250 Do something – Go back to orthogonal polys. Wait

$$\frac{f(w)}{g(w)} = \int \frac{dz}{2\pi i z} \frac{\overline{g(z)}}{1-\bar{w}z} \frac{f(z)}{g(z)} \frac{1}{\cancel{|}g(z)\cancel{|}}$$

$$f(w) = \int \frac{dz}{2\pi i z} \frac{\overline{\overline{g(w)}\, g(z)}}{1-\bar{w}z} f(z) \frac{1}{|g(z)|^2}$$

Considering H^2 with $\|f\|_g^2 = \int \frac{d\theta}{2\pi} \frac{1}{|g(z)|^2} |f(z)|^2$

i.e. $H^2 \longrightarrow H^2$

$f \longmapsto \frac{f}{g}$ $\qquad \|f\|_g^2 = \|\frac{f}{g}\|^2$

Then we find the reproducing kernel for H_g^2:

$$f(w) = (K(w,\cdot), f)_g \qquad K(w,z) = \frac{\overline{g(w)}\, g(z)}{1-\bar{w}z}$$

~~So the observation is trivial that if g has degree n. No.~~

~~orth~~

orth polys. ~~[illegible]~~ $k_n = \sqrt{1-|h_n|^2}$

$$\begin{pmatrix} p_n \\ q_n \end{pmatrix} = \frac{1}{k_n}\begin{pmatrix} 1 & h_n \\ \bar{h}_n & 1 \end{pmatrix}\begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} p_{n-1} \\ q_{n-1} \end{pmatrix} \qquad \begin{pmatrix} z p_{n-1} \\ q_{n-1} \end{pmatrix} = \frac{1}{k_n}\begin{pmatrix} 1 & -h_n \\ -\bar{h}_n & 1 \end{pmatrix}\begin{pmatrix} p_n \\ q_n \end{pmatrix}$$

You want $(1-\bar{w}z) \sum_{j=0}^{n} \overline{p_j(w)}\, p_j(z)$

$$\bar{w}\, \overline{p_j(w)} = k_{j+1}^{-1} \overline{p_{j+1}(w)} - k_{j+1}^{-1} \bar{h}_{j+1} \overline{q_{j+1}(w)}$$

$$z\, p_j(z) = k_{j+1}^{-1} p_{j+1}(z) - k_{j+1}^{-1} h_{j+1} q_{j+1}(z)$$

$$\overline{w} z\, \overline{p_j(w)}\, p_j(z) = k_{j+1}^{-2} \Big(\qquad \text{too hard.}$$

$$\frac{\overline{g(w)}\, g(z)}{1-\overline{w} z} = \frac{\prod (1-a_j \overline{w}) \prod (1-\overline{a}_j z)}{1-\overline{w} z}$$

rational function of z simple pole at $z = \overline{w}^{-1}$
$= w$ if $|w| = 1$

residue is $\underset{-w}{\left(\frac{1}{-\overline{w}}\right)} \prod (1-a_j\overline{w}) \prod (1-\overline{a}_j w)$

$$= -w\, |g(w)|^2$$

What you missed, suppose

$H^2 = Y \overset{\perp}{\oplus} SH^2 \qquad Y = H^2 \cap SH^{2\perp}$

then calculate the pt. evaluator at w for Y
by orthog. projecting $\frac{1}{1-\overline{w} z}$ the pt eval$_w$ for H^2.

$\frac{1}{1-\overline{w} z} = y + S h \qquad h \in H^2$. Now

~~$\cdot$~~ $1-\overline{w} z$ acts invertibly by mult on H^+.
so $h = \frac{g}{1-\overline{w} z}$ $\quad g \in H^2$. Then you want

$\frac{1 - Sg}{1-\overline{w} z} = y$ $\qquad$ Poly situation $S = \frac{f}{g} = \prod_{i=1}^{n} \frac{(z-a_i)}{1-\overline{a}_i z}$
and you want y to be ?

Wait: You know Y has basis $\frac{1}{1-\overline{a}_i z}$ (assuming a_i distinct) Now $\frac{1-S(z)g}{1-\overline{w} z}$ ~~with g~~ has potential singularities at $z = \frac{1}{a_i}, \frac{1}{\overline{w}}$. So take g to be a constant $\ni$ ~~the~~ sing. $\frac{1}{\overline{w}}$ removable.

$$g = S(\bar{\omega}^{-1})^{-1} = \overline{S(\omega)}$$

check $S(\bar{\omega}^{-1}) = \prod \dfrac{\bar{\omega}^{-1} - a_i}{1 - \bar{a}_i \bar{\omega}^{-1}} = \prod \dfrac{1 - a_i\bar{\omega}}{\bar{\omega} - \bar{a}_i} = \prod \overline{\left(\dfrac{\omega - a_i}{1 - \bar{a}_i \omega}\right)}^{-1}$

$= \overline{S(\omega)}^{-1}$

So the point evaluator for Y is

$$K_{\bar{\omega} z} = \frac{1 - \overline{S(\omega)}\, S(z)}{1 - \bar{\omega} z}$$

This thing comes up in connection with interpolation.
Find the geometric picture.
Recall that in H^2 you have

$$\left(\frac{1}{1 - \bar{a}_1 z}, \frac{1}{1 - \bar{a}_2 z} \right) = \frac{1}{1 - \bar{a}_2 a_1}$$

so that the matrix $\dfrac{1}{1 - \bar{\omega} z}$ where ω, z

range over ~~a subset of~~ D is positive definite.
Rational functions with dist. poles are linear indep.
Somehow it looks like a graph construction
Maybe there's a monotonicity ~~problem~~ result.
Look at UHP case.

$$\int \frac{d\lambda}{2\pi i} \frac{1}{\lambda - a} f(\lambda) = f(a)$$

at least if $|f(\lambda)| \leq C (\mathrm{Im}\,\lambda)^{1/2}$

$$|f(a)| \leq \left(\int \frac{d\lambda}{2\pi} \left| \frac{1}{\lambda - a} \right|^2 \right)^{1/2} \|f\|$$

$$\int \frac{d\lambda}{2\pi} \frac{1}{|\lambda - a|^2} = \int \frac{d\lambda}{2\pi} \frac{1}{\lambda^2 + t^2} = \int \frac{d\lambda}{2\pi} \frac{1}{\lambda^2 + 1} \frac{1}{t}$$

$$= \frac{1}{2t} \qquad t = \mathrm{Im}(a).$$

$$|f(a)| \leq 2\,\mathrm{Im}(a)^{-1/2} \|f\|.$$

$$J_a = \frac{i}{\lambda - \bar{a}} \qquad \int \frac{d\lambda}{2\pi} \frac{1}{i} \frac{1}{\lambda - a} f(\lambda) = f(a).$$

$$\|J_a\|^2 = \frac{i}{a-\bar{a}} = \frac{i}{2i\,\mathrm{Im}\,a} = \frac{1}{2\,\mathrm{Im}(a)}.$$

$$\boxed{\left(\frac{i}{\lambda-\bar{a}_1}, \frac{i}{\lambda-\bar{a}_2}\right) = \frac{i}{a_1-\bar{a}_2}}$$

Consider a sequences a_j $j=1,\dots,n$ in D dist points and another b_j and try to solve $f(a_j)=b_j$ $j=1,\dots,n$ with $f\in H^2$

i.e. you want

$$\left(\frac{1}{1-\bar{a}_j z}, f\right) = b_j$$

and you would like $\|f\|\leq 1$.

General question: Given $A=(a_{ij}) > 0$ $N\times N$ matrix, and ?

Basically you have an ind set e_i in a Hilb space E, and a linear functional $\rho(e_i)=b_i$, and you want the norm of f

$$\|\rho\|^2 = \sup \frac{\left|\sum b_i x_i\right|^2}{\left\|\sum' x_i e_i\right\|^2} = \sup \frac{|\rho(x)|^2}{\|x\|^2}$$

$\|\rho\|\leq 1$ means $|\rho(x)|^2 \leq \|x\|^2$ $\forall x$.

i.e. $(x, \rho^*\rho x) \leq (x,x)$

i.e. $1-\rho^*\rho \geq 0$. Want matrix.

$$\frac{1}{1-\bar{a}_i a_j} \geq \bar{b}_i b_j$$

$$\left\| \sum x_i \frac{1}{1-\bar{a}_i z} \right\|^2 = \sum_{i,j} \bar{x}_i \left(\frac{1}{1-\bar{a}_i z}, \frac{1}{1-\bar{a}_j z} \right) x_j$$

$$= \sum_{i,j} \frac{1}{1-\bar{a}_j a_i} \bar{x}_i x_j$$

$$\left(\sum_i x_i \frac{1}{1-\bar{a}_i z} \middle| f \right) = \sum \bar{x}_i f(a_i) = \sum \bar{x}_i b_i$$

$$\left| \left(\sum x_i \frac{1}{1-\bar{a}_i z}, f \right) \right|^2 = \sum_{i,j} \bar{x}_j \bar{b}_j b_i x_i = \sum b_i \bar{b}_j \bar{x}_i x_j$$

so you want $\frac{1}{1-\bar{a}_j a_i} \geq b_i \bar{b}_j$ OKAY

So what are you missing for the interpolation result?? Best possibility is

finite Blashke product = rational fn $f(z)$ of z + $|f(z)| = 1$ for $|z| = 1$. and holom ~~inside~~ for $|z| < 1$.

$$H^2 \supset SH^2 \supset 0$$
$$\cup \qquad \cup$$
$$\Pi(z-a_i)H^2 \supset$$

What does interpolation means. Consider $D \times D$ (z, w) typical points. Can graph f

$\Gamma_f = \{(z, f(z)) \mid |z| < 1\}$. We assume $|f(z)| = 1$

So Γ_f is an effective divisor in $D \times D$. Interpolation amounts to requiring Γ_f to contain the points (a_i, b_i) $i = 1, \ldots, n$.

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$$K_r(s) = \int_0^\infty e^{-r\frac{t+t^{-1}}{2}} t^s \frac{dt}{t} = \int_{-\infty}^{\infty} e^{\overbrace{-r\cosh x + sx}^{y}} dx$$

$$y = -r\cosh x + sx$$

$$\frac{dy}{dx} = -r\sinh x + s = 0$$

$$\frac{d^2y}{dx^2} = -r\cosh x$$

critical point is $\sinh x = \frac{s}{r}$ $\quad x = \sinh^{-1}\left(\frac{s}{r}\right)$

$$\cosh\left(\sinh^{-1}\left(\frac{s}{r}\right)\right) = \sqrt{1+\frac{s^2}{r^2}}$$

$$(\cosh x)^2 - \underbrace{(\sinh x)}_{s/r}{}^2 = 1$$

$$y = -r\sqrt{1+\frac{s^2}{r^2}} + s\sinh^{-1}\left(\frac{s}{r}\right)$$
$$+ \frac{1}{2}\left(-r\sqrt{1+\frac{s^2}{r^2}}\right)\left(x - \sinh^{-1}\left(\frac{s}{r}\right)\right)^2$$

$$K_r(s) = e^{-\sqrt{s^2+r^2} + s\sinh^{-1}\left(\frac{s}{r}\right)} \frac{1}{\sqrt{2\pi}}\left(\sqrt{s^2+r^2}\right)^{-1/2}$$

$y = \sinh^{-1} x$ $\qquad \sinh y = x = \frac{e^y - e^{-y}}{2}$

$$e^{2y} - 2e^y x - 1 = 0$$

$$e^y = x \pm \sqrt{x^2+1}$$

$$\sinh^{-1}(x) = \log\left(x \pm \sqrt{x^2+1}\right)$$

$$-\sqrt{s^2+r^2} + s\log\left(\frac{s}{r} + \sqrt{\frac{s^2}{r^2}+1}\right)$$

256 $-\sqrt{s^2+r^2} + s\log\left(s+\sqrt{s^2+r^2}\right) - s\log r$

$$-s\left(1+\frac{r^2}{s^2}\right)^{1/2} + s\log s + s\log\left(1+\sqrt{1+\frac{r^2}{s^2}}\right) - s\log r$$

$$= -s + s\log s + s\log(2) - s\log r + O\left(\tfrac{1}{s}\right)$$

$$= s\log s + s\left(-1+\log 2 - \log r\right) + O\left(\tfrac{1}{s}\right)$$

From Gaussian get $\log\frac{1}{\sqrt{2\pi}} - \frac{1}{2}\log\left(s\left(1+\frac{r^2}{s^2}\right)\right)$

$$\boxed{s\log s + s(-1+\log 2 - \log r) + \log\left(\frac{1}{\sqrt{2\pi}}\right) + \tfrac{1}{2}\log s + O\left(\tfrac{1}{s}\right)}$$

$$\log\frac{1}{\sqrt{2\pi}\,(s^2+r^2)^{1/2}} = \log\frac{1}{\sqrt{2\pi}\,s\left(1+\frac{r^2}{s^2}\right)^{1/2}}$$

$$= \log\frac{1}{\sqrt{2\pi}} - \log s - \frac{1}{2}\log\left(1+\frac{r^2}{s^2}\right)$$

Actually you should be able to use Legendre transform to get the asymptotics of the curves. These should be the same as for $\pi^{-s/2}\Gamma(s/2)$

Stirling
$\log\Gamma(s+1) = (s+\frac{1}{2})\log s - s + \log\sqrt{2\pi}$

$n! = n^n e^{-n}\sqrt{2\pi n}$

$$\int_0^\infty e^{-t} t^s \frac{dt}{t} = \int_{-\infty}^{\infty} e^{-e^x + xs}\, dx$$

$y = -e^x + xs$

$y' = -e^x + s = 0 \qquad x = +\log s$

$y'' = -e^x$

$$y = \left(-e^{\log x} + s\log s\right) + \tfrac{1}{2}(-s)(x-\log s)^2 + \cdot$$

leads to $\log\Gamma(s) = s\log s - s + \log\left(\sqrt{2\pi/s}\right)$

$= s\log s - s - \frac{1}{2}\log s + \frac{1}{2}\log 2\pi$

same as above for $r=2$

257 Philosophy time. H

Recall: Given Y Hilb space, c contraction on Y $\ni$ $\begin{cases} 1-c^*c & \text{rank } 1 \quad c^n y \to 0 \quad \forall y \\ 1-cc^* & \text{———} \quad (c^*)^n y \to 0 \quad \text{"} \end{cases}$

Then the scattering operator $S(z)$ is an inner function and you have a canonical (up to scalar mod 1) isom

$Y \xrightarrow{\sim} H^2 \ominus SH^2$ such that mult. by z on H^2 compresses to c^*. So you should find an equivalence between inner functions and such contractions.

Any inner function S factors uniquely into a Blashke product (includes scalars modulus 1) and a singular function $f = e^{\log f}$ where the harmonic function $Re(\log f)$ corresponds to a singular measure on S^1.

Example. $Y = \mathbb{C}$, $c = a \in D$. The idea here is that we get a Hilbert space bundle over D, and you want to understand any natural holom. structure wrt a. Problem because $(1-c^*c)^{1/2} = \sqrt{1-|a|^2}$ is not holomorphic in a.

You probably want to use $(1-zc^*)^{-1}$ instead of previous practice. What happens? Also $(1-z^{-1}c)^{-1}$. So it should be the same roughly

$$y \longmapsto \left\| (1-c^*c)^{1/2} \sum_{n\geq 0} z^{-n} c^n y \right\|^2 = \sum_{n \geq 0} \underbrace{\|(1-c^*c)^{1/2} c^n y\|^2}_{\|c^n y\|^2 - \|c^{n+1}y\|^2} .$$

So what am I doing? We send $y \in \mathbb{C}$ to $\sqrt{1-|a|^2} \, \frac{1}{1-z\bar{a}} \, y$. So

Let's try to describe what happens more carefully.

258 Recall this problem arises in connection with viewing ~~the~~ ~~...~~ $\operatorname{Tr}\left(\frac{z^{-1}}{1-z^{-1}c}\delta c\right)$ as $\delta \log\det(1-z^{-1}c)$. I am slowly getting nowhere.

Let's start with the idea that the (Y, c) of this type occurring with $\dim Y = n$ are naturally described by a pos divisor of degree n in D. So you have a holomorphic ~~situation~~ !! The moduli space this pertains to;

So given a poly $p(z) = \prod_{i=1}^{n}(z - a_i)$ $\quad |a_i| < 1$

we ~~associate~~ ~~...~~ form the Hilbert space $Y_p = H^2 \ominus pH^2$ with contraction operator induced by multiplication by z on H^2. Then ~~...~~

$zy - cy = (z-c)y \in pH^2$ so that, the roots of p are eigenvalues for c on Y.
$p = \det(z-c)$. So what we have is a ~~...~~ ~~...~~ trivial holomorphic vector bundle over the space of these p of degree n, equipped with hermitian scalar product. Why is it trivial? because $1, z, \ldots, z^{n-1} \in H^2$ generate a natural complement to pH^2 for any p. What is the inner product? probably

$$(z^i, z^j) = (1, z^{j-i}) = \int \frac{d\theta}{2\pi} \frac{z^{j-i}}{|p^\#|^2}$$

259 Anyway

$y \in \mathrm{Ker}(1-c^*c) \Leftrightarrow y = c^*cy \Rightarrow cy = cc^*cy \Leftrightarrow cy \in \mathrm{Ker}(1-cc^*)$

$c^*y' \in \mathrm{Ker}(1-c^*c) \Leftarrow c^*y' = c^*cc^*y' \Leftarrow y' = cc^*y' \Leftarrow y' \in \mathrm{Ker}(1-cc^*)$

$$y \in \mathrm{Ker}(1-c^*c) \Rightarrow cy \in \mathrm{Ker}(1-cc^*) \Rightarrow \underbrace{c^*cy}_{= y} \in \mathrm{Ker}(1-c^*c)$$

If c is shift, then $c^*c = 1$ so $\mathrm{Ker}(1-cc^*) = \mathrm{Ker}(0) = H$
and cc^* is a projector onto cH

$$\mathrm{Ker}(1-cc^*) = \mathrm{Im}(cc^*) = cH$$

and c, c^* set up an isom between $H \leftrightarrows cH$

~~So~~. In fin. dim. $\mathrm{Ker}(1-c^*c) \xrightarrow{\sim} \mathrm{Ker}(1-cc^*)$

If $1-c^*c$ has rank 1, then it Ker has dim $n-1$
so $1-cc^*$ has rank 1.

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need Groth type theorem relating different pictures

$p = \prod_{i=1}^{n}(z-a_i)$ $a_i \in D$ monic poly with roots in D

$H^+/pH^+ \xleftarrow{\sim} P_{<n} = \mathbb{C}1 + \mathbb{C}z + \cdots + \mathbb{C}z^{n-1}$

gives a holom. vector bundle over $Dw_n(D)$

$$Y_p = H^+ \ominus pH^+ = H^+ \cap \frac{p}{q}H^- \xleftarrow[\sim]{\frac{1}{q}} P_{<n}$$

You should check that $\frac{f}{q}$ is the projection of f onto Y_p, for $f \in P_{<n}$.

$$f = \frac{f}{q} + \frac{p}{q} \ ?$$ Probably No

but you can use

$f \mapsto \frac{f}{q}$ NO because this is not holom. in p.

260 ~~Q~~ $1 \in H^2$ what is its projection in Y

$$H^+ = Y \oplus S\cancel{H}^+$$

$$\frac{1}{1-\bar{w}z} = y + S(z)\frac{g}{1-\bar{w}z} \qquad \exists! y \quad \exists! g \in H^+$$

$$\frac{1-S(z)g}{1-\bar{w}z} = y \qquad g = \frac{1}{S(\bar{w}^{-1})} = \overline{S(w)}$$

$$\frac{1}{1-\bar{w}z} \rightsquigarrow \frac{1-\overline{S(w)}\,S(z)}{1-\bar{w}z}$$

$$1 \rightsquigarrow \cancel{1-\overline{S(0)}S(z)} \quad 1-\overline{S(0)}\,S(z)$$

~~Sit down~~

$$1 \longmapsto 1 - \overline{S(0)}\,\frac{p}{q} \xmapsto{q\cdot} q - \overline{S(0)}\,p$$

Corent dims $\quad Y = \mathbb{C}^n \quad$ How many almost unitary centr. c.

$$\dim_{\mathbb{R}} \mathbb{P}Y = 2(n-1)$$
$$\dim U(n) = n^2 - 1$$
$$\dim D = \underline{\quad 2 \quad}$$

$$n^2 + 2n - 1$$

Remove conjugation by $SU(n)$ $\quad -(n^2-1)$ to get $2n$

$$\overline{S(z)}p(z) = \prod \left(\frac{z^{-1}-\bar{a}_i}{1-a_i z^{-1}} \; z - a_i \right) = \prod \frac{1-\bar{a}_i z}{\cancel{z-a_i}}\,\cancel{z-a_i}$$

~~Q~~ ~~Olililda~~ Can you fit orthog polys of S^1 into what you have been doing? Idea is that ~~you~~ a prob measure on the circle yields a scalar product on the spaces $P_{\le n} = \mathbb{C}1 + \dots + \mathbb{C}z^n$ for all n. How does this relate to what you have been doing?

261 There is a difference because here you have a unitary operator whereas you had a contraction before. But you also have a distinguished line for the unitary. So things are not so bad. So how to proceed?

It appears that given a divisor of degree n on S^1, with distinct roots you get?

Start with a probability measure $d\mu$ on S^1 whence $L^2(S^1, d\mu)$. ~~First suppose $d\mu$ has finite support consisting of n points~~ Important example. Let $p(z) = \prod_{i=1}^{n}(z-a_i)$ be a de Branges fn. i.e. monic polyn. with roots in $\mathbb{D}$. Then $d\mu = \frac{1}{|p(z)|^2}\frac{d\theta}{2\pi}$ / norm.

What did you try earlier?

$$\frac{1}{1-\bar{w}z} = \frac{1-\overline{S(w)}\,S(z)}{1-\bar{w}z} + S(z)\frac{\overline{S(w)}}{1-\bar{w}z}$$

so if we put $w = 0$, we get

$$1 = \underbrace{1-\overline{S(0)}\,S(z)}_{Y} + \underbrace{\overline{S(0)}\,S(z)}_{S\cdot H^+}$$

So you find the orthogonal projection of 1 on Y is

$$1-\overline{S(0)}\,S(z) = 1 - \frac{\overline{p(0)}}{\overline{q(0)}}\frac{p(z)}{q(z)} = \frac{q(z)-\overline{p(0)}\,p(z)}{q(z)}$$

$1 = q(0)$

So if we pass to the deB space picture we get $q(z) - \overline{p(0)}\,p(z)$

But try to see what you can say when $d\mu = \frac{1}{|q|^2}\frac{d\theta}{2\pi}$ / norm. What are you after?

Look at $Y = H^+ \ominus SH^+ \underset{\sim}{\xleftarrow{\frac{1}{q}}} P_{<n}$

$$\begin{array}{ccc} \cap & & \cap \\ H^+ & \underset{\sim}{\xleftarrow{\frac{1}{q}\cdot}} & L^2\left(S^1, \frac{1}{|q|^2}\frac{d\theta}{2\pi}\right) \end{array}$$

 This doesn't work so go back over orthogonal polys. $d\mu$ prob. measure on S^1.

$$\mu_k = \int z^k \, d\mu \qquad k \in \mathbb{Z}.$$

$$(z^k, z^l) = \int z^{-k+l} \, d\mu = \mu_{l-k}$$

matrix of moments
$$\begin{pmatrix} \mu_0 & \mu_1 & \mu_2 & \\ \mu_{-1} & \mu_0 & \mu_1 & \ddots \\ & & \mu_0 & \ddots \\ & & & \ddots \end{pmatrix}$$

is pos. def.

Let $P_{[m,n]} = \mathbb{C} z^m + \cdots + \mathbb{C} z^n \qquad m \le n.$

$$P_n = P_{[0,n]}.$$

Filt $0 \subset P_0 \subset P_1 \subset \cdots \subset P_n \subset \cdots$

have $P_n = \mathbb{C} z^n + P_{n-1}$. Define $p_n \in z^n + P_{n-1}$, $p_n \perp P_{n-1}$. Also have $P_n = \mathbb{C} + zP_{n-1}$. Define $q_n \in (1 + zP_{n-1}) \cap (zP_{n-1})^\perp$. Recursion relations

$$p_{n+1} - \underbrace{z p_n}_{\perp \text{ to } z,\dots,z^n} \in P_n \cap (zP_{n-1})^\perp = \mathbb{C} q_n$$

$$\therefore \quad \boxed{p_{n+1} - z p_n = h_n q_n}$$

$$p_n \in (z^n + P_{n-1}) \cap P_{n-1}^\perp$$

$$\overline{p_n} \in (z^{-n} + P_{[-n+1,0]}) \cap P_{[-n+1,0]}^\perp$$

$$f^\#(z) = \overline{f(\bar z^{-1})}$$

$$z^n \overline{p_n} \in (1 + P_{[1,n]}) \cap P_{[1,n]}^\perp$$

$z^n \overline{p_n} = q_n$ on the circle

$$z^{n+1}\overline{p_{n+1}} - z^n\overline{p_n} = z\,\overline{h}_n z^n \overline{q}_n$$

$$\boxed{q_{n+1} - q_n = \overline{h}_n z p_n}$$

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$$\begin{pmatrix} p_{n+1} \\ q_{n+1} \end{pmatrix} = \begin{pmatrix} 1 & h_n \\ \bar{h}_n & 1 \end{pmatrix} \begin{pmatrix} zp_n \\ q_n \end{pmatrix}$$

$$q_{n+1} - \bar{h}_n z p_n = q_n$$

$$\|q_{n+1}\|^2 + |h_n|^2 \|p_n\|^2 = \|q_n\|^2$$

$$\|q_{n+1}\|^2 = (1-|h_n|^2)\|q_n\|^2$$

$$p_{n+1}(0) = h_n \qquad (q_n, p_n) = \overbrace{(q_n, 1)}^{\|q_n\|^2} p_n(0)$$

This is bare calculation – how can you use it.

<u>Maybe</u> you should be going downwards

Return to your goal. You start with p_n of the appropriate types. Then you get a measure on the circle $\frac{1}{|p|^2} d\theta/\text{norm}$

Starting from $p(z) = \prod_1^n (z - a_i)$ you get $Y = H^+ \cap \frac{p}{q} H^-$ a Hilbert space with c

Let's try to formulate this precisely, namely $p \rightsquigarrow H^+ \cap \frac{p}{q} H^- \Rightarrow H^+/pH^+$, c = mult by z. Conversely given (Y, c) satisfying certain condition you get a p inverse to the preceding. The tricky bit is how to get p, when the naturally occurring thing is an S. In finite dimensions you look at divisor $S=0$, and take the monic polynomial with the same root. But what you understand so far produces an inner function from (Y, c). You need more conditions on c. e.g. $c+1$ is compact. We want to put $p(z) = \det(1 - zc)$, so look for ways to make sense of this. Work in

$$V = \begin{pmatrix} 1 \\ c \end{pmatrix} Y \subset \begin{matrix} Y \\ \oplus \\ Y \end{matrix} \supset \begin{pmatrix} 1 \\ z \end{pmatrix} Y \quad ?$$

264 So what ~~...~~ you are missing is how to bring ~~...~~ prob measures on S^1 into play. At the moment we ~~...~~ can go from p to $S = \frac{p}{q}$ with $q = z^n p^\#$, n large enough, to get

So far you have an S theory, namely an equivalence between inner ~~...~~ functions and pairs (Y, c)

Go over the ~~...~~ equivalence of yesterday

Begin with S an ~~...~~ inner function. same as a unitary operator on $L^2(S^1)$ commuting with z such that $SH^+ \subset H^+$. $S \in L^\infty(S^1)$

New point $L^\infty(S^1) =$ ~~...~~ commutant of Z inside $\mathcal{L}(L^2(S^1))$ is a von Neumann, so have polar decomp. $T \in L^\infty(S^1)$ $\quad T = UP \quad P = (T^*T)^{1/2}$ There may be a problem if T is not invertible - so you don't get an easy explanation of outer functions. ~~...~~ But it's OKAY if T is invertible.

[redacted]

Begin with S inner, equivalent to a unitary op on $L^2(S^1)$ comm. with z $\ni$ $SH^+ \subset H^+$, equiv. also to a closed subspace W of H^+ such that $zW \subset W$ and $zW \neq W$.

Form $Y = H^+ \ominus SH^+ = H^+ \cap SH^-$

$j : Y \hookrightarrow H^+$ inclusion $\quad j^* j = 1. \quad c = j^* \cancel{z}\, j^{\#}$

$$\left.\begin{array}{l} c^n = j^* u^n j \\ (c^*)^n = j^* u^{-n} j \end{array}\right)_{n \geq 0} \qquad Y \to H^+/SH^+$$

$$L^2 = \underbrace{H^-} \oplus Y \oplus SH^+$$

$$\cdots \oplus \mathbb{C} z^{-2} \oplus \mathbb{C} z^{-1} \qquad \cancel{\mathbb{C}} S\cancel{\mathbb{C}} \oplus Sz\mathbb{C} \oplus Sz^2\mathbb{C} \oplus \cdots$$

pt evaluator $\dfrac{1 - \overline{S(\omega)}\, S(z)}{1 - \bar{\omega} z} = J_\omega(z) \in Y$

$\langle J_\omega, J_{\omega'} \rangle = J_{\omega'}(\omega)$

~~done wrong: If $u = z$~~

265 ~~Obgorly~~ ~~From the old~~

Focus on how to connect this S theory to prob measures on S^1! Approach - char function of c, deB stuff. Suppose Y dim n $p = \det(z-c)$, $q = z^n p^\# = z^n \det(z^{-1} - c^*) = \det(1 - zc^*)$. Then

$$Y = H^+ \cap SH^- \xrightarrow{\cdot q} qH^+ \cap pH^- = P_{<n}$$

Y appears as $P_{<n}$ equipped with $\|f\|_p^2 = \int \frac{d\theta}{2\pi} \left|\frac{f}{q}\right|^2$

Your problem is how to connect. ~~Dto~~ deB approach look inside $L^2(S^1, d\mu)$

Maybe the point is the Birkhoff factorization:

$S = \frac{p}{q}$ p ? winding number of S.

Idea here ~~is to sm~~ maybe is to take a simple S i.e. a Blashke product.

But something you should be able to do for a general prob measure on S^1. Szego theory approximates a probability measure with rational measures.

If p_n, q_n are the sequence of orth. polys, then

$\frac{1}{|q_n|^2} d\theta/\text{norm.}$ should have the same moments up to order n?

Consider $d\mu$ prob. measure in S^1. Get orthog sequence $p_0, p_1, \dots, p_n$ of polys at least for $n+1 \leq$ card supp $d\mu$. So what's the point? ~~Yet baled~~ If you fix an n then you get $S_n = \frac{p_n}{q_n}$ and you ought to be able to relate the S_n ~~[illegible]~~ (contraction)

Repeat. Start with a prob measure $d\mu$ on S^1. Then you can form the sequence of (monic) orthogonal polys $p_0, p_1, \dots, q_n, \dots$ ~~and~~ satisfying the recursion relations $\begin{pmatrix} p_n \\ q_n \end{pmatrix} = \begin{pmatrix} 1 & h_n \\ \bar{h}_n & 1 \end{pmatrix} \begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} p_{n-1} \\ q_{n-1} \end{pmatrix}$ where

266 $q_n = z^n \bar{p}_n$ so that $p_n = \prod_{i=1}^{n}(z-a_i)$

$\Rightarrow q_n = \prod_1^n (1-\bar{a}_i z)$

get $S_n = \frac{p_n}{q_n} = \prod_1^n \frac{z-a_i}{1-\bar{a}_i z}$. Check a_i in D.

$S_n = \begin{pmatrix} 1 & h_n \\ \bar{h}_n & 1 \end{pmatrix}(z S_{n-1})$ so by induction. S_n

Now to define $p_0, p_1, p_2, \ldots, p_n$ you need the moments $\mu_j = \int z^j d\mu$ $j = 1, \ldots, n$. Is there a formula for S_n in terms of these moments?

$L^2(S^1, d\mu)$. $d\mu \Longleftrightarrow$ harmonic function ≥ 0 on D.

Consider $P_n = \mathbb{C} + \mathbb{C}z + \cdots + \mathbb{C}z^n \subset L^2(S^1, d\mu)$

You get a partial unitary with domain P_{n-1}

~~You get a partial unitary~~

$Y = P_n$. It seems that if you give moments i.e. you give a hermitian scalar product on P_n such that $z\cdot$ is a partial unitary.

$(z^i, z^j) = \mu_{j-i}$ $0 \leq i, j \leq n$.

~~Note that~~ Condition is that the matrix $\left((\mu_{j-i})_{0\leq i,j\leq n}\right)$ is pos. def. Depends upon $\mu_1, \ldots, \mu_n$ since $\mu_{-k} = \bar{\mu}_k$ open condition so get open subset of $\mathbb{C}^n$ for the possibilities.

Describe what you are doing. You have a prob. measure $d\mu$ on S^1, ~~[illegible]~~ get orthogonal monic poly sequence $p_0, p_1, \ldots, p_n, \ldots\ldots$

$\frac{p_n}{q_n} = S_n$ should be the scattering function associated to the contraction ~~partial unitary~~ on $P_{\leq n}$.

267 You ~~have~~ should have isometric embedding

$$P_{<n} \hookrightarrow H^+ \qquad f \longmapsto \frac{f}{q^n}$$

~~[illegible]~~ Now increase n

$$\begin{array}{ccc} P_{<n} & \xhookrightarrow{\frac{1}{q_n}} & H^+ \\ \cap & \nearrow_{\frac{1}{q_{n+1}}} & \\ P_{\leq n} & & \end{array}$$

~~[illegible]~~ doesn't commute of course

but should be compatible with scalar products, i.e.

$$\int |f|^2 \frac{d\theta}{2\pi |q_n|^2} / \text{norm} = \int |f|^2 \frac{d\theta}{2\pi |q_{n+1}|^2} / \text{norm}.$$

Other important stuff. When $\prod^{\infty} (1 - |h_n|^2) > 0$

Then $\lim (q_n)$ ~~[illegible]~~ $\in 1 +$

What to do next?? ~~I want you to~~ Problem is to connect S stuff with measures.

$H = L^2(S^1, d\mu)$

$$P_n = \mathbb{C} + \mathbb{C}z + \cdots + \mathbb{C}z^n$$

$$p_n \in (z^n + P_{n-1}) \cap P_{n-1}^\perp$$

$$\overline{p_n} \in (z^{-n} + \overline{P_{n-1}}) \cap \overline{P_{n-1}}^\perp$$

$$q_n = z^n \overline{p_n} \in (1 + zP_{n-1}) \cap (zP_{n-1})^\perp$$

$$P_{n-1} \underset{b = za}{\overset{a}{\rightrightarrows}} P_n \quad \text{is a partial unitary}$$

$$\mathbb{C} p_n = V^+ \qquad \mathbb{C} q_n = V^-$$

~~[illegible] $(\lambda a - b)(\lambda) \to S(\lambda)$:~~

suppose $t\, q_n$ and p_n in P_n are congruent modulo $(\lambda - z)P_{n-1}$ i.e.

$$(\lambda - z)x = -p_n + t\, q_n \qquad x \in P_{n-1}$$

Then $t = \dfrac{p_n(\lambda)}{q_n(\lambda)}$.

 Go back. $L^2(S^1, d\mu) \supset P_n$

You get a sequence of partial unitaries.

A partial unitary of type $\mathcal{O}(n)$ is equivalent to an ~~contraction~~ almost unitary contraction (Y, c) with $\dim Y = n$, also equivalent to a divisor of degree n in D.

You should be able to identify $p_n(\zeta) = \det(\zeta - c)$

Go back to $P_{n-1} \xrightarrow[z]{} P_n$. Concentrate on divisors. A partial unitary can be ~~closed up~~ extended to contraction by ~~extending~~ defining $c\,ax = bx$.

$$p_n = z p_{n-1} + h_n q_{n-1} \qquad S_n = \begin{pmatrix} 1 & h_n \\ \bar{h}_n & 1 \end{pmatrix} (z S_{n-1})$$
$$q_n = \bar{h}_n z p_{n-1} + q_{n-1}$$

What are you ~~doing?~~ aiming for?

Claim. A prob measure $d\mu$ on S^1 leads to a sequence of orthogonal polys, equivalently a sequence $h_1, h_2, \ldots$ in D, equivalently a response function $S(z) \in \mathcal{B}$ = unit ball in H^∞. ~~That conversely~~ and the converse is true?

You've made an interesting mistake, because $S(z)$ is not well-defined yet, only a sequence $S_n(z)$. Nevertheless it should be true that $d\mu$ is equivalent to a sequence $h_1, h_2, \ldots$ How?

Suppose given $d\mu$ prob. measure on S^1, form p_n, h_n, Assume $q_\infty = \lim q_n$ exists i.e. $H^2(d\mu) \supset z H^2(d\mu)$ so $q_\infty \in (1 + z H^2(d\mu)) \cap (z H^2(d\mu))^\perp$

n.a.s. that $\prod_{n=1}^{\infty} (1 - |h_n|^2) > 0$

269 So what happens is that the embedding you do for finite n works in general.

Go over finite n case

$X \overset{a}{\underset{b}{\rightrightarrows}} Y$ $\qquad V^+ = \mathrm{Ker}(a^*) = \mathbb{C}p_n$

$P_{n-1} \overset{a}{\underset{za}{\rightrightarrows}} P_n$ $\qquad V^- = \mathrm{Ker}(b^*) = \mathbb{C}q_n$

The contraction should be $c = a^*b$ on P_{n-1}

$c^* = b^*a$

This is a technical point you have to deal with anyway.

$\xi \in X$

$$\begin{array}{ccc} () & \binom{1}{c}X \oplus \binom{1}{z}X \xrightarrow{\sim} & X \oplus X \\ \downarrow & & \\ \binom{1}{c}X & \xrightarrow[\sim]{(z\ \ -1)} & X \end{array}$$

$$\begin{array}{ccccc} V & \xrightarrow{\sim} & X^{\oplus 2}/\binom{1}{z}X & \xrightarrow{\sim} & \ell_z^\vee \otimes X \\ \downarrow & & ? & & \\ W^0/W & & & & \end{array}$$

Given $X \overset{a}{\underset{b}{\rightrightarrows}} Y$ form $W = \binom{a}{b}X \subset Y^{\oplus 2}$

Here it is:

$c = a^*b$

$1 - c^*c = 1 - b^*b$ no

confused

$$\begin{array}{ccc} \binom{1}{c}X & \hookrightarrow & T \otimes X \\ \uparrow s & \searrow \sim & \downarrow (z\ \ -1) \\ X & \xrightarrow[(z-c)]{} & \ell_z^\vee \otimes X \\ \downarrow (1-c^*c)^{1/2} & & \\ V^- & & \end{array}$$

$\underbrace{(1-c^*c)^{1/2}}_{\text{projection on } \mathrm{Ker}(1-b^*b)}(z-c)^{-1}\xi$ $\quad a^*b$

270 Take $Y = P_n$, $X = P_{n-1}$ $\quad a = \text{inc}$, $b = za$.

Then $c = a^*b$ on X

~~no but~~

$$Y = P_{n-1} \oplus \mathbb{C}p_n$$
$$= \mathbb{C}q_n \oplus zP_{n-1}$$

~~It is easier to embed Y~~

Scattering

$$(a\lambda - b)x = -v^+ + v^-$$

$x \in P_{n-1}$ $\qquad (\lambda - z)\,x(z) = -p_n^{(z)} + S^{(\lambda)} q_n^{(z)}$

set $z = \lambda$ to get $S(\lambda) = \dfrac{p_n(\lambda)}{q_n(\lambda)}$.

$$(\lambda - z)\,x(z) = -y(z) + \hat{y}(\lambda)\,q(z)$$

$z = \lambda$ $\qquad \hat{y}(\lambda) = \dfrac{y(\lambda)}{q(\lambda)}$

There are lots of things to look at Szegö, but before this try to describe the sequence of Hilbert spaces P_n, the idea here being point evaluator

Review embedding. Given $P_n = Y$ Let $X = P_{n-1}$ $\quad a$ inclusion $\quad b = za$

$V^+ = (aX)^\perp = P_{n-1}^\perp = \mathbb{C}p_n$, $V^- = \mathbb{C}q_n$. Take contraction $c =$ (should be induced by mult by

$p_n = \det(z - c_n) = \det(z - a^*b) = z^n \det(1 - z^{-1}a^*b)$
$= z^n \det(1 - z^{-1}ba^*) = z^n z^{-n-1} \det(z - ba^*)$

so $\det(z - ba^*) = z\,p_n(z)$. So what? Life goes on.

271 So look at $Y = P_n$ with $c = ba^*$

and you should have $S = \dfrac{\det(z-c)}{\det(1-zc^*)} = \det\left(\dfrac{z-c}{1-zc^*}\right)$

$p_{n+1} = \det(z-c) = z\,p_n(z)$

$q_{n+1} = \det(1-zc^*) = q_n(z).$ $\quad S = z S_n$

So what do you want to check?

~~Consider the eigenvalue equation.~~ You know that you get an isom. embedding $Y \hookrightarrow H^+$ by

$(1-c^*c)^{1/2}(z-c)^{-1}y = (1-aa^*)(z-ba^*)^{-1}y$ analytic for $|z|>1$.

$c^*c = (ba^*)^*(ba^*) = ab^*ba^* = aa^*$. NO you want $(1-cc^*)^{1/2}(1-zc^*)^{-1}y$

$= (1-bb^*)(1-zab^*)^{-1}y. = \hat{y}(z).$ ~~So everything is clean!~~

Work this out in the case of P_n. We know $\hat{y}(\lambda)$ is the number such that $\exists\, x \in P_{n-1} \ni$

$$(a\lambda - b)\,x = -y + \hat{y}(\lambda)\,v^- \qquad v^-(z) = \frac{q(z)}{\|q\|}$$

$$(\lambda - z)\,x(z) = -y(z) + \hat{y}(\lambda)\,v^-(z)$$

Setting $z=\lambda$ gives $-y(\lambda) + \hat{y}(\lambda)\,q(\lambda)\,\|q\|^{-1} = 0$

$$\hat{y}(\lambda) = \frac{y(\lambda)}{q(\lambda)}\|q\|.$$

~~And~~ Important case is when $y = v^-$

then $\hat{y}(\lambda) = 1$, so $1 = \dfrac{\widehat{v^-}(\lambda)}{q(\lambda)}\|q\|$ $\qquad \widehat{v^-}(\lambda) = \dfrac{q(\lambda)}{\|q\|}$

so $\displaystyle\int |\widehat{v^-}(\lambda)|^2 \frac{d\phi}{2\pi} = \int \frac{|q(\lambda)|^2}{\|q\|^2}\frac{d\phi}{2\pi} = 1.$

So all I've checked ~~done~~ is the obvious.

So we have isometric embedding

$$P_n \hookrightarrow H^+ \qquad f \longmapsto \frac{f}{q_n}\|q_n\|$$

272 What sort of relations do I get as n changes?

Organize!!! I seem to be finding that the orthogonal poly ~~approach~~ theory leads to the de Branges situation on the circle. Be more specific.

Suppose you give ~~a sequence of moments~~ μ_n a positive def hermitian ~~scalar~~ form on $\mathbb{C}[z]$ $\ni$ $\hat{z}$ = mult by z is isometry. This should be the same as a positive def. function on $\mathbb{Z}$, hence a measure on S^1. Main question is whether $\hat{z}: \overline{\mathbb{C}[z]} \circlearrowleft$ is onto hence an isom.

Problem: Relate $\mu_n = \int z^n d\mu_{\mathcal{E}}$ to $h_n^{\mathcal{E}}$.

$$S = \Big(\begin{matrix} 1 & h \end{matrix}$$

Recall more formulas.

$$\begin{pmatrix} p_{n+1} \\ q_{n+1} \end{pmatrix} = \begin{pmatrix} 1 & h_{n+1} \\ \bar{h}_{n+1} & 1 \end{pmatrix} \begin{pmatrix} z p_n \\ q_n \end{pmatrix}$$

Assume $h_n = 0$ for $n \gg 0$. You get powers of z.

$$S_n = \frac{p_n}{q_n} = \frac{}{q}$$

Since q_n is not changing you get isom. emb.

$$H^2(S^1, d\mu) \longrightarrow H^2(S^1, \tfrac{d\theta}{2\pi})$$

$$f \in P \longmapsto \frac{f}{q}\|q\|$$

what happens is that $H^2(S^1, d\mu)$ is outgoing so this an isomorphism.

273 Review. Begin with $L^2(S^1, d\mu)$ $d\mu$ inf. supp.

Get p.u. $P_{n-1} \underset{b=z^{-1}a}{\overset{a}{\rightleftarrows}} P_n$ $V^+ = \mathbb{C}\, p_n$

$V^- = \mathbb{C}\, q_n$

~~[illegible]~~ $p_n = \det(\underset{\text{on } P_{n-1}}{z - a^*b}) = z^{-1}\det(z - \underbrace{ba^*}_{c})$

$q_n = \det(1 - zc^*)$ the scattering associated to

Begin again with $P_n \subset L^2(S^1, d\mu)$

Get p.u. $P_{n-1} \underset{b=za}{\overset{a=\text{in}}{\rightleftarrows}} P_n$ $V^+ = \mathbb{C}\, p_n$ $p_n \in (z^n + P_{n-1}) \cap P_{n-1}^{\perp}$

$V^- = \mathbb{C}\, q_n$ $q_n \in (1 + zP_{n-1}) \cap (zP_{n-1})^{\perp}$

On P_{n-1} ~~with~~ have the contraction a^*b = mult by z on P_{n-1} projected back. $p_n = \det(z - a^*b) = z^{-1}\det(z - ba^*)$

On P_n we have $c = ba^*$ $q_n = \det(1 - zb^*a)$

$= \det(1 - zab^*)$

$$z\,p_n(z) = \det(z - ba^*)$$
$$q_n(z) = \det(1 - zab^*)$$

$$\frac{z\,p_n(z)}{q_n(z)} = \frac{\det(z - c)}{\det(1 - zc^*)} = \det\left(\frac{z - c}{1 - zc^*}\right)$$

You probably ~~may~~ have to be careful with $\dfrac{z - c}{1 - zc^*}$ since c, c^* do not commute; the det should be OK.

In fact scattering is

$$\left[(1-c^*c)^{1/2}(1-z^{-1}c)^{-1}\right](1-cc^*)^{1/2}(1-zc^*)^{-1} \ ?$$

$$L^2(S^1, V^+) \longleftarrow H \longrightarrow L^2(S^1, V^-)$$

$$(1-c^*c)^{1/2}(1-z^{-1}c)^{-1}\xi \longleftarrow\!\!| \ \xi \ |\!\!\longrightarrow (1-cc^*)^{1/2}(1-zc^*)^{-1}\xi$$

$$\eta \longmapsto (1-z^{-1}c)(1-c^*c)^{-1/2}\eta \longmapsto (1-cc^*)^{1/2}(1-zc^*)^{-1}(1-z^{-1}c)(1-c^*c)^{-1/2}$$

~~[illegible]~~ Problem: Justify, Explain the meaning of $\dfrac{z-c}{1-zc^*}$ – it should be the scattering operator?

 For ~~[illegible]~~ $(P_n, \underbrace{ba^*}_{c})$ you have a spectral repn.

$$y \longmapsto \underbrace{(1-c^*c)^{1/2}}_{1-aa^*}(z-c)^{-1}y \qquad \text{NO}$$

this is analytic outside.

you want
$$y \longmapsto (1-bb^*)(1-zc^*)^{-1}y$$

interpret as $(v^-|(1-zc^*)^{-1}y = \hat{y}(z)$

We know this is solution of

$$(a\lambda - b)\underset{\substack{n \\ P_{n-1}}}{x} = -y + \hat{y}(\lambda)v^-$$

chk.
$$(1-\lambda b^*a)x = b^*y$$

$$x = (1-\lambda b^*a)^{-1}b^*y = b^*(1-\lambda ab^*)^{-1}y$$

$$\hat{y}(\lambda)v^- = y + (a\lambda - b)b^*(1-\lambda ab^*)^{-1}y$$

$$= \left[1-\lambda ab^* + (a\lambda - b)b^*\right](1-\lambda ab^*)^{-1}y$$

$$= (1-bb^*)(1-\lambda ab^*)^{-1}y \qquad \hat{y}(\lambda) = (v^-|(1-\lambda ab^*)^{-1}y.$$

Now interpret as polys in z.

$$(\lambda - z)x(z) = -y(z) + \hat{y}(\lambda)\frac{q(z)}{\|q\|}$$

$$\therefore \quad \hat{y}(\lambda) = \frac{y(\lambda)}{q(\lambda)}\|q\|$$

So you get
$$P_n \hookrightarrow H^+$$
$$y \longmapsto \hat{y} = \frac{y}{q}\|q\|.$$

~~This is most confusing.~~

Lets review.

27 Review: $L^2(S^1, d\mu)$

$P_n \subset L^2(S^1$ Go back to $Y = aX \oplus V^+ = bX \oplus V^-$

get contraction $c =$

Start again with $Y = P_n \subset L^2(S^1, d\mu)$, $X = P_{n-1}$

$a = \text{inc}$, $b = za$ $\rho = \det(z - a^*b) = z^{-1}\det(z - \underbrace{ba^*}_{c})$

$\det(z - \underbrace{ba^*}_{c}) = z\rho(z)$, $\det(1 - zab^*) = g(z)$. These functions become different ~~in~~ in ∞ dim. Embedding

$\underbrace{(1-cc^*)^{1/2}}_{1-bb^*}\underbrace{(1-zc^*)^{-1}}_{ab^*} : Y \longrightarrow H^+$ isometric. You

$y \longmapsto \tilde{y}$

know this is given by solving $\underset{\lambda - z}{(a\lambda - b)}x = -y + \tilde{y}(\lambda)v^-$

but because of the poly model you get $\tilde{y}(\lambda) = \frac{y(\lambda)}{v^-(\lambda)} = y\frac{\|g\|}{g}$

Put into words what is happening. You have a general isom. embedding result: Given Y, c you embed $Y \hookrightarrow H^+$ $\left\{\begin{array}{l} v^- \longmapsto 1 \\ j^*\bar{z}^n j = c^n \end{array}\right.$

c contraction on $Y \ni$ $1 - cc^*$ rank 1 and $(c^*)^n y \longrightarrow 0$. Suppose

$j(y) = \langle v^- | (1-cc^*)^{1/2}(1 - zc^*)^{-1} y\rangle$? NO this is not correct.

First choice $jy = \sum_{n \geq 0} z^n (1-cc^*)^{1/2} c^{*n} y \in H^2(S^1, \overset{V^-}{\overset{\parallel}{(1-c^*c)^{1/2}Y}})$

$V = \mathrm{Ker}(1-cc^*)^\perp = (1-cc^*)^{1/2}Y$ $Y = \mathrm{Ker}(1-cc^*) \oplus \mathrm{Ker}(1-cc^*)^\perp = (1-cc^*)^{1/2}Y$

$$\begin{array}{ccccc} \mathrm{Ker}(1-c^*c) & \hookrightarrow & Y & \twoheadrightarrow & V^+ \\ \text{inverse} \downarrow\uparrow & & c\downarrow\uparrow c^* & & \downarrow\uparrow \\ \mathrm{Ker}(1-cc^*) & \hookrightarrow & Y & \twoheadrightarrow & V^- \end{array}$$

~~Anyway what happens? Suppose we consider~~ [illegible]

~~Review. Dilate~~ $\cancel{A = \begin{pmatrix} 0 & c^* \\ c & 0 \end{pmatrix}}$ ~~to an~~ [illegible]

$$\begin{tikzcd} X \arrow[dr, hook, "a"] & \\ & Y \\ X' \arrow[ur, hook, "b"'] & \end{tikzcd}$$

$a^*a = 1$
$b^*b = 1$

$$\|ax + bx'\|^2 = \|x\|^2 + \|x'\|^2 + (ax, bx') + (bx', ax)$$

$$= \|x\|^2 + (b^*ax, x') + (x', b^*ax) + \|x'\|^2$$

$$= \underbrace{\|x\|^2 - \|b^*ax\|^2}_{(x,\,(1 - a^*bb^*a)x)} + \|b^*ax + x'\|^2$$

$c = b^*a$

$$\boxed{b^*(ax + bx') = cx + x'}$$

$$= \|(1 - c^*c)^{1/2}x\|^2 + \|cx + x'\|^2$$

$$\|ax + bx'\|^2 = \|x\|^2 + (x, c^*x') + (c^*x', x) + \|x'\|^2$$

$$= \|x + c^*x'\|^2 + \underbrace{(x', (1 - cc^*)x')}_{\|(1 - cc^*)^{1/2}x'\|^2}$$

$$\boxed{a^*(ax + bx') = x + c^*x'}$$

So given $c : X \to X'$ you can factor it
$c = b^*a$

$$\begin{tikzcd} X \arrow[r, hook, "a"] \arrow[dr, "c"'] & Y \arrow[d, "b^*"] \\ & X' \end{tikzcd}$$

~~So you have nothing.~~
Go back - try to understand the difference between S for a contraction and a p.u.

You want to begin with (Y, c) define V^- and V^+.

You need v^- in Y such that

$$|\langle v^- | y \rangle|^2 = \|y\|^2 - \|c^*y\|^2$$

$$|v^-\rangle\langle v^-| = 1 - cc^*$$

where $\|v^-\|^2$ $1 - c^*c$
= value of

277 absurd - ~~choose~~ Assuming $1-c^*c$ has rank 1, say it is $v^-(v^-|$ where $\|v^-\|^2 = tr(1-c^*c)$. $\qquad 1-cc^* = v^-(v^-|$

Then $y \longmapsto \sum_{n\geq 0} z^n (v^-| c^{*n} y = (v^-| \frac{1}{1-zc^*} y$

$$\|(v^-|(1-zc^*)^{-1}y\|^2 = \sum_{n\geq 0} \underbrace{\|(v^-|c^{*n}y\|^2}_{(y,\, \underbrace{v^-(v^-|}_{1-cc^*} c^{*n}y) \,=\, \|c^{*n}y\|^2 - \|c^{*n+1}y\|^2}$$

you could write $\quad 1-cc^* = v^- v^{-*}$

$$y \overset{j}{\longmapsto} v^{-*}(1-zc^*)^{-1}y. \qquad \text{embeds } Y \text{ in } H^2$$

~~Repeat~~ ~~visualize~~

$$\hat{y}(w) = v^{-*}(1-wc^*)^{-1}y = ((1-\bar{w}c)^{-1}v^-, y)$$

pt evaluator is $\frac{1}{1-\bar{w}c} v^- = j^* \frac{1}{1-\bar{w}z} j v^-$

$$j v^- = v^{-*} \frac{1}{1-zc^*} v^-$$

review: $Y, c \quad 1-cc^*$ rank 1 $\quad c^{*n}y \to 0 \ \forall y.$

$Y \longrightarrow H^+ \qquad$ Let $v_- \in Y \ni \ 1-cc^* = v_- v_-^*$

$y \longmapsto v_-^* \frac{1}{1-zc^*} y = \hat{y}$

$$\hat{y} = \sum_{n\geq 0} z^n v_-^* c^{*n} y$$

$$\|\hat{y}\|^2 = \sum_{n\geq 0} |v_-^* c^{*n} y|^2 = \sum_{n\geq 0} (c^{*n}y, \overbrace{v_- v_-^*}^{1-cc^*} c^{*n} y)$$

$$= \sum_{n\geq 0} \|c^{*n}y\|^2 - \|c^{*(n+1)}y\|^2 = \|y\|^2$$

~~Recall~~

$$\hat{v^-} = v_-^* \frac{1}{1-zc^*} v_- \qquad \text{this } \text{seems to cause problems}$$

If $c = ba^*$
$1-cc^* = 1-ba^*ab^*$

 ~~it seems that~~ This needs more work

$$\rho(z) = \sum_{n\geq 0}{}' z^{+n}(c^*)^n + \sum_{n\geq 1}{}' z^{-n}c^n$$

$$= \frac{1}{1-zc^*} + \frac{z^{-1}c}{1-z^{-1}c} = \frac{1}{1-zc^*}\underbrace{\left(1-z^{-1}c+(1-zc^*)z^{-1}c\right)}_{1-c^*c}\frac{1}{1-z^{-1}c}$$

$$= \frac{1}{1-z^{-1}c}\left(1-z^{-1}c+z^{-1}c(1-zc^*)\right)\frac{1}{1-zc^*} = \frac{1}{1-z^{-1}c}(1-cc^*)\frac{1}{1-zc^*}$$

$$\left(\xi, z^n\xi'\right) = \int\frac{d\theta}{2\pi}\left(\xi, z^n\rho(z)\xi'\right)$$

$$= \begin{cases} (\xi, c^n\xi') & n\geq 0 \\ (\xi, (c^*)^{-n}\xi') & n\leq 0\end{cases}$$

$$L^2\left(S^1, \rho(z)\frac{d\theta}{2\pi}\right) \longrightarrow L^2$$

$$\sqrt{1-c^*c}\,\frac{1}{1-z^{-1}c}f(z) \longleftarrow f \longmapsto \sqrt{1-cc^*}\,\frac{1}{1-zc^*}f(z)$$

$$S(z)\cancel{f} = \sqrt{1-c^*c}\,\frac{1}{1-z^{-1}c}(1-zc^*)\frac{1}{\sqrt{1-cc^*}}$$

$$S(z)^* = \frac{1}{\sqrt{1-cc^*}}(1-z^{-1}c)\frac{1}{1-zc^*}\sqrt{1-c^*c}$$

Formally $SS^* = S^*S = 1$.

unclear!!

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try to control passage from X to Y. Roughly X has contraction $\overset{*}{a}b$ which leads to ba^* on Y, but then you have to add: ba^*+h where $h: V^+ \longrightarrow V^-$ is a contraction. $c = ba^*+h$ where $(1-bb^*)h(1-aa^*) = h$.

$$Y = aX \oplus V^+ = V^- \oplus bX$$

$$c = ba^* + h \qquad h : V^+ \to V^-$$

$$c^* = ab^* + h^* \qquad h^* : V^- \to V^+.$$

$$cc^* = bb^* \oplus hh^* \quad \text{on} \quad bX \oplus V^-$$

$$c^*c = aa^* \oplus h^*h \quad \text{on} \quad aX \oplus V^+$$

eigenvalue equation to be formulated in terms of graphs.

$$\underbrace{\begin{pmatrix} a \\ b \end{pmatrix} X}_{W} \subset V = \underbrace{\begin{pmatrix} 1 \\ c \end{pmatrix} Y}_{\begin{pmatrix} a \\ b \end{pmatrix} X + \begin{pmatrix} 1 \\ h \end{pmatrix} V^+} \subset \underbrace{W \oplus \begin{pmatrix} V^+ \\ \oplus \\ V^- \end{pmatrix}}_{W^0} \subset \begin{matrix} Y \\ \oplus \\ Y \end{matrix}$$

first suppose $h = 0$. Then

$$\begin{tikzcd} V = \begin{pmatrix} a \\ b \end{pmatrix} X + \begin{pmatrix} V^+ \\ 0 \end{pmatrix} \arrow[r, hook] \arrow[rr, bend right, "\sim"'] & \begin{matrix} Y \\ \oplus \\ Y \end{matrix} \arrow[r, "(z\ \ -1)"] & Y \end{tikzcd}$$

$$z(ax + v^+) = bx = y$$

$$(za - b)x = -zv^+ + y$$

$$(z - a^*b)x = a^*y \qquad x = (z - a^*b)^{-1} a^* y = a^*(z - ba^*)^{-1} y$$

$$zv^+ = y - (za - b) a^* (z - ba^*)^{-1} y$$

$$= \left(z - ba^* - (za - b)a^*\right)(z - ba^*)^{-1} y$$

$$= z(1 - aa^*)(z - ba^*)^{-1} y$$

$$\boxed{v^+ = (1 - aa^*)(z - ba^*)^{-1} y}$$

$$\begin{tikzcd} V = \begin{pmatrix} 1 \\ ba^* \end{pmatrix} Y \arrow[r, "(z\ \ -1)", "\sim"'] & Y \\ Y \arrow[u] \arrow[ur, "z - ba^*"'] & \end{tikzcd} \qquad \begin{pmatrix} 1 \\ ba^* \end{pmatrix} (z - ba^*)^{-1} y$$

280 Let us consider a p.u. $Y = aX \oplus V^+ = V^- \oplus bX$ ~~...~~ with $V^\pm$ dim 1 and consider $c = ba^* + h$ where $h \in \mathcal{L}(V^+, V^-)$ $= \pi_- \mathcal{L}(V) \pi_+ \subset \mathcal{L}(V)$.

You want the scattering op associated to c.

The basic result is that the scattering S_h assoc. to c on Y and the scattering S_0 assoc. to ba^* are related by $S_h = \begin{pmatrix} 1 & h \\ \bar{h} & 1 \end{pmatrix} S_0$

$$S_{ba^*+h} = \begin{pmatrix} 1 & h \\ \bar{h} & 1 \end{pmatrix} S_{ba^*} \qquad S_{ba^*} = z S_{a^*b}$$

Check this makes sense in the orth poly situation

$$X = P_{n-1} \quad Y = P_n \qquad S_{a^*b} = \frac{\det(z - a^*b)}{\det(1 - zb^*a)} \qquad z^{-1} S_{ba^*}$$

$$= \frac{z^n \det(1 - z^{-1} a^*b)}{\det(1 - z b^*a)} = \frac{z^n \det(1 - z^{-1} ba^*)}{\det(1 - z ab^*)} = z^{-1} \frac{\det(z - ba^*)}{\det(1 - z ab^*)}$$

Check that ~~...~~ $S_{ba^*+h}(0) = h$. ~~...~~

$$S_{ba^*+h} = \frac{p_{n+1}}{q_{n+1}} = \frac{z p_n + h q_n}{q_n + \bar{h} z p_n} \qquad h = h_{n+1}$$

<u>This might be very easy</u>

Go back to S $Y = H^+ \ominus SH^+$ get c on Y induced by $\hat{z}$ on H^+/SH^+. Observe that $1 \in Y$ iff $1 \perp SH^+$ i.e. $S(0) = 0$. $\frac{1}{1-\bar{w}z}$ is the point evaluator

$$\frac{1}{1-\bar{w}z} \overset{pr}{\rightsquigarrow} \frac{1 - \overline{S(w)}\, S(z)}{1 - \bar{w} z} \qquad \text{i.e. } 1 - \overline{S(0)}\, S(z)$$

There is something you still don't understand about a general $c = ba^* + h$, almost unitary. To $\in Y, c$ you assoc. an embed. $Y \hookrightarrow H^+$ $\tilde{y}(z) = v_-^*(1 - zc^*)^{-1} y$

281 ~~[illegible]~~ The philosophy is to focus on the Hilbert space of analytic functions you get in this way. ~~[illegible]~~ These are described by inner functions S, equivalently outgoing subsp.

~~[illegible]~~ Given $S(z)$ inner $Y = H^+ \ominus SH^+ = H^+ \cap SH^-$.

$$\frac{1}{1-\bar{w}z} = y + \frac{S(z)f}{1-\bar{w}z} \qquad f \in H^+$$

$$\frac{1 - S(z)f(z)}{1-\bar{w}z} = y \qquad f = \frac{1}{S(\bar{w}^{-1})} = \overline{S(w)}$$

You work the other way using S unitary.

$$K(w,z) = \frac{1-\overline{S(w)}S(z)}{1-\bar{w}z} \qquad \frac{1}{1-\bar{w}z} = K_w(z) + S(z)g(z)$$

$$(K_w, y) = \underset{y(w)}{\left(\frac{1}{1-\bar{w}z}, y\right)} + \underset{0}{(Sg, y)} \qquad y \perp SH^+$$

$K_w \in H^+$, $(K_w, Sg) = \boxed{\cancel{S(w)}}$

$$(K_w, Sg) = \underbrace{\left(\frac{1}{1-\bar{w}z}, Sg\right)}_{(Sg)(w)} - S(w)\underbrace{\left(S\frac{1}{1-\bar{w}z}, Sg\right)}_{\left(\frac{1}{1-\bar{w}z}, g\right) = g(w)}$$

---

point $\overline{S(w)}\dfrac{1}{1-\bar{w}z}S(z)$ obviously the pt. eval. for SH^+, since S is unitary.

Consider then ~~[illegible]~~ $Y =$ ~~[illegible]~~ $H^+ \cap SH^-$, define C = compression of ~~[illegible]~~ mult. by z. There's a partial unitary with $X = \{y \mid zy \in Y\} = \{\xi \in H^+ \mid \xi, z\xi \perp SH^+\}$

~~[illegible]~~ The issue is that the embedding does what? ~~Embeds X into~~

 Do everything in terms of S. Have contraction c on $Y = H^+ \cap SH^-$ and assoc. p.u. X. $V^\pm$?

$$\frac{1}{1-\bar{\omega} z} \in Y \iff S(\omega) = 0.$$

$1 \in Y \iff S(0) = 0$, so $S(z) = z S_1(z)$

this we recognize

$$\begin{array}{ccccc} & H^+ & \overset{aX}{\supset} & S_1 H^+ & \\ V^- & \cup & & \cup & V^+ = \mathbb{C} S_1 \\ & zH^+ & \underset{bX}{\supset} & SH^+ & \end{array} \qquad S_1 V^+ = V^-$$

So how do you handle $S(0) = h \neq 0$.

Somehow $V^\pm$ ~~come~~ arise out of 1 and $z^{-1}S$

$$1 = \underbrace{1 - \bar{h} S(z)}_{\in Y} + \underbrace{\bar{h} S(z)}_{\in SH^+}$$

$$z^{-1}S = y + Sg \qquad \frac{S(z)-h}{z} \in Y \ ?$$

$$z^{-1}S = z^{-1}h + \frac{S(z)-h}{z} \qquad \frac{1 - hS(z)^{-1}}{z} \in H^-$$

$S(z)^{-1} = \dfrac{g(z)}{p(z)}$ has zeroes outside S^1, poles inside S^1

seems OK.

$$\overline{\left(\frac{1 - hS(z)^{-1}}{z}\right)} = \frac{1 - \bar{h}\, \overline{S(z)}^{\,-1}}{\bar{z}} = \frac{1 - \bar{h} S(z)}{z^{-1}}$$

$$= z\left(1 - \bar{h} S(z)\right) \quad \text{for } |z| = 1.$$

$z^{-1}S$ not in H^+ because $S(0) = h$

$$z^{-1}S = hz^{-1} + \frac{S(z)-h}{z} \quad \text{analytic}$$

is it $\perp$ to SH^+

283 Problem. Given S, form $Y = H^+ \cap SH^- \simeq H^+/SH^+$

c contraction induced by z mult. Assoc. to c is a partial unitary domain $aX = \{y \in Y \mid zy \in SH^-\}$.

$bX = \{y \in Y \mid z^{-1}y \in H^+\} = Y \cap zH^+ = \underline{zH^+ \cap SH^-}$

$$V^- = Y \ominus bX = H^+ \cap SH^- \ominus zH^+ \cap SH^-. \qquad h = \overline{S(0)}$$

$$1 = \frac{1}{1-0z} = 1 - \overline{S(0)}\,S(z) + \overline{S(0)}\,S(z)$$

$$1 = \underbrace{\boxed{1-\bar{h}S}}_{Y} + \underbrace{\bar{h}S}_{\in SH^+}$$

$1 - \bar{h}S(z)$ is pt eval at 0.

$$\therefore \Big(1-\bar{h}S(z),\ \overbrace{zH^+ \cap SH^-}^{Y}\Big) = 0$$

$$\therefore \quad 1 - \bar{h}S(z) \in V^-$$

$$\|1-\bar{h}S(z)\|^2 = 1 - |h|^2$$

$aX = \{y \mid \underbrace{zy \in SH^-}_{Y}\}$ $\qquad V^+ = Y \ominus aX$

$y \in H^+ \cap z^{-1}SH^-$ $\qquad zy \in zH^+ \cap SH^- = bX$

$z^{-1}(S-h) \in H^+$

$z^{-1}(S-h) \overset{?}{\in} z^{-1}SH^-$

$S - h \in SH^-$?

$aX = H^+ \cap z^{-1}SH^-$

Idea should be that

$zv^+ \perp \overset{}{\underset{b}{z a}}X = z(H^+ \cap z^{-1}SH^-)$

$zv^+ \perp zH^+ \cap SH^-$

284 ~~to find $v^+ \in (H^+ \cap SH^-)$~~ $H^+ \cap SH^-$

$aX = H^+ \cap z^{-1}SH^-$ $\quad v^+ \in Y \ominus aX$

$v^+ = H^+ \cap SH^- \ominus H^+ \cap z^{-1}SH^-$

Somehow I feel that I ought to project $z^{-1}S$ into Y. Now $z^{-1}S$ is not ~~analytic~~ in H^+ when $h \neq 0$

$$z^{-1}S = \underbrace{z^{-1}(S-h)}_{\in H^+} + z^{-1}h$$

does $z^{-1}(S-h) \in SH^-$?

i.e. is

$$S-h \in S(\mathbb{C}+H^-)$$
$$1-hS^{-1} \in \mathbb{C}+H^-$$
$$1-h\bar{S} \in \mathbb{C}+H^-$$
$$1-\bar{h}S \in H^+ \qquad \text{OKAY.}$$

$z^{-1}S \in SH^- \qquad z^{-1}h \in SH^- = (SH^+)^\perp$

$$(\underbrace{z^{-1}h}_{H^-}, \underbrace{Sg}_{SH^+ \subset H^+}) = (h, \underbrace{zSg})$$

Review. $Y = H^+ \cap SH^- \quad aX = \{y \mid zy \in Y\} =$

$= H^+ \cap SH^- \cap z^{-1}SH^- = H^+ \cap z^{-1}SH^-$

$bX = zaX = zH^+ \cap SH^-. \quad V^- = Y \ominus bX =$

$H^+ \cap SH^- \ominus zH^+ \cap SH^-$ Let $h = S(0)$. Then

$\left.\dfrac{1-\overline{S(w)}S(z)}{1-\bar{w}z}\right|_{w=0} \in$ $\qquad 1-\bar{h}S(z) \in H^+$

$(1-\bar{h}S(z), Sg)$

$(Sg)(0) - hg(0) = 0.$

285 $\quad V^+ = Y \ominus aX = (H^+ \cap SH^-) \ominus (H^+ \cap z^{-1}SH^-)$

Point maybe is that $aX = H^+ \cap z^{-1}SH^- \quad \perp z^{-1}SH^+$

$z^{-1}S$ is $\perp$ to aX so if you project it onto Y ?

$z^{-1}S$

$\mathcal{H} = aX \oplus V^+ \oplus Y$

$z^{-1}(S-h) \in H^+$

$z^{-1}(S-h) \overset{?}{\in} SH^-$

$S-h \in S(\mathbb{C}+H^-)$

$h \overset{?}{\in} S(\underbrace{\mathbb{C}+H^-}_{(zH^+)^\perp}) =$

$SzH^+ = zSH^+ \subset zH^+$

$S(zH^+)^\perp \supset (zH^+)^\perp \ni h$

Try again. $z^{-1}(S-h) \overset{?}{\in} SH^- = (SH^+)^\perp$

$$\left(z^{-1}(S-h), Sg\right) = (\underset{H^-}{z^{-1}}, \underset{H^+}{g}) - (\underset{\in H^-}{hz^{-1}}, \underset{\in H^+}{Sg}) = 0.$$

Loop group stuff. You need the Szego thm. + Segal ~~stuff~~.
This should concern l^2 sequences $h_1, h_2, \dots$

Problem: Given S inner you can construct $Y = H^+ \cap SH^-$ and c. You want to reverse the process. Given (Y, c)

$1 - cc^*$ [illegible] $v_- v_-^*$, $\quad 1 - c^*c = v_+ v_+^*$. Get isom. emb

$y \longmapsto \tilde{y}(z) = v_-^*(1 - zc^*)^{-1} y \qquad Y \hookrightarrow H^+$

Do this in the case $Y = H^+ \cap SH^-$ where c is induced by z multiplication. What can you do?

Reduce to p.u. $\quad Y = \underbrace{aX}_{\mathrm{Ker}(1-c^*c)} \oplus \underset{\mathbb{C}v_+^*}{\overset{\shortparallel}{V^+}} = \underset{\mathbb{C}v_-}{\overset{\shortparallel}{V^-}} \oplus \underbrace{bX}_{\mathrm{Ker}(1-cc^*)}$

$c = ba^* + v_- \alpha v_+^* \qquad cc^* = bb^* \oplus v_- |\alpha|^2 \|v_+\|^2 v_-^*$

$c^* = ab^* + v_+ \bar{\alpha} v_-^* \qquad c^*c = \underset{1_{aX}}{aa^*} \oplus v_+ |\alpha|^2 \|v_-\|^2 v_+^*$

286 You think that $\|v_-\|^2 = 1-|h|^2 = \|v_+\|^2$

$1-cc^* = v_-v_-^*$ $\quad c^*v_- = hv_+ \quad cv_+ = hv_-$

loop group theory. $L^2(S^1) = H$ form grassmannian.

ψ_n destroys z^n
ψ_n^* creates.

Work on Y, c. Given S form $Y = H^+ \cap SH^-$

$aX = H^+ \cap z^{-1}SH^-$ $\quad bX = zH^+ \cap SH^-$ $\quad h = S(0)$

V^+ spanned by $z^{-1}(S-h)$ norm $\sqrt{1-|h|^2}$ $\quad V^-$ spanned by $1-\bar{h}S$ norm $\sqrt{1-|h|^2}$

Scattering assoc. to the p.u. looks like $S_1 = \dfrac{z^{-1}(S-h)}{1-\bar{h}S}$

If true then $S = \begin{pmatrix} 1 & h \\ \bar{h} & 1 \end{pmatrix}(zS_1)$ as desired.

Given a Y, c you would like an eigenvector equation which is tautological in the case $Y = H^+ \cap SH^-$. When $h = 0$, i.e. $S = zS_1$, $1 \perp S$

Form $V = \begin{pmatrix} 1 \\ c \end{pmatrix} Y = \begin{pmatrix} a \\ b \end{pmatrix} X \oplus \begin{pmatrix} 1 \\ h \end{pmatrix} V^+ \subset \begin{matrix} Y \\ \oplus \\ Y \end{matrix} \supset \begin{pmatrix} 1 \\ z \end{pmatrix} Y$

$$z(ax + v^+) - (bx + hv^+) = y$$

$$(za-b)x + (z-h)v^+ = y$$

$$(z - a^*b)x - a^*hv^+ = a^*y$$

$$(z-a^*b)x = a^*(y + hv^+)$$

$$x = a^*(z-ba^*)^{-1}(y+hv^+)$$

$$(za-b)a^*(z-ba^*)^{-1}(y+hv^+) -$$

287 $$\left((y+hv^+) - (za-b)a^*(z-ba^*)^{-1}(y+hv^+)\right) = zv^+$$

$$\Big(\underbrace{z-\cancel{ba^*}-(za-\cancel{b})\cancel{a^*}}_{z(1-aa^*)}\Big)(z-ba^*)^{-1}(y+hv^+) = zv^+$$

$$v^+ = (1-aa^*)(z-ba^*)^{-1}(y+hv^+)$$

$$(za-b)x + zv^+ = hv^+ + y$$

$$-(1-zb^*a)x + zb^*v^+ = b^*y$$

$$(1-zb^*a)x = b^*(y - zv^+)$$

$$x = b^*(1-zab^*)^{-1}(y - zv^+)$$

$$(za-b)x = (zab^*-bb^*)(1-zab^*)^{-1}(y-zv^+)$$

$$(1-\tilde{S}h)v^+ = \tilde{S}y$$

$$(S-h)v^+ = y \;?$$

Go back to meaning

$$\begin{tikzcd} Y \arrow[r, "\binom{1}{c}"] \arrow[rr, bend right, "z-c"'] & \Gamma_c \arrow[r, "(z\ \ -1)"] & Y. \end{tikzcd}$$

$$W \subset \Gamma_c \subset W^\circ$$

$$\binom{a}{b}X \subset \Gamma_c \subset \binom{a}{b}X \oplus \begin{matrix} V^+ \\ \oplus \\ V^- \end{matrix}$$

$$Y \xleftarrow[\sim]{(z\ \ -1)} \binom{1}{c}Y = \binom{a}{b}X \oplus \binom{1}{h}V^+$$

$$y \longmapsto \binom{1}{c}(z-c)^{-1}y \longmapsto \begin{pmatrix}(1-aa^*)\\(1-bb^*)c\end{pmatrix}(z-c)^{-1}y$$

$$(1-bb^*)(ba^*) = 0$$

$$\binom{1}{h}(1-aa^*)(z-c)^{-1}y$$

You are stupid since you did this calculation before. Start with (Y,c) form the assoc. p.u.

$$Y = \underbrace{aX}_{\mathrm{Ker}(1-c^*c)} \oplus V^+ = \underbrace{V^- \oplus bX}_{\mathrm{Ker}(1-cc^*)}$$

want $v_\pm \in V^\pm$ $\ni$ $1-c^*c = v_+v_+^*$

$1-cc^* = v_-v_-^*$

$$c = ba^* + h$$

$$1-c^*c = 1-aa^*-h^*h = v_+v_+^*$$

$$1-cc^* = 1-bb^*-hh^* = v_-v_-^*$$

288 Your notation is not very good.

Basically $h \in \mathcal{L}(V^+, V^-) = \pi^+ \mathcal{L}(V) \pi^-$
$$= (1-aa^*)\mathcal{L}(V)(1-bb^*)$$

I think you want unit vectors. How?

let $v_\pm \in V^\pm$ be unit vectors. Then

$$v_- h v_+^* + ba^* = c \qquad c^*c = aa^* + v_+ |h|^2 v_+^*$$
$$v_+ \bar{h} v_-^* + ab^* = c^* \qquad (1-c^*c)^{1/2} = v_+ (1-|h|^2)^{1/2} v_+^*$$
$$(1-cc^*)^{1/2} = v_- (1-|h|^2)^{1/2} v_-^*$$

$$y \longmapsto (1-c^*c)^{1/2} \frac{1}{1-z^{-1}c}\, y = v_+ \sqrt{1-|h|^2}\, v_+^* \frac{1}{1-z^{-1}ba^* - z^{-1}\underbrace{v_- h v_+^*}}\, y$$

$$\sqrt{1-|h|^2}\, \pi_+ \frac{1}{1-z^{-1}c_0 - z^{-1}d} \qquad d = v_- h v_+^*$$

$$\sqrt{1-|h|^2}\, \pi_+ \Big(G_0 + G_0 z^{-1} d G_0 + G_0 (z^{-1} d G_0)^2 + \cdots \Big) y$$

$$z^{-1} h v_- v_+^*$$

$$\sqrt{1-|h|^2} \Big(v_+^* G_0 + v_+^* G_0 z^{-1} \underbrace{d}_{v_- h v_+^*} G_0 +$$

$$1 + (v_+^* G_0 z^{-1} h v_-) v_+^* G_0 + (\qquad)^2 v_+^* G_0$$

$$\sqrt{1-|h|^2} \frac{1}{1 - \underbrace{v_+^* G_0 z^{-1} v_-}_{S z^{-1}} h} v_+^* G_0 y$$

$$\sqrt{1-|h|^2} \frac{1}{1 - S(z^{-1}) z^{-1} h} v_+^* G_0 y$$

289 $\quad c = ba^* \oplus v_- h v_+^* \qquad c^* = ab^* \oplus v_+ h^* v_-^*$

$$cc^* = bb^* \oplus v_- h h^* v_-^* \qquad c^*c = aa^* \oplus v_+ h^* h v_+^*$$

$$(1-cc^*)^{1/2} = 0 \oplus v_-(1-hh^*)^{1/2} v_-^* \qquad (1-c^*c)^{1/2} = 0 \oplus v_+(1-h^*h)^{1/2} v_+^*$$

$$y \longmapsto v_-^*(1-cc^*)^{1/2}(1-zc^*)^{-1} y = (1-hh^*)^{1/2} v_-^* (1 - zab^* - \underbrace{z v_+ h^* v_-^*}_{m})^{-1}$$

$$v_-^*\left(G_0 + G_0 m v_-^* G_0 + G_0 m v_-^* G_0 m v_-^* G_0 + \cdots\right)$$

$$v_-^* G_0 + (v_-^* G_0 m v_-^* G_0 + v_-^* G_0 m \qquad = v_-^* G_0 \left(\frac{1}{1 - m v_-^* G_0}\right)$$

$$= \frac{1}{1 - v_-^* G_0 m} v_-^* G_0$$

$$\frac{1}{1 - v_-^* G_0 z v_+ h^*} = \frac{1}{1 - S(z) z h^*}$$

mapping is

$$y \longmapsto (1-hh^*)^{1/2} \frac{1}{1 - S(z) z h^*} v_-^* \frac{1}{1 - zab^*} y$$

embedding associated to ab^*

This is the embedding but what's the scattering?

$$\underbrace{y \longmapsto v_-^*(1-cc^*)^{1/2} \frac{1}{1-zc^*} y}_{\|} \qquad y \longmapsto v_+^*(1-c^*c)^{1/2} \frac{1}{1-z^{-1}c} y$$

$$\left\langle \frac{1}{1-z^{-1}c}(1-cc^*)^{1/2} v_-, y \right\rangle$$

You are confused

The basic problem seems to be ~~that~~ that your embedding $\quad y \longmapsto v_-^*(1-cc^*)^{1/2} \frac{1}{1-zc^*} y \qquad c = ba^*, \; c^* = ab^*$

does not take v_- to 1.

290 a

$S \rightsquigarrow \left| \begin{array}{l} Y = H^+ \cap SH^- \simeq H^+/SH^+ \\ c \text{ ind. by } z \text{ mult.} \end{array} \right.$

$$\Big(\underbrace{\frac{1-\overline{S(w)}S(z)}{1-\bar{w}z}}_{\in Y}, \eta \Big) = \eta(w) \qquad \forall \eta \in Y$$

$$Y = aX \oplus V^+ = bX \oplus V^-$$

$$aX = H^+ \cap z^{-1}SH^- \xrightarrow[z\cdot]{\sim} zH^+ \cap SH^- = bX$$

$h = S(0)$, $\quad 1 - \bar{h}S \in V^-$ norm $(1-|h|^2)^{1/2}$

$\qquad z^{-1}(S-h) \in V^+$

Problem to understand S for a (Y, c).
Your idea is to dilate

$$\cdots \oplus z^{-2}V^- \oplus z^{-1}V^- \oplus Y \oplus zV^+ \oplus z^2V^+ \oplus \cdots$$

$$\|y_0 + zy_1\|^2 = \|y_0 + cy_1\|^2 + \|(1-c^*c)^{1/2}y_1\|^2$$

$$\boxed{\|(z-c)y_1\|^2 = \|(1-c^*c)^{1/2}y_1\|^2}$$

$$\|z^{-1}y_{-1} + y_0\|^2 = \|y_{-1}\|^2 + (c^*y_{-1}, y_0) + (y_0, c^*y_{-1}) + \|y_0\|^2$$

$$= \|c^*y_{-1} + y_0\|^2 + \|(1-cc^*)^{1/2}y_{-1}\|^2$$

$$\|(z^{-1}-c^*)y_{-1}\|^2 = \|(1-cc^*)^{1/2}y_{-1}\|^2$$

$$\|y_0 + zy_1\|^2 = \|z^{-1}y_0 + y_1\|^2$$

$$= \|y_1 + c^*y_0\|^2 + \|(1-cc^*)^{1/2}y_0\|^2$$

$$\boxed{\|(1-zc^*)y_0\|^2 = \|(1-cc^*)^{1/2}y_0\|^2}$$

291 $\quad S$ inner $\rightsquigarrow$ $Y = H^+ \cap SH^- \xrightarrow{\cong} H^+/SH^+$

c induced by $z\cdot$ on H^+: $\quad c^n = j^* z^n j \qquad n \geq 0$

where $j : Y \hookrightarrow H^+$. [redacted] You know that $1 - cc^*$, $1 - c^*c$ have rank 1.

$Y = \underbrace{aX}_{} \oplus V^+ = bX \oplus V^-$

$aX = H^+ \cap z^{-1} SH^- \underset{z^{-1}}{\overset{z}{\rightleftarrows}} zH^+ \cap SH^- = bX$

$h = S(0)$. $\quad 1 - \bar{h}S \in V^-$ $\quad$ $z^{-1}(S - h) \in V^+$ $\quad$ / norm $\sqrt{1-|h|^2}$ $\quad \langle 1 - \bar{h}S, \eta \rangle = 0 \quad \eta \in Y$

$\left(z^{-1}(S-h), \eta\right) = 0 \qquad \eta \in Y = H^+ \cap SH^-$ $\qquad$ $\eta(0)$ so $\eta \in zH^+$ also SH^- $\therefore \eta \in bX$

$z^{-1}h \in H^-$!

$\left(z^{-1}S, \eta\right) = 0 \qquad \eta \perp SH^+$

$\perp Sz^{-1}$ $\qquad \therefore \eta \perp z^{-1}SH^+$ $\quad \therefore \eta \in aX$

Can you see that $c^n y \to 0 \quad \forall y$?

Start with Y, c and construct dilation

H, u, $j : Y \hookrightarrow H \quad \ni \quad j^* u^n j = c^n. \quad \forall n \geq 0$.

$$\|y_0 + zy_1\|^2 = \|y_0 + cy_1\|^2 + \|(1-c^*c)^{1/2} y_1\|^2$$

$$\|(z - c)y_1\|^2 = \|(1-c^*c)^{1/2} y_1\|^2, \text{ so } \quad (1-c^*c)^{1/2} y \longmapsto (z-c)y_1$$

embeds V^+ into H, Also could use $(1-c^*c)^{1/2} y \mapsto (1 - z^{-1}c) y_1$

$$\left((z-c)y, z^n(z-c)y\right) = \left(zy, z^{n+1}y\right) - \left(zy, z^n cy\right) - \left(cy, z^{n+1}y\right) + \left(cy, z^n cy\right) \qquad n \geq 1$$

$$= (y, c^n y) - (y, c^{n-1}cy) - (cy, c^{n+1}y) + (cy, c^{n+1}y)$$

$$= 0.$$

$$j^*(z-c)y = 0$$

$$\|y_0 + zy_1\|^2 = \|z^{-1}y_0 + y_1\|^2$$

$$= \|c^* y_0 + y_1\|^2 + \|(1-cc^*)^{1/2} y_0\|^2$$

$$\|(1 - zc^*)y_0\|^2 = \|(1-cc^*)^{1/2} y_0\|^2$$

292 As you have been calculating inside the full dilation H, + you find

$$\overline{Y + zY} = Y \oplus \overline{(z-c)Y} = zY \oplus \overline{(1-zc^*)Y}$$

$$H: \cdots \oplus z^{-1}\overline{(1-zc^*)Y} \oplus Y \oplus \overline{(z-c)Y} \oplus z\overline{(z-c)Y} \oplus \cdots$$

$$\overline{(1-zc^*)Y} \oplus zY$$

$$\big((1-zc^*)y_1, (z-c)y_2\big) = (y_1, zy_2) - (zc^*y_1, zy_2)$$

$$- (y_1, cy_2) + (zc^*y_1, cy_2)$$

$$= \cancel{(y_1, cy_2)} - \cancel{(c^*y_1, y_2)} - (y_1, cy_2) + (cc^*y_1, cy_2)$$

$$= \big(y_1, (-c + cc^*c)y_2\big)$$ ~~not much help.~~

The scattering appears to be something like

$(z-c)(1-zc^*)^{-1}$: ?

What have you found **relative** to (Y, c)? The dilation H has a certain structure. Outgoing subspace $Y \oplus \overline{(z-c)Y} \oplus z\overline{(z-c)Y} \oplus \cdots$

incoming subspace $\cdots \oplus z^{-1}\overline{(1-zc^*)Y} \oplus Y$. Scattering op between them.

293 Review a bit more. Given Y, c and you construct the dilation H, u, u^{-1}, $Y \xrightarrow[j]{} H$ $\quad j^* u^n j = c^n$.

$$\overline{Y + zY} = Y \oplus \overline{(z-c)Y} = \overline{(1-zc^*)Y} \oplus zY$$

$$(y_1 - zc^* y_1, z y_0) = (y_1, c y_0) - (c^* y_1, y_0) = 0$$

$$\overline{z^{-1}Y + Y} \supset Y \subset \overline{Y + zY} \subset \overline{Y + zY + z^2 Y} \subset \cdots\cdots$$

$$\overline{(z^{-1} - c^*)Y} \quad \overline{(z-c)Y} \quad z\overline{(z-c)Y} \quad z^2\overline{(z-c)Y}$$

$$\|$$

$$z^{-1}\overline{(1-zc^*)Y}$$

$$\cdots \oplus z^{-1}\overline{(1-zc^*)Y} \oplus \underbrace{Y \oplus \overline{(z-c)Y}}_{\| \atop \overline{(1-zc^*)Y} \oplus zY} \oplus z\overline{(z-c)Y} \oplus \cdots$$

Anyway you can now ~~make the dilation~~ calculate the ~~~~ outgoing repn. of y. You have

$$H \longleftarrow L^2(S^1) \quad ?$$

$$z \longleftarrow\!\!\shortmid\ 1$$

$$~~~~ \quad \overline{(z-c)Y} \;\tilde{\longleftarrow}\; \overline{(1-c^*c)^{1/2}Y} \qquad \text{assume 1 dim.}$$

Let ~~~~ v_+ be a unit vector in $\overline{(1-c^*c)^{1/2}Y} = V^+$

Then $\|(z-c)v_+\| = (zv_+, (z-c)v_+) = (v_+, (1-z^{-1}c)v_+) = (v_+, (1-c^*c)v_+) = 1.$

You will want ~~to send y~~ ~~to find~~ to write

$z^N y$ in terms of the $\underbrace{z^k(z-c)v_+}$ orthonormal

You want $\quad y = \sum c_{-n} z^n (z-c) v_{\cancel{+}}$

$$c_{-n} = (z^n(z-c)v_+, y) = ~~~~$$

294 $c_{-n} = \left((z-c)v_+ , z^n y\right) = \left(v_+ , c^{n-1} y\right) - \left(cv_+ , c^n y\right)$

$$= \left(v_+ , (1-c^*c) c^{n-1} y\right).$$

$$Y \oplus \overline{(z-c)Y} \oplus \cdots$$

---

orth sequence (basis in good case).

$$z^k (z-c) v_+ = z^{-n}(1-z^{-1}c) v_+$$

expand $y = \sum a_n z^{-n}(1-z^{-1}c) v_+$

$$a_n = \left(z^{-n}(1-z^{-1}c) v_+ , y\right) =$$

$$\cancel{= \left(v_+ , (1-zc^*) z^n y\right) = \left(v_+ , z^n\right.}$$

$$= \left(z^{-n} v_+ - z^{-n-1} c v_+ , y\right)$$

$$= \left(v_+ , \overset{z^n}{c} y\right) - \left(cv_+ , \overset{z^{n+1}}{c} y\right)$$

$$= \left(v_+ , (1-c^*c) c^n y\right)$$

$$\therefore \quad y = \sum_{n \geq 0} z^{-n}(1-z^{-1}c) v_+ \left(v_+ , (1-c^*c) c^n y\right)$$

Start again. $\quad (y_1 , z^n y_2)_H = \left(y_1 , \begin{pmatrix} c^n \\ c^{*-n} \end{pmatrix} y_2\right)_Y$

$n \geq 1$ $\quad \left((z-c) y_1 , z^n (z-c) y_2\right) = \left(z y_1 , z^{n+1} y_2\right) - \left(z y_1 , \overset{z^n}{c} c y_2\right)$

$$- \left(c y_1 , \underset{c^{n+1}}{z^{n+1}} y_2\right) + \left(c y_1 , \underset{c^n}{z^n} c y_2\right)$$

$$= \cancel{(y_1 , c^n y_2)} - \cancel{(y_1 , c^n y_2)} \quad 0 \qquad n \geq 1$$

$$(y_1, y_2) - (c y_1 , c y_2) \qquad n = 0$$

295 Try $V^- = \overline{(1-zc^*)Y}$

$$\left((1-zc^*)y_1,\ z^n(1-zc^*)y_2\right) = (y_1, z^n y_2) - (y_1, z^{n+1}c^* y_2) - (c^* y_1, z^{n-1} y_2) + (c^* y_1, z^n c^* y_2)$$

$n \geq 1$

Choose $v_- \in \overline{(1-cc^*)^{1/2} Y}$ $\quad \|v_-\| = 1.$

$z^n(1-zc^*)v_-$ is an orthonormal set

$$y = \sum_{n\in\mathbb{Z}} a_n z^n (1-zc^*) v_-$$

$$a_n = \left(z^n(1-zc^*)v_-, y\right) = \left(v_- - zc^* v_-, z^{-n} y\right)$$

$$= (v_-, c^{*n} y) - \underbrace{(c^* v_-, z^{-n-1} y)}_{(v_-,\ cc^{*(n+1)} y)}$$

$$= \left(v_-, (1-cc^*) c^{*n} y\right) \qquad n \geq 0$$

$n \leq -1.$ $\quad = \underbrace{(v_-, c^{-n} y) - (c^* v_-, c^{-n-1} y)}_{=0} \qquad n \leq -1.$

$$\therefore\ y \longmapsto \sum_{n\geq 0} z^n \overset{\text{unit vector}}{(1-zc^*)v_-} \left(v_-, (1-cc^*) c^{*n} y\right)$$

Convert to a function

$$\hat{y}(\lambda) = \left((1-cc^*)^{1/2} v_-,\ (1-cc^*)^{1/2} \frac{1}{1-\lambda c^*} y\right)$$

$$\|(1-zc^*)v_-\|^2 = \left((z^{-1}-c^*)v_-, (z^{-1}-c^*)v_-\right) = \left(z^{-1}v_-, (z^{-1}-c^*)v_-\right)$$
$$= \left(v_-, (1-zc^*)v_-\right) = \left(v_-, (1-cc^*)v_-\right) = 1$$

296 Let us ~~take~~ try again.

~~Start~~ with S inner, get $Y = H^+ \cap SH^- \cong H^+/SH^+$

~~c induced by z mult.~~, Y Hilbert space of functions – the pt eval. is $\dfrac{1-\overline{S(w)}\,S(z)}{1-\bar{w}z}$,

c = contraction, ~~compression~~ on Y ind. by z mult

$$Y = aX \oplus V^+ = \left(H^+ \cap z^{-1}SH^-\right) \oplus \mathbb{C}\,\frac{z^{-1}(S-h)}{\sqrt{1-|h|^2}}\;v_+ \qquad h = S(0).$$

$$= bX \oplus V^- = \left(zH^+ \cap SH^-\right) \oplus \mathbb{C}\left(\frac{1-\bar{h}S}{\sqrt{1-|h|^2}}\right) v_-$$

$$(v_-, v_+) = \frac{\left(z^{-1}(S-h)\right)(0)}{1-|h|^2} = \frac{S'(0)}{1-|h|^2}$$ one level down from S

$$\frac{v_+}{v_-} = \frac{z^{-1}(S-h)}{1-\bar{h}S} = S_1 \qquad S = \frac{zS_1 + h}{1-\bar{h}zS_1}$$

~~that my difficulties is that I should~~

Your difficulties I think are caused by working in Y, instead of $Z = \overline{Y + zY}$

Start with S you get $Y = H^+ \cap SH^-$ etc and $Z = H^+ \cap zSH^+$. ~~So what can you do?~~ ~~etc. So what?~~ You have to concentrate and finish.

Given (Y, c)

Given (Y, c) for Z = completion of $Y \oplus zY$ with norm

$$\|y_0 + zy_1\|^2 = \|y_0 + cy_1\|^2 + \|(1-c^*c)^{1/2}y_1\|^2$$

$$= \|y_1 + c^*y_0\|^2 + \|(1-cc^*)^{1/2}y_0\|^2$$

$\{y_0\}$ $\overline{(u-c)Y}$ $\qquad \{uy_1\}$ $\overline{(1-uc^*)Y}$

$$V^+ = \overline{(u-c)Y} \qquad Z = aY \oplus V^+ = bY \oplus V^-$$

$$aY \oplus V^+ \quad \left((1-uc^*)y_0,\ (u-c)y_1\right)$$

$$V^- \oplus bY = -\left(uc^*y_0,\ (u-c)y_1\right) = -\left(c^*y_0,\ (1-u^{-1}c)y_1\right)$$

$$= -(y_0, cy_1) + (y_0, cc^*c\,y_1)$$

297 $\left((u^{-1}-c^*)y_1, (u-c)y_0\right)$

$$= \underset{\overset{\|}{(y_1, c^2 y_0)}}{(y_1, u^2 y_0)} - \underbrace{(u^{-1}y_1, cy_0)}_{(y_1, c^2y_0)} - \underset{y_1,\, c^2 y_0}{(c^*y_1, uy_0)} + \underset{y_1,\, c^2y_0}{(c^*y_1, cy_0)}$$

$$\overline{u^{-1}Y+Y} \supset Y \subset \overline{Y+uY} \subset \cdots \qquad V^- \ni (z^{-1}-c^*)\xi$$

$$u^{-1}V^- \oplus V^- \oplus Y \oplus V^+ \oplus uV^+ \oplus \cdots$$

form this dilation, you have an outgoing subspace generated by V^+ and an incoming subspace generated by V^- and between these

$$\left((u^{-1}-c^*)\xi_0, (u-c)\xi_0\right) = \left(\xi_0, (u-c)(u-c)\xi_0\right)$$

Given (Y, c), assume $1-cc^*$ rank 1 $\quad c^{*n}\xi \to 0$ all $\xi \in Y$.

Give a nice (Y,c) you want to find S and ~~$\varepsilon : Y \to H$~~ $\varepsilon : Y \to H^+$ $\ni$ $\varepsilon : Y \xrightarrow{\sim} H^+ \cap SH^-$, Original idea want to have v_- in Y go to $1-\bar{h}S$. Your idea is to choose $v_-^* : Y \to \mathbb{C}$ so that $|(v_-^*, \xi)|^2 = \|\xi\|^2 - \|c^*\xi\|^2$, then use $\xi \mapsto \underbrace{v_-^*(1-zc^*)^{-1}\xi}$ to embed Y in H^+.

$$\Big\| \sum_{n\geq 0} z^n v_-^*(c^*)^n \xi \Big\|^2 = \sum \|c^{*n}\xi\|^2 - \|c^{*n+1}\xi\|^2$$

~~Looks a bit hard~~.

Eigenvector equation?

Form $Z = \underbrace{H^+ \ominus SzH^+}_{H^+ \cap SzH^-}$.

If $Y = H^+ \cap SH^-$, then how can you recover S from Y. Then $1 \in Z$ and $S \in Z$

It seems like you are forming $\overline{Y+zY}$. You know $z^{-1}(S-h) \in Y$ so $Y+zY$ contains $S-h$ and $1-\bar{h}S$

$1-\bar{h}S$ hence $S-h+h(1-\bar{h}S) = S(1-|h|^2) \quad \therefore S, 1.$

I guess what's becoming clear is that you want to go from Y, e to $L^2(S^1)$, and somehow calculate the contraction picture from the S picture. But you also want an eigenvector equation for the scattering.

Return to $P_n \subset L^2(S^1, d\mu)$. Have

$$q_n \in (1 + zP_{n-1}) \cap (zP_{n-1})^\perp \qquad p_n \in (z^n + P_{n-1}) \cap P_{n-1}^\perp$$

$$\overline{P_{n-1}} = \mathbb{C}z^{-n+1} + \cdots + \mathbb{C}z^{-1} + \mathbb{C} \qquad \bar p_n \in (z^{-n} + \overline{P_{n-1}}) \cap \overline{P_{n-1}}^\perp$$

$$z^n \overline{P_{n-1}} = \mathbb{C}z + \cdots + \mathbb{C}z^n = zP_{n-1} \qquad z^n \bar p_n \in (1 + zP_{n-1}) \cap (zP_{n-1})^\perp$$

$$\therefore \; q_n = z^n \bar p_n \qquad p_n = \prod_1^n (z - a_i) \qquad \underbrace{z^n \bar p_n}_{q_n} = \prod_1^n \underbrace{z(z^{-1} - \bar a_i)}_{1 - \bar a_i z}$$

On $P_n = Y$ have partial unitary

$$X = P_{n-1} \underset{b=z}{\overset{a}{\subset}} P_n = Y \qquad P_n = Y = aX \oplus V^+ = P_{n-1} \oplus \mathbb{C}p_n$$
$$= bX \oplus V^- = zP_{n-1} \oplus \mathbb{C}q_n$$

This gives an isometric embedding $Y \hookrightarrow H^+$ such that

$v_- = \frac{q_n}{\|q_n\|} \mapsto 1$, $v_+ = \frac{p_n}{\|p_n\|} \mapsto S$. Solve $(a\lambda - b)x = -\hat y(\lambda) v_- + y$

$$(\lambda - z)x(z) = -\hat y(\lambda) v_-(z) + y(z) \implies \hat y(\lambda) = \frac{y(\lambda)}{v_-(\lambda)} \Rightarrow \hat y = \frac{y}{q_n}\|q_n\|$$

$\hat P_{n-1}$

and $S = \hat v_+ = \frac{p_n}{q_n}$.

$$Y = P_n \xrightarrow{\sim} H^+ \cap SzH^- \xrightarrow{\cdot q_n} q_n H^+ \cap p_n z H^-$$
$$f \longmapsto \frac{f}{q_n}\|q_n\| \qquad \| $$
$$H^+ \cap z^{n+1}H^- = P_n.$$

You really find that P_n is the space of polys of degree $\le n$ equipped with $\|f\|^2_{q_n} = \int |f|^2 \frac{d\theta}{|q_n|^2 2\pi} \|q_n\|^2$.

Problem: Can you show directly that q_n and q_{n+1} lead to the same inner product on P_n? You should probably formulate in Hardy space. Take $Y = H^+ \cap S_{n+1} H^-$ where $S_{n+1} = \frac{p_{n+1}}{q_{n+1}}$. Recall two steps $S_n \mapsto zS_n \longmapsto \frac{zS_n + h_{n+1}}{1 + \bar h_{n+1} z S_n}$

299 You want to compare S with $\frac{S-h}{1-\bar{h}S} = zS'$

On one hand you have $Y = H^+ \cap SH^-$

$aX = H^+ \cap z^{-1}SH^-$, $\quad bX = zH^+ \cap SH^-$

$V^- = (1-\bar{h}S)\mathbb{C}$, $\quad V^+ = z^{-1}(S-h)\mathbb{C}$ $\qquad S'V^- = V^+$

I ran into difficulty trying to obtain S directly for an (X, c). Instead you form the partial unitary $X \overset{a}{\underset{b}{\rightrightarrows}} Y$ and take its scattering operator

$$\begin{aligned}\|ax_1 + bx_2\|^2 &= \|x_1 + cx_2\|^2 + \|(1-c^*c)^{1/2}x_2\|^2 \\ &= \|c^*x_1 + x_2\|^2 + \|(1-cc^*)^{1/2}x_1\|^2.\end{aligned}$$

$x_1 + zx_2$

$c = a^*b$

$$a^{-1}V^- \oplus \underbrace{aX \oplus V^+}_{\|} \oplus$$
$$\oplus \overbrace{V^- \oplus bX}^{} \oplus uV^+$$
$$\| \atop Y$$

$\pi_+ = (1-aa^*)$

$$\xi = aa^*\xi + \pi_+\xi$$
$$u\xi = ba^*\xi + u(\pi_+\xi)$$
$$= aa^*ba^*\xi + \pi_+ ba^*\xi + u\pi_+\xi$$
$$u^2\xi = aa^*(ba^*)^2\xi + \pi_+(ba^*)^2\xi + u\pi_+ba^*\xi + u^2\pi_+\xi$$

$u = ba^{-1}$
$u^{-1} = ab^{-1}$

$$\xi = \pi_+\xi + u^{-1}\pi_+ ba^*\xi + u^{-2}\pi_+(ba^*)^2\xi + \cdots$$
$$\mapsto \pi_+(1 - z^{-1}ba^*)^{-1}\xi$$

$$\xi = bb^*\xi + \pi_-\xi$$
$$u^{-1}\xi = ab^*\xi + u^{-1}\pi_-\xi$$
$$= bb^*ab^*\xi + \pi_- ab^*\xi + u^{-1}\pi_-\xi$$
$$u^{-2}\xi = bb^*(ab^*)^2\xi + \pi_-(ab^*)^2\xi + u^{-1}\pi_- ab^*\xi + u^{-2}\pi_-\xi$$
$$\xi = \pi_-\xi + u\pi_- ab^*\xi + u^2\pi_-(ab^*)^2\xi + \cdots$$
$$\mapsto \pi_-(1 - zab^*)^{-1}\xi$$

300 You need to find a way to think about this. You could start with (X, c) and dilate.

$$\oplus u^{-1}V^- \oplus V^- \oplus X \oplus V^+ \oplus uV^+ \oplus \cdots$$

$$u(x) = cx + (u-c)x$$
$$u^{-1}(x) = c^*x + (u^{-1}-c^*)x$$

---

what you can do.

S given put $Y = H^+ \cap SH^- \simeq H^+/SH^+$ c ind. by z.)

mult. pt. eval $\frac{1-\overline{S(w)}\,S(z)}{1-\bar{w}z}$

$aX = \mathrm{Ker}(1-c^*c) = \{\xi \in Y \mid z\xi \in Y\} = H^+ \cap z^{-1}SH^-$

$bX = zH^+ \cap SH^-$. $h = S(0)$

$V^- = \mathbb{C}(1-\bar{h}S)$, $V^+ = z^{-1}(S-h)\mathbb{C}$, $S_1 = \frac{z^{-1}(S-h)}{1-\bar{h}S}$

Maybe you should ~~formulate things~~ put $X = H^+ \cap SH^-$
$Y = H^+ \cap zSH^-$ ~~formulate things what?~~

Given X, c, ~~[illegible]~~ introduce $\overline{(1-cc^*)^{1/2}X} = V^-$

$$X = \mathrm{Ker}(1-cc^*) \oplus \overline{(1-cc^*)^{1/2}X}$$

V^- = completion of X in the norm $\|\xi\|^2 - \|c^*\xi\|^2$

$\pi_-: X \to V^-$ canon. map $\pi_-^*\pi_- = 1-cc^*$.

cbin $(1-cc^*)^{1/2}: X \longrightarrow \overline{(1-cc^*)^{1/2}X} \subset X$.

$$X \longrightarrow H^2(S^1, V^-)$$

$$\xi \longmapsto \hat{\xi}(z) = \pi_-\left(\frac{1}{1-zc^*}\xi\right) = \sum_{n\geq 0} z^n\, \pi_-(c^{*n}\xi)$$

$$\|\hat{\xi}\|^2 = \sum_{n\geq 0} \|c^{*n}\xi\|^2 - \|c^{*n+1}\xi\|^2 = \|\xi\|^2 - \lim_n \|c^{*n}\xi\|^2.$$

301 ~~You want to consider~~

Assuming $c^{*n}\xi \to 0$ $\forall \xi$, you get a closed subspace of H^+. You get $X \rightleftarrows H^+$

$j^* u^n j = c^n$ $\qquad$ $j^*\eta = 0 \overset{?}{\Longrightarrow} j^* z \eta = 0$

$$j^* z\eta = j^* z\big(jj^*\eta + (1-jj^*)\eta\big)$$

$$X + V^+$$

---

dilation of X, c.

$$\cdots \oplus V^- \oplus X \oplus V^+ \oplus \cdots$$

$\pi_- : X \to V^-$ completion $\|x\|^2 - \|c^*x\|^2$

$\pi_+ : X \to V^+$ $\|x\|^2 - \|cx\|^2$

If $H, u, \varepsilon : X \to H$ is $\ni$ $\varepsilon^* u^n \varepsilon = c^n$ $n \geq 0$, then H is the above orthog. direct sum, where $V^+ = (u-c)X$

$V^- = \overline{(u^{-1} - c^*)X}$. Wave operators?

$\pi_+ x = ux - cx$

$$L^2(S^1, V^-) \hookrightarrow H \hookleftarrow L^2(S^1, V^+)$$

You have $V^+ \subset H$, such that $V^+ \perp u^n V$ $\quad n > 0$ $\quad n \neq 0$

Try for a good picture of H. completion of

$$\mathbb{C}[u, u^{-1}] \otimes X = \bigoplus_{n \in \mathbb{Z}} u^n X$$

$$(x_1, u^n x_2)_H = \left(x_1 \begin{pmatrix} c^n \\ c^{*-n} \end{pmatrix} x_2\right)$$

Then prove above orthogonal sum picture of X. where V^+ is the completion. It seems you need to specify V^+ better. Define V^+ by UMP.

V^+ eq. w. $\pi_+ : X \to V^+$ $\qquad$ Comp. wrt $\|\pi_+ x\|^2 = \|x\|^2 - \|cx\|^2$

$$\pi_+^* \pi_+ = 1 - c^*c.$$

$$(\xi_1, \pi_+^* \pi_+ \xi) = (\xi_1, (1-c^*c)\xi)$$

$$(\pi_+ \xi, \pi_+ \pi_+^* \pi_+ \xi) = ((1-c^*c)\xi, (1-c^*c)\xi) \quad ?$$

302 Define $\quad V^+ \longrightarrow H$ $\qquad (u^{-1}-c^*)(u-c)$

$\pi_+ \xi \longmapsto u\xi - c\xi$ $\qquad = 1 - \underbrace{u^{-1}c}_{c^*c\,-\,c^*c} - c^*u + c^*c = 1 - c^*c.$

$$\left(V^+, u^n V^+\right) \ni \left(u\xi_1 - c\xi_1,\ u^n(u\xi_2 - c\xi_2)\right) = 0.$$

~~[illegible]~~. Adjoint of $\quad L^2(S^1, V^+) \xrightarrow{\ \varepsilon_+^{\oplus}\ } H$

$$z^n \pi_+ \xi \longmapsto u^n(u-c)\xi$$

~~[illegible]~~ Picture. Start with (X, c) construct dilation

$$H := H^2_-(S^1, V^-) \oplus X \oplus H^2(S^1, V^+)$$

and map $\quad H \xrightarrow[j_+^*]{} [redacted]\ L^2(S^1, V^+)$ $\qquad$ this is a projection

identity on $H^2(S^1, V^+)$. This maps restricted $\quad$ i.e. $\quad j_+^* j_+ = 1.$

to X sends ξ to $\pi_+\left(\frac{1}{1-z^{-1}c}\xi\right)$ probably.

$$\left(\xi, z^{-n}(z-c)\xi_1\right) \qquad n > 0$$

$$\cancel{[illegible]}$$

$$= \left(z^n\xi, (z-c)\xi_1\right)$$

$$\cancel{=\left(z^{n-1}\xi, (1-z^{-1}c)\xi_1\right)}$$

$$= \left(z^n\xi, z\xi_1\right) - \left(z^n\xi, c\xi_1\right)$$

$$= \left(c^{n-1}\xi, \xi_1\right) - \left(c^n\xi, c\xi_1\right)$$

$$= \left(c^{n-1}\xi, \xi_1 - c^*c\xi_1\right) = \left(\pi_+ c^{n-1}\xi, \pi_+\xi_1\right)$$

$$\qquad\qquad\qquad\qquad\qquad \|$$

so the projection should be $\qquad \left(z^n \pi_+ c^{n-1}\xi, z^{-n}\pi\right.$

$$L^2(S^+, V^+) \xhookrightarrow[j_+]{} H$$

$$\sum_n z^n \pi_+ \xi_n \longmapsto \sum_n u^n (u-c)\xi_n$$

$$\Big(\xi, \sum_n u^n(u-c)\xi_n\Big) = \Big(j_+^*\xi, \sum_n z^n \pi_+ \xi_n\Big)$$

$$\|$$

$$\sum_{n<0}\big(u^{-n}\xi, (u-c)\xi_n\big) = \sum_{n<0}\big(u^{-n-1}\xi, \xi_n\big) - \big(u^{-n}\xi, c\xi_n\big)$$

$$= \sum_{n<0}\big(c^{-n-1}\xi, \xi_n\big) - \big(c^{-n}\xi, c\xi_n\big)$$

$$= \sum_{n<0}\big(c^{-n-1}\xi, (1-c^*c)\xi_n\big)$$

$$= \sum_{n<0}\big((1-c^*c)^{1/2}c^{-n-1}\xi, (1-c^*c)^{1/2}\xi_n\big)$$

$$= \sum_{n<0}\big(\pi_+(c^{-n-1}\xi), \pi_+(\xi_n)\big)$$

$$= \Big(\sum_{n<0} z^n \pi_+(c^{-n-1}\xi), \sum_{n\in\mathbb{Z}} z^n \pi_+(\xi_n)\Big) \qquad n = -1-k$$

$$\therefore\ j_+^*(\xi) = \sum_{n<0} z^n \pi_+(c^{-n-1}\xi) = \sum_{k\geq 0} z^{-1-k}\pi_+(c^k\xi)$$

$$\boxed{j_+^*(\xi) = \pi_+\Big(\frac{z^{-1}}{1-z^{-1}c}\xi\Big)} = \pi_+\Big(\frac{1}{z-c}\xi\Big)$$

conclusion: If $c^n \xi \to 0$ $\forall \xi$, then j_+^* embeds X into $H_-^2(S^1, V^+)$, and $j_+^* X \oplus H_+^2(S^1, V^+)$ is outgoing, i.e. closed under z.

other side

$$H = \cdots \oplus V^- \oplus X \oplus \underset{(z-c)\xi}{V^+} \oplus \cdots$$

now you have $L^2(S^1, V^-) \xrightarrow[j_-]{} H$ and you want to calculate the proj j_-^* rest. to X. Elements of $V^- = (z^{-1} - c^*)X$

$$\pi_- : X \to V_-$$
$$\pi_- \xi = (u^{-1} - c^*)\xi$$

$$\|\pi_- \xi\|^2 = \|u^{-1}\xi - c^*\xi\|^2 =$$

$$((u^{-1} - c^*)\xi, (u^{-1} - c^*)\xi) = (\xi, \underbrace{(u-c)(u^{-1} - c^*)}_{1 - cu^{-1} - uc^* - cc^*}\xi)$$

cu^{-1} not def. $\quad uc^* \to c$

$\|$

$$(\xi - uc^*\xi, \xi - uc^*\xi) = \|\xi\|^2 - (\xi, cc^*\xi) - (cc^*\xi, \xi) + (c^*\xi, c^*\xi)$$
$$= (\xi, (1 - cc^*)\xi).$$

$$\left(j_-^*(\xi), \sum_n z^n \pi_- \xi_n\right) = \left(\xi, \sum_n u^n (u^{-1} - c^*)\xi_n\right)$$

$$= \cancel{\sum_{n \le 0} \left(\xi, ((c^*)^{n+1} - (c^*)^n c^*)\xi\right)} + \sum_{n \ge 1} \left(\xi, (c^{n-1} - c^n c^*)\xi_n\right)$$

$$= \sum_{n \ge 1} \left((c^*)^{n-1}\xi, (1 - cc^*)\xi_n\right) = \sum_{n \ge 1} \left(\pi_-(c^*)^{n-1}\xi, \pi_- \xi_n\right)$$

$$= \left(\sum_{n \ge 1} z^n \pi_- c^{*n-1}\xi, \sum_{n \in \mathbb{Z}} z^n \pi_- \xi_n\right) \qquad \boxed{j_-^*(\xi) = \pi_-\left(\frac{z}{1 - zc^*}\xi\right)}$$

305 Focus on something more conceptual. Namely families. [redacted] Consider the family of pos. divisors of degree n in D. Wait before you get this far you might analyze what happens to the embedding when the partial unitary is fixed. You should get a holom. family of divisors depending on h, $|h|<1$, in $\mathcal{L}(V^+, V^-)$. $\det(z-c)$ $\quad c = ba^* + h$.

$$Y = aX \oplus V^+ = bX \oplus V^-$$

$$c = ba^* \oplus \delta c \qquad\qquad e_- \delta c\, e_+ = \delta c$$

$$\frac{1}{z-c} = \frac{1}{z-c_0} + \frac{1}{z-c_0}\,\delta c\,\frac{1}{z-c_0} + \cdots$$

You are looking at the formula $\pi_+ \frac{1}{z-c}\xi$ for the rational function in H^+ corresponding to $\xi \in X$.

Simplest example. Suppose $\dim X = 1$ then c is a number h. Formula things nicely.

~~[illegible]~~ $X = \mathbb{C}$ $\quad c = h \quad |h|<1$. Dilate

$L^2(S^1, d\mu)$.

$$(z^k, z^l) = (1, z^{l-k}) = \begin{cases} h^{l-k} & l-k \geq 0 \\ \bar{h}^{k-l} & l-k \leq 0. \end{cases}$$

$$d\mu = \rho \frac{d\theta}{2\pi} \qquad\qquad \int z^n \rho \frac{d\theta}{2\pi} = \begin{cases} h^n & n \geq 0 \\ \bar{h}^{-n} & n \leq 0. \end{cases}$$

$$\rho = \sum_{n \geq 0} z^{-n} h^n + \sum_{n<0} z^{-n} \bar{h}^{-n}$$

$$\sum_{n>0} z^n \bar{h}^n$$

$$= \frac{1}{1-z^{-1}h} + \frac{z\bar{h}}{1-z\bar{h}} = =$$

$$\frac{1-z\bar{h}+(1-z^{-1}h)z\bar{h}}{(1-z^{-1}h)(1-z\bar{h})} = \frac{1-|h|^2}{(1-z^{-1}h)(1-z\bar{h})} = \frac{1-|h|^2}{|1-z\bar{h}|^2}$$

Dilation is therefore $L^2\left(S^1, \frac{1-|h|^2}{|1-z\bar{h}|^2}\frac{d\theta}{2\pi}\right) \supset X = \mathbb{C}1$

What is the basic ~~statement~~?? What would you like to do? There should be an equivalence between $\underline{(X,c)}$ almost unitary and outgoing subspaces.

Wait. As the divisor $\prod^n (z-a_i)$ varies, ~~you~~ over the space of divisors (degree n in D), you get a holom hermitian vector bundle $p \mapsto H^+/pH^+$. Can ask about curvature. Examine $n=1$. Over D you get a holom hermitian line bundle. Take a section, i.e. $\mathbf{1}$.

$\mathbf{1} = \frac{p}{q}H^+$ $\qquad\qquad$ $1 = (?) + (z-h)H^+$

$$1 \perp \overline{S(0)}S = 1+\bar{h}S \qquad S = \frac{z-h}{1-\bar{h}z}$$

$$= 1+\bar{h}\frac{z-h}{1-\bar{h}z} = \frac{1-\bar{h}z+\bar{h}z-|h|^2}{1-\bar{h}z}$$

$\text{norm}^2 = (1-|h|^2)$

curvature is $\quad d''d'\log\|s\|^2 = d''d'\log(1-|h|^2)$

$$= -d''\left(\frac{\bar{h}\,dh}{1-|h|^2}\right) = \frac{d\bar{h}\,dh}{1-|h|^2} - \frac{(-h\,d\bar{h})\bar{h}\,dh}{(1-|h|^2)^2}$$

$$= \frac{(1-|h|^2)\,d\bar{h}\,dh + |h|^2 d\bar{h}\,dh}{(1-|h|^2)^2} = \frac{d\bar{h}\,dh}{(1-|h|^2)^2}$$

this probably is the non-Euclidean ~~area its~~ area.

Outgoing subspaces form a complex manifold Blashke products should form an infinite diml manifold. Try for a picture. <u>Subquotients of a polarized Hilbert space should also form a complex manifold.</u>

~~Maybe I should try to decide~~

Outgoing subspaces form a complex manifold. need program to exploit this.

307 How did you arrive at the curvature above? Why consider it? Answer involves determinants. You have $p \mapsto H^+/pH^+$ a map from divisors to almost unitary contractions (X,c). You want to invert this map. If X is fin. dim. you have $p = \det(z-c)$.

Fix the partial unitary part of c allow rest to vary. You really need to clean up.

$-\delta \log \det(z-c) = \operatorname{tr} \frac{1}{z-c}(+\delta c)$ variational δc infinitesimal

$c = c_0 + v_- h v_+^*$

$\delta c = v_- \delta h v_+^*$

$$\delta \frac{1}{z-c} = \frac{1}{z-c}\,\delta c\,\frac{1}{z-c}$$

$$\frac{1}{z-c} = \frac{1}{z-c_0-v_-hv_+^*} = \frac{1}{1-\frac{1}{z-c_0}v_-hv_+^*}\,\frac{1}{z-c_0}$$

$$-\delta \log\det(z-c) = \operatorname{tr}\left(\frac{1}{z-c}v_-\delta h v_+^*\right) = \operatorname{tr}\left(v_+^*\frac{1}{z-c}v_-\delta h\right)$$

$$v_+^*\frac{1}{z-c_0}v_- = S_0^{-1}$$

$$v_+^*\frac{1}{z-c} = v_+^*\frac{1}{1-\frac{1}{z-c_0}v_-hv_+^*}\,\frac{1}{z-c_0} = \frac{1}{1-v_+^*\frac{1}{z-c_0}v_-h}\,v_+^*\frac{1}{z-c_0}$$

$$v_+^*\frac{1}{z-c}v_- = \frac{1}{1-S_0^{-1}h}S_0^{-1}$$

$$-\frac{\delta}{\delta h}\log\det(z-c) = \frac{1}{1-S_0^{-1}h}S_0^{-1} = \frac{1}{S_0-h}$$

$$\frac{\delta}{\delta h}\log(S_0-h) = \frac{1}{S_0-h}(-1)$$

$$\cancel{\det(z-c) = \det(z-c_0)}$$

$$\det(z-c)^{-1}(S_0-h) = \text{constant}. \qquad \text{set } h=0$$
$$= \det(z-c_0)^{-1}S_0$$
$$\frac{\det(z-c)}{\det(z-c_0)} = 1-hS_0^{-1}$$

better might by $\dfrac{\det(z-c)}{S_0-h} = \dfrac{\det(z-c_0)}{S_0}$

$$\boxed{\det(z-c) = \left(1-\frac{h}{S_0}\right)\det(z-c_0)}$$

Put $\det(z-\cancel{c_0}) = \cancel{\,}zp_{n-1} \qquad S_0 = \dfrac{zp_{n-1}}{q_{n-1}}$

$$\det(z-c) = p_n = \cancel{z}p_{n-1}+hq_{n-1}$$

$$\frac{\det(z-c)}{\det(z-c_0)} = \frac{zp_{n-1}+hq_{n-1}}{zp_{n-1}} = \cancel{\quad} 1+hS_0^{-1}$$

~~Consider~~ Go over this again. Maybe reformulate

Given (X,c) and ~~you~~ you form $X \longrightarrow V_-$

Idea: Try for a universal type situation. You would like to specify a class of contractions - mainly via a graph construction. Roughly you would like a parameter space of "potentials" ~~which are~~ which are sequences h_n, $n\geq 1$, where h_n ~~...~~ is real ~~...~~ $\nearrow 1$.

Example. constant coeff case

$$S = \begin{pmatrix} 1 & h \\ h & 1\end{pmatrix}(zS) = \frac{zS+h}{1+hzS}$$

$$S + hzS^2 = zS + h$$

$$\boxed{hzS^2 + (1-z)S - h = 0}$$

$$S^2 + \left(\frac{1}{hz}-\frac{1}{h}\right)S - \frac{1}{z} = 0$$

Better approach: eigenvalues of $\begin{pmatrix} z & h \\ zh & 1\end{pmatrix}$

$$\lambda^2 - (1+z)\lambda + z(1-h^2) = 0$$

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$$\lambda = \frac{1+z \pm \sqrt{(1+z)^2 - 4z(1-h^2)}}{2}$$

$$\lambda \begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} 1 & h \\ h & 1 \end{pmatrix} \begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix}$$

$$\lambda^2 - (1+z)\lambda + z(1-h^2) = 0 \qquad \mu = z^{1/2}\lambda$$

$$z\mu^2 - (1+z)z^{1/2}\mu + z(1-h^2) = 0 \qquad \cancel{\cos 2\theta = \cos^2\theta - \sin^2\theta}$$

$$\mu^2 - (z^{1/2} + z^{-1/2})\mu + 1 - h^2$$

if $|z| = 1$. $$\left(\frac{z^{1/2}+z^{-1/2}}{2}\right)^2 - (1-h^2)$$

$$\delta = \frac{-(1-z) \pm \sqrt{(1-z)^2 + 4h^2 z}}{2}$$

$$(1-z)^2 + 4h^2 z = z^2 - 2z + 1 + 4h^2 z$$
$$= z^2 + (-2+4h^2)z + 1 = 0$$

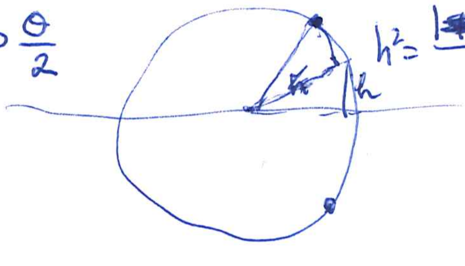
$-1 \leq -1 + 2h^2 \leq 1$. So for $-1 \leq h \leq 1$, the roots are on the circle.

$$\delta = \frac{-(1-z) \pm \sqrt{(1-z)^2 + 4h^2 z}}{2} \qquad -2\cos\theta \qquad -1 \leq -1+2h^2 \leq 1$$

$$(1-z)^2 + 4h^2 z = z^2 + (-2+4h^2)z + 1$$

roots are $$z = (+1-2h^2) \pm \sqrt{(+1-2h^2)^2 - 1}$$
$$= \cos\theta \pm i \sin\theta \qquad \boxed{\cos\theta = +1-2h^2}$$

where $h = \cos\frac{\theta}{2}$ $\qquad h^2 = \frac{1-\cos\theta}{2} = \sin^2\frac{\theta}{2}$



Note

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$$\cancel{S = \frac{zS+h}{1+hzS}} \qquad \cancel{z^{-1/2}S = \frac{z^{1/2}S + z^{-1/2}h}{1+hz^{3/2}z^{-1/2}S}}$$

$$\cancel{S = \frac{zS+h}{hzS+1} = \frac{S+z^{-1}h}{hS+z^{-1}}}$$

$$\cancel{S = \frac{zS+h}{hzS+1}} = \frac{z^{1/2}S + z^{-1/2}h}{hz^{1/2}S + z^{-1/2}}$$

$$\cancel{z^{1/2}S = \frac{S + z^{-1}h}{hz\,z^{-1/2}S +}}$$

$$S = \frac{zS+h}{1+hzS} \qquad S = z^{1/2}T$$

$$z^{1/2}T = \frac{z^{3/2}T + h}{1+hz^{3/2}T} \qquad T = \frac{zT + z^{-1/2}h}{1+hz^{3/2}T}$$

$$T = \begin{pmatrix} 1 & z^{-1/2}h \\ z^{1/2}h & 1 \end{pmatrix}(zT)$$

$$S = \frac{zS+h}{1+hzS} \qquad S + hzS^2 = zS + h$$

$$hzS^2 + (1-z)S - h = 0$$

$$S = \frac{-(1-z) \pm \sqrt{(1-z)^2 + 4h^2 z}}{2hz}$$

$$z^{1/2}S = \frac{-(z^{-1/2} - z^{1/2}) \pm \sqrt{(z^{-1/2}-z^{1/2})^2 + 4h^2}}{2h}$$

$$= \frac{z^{1/2} - z^{-1/2}}{2h} \pm \sqrt{\left(\frac{z^{1/2}-z^{-1/2}}{2h}\right)^2 + 1}$$

$$= \frac{i\sin(\theta/2)}{h} \pm \sqrt{1 - \frac{\sin^2(\theta/2)}{h^2}}$$

311 Constant case $S = \dfrac{zS + h}{1 + hzS}$

$$S + hzS^2 = zS + h$$

$$hzS^2 + (1 - z)S - h = 0$$

$$h(z^{1/2}S)^2 + (z^{-1/2} - z^{1/2})z^{1/2}S - h = 0$$

$$(z^{1/2}S)^2 - 2\left(\frac{z^{1/2} - z^{-1/2}}{2h}\right)(z^{1/2}S) - 1 = 0$$

$$z^{1/2}S = \frac{-(1-z) \pm \sqrt{(1-z)^2 + 4h^2 z}}{2hz^{1/2}}$$

$$= \frac{z^{1/2} - z^{-1/2}}{2h} \pm \sqrt{\left(\frac{z^{1/2} - z^{-1/2}}{2h}\right)^2 + 1}$$

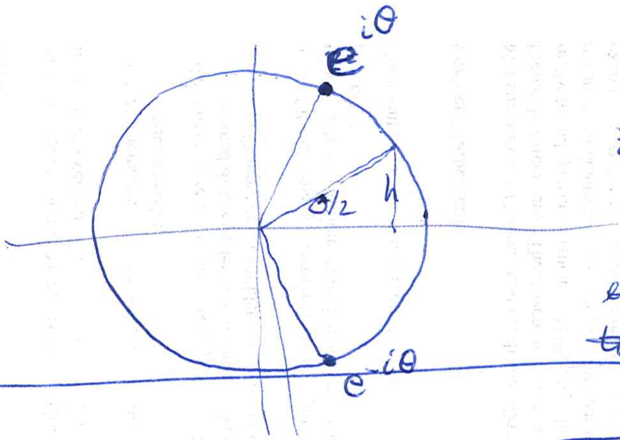
we want $S(z)$ for $|z| = 1$. Put $z = e^{i\theta}$ $\quad \dfrac{z^{1/2} - z^{-1/2}}{2} = i\sin\left(\dfrac{\theta}{2}\right)$

$$z^{1/2}S = \frac{i\sin\frac{\theta}{2}}{h} \pm \sqrt{1 - \left(\frac{\sin\theta/2}{h}\right)^2} \qquad 0 < h < 1$$

for $|\sin\theta/2| \leq h$ $\qquad S = ?$

For $|\sin(\theta/2)| < h$ $\qquad |S| = 1.$

$\sin(\theta/2) = \pm h \qquad z^{1/2}S(z) = \pm i$



The first point maybe is that $|S| = 1$ when $z = e^{i\theta}$ and $|\sin(\frac{\theta}{2})| \leq h$

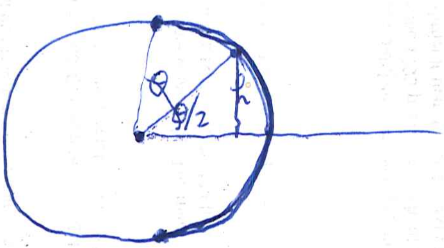
~~eigenvalue~~
~~what S is about~~

$$S = \frac{-1 + z \pm \sqrt{(1-z)^2 + 4h^2 z}}{2hz}$$

$$(1-z)^2 + 4h^2 z = 0.$$

$$\left(\frac{z^{-1/2} - z^{1/2}}{2hi}\right)^2 = 1$$

$$|\sin(\theta/2)| = h.$$



What were you trying to do yesterday?

You looked at $c = c_0 + \Delta c$, $\det(z-c)$.

Setting: Fix $Y = aX \oplus V^+ = bX \oplus V^-$

$$c = \underbrace{ba^*}_{c_0} + \underbrace{v_- h v_+^*}_{\Delta c} \qquad |h| < 1.$$

embedding

$$Y \hookrightarrow H^+ = H^2(S^1)$$

$$y \longmapsto v_+ \quad ?$$

Look at $\det(z-c)$

$$-\delta \log\det(z-c) = -\delta \operatorname{tr} \log(z-c) \underset{\delta h}{=} \operatorname{tr}\left(\frac{1}{z-c}\delta c\right)$$

$$= \operatorname{tr}\left(\frac{1}{z-c} v_- \cancel{\delta} v_+^*\right)$$

$$= \left(v_+^* \frac{1}{z-c} v_-\right)\delta h$$

here is where quasi-det. stuff enters? You have a matrix component of an inverse?

$$v_+^* \frac{1}{z-c} = v_+^* \frac{1}{z-c_0-\Delta c} = v_+^* \frac{1}{1-\frac{1}{z-c_0}\underbrace{\Delta c}_{v_- h v_+^*}} \frac{1}{z-c_0}$$

$$= \frac{1}{1-\left(v_+^* \frac{1}{z-c_0} v_-\right)h} v_+^* \frac{1}{z-c_0}$$

$$\boxed{v_+^* \frac{1}{z-c} y = \frac{1}{1-S_0^{-1}h} v_+^* \frac{1}{z-c_0} y}$$

this formula should relate embeddings.

$$-\delta \log\det(z-c) = \left(v_+^* \frac{1}{z-c} v_-\right)\delta h$$

$$= \frac{1}{1-S_0^{-1}h} S_0^{-1}\delta h = \frac{1}{S_0-h}\delta h = -\delta\log(S_0-h)$$

$$\det(z-c)/\det(z-c_0) = S_0 - h/S_0$$

Ex. $X = P_{n-1}$ $Y = P_n$

$$\frac{p_n}{zp_{n-1}} \overset{?}{=} 1 - h\frac{q_{n-1}}{zp_{n-1}}$$

$$p_n \overset{?}{=} zp_{n-1} - hq_{n-1}$$

OKAY except for sign of h.

$$\|f(z)\|^2 = \|\sum z^n x_n\|^2 = \sum_{k,l}(z^k x_k, z^l x_l)$$

$$= \sum_{k,l}(x_k, z^{l-k}x_l) = \sum_{k,l}\left(x_k, \begin{Bmatrix} c^{l-k} & l \geq k \\ c^{*k-l} & l \leq k \end{Bmatrix} x_l\right)$$

$$= \int \sum_{k,l}\left(z^k x_k, \ \rho(z) \ \ z^l x_l\right)\frac{d\theta}{2\pi}$$

$$\rho(z) = \sum_{n\geq 0} z^{-n}c^n + \sum_{n\geq 1} z^n c^{*n}$$

$$= \frac{1}{1-z^{-1}c} + \frac{zc^*}{1-zc^*} =$$

$$= \frac{1}{1-zc^*}(1-c^*c)\frac{1}{1-z^{-1}c} = \frac{1}{1-z^{-1}c}(1-cc^*)\frac{1}{1-zc^*}$$

take $X = \mathbb{C}$ $c = h \in D$.

$$\rho(z) = \frac{1-|h|^2}{|1-z^{-1}h|^2} = \frac{1-|h|^2}{|z-h|^2} \text{ or } \frac{1-|h|^2}{|1-z\bar{h}|^2}$$

How can I arrange this?

Back to LC circuits. Consider $H = H^+ \oplus H^-$

$\|\xi\|_s^2 = s\|\xi^+\|^2 + s^{-1}\|\xi^-\|^2$, subspaces $W \subset V$ of H.

The herm. scal. prod. $\|\xi\|_s^2$ on H induces one on V/W. Calculate $j_* Q_s$ where $j : H \longrightarrow H/W$

$$(j_* Q_s)\left(\xi_{\bmod W}\right) = \min Q_s(\xi + w)$$

$(j_* Q_s)(\xi')$ = stationary value of Q_s on $j^{-1}(\xi')$. General procedure.

$$\begin{tikzcd} W \arrow[r, hook] & H \arrow[r] \arrow[d, "Q"] & H/W \\ W^* & H^* \arrow[l] & W^\perp \arrow[l] \end{tikzcd}$$

You should do this with herm operators.

$$W \overset{\varepsilon}{\subset} H^+ \oplus H^- \qquad \varepsilon = (\varepsilon_+, \varepsilon_-)$$

$$1 = \varepsilon_+^* \varepsilon_+ + \varepsilon_-^* \varepsilon_- \qquad \varepsilon_+^* \varepsilon = \rho, \ 0 \le \rho \le 1$$

direct sum of $\qquad \varepsilon_+^* \varepsilon_+ = \bigoplus_\omega \rho_\omega E_\omega$

$$W_\lambda = \begin{pmatrix} 1 \\ \lambda \end{pmatrix} H_\lambda^+ \qquad \text{where } 0 \le \rho_\omega \le 1, \quad \rho_\omega = \frac{1}{1+\omega^2}$$

Direct sum decomp. into

$$W_\omega \hookrightarrow H_\omega^+ \oplus H_\omega^-$$

$$\frac{1}{\sqrt{1+\omega^2}} a_+ \qquad a_\pm : W_\omega \xrightarrow{\sim} H_\omega^\pm$$

$$\frac{\omega}{\sqrt{1+\omega^2}} a_-$$

when you restrict $s\|\xi_+\|^2 + s^{-1}\|\xi_-\|^2$ you seem to get

$$s\frac{1}{1+\omega^2} + s^{-1}\frac{\omega^2}{1+\omega^2} = \frac{(s^2+\omega^2)}{s(1+\omega^2)}$$

which gets inverted to $\frac{s(1+\omega^2)}{s^2+\omega^2}$. The end result on the subquotient space is

$$\sum \frac{s(1+\omega^2)}{s^2+\omega^2} a_\omega \xrightarrow[\cancel{\text{Looks}}]{s=1} \sum a_\omega = 1.$$

(a_ω) This should be exactly a ~~symmetric~~ probability measure on S^1.

$$\sum_\omega \left(\frac{1}{s-i\omega} + \frac{1}{s+i\omega} \right) \qquad \text{odd function of } s$$

315 ~~What does one hope for?~~

On V/W you get this response function and its inverse expressed ~~as ratios~~ in the appropriate partial fraction form. Look at the zeroes + the poles of the resp. functions, these should interlace, the pole set ~~to zero~~ should depend on one of V, W and the zero set on the other

Basically you want to see once you fix W the possible functions you get from a choice of V. Recall that $W \subset H^+ \oplus H^-$ yields on $H/W \simeq W^\perp$ a decomp.?

Wait: $W \subset H^+ \oplus H^-$ splits into pieces

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need to distinguish differentials and functions discrete nice spectrum hopefully yields a characteristic function, some kind of analytic function. Take log derivative dlog to get a differential with integral ≥ 0 residues. What happens? You need to take

Look at Blashke products from viewpt of diffs.

$$d\log \prod_n \left(\frac{z-a_n}{1-\bar{a}_n z}\right) = \sum_n \left(\frac{1}{z-a_n} + \frac{+\bar{a}_n}{1-\bar{a}_n z}\right) dz$$

$$= \sum_n \frac{1-\bar{a}_n z + (z-a_n)\bar{a}_n}{(z-a_n)(1-\bar{a}_n z)}\, dz = \sum_n \frac{1-|a_n|^2}{|z-a_n|^2}\,\frac{dz}{z}$$

when does this converge? $|z-a_n|^2$ for $|z|=1$.

$$\sum \frac{1-|a_n|^2}{(z-a_n)(1-\bar{a}_n z)}$$

$(1-|a_n|)(1+|a_n|)$
2

$|z-a_n|$ $\qquad |a_n| \to 1.$

$\geq |a_n| - |z|$. ok if $|z|<1$.

$|1-\bar{a}_n z| \geq 1-|z|.$

 ~~[illegible]~~ intrinsic Hilbert spaces

$L^2 = H^- \oplus H^+$ belonging to a circle.

Idea to develop: coefficient of resolvent = ratio of determinants, bottom should be characteristic poly, numerator something similar. [redacted]

~~[illegible] numerator similar~~

Had idea of distinguishing between functions and differentials. ~~[illegible]~~ ζ is an. function, $d\log\zeta$ merom. simple poles, $\mathbb{R}_{>0}$ residues, compare with coeff of resolvent arising from cyclic vector.

~~Coeff of resolvent~~

Go back to de B functions. $E(\omega)$ entire function such that this

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focus on the pencil ~~[illegible]~~ of divisors idea. You ~~begin with~~ start with the divisor of zeroes, this gives an ~~analytic function~~ oscillatory function

e.g. suppose you take zeroes $\frac{1}{2}+\mathbb{Z}$. Get

$$\prod_{n\geq 0}\left(1-\frac{z^2}{(n+\frac{1}{2})^2}\right) = \cos(\pi z) = \frac{e^{i\pi z}+e^{-i\pi z}}{2}$$

$x+iy$

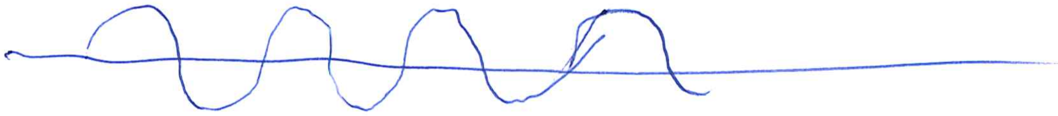
$$|\cos(\pi z)| \leq \frac{e^{\pi y}+e^{-\pi y}}{2} \leq e^{\pi y} \qquad \text{for } y \geq 0$$

~~$|\sin(\pi z)| \leq \frac{e^{\pi y}+e^{-\pi y}}{2}$~~

So the problem becomes clear, namely, how to obtain a natural cyclic vector.

$$\prod_{n\geq 1}\left(1+\frac{y^2}{n^2}\right)$$

~~Have~~ sequence of zeroes $\leadsto$ spectrum, char. poly.



Discuss the problem: You want to start with the divisor of the zeroes of ξ. NO. You want to understand whether you can do something for a class mult. 1 discrete spectra. Thus you give a discrete subset of the line with suitable growth (+ symmetry) properties.

Question. Is there some line bundle..
You need a theory of linearly equivalent divisors of infinite support.

Another idea. Granted that the determinant looks like

can you find a "complementary" imaginary function such that when added to the determinant is a path of the appropriate type in the circle, would yield a 1-parameter family of divisors. This might be a variant of the Hilbert transform.

Take some examples. Start with $\cos x$

Another idea is that KdV methods might help understand the choice of cyclic vector. Think a bit

$$K_s(r) = \int_0^\infty e^{-\frac{r}{2}(t+t^{-1})} t^s \frac{dt}{t} \qquad e^{s \log t}$$

$$\frac{d}{ds} K_s(r) = \int_0^\infty e^{-\frac{r}{2}(t+t^{-1})} t^s \log t \frac{dt}{t}$$

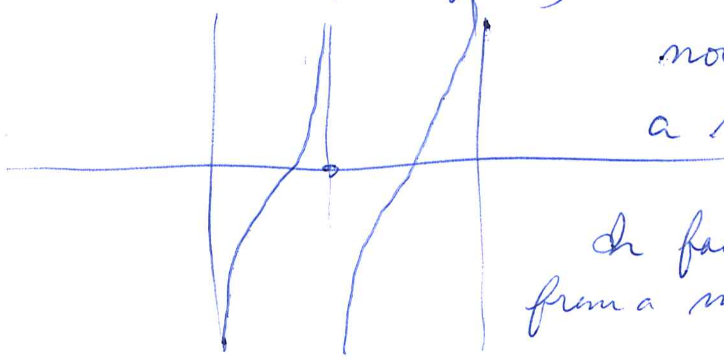
Take a finite subset of the circle and keep in mind Graeme's blip S. First thing you might want is a real valued function with simple zeroes at each point of the subset.

318 ~~How does it work?~~

Take set of zeros. $\pm n \quad n \geq 1$.

Form ~~log~~ $\frac{d \log}{dz} \prod(z-n) = \frac{1}{z} + \sum_{n\geq 1} \frac{1}{z-n} + \frac{1}{z+n} = \frac{1}{z} + 2z \sum_{n \geq 1} \frac{1}{z^2-n^2}$

~~This has simple poles, so its graph~~ This looks like



now take C.T. and you get a nice map to S^1.

In fact the log. deriv. arises from a measure on the line.

dlog

$$\frac{1}{z-\lambda} = \frac{x-\lambda - iy}{(x-\lambda)^2+y^2}$$

$$\int \frac{1}{z-\lambda}\, d\mu(\lambda)$$

~~Need~~ need to assume convergence of the im. part at some λ

$$\int \frac{1}{1+\lambda^2}\, d\mu(\lambda) < \infty \qquad \text{assume.}$$

$$f(\lambda) = \int \left[\frac{1}{\lambda - \omega} + \frac{\omega}{1+\omega^2}\right] d\mu(\omega)$$

converges defining $f(\lambda) \ni \operatorname{Im} f(\lambda) < 0$ in UHP

$$z = \frac{1-s}{1+s} = \begin{pmatrix} -1 & +1 \\ 1 & 1 \end{pmatrix}(s) \qquad s = -i\lambda$$

$$z = \frac{1+i\lambda}{1-i\lambda} = \frac{-\lambda+i}{\lambda+i}$$ takes UHP to $|z|<1$.

~~so~~ so $\frac{1-if}{1+if}$ takes UHP to $|z|<1$.

$$f = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{2z}{z^2-n^2} = \pi \frac{\cos \pi z}{\sin \pi z} \qquad \frac{1 - i\pi \frac{c}{s}}{1+i\pi \frac{c}{s}} = \frac{s-i\pi c}{s+i\pi c}$$

adj π away $\quad \frac{s-ic}{s+ic} = \frac{c+is}{c-is} = e^{2\pi i z}$

319 Review deB functions $|E(\bar\lambda)| < |E(\lambda)|$ for $\mathrm{Im}(\lambda)>0$

avoid zeroes for $\lambda \in \mathbb{R}$

$$|e^{-ia\lambda}| = e^{a\,\mathrm{Im}(\lambda)}$$ so $E(\lambda) = e^{-ia\lambda}$ is deB for $a>0$.

Suppose $E(\lambda+n) = E(\lambda)$ $\forall n$.

$$E(\lambda) = F(\underbrace{e^{2\pi i\lambda}}_{z}) \qquad |e^{2\pi i \lambda}| = e^{-2\pi\,\mathrm{Im}\,\lambda}$$

$E(\lambda)$ entire periodic function. $\therefore$ $F(z)$ analytic except at $z=0,\infty$. Want $|F(z)| > |F(\bar z^{-1})|$ for $|z|<1$.

Are there interesting examples? Suppose F rational.

$F(z) = \frac{p(z)}{q(z)}$ no zeroes for $|z|<1$. So

You ~~to~~ should look for $F(z)$ analytic except at $z=0,\infty$ and such that $|F(z)| < |F(\bar z^{-1})|$ for $|z|<1$. Zeroes of F lie in D. Certainly things ~~look~~ messy if 0 is essential sing. Assume at most a pole. Thus assume $|F(z)| \le C|z|^{-N}$ for $|z| \le \varepsilon$

~~where~~ $\cancel{|F(\bar z^{-1})| < C|z|^{N}}$

$\Longrightarrow$ $|F(z)|$

Start again. Assume F rational and $|F(z)| < |F(\bar z^{-1})|$ for z in D.

Take ~~deB function~~.

$$q(z) = \prod_{i=1}^{n} (1 - \bar a_i z) \qquad |a_i| < 1.$$

$$\overline{q(\bar z^{-1})} = \prod_{i=1}$$

320 $\quad g(z) = 1 - \bar{a} z \qquad \overline{g(\bar{z}^{-1})} = \overline{1 - \bar{a}\bar{z}^{-1}} = 1 - a z^{-1}$

classify periodic inner functions.

$S(\lambda)$ analytic in the UHP $\quad | \ | \le 1$.

$$E(\lambda + n) = E(\lambda) \qquad E(\lambda) = F(\underbrace{e^{2\pi i \lambda}}_{z})$$

$F(z)$ analytic for $z \neq 0, \infty$.

First look at scattering functions $S(\lambda + n) = S(\lambda)$

$S(\lambda) = T(\underbrace{e^{2\pi i \lambda}}_{= z})$. It should be clear that a periodic inner function on the UHP is the same as a scattering function on D.

$$z - a \qquad |\bar{z}^{-1} - a| = |z^{-1} - \bar{a}| \qquad z = \frac{1 + i\lambda}{1 - i\lambda}$$

$$\lambda = i\frac{1-z}{1+z}$$

$$\frac{\lambda - a}{\lambda - \bar{a}} = \begin{pmatrix} 1 & -a \\ 1 & -\bar{a} \end{pmatrix}\begin{pmatrix} -i & i \\ 1 & 1 \end{pmatrix}(z)$$

$$= \begin{pmatrix} -(i+a) & i-a \\ -i-\bar{a} & i-\bar{a} \end{pmatrix}(z) = \frac{-z(i+a) + i - a}{-z(i+\bar{a}) + i - \bar{a}}$$

$$= \frac{-(i+a)\left(z - \frac{i-a}{i+a}\right)}{(i-\bar{a})\left(1 - z\left(\frac{i+\bar{a}}{i-\bar{a}}\right)\right)} = \frac{-(i+a)}{(i-\bar{a})} \frac{z - \alpha}{1 - z\bar{\alpha}}$$

mod. 1

$$\bar{\alpha} = \overline{\frac{i-a}{i+a}} = \frac{-i-\bar{a}}{-i+\bar{a}} = \frac{i+\bar{a}}{i-\bar{a}} \qquad \overline{-i-a} = i - \bar{a}$$

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$$\boxed{\frac{\lambda - a}{\lambda - \bar{a}} = e^{i\theta}\,\frac{z-\alpha}{1 - z\bar{\alpha}}}$$

$$\alpha = \frac{1+ia}{1-ia} = \frac{i-a}{i+a}$$

$$z = \frac{1+i\lambda}{1-i\lambda}$$

$$e^{i\theta} = \frac{-(i+a)}{i-\bar{a}} = \frac{a+i}{\bar{a}-i}$$

Thus Blashke products correspond, more generally ~~the~~ scattering functions inner functions.

But what about dB functions??? There is a problem ~~[illegible]~~ doing the C.T. A rational dB fn. has form ~~[illegible]~~ $c\prod_{i=1}^{n}(\lambda - a_i)$ ~~[illegible]~~ $a_i \in LHP$. rational + entire means polynomial. This goes into

$$\prod_{j=1}^{n}\left(i\,\frac{1-z}{1+z} - a_j\right) = \frac{c}{(1+z)^n}\prod\left(i - iz - a_j - a_j z\right)$$

$$(i-a_j) - z(i+a_j)$$

$$= \frac{c'}{(1+z)^n}\prod_{k=1}^{n}\Bigl(z - \underbrace{\frac{i-a_k}{i+a_k}}_{\alpha_k}\Bigr)$$

~~But the~~ If you have periodic dB functions Blashke products. What is

$$\prod \frac{\lambda - \_\_\_}{\lambda - \_\_\_}$$

You want $\dfrac{z-\alpha}{1-\bar{\alpha}z}$ into $\dfrac{e^{2\pi i\lambda} - \overbrace{e^{2\pi i a}}^{\alpha \; e^{D}}}{1 - \underbrace{e^{-2\pi i\bar{a}}}_{\bar{\alpha}}\,e^{2\pi i\lambda}}$

now ~~[illegible]~~ this gives a scattering function

 ~~The question now is when can we get~~

$e^{2\pi i\lambda} - \alpha$ should be a dB fn. for $|\alpha|<1$.
better probably is $e^{\pi i\lambda} - \alpha e^{-i\pi\lambda}$. How to see this.

$$F(\lambda) = e^{i\pi\lambda} - \alpha e^{-i\pi\lambda}$$

$$\overline{F(\bar\lambda)} = \overline{e^{i\pi\bar\lambda} - \alpha e^{-i\pi\bar\lambda}} = e^{-i\pi\lambda} - \bar\alpha e^{i\pi\lambda}$$

$$\frac{F(\lambda)}{\overline{F(\bar\lambda)}} = \begin{pmatrix} 1 & -\alpha \\ -\bar\alpha & 1 \end{pmatrix}\left(\frac{e^{i\pi\lambda}}{e^{-i\pi\lambda}}\right)$$

$$\frac{e^{i\pi\lambda}}{e^{-i\pi\lambda}} = e^{2i\pi\lambda} < 1$$

It should be possible to make periodic dB fns. by taking products. $e^{2\pi i\lambda} = q$. So you need a product of terms

What about $\prod_p (1-p^{-s})$

$$p^{-s} = e^{-s\log p} = e^{i\log(p)s}$$

wait. Look.

$e^{2\pi i\lambda} - \alpha$ should be a deB function for any $|\alpha|<1$.

I found that ~~$e^{2\pi i\lambda}$~~ $e^{i\pi\lambda} - \alpha e^{-i\pi\lambda}$ is a de B function

You want a complete understanding of periodic dB functions. Let $E(\lambda)$ entire, period 1, no real zeroes, $|E(\lambda)| < |E(\bar\lambda)|$ for $\mathrm{Im}(\lambda) > 0$.

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Put $F(\lambda) = \dfrac{E(\lambda)}{\overline{E(\bar\lambda)}}$ inner function – <1 in UHP

periodic $\Rightarrow$ $F(\lambda) = F_1(e^{2\pi i\lambda})$ F_1 analytic on closed disk except for $z=0$. But bdd $\Rightarrow$ removable sing. So $F_1(z)$ analytic on closed D and $|F_1(z)|=1$ when $|z|=1$. Finitely many zeroes. Reduce to no zeros.

Now the question how to get from F to E.

$z = e^{2\pi i\lambda}$

$$\frac{z-\alpha}{1-z\bar\alpha} = \frac{z^{1/2} - \alpha z^{-1/2}}{-z^{1/2}\bar\alpha + z^{-1/2}}$$

factors $\quad 1 - \dfrac{1}{p^s} = \dfrac{p^{\frac{s}{2}} - p^{-\frac{s}{2}}}{p^{s/2}}$

IDEA Take a product (appropriately) of periodic deB fns. with increasing periods. Basic factor is $e^{it\lambda} - \alpha e^{-it\lambda}$ $\quad t>0$

$$\frac{E(\lambda)}{\overline{E(\bar\lambda)}} = \frac{e^{it\lambda} - \alpha e^{-it\lambda}}{e^{-it\lambda} - \bar\alpha e^{it\lambda}} = \begin{pmatrix} 1 & -\alpha \\ -\bar\alpha & 1 \end{pmatrix}\left(e^{2it\lambda}\right) \qquad \lambda \in UHP$$

$$(e^{i\lambda})^t - \alpha(e^{-i\lambda})^t$$

$$\frac{z^t - \alpha z^{-t}}{z^{-t} - \bar\alpha z^t} = \frac{z^{2t} - \alpha}{1 - \bar\alpha z^{2t}} \in D$$

324 ~~So what!!~~ Basic factor is $z^t - \alpha z^{-t}$ $t>0$. with $|\alpha|<1$. Can you form a product of these which converges nicely. $z = e^{i\lambda}$ or $e^{2\pi i\lambda}$

~~Next point~~ First look at S. So

$$S(\lambda) = \frac{z^t - \alpha z^{-t}}{z^{-t} - \bar\alpha z^{+t}} = \frac{z^{2t} - \alpha}{1 - \bar\alpha z^{2t}}$$

Now you want to take a product of theses, and you want to know when it converges. There should be an easy answer. You need $t \downarrow 0$.

There are some interesting points. There is a result which says when the divisor is realizable by an S, realizable means adjusting phase. Result is the convergence of $\prod |a_n|$

$$\prod \frac{\lambda - a_n}{\lambda - \bar a_n}$$

$$\prod \frac{z - \alpha_n}{1 - \bar\alpha_n z} e^{i\varphi_n} \quad \text{conv.}$$

$$\prod \frac{z - \alpha_n}{1 - \bar\alpha_n z} \cdot \frac{|\alpha_n|}{-\alpha_n}$$

$$\Rightarrow \prod -\alpha_n e^{i\varphi_n} \quad \text{conv.}$$

$$\Rightarrow \prod |\alpha_n| \quad \text{conv.}$$

$$\frac{z - \alpha}{1 - z\bar\alpha} = \frac{i - \bar a}{-(i + a)} \frac{\lambda - a}{\lambda - \bar a}$$

$$\frac{z - \alpha}{1 - \bar\alpha z} = e^{i\varphi} \frac{\lambda - a}{\lambda - \bar a}$$

$$\frac{a - i}{a + i} = e^{i\varphi} \frac{a - i}{\bar a - i}$$

$$e^{i\varphi} = \frac{\bar a - i}{a + i}$$

$$z = \frac{1 + i\lambda}{1 - i\lambda}$$

$$\alpha = \frac{1 + ia}{1 - ia} = \frac{-a + i}{a + i}$$

$$-\alpha = \frac{a - i}{a + i}$$

$$|\alpha| = \left|\frac{a - i}{a + i}\right|$$

$$\frac{i - a}{i - \bar a}$$

$$\prod \frac{\lambda - a_n}{\lambda - \bar a_n} \frac{i - \bar a_n}{i - a_n} \frac{|i - a_n|}{|i - \bar a_n|}$$

325 basic factor is

$$S(\lambda) = \frac{e^{it\lambda} - \alpha e^{-it\lambda}}{e^{-it\lambda} - \bar{\alpha} e^{it\lambda}} \qquad t > 0 \qquad |\alpha| < 1$$

choose t_n, α_n so that the resulting product converges at some convenient point $\lambda = i$. Need ~~conv~~ convergence of $\prod \left|S_n(i)\right|$

$$\frac{e^{-t} - \alpha e^{+t}}{e^{t} - \bar{\alpha} e^{-t}} = \frac{e^{-2t} - \alpha}{1 - \bar{\alpha} e^{-2t}}$$

$$\frac{e^{-2t} - \alpha}{1 - \bar{\alpha} e^{-2t}}$$

~~First suppose α real, then you seem to get e^{-2t} and $\prod e^{-2t_n} = e^{-2\sum t_n}$ converges when $\sum t_n < \infty$.~~

You would like to try for convergence for $\lambda \in \mathbb{R}$

e.g. $\lambda = 0$ $\qquad \dfrac{1-\alpha}{1-\bar{\alpha}}$

So you try. $\quad S(\lambda) = \dfrac{e^{it\lambda} - \alpha e^{-it\lambda}}{e^{-it\lambda} - \bar{\alpha} e^{it\lambda}} \; \dfrac{1-\bar{\alpha}}{1-\alpha}$

So

$$S(\lambda) = \frac{e^{2it\lambda} - \alpha}{1 - \bar{\alpha} e^{2it\lambda}}$$

You want the product of these over t_n, α_n to converge I think you need convergence of $\prod \left|S_n(\lambda)\right|$ for some λ in the UHP

where do the zeros lie $\qquad e^{2it\lambda} = \alpha$

this is a coset of $\lambda_0 + \mathbb{Z}\frac{\pi}{t}$. Real issue is how close $|\alpha|$ gets to 1. You obviously can let

Look as follows. $t_n \downarrow 0$ assumed, so for any $|\lambda| < M$

$e^{2it_n\lambda}$

$$\frac{1-\alpha e^{-2it\lambda}}{1-\bar\alpha e^{2it\lambda}} \quad \frac{1-\bar\alpha}{1-\alpha}$$

$$\frac{1-\alpha e^{-2it\lambda}}{1-\alpha} = 1 - \frac{\alpha(e^{-2it\lambda}-1)}{1-\alpha} = 1 - \frac{\alpha}{1-\alpha}(-2it\lambda) + \cdot$$

$$\frac{1-\bar\alpha e^{2it\lambda}}{1-\bar\alpha} = 1 - \frac{\bar\alpha(e^{2it\lambda}-1)}{1-\bar\alpha} = 1 - \frac{\bar\alpha}{1-\bar\alpha}(2it\lambda) \cdots$$

so you want $\left(\frac{\alpha}{1-\alpha} + \frac{\bar\alpha}{1-\bar\alpha}\right) t$ to form an ℓ^1 seq.

so it should be possible for $|\alpha| \nearrow 1$ provided $t \downarrow 0$ to compensate.

E periodic $|E(\lambda)| < |E(\bar\lambda)|$ on UHP entire no zeroes on $\mathbb{R}$

Put $S = \frac{E(\lambda)}{\overline{E(\bar\lambda)}}$ $\qquad S(\lambda) = S_1(e^{2\pi i\lambda})$

$S_1(z)$ analytic for $\alpha < |z| \leq 1+\varepsilon$ bdd, $z=0$ removable sing.

S_1 has fin. zeroes in D $\qquad S_1 = \prod \frac{z-\alpha_i}{1-\bar\alpha_i z} = \frac{F_1(z)}{F_1(}$?

odd + even stuff. $\qquad S_1(z) = \prod \frac{z^{1/2} - \alpha_i z^{-1/2}}{z^{-1/2} - \bar\alpha_i z^{1/2}} = \frac{F_1(z)}{\overline{F_1(\bar z^{-1})}}$

$$\therefore \frac{E(\lambda)}{\overline{E(\bar\lambda)}} = \frac{F_1(e^{2\pi i\lambda})}{\overline{F_1(e^{2\pi i\bar\lambda})}} \qquad \frac{E(\lambda)}{F_1(e^{2\pi i\lambda})}$$

entire

do it right.

$E(\lambda+1) = E(\lambda)$ $\qquad E(\lambda) = E_1(z)$ $\qquad z = e^{2\pi i\lambda}$

$$\frac{E_1(z)}{\overline{E_1(\bar z^{-1})}} = S_1(z) = \frac{z^{-\frac{n}{2}} p(z)}{z^{-\frac{n}{2}} q(z)}$$

$q(z) = z^n \overline{p(\bar z^{-1})}$

$$\frac{E_1(z)}{z^{-\frac{n}{2}} p(z)} = \frac{\overline{E_1(\bar z^{-1})}}{z^{-\frac{n}{2}} q(z)} \quad \text{Const.}$$

anal. $|z| \leq 1$ $\qquad$ anal $|z| \geq 1$

32-7.

~~[illegible]~~

$$Z(s) = \frac{a_0}{s} + \sum \frac{s(1+\omega_k^2)}{s^2+\omega_k^2} a_k + a_\infty s$$

$$Z(1) = a_0 + \sum a_k + a_\infty$$

$$Z(s) = \frac{1}{s} + \sum_{n=1}^{\infty} \frac{2s}{s^2+n^2} = i\pi \frac{\cos i\pi s}{\sin i\pi s} = i\pi i \frac{e^{-\pi s}+e^{\pi s}}{e^{-\pi s}-e^{\pi s}}$$

$$Z(1) = 1 + \sum_1^{\infty} \frac{2}{1+n^2} \qquad = \pi \frac{\cosh(\pi s)}{\sinh(\pi s)}$$

$$\frac{1}{z} + \sum_{n=1}^{\infty} \frac{2z}{z^2-n^2} = \pi \frac{\cos \pi z}{\sin \pi z}$$

Question: take $Z(s) = \frac{1}{s} + \sum_{n=1}^{\infty} \frac{2s}{s^2+n^2}$. This should comes from a 1-dim subquotient of a polarized Hilbert space.

Suppose you are given $\sum \frac{a_k}{\lambda -}$

Look at $\sum_{n\in\mathbb{Z}} \frac{1}{\lambda - n}$ divergent, but it can summed symmetrically about any point and gives the same result.

~~[illegible]~~

Go back to $Z(s) = \sum_\omega \frac{s(1+\omega^2)}{s^2+\omega^2} a_\omega$

For each ω occurring i.e. $a_\omega > 0$ you will get

For each pole you get a 2 plane $H_\omega = \mathbb{C}^2$ summand together with $W_\omega \subset H_\omega$ graph of

$W_\omega = \binom{1}{\omega}\mathbb{C} \subset H_\omega$. So $H^{\pm} = \bigoplus H^{\pm}_\omega$

$$W \subset H^+ \oplus H^- \supset W^\perp$$

$$W_\omega = \begin{pmatrix} 1/\sqrt{1+\omega^2} \\ \omega/\sqrt{1+\omega^2} \end{pmatrix} \mathbb{C} \subset \underset{\mathbb{C}^2}{\overset{\shortparallel}{H_\omega}} \supset W_\omega^\perp = \begin{pmatrix} -\omega/\sqrt{1+\omega^2} \\ 1/\sqrt{1+\omega^2} \end{pmatrix} \mathbb{C}$$

think of $W = \oplus W_\omega$ and $H^\pm = \oplus H^\pm_\omega$ and

$i = \bigoplus_\omega \iota_\omega$ $\quad \iota_\omega : W_\omega \longrightarrow H_\omega = \begin{matrix} H^+_\omega \\ \oplus \\ H^-_\omega \end{matrix} = \begin{matrix} \mathbb{C} \\ \oplus \\ \mathbb{C} \end{matrix}$

$$\iota_\omega = \begin{pmatrix} 1 \\ \omega \end{pmatrix} (1+\omega^2)^{-1/2}$$

Also have $W^\perp = \bigoplus_\omega W^\perp_\omega$ $\quad j_\omega = \begin{pmatrix} -\omega \\ 1 \end{pmatrix} (1+\omega^2)^{-1/2} : W^\perp_\omega \hookrightarrow H_\omega$

$j = \oplus j_\omega : W^\perp \hookrightarrow H^\pm$

Take $\begin{pmatrix} s & 0 \\ 0 & s^{-1} \end{pmatrix}$ on $H^\pm$ invert it $\begin{pmatrix} s^{-1} & 0 \\ 0 & s \end{pmatrix}$

and for $j^* \begin{pmatrix} s^{-1} & 0 \\ 0 & s \end{pmatrix} j = \bigoplus_\omega (1+\omega^2)^{-1/2} \begin{pmatrix} -\omega \\ 1 \end{pmatrix}^* \begin{pmatrix} s^{-1} & 0 \\ 0 & s \end{pmatrix} \begin{pmatrix} -\omega \\ 1 \end{pmatrix} (1+\omega^2)^{-1/2}$

$$= \bigoplus_\omega \frac{s^{-1}\omega^2 + s}{1+\omega^2} \quad \text{on } \oplus W^\perp_\omega$$

invert to get $\bigoplus \dfrac{s(1+\omega^2)}{s^2+\omega^2}$ on $W^\perp = H/W$

and then you have embed. $V/W \hookrightarrow H/W$,

this is like the choice of cyclic vector

The first part which corresponds to the poles depend on the spectrum + multiplicities. Check this. $\forall \omega$ you have $a_\omega \geq 0$, and I recall $W^\perp_\omega$ being equipped with $V/W \xrightarrow[\iota_\omega]{} W^\perp_\omega$ + $\iota^*_\omega \iota_\omega = a_\omega$

Your idea is that you can handle a general op.

$$\frac{1}{\omega - \lambda} d\mu(\omega)$$ Do for finite dims first.

The idea is that the poles give the ~~set~~ spectrum.

$$\sum \frac{a_\omega}{\omega - \lambda} + \lambda a_\infty \qquad a_\omega \geq 0. \qquad \text{Pick fn.}$$

the poles ~~and~~ give the spectrum + mult.

329 How to fit a Pick function into a similar framework to LC function using subquotients. Function is $\sum \frac{m_k}{\omega_k-\lambda} + m_\infty \lambda$ regularized by an inf. real const. Assumption that the imag. part converges. ~~Now form the Grass~~ Use the ω_k to form the operator, really the graph of the s.a. operator. You have

$$\cancel{H = \oplus\left(H^+_\omega \oplus H^-_\omega\right) \supset \oplus \begin{pmatrix}1\\ \omega\end{pmatrix}(1+\omega^2)^{-1/2}\mathbb{C}}$$

~~For each~~

$W \subset H \oplus H$ graph

$$W = \begin{pmatrix}1\\ A\end{pmatrix}\mathcal{D}_A \subset \begin{matrix}H\\ \oplus\\ H\end{matrix} \supset \begin{pmatrix}1\\ \lambda\end{pmatrix}H$$

$= 1$?

Your assumption is $\sum \frac{m_k}{\omega_k^2+1} + m_\infty < \infty$.

Fit into usual framework. Changing s.a. operator in imag. graph.

You have $W = \begin{pmatrix}1\\ A\end{pmatrix}H \qquad W^\perp = \begin{pmatrix}-A\\ 1\end{pmatrix}H$

What do you need to get a rational function of λ with poles on the spectrum.

$$\begin{tikzcd} H \arrow[r, "\binom{1}{\lambda}"] \arrow[rr, bend right, "\lambda - A"'] & \begin{matrix}H\\ \oplus\\ H\end{matrix} \arrow[r, "(-A\ \ 1)"] & H \end{tikzcd} \qquad \xi = (\sqrt{m_k})$$

$$\begin{tikzcd} \mathcal{D}_A \arrow[r, "\binom{E}{A}"] \arrow[dr, "A-\lambda"'] & \begin{matrix}H\\ \oplus\\ H\end{matrix} \arrow[r] \arrow[d, "(-\lambda\ \ 1)"] & \left(\xi, \frac{1}{A-\lambda}\xi\right) \\ & H & \end{tikzcd}$$

It looks like you are dealing with a sesquilinear form depending on the par. λ.

Passing it to a quotient space, line in quotient space!