1 \\ 1 & -k_1 \end{pmatrix}}}_{\text{involves } e^{\pm i\omega_1 t}} \varepsilon \underbrace{e^{it\begin{pmatrix} k_2 & 1 \\ 1 & -k_2 \end{pmatrix}}}_{\text{involves } e^{\pm i\omega_2 t}} dt$$

$$\left[\begin{pmatrix} k & 1 \\ 1 & -k \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right] = \left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right]$$

$$= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = 2\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$F_t = e^{-it\begin{pmatrix} k & b \\ b & -k \end{pmatrix}} \varepsilon \, e^{it\begin{pmatrix} k & 1 \\ 1 & -k \end{pmatrix}}$$

$$\dot{F}_t = e^{-itA}[-iA, \varepsilon]\, e^{itA} = [-iA, F_t] \quad ?$$

$$e^{itA} = \cos(\omega t)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + i\,\frac{\sin(\omega t)}{\omega}\overbrace{\begin{pmatrix} k & 1 \\ 1 & -k \end{pmatrix}}^{A}$$

$$\varepsilon e^{itA} = \cos(\omega t)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + i\,\frac{\sin(\omega t)}{\omega}\begin{pmatrix} k & 1 \\ -1 & k \end{pmatrix}$$

$$e^{-itA} = \cos(\omega t)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} - i\,\frac{\sin(\omega t)}{\omega}\begin{pmatrix} k & 1 \\ 1 & -k \end{pmatrix}$$

$$e^{itA} = \cos(\omega t) + i\,\frac{\sin(\omega t)}{\omega} A$$

$$e^{-itA}\varepsilon e^{itA} = \left(\cos - i\,\frac{\sin}{\omega} A\right)\left((\cos)\varepsilon + i\,\frac{\sin}{\omega}\varepsilon A\right)$$

$$= \cos(\omega t)^2 \varepsilon + \frac{\sin^2(\omega t)}{\omega^2} A\varepsilon A$$

$$+ i\cos(\omega t)\frac{\sin(\omega t)}{\omega}(\varepsilon A - A\varepsilon)$$

$$A\varepsilon A = \begin{pmatrix} k & 1 \\ 1 & -k \end{pmatrix}\begin{pmatrix} k & 1 \\ -1 & k \end{pmatrix} = \begin{pmatrix} k^2-1 & 2k \\ 2k & 1-k^2 \end{pmatrix}$$

$$\varepsilon A - A\varepsilon = \begin{pmatrix} +k & 1 \\ -1 & +k \end{pmatrix} - \begin{pmatrix} k & -1 \\ 1 & k \end{pmatrix} = \begin{pmatrix} 0 & +2 \\ -2 & 0 \end{pmatrix}$$

Review: You are beginning to understand the continuous grid space situation. You now have a candidate for the grid space, namely, C_0^∞ Cauchy data on the space-like lines $t = \text{const}$.

Propagating from one space line to another should preserve C_0^∞ Cauchy data

You now want to calculate energy + IH.

Energy is easy in this representation:

$$\int \psi^*\psi \, dx \qquad \psi(x,t) = \underbrace{e^{t\begin{pmatrix} \partial_x & i \\ i & -\partial_x \end{pmatrix}}}_{\text{unitary}} \psi_0(x)$$

$x = \text{const}$ time like lines

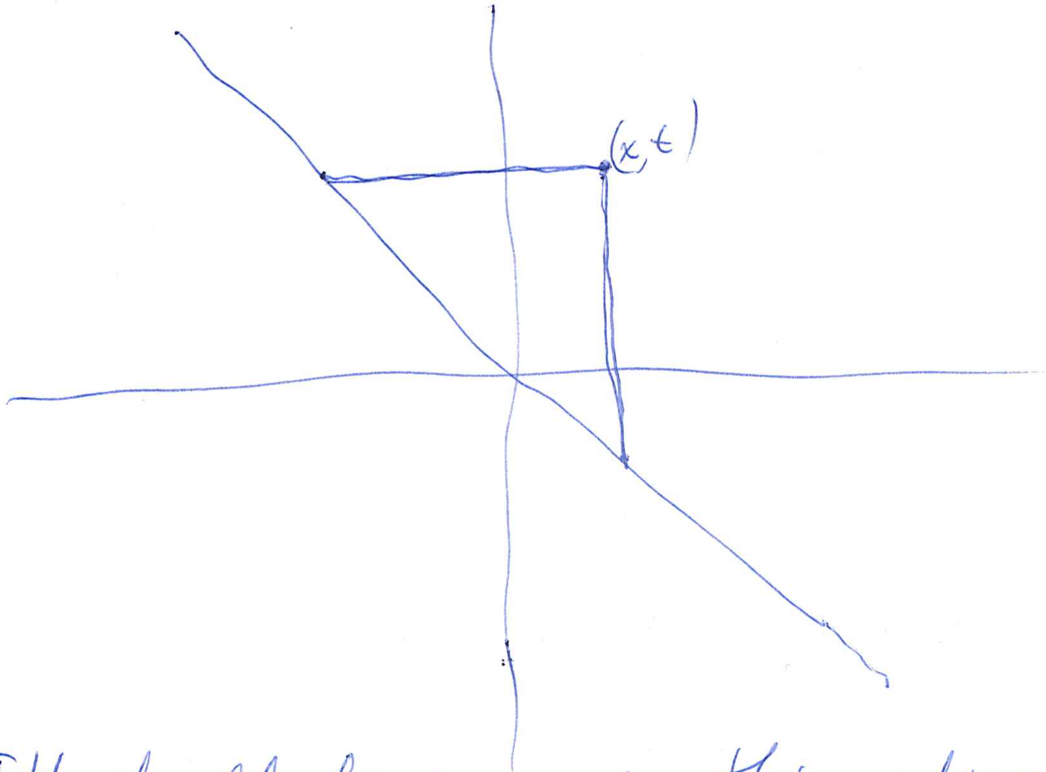
$$\psi(x,t) = e^{ix\begin{pmatrix} \partial_t & -i \\ +i & -\partial_t \end{pmatrix}} \psi(0,t)$$

$$= \int e^{i\omega t} \underbrace{e^{ix\begin{pmatrix} \omega & -1 \\ 1 & -\omega \end{pmatrix}}} \hat{\psi}(0,\omega)\frac{d\omega}{2\pi}$$

$$\underbrace{\cos(kx)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + i\,\frac{\sin(kx)}{k}\begin{pmatrix} \omega & -1 \\ 1 & -\omega \end{pmatrix}}$$

this is an entire function of ω with growth $e^{|x|}$

so again C_0^∞ on the line $x=0$ should be a candidate for your grid space.



IH should be easy in this representation

$$\psi(x,t) = \underbrace{e^{x\begin{pmatrix} \partial_t & -i \\ i & -\partial_t \end{pmatrix}}}_{\in SU(1,1).} \psi(0,t)$$

$\begin{pmatrix} i\omega & -i \\ i & -i\omega \end{pmatrix}$ conjugate diag. imaginary

$$\therefore \int \psi(x,t)^* \varepsilon \psi(x,t)\, dt = \int dt\, \psi(0,t)^* e^{x[\ \ ]^*} \varepsilon\, e^{x(\ \ )} \times \psi(0,t)$$

$$= \int \frac{d\omega}{2\pi}\, \hat{\psi}(0,\omega)^* e^{-ix\begin{pmatrix} \omega & -1 \\ 1 & -\omega \end{pmatrix}} \varepsilon\, e^{ix\begin{pmatrix} \omega & -1 \\ 1 & -\omega \end{pmatrix}} \hat{\psi}(0,\omega)$$

$$= \int \frac{d\omega}{2\pi}\, \hat{\psi}(0,\omega)^* \varepsilon \hat{\psi}(0,\omega) = \int dt\, \psi(0,t)^* \varepsilon \psi(0,t)$$

Now you want to link $t=0$ to $x=0$.
What do you expect to happen?

What is the problem? aim?

$$\partial_t \psi = \begin{pmatrix} \partial_x & i \\ i & -\partial_x \end{pmatrix} \psi \qquad\qquad \partial_x \psi = \begin{pmatrix} \partial_t & -i \\ i & -\partial_t \end{pmatrix} \psi$$

$$\psi(x,t) = e^{t\begin{pmatrix} \partial_x & i \\ i & -\partial_x \end{pmatrix}} \psi(x,0) \qquad\qquad \psi(x,t) = e^{x\begin{pmatrix} \partial_t & -i \\ i & \partial_t \end{pmatrix}} \psi(0,t)$$

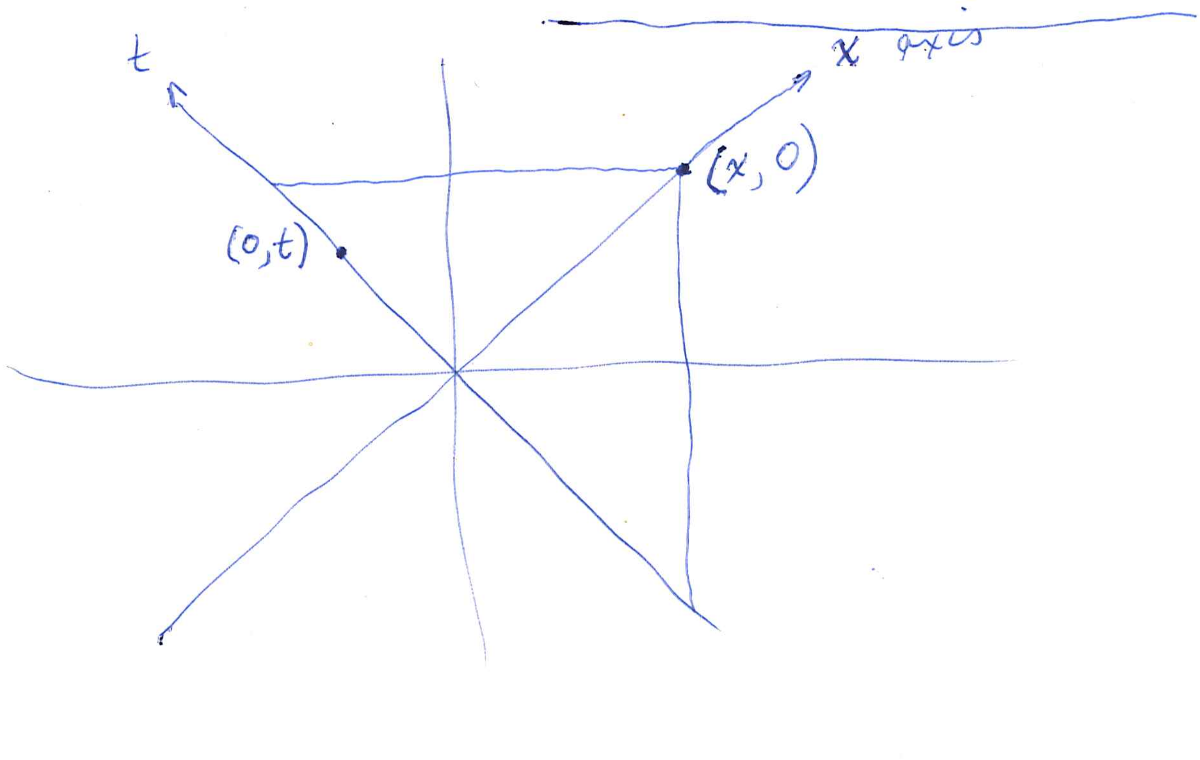
$$= \int \frac{dk}{2\pi} e^{ikx} e^{it\begin{pmatrix} k & 1 \\ 1 & -k \end{pmatrix}} \hat{\psi}(k,0) \qquad\qquad = \int \frac{d\omega}{2\pi} e^{i\omega t} e^{ix\begin{pmatrix} \omega & -1 \\ 1 & -\omega \end{pmatrix}} \hat{\psi}(0,\omega)$$

There should be a correspondence between ~~$\psi(0,t)$~~ Cauchy data $\begin{cases} \psi(0,t) & \text{on the line } x=0 \\ \psi(x,0) & \text{———— } t=0. \end{cases}$

What is the transform between these two?

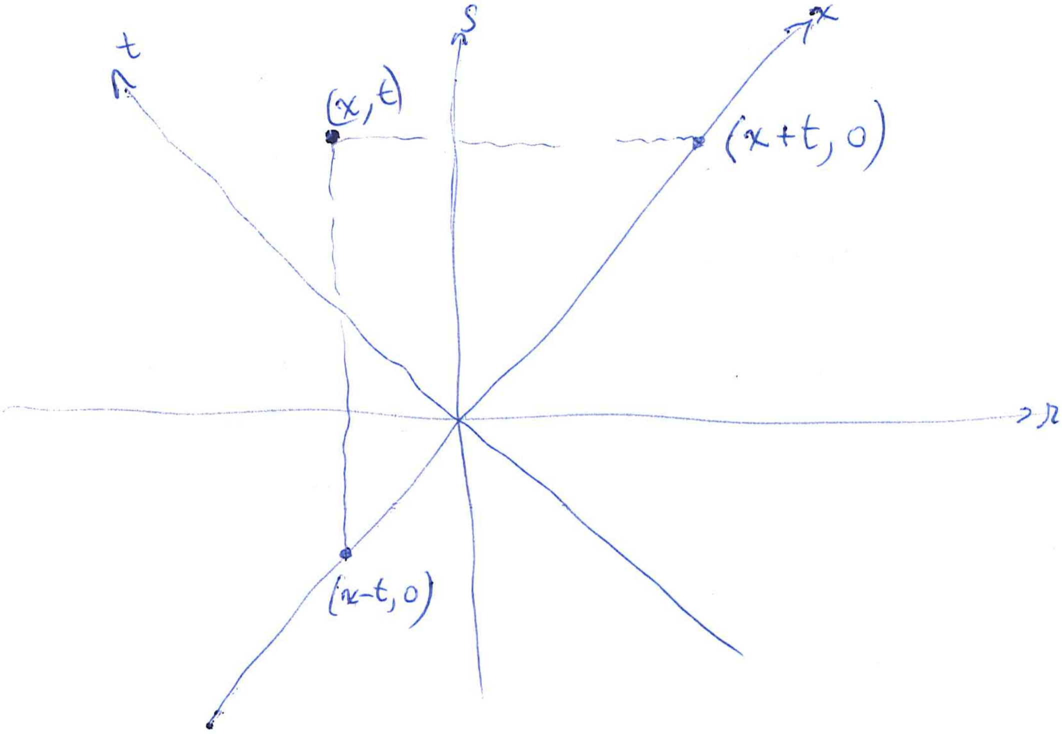
$$\psi(x,0) = \int \frac{d\omega}{2\pi} e^{ix\begin{pmatrix} \omega & -1 \\ 1 & -\omega \end{pmatrix}} \int dt' \, e^{-i\omega t'} \psi(0,t)$$

$$\psi(0,t) = \int \frac{dk}{2\pi} e^{it\begin{pmatrix} k & 1 \\ 1 & -k \end{pmatrix}} \int dx' e^{-ikx'} \psi(x',0)$$



The point is that $\phi(x,t)$ depends on $\phi(x',0)$ for $x-t < x' < x+t$

$x = +r+s$
$t = -r+s$



$\phi(x,t)$ depends on $\phi(x',0)$ for $|x'-x| < |t|$

So $\phi(0,t)$ depends on $\phi(x',0)$ for $|x'| < |t|$, so there's no hope that $\phi(x,0) \in C_c^\infty$ implies $\phi(0,t) \in C_c^\infty$. ~~In fact, if~~ ~~$\phi(0,t)=\delta(t)$~~

if $\phi(x,0) = \delta(x)$, then $\hat{\phi}(k,0) = 1$

$$\phi(x,t) = \int \frac{dk}{2\pi} e^{ikx} e^{it\begin{pmatrix} k & 1 \\ 1 & -k \end{pmatrix}}, \text{ so } \boxed{\phi(0,t) = \int \frac{dk}{2\pi} e^{it\begin{pmatrix} k & 1 \\ 1 & -k \end{pmatrix}}}$$

if $\phi(0,t) = \delta(t)$, then $\hat{\phi}(0,\omega) = 1$ and

$$\phi(x,t) = \int \frac{d\omega}{2\pi} e^{i\omega t} e^{ix\begin{pmatrix} \omega & -1 \\ 1 & -\omega \end{pmatrix}}, \boxed{\phi(x,0) = \int \frac{d\omega}{2\pi} e^{ix\begin{pmatrix} \omega & -1 \\ 1 & -\omega \end{pmatrix}}}$$

$$\psi(x,t) = \int e^{i\left(x\frac{\rho-\rho^{-1}}{2} - t\frac{\rho+\rho^{-1}}{2}\right)} \begin{pmatrix}1\\-\rho\end{pmatrix} f(\rho)\, d\rho$$

$$\psi(x,0) = \int e^{ix\frac{\rho-\rho^{-1}}{2}} \begin{pmatrix}1\\-\rho\end{pmatrix} f(\rho)\, d\rho$$

$$\psi(0,t) = \int e^{+it\frac{\rho+\rho^{-1}}{2}} \begin{pmatrix}1\\-\rho\end{pmatrix} f(\rho)\, d\rho$$

$$k = \frac{\rho-\rho^{-1}}{2}$$

$$\omega = \frac{\rho+\rho^{-1}}{2}$$

$$\rho = \omega + k$$

$$\rho^{-1} = \omega - k$$

$$\psi(x,0) = \int_{-\infty}^{\infty} e^{ix\frac{\rho-\rho^{-1}}{2}} \begin{pmatrix}1\\-\rho\end{pmatrix} f(\rho)\, d\rho$$

$$= \int_{-\infty}^{0} + \int_{0}^{\infty}$$

$$\int_0^\infty d\rho\; e^{ix\frac{\rho-\rho^{-1}}{2}} \begin{pmatrix}1\\-\rho\end{pmatrix} f(\rho) = \int_{-\infty}^{\infty} e^{ikx} \quad ?$$

$$k = \frac{\rho-\rho^{-1}}{2} \qquad dk = \frac{1+\rho^{-2}}{2}\, d\rho = \frac{\rho+\rho^{-1}}{2}\,\frac{d\rho}{\rho}$$

$$\frac{d\rho}{\rho} = \frac{dk}{\omega} \qquad =$$

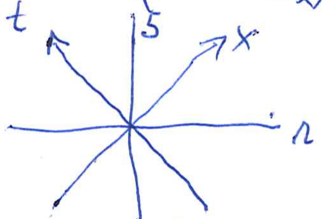
$$\begin{pmatrix}1\\-\omega-k\end{pmatrix} \sim \begin{pmatrix}-\omega+k\\1\end{pmatrix}$$

$$d\rho = \frac{\omega+k}{\omega}\, dk \qquad \begin{pmatrix}1\\-\rho\end{pmatrix} = \begin{pmatrix}1\\-\omega-k\end{pmatrix}$$

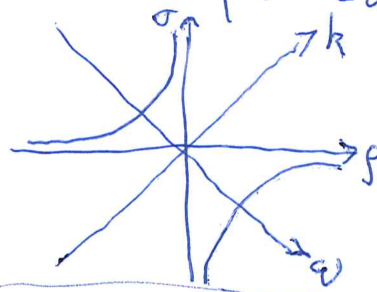
$$\int_{-\infty}^{\infty} \frac{\omega+k}{\omega}\, dk\; e^{ixk} \begin{pmatrix}1\\-\omega-k\end{pmatrix} = \int_{-\infty}^{\infty} dk\, e^{ixk} \Big(\;\omega+k$$

Review. $\partial_t \psi = \begin{pmatrix} \partial_x & i \\ i & -\partial_x \end{pmatrix} \psi \qquad \partial_x \psi = \begin{pmatrix} \partial_t & -i \\ i & -\partial_t \end{pmatrix} \psi$

$$x = r+s \qquad t = -r+s \qquad \partial_r = \partial_x - \partial_t \qquad \partial_s = \partial_x + \partial_t$$



$$r = \frac{x-t}{2} \qquad s = \frac{x+t}{2}$$



$$k = \frac{\rho-\rho^{-1}}{2} \qquad \omega = \frac{\rho+\rho^{-1}}{2} \qquad \rho = \omega + k \qquad \rho^{-1} = \omega - k$$

$$-\partial_r \psi^1 = i\psi^2 \qquad \partial_s \psi^2 = i\psi^1$$

$$\psi = e^{i(r\rho + s\sigma)} \hat{\psi} \qquad -\rho \hat{\psi}^1 = \hat{\psi}^2 \qquad \sigma \hat{\psi}^2 = \hat{\psi}^1$$

Get exp. solutions. $\psi = e^{i(r\rho - s\rho^{-1})} \begin{pmatrix} 1 \\ -\rho \end{pmatrix} = e^{i\left(x\left(\frac{\rho-\rho^{-1}}{2}\right) - t\left(\frac{\rho+\rho^{-1}}{2}\right)\right)} \begin{pmatrix} 1 \\ -\rho \end{pmatrix}$

What's important is the representation of solutions ψ via Cauchy data along $r=0$ or $s=0$. Use $r=0$.

There's an integral formula for the general solution

$$\psi(x,t) = \int \frac{d\rho}{2\pi} e^{i\left(x\frac{\rho-\rho^{-1}}{2} - t\frac{\rho+\rho^{-1}}{2}\right)} \begin{pmatrix} 1 \\ -\rho \end{pmatrix} f(\rho)$$

$$\psi(r,s) = \int \frac{d\rho}{2\pi} e^{i(r\rho - s\rho^{-1})} \begin{pmatrix} 1 \\ -\rho \end{pmatrix} f(\rho)$$

$$\psi^1(r,0) = \int \frac{d\rho}{2\pi} e^{ir\rho} f(\rho) \qquad \psi^2(r,0) = i\partial_r \psi^1(r,0)$$

Try to relate $\psi(x,0) = \int \frac{d\rho}{2\pi} e^{ix\left(\frac{\rho-\rho^{-1}}{2}\right)} \begin{pmatrix} 1 \\ -\rho \end{pmatrix} f(\rho)$ to f. Note $\psi(x,0)$ consists of 2 functions of x. You have a quadratic extension.

$$\rho \longmapsto k = \frac{\rho-\rho^{-1}}{2} \qquad \text{double cover of } k\text{-axis.}$$

$\rho > 0, \quad \rho < 0$

$$\psi(x,t) = e^{t\begin{pmatrix}\partial_x & i \\ i & -\partial_x\end{pmatrix}}\psi(x,0) = \int\frac{dk}{2\pi}\, e^{ikx}\, e^{it\begin{pmatrix}k & 1 \\ 1 & -k\end{pmatrix}}\hat{\psi}_0(k)$$

$$= \int\frac{dk}{2\pi}\, e^{ikx}\left[\frac{e^{i\omega t}}{2\omega}\begin{pmatrix}\omega+k & 1 \\ 1 & \omega-k\end{pmatrix} + \frac{e^{-i\omega t}}{2\omega}\begin{pmatrix}\omega-k & -1 \\ -1 & \omega+k\end{pmatrix}\right]\hat{\psi}_0(k)$$

$$= \int\frac{dk}{2\pi}\, e^{i(kx+\omega t)}\underbrace{\frac{1}{2\omega}\begin{pmatrix}\omega+k & 1 \\ 1 & \omega-k\end{pmatrix}}_{\frac{\omega+A}{2\omega}}\hat{\psi}_0(k) + e^{i(kx-\omega t)}\underbrace{\frac{1}{2\omega}\begin{pmatrix} & \\ & \end{pmatrix}}_{\frac{\omega-A}{2\omega}}\hat{\psi}_0(k)$$

$$\boxed{\psi(x,t) = \int\frac{dk}{2\pi}\, e^{i(kx+\omega t)}\frac{\omega+A}{2\omega}\hat{\psi}_0(k) + \int\frac{dk}{2\pi}\, e^{i(kx-\omega t)}\frac{\omega-A}{2\omega}\hat{\psi}_0(k)}$$

Contrast with

$$\psi(x,t) = \int\frac{d\rho}{2\pi}\, e^{i\left(x\left(\frac{\rho-\rho^{-1}}{2}\right) - t\left(\frac{\rho+\rho^{-1}}{2}\right)\right)}\begin{pmatrix}1 \\ -\rho\end{pmatrix} f(\rho).$$

$$= \int_0^\infty\frac{d\rho}{2\pi}\, e^{i(kx-\omega t)}\begin{pmatrix}1 \\ -\rho\end{pmatrix} f(\rho) + \int_{-\infty}^0\frac{d\rho}{2\pi}\, e^{i(kx-\omega t)}\begin{pmatrix}1 \\ -\rho\end{pmatrix} f(\rho)$$

(in this integral you want to change variable

$$k = \frac{\rho-\rho^{-1}}{2} \qquad \omega = \frac{\rho+\rho^{-1}}{2} \qquad k+\omega = \rho$$

$$\cancel{\omega+A} \qquad \frac{\omega-A}{2\omega} = \frac{1}{2\omega}\begin{pmatrix}\omega-k & -1 \\ -1 & \omega+k\end{pmatrix}$$ projects onto $\begin{pmatrix}1 \\ -\omega-k\end{pmatrix}\mathbb{C}$

$$dk = \frac{1+\rho^{-2}}{2}\, d\rho = \omega\frac{d\rho}{\rho}$$

$$\int_{-\infty}^{0} \frac{d\rho}{2\pi} e^{i(kx-\omega t)} \begin{pmatrix} 1 \\ -\rho \end{pmatrix} f(\rho)$$

$k = \frac{\rho-\rho^{-1}}{2}$

change $\rho \longmapsto -\rho^{-1}$ so as not to change k, but $\omega \mapsto -\omega$

$$= \int^{\infty} \frac{1}{2\pi} d(-\rho^{-1}) e^{i(kx+\omega t)} \begin{pmatrix} 1 \\ \rho^{-1} \end{pmatrix} f(-\rho^{-1})$$

$$= \int_0^{\infty} \frac{1}{2\pi} \frac{1}{\rho^2} d\rho \; e^{i(kx+\omega t)} \begin{pmatrix} 1 \\ \rho^{-1} \end{pmatrix} f(-\rho^{-1})$$

so change f.

$$\int_{-\infty}^{0} \frac{1}{2\pi} \frac{d\rho}{\rho} e^{i(kx-\omega t)} \begin{pmatrix} 1 \\ -\rho \end{pmatrix} f(\rho)$$

$$= \int_0^{\infty} \frac{1}{2\pi} \left(-\frac{d\rho}{\rho}\right) e^{i(kx+\omega t)} \begin{pmatrix} 1 \\ +\rho^{-1} \end{pmatrix} f(-\rho^{-1})$$

$$\psi(x,t) = \int_{-\infty}^{\infty} \frac{d\rho}{2\pi\rho} e^{i(kx-\omega t)} \begin{pmatrix} 1 \\ -\rho \end{pmatrix} f(\rho) \quad = \int_0^{\infty} + \int_{-\infty}^{0}$$

$$\int_0^{\infty} = \int_{-\infty}^{\infty} \frac{1}{2\pi} \frac{dk}{\omega} e^{i(kx-\omega t)} \begin{pmatrix} 1 \\ -\omega-k \end{pmatrix} f(\omega+k)$$

$$\int_{-\infty}^{0} = \int_0^{\infty} \frac{1}{2\pi} \left(-\frac{d\rho}{\rho}\right) e^{i(kx+\omega t)} \begin{pmatrix} 1 \\ +\rho^{-1} \end{pmatrix} f(-\rho^{-1})$$

$$= \int_{-\infty}^{\infty} \frac{1}{2\pi} \frac{dk}{(-\omega)} e^{i(kx+\omega t)} \begin{pmatrix} 1 \\ \omega-k \end{pmatrix} f(-\omega+k)$$

$\therefore$ have

$$\psi(x,t) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ikx} \left(e^{-i\omega t} \frac{1}{\omega} \begin{pmatrix} 1 \\ -\omega-k \end{pmatrix} f(\omega+k) + e^{i\omega t} \frac{1}{-\omega} \begin{pmatrix} 1 \\ \omega-k \end{pmatrix} f(-\omega+k) \right)$$

Repeat. $$\psi(x,t) = \int_{-\infty}^{\infty} \frac{d\rho}{2\pi\rho}\, e^{i(kx-\omega t)} \begin{pmatrix} 1 \\ -\rho \end{pmatrix} f(\rho)$$

Here $k = \frac{\rho - \rho^{-1}}{2}$ $\qquad \{0<\rho<\infty\} \xrightarrow{\sim} \{-\infty<k<\infty\}$

$\omega = \frac{\rho+\rho^{-1}}{2}$ $\qquad \{-\infty<\rho<0\} \xrightarrow{\sim}$

$$dk = \frac{1+\rho^{-2}}{2}\, d\rho = \frac{\rho+\rho^{-1}}{2}\,\frac{d\rho}{\rho} = \omega\,\frac{d\rho}{\rho}$$

$\rho = \omega + k$

$\rho^{-1} = \omega - k$

$$\psi(x,t) = \int_0^\infty + \int_{-\infty}^0$$

$\int_{-\infty}^{0} \frac{1}{2\pi}\frac{d\rho}{\rho} = \int_0^\infty \frac{1}{2\pi}\frac{d\rho}{\rho}(-1)$ $\quad \frac{dk}{-\omega}$

OKAY

$$= \int_{-\infty}^{\infty} \frac{dk}{2\pi} \left\{ \frac{1}{\omega} e^{i(kx-\omega t)} \begin{pmatrix} 1 \\ -\omega-k \end{pmatrix} f(\omega+k) \right.$$
$$\left. + \frac{1}{-\omega} e^{i(kx+\omega t)} \begin{pmatrix} 1 \\ \omega-k \end{pmatrix} f(-\omega+k) \right\}$$

in this formula $\omega = +\sqrt{k^2+1}$, $f(\omega+k) = f(\rho)$ for $\rho>0$ and $f(-\omega+k) = f(\rho)$ for $\rho<0$.

So we know that

$$\frac{\omega+A}{2\omega}\,\hat{\varphi}_0(k) = \frac{-1}{\omega}\begin{pmatrix} 1 \\ \omega-k \end{pmatrix} f(-\omega+k)$$

complementary projections

$$\frac{\omega-A}{2\omega}\,\hat{\varphi}_0(k) = \frac{1}{\omega}\begin{pmatrix} 1 \\ -\omega-k \end{pmatrix} f(\omega+k)$$

Review $$\psi(x,t) = \int \frac{d\rho}{2\pi\rho}\, e^{i(kx-\omega t)} \begin{pmatrix} 1 \\ -\rho \end{pmatrix} f(\rho)$$

$k = \frac{\rho-\rho^{-1}}{2}$

$\omega = \frac{\rho+\rho^{-1}}{2}$

$\frac{dk}{\omega} = \frac{d\rho}{\rho}$

$$= \int_{-\infty}^{\infty} \frac{dk}{2\pi} \left\{ \frac{e^{i(kx-\omega t)}}{\omega} \begin{pmatrix} 1 \\ -\omega-k \end{pmatrix} f(\omega+k) + \frac{e^{i(kx+\omega t)}}{-\omega} \begin{pmatrix} 1 \\ \omega-k \end{pmatrix} f(-\omega+k) \right\}$$

Can I use this to calculate $\int_{-\infty}^{\infty}(\psi^*\psi)(x,t)\,dx$ which should be ind of t.

$$e^{it\begin{pmatrix}k & 1\\ 1 & -k\end{pmatrix}} = e^{i\omega t}\frac{1}{2\omega}(\omega+A) + e^{-i\omega t}\frac{1}{2\omega}(\omega-A)$$

$$= e^{i\omega t}\frac{1}{2\omega}\begin{pmatrix}\omega+k & 1\\ 1 & \omega-k\end{pmatrix} + e^{-i\omega t}\frac{1}{2\omega}\begin{pmatrix}\omega-k & -1\\ -1 & \omega+k\end{pmatrix}$$

$A = \begin{pmatrix}k & 1\\ 1 & -k\end{pmatrix}$ is self-adjoint, so its eigenspaces

$\begin{pmatrix}1\\ \omega-k\end{pmatrix}\mathbb{C}$ and $\begin{pmatrix}+1\\ -\omega-k\end{pmatrix}\mathbb{C}$ are $\perp$

for $\psi^*\psi$: $1-\omega^2+k^2=0$. So

$$\int(\psi^*\psi)(x,t)\,dx = \int_{-\infty}^{\infty}\frac{dk}{2\pi}\frac{1}{\omega^2}\Big(\overbrace{\big(1+(\omega+k)^2\big)}^{1+k^2+2k\omega+\omega^2 \,=\, 2\omega(\omega+k)}|f(\omega+k)|^2 + \underbrace{\big(1+(\omega-k)^2\big)}_{2\omega(\omega-k)}|f(-\omega+k)|^2\Big)$$

$$= \int_{-\infty}^{\infty}\frac{dk}{\pi\omega}\Big((\omega+k)|f(\omega+k)|^2 + (\omega-k)|f(-\omega+k)|^2\Big)$$

$$= \int_0^{\infty}\frac{d\rho}{\pi\rho}\Big(\rho\,|f(\rho)|^2 + \rho^{-1}|f(-\rho^{-1})|^2\Big)$$

$$= \int_0^{\infty}\frac{d\rho}{\pi}|f(\rho)|^2 + \int_{-\infty}^{0}\Big(-\frac{d\rho}{\pi\rho}\Big)\Big(-\rho\,|f(\rho)|^2\Big)$$

$$= \int_{-\infty}^{\infty}\frac{d\rho}{\pi}|f(\rho)|^2 .$$

Change conventions – essentially the sign of f, ~~thus~~. since you have already changed original $f(\rho)$ to $\frac{1}{\rho}f(\rho)$. NOW

$$\psi(x,t) = \int_{-\infty}^{\infty} \frac{d\rho}{2\pi}\, e^{i(kx-\omega t)} \begin{pmatrix} -\rho^{-1} \\ 1 \end{pmatrix} f(\rho)$$

Hopefully this is consistent with your choice in the half continuous case. In any case I seem to have shown that

$$\int \psi^* \psi \, dt = \int_{-\infty}^{\infty} \frac{d\rho}{\pi} |f(\rho)|^2$$

Check the calculation.

$$EN\left(\psi_f(x,t)\right) = EN\left(\psi_{e^{-i\omega t}f}(x,0)\right) = \int \frac{d\rho}{\pi} \left| e^{-i\omega t} f \right|^2 = \int \frac{d\rho}{\pi} |f|^2.$$

$$\psi(x,0) = \int \frac{d\rho}{2\pi}\, e^{ikx} \begin{pmatrix} -\rho^{-1} \\ 1 \end{pmatrix} f(\rho)$$

$$\int dx\, \psi(x,0)^* \psi(x,0) = \int dx \int \frac{d\rho_1}{2\pi} f(\rho_1)^* \begin{pmatrix} -\rho_1^{-1} \\ 1 \end{pmatrix}^* e^{-ik_1 x} \int \frac{d\rho_2}{2\pi} e^{ik_2 x} \begin{pmatrix} -\rho_2^{-1} \\ 1 \end{pmatrix} f(\rho_2)$$

looks too hard

$$\psi(x,t) = \int_0^{\infty} \frac{d\rho}{2\pi}\, e^{ikx} \begin{pmatrix} -\frac{1}{\rho} \\ 1 \end{pmatrix} f(\rho) + \int_{-\infty}^{0} \frac{d\rho}{2\pi}\, e^{ikx} \begin{pmatrix} -\frac{1}{\rho} \\ 1 \end{pmatrix} f(\rho)$$

$$= \int_{-\infty}^{\infty} \frac{dk}{2\pi}\, \frac{k}{\omega}\, e^{ikx} \begin{pmatrix} -1 \\ \omega+k \end{pmatrix} f(\omega+k) + \int_{-\infty}^{\infty} \frac{dk}{2\pi\omega}\, e^{ikx} \begin{pmatrix} -1 \\ -\omega+k \end{pmatrix} f(-\omega+k)$$

$$\rho = -\omega + k$$

$$= \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ikx} \; \underbrace{\frac{1}{\omega}\left[\begin{pmatrix}-1\\ \omega+k\end{pmatrix} f(\omega+k) + \begin{pmatrix}-1\\ -\omega+k\end{pmatrix} f(-\omega+k)\right]}$$

$$\frac{1}{\omega}\begin{pmatrix}-1 & -1\\ \omega+k & -\omega+k\end{pmatrix}\begin{pmatrix}f(\omega+k)\\ f(-\omega+k)\end{pmatrix}$$

$$\begin{pmatrix}-1 & \omega+k\\ -1 & -\omega+k\end{pmatrix}\begin{pmatrix}-1 & -1\\ \omega+k & -\omega+k\end{pmatrix} = \begin{pmatrix}\overset{2\omega^2+2\omega k}{1+(\omega+k)^2} & 0\\ 0 & \underset{2\omega^2-2\omega k}{1+(-\omega+k)^2}\end{pmatrix}$$

$$\therefore \int_{-\infty}^{\infty} (\psi\psi^*)(x,0)\,dx = \int_{-\infty}^{\infty} \frac{dk}{2\pi}\,\frac{1}{\omega^2}\Big(2(\omega^2+\omega k)|f(\omega+k)|^2 + 2(\omega^2-\omega k)|f(\omega+k)|^2\Big)$$

$$= \int_{-\infty}^{\infty} \frac{dk}{\pi}\,\frac{1}{\omega}\Big((\omega+k)|f(\omega+k)|^2 + (\omega-k)|f(-\omega+k)|^2\Big)$$

You should be able to find a ρ proof.
Start with

$$\psi(x,t) = \int_{-\infty}^{\infty} \frac{d\rho}{2\pi} e^{i(kx-\omega t)} \begin{pmatrix}-\rho^{-1}\\ 1\end{pmatrix} f(\rho)$$

better

$$\psi(x,0) = \int_{-\infty}^{\infty} \frac{d\rho}{2\pi} e^{ikx} \begin{pmatrix}-\rho^{-1}\\ 1\end{pmatrix} f(\rho)$$

You want to split this

$$\psi(x,0) = \int_0^{\infty} \frac{d\rho}{2\pi} e^{ikx}\begin{pmatrix}-\frac{1}{\rho}\\ 1\end{pmatrix} f(\rho) + \int_{-\infty}^{0} \frac{d\rho}{2\pi} e^{ikx}\begin{pmatrix}-\frac{1}{\rho}\\ 1\end{pmatrix} f(\rho)$$

$$= \int_0^{\infty} \frac{d(-\rho^{-1})}{2\pi} e^{ikx}\begin{pmatrix}\rho\\ 1\end{pmatrix} f(-\rho^{-1})$$

$$\cancel{\int_0^\infty \frac{1}{2\pi}\frac{d\rho}{\rho^2}\, e^{ikx}\begin{pmatrix}\rho\\ 1\end{pmatrix} f(-\rho^{-1})}$$

$$\cancel{\psi(x,0) = \int_0^\infty \frac{d\rho}{2\pi\rho}\, e^{ikx}\left[\begin{pmatrix}-1\\ \rho\end{pmatrix} f(\rho) + \begin{pmatrix}\rho\\ 1\end{pmatrix} f(-\rho^{-1})\right]}$$

$$\psi(x,0) = \int_0^\infty \frac{d\rho}{2\pi}\, e^{ikx}\begin{pmatrix}-\frac{1}{\rho}\\ 1\end{pmatrix} f(\rho) + \int_{-\infty}^0$$

$$= \int_0^\infty \frac{d\rho}{2\pi\rho}\, e^{ikx}\begin{pmatrix}-1\\ \rho\end{pmatrix} f(\rho) + \int_{-\infty}^0 \underbrace{\frac{d(-\rho^{-1})}{2\pi}}_{\frac{1}{2\pi}\frac{d\rho}{\rho^2}}\, e^{ikx}\begin{pmatrix}\rho\\ 1\end{pmatrix} f(-\rho^{-1})$$

$$= \int_0^\infty \frac{d\rho}{2\pi\rho}\, e^{ikx}\left\{\begin{pmatrix}-1\\ \rho\end{pmatrix} f(\rho) + \begin{pmatrix}1\\ \rho^{-1}\end{pmatrix} f(-\rho^{-1})\right\}$$

orthogonal for $\psi^*\psi$

Rost uses something like

$$\int dx\, e^{ik_1x}\, e^{-ik_2x} = 2\pi\,\delta(k_1-k_2)$$
$$= \rho\,\frac{2\pi}{\omega_{(2)}}\,\delta(\rho_1-\rho_2)$$

$$\int dx \int \frac{d\rho_1}{2\pi\rho_1}\, e^{-ik_1x}\int\frac{d\rho_2}{2\pi\rho_2}\, e^{ik_2x}\,\cancel{\begin{pmatrix}-1\\ \rho_1\end{pmatrix} f(\rho_1)}$$

$$\begin{pmatrix}f(\rho_1)\\ f(-\rho_1^{-1})\end{pmatrix}^{*}\underbrace{\begin{pmatrix}-1 & \rho_1\\ 1 & \rho_1^{-1}\end{pmatrix}\begin{pmatrix}-1 & 1\\ \rho_2 & \rho_2^{-1}\end{pmatrix}}\begin{pmatrix}f(\rho_2)\\ f(-\rho_2^{-1})\end{pmatrix}$$

$$\int \frac{d\rho_1}{2\pi\rho_1}\,\frac{d\rho_2}{2\pi\rho_2}\,\rho\,\frac{2\pi\,\delta(\rho_1-\rho_2)}{\omega}$$

Upshot seems to be.

$$\int_0^\infty \frac{d\rho}{2\pi\rho}\frac{1}{\omega}\left((1+\rho^2)|f(\rho)|^2+(1+\rho^{-2})|f(-\rho^{-1})|^2\right)$$

$$\int_0^\infty \frac{d\rho}{\pi}\left(|f(\rho)|^2\right)+\int_0^\infty \underbrace{\frac{d\rho}{\pi\rho}}\frac{1+\rho^{-2}}{2\omega}|f(-\rho^{-1})|^2$$

$$\frac{d\rho}{\pi\rho^2} = \frac{1}{\pi}d(-\rho^{-1})$$

Seems to justify

$$\int dx\, e^{(-ik_1+ik_2)x} = 2\pi\,\delta(\rho_1-\rho_2)\frac{\rho_2}{\omega_2}$$

So now onward to IH.

$$\psi(x,t)=\int\frac{d\rho}{2\pi}e^{i(kx-\omega t)}\begin{pmatrix}-\rho^{-1}\\1\end{pmatrix}f(\rho)$$

$$B^2=\begin{pmatrix}\omega & -1\\1 & -\omega\end{pmatrix}^2$$

$$\psi(0,t)=\int_{-\infty}^{\infty}\frac{d\rho}{2\pi}e^{-i\omega t}\begin{pmatrix}-\rho^{-1}\\1\end{pmatrix}f(\rho)$$

$$\begin{pmatrix}\omega & -1\\1 & -\omega\end{pmatrix}\begin{pmatrix}\omega & -1\\1 & -\omega\end{pmatrix}=\begin{pmatrix}k^2 & 0\\0 & k^2\end{pmatrix}$$

to see what's going on you need.

$$\psi(x,t)=e^{x\begin{pmatrix}\partial_t & -i\\i & -\partial_t\end{pmatrix}}\psi(0,t)=\int\frac{d\omega}{2\pi}e^{i\omega t}e^{ix\overbrace{\begin{pmatrix}\omega & -1\\1 & -\omega\end{pmatrix}}^{B}}\hat\psi(0,\omega)$$

$B^2=k^2I$.

$$\boxed{e^{ikx}\frac{k+B}{2k}+e^{-ikx}\frac{k-B}{2k}=e^{ixB}}$$

$$e^{ikx}\frac{1}{2k}\begin{pmatrix}k+\omega & -1\\1 & k-\omega\end{pmatrix}+e^{-ikx}\frac{1}{2k}\begin{pmatrix}k-\omega & 1\\-1 & k+\omega\end{pmatrix}$$

~~[illegible]~~ Next you ~~take~~ relate

$$\psi(0,t) = \int_{-\infty}^{\infty} \frac{d\rho}{2\pi}\, e^{-i\omega t}\begin{pmatrix}-\rho^{-1}\\ 1\end{pmatrix} f(\rho)$$

better $$\psi(x,t) = \int \frac{d\rho}{2\pi}\, e^{i(kx-\omega t)}\begin{pmatrix}-\rho^{-1}\\ 1\end{pmatrix} f(\rho)$$

and $$\psi(x,t) = \int \frac{d\omega}{2\pi}\, e^{i\omega t}\left\{ e^{ikx}\frac{1}{2k}\begin{pmatrix}k+\omega & -1\\ 1 & k-\omega\end{pmatrix} + e^{-ikx}\frac{1}{2k}\begin{pmatrix}k-\omega & 1\\ -1 & k+\omega\end{pmatrix}\right\}\hat{\psi}(0,\omega)$$

$$B = \begin{pmatrix}\omega & -1\\ 1 & -\omega\end{pmatrix} = k\,\frac{\cancel{k}+B}{2\cancel{k}} - k\,\frac{\cancel{k}-B}{2\cancel{k}}$$

$$e^{ixB} = e^{ixk}\,\frac{k+B}{2k} + e^{-ixk}\,\frac{k-B}{2k}$$

$$= \cos(kx)\,I + i\,\frac{\sin(kx)}{k}\,B \qquad k = \sqrt{\omega^2-1}$$

IDEA: Can you get all solutions ~~[illegible]~~ $\psi(x,t)$ in the form $\psi(x,t) = \int \frac{d\rho}{2\pi}\, e^{i(kx-\omega t)}\begin{pmatrix}-\rho^{-1}\\ 1\end{pmatrix} f(\rho)$ by including complex ρ. ~~So what~~

~~[illegible]~~ Review: studying $\partial_t\psi = \begin{pmatrix}\partial_x & i\\ i & -\partial_x\end{pmatrix}\psi$ $\quad \partial_x\psi = \begin{pmatrix}\partial_t & -i\\ i & -\partial_t\end{pmatrix}\psi$ $\quad -\partial_x\psi^1 = i\psi^2$, $\partial_t\psi^2 = i\psi^1$

exponential solutions

$$\psi(x,t) = e^{t\begin{pmatrix}\partial_x & i\\ i & -\partial_x\end{pmatrix}}\psi(x,0) = \int_{-\infty}^{\infty}\frac{dk}{2\pi}\, e^{ikx}\, e^{itA}\,\hat{\psi}(k,0)$$

$$A = \begin{pmatrix}k & 1\\ 1 & -k\end{pmatrix} \qquad e^{itA} = e^{i\omega t}\,\frac{\omega+A}{2\omega} + e^{-i\omega t}\,\frac{\omega-A}{2\omega}$$

$$= \cos(\omega t) + i\,\frac{\sin(\omega t)}{\omega}\,A$$

exp. solutions $\quad e^{i(kx+\omega t)}\begin{pmatrix}\omega+k\\ 1\end{pmatrix} \qquad e^{i(kx+\omega t)}\begin{pmatrix}\omega-k\\ 1\end{pmatrix}$

basic exponential solutions

$$e^{i\omega t}\, e^{ikx}\begin{pmatrix}\omega+k\\ 1\end{pmatrix} \qquad e^{i\omega t}\, e^{-ikx}\begin{pmatrix}\omega-k\\ 1\end{pmatrix}$$

$$e^{-i\omega t}\, e^{-ikx}\begin{pmatrix}-\omega-k\\ 1\end{pmatrix} \qquad e^{-i\omega t}\, e^{ikx}\begin{pmatrix}-\omega+k\\ 1\end{pmatrix}$$

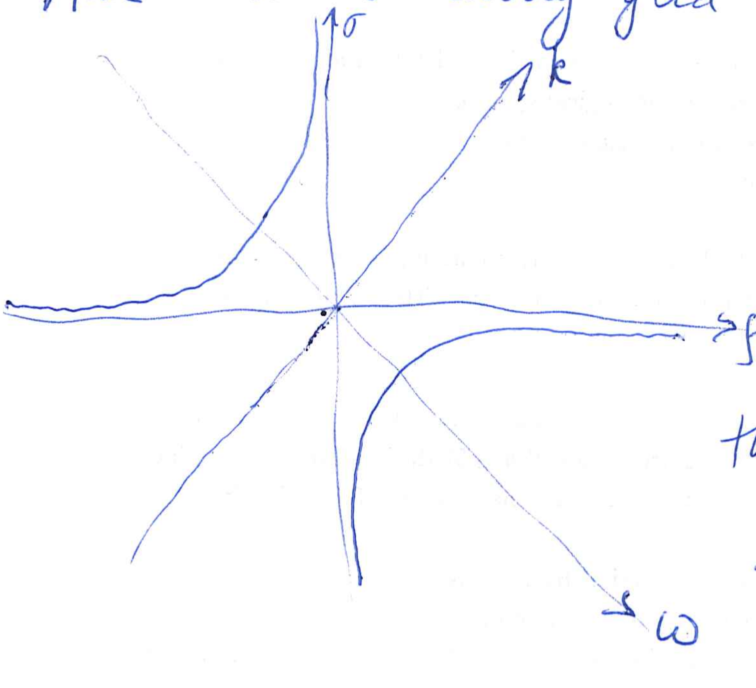
$$e^{i\left(x\left(\frac{\rho-\rho^{-1}}{2}\right)-t\left(\frac{\rho+\rho^{-1}}{2}\right)\right)}\begin{pmatrix}-\rho^{-1}\\ 1\end{pmatrix} f(\rho). \qquad \begin{aligned}\rho&=\omega+k\\ \rho^{-1}&=\omega-k\end{aligned}$$

$$e^{i(kx-\omega t)}\begin{pmatrix}-\omega+k\\ 1\end{pmatrix} f(\omega+k)$$

you get this line of exp. solution for each (ω, k) $\ni$ $\omega^2=k^2+1$, better for each $\rho\in\mathbb{C}^\times$

$(\omega+k)(\omega-k)=1.$

Aim: Understanding grid space, determine IH.



If ρ is required to be real, then $|\omega|\geq 1$, so it's not clear any $\psi(x,t)$ soln. has the form

$$\psi(x,t)=\int_{-\infty}^{\infty} e^{i(kx-\omega t)}\begin{pmatrix}-\rho^{-1}\\ 1\end{pmatrix} f(\rho)\,d\rho$$

So you should add ~~imaginary~~ $\rho=e^{i\theta}$ $\qquad \begin{aligned}\omega&=\cos\theta\\ k&=i\sin\theta\end{aligned}$

Looks good.

contour integral approach to eigenfunction expansion for $\begin{pmatrix} \partial_t & -i \\ i & -\partial_t \end{pmatrix}$ $\qquad \begin{pmatrix} \omega & -1 \\ 1 & -\omega \end{pmatrix}$

spectrum should be $k = \pm\sqrt{\omega^2 - 1}$ $\quad \omega \in \mathbb{R}$,

How to do this? ~~Your problem is to take~~

What happens?

Return to $\quad \psi(x,t) = e^{x\begin{pmatrix} \partial_t & -i \\ +i & -\partial_t \end{pmatrix}} \psi(0,t)$

$$= \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{i\omega t} \underbrace{e^{ix\begin{pmatrix} \omega & -1 \\ 1 & -\omega \end{pmatrix}}} \hat{\psi}(0,\omega)$$

$$e^{ixB} = e^{ikx}\frac{k+B}{2k} + e^{-ikx}\frac{k-B}{2k}$$

$$= \cos(kx) + i\frac{\sin(kx)}{k} B \qquad \text{entire fn of } \omega$$

probably bounded by $\quad e^{|\mathrm{Im}(k) x|} \qquad e^{|\mathrm{Im}(\omega) x|}$

You don't understand Laplace T. theory as well as you should. Use LT on IVP:

$$\partial_x \psi = \begin{pmatrix} \partial_t & -i \\ i & -\partial_t \end{pmatrix} \psi \qquad \psi(0,t) = \psi_0(t)$$

$$\tilde{\psi}(k,t) = \int_0^{\infty} dx\, e^{-ikx} \psi(x,t) \qquad \begin{pmatrix} \partial_t & -i \\ i & -\partial_t \end{pmatrix}$$

$$\int_0^{\infty} dx\, e^{-ikx} \partial_x \psi(x,t) = \int_0^{\infty} dx\, e^{-ikx} D_t \psi(x,t) = D_t \tilde{\psi}(k,t)$$

$$\|$$

$$\left[e^{-ikx}\psi(x,t)\right]_{x=0}^{x=\infty} - \int_0^{\infty} dx(-ik) e^{-ikx}\psi(x,t) = -\psi_0(t) + ik\tilde{\psi}(k,t)$$

so you get $\tilde{\psi}(k,t) = \frac{1}{ik - D_t}\psi_0(t)$. I.L.T.

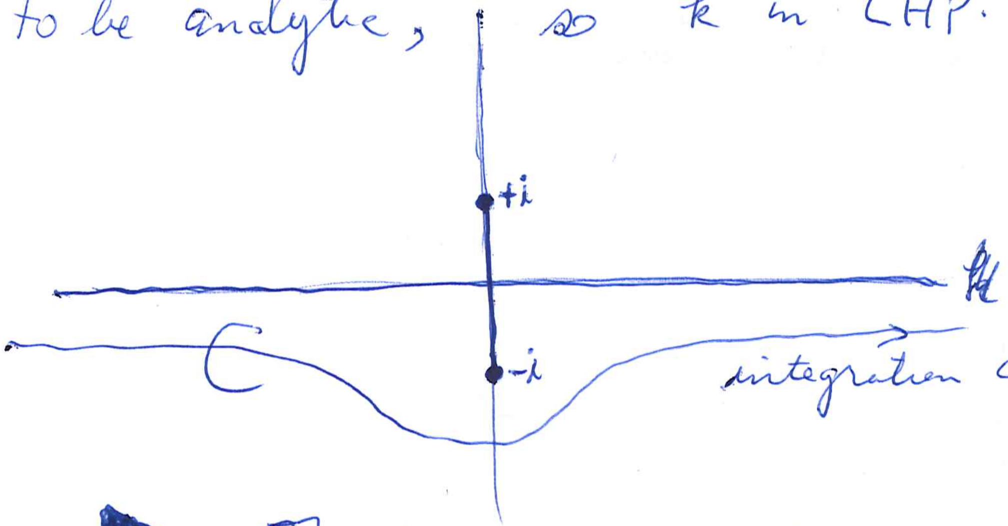
gives

$$\psi(x,t) = \int_{ik=a-i\infty}^{ik=a+i\infty} \frac{d(ik)}{2\pi i} e^{ikx} \frac{1}{ik - D_t}\psi_0(t)$$

a real to the right of singlts.

$a \gg 0$

Reason. ik is a complex variable – you need $\mathrm{Re}(ik) \gg 0$ ik in a RHP for $\tilde{\psi}(k,t)$ to be analytic, so k in LHP.



integration contour for $x > 0$

The key case is probably $\psi_0(t) = \delta(t)\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$ which can be expanded $\int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{i\omega t}$. The corresp. $B = \begin{pmatrix}\omega & -1\\ 1 & -\omega\end{pmatrix}$

$$\psi(x,t) = \int_C \frac{dk}{2\pi} e^{ikx} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{i\omega t} \frac{1}{i(k-B)}$$

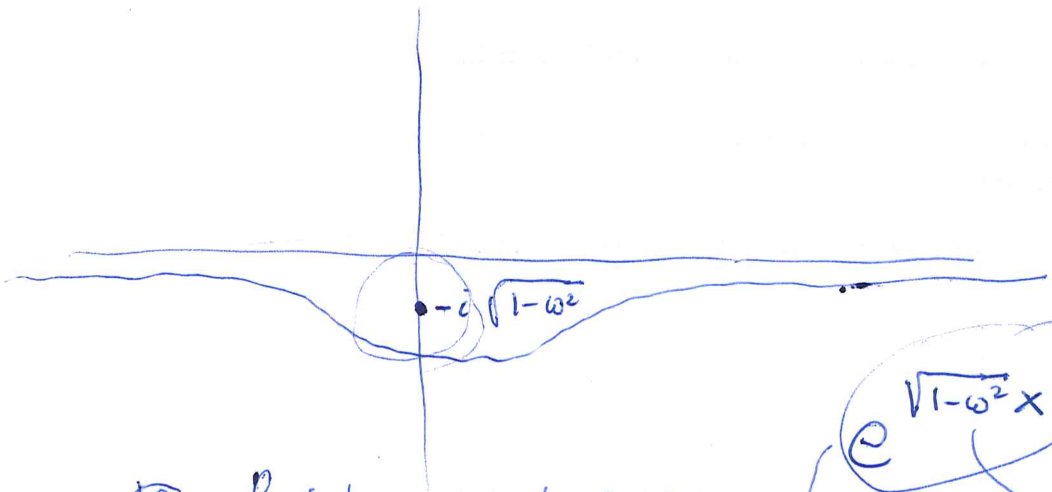
$$= \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{i\omega t} \int_C \frac{dk}{2\pi i} e^{ikx} \frac{1}{k-B}$$

For any ω $(k-B)^{-1}$ has simple pole singularities at $k = \pm\sqrt{\omega^2 - 1}$. So we break the integral over $\omega \in \mathbb{R}$ into $|\omega| < 1$ and $|\omega| > 1$.

$$B_\omega^2 = \omega^2 - 1 = k_\omega^2$$

$$\frac{1}{k-B_\omega} = \frac{1}{k-k_\omega}\,\frac{k_\omega + B_\omega}{2k_\omega} + \frac{1}{k+k_\omega}\,\frac{k_\omega - B_\omega}{2k_\omega} \qquad ?$$

ω is fixed with $-1<\omega<1$, so $k_\omega = \pm i\sqrt{1-\omega^2}$



$$e^{\sqrt{1-\omega^2}\,x} \qquad e^{ik_\omega^- x}$$

Residue contribution

$$\oint_{-i\sqrt{1-\omega^2}} \frac{dk}{2\pi i}\,\frac{e^{ikx}}{k-B_\omega} = \frac{-i\sqrt{1-\omega^2}+B_\omega}{2\left(-i\sqrt{1-\omega^2}\right)} = \frac{k_\omega^- + B_\omega}{2k_\omega^-}$$

$$\int_{-1}^{1}\frac{d\omega}{2\pi}\,e^{i\omega t}\,e^{ik_\omega^- x}\,\frac{k_\omega^- + B_\omega}{2k_\omega^-} \longrightarrow \frac{1}{2k_\omega^-}\begin{pmatrix} k_\omega^- + \omega & -1 \\ 1 & k_\omega^- - \omega \end{pmatrix}$$

Let's go over this again to make it clearer. Basically you solve the IVP on $x=0$. ~~You~~ ~~can assume $\psi_0(t)$ expand~~ One version is

$$\psi(x,t) = e^{xD_t}\psi_0(t) \qquad \text{where } D_t = \begin{pmatrix} \partial_t & -i \\ i & -\partial_t \end{pmatrix}$$

assuming $\psi_0(t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{i\omega t} \hat{\psi}_0(\omega)$

then $\psi(x,t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{i\omega t} e^{x B_\omega} \hat{\psi}_0(\omega) \qquad B_\omega = \begin{pmatrix} \omega & -1 \\ \omega & -\omega \end{pmatrix}$

$$\psi(0,t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{i\omega t} \hat{\psi}_0(\omega)$$

$$\int dt\, \psi(0,t)^* \varepsilon \psi(0,t) = \int dt \int_{-\infty}^{\infty} \frac{d\omega_1}{2\pi} e^{-i\omega_1 t} \hat{\psi}_0(\omega)^* \varepsilon \int_{-\infty}^{\infty} \frac{d\omega_2}{2\pi} e^{i\omega_2 t} \hat{\psi}_0(\omega_2)$$

$$= \int \frac{d\omega}{2\pi} \hat{\psi}_0(\omega)^* \varepsilon \hat{\psi}_0(\omega)$$

It seems that the L.T. approach ends up with what you know already, e.g.

$\tilde{\psi}(k,t) = \int_0^{\infty} e^{-ikx} \psi(x,t)\, dx \qquad \mathrm{Im}(k) \ll 0$ for convergence

Then the IVP for $\partial_x \psi = D_t \psi$, $D_t = \begin{pmatrix} \partial_t & -i \\ i & -\partial_t \end{pmatrix}$, $\psi(0,t) = \psi_0(t)$

~~becomes~~ transforms to $\tilde{\psi}(k,t) = \frac{1}{ik - D_t} \psi_0(t)$ so

$$\psi(x,t) = \int_{ik=a-i\infty}^{ik=a+i\infty} \frac{d(ik)}{2\pi i} e^{ikx} \frac{1}{ik - D_t} \psi_0(t) = \int_{k=-ia-\infty}^{k=ia+i\infty} \frac{dk}{2\pi i} e^{ikx} \frac{1}{k - \frac{1}{i} D_t} \psi_0(t)$$

$$= \int_{-ia-\infty}^{-ia+\infty} \frac{dk}{2\pi i} e^{ikx} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{i\omega t} \frac{1}{k - B_\omega} \hat{\psi}_0(\omega) \qquad B_\omega = \begin{pmatrix} \omega & -1 \\ 1 & -\omega \end{pmatrix}$$

For $x > 0$ e^{ikx} decays as $\mathrm{Im}(k) \uparrow$. The idea is to push the contour $\mathrm{Im}(k) = -a$ upward past the singularities.

Next ~~ω~~ $B_\omega^2 = \omega^2 - 1$, let $k_\omega^2 = \omega^2-1$, Then

$$\frac{1}{k-B_\omega} = \frac{1}{k-k_\omega}\,\frac{k_\omega + B_\omega}{2k_\omega} + \frac{1}{k+k_\omega}\,\frac{k_\omega - B_\omega}{2k_\omega}$$

so have

$$\psi(x,t) = \int_{-\infty}^{\infty}\frac{d\omega}{2\pi}e^{i\omega t}\int_{-ia-\infty}^{-ia+\infty}\frac{dk}{2\pi i}e^{ikx}\left(\frac{1}{k-k_\omega}(\;\cdot\;)+\frac{1}{k+k_\omega}(\;\;)\right)\hat{\psi}_0(\omega)$$

now push the contour $\mathrm{Im}(k) = -a$ upward getting Residues at $k = \pm k_\omega$. $\therefore$

$$\psi(x,t) = \int_{-\infty}^{\infty}\frac{d\omega}{2\pi}e^{i\omega t}\left\{e^{ik_\omega x}\frac{k_\omega+B_\omega}{2k_\omega} + e^{-ik_\omega x}\frac{k_\omega - B_\omega}{2k_\omega}\right\}\hat{\psi}_0(\omega)$$

which is just your

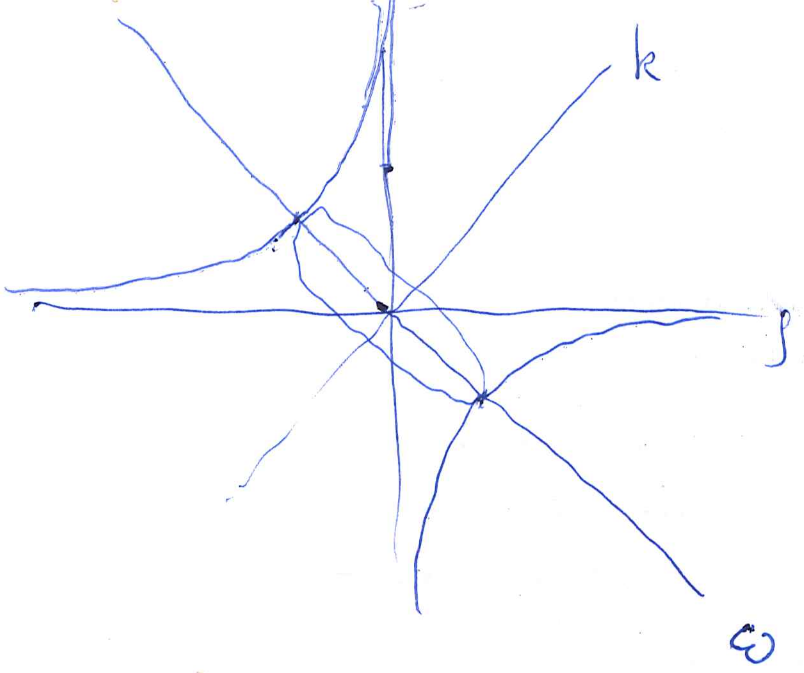
$$\psi(x,t) = e^{xD_t}\psi(0,t) = \int_{-\infty}^{\infty}\frac{d\omega}{2\pi}e^{i\omega t}\,e^{ixB_\omega}\,\hat{\psi}_0(\omega).$$

Nothing has been gained [redacted] except you see clearly that $|\omega| < 1$ must be included in the grid space, so Real ρ are not enough.

$$\rho = \omega + k = \omega \pm i\sqrt{1-\omega^2} = \cos\theta \mp i\sin\theta = e^{-i\theta}$$

So formula should be

$$\psi(x,t) = \int_{-\infty}^{\infty}\frac{d\rho}{2\pi}e^{i(kx-\omega t)}\begin{pmatrix}-\rho^{-1}\\ 1\end{pmatrix}f(\rho)$$
$$+\int_0^{2\pi}\frac{d\theta}{2\pi}e^{+x\sin\theta - it\cos\theta}\begin{pmatrix}-e^{+i\theta}\\ 1\end{pmatrix}f_1(\theta)$$



$$\psi(r,s) = \int \frac{d\rho}{2\pi} e^{i(r\rho - s\rho^{-1})} \begin{pmatrix} -\rho^{-1} \\ 1 \end{pmatrix} f(\rho)$$

First

$$\psi(x,t) = \int_{-\infty}^{\infty} \frac{d\rho}{2\pi}\, e^{i(kx-\omega t)} \begin{pmatrix} -\rho^{-1} \\ 1 \end{pmatrix} f(\rho)$$

$$= \int_{\infty}^{0} \frac{d\rho}{2\pi\rho}\, e^{i(kx-\omega t)} \begin{pmatrix} -1 \\ \rho \end{pmatrix} f(\rho) + \int_0^{\infty}$$

$$= \int_0^{\infty} \frac{1}{2\pi}\frac{d\rho}{\rho}\, e^{i(kx+\omega t)} \begin{pmatrix} +1 \\ +\rho^{-1} \end{pmatrix} f(-\rho^{-1})$$

$$= \int_0^{\infty} \frac{1}{2\pi}\frac{d\rho}{\rho} 2\omega\, e^{ikx} \left\{ \frac{e^{i\omega t}}{2\omega} \begin{pmatrix} 1 \\ \rho^{-1} \end{pmatrix} f(-\rho^{-1}) + \frac{e^{-i\omega t}}{2\omega} \begin{pmatrix} -1 \\ \rho \end{pmatrix} f(\rho) \right\}$$

$$= \int_{-\infty}^{\infty} \frac{1}{2\pi} dk\, e^{ikx} \frac{1}{2\omega} \begin{pmatrix} 1 & -1 \\ \rho^{-1} & \rho \end{pmatrix} \begin{pmatrix} e^{i\omega t} f(-\rho^{-1}) \\ e^{-i\omega t} f(\rho) \end{pmatrix}$$

There should be a ~~simple~~ formula relating $\begin{pmatrix} f(-\rho^{-1}) \\ f(\rho) \end{pmatrix}$ and $\hat{\varphi}_0(k)$

$$\hat{\varphi}_0(k) = \frac{1}{2\omega} \begin{pmatrix} 1 & -1 \\ \rho^{-1} & \rho \end{pmatrix} \begin{pmatrix} f(-\rho^{-1}) \\ f(\rho) \end{pmatrix}$$

$$\hat{\varphi}_0(k)^* \hat{\varphi}_0(k) = \begin{pmatrix} f(-\rho^{-1}) \\ f(\rho) \end{pmatrix}^* \underbrace{\frac{1}{2\omega}\begin{pmatrix} 1 & \rho^{-1} \\ -1 & \rho \end{pmatrix} \frac{1}{2\omega}\begin{pmatrix} 1 & -1 \\ \rho^{-1} & \rho \end{pmatrix}} \begin{pmatrix} f(-\rho^{-1}) \\ f(\rho) \end{pmatrix}$$

$$\frac{1}{2\omega}\begin{pmatrix} \frac{1+\rho^{-2}}{2\omega} & 0 \\ 0 & \frac{1+\rho^2}{2\omega} \end{pmatrix} = \frac{1}{2\omega}\begin{pmatrix} \rho^{-1} & \\ & \rho \end{pmatrix}$$

$$\hat{\varphi}_0(k)^* \hat{\varphi}_0(k) = \frac{1}{2\omega}\left(\rho^{-1} |f(-\rho^{-1})|^2 + \rho\, |f(\rho)|^2 \right)$$

$k = \frac{\rho - \rho^{-1}}{2}$

$$\varphi(x,t) = \int_{-\infty}^{\infty} \frac{d\rho}{2\pi} e^{i(kx-\omega t)} \begin{pmatrix} -\rho^{-1} \\ 1 \end{pmatrix} f(\rho)$$

$$= \underbrace{\int_{-\infty}^{0} \frac{d\rho}{2\pi\rho} e^{i(kx-\omega t)} \begin{pmatrix} -1 \\ \rho^{-1} \end{pmatrix} f(\rho)} + \int_{0}^{\infty} \frac{d\rho}{2\pi\rho} e^{i(kx-\omega t)} \begin{pmatrix} -1 \\ \rho^{-1} \end{pmatrix} f(\rho)$$

$$\int_0^\infty \frac{d\rho}{2\pi\rho} e^{i(kx+\omega t)} \begin{pmatrix} +1 \\ +\rho \end{pmatrix} f(-\rho^{-1})$$

$$= \int_0^\infty \frac{d\rho}{2\pi\rho} e^{ikx} \quad ?$$

Above you got the ρ picture linked to e^{itA_k} $A_k = \begin{pmatrix} k & 1 \\ 1 & -k \end{pmatrix}$

Now you want the ρ-picture linked to e^{ixB_ω} $B_\omega = \begin{pmatrix} \omega & -1 \\ 1 & -\omega \end{pmatrix}$

$$\varphi(x,t) = \int_{-\infty}^{\infty} \frac{d\rho}{2\pi} e^{i(kx-\omega t)} \begin{pmatrix} -\rho^{-1} \\ 1 \end{pmatrix} f(\rho)$$

$$\varphi(x,t) = e^{xD_t} \varphi_0(t) = \int \frac{d\omega}{2\pi} e^{xD_t} e^{i\omega t} \hat{\varphi}_0(\omega)$$

$$= \int \frac{d\omega}{2\pi} e^{i\omega t} \underbrace{e^{ixB_\omega}}_{e^{ik_\omega x}\frac{k_\omega + B_\omega}{2k_\omega} + e^{-ik_\omega x}\frac{k_\omega - B_\omega}{2k_\omega}} \hat{\varphi}_0(\omega) \qquad B_\omega = \begin{pmatrix} \omega & -1 \\ 1 & -\omega \end{pmatrix}$$

$$\psi(x,t) = \int \frac{d\omega}{2\pi}\, e^{i\omega t}\left(e^{ik_\omega x}\left(\frac{k_\omega + B_\omega}{2k_\omega}\right)\hat{\psi}_0(\omega) + e^{-ik_\omega x}\left(\frac{k_\omega - B_\omega}{2k_\omega}\right)\hat{\psi}_0(\omega)\right)$$

$$\psi(x,t) = \int \frac{d\rho}{2\pi}\, e^{i(kx-\omega t)}\begin{pmatrix}-\rho^{-1}\\ 1\end{pmatrix} f(\rho)$$

what you want is how to go between $f(\rho)$ and $\hat{\psi}_0(\omega)$. This ~~...~~ probably means relating $f(\rho)$ $f(\rho^{-1})$ to $\hat{\psi}^j_0(\omega)$ $j = 1, 2.$ where $\omega = \frac{\rho+\rho^{-1}}{2}$

Start again with $\psi(0,t)$ and solution

$$\psi(x,t) = e^{xD_t}\psi_0(t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\, e^{i\omega t}\, e^{ixB_\omega}\hat{\psi}_0(\omega)$$

$$e^{ixB_\omega} = e^{ixk_\omega}\,\frac{k_\omega + B_\omega}{2k_\omega} + e^{-ixk_\omega}\,\frac{k_\omega - B_\omega}{2k_\omega}$$

$$B_\omega = \begin{pmatrix}\omega & -1\\ 1 & -\omega\end{pmatrix} \qquad B_\omega^2 = \omega^2 - 1$$

$$\frac{k_\omega + B_\omega}{2k_\omega} = \frac{1}{2k_\omega}\begin{pmatrix}k_\omega + \omega & -1\\ 1 & k_\omega - \omega\end{pmatrix}$$

Let's rewrite in terms of $\rho = \omega + k_\omega$

Consider $\rho \longmapsto \frac{\rho+\rho^{-1}}{2} = \omega$

For each ω there are two choices ~~for ρ~~

$k_\omega = \pm\sqrt{\omega^2 - 1}$ and so two choices for ρ

map ~~...~~ Φ : function $\underset{k}{\omega}$ $\rightsquigarrow$ function $\frac{\rho+\rho^{-1}}{2}$

Regard ω, k, B as functions of ρ, $\omega = \frac{\rho+\rho^{-1}}{2}$ $\quad k = \frac{\rho-\rho^{-1}}{2}$

$$B = \begin{pmatrix} \omega & -1 \\ 1 & -\omega \end{pmatrix}$$

$$\frac{k+B}{2k} = \frac{1}{2k}\begin{pmatrix} k+\omega & -1 \\ 1 & k-\omega \end{pmatrix} = \frac{1}{2k}\begin{pmatrix} \rho & -1 \\ 1 & -\rho^{-1} \end{pmatrix}$$

$$= \frac{1}{2k}\begin{pmatrix} 1 \\ \rho^{-1} \end{pmatrix}\begin{pmatrix} \rho & -1 \end{pmatrix}$$

$$\frac{k-B}{2k} = \frac{1}{2k}\begin{pmatrix} k-\omega & 1 \\ -1 & k+\omega \end{pmatrix} = \frac{1}{2k}\begin{pmatrix} -\rho^{-1} & 1 \\ -1 & \rho \end{pmatrix}$$

$$= \frac{1}{2k}\begin{pmatrix} -\rho^{-1} \\ -1 \end{pmatrix}\begin{pmatrix} 1 & -\rho \end{pmatrix}$$

$\frac{d\omega}{2\pi} e^{i\omega t}$ lifts to $\frac{1}{2\pi} k \frac{d\rho}{\rho}$ $\qquad \omega = \frac{\rho+\rho^{-1}}{2}$, $\quad d\omega = \frac{\rho-\rho^{-1}}{2}\frac{d\rho}{\rho}$

So

$$\psi(x,t) = \int \frac{1}{2\pi}\frac{d\rho}{\rho} e^{i\omega t}\left(\frac{e^{ikx}}{2}\begin{pmatrix} 1 \\ \rho^{-1} \end{pmatrix}\begin{pmatrix} \rho & -1 \end{pmatrix} + \frac{e^{-ikx}}{2}\begin{pmatrix} +\rho^{-1} \\ +1 \end{pmatrix}\begin{pmatrix} -1 & +\rho \end{pmatrix}\right)\hat{\psi}_0(\omega)$$

$$\psi(0,t) = \int \frac{1}{2\pi} e^{i\omega t}\frac{d\rho}{\rho}\begin{pmatrix} \frac{\rho-\rho^{-1}}{2} & 0 \\ 0 & \frac{\rho-\rho^{-1}}{2} \end{pmatrix}\hat{\psi}_0(\omega)$$

$$\psi(0,t) = \int \frac{1}{2\pi} e^{i\omega t}\,\frac{d\omega}{\cancel{k}}\,\cancel{2k}\,\hat{\psi}_0(\omega)$$

$$\begin{pmatrix} \rho \\ 1 \end{pmatrix}\begin{pmatrix} -1 & \rho^{-1} \end{pmatrix} = \begin{pmatrix} -\rho & 1 \\ -1 & \rho^{-1} \end{pmatrix}$$

Start again with

$$\psi(x,t) = e^{xD_t}\psi_0(t) = \int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\, e^{i\omega t}\, e^{ixB_\omega}\hat{\psi}_0(\omega)$$

Note that $\psi(0,t) = \int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\, e^{i\omega t}\hat{\psi}_0(\omega)$ so

$$\int \psi(0,t)^* \mathcal{E}\,\psi(0,t)\,dt = \int dt\left(\int\frac{d\omega}{2\pi}\, e^{i\omega t}\hat{\psi}_0(\omega)\right)^* \varepsilon\,\psi(0,t)$$

$$= \int dt\int\frac{d\omega}{2\pi}\,\hat{\psi}_0(\omega)^*\varepsilon\, e^{-i\omega t}\psi(0,t) = \int\frac{d\omega}{2\pi}\,\hat{\psi}_0(\omega)^*\varepsilon\,\hat{\psi}_0(\omega).$$

Thus $I\!H(\psi)$ is transparent in this representation.
So what happens next?

$$e^{ixB_\omega} = e^{ixk_\omega}\frac{k_\omega + B_\omega}{2k_\omega} + e^{-ixk_\omega}\frac{k_\omega - B_\omega}{2k_\omega}$$

where $k_\omega^2 = \omega^2 - 1 = B_\omega^2$. You have the soln ψ expressed in terms of $\hat{\psi}_0(\omega)$ a vector function of $\omega \in \mathbb{R}$ with 2 components. The idea now is to split $\hat{\psi}_0(\omega)$ into eigenvectors for B_ω. Parametrize the eigenspaces. Look at $\{(\omega, k_\omega) \mid k_\omega^2 = \omega^2 - 1\}$, such a pair equiv. to $\rho \in \mathbb{C}^\times$ by $k_\rho = \frac{\rho - \rho^{-1}}{2}$, $\omega_\rho = \frac{\rho + \rho^{-1}}{2}$

$$\hat{\psi}_0(\omega) = \underbrace{\frac{k+B}{2k}}_{\substack{\| \\ \frac{1}{2k}\begin{pmatrix} k+\omega & -1 \\ 1 & k-\omega\end{pmatrix} \\ \| \\ \frac{1}{2k_\rho}\begin{pmatrix}\rho & -1 \\ 1 & -\rho^{-1}\end{pmatrix}}}\hat{\psi}_0(\omega) + \underbrace{\frac{k-B}{2k}}_{\substack{\frac{1}{2k}\begin{pmatrix} k-\omega & +1 \\ -1 & k+\omega\end{pmatrix} \\ \frac{1}{2k_\rho}\begin{pmatrix}-\rho^{-1} & 1 \\ -1 & \rho\end{pmatrix}}}\hat{\psi}_0(\omega)$$

Recap. $\psi(x,t) = \int \frac{d\omega}{2\pi} e^{i\omega t} e^{ixB_\omega} \hat{\psi}_0(\omega)$

$$IH(\psi) = \int \frac{d\omega}{2\pi} \hat{\psi}_0(\omega)^* \varepsilon \hat{\psi}_0(\omega) \qquad \text{obvious.}$$

$$\psi(x,t) = \int \frac{d\omega}{2\pi} e^{i\omega t}\left\{ \frac{e^{ik_\omega x}}{2k_\omega}\begin{pmatrix} k_\omega+\omega & -1 \\ 1 & k_\omega-\omega \end{pmatrix} + \frac{e^{-ik_\omega x}}{2k_\omega}\begin{pmatrix} k_\omega-\omega & 1 \\ -1 & k_\omega+\omega \end{pmatrix}\right\}\hat{\psi}_0(\omega)$$

Parametrize pairs $\{(\omega, k_\omega) \mid k_\omega^2 = \omega^2 - 1\}$ by $\omega = \frac{\rho+\rho^{-1}}{2}$ $\quad k_\omega = \frac{\rho-\rho^{-1}}{2}$

$d\omega = \frac{1-\rho^{-2}}{2} d\rho = \frac{k}{\rho} d\rho$

$$\psi(x,t) = \int \frac{1}{2\pi}\frac{d\rho}{\rho} e^{i\omega t}\left\{\frac{e^{ikx}}{2}\begin{pmatrix} \rho & -1 \\ 1 & -\rho^{-1}\end{pmatrix} + \frac{e^{-ikx}}{2}\begin{pmatrix} -\rho^{-1} & 1 \\ -1 & \rho \end{pmatrix}\right\}\hat{\psi}_0(\omega)$$

Somehow you want to rewrite this as

$$\psi(x,t) = \int \frac{d\rho}{2\pi} e^{i(kx-\omega t)} \begin{pmatrix} -\rho^{-1} \\ 1 \end{pmatrix} f(\rho)$$

$x = r+s$
$t = -r+s$

$r = \frac{x-t}{2}$
$s = \frac{x+t}{2}$

$-\partial_r \psi^1 = i\psi^2$
$\partial_s \psi^2 = i\psi^1$

$e^{i(r\rho - s\rho^{-1})}$

$$\psi(x,t) = \int \frac{d\rho}{2\pi} e^{i\left(x\frac{\rho-\rho^{-1}}{2} - t\frac{\rho+\rho^{-1}}{2}\right)} \begin{pmatrix} -\rho^{-1} \\ 1 \end{pmatrix} f(\rho)$$

$$\psi(x,t) = \int \frac{d\rho}{2\pi} e^{-it\omega}$$

---

you probably want to change ρ to $-\rho^{-1}$.

$$k = \frac{\rho-\rho^{-1}}{2} \longmapsto \frac{-\rho^{-1} - (-\rho^{-1})^{-1}}{2} = \frac{\rho-\rho^{-1}}{2} = k$$

$$\omega = \frac{\rho+\rho^{-1}}{2} \longmapsto \frac{-\rho^{-1} + (-\rho^{-1})^{-1}}{2} = -\frac{\rho+\rho^{-1}}{2} = -\omega$$

$$\psi(x,t) = \int \frac{d\rho}{2\pi\rho}\, e^{i(kx-\omega t)} \begin{pmatrix} -1 \\ \rho \end{pmatrix} f(\rho)$$

$$= \int \mp \frac{d\rho}{2\pi\rho}\, e^{i(kx+\omega t)} \begin{pmatrix} +1 \\ +\rho^{-1} \end{pmatrix} f(-\rho^{-1})\; f(\rho)$$

Start again

$$\psi(x,t) = e^{xD_t}\psi_0(t) = e^{xD_t}\int \frac{d\omega}{2\pi}\, e^{-i\omega t}\,\hat{\psi}_0(-\omega)$$

$$= \int \frac{d\omega}{2\pi}\, e^{-i\omega t}\, e^{ixB_{-\omega}}\,\hat{\psi}_0(-\omega) \qquad B_\omega = \begin{pmatrix} \omega & -1 \\ 1 & -\omega \end{pmatrix}$$

$$\psi(x,t) = e^{xD_t}\psi_0(t) = \int \frac{d\omega}{2\pi}\, e^{i\omega t}\, e^{ixB_\omega}\,\hat{\psi}_0(\omega) \qquad B^2 = \omega^2 - 1$$

$$= \int \frac{d\omega}{2\pi}\, e^{i\omega t} \underbrace{\left(e^{ikx}\frac{k+B}{2k} + e^{-ikx}\frac{k-B}{2k} \right)}\hat{\psi}_0(\omega) \qquad k^2 = \omega^2 - 1$$

$$\rho = \omega + k \qquad \rho^{-1} = \omega - k \qquad k = \frac{\rho - \rho^{-1}}{2}$$

$$\frac{e^{ikx}}{2k}\begin{pmatrix} k+\omega & -1 \\ 1 & k-\omega \end{pmatrix} + \frac{e^{-ikx}}{2k}\begin{pmatrix} k-\omega & 1 \\ -1 & k+\omega \end{pmatrix}$$

$$\frac{e^{ikx}}{2k}\begin{pmatrix} \rho & -1 \\ 1 & -\rho^{-1} \end{pmatrix} + \frac{e^{-ikx}}{-2k}\begin{pmatrix} \rho^{-1} & -1 \\ 1 & -\rho \end{pmatrix}$$

$$\psi(x,t) = \int \frac{1}{2\pi}\frac{d\rho}{\rho}\, e^{i\omega t + ikx}\, \underbrace{\frac{1}{2}\begin{pmatrix} \rho & -1 \\ 1 & -\rho^{-1} \end{pmatrix}\hat{\psi}_0(\omega)}$$

$$\frac{1}{2}\begin{pmatrix} \rho \\ 1 \end{pmatrix} f(\rho) \qquad f(\rho) = (1, -\rho^{-1})\,\hat{\psi}_0(\omega)$$

$$\psi(x,t) = \int \frac{1}{4\pi}\, d\rho\, e^{i\omega t + ikx} \begin{pmatrix} 1 \\ \rho^{-1} \end{pmatrix} f(\rho)$$

You know $\mathbb{H}(\psi) = \int \frac{d\omega}{2\pi}\, \hat{\psi}_0(\omega)^{*}\varepsilon\, \hat{\psi}_0(\omega)$

just ~~rewrite~~ write this using $f(\rho)$.

$$\psi(x,t) = \int \frac{d\rho}{2\pi} e^{i(\omega t + kx)} \frac{1}{2}\begin{pmatrix} 1 \\ \rho^{-1} \end{pmatrix} f(\rho)$$

$$\psi_0(t) = \int \frac{d\rho}{2\pi\rho 2} e^{i\omega t} \begin{pmatrix} \rho \\ 1 \end{pmatrix} f(\rho)$$

$$= \int \frac{d\omega}{2\pi k 2} e^{i\omega t}\left[\begin{pmatrix} \rho \\ 1 \end{pmatrix} f(\rho) + \begin{pmatrix} \rho^{-1} \\ 1 \end{pmatrix} f(\rho^{-1})\right]$$

$$\hat{\psi}_0(\omega) = \frac{1}{2k}\left[\begin{pmatrix} \rho \\ 1 \end{pmatrix} f(\rho) + \begin{pmatrix} \rho^{-1} \\ 1 \end{pmatrix} f(\rho^{-1})\right]$$

$$\boxed{\begin{array}{l} \text{Recall if } \psi(x,t) = \int \frac{d\rho}{2\pi\rho} e^{i(kx-\omega t)} \begin{pmatrix} 1 \\ -\rho \end{pmatrix} f(\rho) \\ \text{then } \int \psi^*\psi\, dx = \int \frac{d\rho}{\pi} |f(\rho)|^2 \end{array}}$$

Again $\psi(x,t) = \int e^{i\omega t + ikx} \begin{pmatrix} 1 \\ \rho^{-1} \end{pmatrix} f(\rho)$

$$\partial_t \psi \stackrel{?}{=} \begin{pmatrix} \partial_x & i \\ i & -\partial_x \end{pmatrix} \psi$$

$$\omega \begin{pmatrix} 1 \\ \rho^{-1} \end{pmatrix} \stackrel{?}{=} \begin{pmatrix} k & 1 \\ 1 & -k \end{pmatrix} \begin{pmatrix} 1 \\ \rho^{-1} \end{pmatrix}$$

$$\omega = k + \rho^{-1}$$
$$\omega\rho^{-1} = 1 - k\rho^{-1}$$
$$(\omega + k)\rho^{-1} = 1$$

$$\psi(x,t) = \int_C \frac{d\rho}{2\pi\rho} e^{i\omega t + ikx} \begin{pmatrix} \rho \\ 1 \end{pmatrix} f(\rho)$$ solution f dist.

$$\psi(x,t) = -\int_C \frac{d\rho}{2\pi\rho} e^{i\omega t - ikx} \begin{pmatrix} \rho^{-1} \\ 1 \end{pmatrix} f(\rho^{-1})$$

$f(\rho^{-1})$

$$\psi(x,t) = \int \frac{k\,d\rho}{2\pi\rho} e^{i\omega t}\left\{\frac{e^{ikx}}{2k}\begin{pmatrix} \rho \\ 1 \end{pmatrix} f(\rho) + \frac{e^{-ikx}}{2k}\begin{pmatrix} \rho^{-1} \\ 1 \end{pmatrix}\right\}$$

$$\psi(x,t) = \int \frac{d\omega}{2\pi} e^{i\omega t} \left\{ \frac{e^{ikx}}{2k} \begin{pmatrix} \rho \\ 1 \end{pmatrix} f(\rho) - \frac{e^{-ikx}}{2k} \begin{pmatrix} \rho^{-1} \\ 1 \end{pmatrix} f(\rho^{-1}) \right\}$$

$$\hat{\psi}_0(\omega) = \cancel{\frac{1}{2k} f(\rho)} \; \frac{1}{2k} \left[\begin{pmatrix} \rho \\ 1 \end{pmatrix} f(\rho) - \begin{pmatrix} \rho^{-1} \\ 1 \end{pmatrix} f(\rho^{-1}) \right]$$

$$= \frac{1}{2k} \begin{pmatrix} \rho & +\rho^{-1} \\ 1 & +1 \end{pmatrix} \begin{pmatrix} f(\rho) \\ -f(\rho^{-1}) \end{pmatrix}$$

$$\begin{pmatrix} \rho & 1 \\ \rho^{-1} & 1 \end{pmatrix} \begin{pmatrix} \rho & \rho^{-1} \\ -1 & -1 \end{pmatrix} = \begin{pmatrix} \rho^2 - 1 & 0 \\ 0 & \rho^{-2} - 1 \end{pmatrix}$$

$$\left(\hat{\psi}_0^{*} \, \varepsilon \, \hat{\psi}_0 \right) = \frac{1}{\cancel{4k^2}_{\,2k}} \left(\left(\cancel{\rho^2 - 1}\right)^{\rho} |f(\rho)|^2 - \left(\cancel{1 - \rho^{-2}}\right)^{\rho^{-1}} |f(\rho^{-1})|^2 \right)$$

$$\int \frac{d\omega}{2\pi} \hat{\psi}_0^* \varepsilon \hat{\psi}_0 = \int \frac{d\omega}{2\pi} \frac{1}{2k} \left(\rho |f(\rho)|^2 - \rho^{-1} |f(\rho^{-1})|^2 \right)$$

$$= \int \frac{d\rho}{2\pi\rho} \frac{1}{2} \left(\rho |f(\rho)|^2 - \rho^{-1} |f(\rho^{-1})|^2 \right)$$

You seem to be getting closer.

Repeat. If $\psi(x,t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{i\omega t} e^{ixB_\omega} \hat{\psi}_0(\omega)$

then $IH(\psi) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \left(\hat{\psi}_0^{*} \varepsilon \hat{\psi}_0 \right)(\omega)$. This is the unitarity relation: $\int dt\, f^* g = \int \frac{d\omega}{2\pi} \hat{f}(\omega)^* \hat{g}(\omega)$.

$$k = \pm\sqrt{\omega^2 - 1}$$

$$\psi(x,t) = \int \frac{d\omega}{2\pi} e^{i\omega t} \left\{ e^{ikx} \frac{k+B}{2k} + e^{-ikx} \frac{-k+B}{-2k} \right\} \hat{\psi}_0(\omega)$$

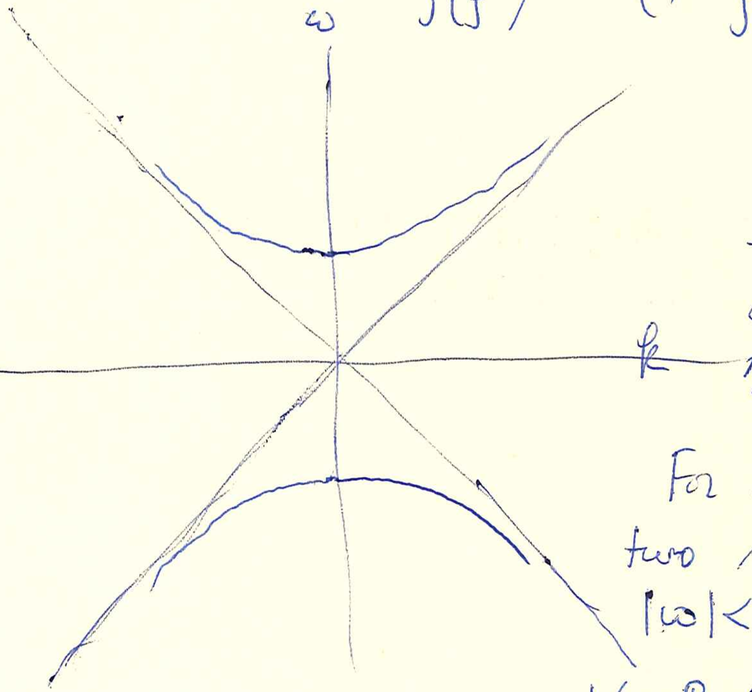
$$= \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{i\omega t} \left\{ \frac{e^{ikx}}{2k} \begin{pmatrix} \omega+k & -1 \\ 1 & \omega-k \end{pmatrix} + \frac{e^{-ikx}}{-2k} \begin{pmatrix} \omega-k & -1 \\ 1 & \omega+k \end{pmatrix} \right\} \hat{\psi}_0(\omega)$$

$$= \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{i\omega t} \left\{ \frac{e^{ikx}}{2k} \begin{pmatrix} \rho & -1 \\ 1 & \rho^{-1} \end{pmatrix} + \frac{e^{-ikx}}{-2k} \begin{pmatrix} \rho^{-1} & -1 \\ 1 & \rho \end{pmatrix} \right\} \hat{\psi}_0(\omega)$$

$$= \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{i\omega t} \left\{ \frac{e^{ikx}}{2k} \begin{pmatrix} \rho \\ 1 \end{pmatrix} f(\rho) + \frac{e^{-ikx}}{-2k} \begin{pmatrix} \rho^{-1} \\ 1 \end{pmatrix} f(\rho^{-1}) \right\}$$

where $\quad f(\rho) = (1 \ \ \rho^{-1})\, \hat{\psi}_0(\omega)$

$\qquad\qquad f(\rho^{-1}) = (1 \ \ \rho)\, \hat{\psi}_0(\omega)$



contour $-\infty < \omega < \infty$

breaks up into 4 pieces

$-\infty < \omega < -1$, $1 < \omega < \infty$

corresp to $\rho < -1$?

k Look at $k^2 = \omega^2 - 1$.

For each real ω $\quad |\omega| > 1$

two roots $\quad k = \pm\sqrt{\omega^2 - 1} \in \mathbb{R}$

$|\omega| < 1 \quad$ two roots $k = \pm i\sqrt{1-\omega^2} \in i\mathbb{R}$

$\forall \omega \in \mathbb{R}$ $|\omega| \neq 1$, have two $\rho = \omega \pm k$

inverses of each other $\qquad \omega > 1 \iff \rho, \rho^{-1} > 0$

$\qquad \omega < -1 \iff \rho, \rho^{-1} < 0$

$$\psi(x,t) = e^{xD_t}\psi_0(t) = \int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\, e^{i\omega t}\, e^{ixB}\,\hat\psi_0(\omega) \qquad B = \begin{pmatrix}\omega & -1\\ 1 & -\omega\end{pmatrix}$$

$$B^2 = \omega^2 - 1$$

$$= \int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\, e^{i\omega t}\left\{\frac{e^{ikx}}{2k}\begin{pmatrix}\omega+k & -1\\ 1 & -\omega+k\end{pmatrix} + \frac{e^{-ikx}}{-2k}\begin{pmatrix}\omega-k & -1\\ 1 & -\omega-k\end{pmatrix}\right\}\hat\psi_0(\omega)$$

$$= \int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\, e^{i\omega t}\left\{\frac{e^{ikx}}{2k}\begin{pmatrix}\rho & -1\\ 1 & -\rho^{-1}\end{pmatrix} + \frac{e^{-ikx}}{-2k}\begin{pmatrix}\rho^{-1} & -1\\ 1 & -\rho\end{pmatrix}\right\}\hat\psi_0(\omega)$$

$$= \int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\, e^{i\omega t}\left(\frac{e^{ikx}}{2k}\begin{pmatrix}\rho\\ 1\end{pmatrix} f(\rho) + \frac{e^{-ikx}}{-2k}\begin{pmatrix}\rho^{-1}\\ 1\end{pmatrix} f(\rho^{-1})\right)$$

$$f(\rho) = (1 \quad -\rho^{-1})\,\hat\psi_0(\omega) \qquad f(\rho^{-1}) = (1 \quad -\rho)\,\hat\psi_0(\omega)$$

$$\begin{pmatrix}f(\rho)\\ f(\rho^{-1})\end{pmatrix} = \begin{pmatrix}1 & -\rho^{-1}\\ 1 & -\rho\end{pmatrix}\hat\psi_0(\omega) \qquad \hat\psi_0(\omega) = \frac{1}{+2k}\begin{pmatrix}+\rho & -\rho^{-1}\\ +1 & -1\end{pmatrix}\begin{pmatrix}f(\rho)\\ f(\rho^{-1})\end{pmatrix} = \frac{1}{2k}\begin{pmatrix}\rho & \rho^{-1}\\ 1 & 1\end{pmatrix}\begin{pmatrix}f(\rho)\\ -f(\rho^{-1})\end{pmatrix}$$

$$\psi_0(\omega)^*\varepsilon\,\psi_0(\omega) = \begin{pmatrix}f(\rho)\\ -f(\rho^{-1})\end{pmatrix}^*\underbrace{\frac{1}{2k}\begin{pmatrix}\rho & 1\\ \rho^{-1} & 1\end{pmatrix}\begin{pmatrix}\rho & \rho^{-1}\\ -1 & -1\end{pmatrix}}\frac{1}{2k}\begin{pmatrix}f(\rho)\\ -f(\rho^{-1})\end{pmatrix}$$

$$\cancel{1/4} \qquad \frac{1}{\cancel{2k}}\begin{pmatrix}\overset{\rho}{\cancel{\rho^2-1}} & 0\\ 0 & \underset{-\rho^{-1}}{\cancel{\rho^{-2}-1}}\end{pmatrix}$$

$$\hat\psi_0(\omega)^*\varepsilon\,\hat\psi_0(\omega) = \frac{1}{2k}\left(\rho|f(\rho)|^2 - \rho^{-1}|f(\rho^{-1})|^2\right)$$

ρ real case
i.e. $|\omega| > 1$

$$\begin{pmatrix}f(\rho)\\ -f(\rho^{-1})\end{pmatrix}^*\frac{1}{2\bar k}\begin{pmatrix}\bar\rho & 1\\ \bar\rho^{-1} & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}\rho & \rho^{-1}\\ 1 & 1\end{pmatrix}\frac{1}{2k}\begin{pmatrix}f(\rho)\\ -f(\rho^{-1})\end{pmatrix}$$

$$\frac{1}{4|k|^2}\begin{pmatrix}\rho^{-1} & 1\\ \rho & 1\end{pmatrix}\begin{pmatrix}\rho & \rho^{-1}\\ -1 & -1\end{pmatrix} = \frac{1}{4(1-\omega^2)}\begin{pmatrix}0 & \rho^{-2}-1\\ \rho^2-1 & 0\end{pmatrix}$$

$|\omega|<1 \qquad k = \pm i\sqrt{1-\omega^2} \qquad \rho = \omega + k = \omega \pm \sqrt{\omega^2-1}$

$\therefore \bar\rho = \rho^{-1}$

$$\frac{1}{2k}\begin{pmatrix} \bar\rho & 1 \\ \bar\rho^{-1} & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} \rho & \rho^{-1} \\ 1 & 1 \end{pmatrix}$$

$$= \frac{1}{2k}\begin{pmatrix} \rho^{-1} & 1 \\ \rho & 1 \end{pmatrix}\begin{pmatrix} \rho & \rho^{-1} \\ -1 & -1 \end{pmatrix} = \frac{1}{2k}\begin{pmatrix} 0 & \rho^{-2}-1 \\ \rho^2-1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & -\rho^{-1} \\ \rho & 0 \end{pmatrix}$$

$$\frac{\rho^{-2}-1}{2k} = \rho^{-1}\frac{\rho^{-1}-\rho}{2k} = -\rho^{-1}.$$

$$\frac{1}{-2k}\begin{pmatrix} f(\rho) \\ -f(\rho^{-1}) \end{pmatrix}^{*}\begin{pmatrix} 0 & -\rho^{-1} \\ \rho & 0 \end{pmatrix}\begin{pmatrix} f(\rho) \\ -f(\rho^{-1}) \end{pmatrix}$$

$$= \frac{-1}{2k}\left(\overline{f(\rho)}\,\rho^{-1}f(\rho^{-1}) - \overline{f(\rho^{-1})}\,\rho\, f(\rho)\right)$$

$$\therefore \quad \hat\psi_0(\omega)^* \varepsilon\, \hat\psi_0(\omega) = \frac{1}{2k}\left(\overline{f(\rho^{-1})}\,\rho\, f(\rho) - \overline{f(\rho)}\,\rho^{-1} f(\rho^{-1})\right)$$

for $|\omega|<1$, i.e, $\rho = \omega \pm i\sqrt{1-\omega^2} \in S^1$

$$\hat\psi_0(\omega)^* \varepsilon\, \hat\psi_0(\omega) = \frac{1}{2k}\left(\rho|f(\rho)|^2 - \rho^{-1}|f(\rho^{-1})|^2\right)$$

for $|\omega|>1$. i.e. $\rho \in \mathbb{R}^\times$

In the next few hours I need to write up notes. Review the calc. of pos. herm. prod.

$$\partial_t\psi = \underbrace{\begin{pmatrix} \partial_x & i \\ i & -\partial_x \end{pmatrix}}_{D_x}\psi \qquad \psi(x,t) = e^{tD_x}\psi_0(x)$$

$$\psi(x,t) = e^{tD_x}\psi_0(x) = \int_{-\infty}^{\infty}\frac{dk}{2\pi}\, e^{ikx}\, e^{itA}\,\hat{\psi}_0(k) \qquad A=\begin{pmatrix}k & 1\\ 1 & -k\end{pmatrix}$$

$A^2 = k^2+1.$ $\quad \omega = \sqrt{k^2+1}$

$$= \int_{-\infty}^{\infty}\frac{dk}{2\pi}\, e^{ikx}\left(e^{i\omega t}\,\frac{\omega+A}{2\omega} + e^{-i\omega t}\,\frac{-\omega+A}{-2\omega}\right)\hat{\psi}_0(k)$$

$$= \int_{-\infty}^{\infty}\frac{dk}{2\pi}\, e^{ikx}\left(\frac{e^{i\omega t}}{2\omega}\begin{pmatrix}\omega+k & 1\\ 1 & \omega-k\end{pmatrix} + \frac{e^{-i\omega t}}{-2\omega}\begin{pmatrix}-\omega+k & 1\\ 1 & -\omega-k\end{pmatrix}\right)\hat{\psi}_0(k)$$

$$\begin{pmatrix}\rho & 1\\ 1 & \rho^{-1}\end{pmatrix} \qquad\qquad \begin{pmatrix}-\rho^{-1} & 1\\ 1 & -\rho\end{pmatrix}$$

$$= \int_{-\infty}^{\infty}\frac{dk}{2\pi}\, e^{ikx}\left\{\frac{e^{i\omega t}}{2\omega}\begin{pmatrix}\rho\\ 1\end{pmatrix} f(\rho) + \frac{e^{-i\omega t}}{-2\omega}\begin{pmatrix}-\rho^{-1}\\ 1\end{pmatrix} f(-\rho^{-1})\right\}$$

$$f(\rho) = \begin{pmatrix}1 & \rho^{-1}\end{pmatrix}\hat{\psi}_0(k), \quad f(-\rho^{-1}) = \begin{pmatrix}1 & -\rho\end{pmatrix}\hat{\psi}_0(k)$$

above integral roughly a sum over $(k,\omega)\in\mathbb{R}^2$ $\quad \omega^2 = k^2+1$

equiv. to a sum over $\rho\in\mathbb{R}^\times$ where $\rho = \omega+k$

$\rho^{-1} = \omega-k$

$-\rho^{-1} = k-\omega$

$k = \frac{\rho-\rho^{-1}}{2}$ double covered

maps $\{\rho>0\} \xrightarrow{\sim} k\in\mathbb{R}$

$$\begin{pmatrix}f(\rho)\\ f(-\rho^{-1})\end{pmatrix} = \begin{pmatrix}1 & \rho^{-1}\\ 1 & -\rho\end{pmatrix}\hat{\psi}_0(k)$$

$$\hat{\psi}_0(k) = \frac{1}{+2\omega}\begin{pmatrix}+\rho & +\rho^{-1}\\ +1 & -1\end{pmatrix}\begin{pmatrix}f(\rho)\\ f(-\rho^{-1})\end{pmatrix}$$

$$\frac{1}{2\omega}\begin{pmatrix}\overset{\rho}{\overset{\|}{\frac{\rho^2+1}{2\omega}}} & 0\\ 0 & \underset{\rho^{-1}}{\frac{\rho^{-2}+1}{2\omega}}\end{pmatrix}$$

$$\hat{\psi}_0^{\,*}\hat{\psi}_0 = \frac{1}{(2\omega)^2}\begin{pmatrix}f(\rho)\\ f(-\rho^{-1})\end{pmatrix}^{*}\begin{pmatrix}+\rho & +1\\ +\rho^{-1} & -1\end{pmatrix}\begin{pmatrix}+\rho & +\rho^{-1}\\ +1 & -1\end{pmatrix}\begin{pmatrix}f(\rho)\\ f(-\rho^{-1})\end{pmatrix}$$

$$\hat\psi_0^*\hat\psi_0 = \frac{1}{2\omega}\left(\rho\,|f(\rho)|^2 + \rho^{-1}|f(-\rho^{-1})|^2\right)$$

$$\cancel{\int_{-\infty}^{\infty}\hat\psi_0^*\hat\psi_0\frac{dk}{2\pi} = \int_0^{\infty}\frac{dk}{2\pi}\frac{\rho}{2\omega}|f(\rho)|^2}$$

$$\int_{-\infty}^{\infty}\frac{dk}{2\pi}\frac{1}{2\omega}\left(\rho\,|f(\rho)|^2 + \rho^{-1}|f(-\rho^{-1})|^2\right)$$

$$\int_{-\infty}^{\infty}\frac{dk}{2\pi}\,\frac{1}{2\omega}\,\rho\,|f(\rho)|^2 \qquad\qquad k = \frac{\rho-\rho^{-1}}{2}, \quad dk = \rho^{-1}\omega\, d\rho$$

$$\int_0^{\infty}\frac{1}{2\pi}\,\frac{\cancel{\omega}}{\rho}\,\frac{d\rho}{\cancel{2\omega}}\,\rho\,|f(\rho)|^2 = \int_0^{\infty}\frac{d\rho}{4\pi}|f(\rho)|^2$$

$$\int_{-\infty}^{\infty}\frac{dk}{2\pi}\,\frac{1}{2\omega}\,\rho^{-1}|f(-\rho^{-1})|^2 = \int_{-\infty}^{0}$$

$$\int_{\rho=0}^{\rho=\infty}\frac{1}{2\pi}\,\underbrace{\frac{dk}{\omega}}_{=\frac{d\rho}{\rho}}\left\{\frac{e^{i\omega t}}{2}\begin{pmatrix}\rho\\ 1\end{pmatrix}f(\rho) + \frac{e^{-i\omega t}}{-2}\begin{pmatrix}-\rho^{-1}\\ 1\end{pmatrix}f(-\rho^{-1})\right\}$$

Return to $\partial_t\psi = \begin{pmatrix}\partial_x & i\\ i & -\partial_x\end{pmatrix}\psi \qquad A = \begin{pmatrix}k & 1\\ 1 & -k\end{pmatrix}$

$$\psi(x,t) = e^{tD}\psi_0(x) = \int\frac{dk}{2\pi}e^{ikx}e^{itA}\hat\psi_0(k) \qquad A^2 = k^2+1$$

$$= \int_{-\infty}^{\infty}\frac{dk}{2\pi}e^{ikx}\left(e^{i\omega t}\frac{\omega+A}{2\omega} + e^{-i\omega t}\frac{-\omega+A}{-2\omega}\right)\hat\psi_0(k)$$

variety of $(k,\omega) \in \mathbb{R}\times\mathbb{R}$ $\quad \omega^2 = k^2+1$.

$$\frac{\omega+A}{2\omega} = \frac{1}{2\omega}\begin{pmatrix} \omega+k & 1 \\ 1 & \omega-k \end{pmatrix} =$$

$$\frac{dk}{\omega} = \frac{d\rho}{\rho}$$

idea proj op. $\frac{\omega+A}{2\omega}$ dep on (k,ω) $\quad \rho = \omega+k$

$\rho^{-1} =$

$$pr_\rho = \frac{1}{2\omega}\begin{pmatrix} \rho & 1 \\ 1 & \rho^{-1} \end{pmatrix} = \frac{1}{2\omega}\begin{pmatrix} \rho \\ 1 \end{pmatrix}\begin{pmatrix} 1 & \rho^{-1} \end{pmatrix}$$

~~$\psi(x,t)$~~ Rewrite in terms of ρ.

$$\int_0^\infty \frac{d\rho}{2\pi\rho} e^{ikx+i\omega t} \frac{1}{2}\begin{pmatrix} \rho & 1 \\ 1 & \rho^{-1} \end{pmatrix} + \int_{-\infty} \frac{d\rho}{2\pi\rho} e^{ikx-i\omega t} \frac{1}{2}\begin{pmatrix} \rho^{-1} & -1 \\ -1 & \rho \end{pmatrix}$$

You want maybe to do a residue calculation somehow involving the whole spectral curve. ~~Titch~~ ~~Titch~~ Titchmarsh method - contour integral of the resolvent.

Situation. In general something like

$$\partial_t \psi = D_x \psi \qquad D_x = \begin{pmatrix} \partial_x & ih \\ i\bar{h} & -\partial_x \end{pmatrix}$$

want eigenvalue expansion for the s.a. op. $\frac{1}{i}D_x$

You briefly had a L.T. approach, which you don't want to forget because it should give you the appropriate spectrum. ~~That~~

$$\tilde{\psi}(x,s) = \int_0^\infty e^{-st}\psi(x,t)\,dt$$

$$(s-D_x)\tilde{\psi} = \psi_0(x) \qquad \tilde{\psi} = \frac{1}{s-D_x}\psi_0(x)$$

$$\tilde{\psi}(x,s) = \frac{1}{s - D_x}\,\psi_0(x)$$

$$\psi(x,t) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} ds\, e^{st}\,\frac{1}{s - D_x}\,\psi_0(x)$$

here $t>0$
a to right
of spectrum

~~pull them past~~ $s = i\omega$ from $a - i\infty$ to $a + i\infty$
ω —— $-ia-\infty$ to $-ia+\infty$.

$$\psi(x,t) = \int_{-ia-\infty}^{-ia+\infty} \frac{d\omega}{2\pi}\, e^{i\omega t}\,\frac{1}{\omega - \frac{1}{i}D_x}\,\frac{1}{i}\psi_0(x)$$

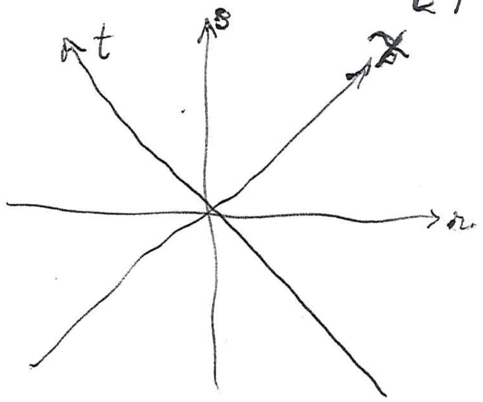
so what you do is to push $\mathrm{Im}(\omega) = -a$ past ~~the~~ the real axis, this means enclosing the jump ~~on~~ crossing ~~the~~ the real axis.

Still seem to be missing an important point which should involve ~~[illegible]~~ complex spectra. Cauchy problem on boundary. Melrose uses Mellin transform

April 25, 2000. Back to $\partial_t \psi = \begin{pmatrix} \partial_x & i \\ i & -\partial_x \end{pmatrix}\psi$

$$(\partial_t - \partial_x)\psi^1 = i\psi^2$$
$$(\partial_t + \partial_x)\psi^2 = i\psi^1$$

$$x = r + s$$
$$t = -r + s$$



$$\frac{\partial f}{\partial r} = \partial_x f\,\underbrace{\frac{\partial x}{\partial r}}_{1} + \partial_t f\,\underbrace{\frac{\partial t}{\partial r}}_{-1}$$

$$\frac{\partial f}{\partial s} = \partial_x f\,\underbrace{\frac{\partial x}{\partial s}}_{1} + \partial_t f\,\underbrace{\frac{\partial t}{\partial s}}_{1}$$

$$\partial_r = -\partial_t + \partial_x$$
$$\partial_s = \partial_t + \partial_x$$

$$-\partial_r \psi^1 = i\psi^2$$
$$\partial_s \psi^2 = i\psi^1$$

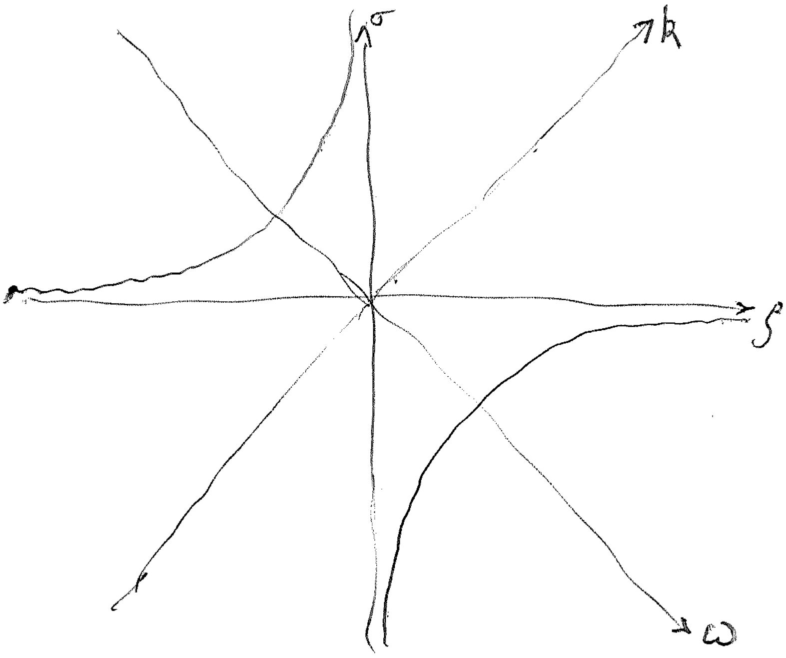
look for exponential solutions $e^{i(\rho r + \sigma t)}\begin{pmatrix}\hat{\psi}^1 \\ \hat{\psi}^2\end{pmatrix}$

$$-\rho\hat{\psi}^1 = \hat{\psi}^2$$
$$\sigma\hat{\psi}^2 = \hat{\psi}^1$$

$$-\rho\sigma = 1$$
$$\sigma = -\rho^{-1}$$

$e^{i(\rho r - \rho^{-1} t)}\begin{pmatrix}1 \\ -\rho\end{pmatrix}\hat{\psi}^1$ are the exp solutions $\rho \in \mathbb{C}^\times$

$$\rho r - \rho^{-1} t = \rho\left(\frac{x-t}{2}\right) - \rho^{-1}\left(\frac{x+t}{2}\right) = x\underbrace{\left(\frac{\rho-\rho^{-1}}{2}\right)}_{k} - t\underbrace{\left(\frac{\rho+\rho^{-1}}{2}\right)}_{\omega}$$

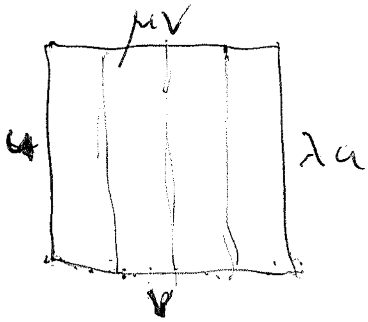


$$\rho = \omega + k$$
$$\rho^{-1} = \omega - k$$

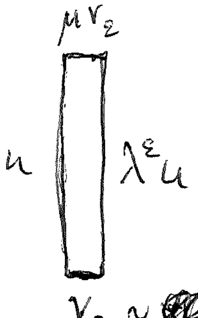
You've gone over these formulas many times. What is your aim? ~~the~~ ~~best~~ I think you want ~~an~~ continuous analog of the grid space for a constant coeff grid. This will be some sort of module over the group of translations in the x,t plane. You want the module to be small. Constant coeff grid you get ~~a~~ a module over the gp ring $\mathbb{C}[\mathbb{Z}^2]$, namely, free module of rank 1 over the quotient $\mathbb{C}[z, z^{-1}, (z-k)^{-1}, (kz-1)^{-1}]$ = alg. functions on the spectral curve $\mu = \frac{\lambda - k}{k\lambda - 1}$. You now want to find a holomorphic function analog over the spectral curve $\sigma = -\rho^{-1}$. Thus you seek a ~~natural~~ nice class of holomorphic functions on $\mathbb{C}^\times = \mathbb{C} - \{0\}$. Now reality properties enter, ~~[illegible]~~ you feel that $\rho \in \mathbb{R}$ and $\rho \in S^1$ are important, real ρ are linked to unitary representations

Let's review ~~one~~ horizontal cont. limit.

$$(k\lambda - 1)\, u = h v \qquad \lambda^{\varepsilon} = e^{i\varepsilon\rho}$$
$$(k\mu - 1)\, v = \bar{h} u$$



$$\begin{pmatrix}\lambda u\\ \mu v\end{pmatrix} = \frac{1}{k}\begin{pmatrix}1 & h\\ \bar{h} & 1\end{pmatrix}\begin{pmatrix}u\\ v\end{pmatrix}$$



$v_\varepsilon \sim \sqrt{\varepsilon}\, v$

$$\begin{pmatrix}\lambda^\varepsilon u\\ \mu (v\sqrt{\varepsilon})\end{pmatrix} = \frac{1}{k}\begin{pmatrix}1 & b\sqrt{\varepsilon}\\ \bar{b}\sqrt{\varepsilon} & 1\end{pmatrix}\begin{pmatrix}u\\ v\sqrt{\varepsilon}\end{pmatrix}$$

$$\frac{k_\varepsilon \lambda^\varepsilon - 1}{\varepsilon}\, u = h v \qquad (k\mu - 1) v = \bar{h} u$$

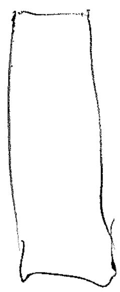
$$k_\varepsilon = \sqrt{1 - |b|^2 \varepsilon} = 1 - a\varepsilon \qquad a = \tfrac{1}{2}|b|^2$$

$$(-a + i\rho) u = b v \qquad (\mu - 1) v = \bar{b} u$$

$$\mu = 1 + \frac{\overset{2a}{|b|^2}}{-a + i\rho} = \frac{a + i\rho}{-a + i\rho} = \frac{\rho - ia}{\rho + ia}$$

$$\lambda^\varepsilon = e^{i\varepsilon\rho}$$

$\psi(r, n)$



$$\frac{k_\varepsilon \psi^1(r+\varepsilon, n) - \psi^1(r, n)}{\varepsilon} = b\psi^2(r, n)$$

$$k_\varepsilon \psi^2(r, n+1) - \psi^2(r, n) = \bar{b}\psi^1(r, n)$$

$$(-a + \partial_r)\psi^1 = b\psi^2$$

$$\psi^2(r, n+1) - \psi^2(r, n) = \bar{b}\psi^1(r, n)$$

Get started with the calculation. For a discrete grid, the grid space has generators corresp to the edges, so ~~you are looking for~~ the hermitian forms are 2-pt. functions, which you might ~~might~~ write

$$\left(\psi^a_{mn} \,\middle|\, \psi^{a'}_{m'n'}\right) \quad , \quad IH\left(\psi^a_{mn} \,\middle|\, \psi^{a'}_{m'n'}\right).$$

A good viewpoint here is that you have a unitary representation of $\mathbb{Z}\times\mathbb{Z}$, the translation group, and a two diml generating 2diml subspace, hence a positive definite function on $\mathbb{Z}\times\mathbb{Z}$ with values in $M_2(\mathbb{C})$.

Somehow these ~~forms~~ should turn out to be Green's functions for the wave eqn., different bdry conditions.

horizontally cont. case.

$$(-a+\partial_r)\psi^1(r,n) = b\,\psi^2(r,n)$$

$$\psi^2(r,n+1)-\psi^2(r,n) = \bar b\,\psi^1(r,n)$$

grid equations

$$\begin{pmatrix}\psi^1(r+\varepsilon,n)\\ \psi^2(r,n+1)\sqrt{\varepsilon}\end{pmatrix}$$

$$\begin{pmatrix}\psi^1(r+\varepsilon,n)\\ \psi^2(r,n+1)\sqrt{\varepsilon}\end{pmatrix} = \frac{1}{k_\varepsilon}\begin{pmatrix}1 & b\sqrt{\varepsilon}\\ \bar b\sqrt{\varepsilon} & 1\end{pmatrix}\begin{pmatrix}\psi^1(r,n)\\ \psi^2(r,n)\sqrt{\varepsilon}\end{pmatrix}$$

$$\frac{k_\varepsilon\,\psi^1(r+\varepsilon,n)-\psi^1(r,n)}{\varepsilon} = b\,\psi^2(r,n)$$

$$k_\varepsilon\,\psi^2(r,n+1)-\psi^2(r,n) = \bar b\,\psi^1(r,n)$$

~~[illegible]~~ You have to compute a Green's functions. You have decided to study the horizontally cont., vert. disc. grid grid equations

$$\begin{cases} (\partial_r - a)\psi^1 = b\psi^2 \\ \Delta\psi^2 = \bar{b}\psi^1 \end{cases} \qquad 2a = |b|^2$$

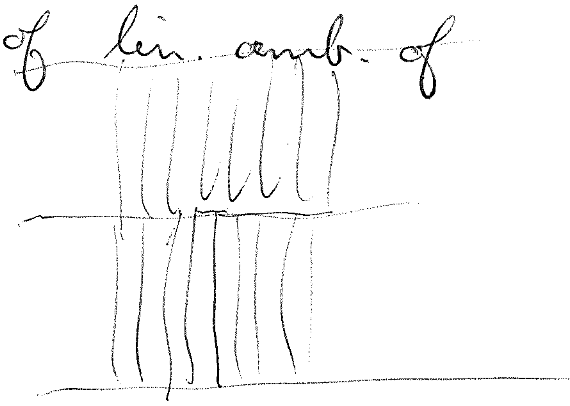
exp. solution $\quad e^{i r\rho}\mu^n \begin{pmatrix}\hat{\psi}^1 \\ \hat{\psi}^2\end{pmatrix} \qquad \begin{aligned}(i\rho - a)\hat{\psi}^1 &= b\hat{\psi}^2 \\ (\mu - 1)\hat{\psi}^2 &= \bar{b}\hat{\psi}^1\end{aligned}$

$$\mu = 1 + \frac{\overset{2a}{\cancel{|b|^2}}}{i\rho - a} = \frac{i\rho + a}{i\rho - a} = \frac{\rho - ia}{\rho + ia}$$

general solution = some type of lin. comb. of

$$e^{ir\rho}\left(\frac{i\rho + a}{i\rho - a}\right)^n \begin{pmatrix}\frac{b}{i\rho - a} \\ 1\end{pmatrix}$$

Picture of grid space



~~[illegible]~~

Cauchy problems. Find solution $\psi(r,n)$ with given $\psi(r,0)$, or $\psi(0,n)$.

$$\psi(r,0) = \int_{-\infty}^{\infty} e^{ir\rho}\,\hat{\psi}_0(\rho)\,\frac{d\rho}{2\pi} \qquad ??$$

Then $$\psi(r,n) = \int_{-\infty}^{\infty} e^{ir\rho}\left(\frac{i\rho + a}{i\rho - a}\right)^n \begin{pmatrix}\frac{b}{i\rho - a} \\ 1\end{pmatrix}$$

missing something like $(\partial_r - a)\psi^1(r,0) = b\psi^2(r,0)$ which determines $\psi^2(r,0)$ from $\psi^1(r,0)$.

~~Grid~~ Grid equations
$$\left|\begin{array}{l}(\partial_r - a)\psi^1(r,n) = b\,\psi^2(r,n)\\ \psi^2(r,n+1) - \psi^2(r,n) = \bar{b}\,\psi^1(r,n)\end{array}\right.$$

$$\therefore\ \psi^2(r,n+1) = \psi^2(r,n) + \frac{\bar{b}b^{\,2a}}{\partial_r - a}\psi^2(r,n) = \frac{\partial_r + a}{\partial_r - a}\psi^2(r,n)$$

$$\boxed{\psi^2(r,n) = \left(\frac{\partial_r + a}{\partial_r - a}\right)^n \psi^2(r,0)} \quad \boxed{\psi^1(r,n) = \frac{b}{\partial_r - a}\psi^2(r,n)}$$

The grid equations describe linear functions on the desired hypothetical grid space. Recall the idea that grid space should split into a horizontal space generated by v under the 1-parameter group $\{\lambda^r\}$ of horizontal translations, and a vertical space generated by u under the discrete group $\mu^{\mathbb{Z}}$ of vertical translations. Model for grid space to consist of ~~...~~ meromorphic functions of ρ

Use $\begin{pmatrix} v^1 \\ v^2 \end{pmatrix}$ for $\begin{pmatrix} \psi^1 \\ \psi^2 \end{pmatrix}$??

Cauchy problems along $n=0$ solved by
$$\psi^2(r,n) = \left(\frac{\partial_r + a}{\partial_r - a}\right)^n \psi^2(r,0), \qquad \psi^1(r,n) = \frac{b}{\partial_r - a}\psi^2(r,n)$$
Now to calculate $(\ |\ )$ and $IH(\ ,\ )$, it suffices to consider a cyclic vector. $v^1 = \psi^1(0,0)$, the universal one, or $v^2 = \psi^2(0,0)$. How to proceed?

~~The only thing you can see is~~ First do the Hilbert space picture, here you expect a unitary equivalence between horizontal + vertical subspaces.

some sort of C.T. equivalence between $L^2(\mathbb{R})$ and $L^2(S^1)$. Can you do this?

Begin with $L^2(\mathbb{R}, \frac{d\rho}{2\pi})$.

~~You see~~ $L^2(\mathbb{R}, dr) \ni \psi^2(r,0) = \int e^{ir\rho}\,\hat\psi_0^2(\rho)\,\frac{d\rho}{2\pi}$

$$\psi^2(r,n) = \int e^{ir\rho}\left(\frac{i\rho+a}{i\rho-a}\right)^n \hat\psi_0^2(\rho)\,\frac{d\rho}{2\pi}\quad, \text{ set } r=0$$

$$\psi^2(0,n) = \int \left(\frac{i\rho+a}{i\rho-a}\right)^n \hat\psi_0^2(\rho)\,\frac{d\rho}{2\pi} = \left(\frac{\partial_r+a}{\partial_r-a}\right)^n \psi^2(r,0)$$

It might be better to use

$$\psi^1(r,n) = \int e^{ir\rho}\left(\frac{i\rho+a}{i\rho-a}\right)^n \frac{b}{i\rho-a}\,\hat\psi_0^2(\rho)\,\frac{d\rho}{2\pi}$$

This resembles an inner product inside $L^2(\mathbb{R}, \frac{d\rho}{2\pi})$ between $\hat\psi_0^2(\rho)$ and $\left(\frac{i\rho+a}{i\rho-a}\right)^n \frac{b}{i\rho-a}$

$$\int_{-\infty}^{\infty}\left|\frac{b}{i\rho-a}\right|^2\frac{d\rho}{2\pi} = \int_{-\infty}^{\infty}\frac{2a}{\rho^2+a^2}\,\frac{d\rho}{2\pi}$$

$$= 2\pi i\,\frac{2a}{2(ia)}\,\frac{1}{2\pi} = 1.$$

~~Repeat ... grid space~~

~~Begin with $L^2(\mathbb{R}$~~

You should get insight into adeles from this situation