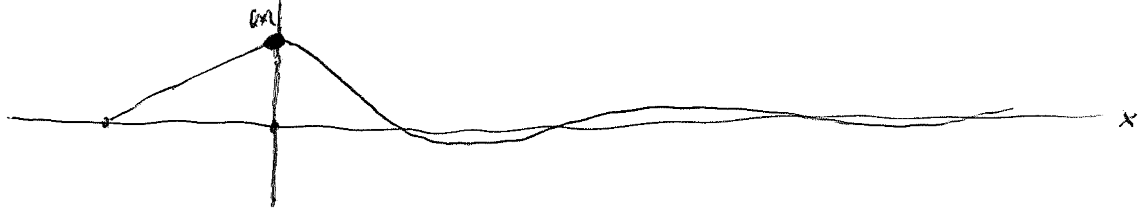
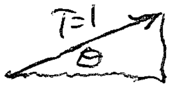
May 14, 1999

I want to study some models for emission and absorption of radiation, given by a simple harmonic oscillator coupled to a photon field. String model:



For $x > 0$ we have a string of uniform density 1 and tension 1; let $u(x,t)$ be the displacement. At $x=0$ the string is attached to a mass m which in turn is joined by a thread (weightless string) to a point on the negative axis. Ignoring $x>0$ the mass is a simple harmonic oscillator whose motion is given by $m(\ddot{y} + \omega_0^2 y) = 0$, where $y(t) = u(0,t)$. The force on the mass due to the string is



$T \sin\theta \sim \tan\theta = \partial_x u|_{x=0}$.

So the eqn of motion is

$$\left\{ \begin{aligned} m(\ddot{y} + \omega_0^2 y) &= \partial_x u(0,t) \\ \partial_t^2 u &= \partial_x^2 u \end{aligned} \right. \qquad y = u(0,t)$$

To solve take the FT in time

$$u(x,t) = \int \frac{d\omega}{2\pi} e^{-i\omega t} \hat{u}(x,\omega)$$

$$\boxed{\begin{aligned} m(-\omega^2 + \omega_0^2)\hat{u}(0,\omega) &= \partial_x \hat{u}(0,\omega) \\ (\partial_x^2 + \omega^2)\hat{u}(x,\omega) &= 0 \end{aligned}}$$

So $\quad \hat{u}(x,\omega) = A e^{i\omega x} - B e^{-i\omega x}$

where A, B are functions of ω satisfying

$$m(-\omega^2+\omega_0^2)(A-B) = i\omega(A+B)$$

which yields

$$\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} A \\ B \end{pmatrix} = \frac{[\text{redacted}]\ \varepsilon i\omega}{-\omega^2+\omega_0^2\ [\text{redacted}]} \qquad \varepsilon = \frac{1}{m}$$

$$\frac{A}{B} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\left(\frac{\varepsilon i\omega}{-\omega^2+\omega_0^2}\right) = \frac{-\omega^2+\omega_0^2+\varepsilon i\omega}{-\omega^2+\omega_0^2-\varepsilon i\omega}$$

So

$$\boxed{S = \frac{A}{B} = \frac{\omega^2-\omega_0^2-\varepsilon i\omega}{\omega^2-\omega_0^2+\varepsilon i\omega}}$$

Note that the poles of S are at

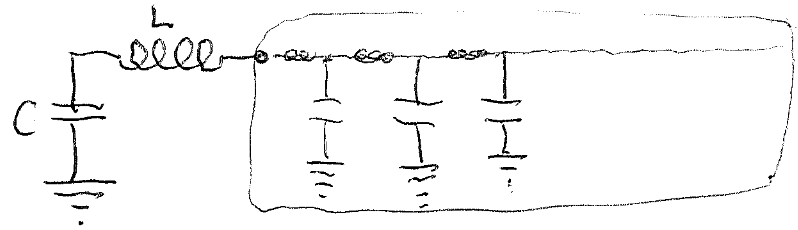
$$\omega = \frac{-i\varepsilon \pm \sqrt{-\varepsilon^2+4\omega_0^2}}{2} = -\frac{i\varepsilon}{2} \pm (\omega_0 + O(\varepsilon))$$ which lies in the LHP

(clearer: $-i\omega = -\frac{\varepsilon}{2} \pm i(\omega_0+O(\varepsilon))$.) As a check note that

$$u(x,t) = \int \frac{d\omega}{2\pi}\Big(\underbrace{A(\omega) e^{i\omega(x-t)}}_{\text{outgoing}} - \underbrace{B(\omega) e^{-i\omega(x+t)}}_{\text{incoming}}\Big)$$

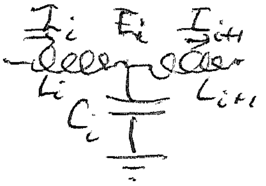
Thus we have a damped harmonic oscillator motion for the mass m.

Next, the transmission line model:



transmission line = a limit of a LC ladder.

Equations for LC ladder are



$$E_i - E_{i-1} = -L_i \partial_t I_i$$

$$I_{i+1} - I_i = -C_i \partial_t E_i$$

Taking the continuous limit with $\frac{L_i}{\Delta x} \to 1$, $\frac{C_i}{\Delta x} \to 1$ yields the equations

$$\partial_x E = -\partial_t I \quad \text{and} \quad \partial_x I = -\partial_t E$$

whence
$$\begin{cases} (\partial_x + \partial_t)(E+I) = 0 \\ (\partial_x - \partial_t)(E-I) = 0 \end{cases}$$
for the transmission line

Look at frequency ω:
$$(E+I)(x,t) = A e^{i\omega(x-t)}$$
$$(E-I)(x,t) = B e^{-i\omega(x+t)}$$

$$\left.\frac{E+I}{E-I}\right|_{x=0} = \frac{A}{B} \qquad \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} E_{x=0} \\ I_{x=0} \end{pmatrix} = \frac{A}{B}$$

$$\frac{E_{x=0}}{I_{x=0}} = -\text{ impedance of the LC circuit} = -\left(Ls + \frac{1}{Cs}\right)$$

$$\frac{A}{B} = \frac{-\left(Ls + \frac{1}{Cs}\right) + 1}{-\left(Ls + \frac{1}{Cs}\right) - 1} = \frac{LCs^2 + 1 - Cs}{LCs^2 + 1 + Cs}$$

Now $\omega_0^2 = \frac{1}{LC}$ and $s = -i\omega$

$$\frac{A}{B} = \frac{s^2 + \omega_0^2 - \frac{1}{L}s}{s^2 + \omega_0^2 + \frac{1}{L}s}$$

poles are at

$$s = \frac{-\varepsilon \pm \sqrt{\varepsilon^2 - 4\omega_0^2}}{2} = -\frac{\varepsilon}{2} \pm i(\omega_0 + O(\varepsilon))$$

where [redacted] $\varepsilon = 1/L$.

May 16, 1999

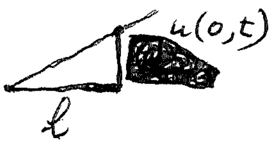
Some simpler examples. First suppose the mass at $x=0$ is zero. Then we have

$$\partial_t^2 u = \partial_x^2 u \quad , \qquad k\,u(0,t) = \partial_x u(0,t)$$

$$\hat{u}(x,\omega) = Ae^{i\omega x} - Be^{-i\omega x} \quad , \qquad k(A-B) = i\omega(A+B)$$

$$\begin{pmatrix}1 & -1\\ 1 & 1\end{pmatrix}\begin{pmatrix}A\\ B\end{pmatrix} = \begin{matrix}i\omega\\ k\end{matrix} \quad , \qquad \frac{A}{B} = \begin{pmatrix}1 & 1\\ -1 & 1\end{pmatrix}\begin{pmatrix}i\omega\\ k\end{pmatrix} = \frac{i\omega+k}{-i\omega+k}$$

$$S = \frac{A}{B} = \frac{-\omega+ik}{\omega+ik}$$ has pole at $\omega = -ik \in$ LHP.

(Note  force $= -\frac{u_{x=0}}{\ell}$ so $k = \frac{1}{\ell}$).

Second suppose $l=\infty$, i.e. no thread attached to the string at $x=0$, and there is a mass $m>0$ here.

$$\partial_t^2 u = \partial_x^2 u \quad , \qquad m\,\partial_t^2 u\big|_{x=0} = \partial_x u\big|_{x=0}$$

Then $m(-\omega^2)(A-B) = (i\omega)(A+B)$, $S = \begin{pmatrix}1 & 1\\ -1 & 1\end{pmatrix}\begin{pmatrix}i\omega\\ -m\omega^2\end{pmatrix} = \frac{-i\omega + m\omega^2}{+i\omega + m\omega^2}$

$$S = \frac{A}{B} = \frac{\omega - i\varepsilon}{\omega + i\varepsilon} \quad , \qquad \varepsilon = \frac{1}{m}$$

Return to the string attached to a simple harmonic oscillator:

$$\boxed{\partial_t^2 u = \partial_x^2 u \ \text{ for } x \geq 0 \ , \quad \left[m(\partial_t^2 u + \omega_0^2 u) - \partial_x u\right]_{x=0} = 0.}$$

Solutions with time dependence $e^{-i\omega t}$ have the form

$$e^{-i\omega t}\hat{u}(x,\omega) = e^{-i\omega t}\left(A(\omega)e^{i\omega x} - B(\omega)e^{-i\omega x}\right)$$

where $\frac{A}{B} = \frac{\omega^2 - \omega_0^2 - i\varepsilon\omega}{\omega^2 - \omega_0^2 + i\varepsilon\omega}$. You get solutions of the equations of motion by taking a suitable linear combination of these for different ω.

Next let's discuss the energy. Because we are considering a harmonic oscillator, the energy gives an inner product preserved by time evolution on the space of finite energy solutions. The energy is

$$E(u,\dot u) = \int_0^\infty \left(\tfrac{1}{2}\dot u^2 + \tfrac{1}{2}(\partial_x u)^2\right)dx + \tfrac{1}{2}m(\dot u_{x=0})^2 + \tfrac{1}{2}m\omega_0^2(u_{x=0})^2$$

$$\partial_t E = \int_0^\infty \underbrace{\left(\dot u \ddot u + \partial_x u\, \partial_x \dot u\right)}_{}dx + m(\dot u \ddot u)_{x=0} + m\omega_0^2(u\dot u)_{x=0}$$

$$= \dot u_{x=0}\left(m\ddot u + m\omega_0^2 u\right)_{x=0}$$

$$\dot u\, \partial_x^2 u + \partial_x \dot u\, \partial_x u = \partial_x(\dot u\, \partial_x u)$$

and $\int_0^\infty \partial_x(\dot u \partial_x u)\,dx = -(\dot u \partial_x u)_{x=0}$, (assuming no contribution at ∞).

But $(\partial_x u)_{x=0} = (m\ddot u + m\omega_0^2 u)_{x=0}$, so $\boxed{\partial_t E = 0.}$

Suppose $\hat u(x,\omega) = \int_{-\infty}^{\infty} dt\, e^{i\omega t} u(x,t)$ and

$$u(x,t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega t}\hat u(x,\omega)$$

where $u(x,t)$ is a sufficiently nice solution of the equations of motion. Then we have seen that $\hat u(x,\omega) = A(\omega)e^{i\omega x} - B(\omega)e^{-i\omega x}$, $\frac{A}{B} = \frac{\omega^2-\omega_0^2 - i\varepsilon}{\omega^2-\omega_0^2+i\varepsilon}$.

The functions A, B should be nice except at $\omega = 0$. My aim is to find the energy $E(u)$ in terms of the pair A, B. (Actually since $S = \frac{A}{B}$ is non-vanishing on $\mathbb{R}$, the functions A, B are equivalent to each other.

$$u(x,t) = \int_{-\infty}^{\infty}\frac{d\omega}{2\pi}A(\omega)e^{i\omega(x-t)} + \int_{-\infty}^{\infty}\frac{d\omega}{2\pi}(-1)B(\omega)e^{-i\omega(x+t)}$$

$$= F(x-t) + G(-x-t)$$

where F and G are the Fourier transforms of A and B respectively. Then

$$u(x+t,t) = F(x) + G(-x-2t) \xrightarrow{t\to+\infty} F(x)$$

$$u(x-t,t) = F(x-2t) + G(-x) \xrightarrow{t\to-\infty} G(-x)$$

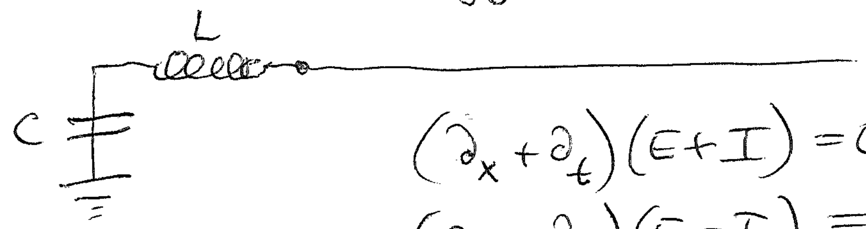
in "good" cases. These limits give the outgoing and incoming representations respectively. In good cases these representations preserve the energy so

$$E(u,\dot u) = \int_{-\infty}^{\infty} \tfrac{1}{2}\{(\partial_t F(x-t))^2 + (\partial_x F(x-t))^2\}\,dx$$

$$= \int_{-\infty}^{\infty} F'(x)^2\,dx = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\,|\omega A(\omega)|^2$$

and similarly $E(u,\dot u) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\,|\omega B(\omega)|^2$, also this follows since $|S(\omega)| = 1$ for ω real.

Next discuss the energy in the electrical example



$$(\partial_x + \partial_t)(E+I) = 0$$

$$(\partial_x - \partial_t)(E-I) = 0$$

time dep $e^{-i\omega t}$

$E+I = Ae^{i\omega x}$ outgoing

$E-I = Be^{-i\omega x}$ incoming

$$S = \begin{pmatrix} A \\ \hline B \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} E_{x=0} \\ \hline I_{x=0} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}(-2$$

where $Z = Ls + \frac{1}{Cs}$ so $S = \dfrac{s^2 + \omega_0^2 - \varepsilon s}{s^2 + \omega_0^2 + \varepsilon s}$ $\qquad \omega_0^2 = \frac{1}{LC}$, $\varepsilon = \frac{1}{L}$

$$EN = \int_0^{\infty} \tfrac{1}{2}(E^2 + I^2)\,dx + \tfrac{1}{2}CE_C^2 + \tfrac{1}{2}LI_L^2$$

$$\partial_t \int_0^\infty \tfrac{1}{2}(E^2+I^2)\,dx = \int_0^\infty \Big(E\overbrace{\partial_t E}^{-\partial_x I} + I\overbrace{\partial_t I}^{-\partial_x E}\Big)\,dx$$

$$= -\int_0^\infty \partial_x(EI)\,dx = E_{x=0}I_{x=0}$$

$$\partial_t\left(\tfrac{1}{2}CE_C^2\right) = CE_C\dot{E}_C = -E_CI_C$$

$$\partial_t\left(\tfrac{1}{2}LI_L^2\right) = LI_L\dot{I}_L = -E_LI_L$$

But $I_{x=0}=I_C=I_L$ and $E_{x=0}=E_L+E_C$ $\therefore\ \partial_t(EN)=0.$

Correction: You once believed that exponential decay is possible classically but not quantum mechanically because of the positive energy condition. By exponential decay you mean $|\langle\psi, e^{-itH}\psi\rangle| \leq Ce^{-\varepsilon|t|}$; since $\langle\psi, e^{-itH}\psi\rangle = \int e^{-it\omega}\langle\psi, dE_\omega\psi\rangle$ = F.T. of the spectrum measure, exp decay $\Longrightarrow$ $\langle\psi, dE_\omega\psi\rangle = \rho(\omega)\,d\omega$, where ρ is analytic in a strip $|\mathrm{Im}(\omega)| < \varepsilon$, and this can't happen if $\rho \equiv 0$ [redacted] $\forall\omega < 0$ unless $\rho \equiv 0$.

But [redacted] classically you have the same problem: say $f \in L^2(\mathbb{R}, \frac{d\omega}{2\pi})$, then $\langle f, e^{-it\omega}f\rangle = \int_{-\infty}^{\infty} e^{-it\omega}|f(\omega)|^2\frac{d\omega}{2\pi}$ decays exponentially iff $|f(\omega)|^2$ is analytic in a strip about the real axis. So [redacted] you need to understand what states, if any, decay exponentially.

Idea: Decay is studied, controlled by the semi-group of contractions on H^+/SH^+.

<u>May 17, 99</u>

Simple harmonic oscillator review.

$H = \frac{p^2}{2m} + \frac{1}{2}kq^2$, Hamilton's eqns. $\dot{q} = \frac{\partial H}{\partial p} = \frac{p}{m}$,

$\dot{p} = -\frac{\partial H}{\partial q} = -kq$, so $\ddot{q} = \frac{\dot{p}}{m} = -\frac{k}{m}q$ and

the frequency is $\omega = \sqrt{\frac{k}{m}}$. To quantize we assume [redacted] $[p,q] = \frac{\hbar}{i}$. Let $a = \lambda ip + \mu q$, $a^* = -\lambda ip + \mu q$ with λ, μ

to be determined so that $[a, a^*] = 1$ and $H = \hbar\omega(a^*a + \frac{1}{2})$

$[a, a^*] = 2\lambda\mu\hbar = 1$. Then ~~[illegible]~~

$a^*a = \lambda^2 p^2 + \mu^2 q^2 + \lambda\mu[-ip, q]$ $-\lambda\mu\hbar = -\frac{1}{2}$. So

$\hbar\omega(a^*a + \frac{1}{2}) = \hbar\omega\lambda^2 p^2 + \hbar\omega\mu^2 q^2 \Rightarrow \hbar\omega\lambda^2 = \frac{1}{2m}, \hbar\omega\mu^2 = \frac{k}{2}$

so $\lambda^2 = \frac{1}{2\hbar(km)^{1/2}}$, $\mu^2 = \frac{(km)^{1/2}}{2\hbar}$. Formulas are complicated!

Classically, the state space for [redacted] a s.h.o. is a 2 dim'l real vector space equipped with positive definite form given by the energy and a skew [redacted] symmetric $\neq 0$ operator which is the infinitesimal generator for time [redacted] evolution. [redacted]

[redacted] A general harmonic oscillator should be given by a real Hilbert space equipped with an invertible skew-symmetric operator. Applying polar decomposition this skew-symm. operator should be product of a complex structure and a positive definite hermitian operator H.

May 21, 1999

Review simple QFT's with one-diml space.

Consider an oriented (smooth) circle. First, there is a real symplectic space given by real functions modulo constants with the skew-form $\int f\,dg$. Next, one has two spin structures on the circle, real line bundles with square given isomorphic to the real cotangent bundle of the circle. Then one has a real vector space with positive inner product given by the sections of the line bundle, where the inner product is $s \mapsto \int s^2$, the integral defined using the orientation.

The QFT's are given by irreducible Hilbert space representations of the CCR (resp. CAR) associated to a real symplectic (resp. orthogonal) vector space. In finite diml such representations (assume even-dimensional in the orthogonal case) are unique. Recall the construction first for the CAR. Given V orthogonal (Euclidean) one wants to represent elements $v \in V$ by self-adjoint operators ϕ_v on a complex Hilbert space such that $\phi_v^2 = |v|^2$. There's a $*$-algebra generated by V with these relation, which is the Clifford algebra over $\mathbb{C}$ generated by $(V, |v|^2)$. When $\dim V = 2n$ this C^* alg is isomorphic to $\mathcal{L}(E)$, where $E \simeq \Lambda\mathbb{C}^n$. The specific construction: Form $V_c = V \otimes_R \mathbb{C}$, extend the quadratic form $|v|^2$ $\mathbb{C}$-bilinearly to V_c, and let W be a maximal isotropic subspace of V_c. Then $V_c = W \oplus \overline{W}$ acts on ΛW by $(w_1 + w_2^*)\xi = e(w_1)\xi + i(w_2)\xi$ as usual.

All that happens here is that we are reducing $O(2n)$ to $U(n)$. What might be relevant would be to look at the situation over the space $O(2n)/U(n)$ of complex structures on $\mathbb{R}^{2n}$. You have a bundle of irreducible representations with cyclic vector of the Clifford algebra. There should be some complex line bundle over $O(2n)/U(n)$, and it probably has a metric + connection. Curvature?

It appears that $O(2n)/U(n) \simeq$ maximal isotropic subspaces of V_c, which is a projective variety over $\mathbb{C}$. $\dim_{\mathbb{R}} O(2n)/U(n) =$
$\frac{1}{2} 2n(2n-1) - n^2 = n(2n-1) - n^2 = n(n-1)$. Find $\dim_{\mathbb{C}}$ of isotropic lines $=$ hypersurface in $\mathbb{P}(V_c)$ so $\dim_{\mathbb{C}} = 2n-2$.
$\dim_{\mathbb{C}}$ of all isotropic flags is $2n-2 + 2n-4 + \cdots + 2 = n(n-1)$.
$\dim_{\mathbb{C}}$ of flags in a given W^n is $n-1 + n-2 + \cdots + 1 = \frac{n(n-1)}{2}$
So $\dim_{\mathbb{C}}$ of max isot. subspaces is $n(n-1) - \frac{n(n-1)}{2} = \frac{n(n-1)}{2}$ which agrees with the $\dim_{\mathbb{R}}$ above.

Next symplectic case. $\dim_{\mathbb{R}} Sp(2n, \mathbb{R})/U(n)$
$= \frac{2n(2n+1)}{2} - n^2 = n^2 + n.$

A polarization of V real symp dim $2n$ is a maximal isotropic subspace W of V_c such that the hermitian form $[w_1, w_2^*]$ is positive definite. (This amounts to the CCR: $[a_i, a_j] = 0$, $[a_i, a_j^*] = \delta_{ij}$.). Calculate $\dim_{\mathbb{C}}$ of the maximal isotropic subspaces of V_c. Any line is isotropic, so $\dim_{\mathbb{C}}$ of isotropic flags is $(2n-1) + (2n-3) + \cdots + 1 = n^2$ and $\dim_{\mathbb{C}}$ of isotropic subspaces is $n^2 - (n-1 + n-2 + \cdots + 1) = n^2 - \frac{n(n-1)}{2}$
$= n(n+1)/2$

So what should be true is that the polarizations form ~~the~~ open set of the symplectic Grassmannian of max isotropic (Lagrangian) subspaces of V_c consisting of the ones with $[\omega, \omega^*] > 0$, for $\omega \neq 0$. There should be a real hypersurface where $W \cap V \neq 0$ and the complement is a union of open sets indexed by the signature.

I have been examing quantizing the above examples of wave equation - string and transmission line. The string situation is usually quantized as a harmonic oscillator. What does this mean?

In finite dimensions a harmonic oscillator is given by a real vector space Q [redacted] of configurations (or positions) together with two positive quadratic forms, kinetic and potential energy. The equation of motion is obtained from the Lagrange equation [redacted] with $L = T - V$. [redacted] For quantization one has the choice of Feynman's path integral (Lagrangian approach) or [redacted] via operators + Hilbert space (Hamiltonian "). Consider the latter. The tangent bundle of Q is identified with the cotangent bundle [redacted] via a Legendre transform (essentially use the metric provided by the kinetic energy). Then we get a real symplectic vector space $V_{\mathbb{R}}$ ("phase space") and a Hamiltonian function H which gives the energy and is a positive quadratic form on $V_{\mathbb{R}}$. Then time evolution on phase space is the vector field X determined by Hamilton's eqns: $\iota_X \omega = dH$ where ω is the symplectic form. In this ~~situation~~ X is the vector field associated to a linear operator on $V_{\mathbb{R}}$.

From the linear algebra viewpoint we have $V_{\mathbb{R}}$ equipped with non-degenerate skew symmetric form ω and pos quadratic form H and time evolution is the operator X on $V_{\mathbb{R}}$ such that $\omega X = H$, ~~[illegible]~~ where you view ω, H as maps $V_{\mathbb{R}} \to V_{\mathbb{R}}^*$.

Choose a basis for V_λ, so that $\omega X = H$ becomes a matrix equation, i.e.

$$\xi_1^t \, \omega X \xi_2 = \xi_1^t H \xi_2 \qquad \begin{array}{l} \forall \xi_2 \in V_\lambda \\ \forall \xi_1^t \in V_\lambda^* \end{array}$$

Let's check that $X = \omega^{-1} H$ is skew-symmetric wrt the inner product associated to H. The adjoint X^* with resp. to H is given by

$$\eta^t H X^* \xi = (X\eta)^t H \xi = \eta^t X^t H \xi$$

$$\Longrightarrow \quad H X^* = X^t H \quad \Longrightarrow \quad X^* = H^{-1} X^t H.$$

But $H^{-1} X^t H = H^{-1} (\omega^{-1} H)^t H = H^{-1} \underbrace{H^t}_{H} \underbrace{(\omega^{-1})^t}_{(-\omega)^{-1}} H$

$= H^{-1} H (-\omega)^{-1} H = -X.$

So X is skew-symmetric + invertible so V_λ splits into orthogonal 2-diml subspaces invariant under X.

March 27, 1999   Harmonic Oscillator Algebra.

Let $A : V \longrightarrow V^*$, $H : V \longrightarrow V^*$ be skew-symmetric and symmetric forms on the vector space V, resp. Assume A invertible and let $X = A^{-1} H : V \longrightarrow V$. Then X preserves A in the sense that $X^t A + AX = 0$ and X —— H —— $X^t H + HX = 0$.

Check: $\quad X^t A + AX = H^t \underbrace{(A^{-1})^t}_{-A^{-1}} A + A A^{-1} H = -H + H = 0$

$$X^t H + HX = H^t (A^{-1})^t H + H A^{-1} H = 0.$$

Here are cleaner statements. 1) Assume $H : V \to V^*$ symmetric and invertible. Then one has a 1-1 corresp between skew-symmetric $A : V \longrightarrow V^*$ and operators X on V such that $X^t H + HX = 0$ given by $A = HX$.

2) Assume $A: V \to V^*$ skew-symmetric and invertible. Then one has a 1-1 corresp. between symmetric $H: V \to V^*$ and operators X such that $X^t A + A X = 0$ given by $H = AX$.

Proof of 2). Given X sat $X^t A + AX = 0$ then $H = AX$ sat $H^t = X^t A^t = -X^t A = AX = H$. Conv. given H symmetric, then $X = A^{-1}H$ is defined since $A^{-1}\ \exists$ and

$$X^t A + AX = H^t (A^{-1})^t A + A A^{-1} H$$
$$= H(-A^{-1})A + H = 0.$$

(Here use $AA^{-1} = I \Rightarrow (A^{-1})^t A^t = I$ and $A^{-1}A = I \Rightarrow A^t (A^{-1})^t = I$. So $(A^{-1})^t = (A^t)^{-1} = -A^{-1}$.)

Consider now a real vector space V, say finite dimensional, equipped with a positive definite quadratic form H, and let X be an operator on V which preserves H, equivalently is skew-symmetric wrt. H: $X^t H + HX = 0$. By spectral theory V splits into invariant 2 planes under X and $\mathrm{Ker}(X)$. If X is invertible then $\mathrm{Ker}(X) = 0$ and $\dim(X)$ is even. Put $A = HX^{-1}$ so that $AX = H$. Then X gives the Hamiltonian flow on V associated the Hamiltonian $\frac{1}{2}v^t H v$ and the symplectic structure A.

Example of simple harm. osc. $V = \mathbb{R}^2 = \left\{ \begin{pmatrix} q \\ p \end{pmatrix} \middle| \, q, p \in \mathbb{R} \right\}$

$$H(v) = \frac{k}{2} q^2 + \frac{p^2}{2m} = \frac{1}{2} \begin{pmatrix} q \\ p \end{pmatrix}^t \begin{pmatrix} k & \\ & \frac{1}{2m} \end{pmatrix} \begin{pmatrix} q \\ p \end{pmatrix} \qquad A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$AX \begin{pmatrix} q \\ p \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \dot q \\ \dot p \end{pmatrix} = \begin{pmatrix} k & \\ & m^{-1} \end{pmatrix} \begin{pmatrix} q \\ p \end{pmatrix} \qquad \therefore\ \dot q = \frac{p}{m} \quad -\dot p = kq.$$

May 31, 1999

List some ideas from scratchwork.

1. Consider the harmonic oscillator situation: real vector space equipped with positive definite quadratic form H (the energy), time evolution operator X, assumed nondegenerate, and nondegenerate skew-symmetric form A (symplectic structure), all these related by $AX = H$. Associated to this structure seems to be both a fermionic and a bosonic quantization. There should be a way to combine these in a "supersymmetric" way. This might provide a model for Witten's supersymmetric quantum mechanics using the free loop space $L(M)$.

2. I have been focussing on wave equations on $\mathbb{R}$, more precisely $L^2(\mathbb{R})$ with time evolution $X = \partial_x$, but there is the compactified version $L^2(S^1)$. This is the usual framework for the Jacobi triple product identity. Note that the Hilbert space constructions, i.e. Fock spaces, depend on a choice of polarization, which is a lot less than a time evolution operator, somehow a part of kinematics rather than dynamics. It might be worthwhile to take your real $L^2(\mathbb{R})$ to to be the intrinsic Hilbert space of L^2-real-sections of $\mathcal{O}(-1)$ over $\mathbb{RP}^1$, then to compare different dynamics: translation $x \mapsto x+t$ and rotation around i. Also to study dividing by $\mathbb{Z}$ translation action.

3. Energy transfer: $\partial_t \int_R \left(\frac{1}{2}\dot{u}^2 + \frac{1}{2}(u')^2\right) dx$

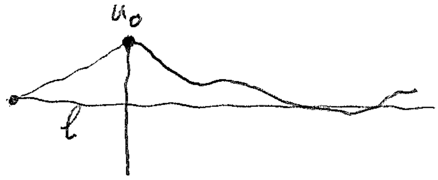
$$\simeq \int_{\partial R} \dot{u}u' \,,$$ where $\dot{u}u'$ looks like $p\,dq$.

June 4, 1999

Consider the wave equation for a light string attached to a simple harmonic oscillator (see p 1, May 14, 1999). This is a harmonic oscillator situation where the configuration space consists of certain real-valued functions $u(x)$ defined for $x \geq 0$. On this space one has quadratic forms giving the KE and the PE:

$$KE = \frac{1}{2} m \dot{u}_0^2 + \int_0^\infty \frac{1}{2} \dot{u}^2 \, dx, \qquad PE = \frac{1}{2\ell} u_0^2 + \int_0^\infty \frac{1}{2} (\partial_x u)^2 \, dx$$

Picture



$$\ddot{u} = \partial_x^2 u \qquad x \geq 0$$

$$m \ddot{u}_0 = -\frac{1}{\ell} u_0 + (\partial_x u)_0$$

Check:
$$\partial_t \int_0^\infty \frac{1}{2} (\dot{u}^2 + \partial_x u^2) \, dx = \int_0^\infty (\dot{u}\ddot{u} + \partial_x u \, \partial_x \dot{u}) \, dx = \int_0^\infty \partial_x (\partial_x u \, \dot{u}) \, dx$$
$$= -(\partial_x u)_0 \dot{u}_0 = -\left(m \ddot{u}_0 + \frac{1}{\ell} u_0\right) \dot{u}_0 = -\partial_t \left(\frac{1}{2} m \dot{u}_0^2 + \frac{1}{2\ell} u_0^2 \right)$$

A standard way to treat a geometric wave equation of the form $\ddot{u} = -\Delta u$, where $-\Delta$ is a Laplacian, is to interpret Δ as a positive self-adjoint operator on an L^2 space of configurations, and then use the spectral theory for Δ. In the present situation the L^2 space of configurations is $L^2(\mathbb{R}_{\geq 0}, d\mu)$ where $d\mu$ is the mass distribution, i.e. a point mass m at $x = 0$ and Lebesgue measure for $x > 0$. The potential energy should determine a positive self-adjoint operator on this Hilbert space which is essentially given by the differential operator $-\partial_x^2$ together with a boundary type condition to handle the point mass. To show self-adjointness one produces the Green's function $(\lambda + \partial_x^2)^{-1}$, and hopefully one can easily get the eigenvalue expansion by contour integration.

June 6, 1999

Recall the Green's function $G_\lambda(x,x')$ for 2nd order DE Sturm-Liouville problem, e.g. $-\partial_x^2 u + qu = \lambda u$ on an interval $\subset \mathbb{R}$, has the form

$$(1) \qquad G_\lambda(x,x') = \begin{cases} \dfrac{u(x)\,v(x')}{W(x')} & x \le x' \\[2ex] \dfrac{u(x')\,v(x)}{W(x')} & x \ge x' \end{cases}$$

where u_λ (resp v) is an eigenfunction satisfying the left (resp right) boundary condition and $W = \begin{vmatrix} u & v \\ u' & v' \end{vmatrix}$ is the Wronskian. $G_\lambda = (\lambda + \partial_x^2 - q)^{-1}$ is analytic in λ off the spectrum and usually one has

$$\oint G_\lambda(x,x')\,\frac{d\lambda}{2\pi i} = \delta(x-x')$$

yielding an eigenfunction expansion (Titchmarsh book).

Example 1. $(\lambda + \partial_x^2)u = 0$ for $x \geq 0$, boundary condition (Dirichlet): $u(0) = 0$. The spectrum is $\mathbb{R}_{\geq 0}$, and $\{\lambda \notin \mathbb{R}_{\geq 0}\} \xrightarrow{\sim} \{\omega \in UHP\}$ via $\lambda = \omega^2$. Then

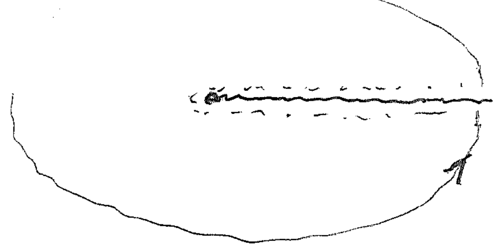
$$u_\omega(x) = \frac{-e^{i\omega x} + e^{-i\omega x}}{2i\omega} = -\frac{\sin(\omega x)}{\omega} \quad (\text{so } S = -1), \quad v_\omega(x) = e^{i\omega x}$$

decays for $\omega \in UHP$ as $x \to +\infty$.

$$W = \begin{vmatrix} -\dfrac{\sin(\omega x)}{\omega} & e^{i\omega x} \\ -\cos(\omega x) & i\omega e^{i\omega x} \end{vmatrix} = e^{i\omega x}(-i\sin\omega x + \cos\omega x) = 1.$$

Comment. (1) means that $G_\lambda(x,x')$ satisfies $(\lambda - \Delta)G_\lambda = 0$ for $x \neq x'$ and the two boundary conds, G_λ is continuous at $x = x'$ and $\partial_x G_\lambda$ jumps by $+1$ at $x = x'$. Note $G_\lambda(x,x')$ is symmetric.

$$\oint G_\lambda(x,x')\,\frac{d\lambda}{2\pi i} =$$



$$= \int_{+\infty}^{-\infty} G_{\omega^2}(x,x')\,\frac{2\omega\,d\omega}{2\pi i}$$

odd fn of ω

$$= \int_{\infty}^{-\infty} -\frac{\sin\omega x}{\cancel{\omega}}\,e^{i\omega x'}\,\frac{\cancel{\omega}\,d\omega}{\pi i} = \int_{-\infty}^{\infty} (\sin\omega x)\,e^{i\omega x'}\,\frac{d\omega}{\pi i}$$

$$= \int_{-\infty}^{\infty} \sin(\omega x)\,\cancel{i}\,\sin(\omega x')\,\frac{d\omega}{\pi \cancel{i}} = \frac{2}{\pi}\int_0^\infty \sin(\omega x)\,\sin(\omega x')\,d\omega$$

Thus $\frac{2}{\pi}\int_0^\infty \sin(\omega x)\sin(\omega x')\,d\omega = \delta(x-x')$, the Fourier sine transform eigenfunction expansion.

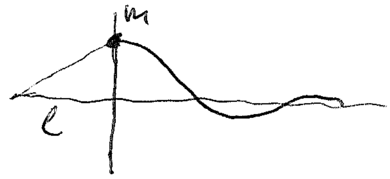
Ex 2. $(\lambda+\partial_x^2)u=0$ for $x \geq 0$, Neumann b.c. $(\partial_x u)_0 = 0$.

Here $u_\omega = e^{i\omega x} + e^{-i\omega x}$ (hence $S=1$), $v = e^{i\omega x}$

$$G_\omega(x,x') = \frac{(e^{i\omega x}+e^{-i\omega x})\,e^{i\omega x'}}{2i\omega} \qquad x<x'$$

$$-\int_{+\infty}^{\infty} G_\omega(x,x')\,\frac{2\omega\,d\omega}{2\pi i} = \int_{-\infty}^{\infty} \overbrace{(e^{i\omega x}+e^{-i\omega x})}^{\text{even in }\omega}\,e^{i\omega x'}\,\frac{d\omega}{2\pi} = 2\int_0^\infty 2\cos(\omega x)\cos(\omega x')\,\frac{d\omega}{2\pi}$$

$$= \frac{2}{\pi}\int_0^\infty \cos(\omega x)\cos(\omega x')\,d\omega = \delta(x-x').$$



Next I want to look at a string with s.h.o. attached. In this situation the equations of motion are $\ddot{u} = \partial_x^2 u$ for $x \geq 0$ and $m\ddot{u}_0 = -\frac{1}{\ell}u_0 + (\partial_x u)_0$. Using the first, the second may be written $\left(m\partial_x^2 u + \frac{1}{\ell}u - \partial_x u\right)_0 = 0$, i.e. a sort of boundary condition with derivatives. Also there are limiting cases of interest: $m=0$, $\ell=\infty$.

Calculate the eigenfunctions: $u_\omega(x) = Ae^{i\omega x} + Be^{-i\omega x}$ where $\left(-m\omega^2 + \frac{1}{\ell}\right)(A+B) = i\omega(A-B)$ and you get

$$S = \frac{A}{B} = (-1)\,\frac{\omega^2-\omega_0^2-i\varepsilon\omega}{\omega^2-\omega_0^2+i\varepsilon\omega} \qquad \omega_0^2 = \frac{1}{m\ell}\ , \quad \varepsilon = \frac{1}{m}$$

$$G_\omega(x,x') = \frac{\left(Se^{i\omega x}+e^{-i\omega x}\right)e^{+i\omega x'}}{2i\omega} \qquad x<x'$$

symm. for $x>x'$.

$$= \left(\frac{S+1}{2i\omega}\right)e^{i\omega(x+x')} - \frac{\sin(\omega x)}{\omega}\,e^{i\omega x'}$$

where $\displaystyle \frac{S+1}{2i\omega} = \frac{1}{2i\omega}\left(1-\frac{\omega^2-\omega_0^2-i\varepsilon\omega}{\omega^2-\omega_0^2+i\varepsilon\omega}\right) = \frac{\varepsilon}{\omega^2-\omega_0^2+i\varepsilon\omega}$

I am uncertain about what this means. The F.T.

$$\int_{-\infty}^{\infty}\frac{\varepsilon}{\omega^2-\omega_0^2+i\varepsilon\omega}\,e^{i\omega x}\,\frac{d\omega}{2\pi} = \begin{cases} 0 & x>0 \\ [redacted]\ \varepsilon\left(\dfrac{e^{ir_1x}}{2ir_1-\varepsilon}+\dfrac{e^{ir_2x}}{2ir_2-\varepsilon}\right) & x<0 \end{cases}$$

r_1, r_2 roots of denom.

$$= -\frac{i\varepsilon}{2} \pm \omega_0' \qquad \omega_0' = \frac{1}{2}\sqrt{-\varepsilon^2+4\omega_0^2}$$

[redacted] $ir_1, ir_2 = \dfrac{\varepsilon}{2} \pm i\omega_0'$ so for $x<0$

$$\int_{-\infty}^{\infty}\frac{\varepsilon}{\omega^2-\omega_0^2+i\varepsilon\omega}\,e^{i\omega x}\,\frac{d\omega}{2\pi} = \varepsilon\,\mathrm{Re}\left\{\frac{e^{ir_1x}}{ir_1-\frac{\varepsilon}{2}}\right\} = \frac{\varepsilon}{\omega_0'}\,\mathrm{Re}\,\frac{e^{\left(\frac{\varepsilon}{2}+i\omega_0'\right)x}}{i}$$

$$= \varepsilon\,e^{\frac{\varepsilon}{2}x}\,\frac{\sin(\omega_0'x)}{\omega_0'}$$

Meaning?

June 9, 1999

Discuss philosophy of harmonic oscillators in infinite dimensions. Begin with global solutions of the equations of motion. For every $\omega \in \mathbb{C}$ you can consider motions with time dependence $e^{-i\omega t}$, but you want to restrict ω to be real and perhaps $\neq 0$, and form what these generate to get phase space. Phase space should decompose according to the eigenvalues of time evolution (frequencies real $\neq 0$). Then splitting into positive and negative frequencies yields a complex structure on phase space, equivalently a polarization.

Next you need the energy function on phase space, which should be conserved under time evolution. You divide energy by time evolution to get the symplectic form. What seems to be emerging is that phase space consists of global solutions of the equation of motion having finite energy; it is a real Hilbert space with the energy norm, and time evolution gives a unitary representation of $\mathbb{R}$. Apply spectral theorem to get frequency analysis and complex structure.

Example of string attached to s.h.o. Solution with pure frequency ω is $e^{-i\omega t}(S(\omega)e^{i\omega x}+e^{-i\omega x})$ up to a constant factor. Note that the Wronskian of $Se^{i\omega x}+e^{-i\omega x}$ with $e^{-i\omega x}$ is $2i\omega$ and that

$$\frac{Se^{i\omega x}+e^{-i\omega x}}{2i\omega} = \frac{\varepsilon}{\omega^2-\omega_0^2+i\varepsilon\omega}e^{i\omega x} - \frac{\sin(\omega x)}{\omega}$$

is nice even at $\omega=0$: $\lim_{\omega\to 0}$ is $\frac{\varepsilon}{-\omega_0^2}-x$. If we use

the representation

$$u(x,t) = \int_{-\infty}^{\infty}\frac{d\omega}{2\pi}e^{-i\omega t}\,\frac{Se^{i\omega x}+e^{-i\omega x}}{i\omega}\,B(\omega)$$

the energy should be $\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}|B(\omega)|^2$ because as $t\to-\infty$

$$u(x-t,t) = \int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\left(e^{i\omega(x-2t)}A(\omega)+e^{-i\omega x}\right)\frac{B(\omega)}{i\omega} \to \frac{\hat{B}}{i\omega}(-x)$$

so the energy $\int \frac{1}{2}(\dot{u}^2+(\partial_x u)^2)\,dx = \int(\partial_x u)^2 dx$
$= \int |B|^2\frac{d\omega}{2\pi}$. It seems that the above representation is a bit complicated for calculation purposes, because of the denominator $i\omega$. Simpler would be to work with

$$\mathring{u}(x,t) = -\int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\left(S e^{i\omega(x-t)} + e^{-i\omega(x+t)}\right) B(\omega)$$

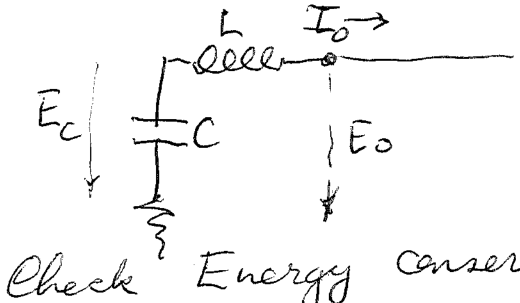
$$\text{Energy} = \|B\|^2 .$$

Electrical example. Wave Eqn. $(\partial_x+\partial_t)(E+I)=0$, $(\partial_x-\partial_t)(E-I)=0$ $\quad x \geq 0 \ \forall t$

$$\frac{E+I}{2} = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} A(\omega)\, e^{i\omega(x-t)} = \hat{A}(x-t)$$

$$\frac{E-I}{2} = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} B(\omega)\, e^{-i\omega(x+t)} = \hat{B}(-x-t)$$

Actually we the transmission line coupled to a series LC circuit



$$C\dot{E}_C = -I_0$$
$$E_0 - E_C = -L\dot{I}_0$$

$$\text{Total energy} = \tfrac{1}{2}CE_C^2 + \tfrac{1}{2}LI_0^2 + \int_0^\infty \tfrac{1}{2}(E^2+I^2)\,dx$$

Check Energy conserved $\quad C\dot{E}_C E_C + L\dot{I}_0 I_0 + \int_0^\infty (E\dot{E} + I\dot{I})\,dx$

$$= -I_0 \cancel{E_C} + (-E_0 + \cancel{E_C})I_0 + \int_0^\infty \underbrace{\left(E(-\partial_x I) + I(-\partial_x E)\right)}_{-\partial_x(EI)} dx = -E_0 I_0 + [-EI]_0^\infty = 0$$

Boundary condition for
$\frac{E+I}{2} = A e^{i\omega(x-t)}$, $\frac{E-I}{2} = Be^{-i\omega(x+t)}$ $\qquad \frac{E_0+I_0}{E_0-I_0} = \frac{A}{B} = S$

$$S = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} E_0 \\ I_0 \end{pmatrix} \qquad \frac{E_0}{I_0} = +\frac{1}{C(+i\omega)} + L(+i\omega)$$

$$S = \frac{\frac{1}{Ci\omega} + Li\omega + 1}{\frac{1}{Ci\omega} + Li\omega - 1} = \frac{-\frac{1}{LC} + \omega^2 - iL^{-1}\omega}{-\frac{1}{LC} + \omega^2 + iL^{-1}\omega} = \frac{\omega^2 - \omega_0^2 - i\varepsilon\omega}{\omega^2 - \omega_0^2 + i\varepsilon\omega}$$

We have
$$E(x,t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\left(S(\omega)e^{i\omega(x-t)} + e^{-i\omega(x+t)}\right)B(\omega)$$
$$I(x,t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\left(S(\omega)e^{i\omega(x-t)} - e^{-i\omega(x+t)}\right)B(\omega)$$

The energy of this solution of the equations of motion should be $\int \frac{d\omega}{2\pi}|B(\omega)|^2$. Why? One has

$$\int_0^\infty \tfrac{1}{2}(E^2+I^2)\,dx = \int_0^\infty \left(\left(\frac{E+I}{2}\right)^2 + \left(\frac{E-I}{2}\right)^2\right)dx = \int_0^\infty \left(\hat{A}(x-t)^2 + \hat{B}(-x-t)^2\right)dx$$

$$= \int_{-t}^{\infty} \hat{A}(x)^2\,dx + \int_{-\infty}^{-t} \hat{B}(x)^2\,dx \longrightarrow \begin{matrix} \|A\|^2 & \text{as } t\to\infty \\ \|B\|^2 & \text{as } t\to-\infty \end{matrix}$$

To make this precise you would want to understand the missing energy terms $\tfrac{1}{2}CE_C^2 + \tfrac{1}{2}LI_0^2$ and to see that they go to zero as $t \to \pm\infty$. So you would like to check that I_0 and $E_C = E_0 + L\mathring{I}_0$ well defined numbers for any $B \in L^2$. But $I_0 = \int \frac{d\omega}{2\pi} (S-1) e^{-i\omega t} B(\omega)$

and $S - 1 = \dfrac{-2i\varepsilon\omega}{\omega^2 - \omega_0^2 + i\varepsilon\omega} \in H^2_+$. Also

$$E_C = E_0 + L\mathring{I}_0 = \int \frac{d\omega}{2\pi} \{(S+1) + L(-i\omega)(S-1)\} e^{-i\omega t} B(\omega)$$

and $(S+1) + L(-i\omega)(S-1) = \Big(\underbrace{\dfrac{S+1}{S-1}}_{E_0/I_0} - Li\omega\Big)(S-1)$

$$= \frac{1}{iC\omega} \frac{-2i\varepsilon\omega}{\omega^2 - \omega_0^2 + i\varepsilon\omega} = \frac{-2\omega_0^2}{\omega^2 - \omega_0^2 + i\varepsilon\omega}. \quad \text{OK.}$$

(In fact $\mathring{E}_C$ is ~~i~~ well-defined, which checks with $C\dot{E}_C = -I_0$.)

June 10, 1999

Let's recall the classical and quantum treatments of a harmonic oscillator (f. diml.). We want to use descriptions which make time flow evident (& hence avoid the Lagrangian picture). Begin with the quantum picture. Here everything is described by a complex Hilbert space E equipped with a positive self-adjoint operator H. Suppose $E = \mathbb{C}^n$ with $H = \mathrm{diag}(\omega_1, \cdots, \omega_n)$, all $\omega_j > 0$. The quantum state space is best described by the holomorphic repn.

$$\left\{ f(z_1, \cdots, z_n) \text{ entire on } \mathbb{C}^n \;\middle|\; \|f\|^2 = \int_{\mathbb{C}^n} e^{-|z|^2} |f(z)|^2 \left(\frac{i}{2\pi}\right)^n \prod_{j=1}^n dz_j d\bar{z}_j < \infty \right\}$$

on which one has the operators $a_j = \partial_{z_j}$, $a_j^* = z_j$ satisfying the CCR. The energy operator is $H = \sum \omega_j a_j^* a_j$. The Hilbert space E can be identified with the 1-particle subspace $\mathbb{C}z_1 \oplus \cdots \oplus \mathbb{C}z_n$, where the z_j are orthonormal. The time flow is $\sum c_j z_j \longmapsto \sum e^{-i\omega_j t} c_j z_j$.

In the classical picture operators become functions on phase space. Consider the real vector space of self-adjoint operators of the form $\sum (c_j a_j^* + \bar{c}_j a_j)$. Call this space V. On this one has a symplectic form given by $\frac{1}{i}[-,-]$. We have a map given by acting on the ground state $|0\rangle = 1$

$$\sum c_j a_j^* + \bar{c}_j a_j \longmapsto \sum c_j a_j^* + c_j a_j |0\rangle = \sum c_j z_j$$

which is a bijection between V and the 1-particle Hilbert space.

The point: Inf. diml harmonic oscillators such as those arising from wave equations seem to yield a real Hilbert space of global solutions with the energy norm and a time flow which preserves norm, hence yields a skew adjoint time-evolution operator. So the natural norm on the space of operators $\sum c_j a_j^* + \bar{c}_j a_j$

appears to be the energy norm^2 $\sum' \omega_j |c_j|^2$, rather than the norm^2 of the associated 1-particle state which is $\sum |c_j|^2$. Formulas.

$$\| \overbrace{c_j a_j^* + \bar{c}_j a_j}^{c_j z_j} |0\rangle \|^2 = \langle 0 | (c_j a_j^* + \bar{c}_j a_j)^2 | 0 \rangle = |c_j|^2$$

Energy: $\underline{\langle \sum c_j z_j | \sum \omega_j a_j^* a_j | \sum c_j z_j \rangle} = \sum_j \omega_j |c_j|^2$.

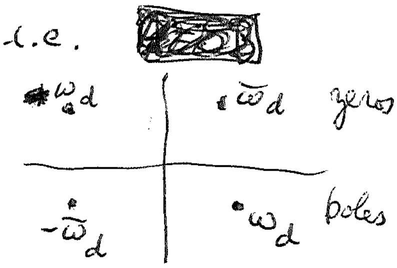
Return to

$$\tfrac{1}{2}(E+I)(x,t) = \int \frac{d\omega}{2\pi} A(\omega) e^{i\omega(x-t)} \qquad = 0 \text{ if } x>t \text{ and } A \in H^2_+$$

$$\tfrac{1}{2}(E-I)(x,t) = \int \frac{d\omega}{2\pi} B(\omega) e^{-i\omega(x+t)} \qquad = 0 \text{ if } x>-t \text{ and } B \in H^2_-$$

You want $A = SB$ with $A \in H^2_+$, $B \in H^2_-$, i.e.

$A = SB \in H^2_+ \cap SH^2_-$. $\qquad S = \dfrac{\omega^2 - \omega_0^2 - i\varepsilon\omega}{\omega^2 - \omega_0^2 + i\varepsilon\omega}$



Here $\omega_d = -\dfrac{i\varepsilon}{2} + \omega_0'$ $\qquad \omega_0' = \dfrac{\sqrt{-\varepsilon^2 + 4\omega_0^2}}{2}$

So the space $H^2_+ \cap SH^2_-$ is spanned by $\dfrac{i}{\omega - \omega_d}$, $\dfrac{i}{\omega + \bar{\omega}_d}$.

Take $A(\omega) = \dfrac{1}{2}\left(\dfrac{i\alpha}{\omega - \omega_d} + \dfrac{i\bar{\alpha}}{\omega + \bar{\omega}_d} \right)$. Then for $x<0$

$$\hat{A}(x) = \frac{-2\pi i}{2\pi} \frac{1}{2} \left(i\alpha e^{i\omega_d x} + i\bar{\alpha} e^{-i\bar{\omega}_d x} \right) = \mathrm{Re}\left(\alpha e^{i\omega_d x} \right),$$

where $\alpha \in \mathbb{C}$ and $i\omega_d = \dfrac{\varepsilon}{2} + i\omega_0'$.

Now you want the quantum version. Our phase space with the energy norm is the space of L^2 functions complex-valued $A(\omega)$ sat. $A(\omega) = \overline{A(-\omega)}$, $\omega \in \mathbb{R}$ and Energy is $2\int_0^\infty \frac{d\omega}{2\pi} |A(\omega)|^2$. The 1-particle Hilbert space will be those $A(\omega)$ sat the same reality condition, but the norm^2 should be $\int_0^\infty \frac{d\omega}{\pi} \frac{|A(\omega)|^2}{\omega}$. Note that the reality condition can be dropped provided we restrict ω to $\omega > 0$. The complex structure is the obvious one for functions on $\omega > 0$. So the phase space and 1-particle space are different completions of a common dense subspace.

Consider now a classical state supported in the s.h.o: $A(\omega) = \frac{1}{2}\left(\frac{i\alpha}{\omega - \omega_d} + \frac{i\bar{\alpha}}{\omega + \bar{\omega}_d}\right)$

If $A(0) \neq 0$ this will now be in the 1 particle space. $A(0) = \frac{1}{2}\left(\frac{i\alpha}{-\omega_d} + \frac{i\bar{\alpha}}{\bar{\omega}_d}\right) = Re\left(\frac{i\alpha}{-\omega_d}\right)$. Taking

$\alpha = \omega_d$ gives $A(\omega) = \frac{1}{2}\,\frac{i\omega_d(\omega + \bar{\omega}_d) + i\bar{\omega}_d(\omega - \omega_d)}{(\omega - \omega_d)(\omega + \bar{\omega}_d)}$

$= \frac{i\omega\, Re(\omega_d)}{\omega^2 - \omega_0^2 + i\varepsilon\omega}$ which was encountered on p 22 up to a scalar; it corresponds to I_0, simpler is $A(\omega) = S - 1$.

Classically

$$\langle A | e^{-i\omega t} A \rangle = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\, e^{-i\omega t} \underbrace{|A(\omega)|^2}_{A(\omega)A(-\omega)}$$

decays exponentially because $A(\omega)A(-\omega)$ is analytic in a strip about $\mathbb{R}$

Quantum

$$\langle A | e^{-i\omega t} A \rangle = \int_0^{\infty} \frac{d\omega}{\pi}\, e^{-i\omega t}\, \frac{|A(\omega)|^2}{\omega}$$

should not decay exponentially, because of the break in analyticity at $\omega = 0$. Here $A = S - 1$ and $\frac{|A(\omega)|^2}{\omega} = \frac{A(\omega)A(-\omega)}{\omega}$ is analytic at $\omega = 0$ and vanishes to first order, which probably means $\langle A | e^{-i\omega t} A \rangle = O\left(\frac{1}{t^2}\right)$ as $t \to \infty$.

June 11, 1999

Ideas from scratchwork.

1. In classical scattering theory the whole Hilbert space + 1-parameter unitary group is determined by the contraction semigroup on H^+ / SH^+. Is there a quantum analog?

2. Planck's observation that $\left(\frac{Gh}{c^3}\right)^{1/2}$ is an absolute unit for distance (independent of the experimenter and his system of units for distance, time, and mass) means that constants such as the mass and lifetime of a particle ~~should~~ be definite real numbers like π, volumes of fundamental domains. Does there exist an arithmetic picture behind quatum mechanics?

3. In the quantization of a wave equation, there are quantities such as the energy, and the imaginary part of the hermitian inner product, which are geometric (integrals of local expressions for example). The real part of the hermitian inner product is non local.

June 14, 1999

Fermion quantization. Recall the algebraic situation first. Given a vector space V and a subspace W there is a variant of ΛV which has lines canonically attached to subspaces of V commensurable with W, namely:

$$\Lambda(V;W) = \Lambda V/W \otimes \Lambda W^*$$

Alternatively, equip $V \oplus V^*$ with the quadratic function $(v+\lambda)^2 = (\lambda, v)$, and form ~~the~~ Clifford algebra $C(V\oplus V^*)$. This acts on $\Lambda(V;W)$ via operators $e_v + \iota_\lambda$ satisfying $(e_v+\iota_\lambda)^2 = (\lambda,v)$, and there is a distinguished vector $|0\rangle$ killed by e_v, ι_λ for $v\in W$ and $\lambda \in (V/W)^* = W^\circ$.

In the Hilbert space situation $H = H_+ \oplus H_-$, $V = H$ and $W = H_-$. The operators $e_v + \iota_{v^*}$, where v^* is the lin. fnl. $(v, -)$, are self adjoint and satisfy $(e_v + \iota_{v^*})^2 = \|v\|^2$. Acting on $|0\rangle$ with these operators gives an $\mathbb{R}$-linear map

$$V \longrightarrow \underset{\underset{W^\perp}{\parallel}}{V/W} \oplus W^* \qquad v \mapsto (v \bmod W) + v^*|W$$

which is a bijection. It's just $V = W^\perp \oplus W$ $\xrightarrow{\sim} W^\perp \oplus W^*$ where the second component is antilinear. Note $W^* \cong \overline{W}$ via the hermitian inner product.

Consider $V = L^2(\mathbb{R}, \frac{d\omega}{2\pi})$ consisting of $(A(\omega))_{\omega\in\mathbb{R}}$ with time flow $A(\omega) \longmapsto e^{-i\omega t}A(\omega)$, let W be the subspace of A's supported on $\omega < 0$. Then the above isom. ($\mathbb{R}$-linear) between V and the 1-particle Hilbert space is

$$(A(\omega))_{\omega\in\mathbb{R}} \longmapsto (A(\omega))_{\omega>0} + \left(\overline{A(\omega)}\right)_{\omega<0}$$

time flow is

$$(e^{-i\omega t}A(\omega))_{\omega\in\mathbb{R}} \longmapsto (e^{-i\omega t}A(\omega))_{\omega>0} + \left(e^{i\omega t}\,\overline{A(\omega)}\right)_{\omega<0}.$$

Thus the frequencies are >0.

<u>Reality condition</u> $\overline{A(-\omega)} = A(\omega)$. ~~[illegible]~~

Restricting to such A, we have the isomorphism $(A(\omega))_{\omega\in\mathbb{R}} \longmapsto (A(\omega))_{\omega>0}$ ~~between~~ between "real" A and all complex L^2 fns defined for $\omega>0$. This means the correlation $(A, e^{-i\omega t}A) = \int_0^\infty \frac{d\omega}{2\pi}\, e^{-i\omega t}|A(\omega)|^2$ should never have exponential decay.

Recall that Graeme constructed a Pfaffian version of fermion Fock space that might be appropriate to this reality condition. I seem to recall that instead of there being only lines corresponding to subspaces, there are Gaussian type states defined using Pfaffians.

<u>June 21, 1999</u>

You want to examine the partition function which gives rise to the Jacobi triple product identity: [redacted]

$$\prod_{n\geq 0}(1+q^n z)\prod_{n\geq 1}(1+q^n z^{-1}) = \frac{\sum_{n\in\mathbb{Z}} q^{\frac{n(n-1)}{2}} z^n}{\prod_{n\geq 1}(1-q^n)}$$

Consider $L^2(S^1)$ ~~with~~ the orthonormal basis $e^{in\theta}$, then form the (fermion) Fock space [redacted] corresponding to the polarization $H^+ =$ span of $e^{in\theta}$ with $n\geq 1$. You want to take the limit as $q \uparrow 1$. The idea is that we are taking the circle $\mathbb{R}/\mathbb{Z}L$ and letting $L \longrightarrow \infty$ (infinite volume limit).

$$\log \prod_{n\geq 0}(1+q^n z) = \sum_{n\geq 0}\left(q^n z - \frac{1}{2}(q^n z)^2 + \frac{1}{3}(q^n z)^3 - \cdots\right)$$

$$= \frac{1}{1-q} z - \frac{1}{1-q^2}\frac{z^2}{2} + \frac{1}{1-q^3}\frac{z^3}{3} - \cdots$$

$$\therefore \; [redacted] (1-q) \log \prod_0^\infty (1+q^n z) \longrightarrow z - \frac{z^2}{4} + \frac{z^3}{3} - \cdots \quad \text{as } q\uparrow 1$$

which is the dilogarithm essentially. Similarly

$$\log \prod_{n\geq 1}(1+q^n z^{-1}) = [redacted] \frac{1}{1-q} z^{-1} - \frac{1}{1-q^2}\frac{z^{-2}}{2} + \cdots$$

$$(1-q)\log \prod_{n\geq 1}(1+q^n z^{-1}) \longrightarrow z^{-1} - \frac{z^{-2}}{4} + \frac{z^{-3}}{9} - \cdots \quad \text{as } q\uparrow 1$$

and

$$(1-q)\log \prod_{n\geq 1}(1-q^n) \longrightarrow -\left(1 + \frac{1}{4} + \frac{1}{9} + \cdots\right) = -\zeta(2) \quad \text{as } q\uparrow 1$$

I want to check this computation with the asymptotics of $\log \sum q^{\frac{n(n-1)}{2}} z^n$ for $z = e^{i\theta}$, $q\uparrow 1$.

The answer should be

$$\left(z+z^{-1}\right) - \frac{(z^2+z^{-2})}{4} + \frac{(z^3+z^{-3})}{9} - \cdots$$
$$-1 - \frac{1}{4} - \frac{1}{9} - \cdots$$

$$= 2\cos\theta - \frac{2\cos 2\theta}{4} + \frac{2\cos 3\theta}{9} - \cdots$$
$$-1 - \frac{1}{4} - \frac{1}{9} - \cdots$$

call this $g(\theta)$.

Then $g'(\theta) = -2\sin\theta + \dfrac{2\sin 2\theta}{2} - \dfrac{2\sin 3\theta}{3} + \cdots$

Better would be $z\dfrac{d}{dz}\left\{ \left(z+z^{-1}\right) - \dfrac{(z^2+z^{-2})}{4} + \dfrac{(z^3+z^{-3})}{9} - \cdots \right\}$

$$= \left(z - \frac{z^2}{2} + \frac{z^3}{3} - \cdots\right) - \left(z^{-1} - \frac{z^{-2}}{2} + \frac{z^{-3}}{3} - \cdots\right)$$

$$= \log(1+z) - \log(1+z^{-1}) = \log\left(\frac{1+z}{1+z^{-1}}\right) = \log z = i\theta$$

So $\dfrac{1}{i}\dfrac{d}{d\theta}\left\{ (z+z^{-1}) - \dfrac{(z^2+z^{-2})}{4} + \cdots \right\} = i\theta$. So

our function $g(\theta)$ should be $-\dfrac{\theta^2}{2}$ + constant made periodic of period 2π. in same way.

Simpler to calculate with

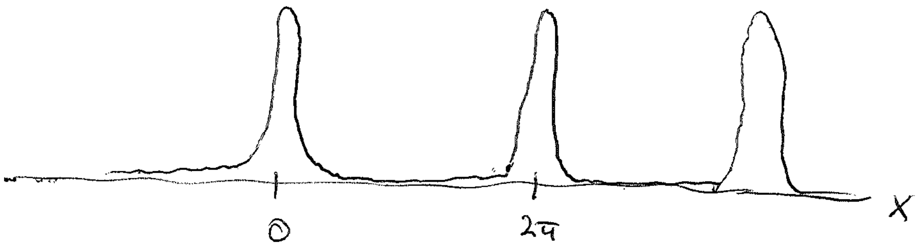
$$\prod_{n\geq 0}\left(1+q^{n+1/2}z\right)\prod_{n\geq 0}\left(1+q^{n+1/2}z^{-1}\right) = \frac{\sum_{n\in\mathbb{Z}} q^{\frac{n^2}{2}} z^n}{\prod_{n\geq 1}(1-q^n)}$$

If $q = e^{-t}$, $z = e^{ix}$, then

$$\sum_{n\in\mathbb{Z}} q^{\frac{n^2}{2}} z^n = \sum_{n\in\mathbb{Z}} e^{-t\frac{n^2}{2} + inx} = \sum_{m\in\mathbb{Z}} \frac{e^{-\frac{(x-2\pi m)^2}{2t}}}{(2\pi t)^{1/2}}$$

fundamental solution of heat eqn. $\partial_t u = \frac{1}{2}\partial_x^2 u$ on $\mathbb{R}/2\pi\mathbb{Z}$.

~~The~~ This looks like



To analyze log of this as $t \downarrow 0$ use

$$\left(a^N + b^N\right)^{1/N} = \left(1 + \left(\frac{b}{a}\right)^N\right)^{1/N} a \to a$$

as $N \to \infty$ when $0 < b < a$. We need to compare

$$e^{-\frac{(x+2\pi)^2}{2t}}, \quad e^{-\frac{x^2}{2t}}, \quad e^{-\frac{(x-2\pi)^2}{2t}}$$

as $t \downarrow 0$. For $-\pi \lesssim x \lesssim \pi$ the maximum is $e^{-\frac{x^2}{2t}}$, so you get

$$t \log\left(\sum_{m \in \mathbb{Z}} \frac{e^{-\frac{(x-2\pi m)^2}{2t}}}{(2\pi t)^{1/2}}\right) \longrightarrow t\left(-\frac{x^2}{2t}\right) = -\frac{x^2}{2} \quad \text{for } -\pi \lesssim x \lesssim \pi$$

and this is extended to be periodic of period 2π.

Note the constant is zero. Check

$$1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \cdots = \left(1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16}\right) - 2\left(\frac{1}{4} + \frac{1}{16} + \frac{1}{36} + \cdots\right)$$

$$= \left(1 - \frac{2}{4}\right)\zeta(2) = \frac{1}{2}\zeta(2).$$

Thus $2\left(1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \cdot\right) - \zeta(2) = \zeta(2) - \zeta(2) = 0.$

June 28, 1999

Consider an infinite chain of coupled pendulums, equivalently a Krein string with unit masses at each $n \in \mathbb{Z}$ connected by massless string segments. $KE = \frac{1}{2} \sum_n \dot{u}_n^2$

$V = PE = \frac{1}{2} \sum_n (u_n - u_{n-1})^2$. The equation of motion is

$$\boxed{\ddot{u}_n = u_{n+1} - 2u_n + u_{n-1}}$$

This has exponential solutions $e^{i\omega t} \lambda^n$, where

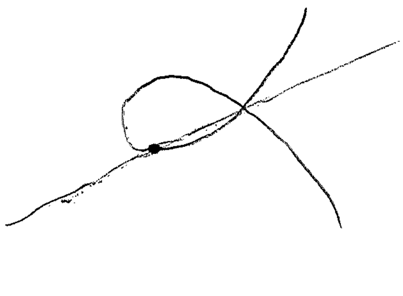
$$\boxed{(i\omega)^2 = \lambda - 2 + \lambda^{-1} = (\lambda^{1/2} - \lambda^{-1/2})^2}$$

Introduce the complex variable $z = \lambda^{1/2}$, and consider the map $z \longmapsto (z - z^{-1}, z^2)$

$$\{z \in \mathbb{C}^\times\} \longrightarrow \{(i\omega, \lambda) \in \mathbb{C} \times \mathbb{C}^\times \mid (i\omega)^2 = \lambda - 2 + \lambda^{-1}\}.$$

Given such a pair $(i\omega, \lambda)$, we can choose z such that $z^2 = \lambda$, then $(i\omega)^2 = (z - z^{-1})^2$, so $i\omega = \pm(z - z^{-1})$. Since the arbitrariness of the choice of z is its sign, there is a unique choice of z satisfying both $z^2 = \lambda$ and $z - z^{-1} = i\omega$, provided $\omega \neq 0$. If $\omega = 0$, then one has two points $z = \pm 1$ mapping to $(0, 1)$. So the above map is bijective except for the fibre $\{\pm 1\}$.

Explanation: We are looking at a singular plane cubic curve $\omega^2 \lambda + \lambda^2 - 2\lambda + 1 = 0$, and the normalized curve is the projective line



<u>July 4, 1999</u>

Consider the inf. diml harmonic oscillator with eqn of motion

$$\ddot{u}_n = u_{n+1} - 2u_n + u_{n-1}$$

This has the family of exponential solutions $e^{i\omega t}\lambda^n$ where $(i\omega, \lambda) \in \mathbb{C} \times \mathbb{C}^\times$ lies on the plane curve $(i\omega)^2 = \lambda - 2 + \lambda^{-1}$, which is a singular plane cubic curve with ordinary double point. The resolution of this singular curve (= normalization) is

$$\mathbb{C}^\times \longrightarrow \{(i\omega, \lambda) \mid (i\omega)^2 = \lambda - 2 + \lambda^{-1}\} \underset{\mathbb{C}\times\mathbb{C}^\times}{\cap}$$

$$z \longmapsto (i\omega, \lambda) = (z - z^{-1}, z^2).$$

Solutions of the equation of motion can be obtained as linear combinations of these exponential functions, where by linear combination we mean something like a distribution or hyperfunction supported on this spectral curves. The obvious thing to look at first are finite energy solutions.

The finite energy solutions should form a Hilbert space on which time flow is a unitary 1-parameter gp. By the spectral thm. we should only need exponential functions with values in S^1, i.e. $\omega \in \mathbb{R}$, $|\lambda| = 1$ hence $|z| = 1$, say $z = e^{i\theta/2}$, whence $\omega = \dfrac{z - z^{-1}}{i} = 2\sin\left(\dfrac{\theta}{2}\right)$, so $0 \leq \omega \leq 2$ and $|\lambda| = 1$.

Let's calculate the energy of

$$u_n(t) = \int_{|z|=1} e^{i\omega t} z^{2n} f(z) \frac{dz}{2\pi i z}$$

$$\dot{u}_n(t) = \int e^{i\omega t}\, i\omega f(z) \; z^{2n} \frac{dz}{2\pi i z}$$

$e^{in\theta} \dfrac{d\theta}{4\pi} = \lambda^n \dfrac{d\lambda}{2\pi i \lambda}$ where $0 \le \theta \le 4\pi$

$$= \int_0^{4\pi} \underbrace{\frac{e^{i\omega t} f(z) - e^{-i\omega t} f(-z)}{2} (i\omega)}_{\text{this has period } 2\pi} e^{in\theta} \frac{d\theta}{4\pi}$$

$$\dot{u}_n(t) = \int_0^{2\pi} \underbrace{\frac{e^{i\omega t} f(z) - e^{-i\omega t} f(-z)}{2} i\omega}\; e^{in\theta} \frac{d\theta}{2\pi}$$

So the sequence $n \mapsto \dot{u}_n(t)$ is essentially the sequence of Fourier coeffs of the function, so we have

$$\frac{1}{2} \sum_n |\dot{u}_n(t)|^2 = \frac{1}{2} \int_0^{2\pi} \left| \frac{e^{i\omega t} f(z) - e^{-i\omega t} f(-z)}{2} \right|^2 |\omega|^2 \frac{d\theta}{2\pi}$$

for the kinetic energy. Next

$$u_{n+1}(t) - u_n(t) = \int e^{i\omega t} f(z)(z^2 - 1)\, z^{2n} \frac{dz}{2\pi i z}$$

$$= \int_0^{2\pi} \frac{e^{i\omega t} f(z) + e^{-i\omega t} f(-z)}{2} (z^2 - 1)\; e^{in\theta} \frac{d\theta}{2\pi}$$

$|z|^2 |z - z^{-1}|^2 = |\omega|^2$

$$\frac{1}{2} \sum_n |u_{n+1}(t) - u_n(t)|^2 = \frac{1}{2} \int_0^{2\pi} \left| \frac{e^{i\omega t} f(z) + e^{-i\omega t} f(-z)}{2} \right|^2 |z^2 - 1|^2 \frac{d\theta}{2\pi}$$

for the potential energy. Thus

$$\text{Total Energy} = \frac{1}{2} \int_0^{2\pi} \frac{|f(z)|^2 + |f(-z)|^2}{2} |\omega|^2 \frac{d\theta}{2\pi}$$

$$= \frac{1}{2}\int_0^{4\pi} \frac{|f(z)|^2 + |f(-z)|^2}{2} |\omega|^2 \frac{d\theta}{4\pi}$$

$$= \frac{1}{2}\int_0^{4\pi} |f(z)|^2 |\omega|^2 \frac{d\theta}{4\pi}$$

$$= \frac{1}{2}\int_{|z|=1} |f(z)|^2 |z - z^{-1}|^2 \frac{dz}{2\pi i z}$$

July 5, 1999

Observe that

$$u_n(t) = \int_{|z|=1} e^{i\omega t} z^{2n} f(z) \frac{dz}{2\pi i z}$$

is defined for $n \in \frac{1}{2}\mathbb{Z}$, and it satisfies

$$* \quad \boxed{\dot{u}_n(t) = u_{n+\frac{1}{2}}(t) - u_{n-\frac{1}{2}}(t)}$$

which gives essentially the flow on phase space.

Let's discuss asymptotics of solutions of our wave equation. Consider

$$\phi_n(t) = \int_{|z|=1} e^{t(z-z^{-1})} z^{2n} \frac{dz}{2\pi i z}$$

which is the solution of $*$ such that $\phi_n(0) = \begin{cases} 1 & n=0 \\ 0 & n \neq 0 \end{cases}$ where $n \in \frac{1}{2}\mathbb{Z}$.

For t fixed $\phi_n(t)$ is essentially the sequence of Laurent series coefficients for $e^{t(z-z^{-1})}$ on $\mathbb{C}^\times$, so $\phi_n(t) = O(R^{-n})$ as $|n| \to \infty$ for any $R > 0$.

Next when $t \to \infty$ and n is fixed you can use steepest descent. The critical points of $i\omega = z - z^{-1}$ are where $1 + z^{-2} = 0$ i.e. $z = \pm i$, whence $\omega = \pm 2$. For $z = e^{i\theta/2}$ one has $\omega = 2\sin\left(\frac{\theta}{2}\right)$ which is real-valued with maximum 2 at $\theta = \pi$, and minimum -2 at $\theta = -\pi$. You should get the asymptotics as $t \to \infty$ without deforming the circle to a steepest descent at the critical points, and get something like

$$\phi_n(t) \sim \left(e^{2it}\, i^{2n} + e^{-2it}(-i)^{2n}\right)\frac{1}{\sqrt{4\pi i t}}$$

up to a constant factor. It seems there is no exponential decay.

Remark that $\oint e^{t(z-z^{-1})} z^{2n} \frac{dz}{2\pi i z}$ should be an (ordinary) Bessel function, and the above asymptotic expansion should easily accessible.

<u>July 8, 1999</u>

I would like now to study inverse scattering, both in the discrete and continuous cases. Let's start with the discrete case using filtration picture I found this spring.

Consider a Hilbert space H with unitary operator u and a filtration by closed subspaces F_n, $n \in \mathbb{Z}$ such that $F_n, uF_n \subset F_{n+1}$ are both of codimension one and

$$
* \qquad
\begin{tikzcd}
 & uF_n \arrow[dr, dash, "q_{n+1}"] & \\
uF_{n-1} \arrow[ur, dash, "up_n"] & & F_{n+1} \\
 & F_n \arrow[ul, dash, "q_n"] \arrow[ur, dash, "p_{n+1}"'] &
\end{tikzcd}
$$

is bicartesian for all n. Unit vectors $p_n \in F_n \ominus F_{n-1}$, $q_n \in F_n \ominus uF_{n-1}$ are given $\forall n$ such that up_n and p_{n+1} agree up to a positive scalar factor under $uF_n / uF_{n-1} \xrightarrow{\sim} F_{n+1}/F_n$, and also q_n and q_{n+1} agree up to a positive scalar factor under $F_n / uF_{n-1} \xrightarrow{\sim} F_{n+1}/uF_n$. In this situation, we have seen that

$$
\begin{pmatrix} p_{n+1} \\ q_{n+1} \end{pmatrix} = \frac{1}{\sqrt{1-|h_n|^2}} \begin{pmatrix} 1 & h_{n+1} \\ \bar{h}_{n+1} & 1 \end{pmatrix} \begin{pmatrix} up_n \\ q_n \end{pmatrix}
$$

where $h_{n+1} = (q_{n+1}, p_{n+1}) = -(q_n, up_n)$.

Proof. $\frac{up_n}{q_{n+1}}$ and $\frac{p_{n+1}}{q_n}$ are both orthonormal bases of $F_{n+1} \ominus uF_{n-1}$, so we have

$$\begin{pmatrix} p_{n+1} \\ q_n \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} up_n \\ q_{n+1} \end{pmatrix}$$

where $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in U(2)$. By the assumptions relating up_n to p_{n+1} and q_n to q_{n+1} we have $a, d > 0$.* But $a^2 + |b|^2 = |b|^2 + d^2 \Rightarrow a = d$, then $0 = a\bar{c} + b\bar{d} \Rightarrow b + \bar{c} = 0$. So putting $h = b$ and $k = \sqrt{1-|h|^2} = a$, we have

$$\begin{pmatrix} p_{n+1} \\ q_n \end{pmatrix} = \begin{pmatrix} k & h \\ -\bar{h} & k \end{pmatrix} \begin{pmatrix} up_n \\ q_{n+1} \end{pmatrix}.$$

~~[illegible]~~ As $p_{n+1} = kup_n + hq_{n+1}$, $q_{n+1} \perp up_n$ one has $(q_{n+1}, p_{n+1}) = (q_{n+1}, hq_{n+1}) = h$. Similarly $q_n = -\bar{h}\,up_n + kq_{n+1}$, $up_n \perp q_{n+1}$ one has $(q_n, up_n) = -h(up_n, up_n) = -h$. Then solving you get $q_{n+1} = \frac{1}{k}(\bar{h}\,up_n + q_n)$ and $p_{n+1} = k\,up_n + \frac{h}{k}(\bar{h}up_n + q_n) = \frac{k^2 + |h|^2}{k} up_n + \frac{h}{k} q_n = \frac{1}{k}(up_n + hq_n)$ as claimed.

* NO. This is OK for $d > 0$ because up_n has constant term 0 so d is a ratio of positive nos., but it doesn't work for a since q_{n+1} can have $\neq 0$ leading coefficient.

July 23, 1999

I want to consider an example of orthogonal polynomials on S^1. Let $c^2+s^2=1$ with $c>0$, $s>0$; let H be the direct sum of the Hilbert spaces $L^2(S^1,\frac{d\theta}{2\pi})$ and $\mathbb{C}$, (where $\|1\|=1$), let $u=\begin{pmatrix} z & 0\\ 0 & 1\end{pmatrix}$ on H, and let $\xi=\begin{pmatrix} c\\ s\end{pmatrix}$, a unit vector in H. Claim ξ is a cyclic vector for (H,u). This follows from

$$f(u)\begin{pmatrix} c\\ s\end{pmatrix} = \begin{pmatrix} f(z)c\\ f(1)s\end{pmatrix} \qquad \text{for } f(z)\in\mathbb{C}[z,z^{-1}].$$

and the fact that the Laurent polynomials vanishing at $z=1$ are dense in $L^2(S^1)$. The closure of $\left\{(u-1)f(u)\begin{pmatrix} c\\ s\end{pmatrix} = \begin{pmatrix}(z-1)f(z)c\\ 0\end{pmatrix}\right\}$ in H is $\begin{pmatrix} L^2(S^1)\\ 0\end{pmatrix}$, etc. The probability measure $d\mu$ associated to this cyclic unit vector ξ is

$$\int f\,d\mu = \xi^* f(u)\xi = \begin{pmatrix} c\\ s\end{pmatrix}^*\begin{pmatrix} f(z) & 0\\ 0 & f(1)\end{pmatrix}\begin{matrix} c\\ s\end{matrix}$$

$$= c^2\int f(z)\frac{d\theta}{2\pi} + s^2 f(1)$$

i.e. $d\mu = c^2\frac{d\theta}{2\pi} + s^2\delta_{z=1}$.

Let's calculate the unnormalized orthogonal polynomial $\tilde{\varphi}_n = \sum_{k=0}^{n} a_k z^k$ where $a_0=1$

satisfying $\left(z^k \middle| \tilde{q}_n \right) = 0$ for $k = 1, \dots, n$

where the inner product is in $L^2(S^1, d\mu)$:

$$\left(z^k \middle| \sum_{k=0}^{n} a_k z^k \right) = c^2 \underbrace{\int z^{-k} \tilde{q}_n(z) \frac{d\theta}{2\pi}}_{a_k} + s^2 \sum_{k=0}^{n} a_k$$

Thus for $1 \le k \le n$ we have $a_k = \alpha$, where

$c^2 \alpha + s^2(1 + n\alpha) = 0$, $(c^2 + s^2 n)\alpha = -s^2$

or $\boxed{\alpha = \frac{-s^2}{c^2 + s^2 n}}$. Thus

$$\tilde{q}_n = 1 - \frac{s^2}{c^2+s^2 n}(z + z^2 + \dots + z^n)$$

$$\tilde{p}_n = z^n \overline{\tilde{q}_n} = z^n - \frac{s^2}{c^2+s^2 n}(z^{n-1} + \dots + 1)$$

Recall $\tilde{p}_n, \tilde{q}_n$ satisfy the recursion relation

$$\begin{pmatrix} \tilde{p}_n \\ \tilde{q}_n \end{pmatrix} = \begin{pmatrix} 1 & h_n \\ \bar{h}_n & 1 \end{pmatrix} \begin{pmatrix} z\tilde{p}_{n-1} \\ \tilde{q}_{n-1} \end{pmatrix}$$

so $\tilde{q}_n - \tilde{q}_{n-1} = \bar{h}_n \, z\tilde{p}_{n-1}$ monic

$$= -\frac{s^2}{c^2+s^2 n}(z + z^2 + \dots + z^n) + \frac{s^2}{c^2+s^2(n-1)}(z + \dots + z^{n-1})$$

whence $\bar{h}_n = -\frac{s^2}{c^2+s^2 n}$. Thus $\sum |h_n|^2 < \infty$

so $\tilde{q}_\infty$ $\exists$ in $L^2(S^1, d\mu)$. In fact

$$\tilde{q}_n \begin{pmatrix} c \\ s \end{pmatrix} = \begin{pmatrix} c\, \tilde{q}_n(z) \\ s\, \tilde{q}_n(1) \end{pmatrix}$$

where $\tilde{q}_n(1) = 1 - \frac{s^2}{c^2+s^2 n} n = \frac{c^2}{c^2+s^2 n}$

converges to zero as $n \to \infty$. Also

$$\|1 - \tilde{q}_n(z)\|^2 = \left\| \frac{s^2}{c^2+s^2 n}(z + \cdots + z^n)\right\|^2 \quad \text{in } L^2(S^1)$$

$$= \left(\frac{s^2}{c^2+s^2 n}\right)^2 n \longrightarrow 0$$

$\therefore$ $\tilde{q}_n\binom{c}{s} \longrightarrow \binom{c}{0}$. Thus we have a situation where $H^2(S^1, d\mu) > zH^2(S^1, d\mu)$ with the orthogonal complement generated by q_∞, which should be the constant function 1 in the summand $L^2(S^1)$ of H. Note that this summand is u-invariant, so q_∞ does not generate $L^2(S^1, d\mu)$ in this example.

Let's discuss the background for the above example.

First recall that given a sequence $(h_n)_{n\in\mathbb{Z}}$ with $|h_n| < 1$ for all $n \in \mathbb{Z}$, we can construct a Hilbert space H equipped with unit vectors p_{mn}, q_{mn} for $(m,n) \in \mathbb{Z}\times\mathbb{Z}$ satisfying the following conditions.

a) Orthogonality. Given (m,n) we have $p_{mn} \perp \begin{matrix} p_{m'n'} \\ q_{m'n'} \end{matrix}$ for $m' \leq m$, $n' < n$ and

$$q_{mn} \perp \left| \begin{matrix} p_{m'n'} \\ q_{m'n'} \end{matrix} \right. \quad \text{for } m' < m \text{ and } n' \le n$$

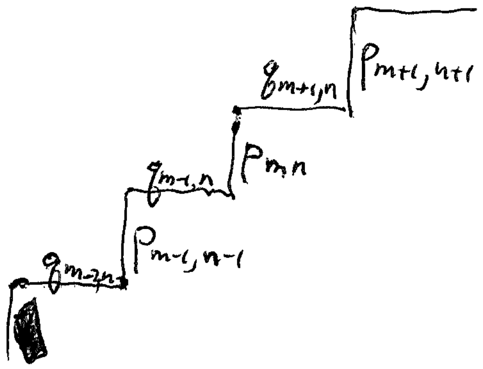
b) For each (m,n) we have

$$\begin{pmatrix} p_{mn} \\ q_{mn} \end{pmatrix} = \frac{1}{k_{mn}} \begin{pmatrix} 1 & h_{mn} \\ \bar{h}_{mn} & 1 \end{pmatrix} \begin{pmatrix} p_{m-1,n} \\ q_{m,n-1} \end{pmatrix}$$

where $h_{mn} = h_{m+n}$, and $k_{mn} = \sqrt{1 - |h_{mn}|^2}$.

c) H is generated by the vectors p_{mn}, q_{mn} for $(m,n) \in \mathbb{Z} \times \mathbb{Z}$.

You should be able to show that the p's, q's along any staircase form an orthonormal basis for H.



Also that $\exists$ a unique unitary operator u on H such that

$$u \left| \begin{matrix} p_{m,n} = p_{m-1,n+1} \\ q_{m,n} = q_{m-1,n+1} \end{matrix} \right. .$$

As at a previous time we tend to work with $p_n = p_{0,n}$ and $q_n = q_{0,n}$ which satisfy the recursion relns.

$$\begin{pmatrix} p_n \\ q_n \end{pmatrix} = \frac{1}{k_n}\begin{pmatrix} 1 & h_n \\ \bar{h}_n & 1 \end{pmatrix}\begin{pmatrix} u p_{n-1} \\ q_{n-1} \end{pmatrix}$$

Now let's examine the convergence of q_n. Writing the above as

$$\begin{pmatrix} 1 & -h_n \\ -\bar{h}_n & 1 \end{pmatrix}\begin{pmatrix} p_n \\ q_n \end{pmatrix} = k_n \begin{pmatrix} u p_{n-1} \\ q_{n-1} \end{pmatrix}$$

we get

$$\begin{aligned} q_n &= \bar{h}_n p_n + k_n q_{n-1} \\ &= \bar{h}_n p_n + k_n \bar{h}_{n-1} p_{n-1} + k_n k_{n-1} q_{n-2} \end{aligned}$$

Start at other end:

$$\begin{aligned} q_1 &= \bar{h}_1 p_1 + k_1 q_0 \\ q_2 &= \bar{h}_2 p_2 + k_2(\bar{h}_1 p_1 + k_1 q_0) \\ &= \bar{h}_2 p_2 + k_2 \bar{h}_1 p_1 + k_2 k_1 q_0 \\ q_3 &= \bar{h}_3 p_3 + k_3 \bar{h}_2 p_2 + k_3 k_2 \bar{h}_1 p_1 + k_3 k_2 k_1 q_0 \end{aligned}$$

In general one has

$$q_n = (k_n k_{n-1} \cdots k_1) q_0 + \sum_{j=1}^{n} (k_n \cdots k_{j+1}) \bar{h}_j p_j$$

Check this, noting that $q_0, p_1, \dots, p_n$ are orthonormal.

$$1 \stackrel{?}{=} |h_n|^2 + k_n^2 |h_{n-1}|^2 + k_n^2 k_{n-1}^2 |h_{n-2}|^2 + \cdots + k_n^2 \cdots \underbrace{k_2^2 |h_1|^2 + k_n^2 \cdots k_1^2}_{1-k_1^2}$$

It telescopes. What happens as $n \to \infty$.

$$\prod_{j=1}^{\infty} k_j^2 = \prod_{j=1}^{\infty} (1 - |h_j|^2) \ \ \underset{\text{and is } \neq 0}{\text{converges}} \iff \sum |h_j|^2 < \infty$$

If $\prod (1-|h_j|^2) = 0$, then ξ_n is a linear combination of the orthonormal sequence $\xi_0, \rho_1, \rho_2, \ldots$ and each coefficient tends to zero. So you have a sequence of unit vectors tending weakly to zero, but not in norm.

You hoped at one point that the scattering data in the case $\sum_{n\geq 1} |h_n|^2 < \infty$, i.e. $\xi_- = \xi_\infty$ and $\xi_+ = \lim_{n\to\infty} u^{-n} \rho_n$ could be used to reconstruct the Hilbert space H and it's structure. The preceding example should show that ξ_-, ξ_+ do not generate H under u.

July 26, 1999 (going home today)

Make a list of points from your scratchwork.

Notion of discrete 1-dim'l Dirac equation:

$$(1) \qquad \begin{pmatrix} p_n \\ q_n \end{pmatrix} = \frac{1}{k_n} \begin{pmatrix} 1 & h_n \\ \bar{h}_n & 1 \end{pmatrix} \begin{pmatrix} z p_{n-1} \\ q_{n-1} \end{pmatrix} \qquad n \in \mathbb{Z}$$

Such an equation is equivalent to the sequence $(h_n)_{n\in\mathbb{Z}}$, (recall $|h_n| < 1$, $\forall n$).

Given such a sequence (h_n) one can construct a pre-Hilbert space H_0 equipped with unitary operator u which is spanned algebraically by unit vectors p_{mn}, q_{mn}, $m,n \in \mathbb{Z}\times\mathbb{Z}$. The Dirac equation $^{(1)}$ is the eigenvector equation for u. Here recall Gelfand's rigged Hilbert space idea: the eigenvectors ~~for~~ for u do not necessarily exist in the Hilbert space H obtained by completing H_0, rather they lie in the dual of the dense space H_0.

Basic viewpoint: The Dirac equation with a spectral parameter corresponds to a wave equation whose phase (or state) space yields a Hilbert space of finite energy states and time evolution is unitary.

Orthogonal polys on S^1 wrt $d\mu$, observe that if you define $\tilde{q}_n \in (1 + zF_{n-1}) \cap (zF_{n-1})^\perp$, then

$$\overline{\tilde{q}_n} \in \left(1 + z^{-1}\overline{F}_{n-1}\right) \cap \left(z^{-1}\overline{F}_{n-1}\right)^\perp = \left(1 + z^{-n}F_{n-1}\right) \cap \left(z^{-n}F_{n-1}\right)^\perp$$

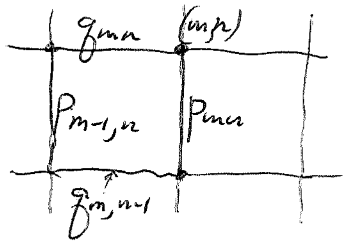
since $z^{-1}\overline{F}_{n-1} = z^{-1}z^{-n+1}F_{n-1} = z^{-n}F_{n-1}$. Thus

$$z^n \overline{\tilde{q}_n} \in (z^n + F_{n-1}) \cap (F_{n-1})^\perp \quad \text{so} \quad z^n \overline{\tilde{q}_n} = \tilde{p}_n .$$

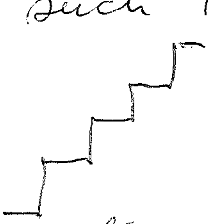
<u>August 28, 1999</u>

Recall the notion of discrete 1diml Dirac equation:
$$\begin{pmatrix}p_n\\ q_n\end{pmatrix} = \frac{1}{k_n}\begin{pmatrix}1 & h_n\\ \bar{h}_n & 1\end{pmatrix}\begin{pmatrix}z p_{n-1}\\ q_{n-1}\end{pmatrix} \qquad n\in\mathbb{Z}$$

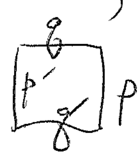
where $|h_n| < 1$, $k_n = \sqrt{1-|h_n|^2}$ $\forall n$. Associated to such a d1dDE is a Hilbert space of finite energy states, which is equipped with a unitary operator u giving the time evolution. The Hilbert space is constructed using an array of unit vectors p_{mn}, q_{mn} for $(m,n)\in\mathbb{Z}\times\mathbb{Z}$ each unit vector belonging to an edge of the graph:



such that the unit vectors in any staircase:



form an orthonormal basis, and such that for any square



one has the equivalent relations

$$\begin{pmatrix}p\\ q\end{pmatrix} = \frac{1}{k}\begin{pmatrix}1 & h\\ \bar{h} & 1\end{pmatrix}\begin{pmatrix}p'\\ q'\end{pmatrix} \qquad\qquad \begin{pmatrix}p\\ q'\end{pmatrix} = \begin{pmatrix}k & h\\ -\bar{h} & k\end{pmatrix}\begin{pmatrix}p'\\ q\end{pmatrix}$$

"transfer" form $\qquad\qquad$ "unitary" form

with $|h| < 1$, $k = \sqrt{1-|h|^2}$. Then $h = (q|p) = -(q'|p')$

We have

$$\begin{pmatrix} p_n \\ q_n \end{pmatrix} = T_{nm} \begin{pmatrix} p_m \\ q_m \end{pmatrix} \qquad \text{for} \quad n > m$$

where

$$T_{nm} = \frac{1}{k_n} \begin{pmatrix} 1 & h_n \\ \bar{h}_n & 1 \end{pmatrix} \begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix} \cdots \frac{1}{k_{m+1}} \begin{pmatrix} 1 & h_{m+1} \\ \bar{h}_{m+1} & 1 \end{pmatrix} \begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix}$$

By induction we have

$$T_{nm} \in \begin{pmatrix} [1, z, \dots, z^{n-m-1}]z & [1, z, \dots, z^{n-m-1}] \\ [1, z, \dots, z^{n-m-1}]z & [1, z, \dots, z^{n-m-1}] \end{pmatrix}$$

Rewrite the dldOE in the form

$$\begin{pmatrix} z^{-n} p_n \\ q_n \end{pmatrix} = \frac{1}{k_n} \begin{pmatrix} 1 & h_n z^{-n} \\ \bar{h}_n z^n & 1 \end{pmatrix} \begin{pmatrix} z^{-n+1} p_{n-1} \\ q_{n-1} \end{pmatrix}$$

Then

$$\begin{pmatrix} z^{-n} p_n \\ q_n \end{pmatrix} = \tilde{T}_{nm} \begin{pmatrix} z^{-m} p_m \\ q_m \end{pmatrix} \qquad n > m$$

where

$$\boxed{\begin{aligned} \tilde{T}_{nm} &= \begin{pmatrix} z^{-n} & 0 \\ 0 & 1 \end{pmatrix} T_{nm} \begin{pmatrix} z^m & 0 \\ 0 & 1 \end{pmatrix} \\ &\in \begin{pmatrix} [1, z^{-1}, \dots, z^{-n+m+1}] & [z^{-n}, \dots, z^{-m-1}] \\ [z^{m+1}, \dots, z^n] & [1, z, \dots, z^{n-m-1}] \end{pmatrix} \end{aligned}}$$

One has

$$\tilde{T}_{nm} = \begin{pmatrix} \bar{d} & \bar{c} \\ c & d \end{pmatrix}$$

where $\bar{c}$ means

where $d\bar{d} - c\bar{c} = 1$, and

conjugation "on S^1"; i.e. conjugation applied to the coefficients and $z \mapsto z^{-1}$.

Let's show now that d does not vanish on $|z| \leq 1$. We have

better notation: c^* for $\bar{c}$

$$T_{nm} = \begin{pmatrix} z^n & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \bar{d} & \bar{c} \\ c & d \end{pmatrix} \begin{pmatrix} z^{-m} & 0 \\ 0 & 1 \end{pmatrix}$$

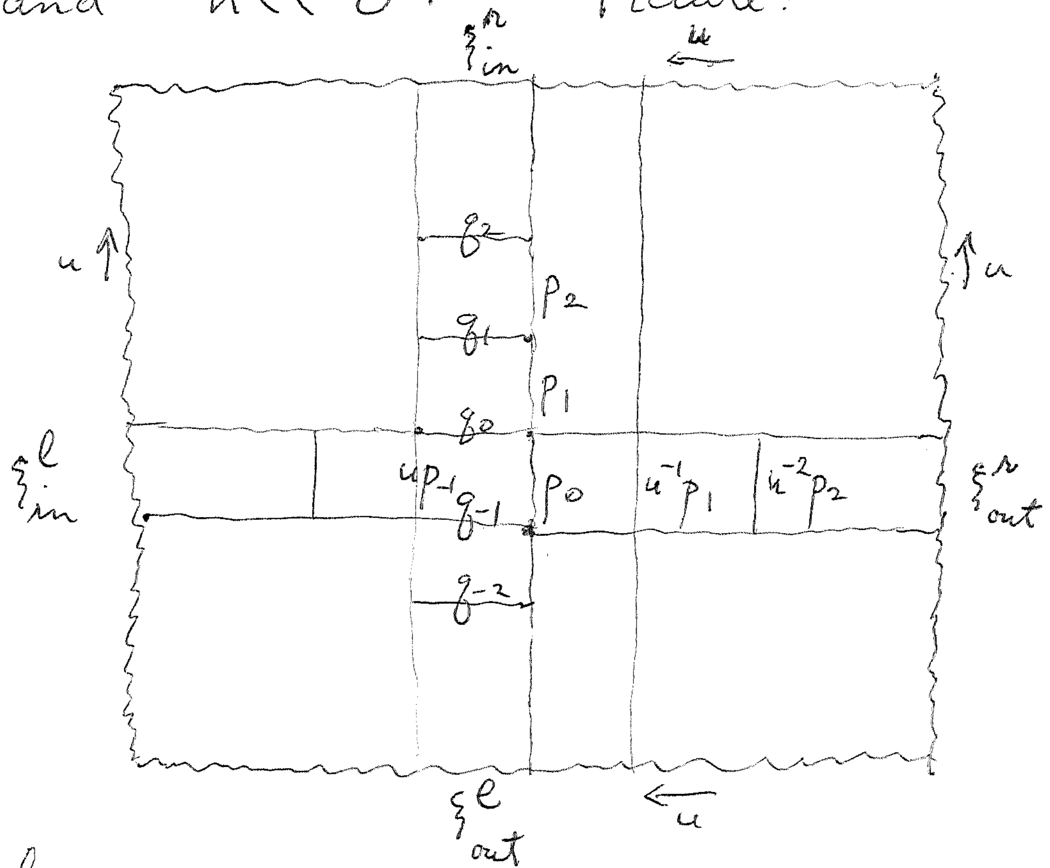
$$= \begin{pmatrix} z^{n-m}\bar{d} & z^n \bar{c} \\ z^{-m} c & d \end{pmatrix}$$

We also know that for $|z| \leq 1$, the matrices $\frac{1}{k}\begin{pmatrix} 1 & h \\ \bar{h} & 1 \end{pmatrix}$ and $\begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix}$ acting on the Riemann sphere carry the closed unit disk into itself. Thus $T_{nm}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} z^n \bar{c}(z) \\ d(z) \end{pmatrix}$ has $\left| \frac{z^n \bar{c}(z)}{d(z)} \right| \leq 1$. Since $d(z)\bar{d}(z) - c(z)\bar{c}(z) = 1$, $\bar{c}(z)$ and $d(z)$ cannot simultaneously vanish, so we see that $d(z) \neq 0$ for $0 < |z| \leq 1$. For $z = 0$ $d \neq 0$ by inspection of the product for T_{nm}.

Since $d(z)$ is a polynomial whose zeroes are outside $\bar{D}$, the condition $|d|^2 = 1 - |c|^2$ on S^1 determines d up to a scalar factor. The point is that $1 - |c|^2$ is a trigonometric polynomial > 0 on S^1, so its roots are closed under reflection through S^1, so the roots of d are the roots (with mult) of $1 - c\bar{c}$ lying outside S^1.

August 29, 1999

Let's analyze two-sided scattering when (h_n) has finite support. In this case $\begin{pmatrix} u^{-n}p_n \\ q_n \end{pmatrix}$ is constant for $n \gg 0$ and $n \ll 0$. Picture:



We have

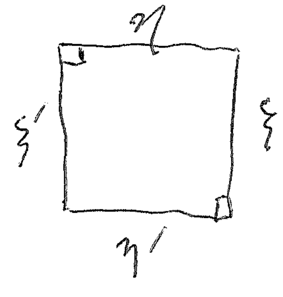
$$\begin{pmatrix} \xi^r_{out} \\ \xi^r_{in} \end{pmatrix} = \tilde{T}_{\infty,-\infty} \begin{pmatrix} \xi^l_{in} \\ \xi^l_{out} \end{pmatrix} = \begin{pmatrix} d^* & b \\ b^* & d \end{pmatrix} \begin{pmatrix} \xi^l_{in} \\ \xi^l_{out} \end{pmatrix}$$

from

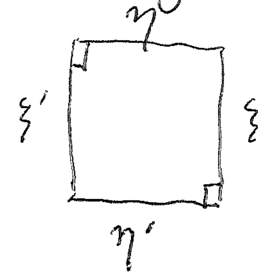
$$\begin{pmatrix} u^{-n}p_n \\ q_n \end{pmatrix} = \tilde{T}_{nm} \begin{pmatrix} u^{-m}p_m \\ q_m \end{pmatrix}$$

and letting $n \to +\infty$, $m \to -\infty$.

Here $b = c^*$. Notice that the scattering is described by new type of square: where the inner products such as $\eta^*\xi$ are elements of the C^*-algebra $C(S^1)$ instead of $\mathbb{C}$.



Let's record the formulas relating the transfer and scattering matrices associated to the "square"



If
$$\begin{pmatrix} \xi \\ \eta' \end{pmatrix} = \underbrace{\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}}_{\text{scattering matrix}} \begin{pmatrix} \xi' \\ \eta \end{pmatrix}$$
scattering matrix – unitary in good cases

$$\begin{pmatrix} \xi \\ \eta \end{pmatrix} = \underbrace{\begin{pmatrix} a & b \\ c & d \end{pmatrix}}_{} \begin{pmatrix} \xi' \\ \eta' \end{pmatrix}$$
transfer matrix $\in U(1,1)$ in good cases

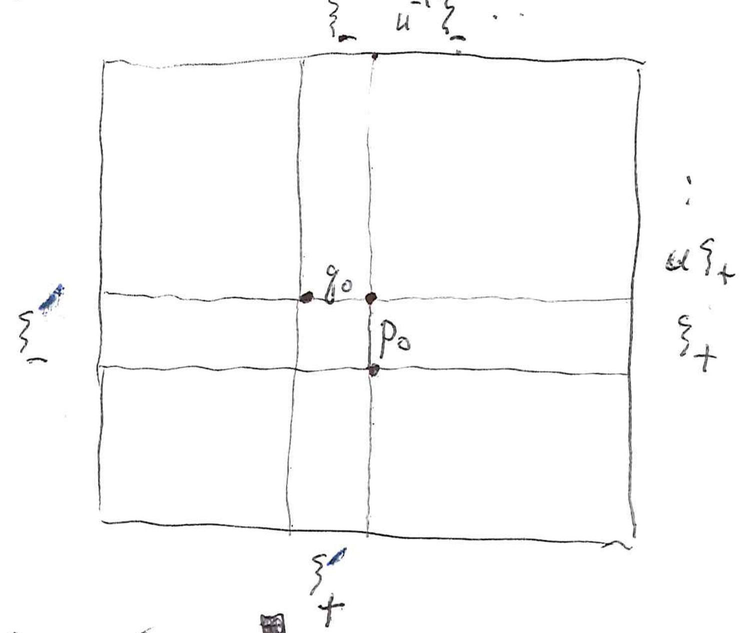
then
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \alpha - \frac{\beta\gamma}{\delta} & \frac{\beta}{\delta} \\ -\frac{\gamma}{\delta} & \frac{1}{\delta} \end{pmatrix}, \quad \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} \frac{ad-bc}{d} & \frac{b}{d} \\ -\frac{c}{d} & \frac{1}{d} \end{pmatrix}$$

Notice that $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \frac{\alpha}{\delta}$, hence the transfer matrix has det $= 1$ iff the two transmission coefficients α, δ of the scattering matrix coincide. (Also clear from second formula).
Notice also that the ~~[illegible]~~ two maps $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \longleftrightarrow \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ are given by the same formula.

Since $|\tilde{T}_{mm}| = 1$, the two transmission coeffs. α, δ coincide.

September 6, 1999

Let's discuss the scattering situation for a disc. 1-dim DE where $(h_n)_{n\in\mathbb{Z}}$ satisfies $\sum' |h_n| < \infty$



Here ξ_+ (= ξ^r_{out} [redacted] in old notation) and $\xi_- = \xi^r_{in}$ are $\xi_+ = \lim_{n\to\infty}$ [redacted] $u^{-n}p_n$, $\xi_- = \lim_{n\to\infty} q_n$ while $\xi'_\pm = \xi^l_{\substack{out\\in}}$ are $\xi'_- = \lim_{n\to-\infty} u^{-n}p_n$, $\xi'_+ = \lim_{n\to-\infty} q_n$.

These limits probably exist when the sequence (h_n) is square summables, but [redacted] the summability condition should be sufficient that $\begin{pmatrix}\xi_+\\ \xi_-\end{pmatrix}$ and $\begin{pmatrix}\xi'_+\\ \xi'_-\end{pmatrix}$ both generate the Hilbert space under the action of the unitary operator u. Why?

$$\begin{pmatrix}\xi_+\\ \xi_-\end{pmatrix} = \lim_{n\to\infty}\begin{pmatrix}u^{-n}p_n\\ q_n\end{pmatrix} = \lim_{n\to\infty} \tilde{T}_{n0}\begin{pmatrix}p_0\\ q_0\end{pmatrix}$$

$$\tilde{T}_{n0} = \frac{1}{k_n}\begin{pmatrix}1 & h_n z^{-n}\\ \bar{h}_n z^n & 1\end{pmatrix} \cdots \frac{1}{k_1}\begin{pmatrix}1 & h_1 z^{-1}\\ \bar{h}_1 z & 1\end{pmatrix}$$

The infinite product $\lim_{n\to\infty} \tilde{T}_{n0}$ takes place in a Banach Lie group, namely continuous functions on S^1 with values in $SU(1,1)$. It has the form

$$\lim_{n\to\infty} (1+X_n)(1+X_{n-1})\cdots(1+X_1)$$

where $\sum_n \|X_n\| < \infty$, so the limit should exist and be invertible. (Invertible in this case also results from $\det = 1$.)

Thus $\begin{pmatrix} \xi_+ \\ \xi_- \end{pmatrix} = \tilde{T}_{\infty,0} \begin{pmatrix} p_0 \\ q_0 \end{pmatrix}$ $\qquad \tilde{T}_{\infty,0} = \begin{pmatrix} d^* & c^* \\ c & d \end{pmatrix}$

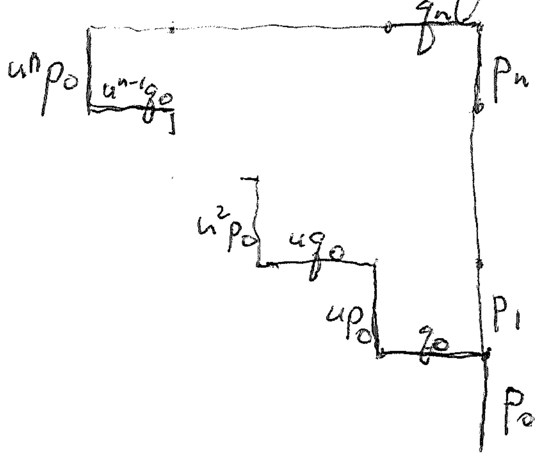
with $c, d \in \mathbf{C}(S^1)$ satisfying $d^*d - c^*c = 1$. So

$$* \quad \boxed{\begin{pmatrix} p_0 \\ q_0 \end{pmatrix} = \begin{pmatrix} d & -c^* \\ -c & d^* \end{pmatrix} \begin{pmatrix} \xi_+ \\ \xi_- \end{pmatrix}.}$$

From p 47 we have

$$\begin{pmatrix} p_n \\ q_n \end{pmatrix} \in \begin{pmatrix} [u, \ldots, u^n] & [1, \ldots, u^{n-1}] \\ [u, \ldots, u^n] & [1, \ldots, u^{n-1}] \end{pmatrix} \begin{pmatrix} p_0 \\ q_0 \end{pmatrix}$$

which is also clear from the picture



Thus $c(z)$, $d(z)$ extend analytically to D.

From $*$ on the previous page the Hilbert space E of finite energy states is generated by ξ_+, ξ_- under the u-action.

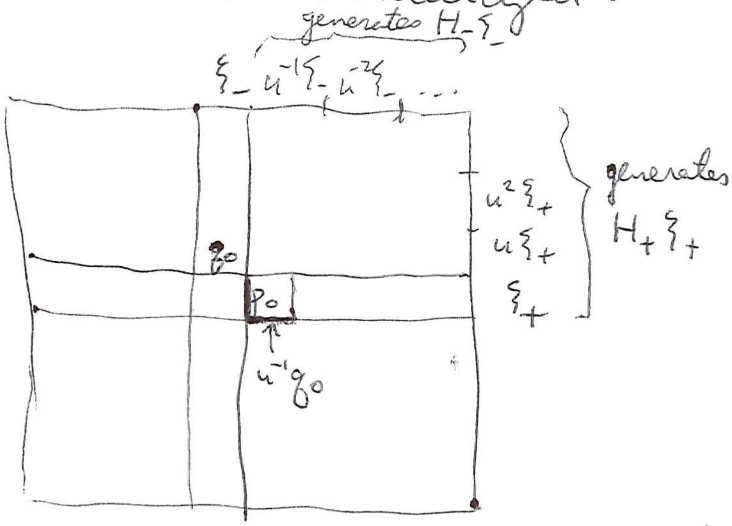
One has $u^n q_k \perp q_{n+k}$ (for $n \neq 0$), so $u^n \xi_- \perp \xi_-$ for $n \neq 0$, thus the u-invariant subspace generated by ξ_- is $L^2(S^1)\xi_-$. Similarly $p_n \perp p_{n-k}$ for $k \neq 0$, so $u^{k-n} p_n \perp u^{-n} p_{n-k}$, and letting $n \to \infty$ yields $u^k \xi_+ \perp \xi_+$ for $k \neq 0$.

~~From $*$ we have~~

The Hilbert space E is thus obtained by glueing $L^2(S^1)\xi_+$ and $L^2(S^1)\xi_-$ together, where the glueing is given by a contraction operator $S : L^2(S^1)\xi_+ \longrightarrow L^2(S^1)\xi_-$, which is the orthogonal projection of the former to the latter inside E. S has matrix coefficients $(u^{-k}\xi_- \mid u^j \xi_+) = (\xi_- \mid u^{j+k}\xi_+)$. Because S commutes with u it is a function $S(u)$ of u, where $S(z)$ is a function of abs. value ≤ 1 in general. So $S : \xi_+ \longmapsto S(u)\xi_-$ and $(\xi_- \mid u^n \xi_+) = (\xi_- \mid u^n S(u) \xi_-) \simeq \int z^n S(z) \frac{d\theta}{2\pi}$, call this S_n, whence $S(z) = \sum_{n \in \mathbb{Z}} z^{-n} S_n$.

Next from $*$ on the previous page we have $p_0 \in H_+\xi_+ + H_-\xi_-$, $\quad q_0 \in uH_+\xi_+ + uH_-\xi_-$

This can be visualized:



so
$$p_0 \in (H_+\xi_+ + H_-\xi_-) \cap (zH_+\xi_+ + H_-\xi_-)^\perp$$
$$u^{-1}q_0 \in (H_+\xi_+ + H_-\xi_-) \cap (H_+\xi_+ + u^{-1}H_-\xi_-)^\perp$$
$$q_0 \in (uH_+\xi_+ + uH_-\xi_-) \cap (uH_+\xi_+ + H_-\xi_-)^\perp$$

which agrees with
$$\begin{pmatrix} p_0 \\ q_0 \end{pmatrix} = \begin{pmatrix} d & -c^* \\ -c & d^* \end{pmatrix} \begin{pmatrix} \xi_+ \\ \xi_- \end{pmatrix}$$

since $c \in \cancel{z}H_+$, $d \in H_+$, $c^* \in H_-$, $d^* \in \cancel{z}H_-$.

These properties allow us to construct p_0, q_0 from $S(z)$, namely

$$p_0 = \sum_{j \geq 0} d_j u^j \xi_+ - \sum_{k \geq 1} b_k u^{-k} \xi_-$$

$$q_0 = -\sum_{j \geq 1} c_j u^j \xi_+ + \sum_{k \geq 0} a_k u^{-k} \xi_-$$

$$0 = (u^j \xi_+ | p_0) = d_j - \sum_{k \geq 1} b_k \overline{S}_{j+k} \qquad j \geq 1$$

$$0 = (u^{-k} \xi_- | p_0) = \sum_{j \geq 0} d_j S_{k+j} - b_k \qquad k \geq 1$$

$$0 = (u^j \xi_+ | q_0) = - c_j + \sum_{k \geq 0} a_k \overline{S}_{j+k} \qquad j \geq 1$$

$$0 = (u^{-k} \xi_- | q_0) = - \sum_{j \geq 1} c_j S_{j+k} + a_k \qquad k \geq 1$$

Write these in terms of $a = (a_k)_{k \geq 1}$, similarly for b, c, d, and the matrix

$$S_{kj} = \left(u^{-k}\xi_- \,\middle|\, u\delta\xi_+\right) = S_{k+j}. \quad \text{Then}$$

$$(S^*)_{j,k} = \overline{S_{k,j}} = \overline{S_{k+j}}$$ so we have the equations

$$d = S^* b \qquad b - Sd = d_0 (S_k)_{k \geq 1}$$

$$a = Sc \qquad c - S^* a = a_0 (\overline{S_k})_{k \geq 1}$$

i.e.

$$\boxed{\begin{aligned} (1 - SS^*)\, b &= d_0 (S_k)_{k \geq 1} \\ (1 - S^*S)\, c &= a_0 (\overline{S_k})_{k \geq 1} \end{aligned}}$$

September 7, 1999

I propose now to start with the scattering side. Let $S(z) = \sum_{n\in\mathbb{Z}} z^{-n} S_n$ be a smooth function on S^1 such that $|S(z)| < 1$, whence $|S(z)| \leq 1-\varepsilon$ for some $\varepsilon > 0$ and all $z \in S^1$. Have $S_n = \int z^n S \frac{d\theta}{2\pi}$. Define E to be the Hilbert space with unitary u generated by two copies of $L^2(S^1)$, namely $L^2\xi_+$ and $L^2\xi_-$; thus $(\xi_+, u^n\xi_+) = \delta_{n0}$ and also for ξ_-. Also in E we put $(\xi_- | u^n \xi_+) = S_n$. We can describe E as the completion of the space of elements $f_1\xi_+ + f_2\xi_-$ with inner product

$$\|f_1\xi_+ + f_2\xi_-\|^2 = \int \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}^* \begin{pmatrix} 1 & S^* \\ S & 1 \end{pmatrix} \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \frac{d\theta}{2\pi}$$

Check: $(u^{-k}\xi_- | u^j \xi_+) = \int \overline{z^{-k}}\, S(z)\, z^j \frac{d\theta}{2\pi} = S_{j+k}$.

The completion is unnecessary as $\begin{pmatrix} 1 & S^* \\ S & 1 \end{pmatrix}$ is bdd invertible.

In E the orthogonal projection from $L^2\xi_+$ to $L^2\xi_-$ is $f\xi_+ \longmapsto f \sum_{k\in\mathbb{Z}} u^{-k}\xi_- \underbrace{(u^{-k}\xi_- | \xi_+)}_{S_k} = f\sum_k S_k u^{-k}\xi_- = fS(u)\xi_-$

$\therefore$ The orth projection map $L^2\xi_+ \longrightarrow L^2\xi_-$ in E is the operator commuting with u such that $\xi_+ \mapsto S(u)\xi_-$.

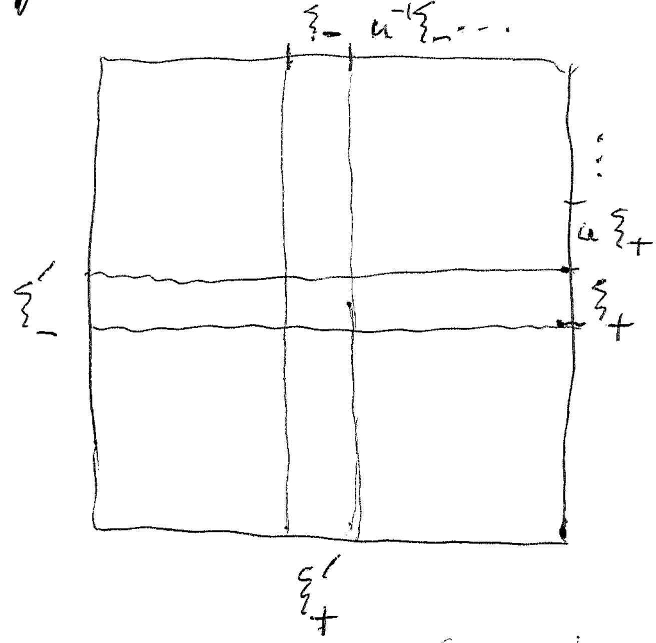
So we have

$$\boxed{(f_2\xi_- | f_1\xi_+) = (f_2\xi_- | f_1 S\xi_-) = (f_2 | f_1 S) = \int f_2^* S f_1 \frac{d\theta}{2\pi}}$$

Next discuss $\xi_\pm^\ell$.

<u>September 10, 1999</u>

I've made some progress with the orthogonality relations. Suppose given $\beta(z) = \sum_n \beta_n z^n$ on S^1 with $|\beta(z)| \leq 1-\varepsilon$, $\varepsilon > 0$, and form $E = L^2\xi_+ + L^2\xi_-$ with hermitian scalar product $(f\xi_\pm | g\xi_\pm) = \int f^* g$, $(f\xi_- | g\xi_+) = \int f^* g\beta$. Thus $g\xi_+ \longmapsto g\beta\xi_-$ is the orthogonal projection of $L^2\xi_+$ to $L^2\xi_-$. We aim to construct the scattering on the left, i.e. the vectors $\xi'_- = \xi^l_{in}$, $\xi'_+ = \xi^l_{out}$; recall $\xi_+ = \xi^r_+$, $\xi_- = \xi^r_-$ and we have the picture



<u>First approach:</u> $(L^2\xi_-)^\perp = \{f(\xi_+ - \beta\xi_-) \mid f \in L^2\}$.

$\eta = \xi_+ - \beta\xi_-$ is a cyclic vector in $(L^2\xi_-)^\perp$ wrt u.

The corresponding spectral measure is

$$(\eta | f\eta) = (\xi_+ | f(\xi_+ - \beta\xi_-)) = \int f(1-\beta\bar\beta)$$

Now doing Szegö theory for this measure (aka prediction theory), i.e. $\log(1-\beta\bar\beta) = \phi + \bar\phi$ with

ϕ analytic in the disk, $\phi(0) \in \mathbb{R}$ one gets $\alpha = e^{\phi}$ analytic invertible in the disk satisfying

$$|\alpha|^2 = 1 - |\beta|^2 \qquad \alpha(0) > 0 .$$

Then $\boxed{\xi'_- = \frac{1}{\alpha}\left(\xi_+ - \beta \xi_-\right)} = \frac{1}{\alpha}\eta$

is a cyclic vector for $(L^2\xi_-)^\perp$ with spectral measure $\frac{1}{|\alpha|^2}(1-|\beta|^2)\frac{d\theta}{2\pi} = \frac{d\theta}{2\pi}$. So

$$\xi_+ = \alpha \xi'_- + \beta \xi_-$$
$$= \sum_{n \geq 0} \alpha_n u^n \xi'_- + \sum_{n \in \mathbb{Z}} \beta_n u^n \xi_-$$

is the expansion of ξ_+ in terms of the natural incoming orthonormal basis. In particular we have

$$1 = \sum_{n \geq 0} |\alpha_n|^2 + \sum_{n \in \mathbb{Z}} |\beta_n|^2$$

Similarly with the same $\alpha(z)$ (since $1 - \beta\bar{\beta} = 1 - \bar{\beta}\beta$) we find that

$$\boxed{\xi'_+ = \frac{1}{\bar{\alpha}}\left(\xi_- - \bar{\beta}\xi_+\right)}$$

is a cyclic vector for $(L^2\xi_+)^\perp$ with spectral measure $\frac{d\theta}{2\pi}$ and we have

$$\xi_- = \bar{\beta}\xi_+ + \bar{\alpha}\xi'_+$$
$$= \sum_{n \in \mathbb{Z}} \bar{\beta}_n u^{-n} \xi_+ + \sum_{n \geq 0} \bar{\alpha}_n u^{-n} \xi'_+$$

Thus we have the transfer matrix

$$\begin{pmatrix} \xi_-' \\ \xi_+' \end{pmatrix} = \underbrace{\begin{pmatrix} \frac{1}{\alpha} & -\frac{\beta}{\alpha} \\ -\frac{\bar\beta}{\bar\alpha} & \frac{1}{\bar\alpha} \end{pmatrix}}_{\in SU(1,1)} \begin{pmatrix} \xi_+ \\ \xi_- \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} \xi_+ \\ \xi_- \end{pmatrix} = \underbrace{\begin{pmatrix} \frac{1}{\alpha} & \frac{\beta}{\alpha} \\ \frac{\bar\beta}{\bar\alpha} & \frac{1}{\bar\alpha} \end{pmatrix}}_{\begin{pmatrix} a & b \\ c & d \end{pmatrix}} \begin{pmatrix} \xi_-' \\ \xi_+' \end{pmatrix}$$

and the scattering matrix

$$\begin{pmatrix} \xi_+ \\ \xi_+' \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ -\frac{\alpha}{\bar\alpha}\bar\beta & \alpha \end{pmatrix} \begin{pmatrix} \xi_-' \\ \xi_- \end{pmatrix}$$

<u>September 11, 1999</u>

<u>Second approach</u> via the orthogonality relations. Notation: You want $\begin{pmatrix} \xi_-' \\ \xi_+' \end{pmatrix} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \begin{pmatrix} \xi_+ \\ \xi_- \end{pmatrix}$ so write

$$\xi_-' = \sum_{j \geq 0} d_j u^j \xi_+ - \sum_{k \in \mathbb{Z}} b_k u^k \xi_-$$

replace by $j \in \mathbb{Z}$, but impose $\left(\begin{matrix} d_j = 0, \ j < 0. \\ d_0 > 0 \end{matrix} \right.$

Then $\xi_-' \perp uH_+ \xi_+ + H_- \xi_-$ yields

$$0 = (u^k \xi_- | \xi_-') = \sum_j d_j \underbrace{(u^k \xi_- | u^j \xi_+)}_{\beta_{k-j}} - b_k$$

$$0 = (u^j \xi_+ | \xi_-') = d_j - \sum_k b_k \underbrace{(u^j \xi_+ | u^k \xi_-)}_{\bar\beta_{k-j}} \quad \text{for } j \geq 1$$

So

$$b(z) = \sum_k b_k z^k = \sum_k \sum_j d_j \overset{z^j}{\beta_{k-j} z^{k-j}} = \sum_j d_j z^j \overbrace{\sum_k \beta_{k-j} z^{k-j}}^{\text{ind of } j}$$

Thus $\boxed{b(z) = d(z)\beta(z)}$

Next $\displaystyle \sum_j \sum_k b_k \bar\beta_{k-j} z^k z^{j-k} = \sum_k b_k z^k \overbrace{\sum_j \bar\beta_{k-j} z^{j-k}}^{\text{ind of } k}$

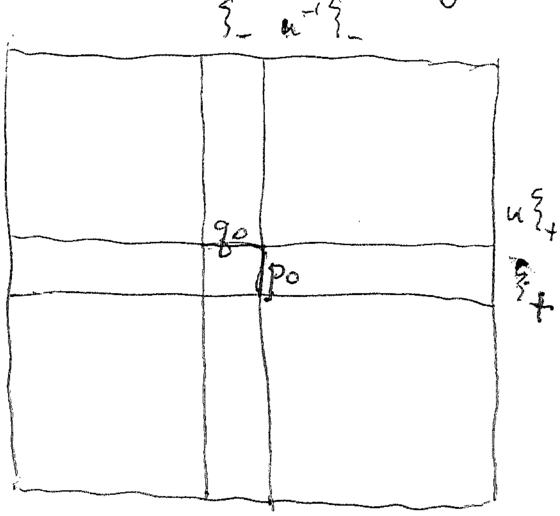
$\displaystyle = b(z) \sum_j \bar\beta_{-j} z^j = b(z) \sum_j \bar\beta_j z^{-j} = b(z)\overline{\beta(z)}.$

So the second condition says

$$\boxed{d(z) - b(z)\overline{\beta(z)} = \sum_{j\le 0} t_j z^j \qquad \text{some } t_j}$$

Thus ~~[illegible]~~ $d(1-|\beta|^2)$ is analytic outside D.
So if $1-|\beta|^2 = |\alpha|^2$ with α analytic invertible inside D, then $d\alpha = \frac{1}{\bar\alpha}\sum_{j\le 0} t_j z^j$ ~~[illegible]~~ is analytic both inside and outside D, so $d\alpha$ is a constant.

Next examine the orthogonality relations for p_0



$p_0 \in H_+\xi_+ + H_-\xi_-$

$\perp uH_+\xi_+ + H_-\xi_-$

$$\begin{pmatrix}\xi_+\\ \xi_-\end{pmatrix} = \begin{pmatrix} a & b\\ c & d\end{pmatrix}\begin{pmatrix} p_0\\ q_0\end{pmatrix}$$

$$\begin{pmatrix}p_0\\ q_0\end{pmatrix} = \begin{pmatrix} d & -b\\ -c & a\end{pmatrix}\begin{pmatrix} \xi_+\\ \xi_-\end{pmatrix}$$

$d \in H_+$, $b \in$ ~~[illegible]~~ H_-

$a = d^* \in zH_-$, $c = b^* \in$ ~~z~~H_+

$q_0 = -c\xi_+ + a\xi_-$ ~~[illegible]~~ $\in uH_+\xi_+ + uH_-\xi_-$

$$p_0 = \sum_j d_j u^j \xi_+ - \sum_k b_k u^k \xi_-$$

$d_j = 0$ if $j<0$ $\qquad b_k = 0$ if $k \geq 0$.

$$0 = (u^k \xi_- | p_0) = \sum_j d_j \beta_{k-j} - b_k \qquad \text{for } k<0.$$

$$0 = (u^j \xi_+ | p_0) = d_j - \sum_k b_k \overline{\beta_{k-j}} \qquad \text{for } j>0.$$

The orthogonality conditions say

$$\boxed{\begin{array}{ll} b - d\beta \in H_+ & d - b\overline{\beta} \in zH_- \\ \text{with } \quad d \in H_+ & b \in H_- \end{array}}$$

Next $\quad q_0 = -\sum_j c_j u^j \xi_+ + \sum_k a_k u^k \xi_-$ $\qquad c_j = 0 \quad j \leq 0$, $\quad a_k = 0 \quad k > 0$

$$0 \overset{k<0}{=} (u^k \xi_- | q_0) = -\sum_j c_j \beta_{k-j} + a_k$$

$c \in zH_+$

$a \in zH_-$

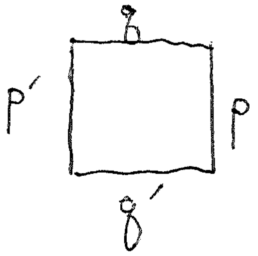
$$0 \overset{j>0}{=} (u^j \xi_+ | q_0) = -c_j + \sum_k a_k \overline{\beta_{k-j}}$$

So you get

$$\boxed{\begin{array}{ll} a - c\beta \in H_+ & c - a\overline{\beta} \in zH_- \\ a \in zH_- & c \in zH_+ \end{array}}$$

September 15, 1999

Recall the formulas relating 2×2 transfer and scattering matrices:

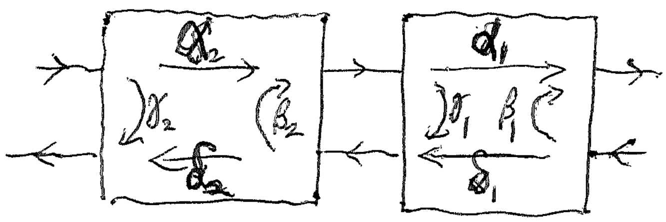


$$\begin{pmatrix} p \\ q \end{pmatrix} = \underbrace{\begin{pmatrix} a & b \\ c & d \end{pmatrix}}_{\begin{pmatrix} \frac{\alpha\delta-\beta\gamma}{\delta} & \frac{\beta}{\delta} \\ -\frac{\gamma}{\delta} & \frac{1}{\delta} \end{pmatrix}} \begin{pmatrix} p' \\ q' \end{pmatrix} \qquad \begin{pmatrix} p \\ q' \end{pmatrix} = \underbrace{\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}}_{\begin{pmatrix} \frac{ad-bc}{d} & \frac{b}{d} \\ -\frac{c}{d} & \frac{1}{d} \end{pmatrix}} \begin{pmatrix} p' \\ q \end{pmatrix}$$

Suppose now that

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}\begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} = \begin{pmatrix} a_1a_2+b_1a_2 & a_1b_2+b_1d_2 \\ c_1a_2+d_1c_2 & c_1b_2+d_1d_2 \end{pmatrix}$$

Then the scattering matrix associated to the product is related to the scattering matrices of the factor as follows. Picture:



$$\beta = \beta_1 + \alpha_1\beta_2\delta_1 + \alpha_1\beta_2\gamma_1\beta_2\delta_1 + \cdots = \beta_1 + \alpha_1\beta_2\frac{1}{1-\gamma_1\beta_2}\delta_1$$

$$\delta = \delta_2\delta_1 + \delta_2\gamma_1\beta_2\delta_1 + \cdots = \delta_2\frac{1}{1-\gamma_1\beta_2}\delta_1$$

$$\alpha = \alpha_1\alpha_2 + \alpha_1\beta_2\gamma_1\alpha_2 + \cdots = \alpha_1\frac{1}{1-\beta_2\gamma_1}\alpha_2$$

$$\gamma = \gamma_2 + \delta_2\gamma_1\alpha_2 + \delta_2\gamma_1\beta_2\gamma_1\alpha_2 + \cdots = \gamma_2 + \delta_2\gamma_1\frac{1}{1-\beta_2\gamma_1}\alpha_2$$

Check β, δ:

$$\delta = \frac{1}{d} = \frac{1}{d_1 d_2 + c_1 b_2} = \frac{1}{d_2} \frac{1}{1 + \frac{c_1}{d_1}\frac{b_2}{d_2}} \frac{1}{d_1} = \delta_2 \frac{1}{1-\gamma_1 \beta_2} \delta_1$$

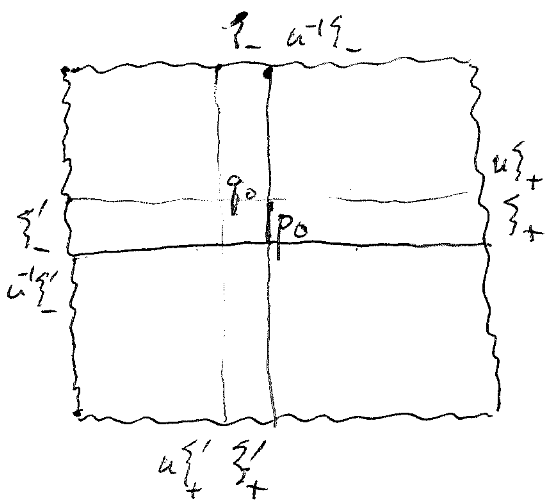
$$\beta = \frac{b}{d} = \frac{a_1 b_2 + b_1 d_2}{d_1 d_2 + c_1 b_2} = \frac{\frac{a_1}{d_1}\beta_2 + \beta_1}{1 - \gamma_1 \beta_2}$$

$$\beta - \beta_1 = \frac{1}{1-\gamma_1\beta_2}\left(\frac{a_1}{d_1}\beta_2 + \cancel{\beta_1} - \beta_1(\cancel{1} - \gamma_1 \beta_2)\right)$$

$$= \frac{1}{1-\gamma_1\beta_2}\beta_2 \Bigg(\underbrace{\frac{a_1}{d_1} + \frac{b_1}{d_1}\left(\frac{-c_1}{d_1}\right)}_{\frac{a_1 d_1 - b_1 c_1}{d_1^2} = \alpha_1 \delta_1}\Bigg)$$

$$= \alpha_1 \beta_2 \frac{1}{1-\gamma_1\beta_2} \delta_1 .$$

September 16, 1999



$$\begin{pmatrix}\xi_+\\ \xi_-\end{pmatrix} = \begin{pmatrix} a & b \\ c & d\end{pmatrix}\begin{pmatrix}\xi'_-\\ \xi'_+\end{pmatrix}$$

$$= \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1\end{pmatrix}\underbrace{\begin{pmatrix} a_2 & b_2 \\ c_2 & d_2\end{pmatrix}\begin{pmatrix}\xi'_-\\ \xi'_+\end{pmatrix}}_{\begin{pmatrix}p_0\\ q_0\end{pmatrix}}$$

$$\begin{pmatrix}p_0\\ q_0\end{pmatrix} = \begin{pmatrix} d_1 & -b_1 \\ -c_1 & a_1\end{pmatrix}\begin{pmatrix}\xi_+\\ \xi_-\end{pmatrix} = \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2\end{pmatrix}\begin{pmatrix}\xi'_-\\ \xi'_+\end{pmatrix}$$

$p_0, u^{-1}q_0 \in H_+\xi_+ + H_-\xi_- \implies \boxed{\begin{matrix} d_1 \in H_+ , & b_1 \in H_- \\ c_1 \in zH_+ & a_1 \in zH_- \end{matrix}}$

$p_0, q_0 \in zH_-\xi'_- + H_+\xi'_+ \implies \boxed{\begin{matrix} a_2 \in zH_- & b_2 \in H_+ \\ c_2 \in zH_- & d_2 \in H_+ \end{matrix}}$

Thus

$$\beta = \beta_1 + \alpha_1 \beta_2 \frac{1}{1-\gamma_1\beta_2}\delta_1 \implies \beta - \beta_1 \in H_+$$

$$\gamma = \gamma_2 + \delta_2 \gamma_1 \frac{1}{1-\beta_2\gamma_1}\alpha_2 \implies \gamma - \gamma_2 \in zH_+$$

Here's another version of the orthogonality relations for $p_0 = d_1\xi_+ - b_1\xi_-$.

The projection into $L^2\xi_-$ of ξ_+ is $\beta\xi_-$, so applying this projection to p_0 yields

$$(d_1\beta - b_1)\xi_- = \text{proj of } p_0 \in H_+\xi_-.$$

Thus $\boxed{d_1\beta - b_1 \in H_+}$. As the projection of ξ_- into $L^2\xi_+$ is $\bar\beta\xi_+$ we get $\boxed{d_1 - b_1\bar\beta \in zH_-}$

Since $\xi_+ = \alpha\xi'_- + \beta\xi'_+ = \frac{1}{d}\xi'_- + \frac{b}{d}\xi'_+$, the projection of p_0 into $L^2\xi'_- = (L^2\xi_-)^\perp$ ~~yields~~ is

$p_0 \longmapsto d_1\frac{1}{d}\xi'_- \in H_+\xi'$, so $\boxed{d_1\frac{1}{d} \in H_+ \text{ equivalently } d_1 \in H_+}$

Since $\boxed{\xi_- = \frac{c}{a}\xi_+ + \frac{1}{a}\xi'_+}$ the projection into $L^2\xi'_+ = (L^2\xi_+)^\perp$ yields $p_0 \longmapsto -b_1\frac{1}{a}\xi'_+$ so

$\boxed{\frac{b_1}{a} \in H_- \text{ equiv. } b_1 \in H_-}$. Formulas used:

$$\boxed{\begin{pmatrix}\xi'_- \\ \xi'_+\end{pmatrix} = \begin{pmatrix} d & -b \\ -c & a\end{pmatrix}\begin{pmatrix}\xi_+ \\ \xi_-\end{pmatrix} \qquad \begin{pmatrix}\xi'_- \\ \xi_-\end{pmatrix} = \begin{pmatrix}\frac{1}{a} & -\frac{b}{a} \\ \frac{c}{a} & \frac{1}{a}\end{pmatrix}\begin{pmatrix}\xi_+ \\ \xi'_+\end{pmatrix}}$$

Assume known that the matrix $\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}$ relating $\begin{pmatrix} \xi_+ \\ \xi_- \end{pmatrix}$ to $\begin{pmatrix} p_0 \\ q_0 \end{pmatrix}$ is a loop in $SU(1,1)$, i.e. $a_1 = \bar{d}_1$, $b_1 = \bar{c}_1$, $\det = 1$. Then

$$d_1 - b_1\bar{\beta} \in zH_- \iff a_1 - c_1\beta \in H_+,$$

and the orthogonality conditions become

$$\boxed{\begin{array}{l} d_1\beta - b_1 \in H_+ \\ -c_1\beta + a_1 \in H_+ \end{array}}$$

These follow from

$$\begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} = \begin{pmatrix} d_1 & -b_1 \\ -c_1 & a_1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

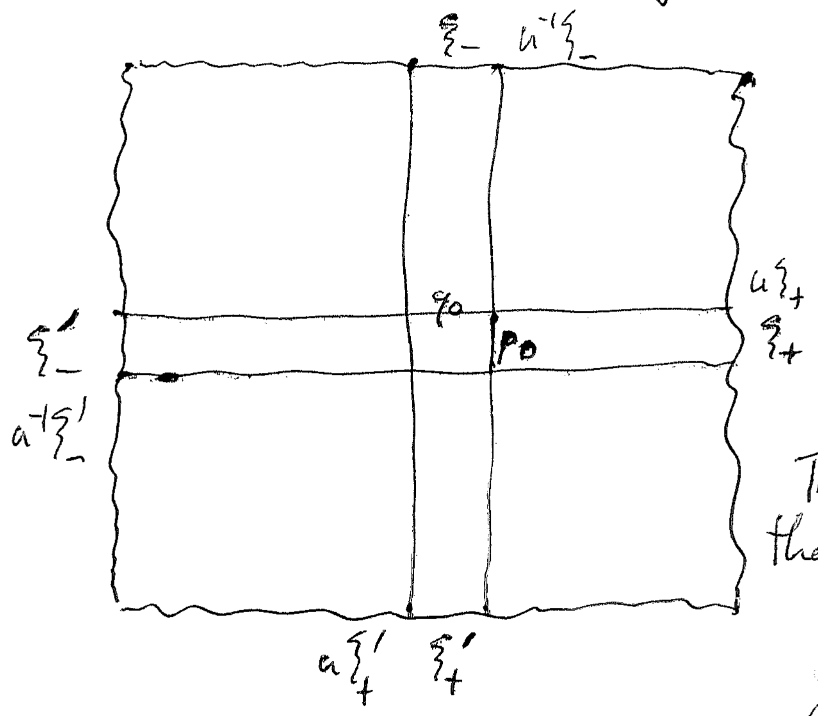
$$\Longrightarrow \quad \frac{b_2}{d} = \frac{d_1 b - b_1 d}{d} = d_1\beta - b_1$$

$$\frac{d_2}{d} = \frac{-c_1 b + a_1 d}{d} = -c_1\beta + a_1$$

and the left sides are in H_+ since $b_2 \in H_+$ and d_2, d are in H_+ and invertible.

September 27, 1999

Consider a dIdDE with summable $(h_n)_{n\in\mathbb{Z}}$, so that the propagators to $n \to \pm\infty$ exist and are invertible. Scattering situation:



$$\begin{pmatrix}\xi_+\\ \xi_-\end{pmatrix} = \begin{pmatrix} a & b\\ c & d\end{pmatrix}\begin{pmatrix}\xi'_-\\ \xi'_+\end{pmatrix}$$

$$\begin{pmatrix}\xi_+\\ \xi'_+\end{pmatrix} = \begin{pmatrix} \frac{1}{d} & \frac{b}{d}\\ -\frac{c}{d} & \frac{1}{d}\end{pmatrix}\begin{pmatrix}\xi'_-\\ \xi_-\end{pmatrix}$$

The picture shows that $\xi_+ \in zH_-\xi'_- + L^2\xi'_+$

$\xi_- \in L^2\xi'_- + H_+\xi'_+$

so $\begin{pmatrix} a & b\\ c & d\end{pmatrix} \in \begin{pmatrix} zH_- & L^2\\ L^2 & H_+\end{pmatrix}$

Now factor

$$\begin{pmatrix}\xi_+\\ \xi_-\end{pmatrix} = \underbrace{\begin{pmatrix} a_> & b_>\\ c_> & d_>\end{pmatrix}}_{\lim\limits_{n\to\infty} \frac{1}{k_n}\begin{pmatrix} 1 & h_n z^{-n}\\ \bar{h}_n z^n & 1\end{pmatrix}\cdots\frac{1}{k_1}\begin{pmatrix} 1 & h_1 z^{-1}\\ \bar{h}_1 z & 1\end{pmatrix}}\begin{pmatrix}p_0\\ q_0\end{pmatrix} = \begin{pmatrix} a_> & b_>\\ c_> & d_>\end{pmatrix}\begin{pmatrix} a_0 & b_0\\ c_0 & d_0\end{pmatrix}\begin{pmatrix}\xi'_-\\ \xi'_+\end{pmatrix}$$

$$\begin{pmatrix} a_> & b_>\\ c_> & d_>\end{pmatrix} \in \begin{pmatrix} zH_- & H_-\\ zH_+ & H_+\end{pmatrix}, \qquad \begin{pmatrix} a_0 & b_0\\ c_0 & d_0\end{pmatrix} \in \begin{pmatrix} zH_- & H_+\\ zH_- & H_+\end{pmatrix}$$

Check with picture $p_0 = a_0\xi'_- + b_0\xi'_+ \in zH_-\xi'_- + H_+\xi'_+$

$q_0 = c_0\xi'_- + d_0\xi'_+ \in zH_-\xi'_- + H_+\xi'_+$

$$\begin{pmatrix}p_0\\ q_0\end{pmatrix} = \begin{pmatrix} d_> & -b_>\\ -c_> & a_>\end{pmatrix}\begin{pmatrix}\xi_+\\ \xi_-\end{pmatrix} \in \begin{pmatrix} H_+\xi_+ + H_-\xi_-\\ zH_+\xi_+ + zH_-\xi_-\end{pmatrix} = \begin{pmatrix} H_+ & H_-\\ zH_+ & zH_-\end{pmatrix}\begin{pmatrix}\xi_+\\ \xi_-\end{pmatrix}$$

One has

$$\begin{pmatrix} a_0 & b_0 \\ c_0 & d_0 \end{pmatrix} = \begin{pmatrix} a_> & b_> \\ c_> & d_> \end{pmatrix}^{-1} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad \text{so}$$

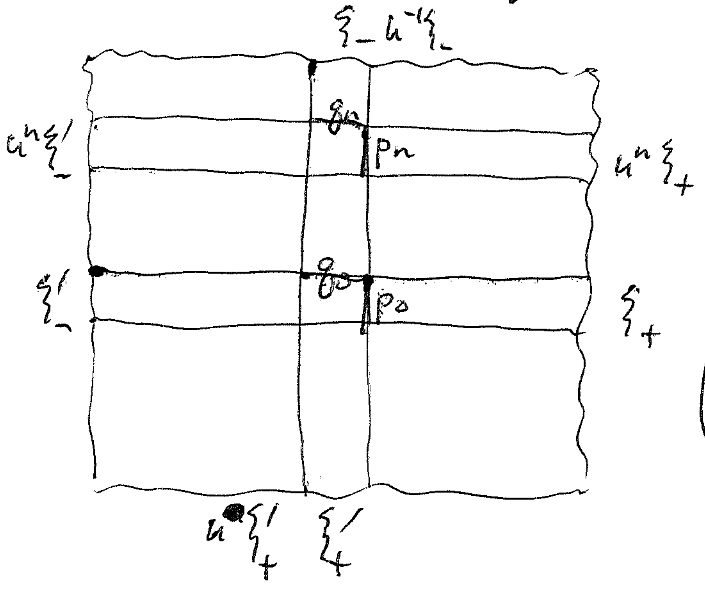
$$\begin{pmatrix} d_> & -b_> \\ -c_> & a_> \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} d_>a - b_>c & d_>b - b_>d \\ -c_>a + a_>c & -c_>b + a_>d \end{pmatrix} \in \begin{pmatrix} zH_- & H_+ \\ zH_- & H_+ \end{pmatrix}$$

hence

$$\begin{array}{lll} d_>a - b_>c \in zH_- & \text{equiv.} & \boxed{\begin{array}{l} d_> - b_> \overset{\bar\beta}{\left(\frac{c}{a}\right)} \in zH_- \\ d_>\beta - b_> \in H_+ \end{array}} \\ d_>b - b_>d \in H_+ & \quad " & \\ -c_>a + a_>c \in zH_- & \quad " & \boxed{\begin{array}{l} -c_> + a_>\bar\beta \in zH_- \\ -c_>\beta + a_> \in H_+ \end{array}} \\ -c_>b + a_>d \in H_+ & \quad " & \end{array}$$

These relations in boxes are the orthogonality relations holding for p_0 and q_0.

Next do the formulas for p_n, q_n



$$\begin{pmatrix} z^{-n}p_n \\ q_n \end{pmatrix} = \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix} \begin{pmatrix} \xi'_- \\ \xi'_+ \end{pmatrix}$$

$$\begin{pmatrix} p_n \\ q_n \end{pmatrix} \in \begin{pmatrix} z^{n+1}H_-\xi'_- + H_+\xi'_+ \\ z^{n+1}H_-\xi'_- + H_+\xi'_+ \end{pmatrix}$$

$$\therefore \quad \boxed{\begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix} \in \begin{pmatrix} zH_- & z^{-n}H_+ \\ z^{n+1}H_- & H_+ \end{pmatrix}}$$

$$\begin{pmatrix} z^{-n}p_n \\ q_n \end{pmatrix} = \begin{pmatrix} d_> & -b_> \\ -c_> & a_> \end{pmatrix} \begin{pmatrix} \xi_+ \\ \xi_- \end{pmatrix} \qquad \begin{pmatrix} p_n \\ q_n \end{pmatrix} \in \begin{pmatrix} z^nH_+\xi_+ + H_-\xi_- \\ z^{n+1}H_+\xi_+ + zH_-\xi_- \end{pmatrix}$$

$$\begin{pmatrix} d_> & -b_> \\ -c_> & a_> \end{pmatrix} \in \begin{pmatrix} H_+ & z^{-n}H_- \\ z^{n+1}H_+ & zH_- \end{pmatrix} \qquad \text{so} \quad \boxed{\begin{pmatrix} a_> & b_> \\ c_> & d_> \end{pmatrix} \in \begin{pmatrix} zH_- & z^{-n}H_- \\ z^{n+1}H_+ & H_+ \end{pmatrix}}$$

$$\begin{pmatrix} zH_- & z^{-n}H_+ \\ z^{n+1}H_- & H_+ \end{pmatrix} \ni \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix} = \begin{pmatrix} d_> & -b_> \\ -c_> & a_> \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} d_>a - b_>c & d_>b - b_>d \\ -c_>a + a_>c & -c_>b + a_>d \end{pmatrix}$$

$$\begin{array}{ll} d_>a - b_>c \in zH_- & \text{equiv.} \\ d_>b - b_>d \in z^{-n}H_+ & \text{''} \\ -c_>a + a_>c \in z^{n+1}H_- & \text{''} \\ -c_>b + a_>d \in H_+ & \text{''} \end{array} \qquad \boxed{\begin{array}{l} d_> - b_>\bar{\beta} \in zH_- \\ d_>\beta - b_> \in z^{-n}H_+ \\ -c_> + a_>\bar{\beta} \in z^{n+1}H_- \\ -c_>\beta + a_> \in H_+ \end{array}}$$

$$\begin{pmatrix} a_> & b_> \\ c_> & d_> \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} d_n & -b_n \\ -c_n & a_n \end{pmatrix} = \begin{pmatrix} ad_n - bc_n & -ab_n + ba_n \\ cd_n - dc_n & -cb_n + da_n \end{pmatrix} \in \begin{pmatrix} zH_- & z^{-n}H_- \\ z^{n+1}H_+ & H_+ \end{pmatrix}$$

$$\begin{array}{ll} ad_n - bc_n \in zH_- & \text{equiv.} \\ -ab_n + ba_n \in z^{-n}H_- & \text{''} \\ cd_n - dc_n \in z^{n+1}H_+ & \text{''} \\ -cb_n + da_n \in H_+ & \text{''} \end{array} \qquad \boxed{\begin{array}{l} d_n - \frac{b}{a}c_n \in zH_- \\ -b_n + \frac{b}{a}a_n \in z^{-n}H_- \\ \frac{c}{d}d_n - c_n \in z^{n+1}H_+ \\ -\frac{c}{d}b_n + a_n \in H_+ \end{array}}$$

$$= \frac{1}{a}\begin{pmatrix} d_> a - b_> c & -b_> \\ -c_> a + a_> c & a_> \end{pmatrix}\begin{pmatrix} \xi_+ \\ \xi'_+ \end{pmatrix} = \underbrace{\frac{1}{a}\begin{pmatrix} a_0 & -b_> \\ c_0 & a_> \end{pmatrix}}_{\in \begin{pmatrix} zH_- & H_- \\ zH_- & zH_- \end{pmatrix}}\begin{pmatrix} \xi_+ \\ \xi'_+ \end{pmatrix}$$

So

$$\boxed{\begin{pmatrix} p_0 \\ q_0 \end{pmatrix} = \frac{1}{a}\begin{pmatrix} a_0 & -b_> \\ c_0 & a_> \end{pmatrix}\begin{pmatrix} \xi_+ \\ \xi'_+ \end{pmatrix}} \qquad \boxed{\begin{pmatrix} \xi_+ \\ \xi'_+ \end{pmatrix} = \begin{pmatrix} a_> & b_> \\ -c_0 & a_0 \end{pmatrix}\begin{pmatrix} p_0 \\ q_0 \end{pmatrix}}$$

Then we have

$$\begin{pmatrix} p_0 \\ q_0 \end{pmatrix} = \frac{1}{a}\begin{pmatrix} a_0 & -b_> \\ c_0 & a_> \end{pmatrix}\begin{pmatrix} \xi_+ \\ \xi'_+ \end{pmatrix} = \frac{1}{d}\begin{pmatrix} d_> & b_0 \\ -c_> & d_0 \end{pmatrix}\begin{pmatrix} \xi'_- \\ \xi_- \end{pmatrix}.$$

So one has the factorization of the S-matrix

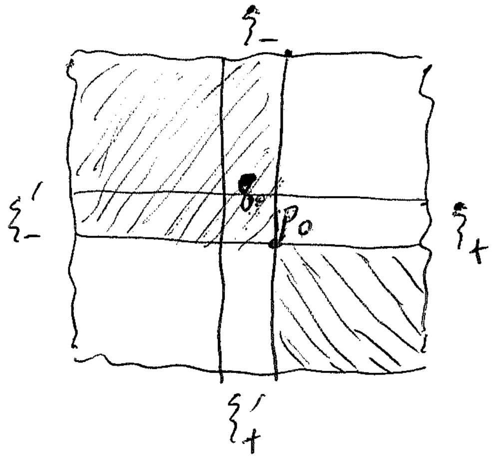
$$\begin{pmatrix} \frac{1}{d} & \frac{b}{d} \\ -\frac{c}{d} & \frac{1}{d} \end{pmatrix} = \underbrace{\begin{pmatrix} a_> & b_> \\ -c_0 & a_0 \end{pmatrix}}_{\in \begin{pmatrix} zH_- & H_- \\ zH_- & zH_- \end{pmatrix}} \underbrace{\frac{1}{d}\begin{pmatrix} d_> & b_0 \\ -c_> & d_0 \end{pmatrix}}_{\begin{pmatrix} zH_+ & H_+ \\ zH_+ & H_+ \end{pmatrix}}$$

which checks as $a_> d_> - b_> c_> = a_0 d_0 - b_0 c_0 = 1$ and

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a_> & b_> \\ c_> & d_> \end{pmatrix}\begin{pmatrix} a_0 & b_0 \\ c_0 & d_0 \end{pmatrix} \Rightarrow \begin{array}{l} a_> b_0 + b_> d_0 = b \\ c_> a_0 + d_> c_0 = c \end{array}$$

Thus $\boxed{\begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix} \in \begin{pmatrix} zH_- & z^{-n}H_+ \\ z^{n+1}H_- & H_+ \end{pmatrix} \quad \begin{pmatrix} a_> & b_> \\ c_> & d_> \end{pmatrix} \in \begin{pmatrix} zH_- & z^{-n}H_- \\ z^{n+1}H_+ & H_+ \end{pmatrix}}$

I now want to discuss the observation that the subspace closed under multiplication by u generated by p_0, q_0 is a forward light cone.



In fact each vertex in the grid gives rise to a forward and a backward light cone, e.g.

$$H_+\xi_-' + H_+\xi_- \quad , \quad H_-\xi_+ + H_-\xi_+'$$

which are complementary.

$$\begin{pmatrix} p_0 \\ q_0 \end{pmatrix} = \begin{pmatrix} a_0 & b_0 \\ c_0 & d_0 \end{pmatrix} \begin{pmatrix} \xi_-' \\ \xi_+' \end{pmatrix} = \begin{pmatrix} a_0 & b_0 \\ c_0 & d_0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\frac{c}{d} & \frac{1}{d} \end{pmatrix} \begin{pmatrix} \xi_-' \\ \xi_- \end{pmatrix}$$

$$= \frac{1}{d} \begin{pmatrix} a_0 d - b_0 c & b_0 \\ c_0 d - d_0 c & d_0 \end{pmatrix} \begin{pmatrix} \xi_-' \\ \xi_- \end{pmatrix} = \boxed{\frac{1}{d} \begin{pmatrix} d_> & b_0 \\ -c_> & d_0 \end{pmatrix} \begin{pmatrix} \xi_-' \\ \xi_- \end{pmatrix} = \begin{pmatrix} p_0 \\ q_0 \end{pmatrix}}$$

From the picture the matrix $\in \begin{pmatrix} H_+ & H_+ \\ zH_+ & H_+ \end{pmatrix}$, which checks, and it has the determinant $\frac{1}{d}$, whence

$$\boxed{\begin{pmatrix} \xi_-' \\ \xi_- \end{pmatrix} = \begin{pmatrix} d_0 & -b_0 \\ c_> & d_> \end{pmatrix} \begin{pmatrix} p_0 \\ q_0 \end{pmatrix}}$$

so it's clear that $H_+ p_0 + H_+ q_0 = H_+ \xi_-' + H_+ \xi_-$

Next

$$\begin{pmatrix} p_0 \\ q_0 \end{pmatrix} = \begin{pmatrix} d_> & -b_> \\ -c_> & a_> \end{pmatrix} \begin{pmatrix} \xi_+ \\ \xi_- \end{pmatrix} = \begin{pmatrix} d_> & -b_> \\ -c_> & a_> \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \frac{c}{a} & \frac{1}{a} \end{pmatrix} \begin{pmatrix} \xi_+ \\ \xi_+' \end{pmatrix}$$

<u>September 28, 1999</u>

Recall the relations between the transfer and scattering matrices:

$$\begin{pmatrix}\xi_+\\ \xi_-\end{pmatrix} = \begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}\xi'_-\\ \xi'_+\end{pmatrix} \qquad \begin{pmatrix}\xi_+\\ \xi'_+\end{pmatrix} = \begin{pmatrix}\frac{1}{d} & \frac{b}{d}\\ -\frac{c}{d} & \frac{1}{d}\end{pmatrix}\begin{pmatrix}\xi'_-\\ \xi_-\end{pmatrix}$$

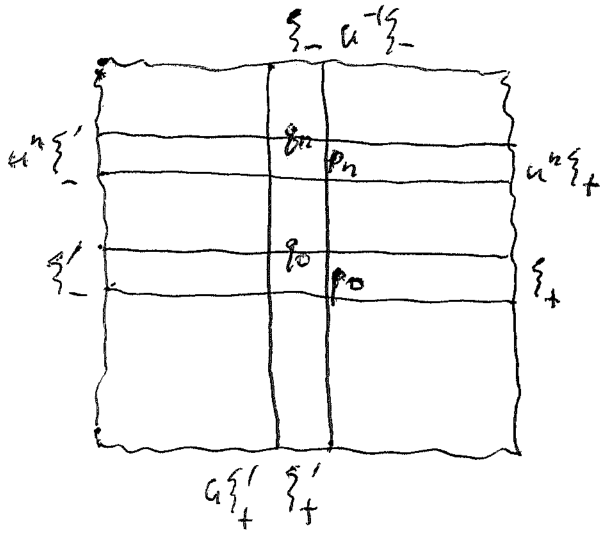
$$\begin{pmatrix}\xi'_-\\ \xi'_+\end{pmatrix} = \begin{pmatrix}d & -b\\ -c & a\end{pmatrix}\begin{pmatrix}\xi_+\\ \xi_-\end{pmatrix} \qquad \begin{pmatrix}\xi'_-\\ \xi_-\end{pmatrix} = \begin{pmatrix}\frac{1}{a} & -\frac{b}{a}\\ \frac{c}{a} & \frac{1}{a}\end{pmatrix}\begin{pmatrix}\xi_+\\ \xi'_+\end{pmatrix}$$

Consider the factorization of the transfer matrix:

$$\begin{pmatrix}\xi_+\\ \xi_-\end{pmatrix} = \begin{pmatrix}a_> & b_>\\ c_> & d_>\end{pmatrix}\begin{pmatrix}u^{-n}p_n\\ q_n\end{pmatrix} \qquad \begin{pmatrix}u^{-n}p_n\\ q_n\end{pmatrix} = \begin{pmatrix}a_n & b_n\\ c_n & d_n\end{pmatrix}\begin{pmatrix}\xi'_-\\ \xi'_+\end{pmatrix}$$

$$\begin{pmatrix}a_> & b_>\\ c_> & d_>\end{pmatrix} = \begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}d_n & -b_n\\ -c_n & a_n\end{pmatrix} = \begin{pmatrix}ad_n - bc_n & -ab_n + ba_n\\ cd_n - dc_n & -cb_n + da_n\end{pmatrix}$$

$$\begin{pmatrix}a_n & b_n\\ c_n & d_n\end{pmatrix} = \begin{pmatrix}d_> & -b_>\\ -c_> & a_>\end{pmatrix}\begin{pmatrix}a & b\\ c & d\end{pmatrix} = \begin{pmatrix}d_>a - b_>c & d_>b - b_>d\\ -c_>a + a_>c & -c_>b + a_>d\end{pmatrix}$$



$$p_n, q_n \in \cancel{z}^{n+1}H_-\xi'_- + H_+\xi'_+$$

$$\begin{pmatrix}u^{-n}p_n\\ q_n\end{pmatrix} \in \underbrace{\begin{pmatrix}zH_- & z^{-n}H_+\\ z^{n+1}H_- & H_+\end{pmatrix}}\begin{pmatrix}\xi'_-\\ \xi'_+\end{pmatrix}$$

so $\begin{pmatrix}a_n & b_n\\ c_n & d_n\end{pmatrix} \in$

$$\begin{pmatrix}u^{-n}p_n\\ q_n\end{pmatrix} = \begin{pmatrix}d_> & -b_>\\ -c_> & a_>\end{pmatrix}\begin{pmatrix}\xi_+\\ \xi_-\end{pmatrix}$$

$$p_n \in \cancel{z}^n H_+\xi_+ + H_-\xi_-$$

$$q_n \in z^{n+1}H_+\xi_+ + zH_-\xi_-$$