202 Basic question is what sort of zeroes a scattering function has. What kind of zeroes.

You should try to eliminate $x+iy = in$

Suppose F scattering with simple zeroes at $\lambda = in$ $n = 1, 2, \ldots$.

want $f(z)$ analytic for $|z| < 1$
$|f(z)| < 1$

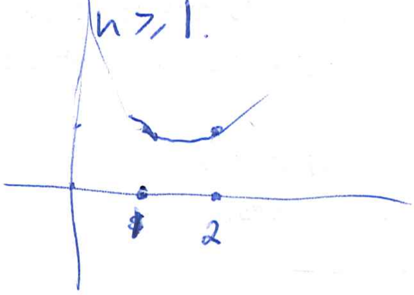
want $f(z)$ anal. & $|f(z)| = 1$ for $z \neq -1$, $|z| = 1$.

$-\log|f(z)| = -\mathrm{Re}\log f(z)$ harmonic with $\log r$ singularities at zeroes of f.

$f(z) = z - a$

$f(z)$ analytic in the disk

$d \log f(z) = \frac{f'(z)}{f(z)}$ meromorphic, residues $2\pi i n$ $n \geq 1$.



$$\frac{1}{\Gamma(s)} = e^{+\gamma s} s \prod_{n=1}^{\infty} \left(1 + \frac{s}{n}\right) e^{-s/n}$$

$$-\frac{\Gamma'(s)}{\Gamma(s)} = +\gamma + \frac{1}{s} + \sum_{n \geq 1} \left(\frac{1}{s+n} - \frac{1}{n}\right) \qquad \Gamma'(1) = -\gamma$$

You want to see an obstruction. There should be a connection between the zeroes and growth – Ahlfors

Idea from Bott–Chern: First thm. of dist. theory results from a transgression formula for c_1. There's a line bundle, holom., with herm. metric, holom. herm. conn., c_1 form type $(1,1)$, killed by d' and d'', arb. up to Img $d'd''$.

203 Consider $f(z)$ entire, consider trivial line bundle L, f is holom section, take flat metric on L, then curvature should be $\bar\partial\partial \log \|f\|^2$, why. In general suppose $s_1, \dots, s_n$ local frame of holom. sections $(s_i, s_j) = N_{ij}$

$N = N^*$ pos. ~~$ds = (d+A)s$~~ $Ds = (d+A)s$

Conditions are that D resp. metric, D holom section is type 0,1. A type 1,0.

$Ds = As$ ~~$\bar\partial s + As = 0$~~

$$d(s_i, s_j) = (Ds_i, s_j) + (s_i, Ds_j)$$

$$dN_{ij} = (A_{ik}s_k, s_j) + (s_i, A_{jl}s_l)$$

$$= \underbrace{\bar A_{ik} N_{kj}}_{\text{type } 0,1} + \underbrace{N_{il} A_{jl}}_{\text{type } 1,0}$$

$$\partial N_{ij} = N_{il} A_{jl} \qquad A = \partial N \cdot N^{-1} = \partial \log N$$

curvature is $d(\partial \log N) = \bar\partial\partial \log N$. So what do you learn.

$f(z)$ entire

$|\alpha| < 1$

$e^{2\pi i s} - \alpha$ ~~$z \in UHP$~~

is this a de B function?

~~$\frac{e^{2\pi i w} - \alpha}{e^{2\pi i s} - \bar\alpha} = \frac{z-\alpha}{z-\bar\alpha}$~~ $\qquad \frac{e^{2\pi i w} - \alpha}{e^{+2\pi i \bar w} - \alpha}$

$$\frac{E(w)}{E(\bar w)} = \frac{e^{2\pi i w} - \alpha}{e^{-2\pi i \bar w} - \bar\alpha} = e^{2\pi i w}\,\frac{e^{2\pi i w} - \alpha}{1 - \bar\alpha e^{2\pi i w}}.$$

204

$$\left(1-\frac{\lambda}{\alpha}\right)\left(1+\frac{\lambda}{\bar\alpha}\right) = 1+\left(\frac{1}{\bar\alpha}-\frac{1}{\alpha}\right)\lambda - \frac{\lambda^2}{|\alpha|^2}$$
$$= 1+\underbrace{\left(\frac{\alpha-\bar\alpha}{|\alpha|^2}\right)}_{\frac{2iy}{x^2+y^2}}\lambda - \frac{\lambda^2}{|\alpha|^2}$$

$\alpha = x+iy$
$\bar\alpha = x-iy$

scattering
$$1-\frac{1-\frac{\lambda}{\alpha}}{1-\frac{\lambda}{\bar\alpha}} = \frac{1-\frac{\lambda}{\bar\alpha}-1+\frac{\lambda}{\alpha}}{1-\frac{\lambda}{\bar\alpha}} = \left(\frac{1}{\alpha}-\frac{1}{\bar\alpha}\right)\lambda \Big/ 1-\frac{\lambda}{\bar\alpha}$$

So you can construct a de Branges function

~~Take ret mon~~ First thing to check. ~~Let~~ ~~the~~

Describe prob measures on $\mathbb{R}\cup\infty$ ~~corresp. to~~

$$\overset{\mathbb{R}}{\lambda} \longmapsto \frac{1-(-i\lambda)}{1+(-i\lambda)} = \frac{1+i\lambda}{1-i\lambda} = \frac{\lambda-i}{\lambda+i} = z$$

$$\frac{dz}{z} = d\lambda\left(\frac{1}{\lambda-i}-\frac{1}{\lambda+i}\right) = \frac{2i\,d\lambda}{\lambda^2+1}$$

$$d\theta = \frac{dz}{2\pi i z} = \frac{1}{\lambda^2+1}\frac{d\lambda}{\pi} \qquad \text{OKAY}$$

---

value distribution - you want to generalize the following ideas.

$$\text{no. of zeroes} = \frac{1}{2\pi i}\oint \frac{f'}{f}\,dz \qquad d\log f$$

Also want to use $\log f = \operatorname{Re}\log f + i\operatorname{Im}\log f$
So what ideas are around? What you would like to do is to treat the line bundle. Consider

$$\sum_{k=1}^{\infty}\frac{s(1+\omega_k^2)}{s^2+\omega_k^2}\;\frac{2}{1+\omega_k^2}$$

where ω_k is an ℓ^2 sequence

205 you want to consider the graph of this

$$\prod_{k=1}^{\infty}\left(1+\frac{s^2}{\omega_k^2}\right)$$

Commutes with $s \mapsto \bar{s}$), fixed by $s \to -s$

Review theory. Given p.u. $X \underset{b}{\overset{a}{\rightrightarrows}} Y$ $O(n)$ type

$W = \begin{pmatrix} a \\ b \end{pmatrix} X \subset \begin{matrix} Y \\ \oplus \\ Y \end{matrix}$ equipped with $\|y_1\|^2 - \|y_2\|^2$ $\quad \begin{pmatrix} a \\ b \end{pmatrix} x = \begin{pmatrix} 1 \\ z \end{pmatrix} y$ $\quad \mathrm{Ker}(az-b)$

$a^* y_1 = b^* y_2$

$W^0 = W \oplus \begin{matrix} \mathrm{Ker}\, a^* \\ \oplus \\ \mathrm{Ker}\, b^* \end{matrix}$ $\quad \Big| \; W \cap \begin{pmatrix} 1 \\ z \end{pmatrix} Y = 0$ $\quad \forall z$ incl. ∞.

$$L_z = W^0 \cap \begin{pmatrix} 1 \\ z \end{pmatrix} Y = \left\{ \begin{pmatrix} y \\ zy \end{pmatrix} \,\middle|\, a^* y = z b^* y \right\} \simeq \mathrm{Ker}(a^* - z b^*).$$

$$\begin{pmatrix} 1 \\ ba^* \end{pmatrix} Y = \begin{pmatrix} a \\ b \end{pmatrix} X \oplus \begin{pmatrix} \mathrm{Ker}\, a^* \\ 0 \end{pmatrix}$$

$$L_z \hookrightarrow W^0/W = \begin{matrix} \mathrm{Ker}\, a^* \\ \oplus \\ \mathrm{Ker}\, b^* \end{matrix}$$

$$K_s(r) = \int_0^\infty e^{-\frac{r}{2}(t+t^{-1})}\, t^s \frac{dt}{t}$$

$$\partial_r K_s(r) = -\frac{1}{2}\left(K_{s+1} + K_{s-1}\right)$$

$$s K_s(r) = \int_0^\infty e^{-\frac{r}{2}(t+t^{-1})}\, d(t^s)$$

$$= \int_0^\infty e^{-\frac{r}{2}(t+t^{-1})}\frac{r}{2}(1-t^{-2})\, t^s\, dt = \frac{r}{2}\left(K_{s+1} - K_{s-1}\right)$$

206

$$\partial_r K_s = -\frac{1}{2}(K_{s+1} + K_{s-1})$$

$$\frac{s}{r} K_s = \frac{1}{2}(K_{s+1} - K_{s-1})$$

$$\left(\partial_r + \frac{s}{r}\right) K_s = - K_{s-1}$$

$$\left(\partial_r - \frac{s}{r}\right) K_s = - K_{s+1}$$

$$\cancel{\left(\partial_r + \frac{s+\frac{1}{2}}{r}\right) K_{s+\frac{1}{2}} = - K_{s-\frac{1}{2}}}$$

asymptotics in s

$$f(t) = -\frac{r}{2}(t+t^{-1}) + s \log t$$

$$f'(t) = -\frac{r}{2}(1-t^{-2}) + \frac{s}{t} = 0$$

$$\frac{r}{2}(t-t^{-1}) = s$$

$$\frac{t-t^{-1}}{2} = \frac{s}{r}$$

$$\int_{-\infty}^{\infty} e^{-r(\cosh x) + sx} \, dx$$

$$\cosh \approx \sqrt{1+\sinh^2}$$

$$s = r \sinh x$$

$$e^{2x} - 1 = \frac{2s}{r} e^x$$

$$e^{2x} - \frac{2s}{r} e^x - 1 = 0$$

$$\underbrace{-r\sqrt{1+\frac{s^2}{r^2}}}_{\sqrt{r^2+s^2}} + s \sinh^{-1}\left(\frac{s}{r}\right)$$

$$e^x = \frac{s}{r} \pm \sqrt{\frac{s^2}{r^2}+1}$$

$$x = \log\left(\frac{s}{r} \pm \sqrt{\frac{s^2}{r^2}+1}\right)$$

$$-\sqrt{r^2+s^2} + s \log\left(\frac{s}{r} + \sqrt{\frac{s^2}{r^2}+1}\right)$$

dominant term is $s \log s$, $s\left(\log \frac{2}{r} - 1\right)$

207 Anyway, what happens?

Recall

$$\mathcal{O}(-1) \longrightarrow \mathcal{O}\otimes T \longrightarrow \mathcal{O}(1)$$

$$\begin{array}{ccccc}
\mathcal{O}(-1)\otimes Y & \longrightarrow & \mathcal{O}\otimes T\otimes Y & \longrightarrow & \mathcal{O}(1)\otimes Y \\
\| & & & & \| \\
\mathcal{E} & \hookrightarrow & \mathcal{O}\otimes T\otimes Y^{n+1} & \longrightarrow & \mathcal{E}^{\vee *} \\
 & & \cup & & \\
\mathcal{E}\cap(\mathcal{O}\otimes W^0) & \hookrightarrow & \mathcal{O}\otimes W^{0} & \longrightarrow & \check{\mathcal{E}} \\
 & & \downarrow & & \\
 & & \mathcal{O}\otimes(W^0/W)^2 & \longrightarrow & \check{\mathcal{L}}
\end{array}$$

rank 1, $d=-n-1$; $n+2$; $\vee\, n+1$; $\mathcal{E}\cap(\mathcal{O}\otimes W^0) = \mathcal{L}$; $r=1$ $d=n+1$

$$\mathcal{E}\cap \mathcal{O}^{(-1)}\otimes W = 0$$
$$\Updownarrow$$
$$\mathcal{E} + \mathcal{O}(-1)\otimes W^0 = \mathcal{O}\otimes T\otimes Y$$

Solicitors: John Deech, Ken Brooks, firm Linnell + Murphy, 12 King Edward, 248607

£6000 fees

You need a quaternionic version. Now $T \simeq \mathbb{C}^2$ equipped with σ $\sigma^2=-1$ and $\Lambda^2 T \simeq \mathbb{C}$

It would seem that [redacted] Y [redacted] should be only. To be able to tensor the basic [redacted]

$\mathcal{O}(-1) \longrightarrow \mathcal{O}\otimes T \longrightarrow \mathcal{O}(1)$ you need Y to have a σ s.t. $\sigma^2=-1$. which means $T\otimes Y$ will have a real structure

---

$$\cancel{\prod\left(1-\frac{z}{a_k}\right)}$$

$$\left|\frac{z-a_k}{1-\bar a_k z}\right| = 1 \quad \text{if } |a_k|=1.$$

When does $\prod_{k=1}\left(\frac{z-a_k}{1-\bar a_k z}\right)$ converge on $|z|<1$.

$z=0$ $\quad -a_k$

$$\frac{z-a}{1-\bar a z}\,\frac{-1/\bar a}{-1/\bar a} = \frac{1-\frac{z}{a}}{\bar z - \frac{1}{\bar a}}$$

for $|z|<1$. $\quad$ ~~$\prod(1-\bar a_k z)$ converges easily~~

No

$$\frac{\lambda - a}{\lambda - \bar{a}} \sim \frac{a}{\bar{a}} \qquad \text{as } |a| \to \infty.$$

$$1 - \frac{1 - \frac{\lambda}{a}}{1 - \frac{\lambda}{\bar{a}}} = \frac{\cancel{1} - \frac{\lambda}{\bar{a}} - \cancel{1} + \frac{\lambda}{a}}{1 - \frac{\lambda}{\bar{a}}} = \left(-\frac{1}{\bar{a}} + \frac{1}{a}\right) \frac{\lambda}{1 - \frac{\lambda}{\bar{a}}}$$

so need $-\frac{1}{\bar{a}} + \frac{1}{a}$ to be an ℓ^1 sequence

From the circle viewpoint you are looking at a sequence $a_n \to -1$.

$$\frac{z - a}{1 - \bar{a} z} e^{i\theta} \qquad \text{at } z=0 \text{ this } = -a e^{i\theta}$$

you want this $= |a|$

$$\frac{z - a}{1 - \bar{a} z} \frac{|a|}{-a} = \frac{\left(1 - \frac{z}{a}\right)|a|}{1 - \bar{a} z} \qquad \text{at } z = 0 \quad |a|.$$

approaching 1.

apply $-\log$

$$-\log|a| - \log\left(1 - \frac{z}{a}\right) + \log(1 - \bar{a} z)$$

$$1 - |a| \qquad \frac{z}{a} - \bar{a} z$$

$$z\left(\frac{1}{a} - \bar{a}\right) = z\left(\frac{1 - |a|^2}{a}\right) = z \frac{1 + |a|}{a}(1 - |a|)$$

$$1 - \frac{\left(1 - \frac{z}{a}\right)|a|}{1 - \bar{a} z} = \frac{1 - \bar{a} z - \left(1 - \frac{z}{a}\right)|a|}{1 - \bar{a} z}$$

$$= \frac{1 - |a| + z\left(-\bar{a} + \frac{|a|}{a}\right)}{1 - \bar{a} z} = 1 - |a| + z$$

209 $$1-\frac{\left(1-\frac{z}{a}\right)|a|}{1-\bar{a}z}$$

$$=\frac{1-\bar{a}z-|a|+z\frac{|a|}{a}}{1-\bar{a}z}=\frac{1-|a|+\overbrace{\left(-\bar{a}+\frac{|a|}{a}\right)}^{\frac{-|a|^2+|a|}{a}}z}{1-\bar{a}z}$$

$$=\frac{1-|a|+\frac{|a|}{a}(1-|a|)z}{1-\bar{a}z}=(1-|a|)\left(\frac{1+\frac{|a|}{a}z}{1-\bar{a}z}\right)$$

$$|1-\bar{a}z|\geq 1-|a||z|\geq 1-|z|$$

so ~~[illegible]~~ you need $1-|a_n|$ to be ℓ^1.

~~[illegible]~~ $$\left(\frac{z-a}{-a}\right)|a|=\left(1-\frac{z}{a}\right)|a|$$

$$1-\left(1-\frac{z}{a}\right)|a|=1-|a|+z\frac{|a|}{a}$$

basic idea $\frac{z-a}{1-\bar{a}z}$ has value $-a$ at $z=0$

you adjust phase so there's a chance of convergence

$\theta_a(z)=\frac{z-a}{1-\bar{a}z}\,\frac{|a|}{-a}$. Given ~~[illegible]~~ $F(z)$ analytic

for $|z|<1$, and bounded $|F(z)|\leq M=\sup_{|z|<1}|F(z)|$

~~arrange~~ let a_n be zeroes of F ~~arranged so~~

countable according to mult.

~~Schwarz lemma~~

set $F_n(z)=F(z)/\prod_{j=1}^{n}\theta_{a_j}(z)$

21 Know $|F_n(z)| \leq M$. Remove $a_k = 0$

$$F_n(0) = F(0) \Big/ \prod_{j=1}^{n} |a_j|$$ ~~But $|F_n(0)|$~~

~~is increasing~~ ~~But~~ we know that

$|F_n(z)|$ increasing in n, since

$$F_n(z) = F_{n-1}(z) / \theta_{a_n}(z)$$

$$|F_n(z)| \, \underbrace{|\theta_{a_n}(z)|}_{|a_n| < 1} = |F_{n-1}(z)|$$

$\therefore$ $|F_n(z)|$ increasing and bounded by M

$$\prod_{j=1}^{n} \overbrace{|a_n|}^{<1} = \frac{|F(0)|}{|F_n(0)|} \geq \frac{|F(0)|}{M}$$

Use UHP model.

$$\frac{\lambda - \alpha}{\lambda - \bar{\alpha}}$$

Construct the function $\prod \frac{\lambda - \alpha_k}{\lambda - \bar{\alpha}_k} e^{i\theta_k}$

to simplify assume $|\alpha_k| \longrightarrow \infty$.

212

Question. You are missing a point namely whether $\prod_{k=1}^{\infty} \frac{z-a_k}{1-\bar a_k z}\,\frac{|a_k|}{-a_k}$

is of modulus 1 a.e. on the boundary

$$1-\frac{z-a}{1-\bar a z}\,\frac{|a|}{-a} = \frac{-a+\overset{|a|^2}{a\bar a}\, z - |a|z + a|a|}{(1-\bar a z)(-a)}$$

$$= \frac{a(-1+|a|)+|a|z(|a|-1)}{(1-\bar a z)(-a)}$$

$$= (1-|a|)\underbrace{\frac{a+|a|z}{(1-\bar a z)\,a}}$$

$$\frac{a+|a|z}{a-|a|^2 z}$$

~~It's OK because of the routine ...~~

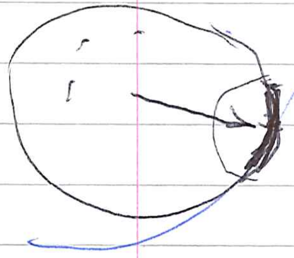
~~Simple enough. Yes~~

Assume $a_n \longrightarrow {}^{+}1$, and $1-|a_n|$ is ℓ^1

OKAY

You know that $\left|\frac{z-a}{1-\bar a z}\,\frac{|a|}{-a}\right| \longrightarrow 1$ uniformly as $|z|\to 1$

z outside ?



$$\frac{z-a}{1-\bar a z} = \frac{z-a}{z(z^{-1}-\bar a)}$$

$$1-\frac{1-\frac{\lambda}{\alpha}}{1-\frac{\lambda}{\bar\alpha}} = \left(\cancel{1}-\frac{\lambda}{\alpha}\cancel{-1}+\frac{\lambda}{\bar\alpha}\right)\frac{1}{1-\frac{\lambda}{\bar\alpha}}$$

$$= \left(\frac{1}{\bar\alpha}-\frac{1}{\alpha}\right)\frac{\lambda}{1-\frac{\lambda}{\bar\alpha}} \quad \text{bdd.}$$

211

$$z = \frac{1+i\lambda}{1-i\lambda} = \frac{1-(-i\lambda)}{1+(-i\lambda)}$$

$$-i\lambda = \frac{1-z}{1+z} \qquad \lambda = i\,\frac{1-z}{1+z}$$

$$\frac{\lambda-\alpha}{\lambda-\bar\alpha} = \frac{i\,\frac{1-z}{1+z}-\alpha}{i\,\frac{1-z}{1+z}-\bar\alpha} = \frac{i(1-z)-\alpha(1+z)}{i(1-z)-\bar\alpha(1+z)}$$

$$= \frac{(i-\alpha)-(i+\alpha)z}{(i-\bar\alpha)-(i+\bar\alpha)z} = \frac{i+\alpha}{i-\bar\alpha}\,\frac{\left(\frac{i-\alpha}{i+\alpha}\right)-z}{1-\frac{i+\bar\alpha}{i-\bar\alpha}z}$$

$$\beta = \frac{i-\alpha}{i+\alpha} = \frac{1+i\alpha}{1-i\alpha} \qquad \frac{i+\bar\alpha}{i-\bar\alpha} = \frac{1-i\bar\alpha}{1+i\bar\alpha} = \bar\beta$$

$$\boxed{\frac{\lambda-\alpha}{\lambda-\bar\alpha} = \frac{i+\alpha}{-i+\bar\alpha}\,\frac{-\beta+z}{1-\bar\beta z}}$$

in order to get convergence factor you had $\frac{|\beta|}{-\beta}$ phase of $-\beta$

$$\frac{i-\alpha}{i-\bar\alpha} = \frac{i+\alpha}{-i+\bar\alpha}(-\beta) \qquad \frac{|\beta|}{-\beta}\frac{\bar\beta}{\bar\beta} = \frac{|\beta|\bar\beta}{-|\beta|^2} = \frac{-\bar\beta}{|\beta|}$$

$$\alpha = i\,\frac{1-a}{1+a} \qquad \frac{1}{\alpha}-\frac{1}{\bar\alpha} = \frac{1}{i}\frac{1+a}{1-a}+\frac{1}{i}\frac{1+\bar a}{1-\bar a}$$

$|\alpha| \to \infty$

$\therefore a \to -1$

$$= \frac{1}{i}\,\frac{(1+a)(1-\bar a)+(1-a)(1+\bar a)}{|1-a|^2}$$

$$= \frac{1}{i}\,\frac{1+\cancel{a}-\cancel{\bar a}-|a|^2+1-\cancel{a}+\cancel{\bar a}-|a|^2}{}$$

$$= \frac{2}{i}\,\frac{1-|a|^2}{|1-a|^2}$$

~ 2

213 You have an analytic problem.

Suppose $a_n \to 1$; $1-|a_n|$ is ℓ^1. Is it possible to see that
$$F(z) = \prod \frac{z-a_n}{1-\bar{a}_n z} \frac{|a_n|}{-a_n},$$
which we know converges for $|z|<1$, also converges for $|z|=1$ $z \neq 1$.

$$1 - \frac{z-a}{1-\bar{a}z}\frac{|a|}{-a} = \frac{-a+|a|^2 z - |a|z + |a|a}{(1-\bar{a}z)(-a)}$$

$$= \frac{(-1+|a|)(a+|a|z)}{(1-\bar{a}z)(-a)} = (1-|a|)\frac{a+|a|z}{a(1-\bar{a}z)}$$

If $a_n \to 1$, then $\frac{a_n + |a_n| z}{a_n(1-\bar{a}_n z)} \longrightarrow \frac{1+z}{1-z}$

Invariantly given a disk and an int. point α there's a unique degree 1 rational function ~~with~~ f_α with $|f_\alpha| = 1$ on circle and $f_\alpha = 0$ at α, up to scalar of modulus !

$|z|<1$. $f(z) = \frac{z-\alpha}{1-\bar{\alpha}z}$

$\mathrm{Im}(\lambda) > 0$ $f(\lambda) = \frac{\lambda - \alpha}{\lambda - \bar{\alpha}}$

$\mathrm{Re}(s) > 0$ $f(s) = \frac{s-\alpha}{s+\bar{\alpha}}$

Now given a sequence α_k form

$\prod_{k=1}^{n} f_{\alpha_k}(s)$ normalized by requiring value at z_0 to be >0

214 basic question is then convergence of $\prod_k \rho_{\alpha_k}(z_0)$.

So in $\mathrm{Im}(\lambda) > 0$ picture $z_0 = 0$

$$\frac{\lambda - \alpha}{\lambda - \bar{\alpha}} \left(\text{phase of } \frac{i - \alpha}{i - \bar{\alpha}}\right)^{-1}$$

so you need the convergence of $\prod_k \left|\frac{i - \alpha_k}{i - \bar{\alpha}_k}\right|$

Should do $|z| < 1$ first.

$$\frac{z - a}{1 - \bar{a} z} \left(\text{phase of } -a\right)^{-1} = \frac{z - a}{1 - \bar{a} z} \frac{|a|}{-a}$$

want convergence of $\prod |a_k|$.

$$\left|\frac{i - \alpha}{i - \bar{\alpha}}\right| = \left|\frac{i - \alpha}{i + \alpha}\right| = \left|\frac{1 - \frac{i}{\alpha}}{1 + \frac{i}{\alpha}}\right|$$

$$= \left|1 - \frac{2i}{\alpha} + O\left(\frac{1}{\alpha^2}\right)\right|$$

assuming $|\alpha_k| \longrightarrow \infty$

$$= \left(1 - \frac{2i}{\alpha}\right)\left(1 + \frac{2i}{\alpha}\right)^{1/2}$$

$$= 1 + \left(\frac{i}{\alpha} - \frac{i}{\alpha}\right) + O\left(\frac{1}{\alpha^2}\right)$$

$\therefore$ get n.a.s.c.

215 Let's review. Suppose $F(\lambda)$ analytic in UHP & bdd by M.
Let α_k be the zeroes of F in the UHP, counted w mult. Let $\rho_\alpha(\lambda) = \frac{\lambda-\alpha}{\lambda-\bar\alpha}\left(\text{phase of } \frac{i-\alpha}{i-\bar\alpha}\right)^{-1}$ $\lambda \neq \alpha$

Assume $F(i) \neq 0$. Then

$$\frac{F(\lambda)}{\prod_{k=1}^{n} \rho_{\alpha_k}(\lambda)}$$ remov. sing. giving analy function $F_n(\lambda)$ in UHP, also bdd by M.

$$F(\lambda) = \prod_{k=1}^{n} \rho_{\alpha_k}(\lambda)\, F_n(\lambda)$$

$$|F(i)| \leq \prod_{k=1}^{n} \left|\frac{i-\alpha_k}{i-\bar\alpha_k}\right| M$$

$\boxed{\cancel{4{:}30}}$

$$\therefore \prod_{k=1}^{\infty} \left|\frac{i-\alpha_k}{i-\bar\alpha_k}\right| > 0.$$

$$\left|\frac{i-\alpha}{i-\bar\alpha}\right| = \left|\frac{1-\frac{i}{\alpha}}{1-\frac{i}{\bar\alpha}}\right| = \left(1-\frac{i}{\alpha}\right)\left(1+\frac{i}{\bar\alpha}+O\left(\frac{1}{|\alpha|^2}\right)\right)$$

$$= \left|1 + \left(\frac{i}{\bar\alpha}-\frac{i}{\alpha}\right) + O\left(\frac{1}{|\alpha|^2}\right)\right|$$

$\alpha = x+iy$

$$= 1 + 2\,\underbrace{\mathrm{Re}\left(\frac{i}{\bar\alpha}\right)}_{-\mathrm{Im}\left(\frac{1}{\bar\alpha}\right) = \mathrm{Im}\left(\frac{1}{\alpha}\right)} + O\,\frac{1}{|\alpha|^2}$$

$$\frac{1}{\alpha} = \frac{x-iy}{x^2+y^2}$$

$$-\mathrm{Im}\left(\frac{1}{\alpha}\right) = \frac{y}{x^2+y^2}$$

$$= 1 - 2\,\mathrm{Im}\left(\frac{1}{\alpha}\right) + O\left(\frac{1}{|\alpha|^2}\right)$$

Conversely suppose α_k given in UHP such that $|\alpha_k| \longrightarrow \infty$ and $\mathrm{Im}\left(\frac{1}{\alpha_k}\right)$ is summable.

You can form $$\prod_{k=1}^{\infty} \frac{1-\frac{\lambda}{\alpha_k}}{1-\frac{\lambda}{\bar\alpha_k}}$$

216 Convergence?

$$1-\frac{1-\frac{\lambda}{\alpha}}{1-\frac{\lambda}{\bar\alpha}} = \left(\cancel{1}-\frac{\lambda}{\bar\alpha}-\cancel{1}+\frac{\lambda}{\alpha}\right)\frac{1}{1-\frac{\lambda}{\bar\alpha}}$$

$$=\left(\frac{1}{\bar\alpha}-\frac{1}{\alpha}\right)\frac{\lambda}{1-\frac{\lambda}{\bar\alpha}}$$

Observe that if the roots outside of $|\lambda|<R$. then this merom. function analytic in $|\lambda|<R$.

Do the same in the circle.

Suppose $a_k \ni 1-|a_k|$ ℓ^1.

$$\prod \frac{z-a_k}{1-\bar a_k z}\frac{|a_k|}{(-a_k)}$$

$$1-\frac{z-a}{1-\bar a z}\frac{|a|}{-a} = \frac{-a+|a|^2 z - z|a|+a|a|}{(1-\bar a z)(-a)}$$

$$=\left(\frac{-1+|a|}{-a}\right)\frac{a+|a|z}{1-\bar a z} = (1-|a|)\frac{a+|a|z}{a(1-\bar a z)}$$

assume $a_k \longrightarrow -1$. then $\frac{a_k+|a_k|z}{a_k(1-\bar a_k z)} \longrightarrow \frac{-1+z}{-(1+z)} = \frac{1-z}{1+z}$

So if we stay away from $|1+z|\lesssim\varepsilon$ the product will be merom.

Start with a measure of the type $\sum \frac{2s}{s^2+\omega_k^2}$ you get a Hilbert sp^t operator + cyclic vector. Consider the partial unitary operator you get by "removing" the cyclic vector. Look at 1-param. family of contractions, from different bdry conditions. Can you say anything about the resulting

Return to partial unitary $W = \begin{pmatrix} a \\ b \end{pmatrix} X \subset \begin{matrix} Y \\ \oplus \\ Y \end{matrix}$

$W^0 = W \oplus \begin{pmatrix} \operatorname{Ker} a^* \\ \oplus \\ \operatorname{Ker} b^* \end{pmatrix}$ $\quad L_z = W^0 \cap \begin{pmatrix} 1 \\ z \end{pmatrix} Y = \left\{ \begin{pmatrix} y \\ zy \end{pmatrix} \middle| \ a^* y = z b^* y \right\}$

$\simeq \operatorname{Ker}(a^* - z b^*)$

$L_z \hookrightarrow W^0 / W$

$\begin{pmatrix} a \\ b \end{pmatrix} x + \begin{pmatrix} v^+ \\ v^- \end{pmatrix} = \begin{pmatrix} y \\ zy \end{pmatrix}$

$z(ax + v^+) = bx + v^-$ $\qquad (az - b)x = -zv^+ + v^-$

say $(S\, z) v^+ = v^-$

$S \simeq (1 - bb^*)(1 - zab^*)^{-1}$

You want [redacted] what. Explain what happens.

You have for each boundary condition i.e. contraction γ with ~~...~~ $\gamma a = b$, $b^* \gamma = a^*$

$\left(\gamma a a^* = b a^*, \quad b^* \gamma (1 - a a^*) = b^* \gamma - b^* \underbrace{\gamma a}_{b} a^* = \underbrace{b^* \gamma - a^*}_{= 0} \right.$

Such γ are described by $\begin{pmatrix} 1 \\ \gamma \end{pmatrix} Y = W + \begin{pmatrix} 1 \\ \tau \end{pmatrix} \operatorname{Ker}(a^*)$

where $\tau : \operatorname{Ker}(b^*) \longrightarrow \operatorname{Ker} a^*$ has $\|\tau\| \le 1$.

Pencil of contractions. Anyway, what next.

Puzzle. Compare $t(\lambda) = \sum_{k=1}^{\infty} \left(\frac{1}{\lambda - a_k} + \frac{1}{\lambda + a_k} \right) = 2\lambda \sum_{k=1}^{\infty} \frac{1}{\lambda^2 - a_k^2}$

with $\prod_{k=1}^{\infty} \left(1 - \frac{\lambda}{a_k}\right)\left(1 + \frac{\lambda}{a_k}\right) = \prod_{k=1}^{\infty} \left(1 - \frac{\lambda^2}{a_k^2}\right) = g(\lambda)$

one is the log der. of the other.

$$\frac{d}{d\lambda} \log \det(\lambda - A) = \operatorname{Tr}\left(\frac{1}{\lambda - A}\right)$$

What about C.T. ~~...~~ Examine the function $\lambda \to t(\lambda)$

$\mathbb{R} \longrightarrow \mathbb{R} \cup \infty. = S^1$

$x \longmapsto \frac{-i + x}{i + x}$ $\qquad \frac{g'(\lambda)}{g(\lambda)} = t(\lambda)$

218

$$\frac{1+it}{1-it} \qquad \frac{g+ig'}{g-ig'} \quad \text{scattering}$$

$\cdot 1+it$

$\cdot 1$ So $g+ig'$ is de B

$\cdot 1-it$

Begin with a partial unitary fin. diml rank 1.

$$\begin{pmatrix}1\\ba^*\end{pmatrix}Y = \begin{pmatrix}a\\b\end{pmatrix}X \oplus \begin{matrix}\mathrm{Ker}(a^*)\\ \oplus\\ 0\end{matrix}$$

$$L_z = \left[\begin{pmatrix}a\\b\end{pmatrix}X + \begin{pmatrix}\mathrm{Ker}(a^*)\\ \oplus \\ \mathrm{Ker}(b^*)\end{pmatrix}\right] \cap \begin{pmatrix}1\\z\end{pmatrix}Y$$

$$z(ax+v^+) = bx+v^-$$

$$(az-b)x = -zv^+ + v^- \qquad x = (1-zb^*a)^{-1}zb^*v^+$$

$$v^- = Szv^+ \qquad S = (1-bb^*)(1-zab^*)^{-1}$$

$$\begin{pmatrix}1\\ba^*\end{pmatrix}Y + \begin{pmatrix}1\\z\end{pmatrix}Y \qquad Y^2$$

$$Y = \begin{pmatrix}1\\ba^*\end{pmatrix}Y \xrightarrow{(z-1)} Y$$

$$z - ba^*$$

$V = \Gamma_{ba^*}$ off spec

Review mechanism

$$l_z \otimes Y \longrightarrow T \otimes Y \longrightarrow \check{l}_z \otimes Y$$

$$V \xrightarrow{S} \check{l}_z \otimes Y$$

So canon. map is $V/W \longleftarrow V \xrightarrow[\sim]{\text{off spec}} \check{l}_z \otimes Y$

219 So you need to understand the fin. dim'l case better. ~~[illegible]~~ You are still ~~[illegible]~~ missing how de B orders things.

Exercise. Find the reproducing kernel in the circle situation.

$$W^0 = \underbrace{\begin{pmatrix} a \\ b \end{pmatrix} X}_{W} \oplus \begin{matrix} \operatorname{Ker} a^* \\ \oplus \\ \operatorname{Ker} b^* \end{matrix} \subset T \otimes Y = \begin{matrix} Y \\ \oplus \\ Y \end{matrix}$$

$$L_z = W^0 \cap \begin{pmatrix} 1 \\ z \end{pmatrix} Y = \left\{ (x, v^+, v^-) \;\middle|\; \begin{matrix} z(ax + v^+) = bx + v^- \\ (za - b)x = -zv^+ + v^- \end{matrix} \right\}$$

Let $W \subset V \subset W^0$ middle dim. e.g. $\begin{pmatrix} 1 \\ \gamma \end{pmatrix} Y$

~~[illegible]~~ $\gamma a = b$ $\quad b^* \gamma a = 1$

$\gamma(1 - aa^*) = \gamma - ba^*$ Puzzle $\Gamma = \begin{pmatrix} 1 \\ \gamma \end{pmatrix} Y$. Then

$b^* \gamma (1 - aa^*) = b^* \gamma - a^*$. You need $\Gamma \subset W^0$

i.e. $a^* - b^* \gamma = 0$.

$$V/W \longleftarrow V \subset T \otimes Y \xrightarrow{(z \; -1) \otimes 1} \ell_z^\vee \otimes Y$$

claim isom off spec.

$\operatorname{spec} = \{ z \mid V \cap \begin{pmatrix} 1 \\ z \end{pmatrix} Y \neq 0 \}$. $\simeq \operatorname{Ker}(z - \gamma)$. better is to compute

$\begin{pmatrix} y \\ \gamma y \end{pmatrix} = \begin{pmatrix} y \\ zy \end{pmatrix}$ $\quad V \to \ell_z^\vee \otimes Y$ as $z - \gamma : Y \to Y$.

So you end up with $\quad Y \longrightarrow \ell_z \otimes V/W \simeq \operatorname{Ker} a^*$

$$y \longmapsto (1 - aa^*)(z - \gamma)^{-1} y$$

e.g. if you take $\gamma(1 - aa^*) = 0$. i.e. $\gamma = ba^*$, then you get $(1 - aa^*)(z - ba^*)^{-1} y$ rational function w poles inside the disk.

Maybe go back to scattering picture.

220 ~~What Next?~~

$$Y = aX + V^+ = V^- \oplus bX \quad \text{given}$$

$$\cdots \oplus u^{-1}V^- \oplus \underbrace{aX \oplus V^+}_{\parallel} \oplus uV^+ \oplus \cdots$$

$$\cdots \oplus u^{-1}V^- \oplus \overbrace{V^- \oplus bX} \oplus uV^+ \oplus \cdots$$

defines $H,\ u,\quad Y \overset{j}{\hookrightarrow} H$

You need to study γ on Y dilating to a unitary. $\quad j : Y \hookrightarrow H\ \circlearrowleft \qquad j^* u^n j = \begin{cases} \gamma^n & n \geq 0 \\ (\gamma^*)^{-n} & n \leq 0 \end{cases}$

$$H_+ = \overline{\sum_{n \geq 0} u^n j Y}$$

$$\Big\| \sum_{n \geq 0} u^n j y_n \Big\|^2 = \Big\| j y_0 + u \sum_{n \geq 1} u^{n-1} j y_n \Big\|^2$$

$$= \|y_0\|^2 + \Big(j y_0, \sum_{n \geq 1} u^n j y_n \Big) + \Big(\sum_{n \geq 1} u^n j y_n, j y_0 \Big) + \Big\| \sum_{n \geq 1} u^{n-1} j y_n \Big\|^2$$

$$= \|y_0\|^2 + \Big(y_0, \sum_{n \geq 1} \gamma^n y_n \Big) + \Big(\sum \gamma^n y_n, y_0 \Big) + \Big\| \sum_{n \geq 0} u^n j y_{n+1} \Big\|^2$$

$$= \Big\| y_0 + \sum_{n \geq 1} \gamma^n y_n \Big\|^2 - \Big\| \sum_{n \geq 1} \gamma^n y_n \Big\|^2 + \Big\| \sum_{n \geq 1} u^{n-1} j y_n \Big\|^2$$

$$\Big\| \sum_{n \geq 0} u^n j y_n \Big\|^2 - \Big\| \sum_{n \geq 0} \gamma^n y_n \Big\|^2 = \Big\| \sum_{n \geq 0} u^n j y_{n+1} \Big\|^2 - \Big\| \sum_{n \geq 0} \gamma^n y_{n+1} \Big\|^2$$

$$= \Big\| \sum_{n \geq 0} u^n j y_{n+1} \Big\|^2 -$$

$$u y = \gamma y$$

$$y = \gamma y +$$

$$y = j^* j y + (1 - j^* j) y$$

221

$$\Big\|\sum_{n\geq 0} u^n \gamma y_n\Big\|^2 = \Big\|\gamma y_0 + \sum_{n\geq 1} u^n \gamma y_n\Big\|^2$$

$$= \|y_0\|^2 + \Big(y_0, \sum_{n\geq 1}\gamma^n y_n\Big) + \Big(\sum_{n\geq 1}\gamma^n y_n, y_0\Big) + \Big\|\sum_{n\geq 1}\gamma^n y_n\Big\|^2 + \Big\|\sum_{n\geq 0} u^n \gamma y_{n+1}\Big\|^2 - \Big\|\gamma\sum_{n\geq 0}\gamma^n y_{n+1}\Big\|^2$$

$$\Big\|\sum_{n\geq 0} u^n \gamma y_n\Big\|^2 - \Big\|\sum_{n\geq 0} u^n \gamma y_{n+1}\Big\|^2 = \Big\|\sum_{n\geq 0}\gamma^n y_n\Big\|^2 - \Big\|\sum_{n\geq 0}\gamma^n y_{n+1}\Big\|^2 + \Big\|(1-\gamma^*\gamma)^{1/2}\sum \gamma^n y_{n+1}\Big\|^2$$

$$\Big\|\sum_{n\geq 0} u^n \gamma y_n\Big\|^2 = \Big\|\sum_{n\geq 0}\gamma^n y_n\Big\|^2 + \Big\|\sum_{n\geq 0} u^n\gamma y_{n+1}\Big\|^2 - \Big\|\gamma\sum_{n\geq 0}\gamma^n y_{n+1}\Big\|^2$$

$$= \Big\|\sum_{n\geq 0}\gamma^n y_{n+1}\Big\|^2 + \Big\|\sum u^n\gamma y_{n+2}\Big\|^2 - \Big\|\gamma\sum \gamma^n y_{n+2}\Big\|^2$$

$$= \Big\|\sum_{n\geq 0}\gamma^n y_n\Big\|^2 + \sum_{k\geq 1}^{N-1}\Big\|(1-\gamma^*\gamma)^{1/2}\sum\gamma^n y_{n+k}\Big\|$$

$$+ \Big\|\sum u^n\gamma y_{n+N}\Big\|^2 - \Big\|\gamma\sum_{n\geq 0}\gamma^n y_{n+N}\Big\|^2$$

222

$s = \frac{1-z}{1+z}$ maps $|z|<1$ to $Re(s)>0$

$\lambda = i\frac{1-z}{1+z}$ $\in$ UHP for $|z|<1$
$\in \mathbb{R} \cup \infty$ for $|z|=1.$

pole at $z=-1.$

$\lambda = i\frac{1+z}{1-z}$ pole at $z=1.$

$\lambda = i\frac{1+z/\zeta}{1-z/\zeta}$ pole at $z=\zeta=e^{i\theta}$

$$\mathrm{Im}\left(i\frac{1+z/\zeta}{1-z/\zeta}\right) = \mathrm{Re}\left(\frac{1+w}{1-w}\right)$$

$$= \frac{1}{2}\left(\frac{1+w}{1-w} + \frac{1+\bar{w}}{1-\bar{w}}\right) = \frac{1-|w|^2}{|1-w|^2}$$

$$f(z) = \int_0^{2\pi} i\frac{1+z\zeta^{-1}}{1-z\zeta^{-1}} \overset{\rho(\theta)\frac{d\theta}{2\pi}}{d\rho(\theta)}$$

suppose you want poles on

$$\mathrm{Im}\, f(z) = \int \frac{1-|z|^2}{|1-z\zeta^{-1}|^2}\, d\rho(\theta)$$

$i\frac{1+z\zeta^{-1}}{1-z\zeta^{-1}}$ has Res $= i\frac{1+\zeta\zeta^{-1}}{-\zeta^{-1}} = \frac{2\zeta}{i}$

Residue not preserved

223 $\left\| \sum_{n\geq 0} z^n (1-\gamma^*\gamma)^{1/2}\gamma^n x \right\|^2 = \sum_{n\geq 0} \|(1-\gamma^*\gamma)^{1/2}\gamma^n x\|^2$

$$= \sum_{n\geq 0} \left(\|\gamma^n x\|^2 - \|\gamma^{n+1}x\|^2\right) = \|y\|^2 - \lim_{n} \|\gamma^n x\|^2$$

shows that $x \longmapsto (1-\gamma^*\gamma)^{1/2}(1-z\gamma)^{-1}x$ is an isometric embedding $X \hookrightarrow H^2\left(S^1, \overline{(1-\gamma^*\gamma)^{1/2}X}\right)$ provided $\gamma^n x \longrightarrow 0$ for all x.

---

analytic functions on the UHP with $\mathrm{Im} > 0$

e.g. $\frac{1}{t-\lambda}$ $\quad t\in\mathbb{R}$ $\qquad \mathrm{Im}(\lambda) > 0$

$\Rightarrow \mathrm{Im}(-\lambda) < 0$

and $a\lambda$ $\quad a > 0.$ $\qquad \Rightarrow \mathrm{Im}(t-\lambda) < 0$

$\Rightarrow \mathrm{Im}\left(\frac{1}{t-\lambda}\right) > 0.$

Take a positive linear combination i.e. integrate wrt a $\overset{\geq 0}{\text{pos.}}$ measure $d\mu(t)$

$$\frac{1}{t-\lambda} = \frac{1}{t-x-iy} = \frac{t-x+iy}{(t-x)^2+y^2} \qquad \mathrm{Im}\left(\frac{1}{t-\lambda}\right) = \frac{\mathrm{Im}(\lambda)}{|t-\lambda|^2}$$

You need

$$\mathrm{Im}\int \frac{1}{t-\lambda}\,d\mu(t) = \int \frac{\mathrm{Im}(\lambda)}{|t-\lambda|^2}\,d\mu(t)$$

to converge i.e. $\int \frac{d\mu(t)}{1+t^2}\, d\mu(t) < \infty.$ (corresp to convergence at $\lambda = i$)

Assuming this you can additively renormalize by a real const.

$$f(\lambda) = \int\left(\frac{1}{t-\lambda} - \frac{t}{1+t^2}\right)d\mu(t) + \mu_\infty\lambda + \text{real const.}$$

What about S^1, $\lambda = i\frac{1-z}{1+z} \in$ $\mathrm{Im}(\lambda) > 0$ for $|z|<1$

$\mathrm{Im}(\lambda) = 0$ $\quad |z|=1$

so $\lambda = i\frac{1+z/\zeta}{1-z/\zeta}$ has $\mathrm{Im}(\lambda) > 0$ for $|z|<1$

$= 0$ $\quad |z|=1$

$\zeta = e^{i\theta}\in S^1$ $\quad$ pole at $z=\zeta$

224 $\operatorname{Im}\left(i\,\frac{1+z/\zeta}{1-z/\zeta}\right) = \operatorname{Re}\left(\frac{(1+z/\zeta)(1-\bar z/\bar\zeta)}{|1-z/\zeta|^2}\right) = \frac{1-|z|^2}{|1-z/\zeta|^2}$

~~[illegible]~~

Consider a partial unitary of $\mathcal{O}(n)$ type

Focus on doing orthogonal polys on the circle with the partial unitary obtained by removing the cyclic vector. This might work for a general measure

$d\log g = \frac{dg}{g}$ is a differential so it has intrinsic meaning but it isn't a function

Anyway, consider a probability measure $d\mu$ on S^1, let $H = L^2(S^1, d\mu)$, u = mult by z, v = cyclic vector. Suppose $d\mu$ supported on n points so that $\dim(Y) = n+1$

First get the algebraic structure straight. You have a partial unitary $Y = aX \oplus V^+ = V^- \oplus bX$ of type $\mathcal{O}(n)$: X dim n, Y dim $n+1$, $a - zb$ inj $\forall z \in \mathbb{C}_\infty$. Y has scalar prod. $\ni$ $\|ax\| = \|bx\|$ $\forall x$. Then get

$$0 \longrightarrow \mathcal{O}(-1)\otimes X \longrightarrow \mathcal{O}\otimes Y \longrightarrow \underset{\underset{\mathcal{O}(n)}{\wr\wr}}{\mathcal{E}} \longrightarrow 0$$

canon. res. of reg. sheaf $\mathcal{O}(n)$. We work [illegible] in the unit circle picture, so $z=0$, $z=\infty$ are distinguished points. ~~Get [illegible] structure~~ By considering vanishing order at 0 and at ∞ you get 2 complementary filtrations

Review. The ~~[illegible]~~ problem somehow is to correlate scalar product and the algebraic splitting which ~~[illegible]~~ reflects a $\mathbb{G}_m$ action. The algebraic splitting gives two ends, the 0 end & ∞ end. To get a spectrum we need to choose a line in Y. [illegible]

225 ~~Take ν a prob measure on $\mathbb{R}$~~

Various objects to relate.

1) Hilbert space, unity op, cyclic vector of $\| \| = 1$. (H, u, v)

2) partial unitary $Y = aX \oplus V^+ = V^- \oplus bX$ with no bound states and $V^\pm$ dim 1.

3) (H, γ) γ contraction $\ni$ $1-\gamma^*\gamma$ and $1-\gamma\gamma^*$ rank 1, no bound states

You want to start with 1) $Y = H$, take the cyclic vector line $\mathbb{C}v$ to be either V^+ or V^-.

Given $\cancel{Y}$, u, v you get a partial unitary by restricting u to $(\mathbb{C}v)^\perp$, ~~[illegible]~~

Idea, suppose you are interested in the $\det(z-\gamma)$ where γ ranges over contractions extending ba^{-1}, then

$$\delta \log \det(z-\gamma) = \mathrm{tr}\left((z-\gamma)^{-1}\delta\gamma\right)$$

where $\delta\gamma$ is a rank 1 operator, ~~[illegible]~~ more precisely a map $\mathrm{Ker}(a^*) \longrightarrow \mathrm{Ker}(b^*)$. Specifically it seems to involve $(1-aa^*)(z-\gamma)^{-1}(1-bb^*)$

$$\mathrm{tr}\left(A \circ |v\rangle\langle\phi|\right) = \mathrm{tr}\left(A|v\rangle\langle\phi|\right) = \langle\phi|A|v\rangle$$

---

go back to (Y, u, v), whence you have the filtration $\mathbb{C}v$, $\mathbb{C}v + \mathbb{C}u(v)$, $\mathbb{C}(v) + \mathbb{C}u(v) + \mathbb{C}u^2(v)$. It looks like you want aX = span of $v, u(v), \dots, u^{n-1}(v)$

You need to find exact relation between the cyclic vector and the partial unitary, which leads to the best understanding.

226 Consider a finite prob measure on S^1, $u =$ mult by z, ~~...~~ $Y = \mathbb{C} + \mathbb{C}z + \cdots + \mathbb{C}z^n$

It seems you have two choices. You have to take

$$aX = \mathbb{C} + \cdots + \mathbb{C}z^{n-1} \qquad z = ba^{-1}$$
$$bX = \mathbb{C}z + \cdots + \mathbb{C}z^n$$

~~...~~ No because this depends on cyclic vector. There's a specific problem here. Suppose ~~...~~ given a partial unitary of type $O(n)$. Then there are canonically defined lines $L_0, \ldots, L_n$ such that $ba^{-1}L_i = L_{i+1}$. L_0 consists of y vanishing to order n at $z = \infty$. Then $aX = L_0 + \cdots + L_{n-1}$ $bX = L_1 + \cdots + L_n$. There seems to be two choices for cyclic vector, maybe more if there's some sort of symmetry.

What can you do? If you ~~are~~ given the partial unitary, then you have two obvious cyclic lines namely V^+, V^-

Perhaps you need to mimic the expansion

$$S(z) = \begin{pmatrix} 1 & h_0 \\ \bar{h}_0 & 1 \end{pmatrix} \begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix} \cdots$$

for an analytic function in $|z| < 1$ bdd by 1.

~~...~~ Consider X with contraction γ

First take case $Y = aX \oplus V^+ = V^- \oplus bX$ $\qquad \gamma = ba^*$

~~cyclic vector~~ $H = \cdots \oplus u^{-1}V^- \oplus aX \oplus V^+ \oplus \cdots$

$$V^- \oplus bX$$

Does V^- generate Y wrt γ action.

$$\sum_{n \geq 0} \gamma^n V^- = \sum_{n \geq 0} \gamma^* u^n V^- = \sum_{n \in \mathbb{Z}} \gamma^* u^n V^-$$

227 Suppose $y \perp$ all $\gamma^n V^-$, i.e.

$$(1-\gamma\gamma^*)^{1/2} \gamma^{*n} y = 0 \quad \forall n \geq 0.$$

then you know that $\|y\|^2 = \|\gamma^{*n} y\|^2 \quad \forall n \geq 0.$

$$y \in \bigcap_{n \geq 0} \operatorname{Ker}(1-\gamma^n \gamma^{*n})$$

O

$$\underbrace{1-\gamma^n\gamma^{*n}}_{\geq 0} + \underbrace{\gamma^n(1-\gamma\gamma^*)\gamma^{*n}}_{\geq 0} = 1-\gamma^{n+1}\gamma^{n+1*}$$

Note $1-\gamma^n\gamma^{*n}$ increasing family of s.a. ops $0 \leq 1-\gamma^n\gamma^{*n} \leq 1$.

You want to assume $\|\gamma^{*n} y\| \to 0$ all y.

so it's clear.

Consider a contraction c on Y, $Y = aX \oplus V^+ = V^- \oplus bX$ the assoc. partial unitary. Compare the S-operators

$$(1-aa^*)(1-zba^*)^{-1} : V^- \longrightarrow V^+$$

$$(1-\gamma^*\gamma)^{1/2}(1-z\gamma)^{-1}(z-\gamma^*)(1-\gamma\gamma^*)^{-1/2} : V^- \longrightarrow V^+$$

need formulas

$$\operatorname{Ker}(1-\gamma^*\gamma) \oplus \overline{(1-\gamma^*\gamma)^{1/2} Y}$$

$$\|$$

$$\overline{(1-\gamma\gamma^*)^{1/2} Y} \oplus \operatorname{Ker}(1-\gamma\gamma^*)$$

$$\|y\|^2 = \underbrace{\|y\|^2 - \|\gamma y\|^2}_{\|(1-\gamma^*\gamma)y\|^2} + \|\gamma y\|^2$$

$$\|(1-\gamma^*\gamma)y\|^2 + \|(1-\gamma^*\gamma)\gamma y\|^2 + \|\gamma^2 y\|^2$$

$$\|y\|^2 = \|y\|^2 - \|\gamma^* y\|^2 + \|\gamma^* y\|^2$$

$$y \longmapsto \sum_{n\geq 0} z^n (1-g^*g)^{1/2} g^n y$$

$$y \longmapsto \sum z^{-n} (1-gg^*)^{1/2} (g^*)^n y$$

tomorrow - you want to look at $\delta \log\det c$ for variations in the boundary conditions

scattering formulas for a contraction c

$$g^* u^n g = \begin{matrix} c^n & n \geq 0 \\ (c^*)^{-n} & n \leq 0. \end{matrix}$$

to find ρ on circle ~~such that~~ $\mathcal{L}(Y)$ valued

$$\int (z^\ell y', \rho z^k y) = \left(y', \begin{pmatrix} c^{k-\ell} & k \geq \ell \\ c^{*\ell-k} & k \leq \ell \end{pmatrix} y\right)$$

$$\rho = \sum_{n\geq 0} c^n z^{-n} + \sum_{n>0} c^{*n} z^n$$

$$= \frac{1}{1-z^{-1}c} + \frac{c^*z}{1-zc^*} = \frac{1}{1-zc^*}\left(1-zc^* + c^*z(1-z^{-1}c)\right)\frac{1}{1-z^{-1}c}$$

$$\frac{1}{1-z^{-1}c}\left(1-zc^* + (1-z^{-1}c)c^*z\right)\frac{1}{1-zc^*}$$

$$\| $$

$$\frac{1}{1-z^{-1}c}(1-c\underset{|z|^2}{c^*})\frac{1}{1-zc^*}$$

$$= \frac{1}{1-zc^*}(1-c^*\underset{|z|^2}{\hat{}}c)\frac{1}{1-z^{-1}c}$$

$$\|$$

$$\left((1-c^*c)^{1/2}\frac{1}{1-z^{-1}c}\right)^*\left((1-c^*c)^{1/2}\frac{1}{1-z^{-1}c}\right)$$

Take a contraction c on Y, get

$$Y = \operatorname{Ker}(1-c^*c) \oplus \overline{(1-cc^*)^{1/2}Y}$$

$$= \operatorname{Ker}(1-cc^*) \oplus (1-cc^*)^{1/2}Y$$

$$\|y\|^2 = \|cy\|^2 + \|(1-c^*c)^{1/2}y\|^2$$

229 $\quad 1 = c^*c + \left((1-c^*c)^{1/2}\right)^2$

$$Y \xrightarrow{\begin{pmatrix} c \\ (1-c^*c)^{1/2} \end{pmatrix}} \begin{array}{c} Y \\ \oplus \\ \overline{(1-c^*c)^{1/2} Y} \end{array}$$

don't understand

Let c be a contraction on Y f.d.

$$Y = \mathrm{Ker}(1-c^*c) \oplus \overline{(1-c^*c)^{1/2}Y} = aX \oplus V^+ \qquad \pi^+ = 1-aa^*$$

$$= \mathrm{Ker}(1-cc^*) \oplus (1-cc^*)^{1/2}Y = bX \oplus V^- \qquad \pi^- = 1-bb^*$$

$c = ba^* + h \qquad h \in \mathcal{L}(V^+, V^-) \hookrightarrow \mathcal{L}(V)$

$c^* = ab^* + \underset{\pi^+ h^* \pi^-}{\overset{\shortparallel}{h^*}} \qquad h = \pi^- h \pi^+$

$$-\delta \log \det(1-z^{-1}c) = \mathrm{tr}_Y (1-z^{-1}c)^{-1} z^{-1}\delta c = \mathrm{tr}\, (1-z^{-1}c)^{-1} z^{-1} \pi^- \delta h\, \pi^+$$

$$= \mathrm{tr}\left[(1-z^{-1}c)^{-1} z^{-1}\delta h\right] \qquad \pi^-\delta h\, \pi^+ = \delta h$$

$c^*c = aa^* + h^*h$

$1-c^*c = 1-aa^*-h^*h = \pi^+(1-h^*h)\pi^+$

$(1-c^*c)^{1/2} = \pi^+(1-h^*h)^{1/2}\pi^+$

$$\mathrm{tr}\left[\frac{1}{1-z^{-1}c} z^{-1}\delta h\right] = \mathrm{tr}_{V^+}\left[\pi^+ \frac{1}{1-z^{-1}c} z^{-1}\pi^- \delta h\right]$$

$$= \mathrm{tr}_{V^+}\left[\underbrace{(1-h^*h)^{1/2}\pi^+ \frac{1}{1-z^{-1}c} z^{-1}\pi^-}_{S(z^{-1})z^{-1}} \delta h\, (1-h^*h)^{-1/2}\right]$$

$$-\delta \log \det(1-z^{-1}c)$$

$$= \mathrm{tr}_{V^+}\left[S(z^{-1}) z^{-1}\, \delta h (1-h^*h)^{-1/2}\right]$$

$c = ba^* + h$ is holomorphic in h

So $\det(z-c)$

so $-\delta \log \det(z-c) = \mathrm{tr}\left[\frac{1}{z-c}\, \underbrace{\delta c}_{\pi^-\delta h\,\pi^+}\right]$

$$= \mathrm{tr}_{V^+}\left[\pi^+ \frac{1}{z-c}\pi^- \delta h\right]$$ this is correct.

but you want to write it using the scattering $S(z^{-1}) = (1-c^*c)^{1/2} \frac{1}{1-z^{-1}c}\pi^- = (1-h^*h)^{1/2}\pi^+\frac{1}{1-z^{-1}c}\pi^-$ which is not holom. in h. But there should be some line bundle stuff happening.

Something here is very puzzling

Consider $R_c = (1-c^*c)^{1/2}(1-z^{-1}c)^{-1} : Y \hookrightarrow H^2(V^+)$

Not holom. in c. But $c_0 = ba^*$

$$R_c y = \pi^+(1-h^*h)^{1/2}\pi^+\left[\frac{1}{1-z^{-1}c_0}y + \frac{1}{1-z^{-1}c_0}z^{-1}h\frac{1}{1-z^{-1}c_0}y + \cdots\right.$$

Yesterday I ran into problem that the ~~matrix~~ ~~S-matrix~~ operator $(1-c^*c)^{1/2}(1-z^{-1}c)^{-1}$ which is something like a row of the operator $(1-z^{-1}c)$ is not holomorphic in c.

231 Review. You have $Y = aX \oplus V^+ = V^- \oplus bX$

and $c = ba^* + h$ $\quad h \in \mathcal{L}(V^+, V^-) \simeq$

$\pi^- \mathcal{L}(Y) \pi^+ \subset \mathcal{L}(Y)$. Think of V^+, V^- as dim 1

Then h is equiv. to a complex no of $|\ | < 1$, and c is holom. in h. $\quad -\delta \log \det(z-c) = tr\left(\frac{1}{z-c}\delta c\right)$

$$= tr\left(\frac{1}{z-c}\pi^- \delta h \pi^+\right) = tr\left(\pi^+ \frac{1}{z-c}\pi^- \delta h\right)$$

$$\frac{1}{z-c} = \frac{1}{z-c_0-h} = \frac{1}{z-c_0} + \frac{1}{z-c_0}\pi^- h \pi^+ \frac{1}{z-c_0} +$$

$$\pi^+ \frac{1}{z-c}\pi^- = \underbrace{\left(\pi^+ \frac{1}{z-c_0}\pi^-\right)}_{S(z^{-1})z^{-1}} + \left(\pi^+ \frac{1}{z-c_0}\pi^-\right) h \left(\pi^+ \frac{1}{z-c_0}\pi^-\right) + \cdots$$

$$= S(z^{-1})z^{-1} \frac{1}{1-hS(z^{-1})z^{-1}} = \frac{1}{1-S(z^{-1})z^{-1}h} S(z^{-1})z^{-1}$$

$$-\delta \log\det(z-c) = tr\left(\frac{1}{1-S(z^{-1})z^{-1}h} S(z^{-1})z^{-1}\delta h\right)$$

$$-\delta \log\det(z-c) = tr\left(\pi^+ \frac{1}{z-c}\pi^- \delta h\right)$$

$$= \sum_{n \geq 0} tr\left(\left(\pi^+ \frac{1}{z-c_0}\pi^- h\right)^n \pi^+ \frac{1}{z-c_0}\pi^{-1}\delta h\right)$$

$$-\log\det(z-c) = \sum_{n\geq 0} \frac{1}{n+1} tr\left(\pi^+ \frac{1}{z-c_0}\pi^- h\right)^{n+1}$$

$$= -\cancel{tr}\log\left(1 - \left(S(z^{-1})z^{-1}\right)h\right)$$

$$\boxed{\det(1-z^{-1}c) = 1 - S(z^{-1})z^{-1}h}$$ except there can be a z dependent factor ind of h. In fact it should be the denominator of $S(z^{-1})$. You need some conjectures!!

$$\frac{\det(1-z^{-1}c)}{\det(1-z^{-1}ba^*)} = 1 - S(z^{-1})z^{-1}h$$ quasi determinants again. You need to learn

tomorrow try for Grothendieck type completeness

e.g. given Y, c $\quad c^n \to 0$ on any y

get $\quad Y \xrightarrow{\mathcal{F}} H^2(S^1, V)$, conversely

also two sided version: $\quad Y \xrightarrow{\sim} H^+ \cap SH^-$

E version $\quad S = E/E^\#$ $\quad \theta$

Other idea Mumford explanation of KdV uses point ∞ on a curve and e^{tz}, where z is a uniformizing parameter at ∞.

Is it possible that dB theory & KP heirarchy - does this relate, provide insight to writing the appropriate hull of a curve

---

Anyway consider zilch

~~Consider the~~ Given Y, c contraction, form completion of $\bigoplus_{n\in\mathbb{Z}} z^n Y$ with scalar product

$$(z^k y_1, z^l y_2) = \left(y_1, \begin{pmatrix} c^{l-k} & l \geq k \\ (c^*)^{k-l} & l \leq k \end{pmatrix} y_2\right)$$

I need to know the picture

$$Y = \operatorname{Ker}(1-c^*c) \oplus V^+ = \operatorname{Ker}(1-cc^*) \oplus V^-$$

two things: completion of Y for $\|y\|^2 - \|cy\|^2 = (y, (1-c^*c)y)$

$$= \|(1-c^*c)^{1/2} y\|^2$$

$y \longrightarrow$

$$(1-cc^*)^{1/2} \frac{1}{1-z^{-1}c^*} y \longleftarrow y \longmapsto (1-c^*c)^{1/2} \frac{1}{1-zc} y$$

$c = bq^*$ $\quad c^*c = aq^*$ $\quad cc^* = bb^*$

$$S = (1-c^*c)^{1/2} \frac{1}{1-zc} (1-z^{-1}c^*)(1-cc^*)^{-1/2}$$

$$(1-aa^*) \frac{1}{1-zbq^*} (1-bb^*)$$

233 $\prod_k (z-a_k)$ to renormalize to make converge for $|z|<1$. Here $|a_k|<1$. and $|a_k| \nearrow 1$. Need convergence at $z=0$.

$$\prod_k \frac{(z-a_k)}{-a_k} = \prod_k \left(1-\frac{z}{a_k}\right)$$

scattering function

$$\prod \frac{z-a_k}{1-\bar{a}_k z} \frac{|a_k|}{-a_k}$$

$$1-\frac{z-a}{1-\bar{a}z}\frac{|a|}{-a} = \frac{-a(1-\bar{a}z)-(z-a)|a|}{(1-\bar{a}z)(-a)}$$

$$= \frac{a-|a|^2 z+|a|z-a|a|}{(1-\bar{a}z)a} = \frac{(1-|a|)(a+|a|z)}{(1-\bar{a}z)a}$$

$$\left| \qquad \right| \le \frac{(1-|a|)(1+|az|)}{(1-|a||z|)}$$

Anyway what to do? You want to understand when whether h

~~do what I need to~~

Amazing. degrees of freedom in a partial $\boxed{\text{moduli space}}$ for partial unitaries of type $O(n)$ has real dimension $2n$. ~~Real dim 2n~~ Why?

A partial unitary on Y is the same as a contraction c such that c^*c and cc^* are idemp. equiv. $c^*cc^* = c^*$, $cc^*c = c$

234 $\quad Y = \mathrm{Ker}(1 - \overset{*}{c}c^*) +$

If c^*c is idemp. then $\mathrm{Ker}(1-\overset{*}{c}c^*) = \mathrm{Im}(c^*c$

$$cc^*c = \underbrace{cc^*cc^*c} = c$$

So $Y = aX \oplus V^+ = bX \oplus V^- \qquad c = ba^*$

~~Look det$(1-zc)$~~ Look at char poly of c spectrum of c with mult. [illegible]

Idea is that the characteristic poly is the denominator of the resolvo

---

Hardy space for the disk. $\qquad z = e^{i\theta}$

$$f(z_0) = \oint \frac{z f(z)}{z - z_0}\frac{dz}{2\pi i z} = \int_{-\pi}^{\pi} \frac{1}{1 - \bar{z} z_0} f(z)\frac{d\theta}{2\pi}$$

$$= \int_{-\pi}^{\pi} \overline{\frac{1}{1-\bar{z}_0 z}} f(z) \frac{d\theta}{2\pi} = \left(\frac{1}{1-\bar{z}_0 z}, f\right)$$

$$e v_{z_0} = \frac{1}{1-\bar{z}_0 z} \qquad (ev_{z_0}, ev_{z_0}) = \frac{1}{1-|z_0|^2}$$

and so $\quad |f(z_0)| \leq \|ev_{z_0}\| \|f\| = \frac{1}{\sqrt{1-|z_0|^2}}\|f\|.$

~~So Suppose f analytic for $|z|<1$ and satisfies $|f(z)| \leq \frac{1}{\sqrt{1-|z|^2}}$ for $|z| < 1$.~~

Suppose $f = \sum_{n\geq 0} a_n e^{in\theta}$, where $\{a_n\}$ is in ℓ^2

Start with ~~$f = \sum a_n z^n$~~

$$|f(z_0)| = \left|\sum_{n\geq 0} a_n z_0^n\right| \leq \left(\sum |a_n|^2\right)^{1/2} \left(\sum_{n\geq 0} |z_0|^n\right)^{1/2}$$

235

Forget the analysis - find the formulas

Given $Y = aX \oplus V^+ = bX \oplus V^-$

you get $c = ba^*$ $\quad 1 - c^*c = 1 - ab^*ba^* = 1 - aa^*$

and assuming $c^n y \to 0$ all y you get

$$Y \hookrightarrow L^2_+(S^1, V^+)$$

$$y \longmapsto (1-c^*c)^{1/2}(1-zc)^{-1}y$$

$$\|$$

$$\Big\| \sum_{n \geq 0} z^n (1-c^*c)^{1/2} c^n y \Big\|^2$$

$$= \sum_{n\geq 0} \|c^n y\|^2 - \|c^{n+1}y\|^2 = \|y\|^2$$

Question: Can you characterize Y's arising in this way. Thus if you have $Y \subset L^2_+(S^1)$ when might it occur. Should probably have $j^* u^n j = c^n$

$$\Big(j_*(y), \sum_k z^k v\Big) = \sum_n \Big((1-c^*c)^{1/2} z^n c^n y, \sum_k z^k v\Big)$$

$$= \sum_k \Big((1-c^*c)^{1/2}c^k y, v\Big) = \Big(y, \sum_k (c^*)^k (1-c^*c)^{1/2} v\Big)$$

$$y \longmapsto u^\ell j_*(y) = \sum_k z^{k+\ell}(1-c^*c)^{1/2} c^k y \longmapsto \sum_k (c^*)^{k+\ell}(1-c^*c)c^k y$$
$$= (c^*)^\ell y.$$

236 $\quad Y = aX \oplus V^+ = bX \oplus V^-$

$c = ba^*$.

$$j : Y \longrightarrow H^2(S^1, V^+)$$

$$j(y) = \pi^+(1 - zc)^{-1} y = \sum_{n \geq 0} z^n \pi^+ c^n y$$

$$\|jy\|^2 = \sum_{n \geq 0} \|\pi^+ c^n y\|^2 = \sum_{n\geq 0} \underbrace{(c^n y, (1 - c^* c) c^n y)}_{\|c^n y\|^2 - \|c^{n+1} y\|^2} = \|y\|^2$$

$$\left(jy, \sum_{n\geq 0} z^n v_n\right) = \left(\sum_{n\geq 0} z^n \pi^+ c^n y, \sum_{n \geq 0} z^n v_n\right)$$

$$= \sum_{n\geq 0} (\pi^+ c^n y, v_n) = \sum_{n \geq 0} (c^n y, v_n)$$

$$= \sum (y, (c^*)^n v_n) = \left(y, \sum_{n\geq 0} (c^*)^n v_n\right)$$

$$\therefore \quad j^*\left(\sum_{n\geq 0} z^n v_n\right) = \sum_{n \geq 0} (c^*)^n v_n$$

Now take

$$\cancel{\left(j^*\left(\frac{1}{1-\bar z_0 z} v\right), y\right) = \left(\frac{1}{1 - \bar z_0 z} v^+\right)}$$

$$j^*\left(\frac{1}{1-\bar z_0 z} v^+\right) = j^*\left(\sum_{n \geq 0} z^n \underbrace{\bar z_0^{\,n} v^+}_{v_n}\right)$$

$$= \sum_{n\geq 0} (c^*)^n \bar z_0^{\,n} v^+ = \frac{1}{1 - \bar z_0 c^*} v^+$$

$$y \longmapsto \left\langle v^+ \middle| \frac{1}{1 - z_0 c} y \right\rangle \quad \text{at } z_0$$

$$\parallel$$

$$\left\langle \frac{1}{1 - \bar z_0 c^*} v^+ \middle| y \right\rangle$$

Point eval. is $\frac{1}{1 - \bar z_0 c^*} v^+$.

23 Now suppose you know $\underset{\|}{H^+} \cap z^n H^- = \mathbb{C}1 + \cdots + \mathbb{C}z^{n-1}$

$$Y \xrightarrow{\sim} H^+ \cap \overset{g_{\|}}{S} H^- \xrightarrow{\cdot g} \overline{gH^+ \cap pH^-}$$

$$y \longmapsto \langle v^+ | \frac{1}{1-zc} y \rangle$$

Reproducing kernel seems to be

$$\langle v^+ | \frac{1}{1-zc} \frac{1}{1-\bar{z}_0 c^*} v^+ \rangle$$

unit v.
v^+ spans V^+

~~Anyway things are better~~
~~to what should I do now.~~
~~to what How are they~~

Let c be a contraction on Y such that $c^n y \to 0$ $\forall y$. ~~Make c into~~

$$Y = \operatorname{Ker}(1-c^*c) \oplus \underbrace{\overline{(1-c^*c)^{1/2} Y}}_{\text{completion of } Y \text{ wrt } \|y\|^2 - \|cy\|^2}$$

$$j : Y \longrightarrow \overset{u}{H} \qquad j^* u^n j = c^n \qquad n \geq 0. \qquad u^* u = 1.$$

~~First thing you do is to form.~~ Try again

Dilate c to a unitary u.

$$j^* u^n j = c^n \qquad j u^{-n} y = (c^*)^{-n} \qquad n \geq 0$$

$$(u^k y, u^l y') = (y, u^{l-k} y')$$

$$= \left(y, \begin{cases} c^{l} & l \geq 0 \\ (c^*)^{(0-l)} & l \leq 0 \end{cases} y'\right)$$

$$= \int \frac{d\theta}{2\pi} z^{l} \left(y, \left(\sum_{n \geq 0} z^{-n} c^n + \sum_{n > 0} z^n (c^*)^n\right) y'\right)$$

$$\frac{1}{1-z^{-1}c} + \frac{zc^*}{1-zc^*} = \frac{1}{1-zc^*} \Big(\underbrace{1-zc^* + zc^*(1-z^{-1}c)}_{1-c^*c} \Big) \frac{1}{1-z^{-1}c}$$

$$\|f\|^2$$

$$f = \sum u^n y_n$$

238 $\qquad \xi = \sum u^n y_n \qquad \xi(z) : S^1 \longrightarrow Y$ Laurent poly

$$\|\xi\|_H^2 = \int \frac{d\theta}{2\pi} \left\| (1-c^*c)^{1/2} \frac{1}{1-z^{-1}c} \xi(z) \right\|_Y^2$$

$$= \int \frac{d\theta}{2\pi} \left\| (1-cc^*)^{1/2} \frac{1}{1-zc^*} \xi(z) \right\|_Y^2$$

$$Y \hookrightarrow H^2(S^1, V^+)$$

$$\pi^+ : Y \longrightarrow V^+ \qquad \|\pi^+ y\|^2 = \|y\|^2 - \|cy\|^2$$

$$(\pi^+)^* : V^+ \longrightarrow Y$$

$$(\pi^+ y_1, \pi^+ y_2) = (y_1, (\pi^+)^* \pi^+ y_2)$$

$$\parallel$$

$$(y_1, y_2) - (cy_1, cy_2) = (y_1, (1-c^*c) y_2)$$

$$\therefore \ (\pi^+)^* (\pi^+ y_2) = ((1-c^*c)) y_2 \quad ?$$

$\pi : Y \longrightarrow V$ $\qquad$ dense image $\qquad \|\pi y\|_Y^2 = \|y\|^2 - \|cy\|^2$

$$= \|(1-c^*c)^{1/2} y\|_Y^2$$

so (V, π) can be id w. $\overline{(1-c^*c)^{1/2} Y}$, $(1-c^*c)^{1/2}$

Then $(y_1, \pi^* \pi y)_Y = (\pi y_1, \pi y)_V = (y_1, (1-c^*c) y)_Y$

$$\pi^* \underbrace{\left((1-c^*c)^{1/2} y\right)}_{V} = (1-c^*c)^{1/2} \underbrace{(1-c^*c)^{1/2} y}_{V}.$$

Begin again. $\quad Y, c \quad$ define V^+

Wait: For scattering purposes you want $\pi : Y \to V$

$\|\pi y\|^2 = \|y\|^2 - \|cy\|^2 = \|(1-c^*c)^{1/2} y\|^2$. $\quad$ and $V = \overline{\pi Y}$

But then $\pi^* : V \longrightarrow Y$

$\pi^*\pi = (1-c^*c)$ etc. so you get $V \simeq \overline{(1-c^*c)^{1/2} Y}$

Anyway you get

$$Y \xhookrightarrow[j]{} H^2(S^1, V)$$
$$y \longmapsto \pi \frac{1}{1-zc} y$$

$$\left\| \pi \frac{1}{1-zc} y \right\|^2_{H^2(S^1,V)} = \sum_n \| \pi c^n y \|^2_V = \sum_n (c^n y, (1-c^*c)c^n y)_Y$$
$$= \|y\|^2 - \lim \|c^n y\|^2$$

$$\left(y, j^*\left(\sum_n z^n v_n\right)\right)_Y = \left(\sum_{n\geq 0} z^n \pi(c^n y), \sum_{n\geq 0} z^n v_n\right)_{H^2(S^1,V)}$$
$$= \sum_{n\geq 0} (\pi c^n y, v_n)_V = \left(y, \sum_{n\geq 0} (c^*)^n \pi^* v_n\right)$$

$$j^*\left(\sum_n z^n v_n\right) = \sum_n (c^*)^n \pi^* v_n$$ inclusion of

Clearly $j^*(z^k \xi) = (c^*)^k j^*(\xi)$ $k \geq 0$.

~~So what you want is the end~~

$$j(y) = \pi \left(\frac{1}{1-zc}\right) y$$
$$j^*(\xi(z)) = j^*\left(\sum z^p v_p\right)$$
$$= \sum_{p\geq 0} (c^*)^p \pi^* v_p = \sum_{p\geq 0} (c^*)^p \int \frac{d\theta}{2\pi} z^{-p} \xi(z)$$
$$= \int \frac{d\theta}{2\pi} \frac{1}{1-z^{-1}c^*} \pi^* \xi(z)$$

$$j j^* \xi = \pi\left(\frac{1}{1-zc}\right) \int \frac{d\theta_0}{2\pi} \left(\frac{1}{1-\bar{z}_0 c^*}\right) \pi^* \xi(z_0)$$

$$(jj^*\xi)(z) = \int \frac{d\theta_0}{2\pi} \underbrace{\left(\pi \frac{1}{1-zc}\frac{1}{1-\bar{z}_0 c^*}\pi^*\right)}_{\text{reproducing kernel}} \xi(z_0)$$

for the closed subspace Y

Note

$$\frac{1}{1-\bar{z}_0 c^*}\pi^*\pi\frac{1}{1-zc} = \frac{1}{1-\bar{z}_0 c^*}(1-c^*c)\frac{1}{1-zc}$$

~~Scattering operator~~ Measure on the circle is

$$\sum_{n\geq 0} z^n c^n + \sum_{n>0} \bar{z}_0^n (c^*)^n$$

$$= \frac{1}{1-zc} + \frac{\bar{z}_0 c^*}{1-\bar{z}_0 c^*} = \frac{1}{1-\bar{z}_0 c^*}\left(1-\bar{z}_0 c^* + \bar{z}_0 c^*(1-zc)\right)\frac{1}{1-zc}$$

$$= \frac{1}{1-\bar{z}_0 c^*}\left(1 - \bar{z}_0 z c^* c\right)\frac{1}{1-zc}$$

---

try again. $j : Y \hookrightarrow H^2(S^1, V)$ $\qquad \pi : Y \longrightarrow V$

$$j(y) = \sum_{n\geq 0} z^n \, \pi(c^n y) \qquad \overline{\pi Y} = V$$

$$\|\pi y\|^2 = \|y\|^2 - \|cy\|^2$$

$$\left(j(y), \sum_{n\geq 0} z^n v_n\right)_{H^2} = \sum_n (\pi c^n y, v_n)_V$$

$$= \sum_n (c^n y, v_n)_Y = \sum_n (y, (c^*)^n \pi^* v_n)_Y$$

$$j^*\left(\sum_{n\geq 0} z^n v_n\right) = \sum_{n\geq 0} (c^*)^n \pi^* v_n$$

$$= \int \frac{d\theta}{2\pi} \sum_{n\geq 0} z^{-n}(c^*)^n \sum_{n\geq 0} z^n \pi^* v_n$$

$$= \int \frac{dz}{2\pi i z} \frac{1}{1-z^{-1}c^*}\pi^*\xi(z)$$

241 What next?

$$j^*(\xi(z)) = \int \frac{dz}{2\pi i z} \frac{1}{1-zc^*} \pi^* \xi(z)$$

$$= \int \frac{dz}{2\pi i} \frac{1}{z-c^*} \pi^* \xi(z)$$

$$= \pi^* \xi(c^*)$$

Take $\xi(z) = \frac{1}{1-\bar{z}_0 z} v^+$ $\qquad v^+$ base for V^+

so that

$$\int \frac{dz}{2\pi i z} \overline{\frac{1}{1-\bar{z}_0 z}} f(z) = \int \frac{dz}{2\pi i} \frac{1}{z-z_0} f(z) = f(z_0).$$

$$\overline{\frac{1}{1-\bar{z}_0 z}} = \frac{1}{1-z_0 \bar{z}}$$

$$j^*\left(\frac{1}{1-\bar{z}_0 z} v^+\right) = \int \frac{dz}{2\pi i} \frac{1}{z-c^*} \frac{1}{1-\bar{z}_0 z} v^+ = \frac{1}{1-\bar{z}_0 c^*} v^+$$

$$\left(\frac{1}{1-\bar{z}_0 c^*} v^+, \pi^+\left(\frac{1}{1-zc}\right) y\right)$$

$\|v^+\| = 1.$ $\qquad v^+$

Start again Y, c contraction, say ba^*

$Y = aX \oplus V^+ = bX + V^-$ $\qquad a$ inc. of X

$X = \operatorname{Ker}(1-c^*c)$

$= \operatorname{Ker}(1-aa^*)$

$$Y \overset{j}{\hookrightarrow} H^2(S^1, V^+)$$

$$y \longmapsto \overbrace{(1-aa^*)}^{1-a^*c} \frac{1}{1-zba^*} y$$

$$\left\|(1-aa^*)\frac{1}{1-zba^*} y\right\|^2 = \sum_{n\geq 0} \|(1-aa^*) c^n y\|^2 = \sum \|c^n y\|^2 - \|c^{n+1} y\|^2 = \|y\|$$

$$(j(y), \xi(z)) = \Big(\sum_{n\geq 0} z^n (1-aa^*) c^n y, \sum_{n\geq 0} z^n v_n\Big)_{H^2}$$

$\xi(z) = \sum_{n\geq 0} z^n v_n \qquad v_n \in V^+$

$c = ba^*$

$c^* = ab^*$

$$= \sum_{n\geq 0} \big((1-aa^*) c^n y, v_n\big)_{V^+}$$

$$= \sum_{n\geq 0} \big(y, (c^*)^n v_n\big)_Y = \Big(y, \sum_{n\geq 0} (c^*)^n v_n\Big)_Y$$

$$\therefore\ j^* \xi(z) = \xi(c^*) = \int \frac{dz}{2\pi i z} \frac{1}{1-\bar{z}c^*} \xi(z)$$

Now take $\xi(z) = \frac{1}{1-\bar{z}_0 z} v^+$ $\qquad$ v^+ unit vector gen. V^+

$$j^* \frac{1}{1-\bar{z}_0 z} v^+ = \frac{1}{1-\bar{z}_0 c^*} v^+ \qquad 1-aa^* = |v^+\rangle\langle v^+|$$

$$jj^* \frac{1}{1-\bar{z}_0 z} v^+ = (1-aa^*) \frac{1}{1-zc} \frac{1}{1-\bar{z}_0 c^*} v^+$$

$$= v^+ \Big(v^+, \frac{1}{1-zc} \frac{1}{1-\bar{z}_0 c^*} v^+\Big)$$

suppose you use $\langle v^+| : V^+ \xrightarrow{\sim} \mathbb{C}$ systematically.

$$j(y) = \langle v^+| \frac{1}{1-zc} y$$

$$j^* \xi(z) = \xi(c^*) v^+$$

$$j^* \frac{1}{1-\bar{z}_0 z} = \frac{1}{1-\bar{z}_0 c^*} v^+$$

$$jj^* \frac{1}{1-\bar{z}_0 z} = \langle v^+| \frac{1}{1-zc} \frac{1}{1-\bar{z}_0 c^*} v^+$$

Conclusion is that the reproducing kernel for $jY \subset H^2$ is $\langle v^+| \frac{1}{1-zc} \frac{1}{1-\bar{z}_0 c^*} v^+ = \Big(\frac{1}{1-\bar{z}c^*} v^+, \frac{1}{1-\bar{z}_0 c^*} v^+\Big)$

243 Still missing something important.

You have a formula for $K(z, \bar{z}_0)$ depending on Y, c, yet you know it should depend only on the divisor, i.e. the ~~roo~~ eigenvalues of c. Moduli space of

Try to proceed from the roots.

Let $p(z) = \prod_{i=1}^{n} (z - a_i)$ $\quad q(z) = z^n p^{\#} = z^n \prod (z^{-1} - \bar{a}_i) = \prod_1^n (1 - \bar{a}_i z)$

$$\frac{p}{q} = \prod \left(\frac{z - a_i}{1 - \bar{a}_i z} \right) \qquad Y = H^2 \ominus pH^2 = H^2 \cap \frac{p}{q} H^2_-$$

can you find the reproducing kernel. Take $\frac{1}{1 - \bar{z}_0 z}$ and find its orthogonal projection onto Y.

$$\frac{1}{1 - \bar{z}_0 z} = jy + Sf \qquad y \in Y, \quad m \in SH^2, \quad m = Sf$$

Example. $\quad p = z - a \qquad q = 1 - z\bar{a}$

$$H^2 = jY \oplus SH^2$$ y has the form —

$\langle v^+ | \frac{1}{1 - zc} y$ $\quad$ ~~Rat~~ rational function with denom $p_n = \prod (z - a_i)$

You need to do an example. Interpolation was one of the ideas you were missing. Start with Y, $c = b a^*$. Then $Y \xhookrightarrow{j} H^2$

$$y \longmapsto \langle v^+ | \frac{1}{1 - zc} y \qquad \text{isometric embedding}$$

$j^* \xi(z) = \xi(c^*) v^+$ $\quad$ Why is $j^* j = 1$?

$$\int \frac{dz}{2\pi i} \frac{1}{z - c^*} |v^+\rangle \langle v^+| \frac{1}{1 - zc} y = \int \frac{dz}{2\pi i} \frac{1}{z - c^*} (1 - c^* c) \frac{1}{1 - zc} y$$

What about interpolation? Forget this, you want to under the Hilbert space

$$Y = H^2 \ominus \prod_{k=1}^{n} (z-a_k) H^2$$

To simplify suppose $a_1, \dots, a_k$ are distinct.

$$\frac{1}{1-\bar{z}_0 z} \quad \underbrace{\text{evaluator}}_{\text{in } H^2} \text{ for } z_0, \quad f^* \frac{1}{1-\bar{z}_0 z} = \frac{1}{1-\bar{z}_0 C^*} v^+$$

$$f f^* \frac{1}{1-\bar{z}_0 z} = \langle u^+ | \frac{1}{1-zC} \frac{1}{1-\bar{z}_0 C^*} v^+$$

Example $\quad Y = H^2 \ominus (z-a) H^2 \qquad \frac{1}{1-\bar{a} z}$

$$\left(\frac{1}{1-\bar{a} z}, f\right) = f(a) = 0 \quad \text{for } f \in (z-a) H^2$$

$$\left(\frac{1}{1-\bar{a}_i z}, f\right) = f(a_i) \qquad \text{So clearly } \frac{1}{1-\bar{a}_i z} \quad i = 1, \dots, n$$

are in Y and form a basis because of distinct roots. So what is the interpolation condition?

$$\frac{1}{1-\bar{a} z} \qquad \frac{1}{1-\bar{b} z}$$

You want to express $\frac{1}{1-\bar{z}_0 z}$ as a sum of an elt of Y and an elt of pH^2 $\quad \phi \in H^2$

$$f(z) \rightsquigarrow \frac{1}{1-\bar{z}_0 z} = \frac{c_1}{1-\bar{a}_1 z} + \frac{c_2}{1-\bar{a}_2 z} + \frac{(z-a_1)(z-a_2)\phi(z)}{(1-\bar{a}_1 z)(1-\bar{a}_2 z)}$$

More generally for any $f(z) \in H^2$. You have to solve for c_1, c_2

$$Y = H^2 \ominus (z-a_1)(z-a_2)H^2 = SH^2$$

$$S = \prod \frac{z-a_i}{1-\bar{a}_i z} = \frac{p(z)}{q(z)}$$

Given $f \in H^2$ you want to project f onto Y i.e. write $f = y + Sf'$ with $y \in Y$, $f' \in H^2$

~~.....~~ $SH^2 = \{ f \in H^2 \mid f(a_i) = 0 \quad i = 1, 2 \}$.

$$Y = \sum \mathbb{C} \frac{1}{1-\bar{a}_i z} = \frac{\mathbb{C}1 + \dots + \mathbb{C}z^{n-1}}{q(z)}$$

$$f(z) \cong \frac{c_1}{1-\bar{a}_1 z} + \frac{c_2}{1-\bar{a}_2 z} \quad \text{mod } (z-a_1)(z-a_2)H^2$$

$$f(a_1) = \frac{c_1}{1-a_1\bar{a}_1} + \frac{c_2}{1-a_1\bar{a}_2}$$

$$f(a_2) = \frac{c_1}{1-a_2\bar{a}_1} + \frac{c_2}{1-a_2\bar{a}_2}$$

apparently the matrix $\dfrac{1}{1-a_i\bar{a}_j}$ is invertible for $|a_i| < 1$ (Schur inverse?)

It's clear because $\xi_j = \dfrac{1}{1-z\bar{a}_j}$ are indep. vectors in H^2 and $(\xi_i, \xi_j) = \dfrac{1}{1-a_i\bar{a}_j}$

I want to do this explicitly for $f(z) = \dfrac{1}{1-\bar{w}z}$ $n = 1$.

$$f(z) \equiv \overbrace{\frac{c}{1-z\bar{a}}}^{y} + (z-a)(H^2)$$

$$f(a) = \frac{c}{1-|a|^2} \Rightarrow y = \frac{c}{1-z\bar{a}} = \frac{1}{1-z\bar{a}}(1-|a|^2) f(a)$$

so for $\dfrac{1}{1-z\bar{w}}$ get $\dfrac{1}{1-z\bar{a}}(1-|a|^2)\dfrac{1}{1-a\bar{w}}$

$p = \prod_1^n (z - a_i) \qquad q = \prod_1^n (1 - z\bar{a}_i)$

$Y \simeq H^2 \ominus \frac{p}{q} S \xrightarrow[\sim]{q\cdot} qH^2 \cap p\cancel{H}H^2_- = H^2 \cap z^n H^2_- =$ polys deg $< n$

$\|f\|_q^2 = \int \frac{d\theta}{2\pi} \left|\frac{f}{q}\right|^2$ defines norm in $L^2(S^1)$

in particular for f a poly in z.

$$\frac{f(a)}{q(a)} = \int \frac{d\theta}{2\pi} \overline{\frac{1}{1-\bar{a}z}} \frac{f(z)}{q(z)} \qquad \frac{d\theta}{2\pi} = \frac{dz}{2\pi i z} \times \frac{1}{1-az^{-1}} = \frac{dz}{2\pi i} \frac{1}{z-a}$$

$$f(a) = \int \frac{d\theta}{2\pi} \overline{\frac{q(a)\, q(z)}{1-\bar{a}z}} \frac{f(z)}{|q(z)|^2}$$

$\therefore f(a)$ is the inner product wrt $\| \, \|_q$ of f with $\frac{\bar{q}(a)\, q(z)}{1-\bar{a}z}$. Is this a poly in z of degree $< n$

This is true iff $a = a_i$ for some i equiv $p(a) = 0$. Examine case $a = 0$, where $p(0) \neq 0$.

How to modify

$$f(0) = \int \frac{d\theta}{2\pi} \overline{q(z)} \frac{f(z)}{|q(z)|^2} \qquad \text{note } q(0) = 1$$

You need somehow to replace $q(z)$ by a lower degree polynomial.

$$z^{-n} \frac{z^i}{\prod(1-z\bar{a}_i) \prod(1-z^{-1}a_i)}$$

$$\frac{f(z)}{z^{-n} p(z)\, q(z)}$$

247 $\quad p = \prod_1^n (z - a_i) \qquad a_1, \dots, a_n \in \mathbb{D}$

$q = \prod_1^n (1 - \bar{a}_i z) \qquad S = \frac{p}{q}$

$Y = H^2 \ominus S H^2 = H^2 \cap S H^2_- \xrightarrow{\cdot q} \overbrace{q H^2}^{H^2} \cap \overbrace{p H^2_-}^{z^n H^2_-} = \begin{array}{l}\text{poly}\\ \deg < n\end{array}$

$Y \simeq \mathbb{C} + \mathbb{C} z + \dots + \mathbb{C} z^{n-1} \qquad \|f\|_q^2 = \int \frac{d\theta}{2\pi} \left|\frac{f}{q}\right|^2$

You see reproducing kernel for Y, $K_a(z)$ poly of degree n

$f(a) = (K_a, f)_q = \int \frac{d\theta}{2\pi} \overline{K_a(z)} f(z) \frac{1}{|q(z)|^2}$

If we ask this to hold for all $f \in H^2$, then K_a is essentially the rep. kernel for H^2.

$$f(a) = \int \frac{dz}{2\pi i z} \overline{\frac{1}{1 - \bar{a} z}} f(z) = \int \frac{dz}{2\pi i} \underbrace{\frac{1}{z(1 - a\bar{z})}}_{\frac{1}{z-a}} f(z)$$

$$f(a) = \int \frac{d\theta}{2\pi} \overline{\frac{\overline{q(a)}\, q(z)}{1 - \bar{a} z}} \frac{f(z)}{q(z)\, \overline{q(z)}}$$

so $K_a(z) = \frac{\overline{q(a)}\, q(z)}{1 - \bar{a} z}$, But this is not always in Y. $K_a(z)$ is a poly of deg $< n$ $\iff$ a is one of $a_1, \dots, a_n$ equivalently $p(a) = \prod_1^n (a - a_i) = 0$

Note: $K_a(z) = \overline{q(a)} \underset{\underset{\mathcal{P}_{n-1}}{\cap}}{\frac{q(z)}{1 - \bar{a} z}} = \; ? \qquad$ symmetry

~~So we have a~~ Wait: The reproducing kernel probably should be hermitian symmetric if done properly. Yes

$$\frac{\overline{q(a)}\, q(z)}{1 - \bar{a} z} = \frac{\prod_1^n (1 - a_i \bar{a}) \prod_{i=1}^n (1 - \bar{a}_i z)}{1 - \bar{a} z}$$ is herm. symm. under $a \leftrightarrow z$

248 We would ^{like} $K_a(z) = \sum_{0\le k,l<n} t_{kl}\,\bar a^k z^l$

~~If so you have.~~ You know

$K_a(z) = \dfrac{\overline{g(a)}\,g(z)}{1-\bar a z}$ for $a = a_1,\dots,a_n$ $\forall z$
and $z = a_1,\dots,a_n$ $\forall a$

Take $n=1$.

$$\frac{(1-\bar a a_1)(1-\bar a_1 z)}{1-\bar a z}$$

If $a=a_1$, then $K_{a_1}(z) = 1-|a_1|^2$ $\forall z$
If $z=a_1$ then $K_a(a_1) = 1-|a_1|^2$ $\forall a$

Take $n=2$.

$$K_{\bar w}(z) = \frac{(1-a_1\bar w)(1-a_2\bar w)(1-\bar a_1 z)(1-\bar a_2 z)}{1-\bar w z}$$

If $a=a_1$ $K_{a_1}(z) = (1-|a_1|^2)(1-a_2\bar a_1)(1-\bar a_2 z)$

$a=a_2$ $K_{a_2}(z) = (1-a_1\bar a_2)(1-|a_2|^2)(1-\bar a_1 z)$

$z=a_1$ $K_w(a_1) = (1-a_2\bar w)(1-|\bar a_1|^2)(1-\bar a_2 a_1)$

$z=a_2$ $K_w(a_2) = (1-a_1\bar w)(1-\bar a_1 a_2)(1-|a_2|^2)$

You would like $K_w(z) = \alpha + \beta\bar w + \gamma z + \delta\,\bar w z$

$K_w(a_1) = \alpha+\beta\bar w + \gamma a_1 + \delta\bar w a_1 = (\alpha+\gamma a_1) + (\beta+\delta a_1)\bar w$
$K_w(a_2) = \alpha + \beta\bar w + \gamma a_2 + \delta a_2\bar w = (\alpha+\gamma a_2)+(\beta+\delta a_2)\bar w$

$\cancel{\beta+\delta a_1 = \ldots}$

$\alpha+\gamma a_1 = (1-|a_1|^2)(1-\bar a_2 a_1)$
$\alpha+\gamma a_2 = (1-|a_2|^2)(1-\bar a_1 a_2)$

249 $\gamma(a_1-a_2) = (1-|a_1|^2)(1-\bar{a}_2 a_1) - (1-|a_2|^2)(1-\bar{a}_1 a_2)$

$$= \cancel{1} - \bar{a}_2 a_1 - a_1\bar{a}_1 + a_1^2 \bar{a}_1 \bar{a}_2$$
$$- \cancel{1} + \bar{a}_1 a_2 + a_2 \bar{a}_2 - a_2^2 \bar{a}_2 \bar{a}_1$$

$$= \bar{a}_2(-a_1+a_2) + \bar{a}_1(-a_1+a_2) + \bar{a}_1\bar{a}_2 \underbrace{(a_1^2-a_2^2)}_{(a_1-a_2)(a_1+a_2)}$$

$$= (a_1-a_2)\left(-\bar{a}_1 - \bar{a}_2 + (a_1+a_2)\bar{a}_1\bar{a}_2\right)$$

$$\boxed{\gamma = -\bar{a}_1 - \bar{a}_2 + (a_1+a_2)\bar{a}_1\bar{a}_2}$$

$$\alpha = -\gamma a_1 + \overset{①}{1-|a_1|^2} \overset{②}{-\bar{a}_2 a_1} + \overset{③}{|a_1|^2 \bar{a}_2 a_1}$$

$$-\gamma a_1 = \underset{①}{a_1\bar{a}_1} + \underset{②}{\bar{a}_2 a_1} - \underset{③}{(a_1+a_2)\bar{a}_1\bar{a}_2 a_1}$$

$$\boxed{\alpha = |a_1|^2|a_2|^2}$$

$$a_1(\beta+\delta a_1) = -a_2(1-|a_1|^2)(1-\bar{a}_2 a_1)$$
$$a_2(\beta+\delta a_2) = -a_1(1-\bar{a}_1 a_2)(1-|a_2|^2)$$

$$(a_1-a_2)\beta = -a_1 a_2\begin{pmatrix} \cancel{1} - |a_1|^2 - \bar{a}_2 a_1 + |a_1|^2 a_1 a_2 \\ \cancel{-1} + |a_2|^2 + \bar{a}_1 a_2 - \bar{a}_1 a_2 |a_2|^2 \end{pmatrix}$$

$$=$$