121 $\quad \Gamma_{\varepsilon A^*} \oplus \begin{pmatrix} 0 \\ \mathrm{Ker}\,\varepsilon^* \end{pmatrix} = W^\circ$

So what can we do??

Go back to $\left(\begin{array}{c|c} \alpha & \gamma \\ \hline \gamma^* & \beta \end{array}\right)$.

What to do: You want to calculate the image of $\mathrm{Ker}(\omega\varepsilon^* - A^*)$

$$W = \begin{pmatrix} \varepsilon \\ A \end{pmatrix} X \subset W^\circ \subset \begin{matrix} Y \\ \oplus \\ Y \end{matrix} \supset \begin{pmatrix} 1 \\ \omega \end{pmatrix} Y$$

$$\mathcal{L}_\omega = W^\circ \cap \begin{pmatrix} 1 \\ \omega \end{pmatrix} Y = \begin{pmatrix} 1 \\ \omega \end{pmatrix} \mathrm{Ker}(\omega\varepsilon^* - A^*).$$

y_1 $\quad A^* y_1$

y_1 $\quad \varepsilon^* \omega y_1$

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{L}_\omega \arrow[r] \arrow[dr] & W^\circ \arrow[r, "(\omega\ -1)"] \arrow[d] & Y \arrow[r] & 0 \\ & & W^\circ/W & & \end{tikzcd}$$

You want to calculate the image of $\mathcal{L}_\omega$ in W°/W. The answer should basically be the graph of $j^* \frac{1}{\omega - \tilde{A}} j$, an operator on $\mathrm{Ker}(\varepsilon^*)$. In order for this to be meaningful, you need to identify W°/W with $\begin{matrix} \mathrm{Ker}\,\varepsilon^* \\ \oplus \\ \mathrm{Ker}\,\varepsilon^* \end{matrix}$ somehow. Now you definitely have $\begin{matrix} 0 \\ \oplus \\ \mathrm{Ker}\,\varepsilon^* \end{matrix}$ as a subspace of W° complementary to W

Is $W \cap \begin{matrix} 0 \\ \oplus \\ \mathrm{Ker}\,\varepsilon^* \end{matrix} = 0$ $\quad \begin{pmatrix} \varepsilon x \\ Ax \end{pmatrix} = \begin{pmatrix} 0 \\ z \end{pmatrix} \Rightarrow x = 0.$

embed $\quad \begin{pmatrix} 1 \\ \varepsilon A^* \end{pmatrix} \mathrm{Ker}\,\varepsilon^* \subset W^\circ$

Is $W \oplus \begin{pmatrix} 1 \\ \varepsilon A^* \end{pmatrix} \mathrm{Ker}\,\varepsilon^* \oplus \begin{pmatrix} 0 \\ 1 \end{pmatrix} \mathrm{Ker}\,\varepsilon^* = W^\circ$? YES

$$\underbrace{\boxed{\begin{pmatrix} \varepsilon x \\ Ax \end{pmatrix} + \begin{pmatrix} z \\ \varepsilon A^* z \end{pmatrix}}}_{\Gamma_{\tilde{A}}} + \begin{pmatrix} 0 \\ z' \end{pmatrix} = 0 \quad \Longrightarrow \quad \begin{matrix} \varepsilon x + z = 0 \\ x + \cancel{\varepsilon^* z} = 0 \end{matrix}$$

$$y = \varepsilon\varepsilon^* y + (1-\varepsilon\varepsilon^*) y$$

$$\tilde{A} y = A\varepsilon^* y + \varepsilon A^*(1-\varepsilon\varepsilon^*) y$$

$$= \left(A\varepsilon^* + \varepsilon A^* - \frac{\varepsilon A^*\varepsilon\varepsilon^*}{\varepsilon\varepsilon^* A\varepsilon^*} \right) y$$

$$\tilde{A} = A\varepsilon^* + \varepsilon A^*(1-\varepsilon\varepsilon^*)$$

$$= (1-\varepsilon\varepsilon^*)A\varepsilon^* + \varepsilon A^*$$

So now go through the process. You need to look at $W^0 + \binom{1}{\omega} Y = Y \oplus Y$, you have

$$\Gamma_{\tilde{A}} \oplus \binom{1}{\omega} Y = Y \oplus Y \quad \text{for } \omega \notin \mathrm{sp}\, \tilde{A}$$

$$\Gamma_{\tilde{A}} \oplus \binom{0}{1} \operatorname{Ker} \varepsilon^* = W^0 \qquad Z = \operatorname{Ker} \varepsilon^*$$

$$\underbrace{W \oplus \binom{1}{\varepsilon A^*} Z}_{\Gamma_{\tilde{A}} = \binom{1}{\tilde{A}} Y} \oplus \binom{0}{1} Z = W^0$$

To ~~also~~ solve $\binom{\varepsilon x}{Ax} + \binom{1}{\varepsilon A^*} z_1 + \binom{0}{1} z_2 \in \binom{1}{\omega} Y$

better $\binom{y}{\tilde{A} y} + \binom{0}{1} z_2 \in \binom{1}{\omega} Y$

$$\omega y = \tilde{A} y + z_2 \qquad y = (\omega - \tilde{A})^{-1} z_2$$

Then you ~~can~~ find z_1 by applying π

$$z_1 = \pi(\omega - \tilde{A})^{-1} z_2 = j^*(\omega - \tilde{A})^{-1} j z_2$$

123 $\begin{pmatrix} \varepsilon x \\ Ax \end{pmatrix} + \begin{pmatrix} 1 \\ \varepsilon A^* \end{pmatrix} j z_1 + \begin{pmatrix} 0 \\ 1 \end{pmatrix} j z_2 = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$

$x = \varepsilon^* y_1 \qquad z_1 = j j^* y_1$

$y_2 - Ax - \varepsilon A^* j z_1 = y_2 - A\varepsilon^* y_1 - \varepsilon A^* j^* y_1$

$\begin{pmatrix} \varepsilon x \\ Ax \end{pmatrix} + \begin{pmatrix} 1 \\ \varepsilon A^* \end{pmatrix} z_1 + \begin{pmatrix} 0 \\ 1 \end{pmatrix} z_2 = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$

$\varepsilon x + z_1 = y_1 \implies x = \varepsilon^* y_1 \qquad z_1 = j^* y_1$

$Ax + \varepsilon A^* z_1 + z_2 \overset{?}{=} y_2$

$\underbrace{\varepsilon^* A x}_{A^*(\varepsilon x + z_1)} + A^* z_1 + \overset{?}{=} \varepsilon^* y_2 = A^* y_1$

$z_2 = j^* y_2 - j^* A \varepsilon^* y_1$

$W^0 = \underbrace{\begin{pmatrix} \varepsilon \\ A \end{pmatrix} X \oplus \begin{pmatrix} 1 \\ \varepsilon A^* \end{pmatrix} Z}_{\begin{pmatrix} 1 \\ \tilde{A} \end{pmatrix} Y} \oplus \begin{pmatrix} 0 \\ 1 \end{pmatrix} Z$

$W^0 \xrightarrow{\begin{pmatrix} y^* \\ j^* \end{pmatrix}} \begin{matrix} Z \\ + \\ Z \end{matrix}$

$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{matrix} \varepsilon x + z_1 \\ Ax + \varepsilon A^* z_1 + z_2 \end{matrix}$

$x = \varepsilon^* y_1 \quad z_1 = j^* y_1$

$z_2 = \cancel{j^*(y_2 - Ax)}.$

$y_2 - \tilde{A} y_1$

$z_2 = y_2 - Ax - \varepsilon A^* z_1$

$= y_2 - A\varepsilon^* y_1 - \varepsilon A^* j j^* y_1$

$z_2 = y_2 - \tilde{A} y_1$

 Not yet clear.

$$W^0 = \Gamma_{\tilde A} \oplus \begin{pmatrix}0\\1\end{pmatrix} Z$$

$$\Gamma_{\tilde A} \oplus \begin{pmatrix}1\\ \omega\end{pmatrix} Y = \begin{matrix}Y\\ \oplus\\ Y\end{matrix}$$

$$W^0 \cap \begin{pmatrix}1\\ \omega\end{pmatrix} Y \ni \begin{pmatrix} y_1 \\ \tilde A y_1 + z_2\end{pmatrix}$$ where $\omega y_1 = \tilde A y_1 + z_2$

$y_1 = (\omega - \tilde A)^{-1} z_2$

$$y_1 = \varepsilon x + z_1$$
$$\tilde A y_1 = Ax + \varepsilon A^* z_1$$

You need to explain the map $W^0 \longrightarrow \begin{matrix}Z\\ \oplus\\ Z\end{matrix}$

it is $\begin{pmatrix} y_1 \\ \tilde A y_1 + z_2\end{pmatrix} \longleftrightarrow \begin{matrix} z_1 = j^* y_1 \\ z_2 = y_2 - \tilde A y_1\end{matrix}$

---

get $\begin{pmatrix} (\omega - \tilde A)^{-1} z_2 \\ z_2 \end{pmatrix} \in \begin{matrix}Z\\ \oplus\\ Z\end{matrix}$

However it might be easier to show that

$$\begin{pmatrix} 1 \\ \omega - C^*(\omega - B)^{-1} C\end{pmatrix} Z$$ is the response

where $\tilde A = \begin{pmatrix} B & C \\ C^* & 0 \end{pmatrix}$

M2 5 Review. $T=\mathbb{R}^2$ with $(\ )^*\begin{pmatrix}0&1\\-1&0\end{pmatrix}(\ )$

Y Hilbert space $T\otimes Y = \begin{matrix}Y\\ \oplus\\ Y\end{matrix}$ $\begin{pmatrix}y_1\\y_2\end{pmatrix}^*\begin{pmatrix}0&1\\-1&0\end{pmatrix}\begin{pmatrix}y_1'\\y_2'\end{pmatrix} = y_1^*y_2' - y_2^*y_1'$

$l_\omega = \binom{1}{\omega}\mathbb{R} \in T$. Given $W\subset T\otimes Y$ isotropic and no bound states $W\cap\binom{1}{\omega}Y=0$ $\forall\omega\in\mathbb{R}\cup\infty$.

$W=\binom{\varepsilon}{A}X$ $\underbrace{\quad}_{\mathrm{Ker}(\varepsilon)\cap\mathrm{Ker}(A)=0}$, $W^\circ = \left\{\binom{y_1}{y_2} \,\middle|\, \cancel{y_1^*Ax = y_2^*\varepsilon x}\ \forall x.\right.$

$(Ax)^*y_1 = (\varepsilon x)^*y_2$

$x^*(A^*y_1-\varepsilon^*y_2)=0$

$W^\circ = \left\{\binom{y_1}{y_2} \,\middle|\, A^*y_1=\varepsilon^*y_2\right\}$.

$W\subset W^\circ \iff A^*\varepsilon=\varepsilon^*A$ $\quad W\cap\binom{1}{\omega}Y = \binom{1}{\omega}\mathrm{Ker}(\omega\varepsilon-A)$

no bdd states means $\mathrm{Ker}(\omega\varepsilon-A)=0$ $\forall\omega\in\mathbb{R}\cup\infty$

$\mathcal{L}_\omega = W^\circ\cap\binom{1}{\omega}Y \hookrightarrow W^\circ/W$. To calculate W°/W and the image of $\mathcal{L}_\omega$. Adjust $\|\varepsilon x\|=\|x\|$

or $\varepsilon^*\varepsilon=1$, $Z=\mathrm{Ker}(\varepsilon^*)$. $Y = \varepsilon X\oplus_j Z$

$$W^\circ = W\oplus\binom{1}{\varepsilon A^*}Z\oplus\binom{0}{1}Z$$

$$y_1 = \varepsilon x + z_1 \iff x=\varepsilon^*y_1,\quad z_1=j^*y_1$$

$$y_2 = Ax+\varepsilon A^*_j z_1 + z_2 \iff z_2 = \underbrace{y_2 - A\varepsilon^*y_1 - \varepsilon A^*_j j^* y_1}$$

To get existence you must check ε^* kills

$$\varepsilon^*y_2 - \underbrace{\varepsilon^*A}_{A^*\varepsilon}\varepsilon^*y_1 - \cancel{\varepsilon^*\varepsilon A_j^* j^*}y_1 = \varepsilon^*y_2 - A^*\underbrace{(\varepsilon\varepsilon^*+jj^*)}_{1}y_1 = 0.$$

Calculate $W^\circ\cap\binom{1}{\omega}Y \ni \begin{pmatrix}\varepsilon x+z_1\\ Ax+\varepsilon A_j^*z_1+jz_2\end{pmatrix}$

$$\omega(\varepsilon x+z_1) = Ax+\varepsilon A_j^*z_1+z_2$$

$$(\omega\varepsilon-A)x = (\omega+\varepsilon A_j^*)z_1 + z_2$$

$$(\omega-\varepsilon^*A)x = A^*z_1 \Rightarrow \cancel{x=(\omega-\varepsilon^*A)^{-1}A^*z_1}$$

$$z_2 = (\omega\varepsilon-A)(\omega-\varepsilon^*A)^{-1}A^*z_1 - (\omega+\varepsilon A^*)\varepsilon A^*z_1$$

$$= (\omega\varepsilon-A) -$$

126 Suppose $\begin{pmatrix} \varepsilon x + z_1 \\ Ax + \varepsilon A^* z_1 + z_2 \end{pmatrix} \in W^0 \cap \begin{pmatrix} 1 \\ \omega \end{pmatrix} Y$ i.e.

$$\omega(\varepsilon x + z_1) = Ax + \varepsilon A^* z_1 + z_2$$

$$\omega x = \varepsilon^* A x + A^* z_1 \qquad x = (\omega - \varepsilon^* A)^{-1} A^* z_1$$

$$\begin{aligned} z_2 &= (\omega\varepsilon - A)x + (\omega - \varepsilon A^*) z_1 \\ &= \omega z_1 + (\omega\varepsilon - A)(\omega - \varepsilon^* A)^{-1} A^* z_1 \\ &\quad - \varepsilon(\omega - \varepsilon^* A)(\omega - \varepsilon^* A)^{-1} A^* z_1 \\ &= \omega - (1 - \varepsilon\varepsilon^*)A(\omega - \varepsilon^* A)^{-1} A^* z_1 \end{aligned}$$

$$\tilde{A} = \left(\begin{array}{c|c} \varepsilon^* A & A^* j \\ \hline j^* A & 0 \end{array}\right)$$

Review this. $T = \mathbb{C}^2$, $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}^* \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x_1' \\ x_2' \end{pmatrix}$

Y Hilbert, $T \otimes Y = \begin{matrix} Y \\ \oplus \\ Y \end{matrix}$ $\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}^* \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = y_1^* y_2 - y_2^* y_1$

$$\left(\begin{pmatrix} 1 \\ f \end{pmatrix} Y\right)^0 = \left\{ \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \middle| \; y^* y_2 = \underbrace{(fy)^* y_1}_{y^* f^* y_1} \quad \forall y \right\} = \begin{pmatrix} 1 \\ f^* \end{pmatrix} Y$$

W isot. $W = \begin{pmatrix} \varepsilon \\ \alpha \end{pmatrix} X$ $\quad \operatorname{Ker}\varepsilon \cap \operatorname{Ker}\alpha = 0.$

$$W^0 = \left\{ \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \middle| \; (\varepsilon x)^* y_2 = (\alpha x)^* y_1 \quad \forall x \right\}$$

$$\alpha^* y_1 = \varepsilon^* y_2$$

$W < W^0 \Longleftrightarrow \alpha^* \varepsilon = \varepsilon^* \alpha$

$$W^0 = W \oplus \begin{pmatrix} 1 \\ \varepsilon\alpha^* \end{pmatrix} Z + \begin{pmatrix} 0 \\ 1 \end{pmatrix} Z$$

$Z = \operatorname{Ker}\varepsilon$

$\varepsilon^* \varepsilon = 1.$

197 to compute $\mathrm{Img}\{\underbrace{W^0 \cap \binom{1}{\omega} Y}\} \subset W^0/W$

$$\begin{pmatrix} \varepsilon x + z_1 \\ \alpha x + \varepsilon\alpha^* z_1 + z_2 \end{pmatrix}$$

to solve

$$\boxed{(\omega\varepsilon - \alpha)x = (-\omega + \varepsilon\alpha^*) z_1 + z_2}$$

$$(\omega - \varepsilon^*\alpha)x = \alpha^* z_1 \implies x = (\omega - \varepsilon^*\alpha)^{-1}\alpha^* z_1$$

$$z_2 = \omega z_1 - \varepsilon\underbrace{\alpha^* z_1} + (\omega\varepsilon - \alpha)(\omega - \varepsilon^*\alpha)^{-1}\underbrace{\alpha^* z_1}$$

$$\cancel{= \{\omega z_1 + \{-\varepsilon(\omega - \varepsilon^*\alpha)(\omega - \varepsilon^*\alpha)^{-1}\alpha^* z_1}$$

$$= \omega z_1 + \{\underbrace{(-\varepsilon)(\omega - \varepsilon^*\alpha) + (\omega\varepsilon - \alpha)}_{-(1-\varepsilon\varepsilon^*)\alpha}\}(\omega - \varepsilon^*\alpha)^{-1}\alpha^* z_1$$

$$= \{\omega - jj^*(\omega - \varepsilon^*\alpha)^{-1}\alpha^*\} z_1$$

Other method: Introduce A

$$A = \begin{array}{cc} \overbrace{\quad}^{\varepsilon X} & \overbrace{\quad}^{jZ} \\ \left[\begin{array}{c|c} \varepsilon^*\alpha & \alpha^* j \\ \hline j^*\alpha & \end{array}\right] \end{array}$$

$\ni \binom{1}{A} y = W + \binom{1}{\varepsilon\alpha^*} Z$

$\underbrace{W^0 \cap \binom{1}{\omega} Z} \ni \begin{pmatrix} y \\ Ay + z_2 \end{pmatrix}$

$$\omega y = Ay + z_2$$

$$y = (\omega - A)^{-1} z_2$$

$$y = \varepsilon\varepsilon^* x + \underbrace{jj^* y}_{jz_1}$$

$$z_1 = j^*(\omega - A)^{-1} j z_2$$

$$\left(\omega - jj^*(\omega - \varepsilon^*\alpha)^{-1}\alpha^*\right)^{-1} = j^*(\omega - A)^{-1} j \qquad \boxed{\text{YES!}}$$

128. ~~Atiyah~~ Quaternionic version of orth. polys.
Atiyah problem $\mathbb{R}^3 = \mathbb{R}i + \mathbb{R}j + \mathbb{R}k \subset \mathbb{H}$. Try to fit this into the $\mathcal{O}(-1)\otimes Y \hookrightarrow \mathcal{O}\otimes T\otimes Y \twoheadrightarrow \mathcal{O}(1)\otimes Y$ framework. In order that $\mathcal{O}(-1)\otimes Y$ makes sense over the quaternion sphere you need an anti linear autom on Y of square -1.

Go over the $\mathbb{C}$ case before you handle $\mathbb{H}$

$T = \mathbb{C}^2$, $v^*\begin{pmatrix} 0 & 1 \\ -1 & 0\end{pmatrix}v$ $\qquad T\otimes Y = \begin{matrix} Y \\ \oplus \\ Y\end{matrix}$ $\qquad \begin{pmatrix} y_1 \\ y_2\end{pmatrix}^t\begin{pmatrix} 0 & 1 \\ -1 & 0\end{pmatrix}\begin{pmatrix} y_1' \\ y_2'\end{pmatrix}$

graphs $\begin{pmatrix} 1 \\ A\end{pmatrix}Y$ isotropic $\iff A^t = A$ ~~so if you want~~

$W = \begin{pmatrix} \varepsilon \\ \alpha\end{pmatrix}X \implies W^0 = \left\{ \begin{pmatrix} y_1 \\ y_2\end{pmatrix} \,\middle|\, \begin{pmatrix} \varepsilon x \\ \alpha x\end{pmatrix}^t\begin{pmatrix} & 1 \\ -1 & \end{pmatrix}\begin{pmatrix} y_1 \\ y_2\end{pmatrix} = 0 \ \forall x \right\}$

$W \subset W^0$, $\varepsilon^t\alpha = \alpha^t\varepsilon$. $\quad \begin{pmatrix} 1 \\ \omega\end{pmatrix} \operatorname{Ker}(\omega\varepsilon - \alpha) \quad \boxed{\varepsilon^t y_2 = \alpha^t y_1}$

Assume no bound states $\quad W \cap \begin{pmatrix} 1 \\ \omega\end{pmatrix}Y = 0 \quad \forall \omega \in \mathbb{P}T$

~~e.g.~~ in particular ε, α injective. So can assume $X \subset Y$ and ε inclusion. Good case will be when X is non degenerate subspace, in which case

$$\begin{array}{ccccc} X & \longrightarrow & Y & \longrightarrow & Y/X \\ \wr\!\downarrow & & \downarrow & & \\ X^* & \longleftarrow & Y^* & \longleftarrow & X^\perp \end{array}$$

~~you get an ε^* such that~~

you can define ε^* by $(\varepsilon^* y, x) = (y, \varepsilon x)$.

$\qquad\qquad\qquad\qquad\qquad\qquad \parallel$

~~$(\varepsilon x)^t y = x^t(\varepsilon^* y)$~~ $(\varepsilon\varepsilon^* y, \varepsilon x)$

Conclusion seems to be that in the complex setting you need to ~~assume~~ strengthen the assumption of no bound states to ~~inclusion~~ include $(\varepsilon x_1, \varepsilon x_2)$ is non degenerate on X. In this case you should ~~then look at~~ have an analog of $\begin{pmatrix} \varepsilon^*\alpha & \alpha^* j \\ j^*\alpha & 0\end{pmatrix}$

Now move on to quaternions, putting a σ on things. You just realized where the no bdd states ass. is needed, namely, to connect the bundle and K-modules. ~~Set up for the gen~~ Point recall is

$$0 \longrightarrow \mathcal{L} \longrightarrow \mathcal{O}\otimes W^0 \twoheadrightarrow \mathcal{O}(1)\otimes Y \longrightarrow 0$$

$$\begin{tikzcd}
 & \mathcal{L} \arrow[d] & \\
\mathcal{O}\otimes W \arrow[r, hook] & \mathcal{O}\otimes W^0 \arrow[dl] \arrow[dr] & \mathcal{O}\otimes W^0/W \\
\mathcal{O}(-1)\otimes Y \arrow[r, hook] & \mathcal{O}\otimes T\otimes Y \arrow[r, two heads] & \mathcal{O}(1)\otimes Y
\end{tikzcd}$$

You have filtration $0 \subset W \subset W^0 \subset T\otimes Y$ and $\mathcal{O}(-1)\otimes Y \subset \mathcal{O}\otimes T\otimes Y$. Get chains of lengths 6 except

$$\mathcal{O}\otimes W \cap \mathcal{O}(-1)\otimes Y = 0$$
$$\mathcal{O}\otimes W^0 + \mathcal{O}(-1)\otimes Y = \mathcal{O}\otimes T\otimes Y$$

$$0 \longrightarrow \mathcal{L} \longrightarrow \mathcal{O}\otimes W^0 \longrightarrow \mathcal{O}(1)\otimes Y \longrightarrow 0$$
$$0 \longrightarrow \mathcal{O}(-1)\otimes Y^* \longrightarrow \mathcal{O}\otimes W^{0*} \longrightarrow \mathcal{L}^* \longrightarrow 0$$

$$Y \xrightarrow{\sim} H^1(\mathcal{L}(-1)) \qquad H^0(\mathcal{L}^*(-1)) \xrightarrow{\sim} Y^*$$

$$0 \longrightarrow W^0 \longrightarrow T\otimes Y \longrightarrow H^1(\mathcal{L}) \longrightarrow 0$$

$$H^1(\mathcal{L}(-2)) \xrightarrow{\sim} W^0$$ so $\sigma^2 = 1$ on W^0

What you are trying to get is an H K-mod.

$$0 \longrightarrow \mathcal{L} \longrightarrow \mathcal{O}\otimes W^0/W \longrightarrow \mathcal{L}^* \longrightarrow 0$$

Y complex n diml equipped with a σ antilinear of square -1. Then $Y \otimes Y$ and $S^2 Y$ have $\sigma^2 = 1$. Real st. space of quadratic forms is a real vector space of dim $\frac{n(n+1)}{2}$ $n = 2 \Rightarrow 3$. What are autos of Y commuting with σ.

$$\sigma(z_1 + z_2 j) = -\bar{z}_2 + \bar{z}_1 j$$

$$Y = \mathbb{C}^2 \quad \text{with} \quad \sigma\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} -\bar{z}_2 \\ \bar{z}_1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \bar{z}_1 \\ \bar{z}_2 \end{pmatrix}$$

$$Q\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}^t \begin{pmatrix} a & b \\ b & c \end{pmatrix}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \qquad Q\begin{pmatrix} -\bar{z}_2 \\ \bar{z}_1 \end{pmatrix} = \begin{pmatrix} -\bar{z}_2 \\ \bar{z}_1 \end{pmatrix}^t \begin{pmatrix} a & b \\ b & c \end{pmatrix}\begin{pmatrix} -\bar{z}_2 \\ \bar{z}_1 \end{pmatrix}$$

$$= \left[\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \bar{z}_1 \\ \bar{z}_2 \end{pmatrix}\right]^t \begin{pmatrix} a & b \\ b & c \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \bar{z}_1 \\ \bar{z}_2 \end{pmatrix}$$

$$= \begin{pmatrix} \bar{z}_1 \\ \bar{z}_2 \end{pmatrix}^t \begin{pmatrix} 0 & +1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} b & -a \\ c & -b \end{pmatrix}\begin{pmatrix} \bar{z}_1 \\ \bar{z}_2 \end{pmatrix} = \begin{pmatrix} \bar{z}_1 \\ \bar{z}_2 \end{pmatrix}^t \begin{pmatrix} +c & -b \\ -b & +a \end{pmatrix}\begin{pmatrix} \bar{z}_1 \\ \bar{z}_2 \end{pmatrix}$$

So it appears that we have $\begin{pmatrix} a & i\beta \\ i\beta & \bar{a} \end{pmatrix}$ $\beta \in \mathbb{R}$.

Question: Is there a meaning for quaternionic Hilbert space. ~~[illegible]~~ On $\mathbb{H}$ you have $*$ and τ. Real $*$ alg. need a dual pair!

Consider $x^* y = (x_0 - x_1 i - x_2 j - x_3 k)\cdot(y_0 + y_1 i + y_2 j + y_3 k)$

$$\mathbb{H} \otimes_{\mathbb{R}} \mathbb{H} \longrightarrow \mathbb{H} \qquad x \otimes y \longmapsto x^* y$$

right $\mathbb{H}$ linear and left $*$ linear

$$x^* x = x_0^2 + \cancel{x_0(x_1 i + x_2 j + x_3 k)} + x_1^2 + x_2^2 + x_3^2$$

131 So what to do? Go back to a hermitian version. $T = \mathbb{C}^2$ with $\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}^* \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} z_1' \\ z_2' \end{pmatrix} Z.$

$= z_1^* z_1' - z_2^* z_2'$. $\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \longmapsto |z_1|^2 - |z_2|^2$.

Y Hilb. space $T \otimes Y = \begin{matrix} Y \\ \oplus \\ Y \end{matrix}$ $\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}^* \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \|y_1\|^2 - \|y_2\|^2$

Graph $\begin{pmatrix} 1 \\ \alpha \end{pmatrix} Y$ is isotropic $\Leftrightarrow$ α unitary.

Isotropic: $\begin{pmatrix} a \\ b \end{pmatrix} X$ $a^*a = b^*b = 1$. Is there a quaternionic version? Can you descend to SV

Look for a quaternionic version of

$$0 \longrightarrow \mathcal{L} \longrightarrow \mathcal{O} \otimes W^0/W \longrightarrow \mathcal{L}^* \longrightarrow 0$$

or
$$0 \longrightarrow \mathcal{O}(-2) \otimes \mathcal{E}^* \longrightarrow \mathcal{O}(-1) \otimes W^0/W \longrightarrow \mathcal{E} \longrightarrow 0$$

Go back to partial operators. Coupling with a transmission line. have 1-port, space of $\begin{pmatrix} E \\ I \end{pmatrix}$ with herm. form $\begin{pmatrix} E \\ I \end{pmatrix}^* \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} E \\ I \end{pmatrix} = 2\,\mathrm{Re}(\bar{E} I)$ and a response function which is a rational mfs

---

$$W^0 = \underbrace{\begin{pmatrix} \varepsilon \\ \alpha \end{pmatrix} X \oplus \begin{pmatrix} 1 \\ \varepsilon\alpha^* \end{pmatrix} Z}_{\begin{pmatrix} 1 \\ A \end{pmatrix} Y} \oplus \begin{pmatrix} 0 \\ 1 \end{pmatrix} Z \qquad Z = \mathrm{Ker}\,\varepsilon$$

how does this look in the other picture

$T \otimes Y$ $\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}^* \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \|y_1\|^2 - \|y_2\|^2$

C.T. $-i\omega = \dfrac{1-z}{1+z}$ $z = 1$ $a - b$
$z = -1$ $-(a+b)$

$$az - b = a\frac{1+i\omega}{1-i\omega} - b \sim a(1+i\omega) - b(1-i\omega)$$
$$= a - b + \omega i(a+b) \sim -i(a-b) + \omega(a+b)$$

Review. Begin with T 2 diml over $\mathbb{C}$ equipped with a hermitian form of sign $+,-$ Y cx Hilbert space, then $T \otimes Y$ is Krein, given $W \subset T \otimes Y$ isotropic. (Idea: take Y infinite diml and try building W ~~successively~~ stepwise with the aim of understanding the difference between the self-adjoint and unitary cases.) $\omega \in PT$ l_ω corres. line in T, then hermitean form rest. to l_ω is $>0, =0, <0$ dividing PT into circle + complem. disks. bound states are given by $\omega \ni W \cap l_\omega \otimes Y \neq 0$, these can be split off. Assume no bound states, then $W^\circ + l_\omega \otimes Y = T \otimes Y$ $\forall \omega$ and $\mathcal{L}_\omega = W^\circ \cap (l_\omega \otimes Y) \hookrightarrow W^\circ / W$.

Details: Take $T = \mathbb{C}^2$ $\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}^* \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = |z_1|^2 - |z_2|^2$

$T \otimes Y = \begin{matrix} Y \\ \oplus \\ Y \end{matrix}$ $\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}^* \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = y_1^* y_1 - y_2^* y_2$. $W \subset \begin{matrix} Y \\ \oplus \\ Y \end{matrix}$

isotropic $W = \begin{pmatrix} a \\ b \end{pmatrix} X$ $a^*a = b^*b = 1$ on X, $W^\circ = \Big\{ \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \Big|$ $(ax, y_1) = (bx, y_2)$ $\forall x$ i.e. $a^* y_1 = b^* y_2 \Big\}$. Given such $\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$ put $x = a^* y_1 = b^* y_2$, then $\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} - \begin{pmatrix} a \\ b \end{pmatrix} x = \begin{pmatrix} y_1 - aa^* y_1 \\ y_2 - bb^* y_2 \end{pmatrix} \in \begin{matrix} \operatorname{Ker} a^* \\ \oplus \\ \operatorname{Ker} b^* \end{matrix}$

so find $W^\circ = W \oplus \begin{matrix} \operatorname{Ker} a^* \\ \oplus \\ \operatorname{Ker} b^* \end{matrix}$ $l_z = \begin{pmatrix} 1 \\ z \end{pmatrix} \mathbb{C} \subset T$ herm. form is $1 - |z|^2$. No bound states means

$\begin{pmatrix} a \\ b \end{pmatrix} X \cap \begin{pmatrix} 1 \\ z \end{pmatrix} Y = \begin{pmatrix} 1 \\ z \end{pmatrix} \operatorname{Ker}(az - b) = 0$ $\forall z$

Calculate $W^\circ \cap \begin{pmatrix} 1 \\ z \end{pmatrix} Y$ $\ni$ $\begin{pmatrix} a \\ b \end{pmatrix} x + \begin{pmatrix} v^+ \\ v^- \end{pmatrix}$ $\ni$ $z(ax + v^+) = bx + v^-$

$(az - b)x = -zv^+ + v^-$ $\quad$ $y = ax + v^+$

$W^\circ = \begin{pmatrix} 1 \\ ba^* \end{pmatrix} Y \oplus \begin{pmatrix} 0 \\ \operatorname{Ker} b^* \end{pmatrix}$ $\quad$ $zy = (ba^*) y + v^-$

$y =$

$$W^0 \cap \binom{1}{z} Y \ni \binom{ax+v^+}{bx+v^-} \qquad (za-b)x = -zv^+ + v^-$$

Put $y = ax + v^+$, then $ba^* y = bx$

$$\binom{y}{ba^*y + v^-} \in W^0 \cap \binom{1}{z} Y \qquad zy = ba^*y + v^-$$

$$y = (z-ba^*)^{-1}v^-$$

$$v^+ = (1-aa^*)(z-ba^*)^{-1}v^-$$

$$(1 - zb^*a)x = zb^*v^+ \qquad x = (1-zb^*a)^{-1} b^* v^+$$

$$v^- = z(v^+ + (za-b)(1-zb^*a)^{-1}b^*v^+)$$

$$= z(1-zab^* + (za-b)b^*)(1-zab^*)^{-1} v^+$$

$$= (1-bb^*)(1-zab^*)^{-1} z v^+$$

From 131. $az-b = a\frac{1+i\omega}{1-i\omega} - b \sim a(1+i\omega) + b(-1+i\omega)$

$$= \underbrace{a-b}_{z=1} + \omega i \underbrace{(a+b)}_{z=-1}$$

$\omega = \infty \qquad z=-1.$

$\varepsilon\omega - \alpha \qquad \varepsilon \sim (a+b)$

$\qquad\qquad\qquad \alpha \sim i(a-b)$

$$(\varepsilon x, \alpha x) = \frac{1}{i}\{((a+b)x, (a-b)x)\} = \frac{1}{i}\begin{pmatrix}-(ax,bx) \\ +(bx,ax)\end{pmatrix}$$

$$(\alpha x, \varepsilon x) = \frac{1}{i}\{((a-b)x, (a+b)x)\} = i\begin{pmatrix}(ax,bx) \\ -(bx,ax)\end{pmatrix}$$

$\therefore (\varepsilon x, \alpha x)$ is real

$$\begin{pmatrix}\varepsilon \\ \alpha\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1 \\ +i & -i\end{pmatrix}\begin{pmatrix}a \\ b\end{pmatrix}$$

$$\begin{pmatrix}a \\ b\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & -i \\ 1 & +i\end{pmatrix}\begin{pmatrix}\varepsilon \\ \alpha\end{pmatrix}$$

134 $\quad \varepsilon = a+b \qquad \begin{pmatrix} \varepsilon \\ \alpha \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ +i & -i \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}$

$\alpha = \frac{i\boxed{a-b}}{\cancel{2}}$

$$\cancel{\begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix} = \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix} \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}}$$

$$\begin{pmatrix} a \\ b \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ 1 & +i \end{pmatrix} \begin{pmatrix} \varepsilon \\ \alpha \end{pmatrix}$$

$$\frac{1}{2} \begin{pmatrix} 1 & 1 \\ +i & -i \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & -i \\ 1 & +i \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ +i & -i \end{pmatrix} \begin{pmatrix} 1 & -i \\ -1 & -i \end{pmatrix}$$

$$= \frac{1}{2} \begin{pmatrix} 0 & -2i \\ +2i & 0 \end{pmatrix} = \begin{pmatrix} 0 & -i \\ +i & 0 \end{pmatrix}$$

This is a hermitian form, but it has the ~~same~~ same W^0 as $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ which is skew-herm.

What do you get?

Go back to p.m. $\begin{pmatrix} a \\ b \end{pmatrix} X < \begin{matrix} Y \\ \oplus \\ Y \end{matrix}$ $\qquad$ set $\varepsilon = a+b/2$

$\cancel{\alpha = i(a-b)}$

$\alpha = i(a-b)/2$

$$z = \frac{1-(-i\omega)}{1+(-i\omega)} = \frac{1+i\omega}{1-i\omega}$$

$a = \varepsilon - i\alpha$

$b = \varepsilon + i\alpha$

$$az - b \approx a(1+i\omega) - b(1-i\omega)$$

$$= a-b + i(a+b)\omega$$

$$\sim (a+b)\omega - i(a-b) \sim \varepsilon\omega - \alpha.$$

$ia = i\varepsilon + \alpha$

$ib = i\varepsilon - \alpha$

 Start again. $W = \begin{pmatrix} a \\ b \end{pmatrix} X \subset \begin{matrix} Y \\ \oplus \\ Y \end{matrix}$ equipped with $\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}^* \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$. Make C.T. $z = \frac{1+i\omega}{1-i\omega}$

$$az - b \sim a(1+i\omega) - (1-i\omega)b = a-b + i(a+b)\omega$$
$$\sim (a+b)\omega - i(a-b)$$

$$\begin{aligned} \varepsilon &= a+b \\ \alpha &= i(a-b) \end{aligned} \qquad \begin{pmatrix} \varepsilon \\ \alpha \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} \qquad \begin{pmatrix} a \\ b \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix} \begin{pmatrix} \varepsilon \\ \alpha \end{pmatrix}$$

$$\frac{1}{2} \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} \begin{pmatrix} 1 & -i \\ -1 & -i \end{pmatrix} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$= \frac{1}{i} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

So the transform

$\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}$ transforms $\begin{pmatrix} a \\ b \end{pmatrix} X$ which is isot. for $\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}^* \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$ to the subspace $\begin{pmatrix} \varepsilon \\ \alpha \end{pmatrix} X$ which is isotropic for the herm. form $\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, which has ~~an~~ same annihilator as $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. If $W = \begin{pmatrix} \varepsilon \\ \alpha \end{pmatrix} X$, then

$$W^0 = \left\{ \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \middle| \begin{pmatrix} \varepsilon \\ \alpha \end{pmatrix}^* \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = 0 \right\}$$

$$-i\varepsilon^* y_2 + i\alpha^* y_1 = 0 \qquad \therefore \varepsilon^* y_2 = \alpha^* y_1$$

$W \subset W^0 \qquad \varepsilon^*\alpha = \alpha^*\varepsilon$. Calculate the nice hermitian extension

$$\Gamma_{\tilde{\alpha}} = \begin{pmatrix} 1 \\ \varepsilon\alpha^* \end{pmatrix}^{j} K \qquad \tilde{\alpha} = \left(\begin{array}{c|c} \varepsilon^*\alpha & \alpha^* j \\ \hline j^*\alpha & 0 \end{array} \right) \text{ on } \begin{matrix} X \\ \oplus \\ K \end{matrix}$$

$K = \mathrm{Ker}(\varepsilon^*)$

136 Calculate $W^0 \cap \binom{1}{\omega} Y$

$$W^0 = \underbrace{W \oplus \binom{1}{\varepsilon\alpha^*} jK}_{\binom{1}{\tilde{\alpha}} Y} \oplus \binom{0}{1} jK$$

$$\overset{y}{\omega y} = \tilde{\alpha} y + j z_2$$

$$y = (\omega - \tilde{\alpha})^{-1} j z_2$$

$$z_1 = j^* (\omega - \tilde{\alpha})^{-1} j z_2$$

On the other hand the unitary model gives $W^0 = \binom{a}{b} X \oplus \binom{\operatorname{Ker} a^*}{\operatorname{Ker} b^*} \supset \binom{1}{ba^*} Y$

$$W^0 = \binom{1}{ba^*} Y \oplus \binom{0}{1} \operatorname{Ker} b^*$$

$$\binom{y}{ba^* y + v^-} \in W^0 \cap \binom{1}{z} Y \qquad zy = ba^* y + v^-$$

$$y = (z - ba^*)^{-1} v^- \qquad v^+ = (1 - aa^*)(z - ba^*)^{-1} v^-$$

from p140

Goal: to understand how deBranges goes from ~~a~~ a scalar product on the space of polys of degree < n to an isom. embedding into the Hardy space, somehow it ~~...~~ involves choosing a point in the UHP. Somehow you need

Connecting to a transmission line ~~a~~ is to be interpreted as polarizing ~~...~~ the gate W^0/W.

137 Review : $W = \binom{a}{b} X \subset \begin{matrix} Y \\ \oplus \\ Y \end{matrix}$ $\qquad \binom{y_1}{y_2}^* \begin{pmatrix}1&0\\0&-1\end{pmatrix}\binom{y_1}{y_2} = \|y_1\|^2 - \|y_2\|^2$ $\qquad a$

$$W^\circ = \left\{ \binom{y_1}{y_2} \,\middle|\, a^* y_1 = b^* y_2 \right\} = W \oplus \begin{matrix}\mathrm{Ker}\, a^* \\ \oplus \\ \mathrm{Ker}\, b^*\end{matrix}$$

where W isot. i.e. $a^*a = b^*b = 1$

$$W^\circ = \underbrace{W \oplus \binom{1}{0}\mathrm{Ker}\,a^*}_{\binom{1}{ba^*}Y} \oplus \binom{0}{1}\mathrm{Ker}\, b^*$$

$ba^*ax = bx$.

$$\binom{y}{ba^*y + v^-} \in W^\circ \cap \binom{1}{z} Y$$

$$zy = ba^*y + v^- \Rightarrow y = (z-ba^*)^{-1} v^-$$

$$v^+ = (1-aa^*)y = (1-aa^*)^{-1}(z-ba^*)^{-1}v^-$$

Do C.T. $\qquad z = \dfrac{1 - (-i\omega)}{1+(-i\omega)} = \dfrac{1+i\omega}{1-i\omega}$

$$az - b \sim a(1+i\omega) - b(1-i\omega)$$
$$= a-b + i(a+b)\omega \sim \overbrace{i(a-b)}^{\alpha} - \omega\overbrace{(a+b)}^{\varepsilon}$$

$$\binom{\varepsilon}{\alpha} = \underbrace{\begin{pmatrix}1&1\\i&-i\end{pmatrix}}_{g}\binom{a}{b} \qquad g\binom{a}{b}X = \binom{\varepsilon}{\alpha}X$$

$$\left(g^{-1}\binom{y_1}{y_2}\right)^* \begin{pmatrix}1&0\\0&-1\end{pmatrix} g^{-1}\binom{y_1}{y_2} = \binom{y_1}{y_2}^*\begin{pmatrix}0&-i\\i&0\end{pmatrix}\binom{y_1}{y_2}$$

$$\frac12 \begin{pmatrix}1&1\\i&-i\end{pmatrix}\begin{pmatrix}1&0\\0&-1\end{pmatrix}\begin{pmatrix}1&-i\\1&i\end{pmatrix} = \frac12\begin{pmatrix}1&1\\i&-i\end{pmatrix}\begin{pmatrix}1&-i\\-1&-i\end{pmatrix} = \begin{pmatrix}0&-i\\i&0\end{pmatrix}$$

So $\binom{\varepsilon}{\alpha}X$ isot. for $\binom{y_1}{y_2}^*\begin{pmatrix}0&1\\-1&0\end{pmatrix}\binom{y_1}{y_2} = y_1^*y_2 - y_2^*y_1$

i.e. $\varepsilon^*\alpha = \alpha^*\varepsilon$ ind. of metric on X, so you can take $\varepsilon^*\varepsilon = 1$ provided $\varepsilon = a+b$ inj. i.e. $z = -1$ not bdd state ev.

$$\varepsilon = \frac{a+b}{2} \qquad i\varepsilon + \alpha = ia$$

$$\alpha = \frac{ia - ib}{2} \qquad i\varepsilon - \alpha = ib$$

You need to get the remaining steps. What are the remaining steps? Something involving transmission line - or maybe the spectral repn. associated to a nearly hermitian operator. Recall, given β an operator on Y such that $sp(\beta) \subset LHP$ and $\rho = \left(\frac{\beta^* - \beta}{i}\right) \geq 0$ that you get an isometric embedding $y \longmapsto \rho^{1/2}(\omega - \beta)^{-1} y$. How?

$$\tilde{y}(\omega) = \rho^{1/2}(\omega - \beta)^{-1} y \qquad \text{analytic on UHP of } sp(\beta)$$

$$\int_{-\infty}^{\infty} \tilde{y}(\omega)^* \tilde{y}(\omega) \frac{d\omega}{2\pi} = \int_{-\infty}^{\infty} y^*(\omega - \beta^*)^{-1} \rho (\omega - \beta)^{-1} y \frac{d\omega}{2\pi}$$

$$\frac{i}{\beta^* - \beta}\rho = 1 \qquad = \int \text{———} = \frac{2\pi i}{2\pi} y^* \rho (\beta^* - \beta)^{-1} y = y^* y$$

$$\rho = \frac{\beta^* - \beta}{i}$$

This residue β^* calculus is slightly non rigorous but is OKAY in finite dimensions.

You want to think of this mathematical situation as "coupling with a transmission line"!!

You use this process in the case of a partial herm. op.

For $W = \begin{pmatrix}\varepsilon \\ \alpha\end{pmatrix} X \subset \begin{matrix} Y \\ \oplus \\ Y \end{matrix}$, $W^0 = \begin{pmatrix}\varepsilon \\ \alpha\end{pmatrix} X \oplus \begin{pmatrix} 1 \\ \varepsilon\alpha^* \end{pmatrix} jK \oplus \begin{pmatrix} 0 \\ 1 \end{pmatrix} jK$

So what kind of thing do you put in the port? Go to unitary situation. You know that you should use a contraction $ba^* = ba^{-1}$ on aX. ~~Recall the situation in great detail~~ Keep in mind the case where $\operatorname{Ker} \varepsilon^*$ 1-diml. ~~So what comes nex~~

You have γ extending ba^{-1}.

$$\tilde{y}(z) = \rho^{1/2} \frac{1}{1 - z\gamma^*} y \qquad \rho = (1-\gamma\gamma^*) \qquad sp(\gamma) \subset |z| < 1$$

$$\int \frac{d\theta}{2\pi} \tilde{y}(z)^* \tilde{y}(z) = \int \frac{dz}{2\pi i z} y^* \frac{1}{\underset{z-\gamma}{\cancel{1 - z^{-1}\gamma}}} \rho \underbrace{\frac{1}{1 - z\gamma^*} y}_{\text{analytic inside}}$$

$$= y^* \rho \frac{1}{1 - \gamma\gamma^*} y = y^* y$$

You are aiming for ~~a picture~~ an interpretation of $\gamma : \operatorname{Ker}(a^*) \longrightarrow \operatorname{Ker}(b^*)$ as impedance of a transmission line. The point is that any subspace of $\begin{matrix}\operatorname{Ker} a^* \\ \oplus \\ \operatorname{Ker} b^*\end{matrix}$ of form $\begin{pmatrix} 1 \\ \gamma \end{pmatrix} \operatorname{Ker} a^*$ is > 0

i.e. $\forall v \in \operatorname{Ker} a^*$ $\quad \begin{pmatrix} v \\ \gamma v \end{pmatrix}^* \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} v \\ \gamma v \end{pmatrix} = \|v\|^2 - \|\gamma v\|^2 > 0$

hence $\underline{1 - \gamma^*\gamma > 0}$. This should be the impedance in the unitary picture. In the hermitian picture $\frac{1}{i} y^* \begin{pmatrix} 1 \\ \beta \end{pmatrix}^* \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ \beta \end{pmatrix} y$

$$= \frac{1}{i} y^* \begin{pmatrix} 1 & \beta^* \end{pmatrix} \begin{pmatrix} \beta \\ -1 \end{pmatrix} y = y^* \left(\frac{\beta - \beta^*}{i} \right) y$$

So ~~when~~ what happens in the hermitian picture is ~~on~~ the ~~first~~ gate $\begin{matrix} K \\ \oplus \\ K \end{matrix}$ you have ~~a~~ subspace $\begin{pmatrix} 1 \\ \beta \end{pmatrix} K$ where $\frac{\beta - \beta^*}{i} > 0$. How is this related to the impedance of a transm. line?

Remaining is deBranges proof.

140

Today I want to understand ~~better~~ d better how de Branges uses the point $\pm i$ to get an embedding a family of orth polys into Hardy space.

Abstract situation is ~~isotropic~~ a K-module $W \subset T \otimes Y$ of type $\mathcal{O}(n)$, where ~~Krein~~ W is isotropic rel Krein form on $T \otimes Y$ tens product of Krein f on T and scal. prod. on Y. The Kff on T gives $\mathbb{P}T$ a circle, inside and out disks. The circle carries an intrinsic Hilbert space of L^2 sections of $\mathcal{O}(-1)$. Choose $W^{\circ} \subset V \subset W^0$ then you get a "spectrum" where $\dim V = \dim Y$

$V \cap l_\omega \otimes Y = 0$ off spec. whence $V \oplus l_\omega \otimes Y = T \otimes Y$ and $\mathcal{O} \otimes V \xrightarrow{\sim} \mathcal{O}(1) \otimes Y$ off spectrum, so off spectrum you get $Y \xleftarrow{\sim} \mathcal{O}(-1) \otimes V \longrightarrow \mathcal{O}(-1) \otimes W/W$

Q: Can Atiyah's divisors be improved to operators?

Consider p.h. case $\quad Y = \varepsilon X \oplus jK \simeq X \oplus K$

$$\tilde{A} = \begin{pmatrix} \varepsilon^* \alpha & \alpha^* j \\ j^* \alpha & 0 \end{pmatrix} = \begin{pmatrix} a & b \\ b^* & 0 \end{pmatrix} \qquad \text{~~Consider~~}$$

Recall that
$$W^0 = \underbrace{\begin{pmatrix} \varepsilon \\ \alpha \end{pmatrix} X \oplus \begin{pmatrix} 1 \\ \varepsilon \alpha^* \end{pmatrix} jK}_{\begin{pmatrix} 1 \\ A \end{pmatrix} Y} \oplus \begin{pmatrix} 0 \\ 1 \end{pmatrix} jK$$

Consider a different $\tilde{A} = \begin{pmatrix} a & b \\ b^* & \gamma \end{pmatrix}$ arranged so that i is an eigenvalue

$$\omega - \tilde{A} = \begin{pmatrix} \omega - a & -b \\ -b^* & \omega - \gamma \end{pmatrix}$$ want this to be ~~sing~~ singular when $\omega = i$, actually you $j^*(\omega - \tilde{A})^{-1} j$ to blow up at $\omega = i$ which means the quasi-det $\omega - \gamma - b^*(\omega - a)^{-1} b$ should vanish somewhat at $\omega = i$. ~~Ship~~ If $\dim K = 1$ this

141 means that $\gamma = i - b^*(i-a)^{-1}b$. So apparently you take this choice of γ when you ~~worry about~~ follow scattering.

$$\cancel{\begin{pmatrix} i-a & -b \\ -b^* & i-\gamma \end{pmatrix}\begin{pmatrix} t \\ 1 \end{pmatrix}} =$$

$$j^* \begin{pmatrix} \omega - a & -b \\ -b^* & \omega-\gamma \end{pmatrix}^{-1} j = \frac{1}{\omega - i} + \text{lower}$$

$$\begin{pmatrix} \omega - a & -b \\ -b^* & \omega-\gamma \end{pmatrix}\begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} h \\ 1 \end{pmatrix}$$

$$(\omega-a)f - bg = h$$
$$-b^*f + (\omega-\gamma)g = 1$$

$$f = (\omega-a)^{-1}(bg+h)$$

$$1 = (\omega-\gamma)g - b^*(\omega-a)^{-1}(bg+h)$$

$$1 = \left(\omega - \gamma - b^*(\omega-a)^{-1}b\right)g + b^*(\omega-a)^{-1}h$$

so if I take $\gamma = i - b(\cancel{\omega}\, i-a)^{-1}b$;

If you want $j^*(\omega-\tilde{A})^{-1}j$ you take $h=0$,

and you get $g = \dfrac{1}{\omega-\gamma-b^*(\omega-a)^{-1}b}$

Review. $W = \begin{pmatrix} \varepsilon \\ \alpha \end{pmatrix} X \subset \begin{matrix} Y \\ \oplus \\ Y \end{matrix}$ $\qquad W^0 = \begin{pmatrix} \varepsilon \\ \alpha \end{pmatrix} X \oplus \begin{pmatrix} 1 \\ \varepsilon\alpha^* \end{pmatrix} jK \oplus \begin{pmatrix} 0 \\ 1 \end{pmatrix} jK$

$$A = \left(\begin{array}{c|c} \varepsilon^*\alpha & \alpha^* j \\ \hline j^*\alpha & 0 \end{array}\right) : \begin{matrix} X \\ \oplus \\ K \end{matrix} \longrightarrow \begin{matrix} X \\ \oplus \\ K \end{matrix}$$

So what do you do? Shift to the p.u. case

$$W^0 = \begin{pmatrix} a \\ b \end{pmatrix} X \oplus \begin{matrix} \operatorname{Ker} a^* \\ \oplus \\ \operatorname{Ker} b^* \end{matrix} \qquad \text{gate is } \begin{matrix} \operatorname{Ker} a^* \\ \oplus \\ \operatorname{Ker} b^* \end{matrix}$$

and you extend the partial unitary ba^{-1} to the contraction ba^*.

$$\begin{pmatrix} 1 \\ ba^* \end{pmatrix} Y = \begin{pmatrix} a \\ b \end{pmatrix} X \oplus \begin{pmatrix} \operatorname{Ker} a^* \\ 0 \end{pmatrix}$$

$$W^0 = \left\{ \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \,\middle|\, a^* y_1 = b^* y_2 \right\}$$

Gate is $W^0/W = \begin{matrix} \operatorname{Ker} a^* \\ \oplus \\ \operatorname{Ker} b^* \end{matrix}$ $\qquad$ response line is the image in W^0/W of $W^0 \cap \begin{pmatrix} 1 \\ z \end{pmatrix} Y =$

$$\mathcal{L}_z = \begin{pmatrix} 1 \\ z \end{pmatrix} \operatorname{Ker}(b^* z - a^*)$$

two descriptions

$$W^0 \cap \begin{pmatrix} 1 \\ z \end{pmatrix} Y \ni \begin{pmatrix} y \\ ba^* y + v^- \end{pmatrix}$$

$$\operatorname{Img}(\mathcal{L}_z) = \left\{ \begin{pmatrix} (1-aa^*)(z-ba^*)^{-1} v^- \\ v^- \end{pmatrix} \right\}$$

Other $\quad W^0 \cap \begin{pmatrix} 1 \\ z \end{pmatrix} Y \ni \begin{pmatrix} ab^* y + v^+ \\ y \end{pmatrix}$

$$y = zab^* y + zv^+$$

$$y = (1 - zab^*)^{-1} zv^+ \qquad \operatorname{Im} \mathcal{L}_z = \left\{ \begin{pmatrix} v^+ \\ (1-bb^*)(1-zab^*)^{-1} zv^+ \end{pmatrix} \right\}$$

$$z = 0 \implies \operatorname{Im} \mathcal{L}_z = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \operatorname{Ker} a^*$$

$$z = \infty \implies \operatorname{Im} \mathcal{L}_z = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \operatorname{Ker} b^*$$

143 The details require patience. ~~$(a^* y_1 = b^*(y_2) \neq z y_1$~~?

$$\cancel{\mathcal{L}_z = \binom{1}{z} \operatorname{Ker}(a^* z - b^*)}$$

$$W^0 = \left\{ \binom{y_1}{y_2} \,\middle|\, \begin{array}{l} (ax)^* y_1 = (bx)^* y_2 \quad \forall x \\ \text{i.e. } a^* y_1 = b^* y_2 \end{array} \right\}$$

If $y_2 = z y_1$, then $a^* y_1 = z b^* y_1$

$$\therefore \mathcal{L}_z = W^0 \cap \binom{1}{z} Y = \binom{1}{z} \operatorname{Ker}(b^* z - a^*)$$

$\mathcal{L}_0 = \binom{1}{0} \operatorname{Ker} a^*$. Therefore your extension ba^* of ba^{-1} adds $\mathcal{L}_0$ to W, if I shift to the partial herm. picture, then $z = 0$ becomes $\omega = i$.

Shift to herm. picture $W = \binom{\varepsilon}{\alpha} X$.

carry out calculations again.

$$z = \frac{1 - (-i\omega)}{1 + (-i\omega)} = \frac{1 + i\omega}{1 - i\omega}$$

$$az - b \sim a(1 + i\omega) - b(1 - i\omega) = a - b + i(a+b)\omega$$

$$\sim \underbrace{i(a-b)}_{2\alpha} - \underbrace{(a+b)}_{2\varepsilon}\omega \sim \omega\varepsilon - \alpha$$

$$az - b \sim \omega\varepsilon - \alpha.$$

Review. In the unitary picture. You choose to extend ba^{-1} by ba^* getting $V = \binom{1}{ba^*} Y$. Note that ba^* has the eigenvalue 0.

You want to check that if you extend ba^{-1} by adding $\operatorname{Im} \mathcal{L}_0 = \emptyset$? What is the meaning of adding $\mathcal{L}_z$ to $\binom{a}{b} X$? $\quad \mathcal{L}_z = \binom{1}{z} \operatorname{Ker}(b^* z - a^*)$

$W + \mathcal{L}_z = V \subset W^0$. If $V = \binom{1}{\gamma} Y$, then $\binom{1}{\gamma} Y \cap \binom{1}{z} Y \neq 0$. $\quad \gamma y = z y$.

 Return to herm. pictures.

Puzzle about the two ends in the case of a J matrix.

Let's try to improve understanding of partial skew-symmetric + partial orthogonal operators. Let Y be a real Hilbert space, can form $\begin{matrix}Y\\ \oplus\\ Y\end{matrix}$ with the symmetric Krein form $\begin{pmatrix}y_1\\ y_2\end{pmatrix}^*\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}y_1'\\ y_2'\end{pmatrix}$ i.e. det. by quad form $y_1^*y_1 - y_2^*y_2$ isotropic $W \subset \begin{matrix}Y\\ \oplus\\ Y\end{matrix}$ are partial orthogonal operators: $\begin{pmatrix}a\\ b\end{pmatrix}X$ $a^*a = b^*b = 1$. $W^0 = \left\{\begin{pmatrix}y_1\\ y_2\end{pmatrix} \middle| \; y_1^* ax = y_2^* bx \quad \forall x\right\} = W \oplus \begin{matrix}\mathrm{Ker}\, a^*\\ \oplus\\ \mathrm{Ker}\, b^*\end{matrix}$

$a^*y_1 = b^*y_2$.

What about spectrum — you complexify. $W^0 \cap \begin{pmatrix}1\\ z\end{pmatrix}Y = L_z$

This situation should be the same as the ~~case~~ partial unitary operator case with σ added, σ antilinear $\sigma^2 = +1$. Should be nothing new. You should have ba^* ~~[illegible]~~ contraction etc. Eigenvalues closed under σ.

$$X \qquad \begin{pmatrix}\varepsilon\\ \chi\end{pmatrix} = \frac{1}{\sqrt 2}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\begin{pmatrix}a\\ b\end{pmatrix}$$

$$\frac{1}{2}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\cancel{\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}}\begin{pmatrix}1 & 1\\ -1 & +1\end{pmatrix} = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$$

$$W = \begin{pmatrix}\varepsilon\\ \chi\end{pmatrix}X \subset \begin{matrix}Y\\ \oplus\\ Y\end{matrix} \qquad W^0 = \left\{\begin{pmatrix}y_1\\ y_2\end{pmatrix} \middle| (\varepsilon x)^* y_2 + (\chi x)^* y_1 = 0\right.$$

$$\varepsilon^* y_2 + \chi^* y_1 = 0$$

$$W \subset W^0 \iff \varepsilon^*\chi + \chi^*\varepsilon = 0$$

$$\left[\begin{array}{c|c} \begin{matrix}\varepsilon^*\chi = \\ -\chi^*\varepsilon\end{matrix} & -\chi_j^* \\ \hline j^*\chi & 0 \end{array}\right] \qquad W^0 = \begin{pmatrix}\varepsilon\\ \chi\end{pmatrix}X \oplus \begin{pmatrix}1\\ -\varepsilon\chi^*\end{pmatrix}jK \oplus \begin{pmatrix}0\\ 1\end{pmatrix}jK$$

 main point now is to understand the deBranges calculation, consider Y = polys of degree $\leq n$, X = polys deg $< n$, $\varepsilon, \alpha : X \to Y$ inclusion + mult by λ, assume scalar product given on Y such that $\begin{pmatrix}\varepsilon\\ \alpha\end{pmatrix} X$ isotropic for $\begin{pmatrix}y_1\\ y_2\end{pmatrix}^* \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\begin{pmatrix}y_1'\\ y_2'\end{pmatrix}$

i.e. $\varepsilon^*\alpha = \alpha^*\varepsilon$ for any scal. prod. on X, take $\varepsilon^*\varepsilon = 1$.

deBranges calculates the Bargman kernel in this situation, i.e. the element of ~~Y~~ representing the linear functional: evaluation at ω_0.

abstractly, start with $W = \begin{pmatrix}\varepsilon\\ \alpha\end{pmatrix} X \subset \begin{matrix}Y\\ \oplus\\ Y\end{matrix}$ and construct a map of Y to polys of degree $\leq n$.

old viewpoint: Given ~~assume~~ a suitable scalar product on Y = polys deg $\leq n$, the point evaluators are defined and de Branges gives a formula. ~~[illegible]~~ The eval. at ξ must be orthogonal to $(\omega - \xi) X = (\xi\varepsilon - \alpha) X$

Let's begin with Y = polys deg $\leq n$ $= \mathbb{C} + \mathbb{C}\lambda + \cdots + \mathbb{C}\lambda^n$
X = polys of deg $< n$. $\varepsilon : X \to Y$ the inclusion α = mult by λ
suppose scalar product $y_1^* y_2$ given on Y. You want ~~α~~

~~[illegible]~~ $(y_1, \lambda y_2) = (\lambda y_1, y_2)$ deg $y_1, y_2 < n$

i.e. $(\varepsilon x_1, \underbrace{\lambda}_{\alpha} \varepsilon x_2) = (\underbrace{\lambda}_{\alpha}\varepsilon x_1, \varepsilon x_2)$ $\forall x_1, x_2$

which means $\begin{pmatrix}\varepsilon\\ \alpha\end{pmatrix} X \subset \begin{matrix}Y\\ \oplus\\ Y\end{matrix}$ is isol. for $\begin{pmatrix}y_1\\ y_2\end{pmatrix}^* \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\begin{pmatrix}y_1'\\ y_2'\end{pmatrix}$

We now ~~get~~ have a partial herm. op. Count dims. ~~[illegible]~~

~~[illegible]~~ Consider an extension A of $\alpha\varepsilon^{-1}$

First understand ~~the situation~~ without the ~~[illegible]~~ scalar product on Y. Take an operator γ on Y ~~[illegible]~~ such that ξ is a cyclic vector. Then get

$\mathbb{C}[\lambda] \twoheadrightarrow Y$ $\quad f(\lambda) \mapsto f(\gamma)\xi$ and kernel is ideal gen. by char. poly $\det(\lambda - \gamma)$. Thus get $\underset{\deg < n}{\text{polys}} \xrightarrow{\sim} Y$

 On the other hand given y can form $\xi^*(\lambda-\gamma)^{-1}y$ where ξ^* is a linear functional. This gives a rational function with denominator $\det(\lambda-\gamma)$ and numerator $\xi^* \mathrm{Cof}(\lambda-\gamma)y$ a poly of degree $< n$. Relate these maps. The former is poly of deg $\le n$

is an isom. when ξ is a cyclic vector for γ on Y, and the latter should be an isom. when ξ^* is cyclic vector for contragredient rep. γ^t on Y^*. E.g. $y \longmapsto \xi^*(\lambda-\gamma)^{-1}y$ is injective iff $\xi^*\gamma^k y = 0 \ \forall k \Rightarrow y=0$ iff $\xi^*\gamma^k$ $k\ge 0$ span Y^*.

Do it possible for these maps to be inverse?

$$\mathbb{C}[\lambda]/(d(\lambda)) \xrightarrow{\sim} Y \qquad \lambda^j \longmapsto \gamma^j\xi$$

$$\mathbb{C}[\lambda]/(d(\lambda)) \xrightarrow{\sim} Y^* \qquad \lambda^j \longmapsto \xi^*\gamma^j$$

have canonical pairing

$$\langle f(\lambda), g(\lambda) \rangle = \xi^* f(\gamma) g(\gamma) \xi \qquad ?$$

You would like to solve the equation

$$\lambda^j \longmapsto \gamma^j \xi$$

$$y \longmapsto \xi^*(\lambda-\gamma)^{-1}y$$

compose $\quad \lambda^j \longmapsto \xi^*(\lambda-\gamma)^{-1}\gamma^j\xi$

Why is $\oint \frac{dz}{az-b}$ a projector

$$\frac{1}{az-b}\,\frac{1}{aw-b} \qquad \frac{1}{az-b} \qquad \frac{1}{aw-b}$$

$$\frac{1}{az-b}\quad \frac{1}{aw-b} \qquad \frac{1}{az-b}\left(aw-b \qquad az-b\right)\frac{1}{aw-b}$$

147

$$\frac{1}{2\pi i}\oint \frac{dz}{z-A} \qquad \frac{1}{2\pi i}\int \frac{dw}{w-A}$$

$$\frac{1}{z-A}\,\frac{1}{w-A} = \frac{1}{-z+w}\left\{\frac{1}{z-A} - \frac{1}{w-A}\right\}$$

$$|z|>|w| \qquad = \int \frac{dz}{2\pi i}\underbrace{\int \frac{dw}{2\pi i}\,\frac{1}{w-z}}_{=0}\,\frac{1}{z-A} \qquad \int \frac{dw}{2\pi i}\overbrace{\left(\int \frac{dz}{2\pi i}\,\frac{1}{z-w}\right)}^{1}\frac{1}{w-A}$$

$$\frac{1}{az-b} \qquad a\,\frac{1}{aw-b}$$

$$= \frac{1}{z-w}\left[\frac{1}{az-b}\left(az-b-aw+b\right)\frac{1}{aw-b}\right]$$

$$= \frac{1}{z-w}\left[\frac{1}{aw-b} - \frac{1}{az-b}\right]$$

$$\cancel{\frac{1}{\lambda-A}\oint\frac{dz}{2\pi i}\,\frac{1}{z-A} = \oint\frac{dz}{2\pi i}\,\frac{1}{z-A}\left[\frac{1}{\lambda-A}-\frac{1}{\lambda-A}\right]}$$

back to orth. polys. Suppose given ~~scalar~~ moments through ~~dim~~ μ_{2n}.

$$\begin{bmatrix} b_1 & a_1 & & & \\ a_1 & b_2 & \ddots & & \\ & \ddots & \ddots & & \\ & & & b_n & a_n \\ & & & a_n & 0 \end{bmatrix}$$

$$\lambda p_k = a_k p_{k+1} + b_k p_k + a_{k-1} p_{k-1}$$

$$\lambda p_1 = a_1 p_2 + b_1 p_1 \qquad \frac{a_1 p_2}{p_1} = \lambda - b_1$$

$$\frac{a_k p_{k+1}}{p_k} + \frac{a_{k-1} p_{k-1}}{p_k} = \lambda - b_k$$

$$f_k = \lambda - b_k - \frac{a_{k-1}^2}{\left(\frac{a_{k-1} p_k}{p_{k-1}}\right) f_{k-1}}$$

observe that if $p_k = c_k \lambda^{k-1} +$ lower then

$$\frac{a_k c_{k+1}}{c_k} = 1 \qquad \text{so} \qquad c_k = \frac{1}{a_{k-1}\cdots a_1}$$

148

$$d_{k+1} = \det \begin{bmatrix} \lambda - b_1 & -a_1 & & \\ -a_1 & & & \\ & & \lambda - b_k & -a_k \\ & & -a_k & \lambda - b_{k+1} \end{bmatrix}$$

$$d_{k+1} = (\lambda - b_{k+1}) d_k - a_k^2 d_{k-1}$$

$$a_{k+1} \underbrace{\frac{d_{k+1}}{a_{k+1} \cdots a_1}}_{p_{k+2}} = (\lambda - b_{k+1}) \underbrace{\frac{d_k}{a_k \cdots a_1}}_{p_{k+1}} - a_k \underbrace{\frac{d_{k-1}}{a_{k-1} \cdots a_1}}_{p_k}$$

$$p_{k+1} = \frac{d_k}{a_k \cdots a_1}$$

You need to get control of Bargman kernel idea. With polys. Given a f.d Hilbert space H hermitian operator A and cyclic vector ξ, you get a map $\mathbb{C}[\lambda] \twoheadrightarrow H$, $p(\lambda) \longmapsto p(A)\xi$ and an induced scalar product on $\mathbb{C} + \mathbb{C}\lambda + \cdots + \mathbb{C}\lambda^{n-1}$ $n = \dim H$, whence pt evaluators and Bargman kernel.

There's a problem of ends.

You just reviewed the derivation you know of the Bargman kernel for orthogonal polys:

$$e_{\omega}(\lambda) = \frac{a_{n+1}}{\bar{\omega} - \lambda} \begin{vmatrix} p_{n+2}(\bar{\omega}) & p_{n+2}(\lambda) \\ p_{n+1}(\bar{\omega}) & p_{n+1}(\lambda) \end{vmatrix}$$

Here $p_{k+1} = \dfrac{d_k}{a_1 \cdots a_k}$ where $d_k = \det(\lambda - A_k)$

The meaning of this formula is very obscure at the moment.

149 List ideas: 1) Can you relate the formula for $e_\omega(\lambda)$ to a resolvent? 2) Find unitary analog of $e_\omega(\lambda)$. How might I proceed? Probably start from ~~...~~ Stieltjes theory. You decide that ~~...~~ basis of orthogonal polys isn't very important. Certain things are not so important e.g. probably the process.

$$d\mu \underset{\text{Stieltjes transform}}{\rightsquigarrow} \int \frac{1}{x-\lambda}\, d\mu(x) = \frac{-1}{|\lambda - b_1} - \frac{a_1^2}{|\lambda - b_2} - \frac{a_2^2}{|}$$

which should probably correspond to doing Gram-Schmidt process. ~~...~~ You probably have the ~~...~~ wrong end

Proceed as follows. ~~...~~ Basic object ~~...~~ admits ~~...~~ various descriptions:

1) partial herm. op of type $O(n)$

2) An scalar product on polys $\mathbb{C} + \mathbb{C}\lambda + \cdots + \mathbb{C}\lambda^n$ such that $(1,1) = 1$ and $\mu_{i+j} = (\lambda^i, \lambda^j)$ depends only on $i+j$ for $0 \le i,j \le n$. (Thus have $2n$ real nos $1 = \mu_0, \mu_1, \ldots, \mu_n$ $\ni$ matrix is > 0).

Consider $W = \begin{pmatrix} \varepsilon \\ \alpha \end{pmatrix} X \subset \begin{matrix} Y \\ \oplus \\ Y \end{matrix}$ equipped with $\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}^* \begin{pmatrix} & 1 \\ -1 & \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$

$W^0 \qquad (\varepsilon x)^* y_2 = (\alpha x)^* y_1 \quad \forall x \qquad \alpha^* y_1 = \varepsilon^* y_2$

$W \subset W^0$ means $\alpha^*\varepsilon = \varepsilon^*\alpha$. Assume ε inj, choose scal. prod on $X \ni \varepsilon^*\varepsilon = 1$. Recall calc. of W^0

$W^0 = \begin{pmatrix} \varepsilon \\ \alpha \end{pmatrix} X \oplus \begin{pmatrix} 1 \\ \varepsilon\alpha^* \end{pmatrix} jK \oplus \begin{pmatrix} 0 \\ 1 \end{pmatrix} jK$

$\alpha^* y_1 = \varepsilon^* y_2$

$y_1 = \varepsilon x + jk_1$

$\varepsilon^* y_2 = \underbrace{\alpha^*\varepsilon x}_{\varepsilon^*\alpha x} + \alpha^* j k_1$

$y_1 = \varepsilon x + jk_1$

$y_2 = \alpha x +$

$y_2 - \alpha$

Avoid calculation.

What's ~~important~~ new is the cyclic vector idea. ~~The~~ old thought pattern puts the cyclic vector first and the gate last, you need to straighten out the ideas. Given A self adj and ξ cyclic $\|\xi\|=1$. get $\left(\xi, \frac{1}{\omega - A}\xi\right) = \int \frac{1}{\omega - x}\, d\mu(x)$

Stieltjes transform of the measure. If finite support this is a rational function poles on R pos. imag. parts. Continued fraction. $\operatorname{Im}\omega > 0 \implies \operatorname{Im}\left(\frac{1}{\omega - x}\right) < 0$

$$\frac{1}{z} = \frac{1}{x+iy} = \frac{x-iy}{x^2+y^2} \qquad \operatorname{Im}\left(\frac{1}{z}\right) = \frac{-1}{|z|^2}\operatorname{Im}(z)$$

$$\operatorname{Im}\left(\frac{az+b}{cz+d}\right) = \frac{\operatorname{Im}\left((az+b)(c\bar z+d)\right)}{|cz+d|^2} = (ad-bc)\frac{\operatorname{Im}(z)}{|cz+d|^2}$$

$$ac|z|^2 + bd + adz + bc\bar z$$

The Stieltjes transf. of a f. meas has a C.F. expansion.

It seems that the picture, point of view, point of departure arising from ~~the~~ orthogonal polys is misleading. Apparently you don't want to start with ~~the~~ $\xi, A\xi, A^2\xi, \ldots$??

~~What can we do?~~

Start where?

Begin with Y, A, ξ

~~Question Consider a real~~

$$\begin{pmatrix} 1 & h \\ h & 1 \end{pmatrix}\begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix}$$

$$\left(\frac{1+X}{\sqrt{1-X^2}}\right)^2 = \frac{1+X}{1-X}$$

If $X = \begin{pmatrix} 0 & s \\ s & 0 \end{pmatrix}$

$$\frac{1-X}{1+X} = \left(\frac{1-X}{\sqrt{1-X^2}}\right)^2$$

$$\frac{1+X}{\sqrt{1-X^2}} = \frac{1}{\sqrt{1-s^2}}\begin{pmatrix} 1 & s \\ s & 1 \end{pmatrix}$$

151 Consider a partial unitary $W = \binom{a}{b} X \subset \begin{smallmatrix} Y \\ \oplus \\ Y \end{smallmatrix}$ of type $O(n)$. On X you have a partial unitary assoc. to the contraction of a^*b

$$\begin{pmatrix}1 & 0\\ -s & 1\end{pmatrix}\begin{pmatrix}1 & s\\ s & 1\end{pmatrix}\begin{pmatrix}1 & -s\\ 0 & 1\end{pmatrix} = \begin{pmatrix}1 & 0\\ -s & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ s & 1-s^2\end{pmatrix}$$

$$= \begin{pmatrix}1 & 0\\ 0 & 1-s^2\end{pmatrix}$$

Can you organize the relation between partial unitaries + pherms. better?

$$\begin{array}{c} \binom{a}{b} X \subset Y \oplus Y \\ \Big\downarrow \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ i & -i\end{pmatrix} \\ \binom{\varepsilon}{\alpha} X \subset Y \oplus Y \end{array}$$

In each picture there is a natural extension of the partial operator. In the bottom the operator is herm, in the top the operator is ba^*.

$$\binom{a}{b} = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & -i\\ 1 & i\end{pmatrix}\binom{\varepsilon}{\alpha} = \begin{pmatrix}(\varepsilon - i\alpha)/\sqrt{2}\\ (\varepsilon + i\alpha)/\sqrt{2}\end{pmatrix}$$

$$ba^* = \frac{1}{2}\left((\varepsilon + i\alpha)(\varepsilon^* + i\alpha^*)\right)$$

$$a^*b = \frac{1}{2}(\varepsilon^* + i\alpha^*)(\varepsilon + i\alpha) = \frac{1}{2}\left(\varepsilon^*\varepsilon - \alpha^*\alpha + i(\alpha^*\varepsilon + \varepsilon^*\alpha)\right)$$

$$b^*a = (\varepsilon^* - i\alpha^*)(\varepsilon - i\alpha)$$

$$2\alpha^*\varepsilon = -i(a^* - b^*)(a + b) = -i(1$$

152

$$b = \frac{1}{\sqrt{2}}(\varepsilon + i\alpha)$$

$$b^* = \frac{1}{\sqrt{2}}(\varepsilon^* - i\alpha^*)$$

?

$$b^*(\lambda\varepsilon - \alpha)$$

$$= \frac{b^*}{\sqrt{2}}(\lambda(a+b) - i(a-b)) = \frac{\lambda(b^*a+1) - i(b^*a-1)}{\sqrt{2}}$$

Clean up resolvent.
Abstract version.

$$\begin{pmatrix} a \\ b \end{pmatrix} X \subset Y \oplus Y \qquad \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}^* \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} y_1' \\ y_2' \end{pmatrix}$$

$$\Big\downarrow \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}$$

$$\begin{pmatrix} \varepsilon \\ \alpha \end{pmatrix} X \subset Y \oplus Y \qquad \frac{1}{2}\begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}$$

$$\frac{1}{2}\begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}\begin{pmatrix} 1 & -i \\ -1 & -i \end{pmatrix} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$W^\circ = W \oplus \begin{matrix} \operatorname{Ker} a^* \\ \operatorname{Ker} b^* \end{matrix} \supset W + \begin{pmatrix} \operatorname{Ker} a^* \\ 0 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 \\ ba^* \end{pmatrix} Y}_{V}$$

Then

$$\begin{tikzcd} \mathcal{O} \otimes V \arrow[r, "\sim"] \arrow[d] & \mathcal{O}(1) \otimes Y \\ \mathcal{O} \otimes \underbrace{V/W}_{\operatorname{Ker} a^*} & \end{tikzcd}$$

$$\begin{pmatrix} 1 \\ ba^* \end{pmatrix} y \longmapsto (z - ba^*) y$$

$$V \longrightarrow Y$$

$$(1 - aa^*)(z - ba^*)^{-1} y \longleftarrow y$$

$$\begin{matrix} Y \\ \oplus \\ Y \end{matrix} = \begin{pmatrix} a \\ b \end{pmatrix} X \oplus \begin{pmatrix} 1 \\ 0 \end{pmatrix} V^+ \oplus \begin{pmatrix} 1 \\ z \end{pmatrix} Y \qquad \text{off spec.}$$

$$W^\circ = \begin{pmatrix} a \\ b \end{pmatrix} X \oplus \begin{pmatrix} 1 \\ 0 \end{pmatrix} V^+ \oplus \begin{pmatrix} 0 \\ 1 \end{pmatrix} V^- \ni \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

$$y = ax + v^+.$$

$$zy = bx + v^-$$

$$zv^+ = v^- - (az-b)(z-a^*b)^{-1} a^* v^-$$

$$(az - b)x = -zv^+ + v^-$$

$$x = (z - a^*b)^{-1} a^* v^-$$

$$zv^+ = (1 - (az-b)(z-a^*b)^{-1} a^*) v^-$$

$$1-(az-b)a^*(z-ba^*)^{-1}$$

$$=\left((z-\cancel{ba^*})-(az-\cancel{b})a^*\right)(z-ba^*)^{-1} = z(1-aa^*)(z-ba^*)^{-1}$$

Important should be the quasi-det. link between the responses for a^*b and ba^*.

Idea - maybe having operators ~~is~~ not as important as correspondences.

Potentially interesting idea: Compactifying ~~the~~ operators ~~the~~ via their graphs.

skew ~~adjoint~~ symmetric operators in a real context.

Y [illegible] Euclidean space $\begin{pmatrix} a \\ b \end{pmatrix} X \subset \begin{matrix} Y \\ \oplus \\ Y \end{matrix}$ partial orth. ~~[illegible]~~ operator: $a^*a = b^*b$. Have a quadratic (symm. bilinear) form. $\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}^* \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = y_1^* y_1' - y_2^* y_2'$

If $W = \begin{pmatrix} a \\ b \end{pmatrix} X$, then $W^0 = \left\{ \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \middle| \cancel{y_1^* ax - y_2^* bx \;\; \forall x} \;\; a^* y_1 = b^* y_2 \right\}$

and $W \subset W^0$ means $a^*a = b^*b$. Now perform C.T.

$$W = \begin{pmatrix} a \\ b \end{pmatrix} X \subset \begin{matrix} Y \\ \oplus \\ Y \end{matrix}$$

$$\Big\downarrow S \quad \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

carries $\xi^* \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \xi$ into

$$\underbrace{\frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 & \\ & -1 \end{pmatrix}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$\frac{1}{2}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 0 & +1 \\ 1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} \varepsilon \\ \alpha \end{pmatrix} X \subset \begin{matrix} Y \\ \oplus \\ Y \end{matrix}$$

$$\left[\begin{pmatrix} \varepsilon \\ \alpha \end{pmatrix} X\right]^0 = \left\{ \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \middle| \cancel{(\varepsilon x)^* y_2 + (\alpha x)^* y_1 \;\; \forall x} = 0 \;\; \varepsilon^* y_2 = -\alpha^* y_1 \right\} \supset \begin{pmatrix} \varepsilon \\ \alpha \end{pmatrix} X$$

iff $\varepsilon^*\alpha = -\alpha^*\varepsilon$

Assume ε injective and define $\|x\| = \|\varepsilon x\|$, i.e. $\varepsilon^*\varepsilon = 1$.

graphs $\begin{pmatrix} 1 \\ m \end{pmatrix}$ are isot. for $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ if $m + m^* = 0$.

Focus on the Galois theory.

$$\begin{pmatrix} a \\ b \end{pmatrix} X \subset \begin{matrix} Y \\ \oplus \\ Y \end{matrix} \xrightarrow{\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}} \begin{matrix} Y \\ \oplus \\ Y \end{matrix} \supset \begin{pmatrix} a+b \\ ia-ib \end{pmatrix} X \qquad \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

assume ε^{a+b} injective with closed image

Then can assume $\varepsilon^*\varepsilon = 1$ and split

$$Y \xrightarrow{\begin{pmatrix} \varepsilon^* \\ j^* \end{pmatrix}} \begin{matrix} X \\ \oplus \\ K \end{matrix} \xrightarrow{(\varepsilon \ j)} Y$$

and you find

$$W^0 = \underbrace{\begin{pmatrix} \varepsilon \\ \alpha \end{pmatrix} X \oplus \begin{pmatrix} 1 \\ \varepsilon\alpha^* \end{pmatrix} jK}_{\begin{pmatrix} 1 \\ A \end{pmatrix} Y} \oplus \begin{pmatrix} 0 \\ 1 \end{pmatrix} jK$$

You get picture of A

$$A = \left[\begin{array}{c|c} 0 & j^*\alpha \\ \hline \alpha^* j & \varepsilon^*\alpha = \alpha^*\varepsilon \end{array}\right]$$

and you have the quasi-det. reln.

$$j^*(\lambda - A)^{-1} j = \frac{1}{\lambda - j^*\alpha \frac{1}{\lambda - \varepsilon^*\alpha} \alpha^* j}$$

If you arrange the basis in X such that $\alpha^* j = \begin{pmatrix} a_1 \\ 0 \end{pmatrix}$ then

$$A = \left(\begin{array}{c|c} 0 & a_1 \\ \hline a_1 & \varepsilon^*\alpha \end{array}\right) \qquad \left(j^*(\lambda - A)^{-1} j\right)^{-1} = \lambda - a_1^2 \, j_1^*(\lambda - \varepsilon^*\alpha)^{-1} j.$$

giving the recursion relation $R_1 = \dfrac{1}{\lambda - a_1^2}$

In general there should be a b_1

$$R_1 = \frac{1}{\lambda - b_1 -} \frac{a_1^2}{\lambda - b_2 -} \frac{a_2^2}{\lambda - b_3 -} = \xi_1^* \frac{1}{\lambda - A} \xi_1$$

Spectral representation. How?

$$\begin{pmatrix} l_\omega \end{pmatrix}^0 = \left(\begin{pmatrix} 1 \\ \omega \end{pmatrix} \mathbb{C}\right)^0 = \begin{pmatrix} 1 \\ \bar\omega \end{pmatrix} \mathbb{C}$$

$$W \cap l_\omega \otimes Y = 0 \iff W^0 + l_{\bar\omega} \otimes Y = T \otimes Y.$$

$$\mathcal{L}_\omega = W^0 \cap l_\omega \otimes Y \longrightarrow$$

$$V^0 \cap l_{\bar\omega} \otimes Y = 0$$

$$\iff V + l_\omega \otimes Y = 0$$

Need $W \subset V \subset W^0$, $V = \begin{pmatrix} 1 \\ \alpha \end{pmatrix} Y$

+ if so you get

$$V \xrightarrow{\sim} T/l_\omega \otimes Y$$

$$\downarrow$$

$$V/W \subset W^0/W$$

155 So what is the real point

$$V = \begin{pmatrix}1\\ \tilde{\alpha}\end{pmatrix} Y \xrightarrow{(\omega \ \ -1)} Y \qquad \begin{pmatrix}1\\ \tilde{\alpha}\end{pmatrix} y \longmapsto (\omega - \tilde{\alpha}) y$$

so the map is $\quad y \longmapsto \begin{pmatrix}(\omega-\tilde{\alpha})^{-1} y \\ \varepsilon\alpha^*(\omega-\tilde{\alpha})^{-1} y\end{pmatrix} \longmapsto j^*(\omega-\tilde{\alpha})^{-1} y.$

$$V/W = \begin{pmatrix}1\\ \varepsilon\alpha^*\end{pmatrix} jK$$

Review. $\begin{pmatrix}\varepsilon\\ \alpha\end{pmatrix} X \subset \begin{matrix}Y\\ \oplus\\ Y\end{matrix}$ part. hom.
Is there something more to be found about the resolvent, L^2 embeddings, maybe on the level of the double and subspaces instead of operators. Somehow the theme should be to compactify operators via their graphs — Grass. graph construction exploited by MacPherson, Fulton, et al.

Above you have an intrinsic situation, namely W isotropic in $T \otimes Y$, and a map sending $y \in Y$ to rational sections of $\mathcal{O}(-1)$

$$\begin{tikzcd}
\mathcal{O}(-1)\otimes Y \arrow[r, hook] & \mathcal{O}\otimes T \otimes Y \arrow[r, two heads] & \mathcal{O}(1)\otimes Y \\
& \mathcal{O}\otimes V \arrow[u] \arrow[ur, "\sim \text{ off spectrum}"'] &
\end{tikzcd}$$

effect is a natural map $\quad Y \longrightarrow \mathcal{O}(-1)\otimes V$
off the spectrum.

You want to restrict to a circle.

Look at the case when V is isotropic i.e. $V = \begin{pmatrix}1\\ A\end{pmatrix} Y$ where $A^* = A$. You still have the cyclic vector $e_1 = \xi$. $\quad j^* \dfrac{1}{\lambda - A} j = e_1^* \dfrac{1}{\lambda - A} e_1$

$$= \sum_{j=1}^{n+1}$$

156 Consider $W = \begin{pmatrix} a \\ b \end{pmatrix} X \subset \begin{matrix} \mathcal{Y} \\ \oplus \\ \mathcal{Y} \end{matrix}$ $\qquad \|y_1\|^2 - \|y_2\|^2$

Have $\mathcal{Y} = aX \oplus V^+$

$\quad = bX \oplus V^-$

eigenvector $\quad z(ax + v^+) = bx + v^-$

$\qquad (za - b)x = -zv^+ + v^-$

scattering picture

$$y = aa^* y + \overbrace{(1-aa^*)}^{\pi} y$$

$$uy = aa^* ba^* y + \pi ba^* y + u\pi y$$

$$u^2 y = aa^*(ba^*)^2 y + \pi (ba^*)^2 y + u\pi(ba^*) y + u^2 \overset{(1-aa^*)}{\pi} y$$

$$y \longmapsto \sum_{n \geq 0} z^{-n} \pi (ba^*)^n y = \frac{1}{1 - z^{-1} ba^*} y$$

Basically you need to understand the transf.

$k_0 = \sqrt{1-h_0^2}$ $\qquad S_0(z) = \frac{1}{k_0}\begin{pmatrix} 1 & h_0 \\ h_0 & 1 \end{pmatrix}\begin{pmatrix} z^{1/2} & 0 \\ 0 & z^{-1/2} \end{pmatrix} S_1(z) \qquad \begin{pmatrix} 1+h_0 & 0 \\ 0 & 1-h_0 \end{pmatrix}$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \|$

to $\frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 & h_0 \\ h_0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1+h_0 & 1-h_0 \\ 1+h_0 & -1+h_0 \end{pmatrix}$

$$\frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} z^{1/2} & z^{1/2} \\ z^{-1/2} & -z^{-1/2} \end{pmatrix} = \begin{pmatrix} \frac{z^{1/2}+z^{-1/2}}{2} & \frac{z^{1/2}-z^{-1/2}}{2} \\ \frac{z^{1/2}-z^{-1/2}}{2} & \frac{z^{1/2}+z^{-1/2}}{2} \end{pmatrix} = \begin{pmatrix} \cos & i\sin \\ i\sin & \cos \end{pmatrix}$$

$$= \frac{1}{\sqrt{1-s^2}}\begin{pmatrix} 1 & s \\ s & 1 \end{pmatrix}$$ OKAY

where $s = \frac{i\sin}{\cos} = i\tan\theta$

when $z = e^{i\theta}$

You believe that provided the $h_n \not\to -1$ so that $\overset{\rho_n}{\frac{1+h_n}{1-h_n}} \not\to 0$

that then $R_0(s) = \begin{pmatrix} \rho_0^2 & \\ & 1 \end{pmatrix}\begin{pmatrix} 1 & s \\ s & 1 \end{pmatrix}\begin{pmatrix} \rho_1^2 & \\ & 1 \end{pmatrix}\begin{pmatrix} 1 & s \\ s & 1 \end{pmatrix}$

converges to an merom. fn. of s.

157

OKAY

$$\begin{pmatrix} 1 & 1+\varepsilon \\ 1+\varepsilon & 1 \end{pmatrix} \Big(\underbrace{1-(1+\varepsilon)^2}_{-2\varepsilon-\varepsilon^2}\Big)^{-1/2}$$

$z \neq z$

$$\frac{1}{s+2i} + \frac{1}{s-2i} = \frac{2s}{s^2+4}$$

Basic ~~recursion~~ type of function

$$f(s) = \int_0^\infty \frac{s(1+\omega^2)}{s^2+\omega^2}\, d\mu(\omega) + \mu_\infty s$$

where $d\mu$ is a probability measure on $[0,\infty]$.

discrete case $f(s) = \sum \frac{s(1+\omega^2)}{s^2+\omega^2} a_\omega$

$$W_1 \xrightarrow{j_1} H^+ \oplus H^-$$

$$\downarrow p$$

$$W_1/W_0$$

$$s\pi^+ \oplus s^{-1}\pi^-$$

$$\rho = j_1^* \pi^+ j_1 = \bigoplus_\omega$$

$$W_1 = \bigoplus_\omega W_{1,\omega}$$

$$j^*\pi^+ j = \bigoplus \frac{1}{1+\omega^2}\pi_{1,\omega}$$

$$j^*\pi^- j = \bigoplus \frac{\omega^2}{1+\omega^2}\pi_{1,\omega}$$

$$j^*(s\pi^+ + s^{-1}\pi^-)j = \bigoplus \frac{s+s^{-1}\omega^2}{1+\omega^2}\pi_{1,\omega}$$

$$\big(j^*(s\pi^+ + s^{-1}\pi^-)j\big)^{-1} = \bigoplus \frac{s(1+\omega^2)}{s^2+\omega^2}\pi_{1,\omega}$$

$$p\big(j^*(\quad)^{-1}j\big)^{-1}p^* = \sum \frac{s(1+\omega^2)}{s^2+\omega^2}\, p\pi_{1,\omega}p^*$$

158 $\begin{pmatrix} 1 & -s \\ -s & 1 \end{pmatrix} \dfrac{s(1+\omega^2)}{s^2+\omega^2} = \dfrac{s+\cancel{s\omega^2}-s^3-\cancel{s\omega^2}}{\cancel{s^2}-s^2\omega^2\cancel{+s^2}+\omega^2} = \dfrac{s(1-s^2)}{\omega^2(1-s^2)} = \dfrac{s}{\omega^2}$

$$\frac{s(1+\omega^2)}{s^2+\omega^2} = \begin{pmatrix} 1 & s \\ s & 1 \end{pmatrix}\begin{pmatrix} \frac{1}{\omega^2} & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & s \\ s & 1 \end{pmatrix}(0)$$

$$\begin{pmatrix} 1 & -s \\ -s & 1 \end{pmatrix} \frac{s^2+\omega^2}{s(1+\omega^2)} = \frac{\cancel{s^2}+\omega^2-\cancel{s^2}-s\omega^2}{-s^3-\cancel{s\omega^2}+s+\cancel{s\omega^2}} = \frac{\omega^2(1-s)}{s(1-s)} = \frac{\omega^2}{s}$$

$$\frac{s^2+\omega^2}{s(1+\omega^2)} = \begin{pmatrix} 1 & s \\ s & 1 \end{pmatrix}\begin{pmatrix} \omega^2 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & s \\ s & 1 \end{pmatrix}(\infty)$$

What else? Need to calculate ~~some~~ examples.

Start with

$$\omega = \rho\frac{\omega+s}{s\omega+1}$$

$$s\omega^2+\omega = \rho\omega + \rho s$$

$$s\omega^2 + (1-\rho)\omega - \rho s = 0$$

$$\omega = \frac{-(1-\rho) \pm \sqrt{(1-\rho)^2+4\rho s}}{2s}$$

$a^2-2a+1+4as^2 = (a-1)^2+4as^2$

START AGAIN

$a^2+2a+1-4a+4as^2$

$(a-1)^2+4as^2$

$$\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & s \\ s & 1 \end{pmatrix} = \begin{pmatrix} a & as \\ s & 1 \end{pmatrix}$$

$$\lambda^2 - (a+1)\lambda + a(1-s^2) = 0$$

$$\lambda = \frac{a+1 \pm \sqrt{(a+1)^2-4a(1-s^2)}}{2}$$

Fix $s \in i\mathbb{R}$ let $a > 0$

thus $(a-1)^2+4as^2 < (a-1)^2$

so $\sqrt{(a-1)^2+4as^2} < 1-a$ assuming $0<a<1$

and so ~~both roots~~ ~~$a+$~~

$$\lambda_1 < \frac{a+1+1-a}{2} = 1 \qquad \lambda_2 > \frac{a+1-(1-a)}{2} = a$$

159 So what to write.

$$\begin{pmatrix}1 & -s\\ -s & 1\end{pmatrix}\frac{s(1+\omega^2)}{s^2+\omega^2} = \frac{s+\cancel{s\omega^2}-s^3-\cancel{s\omega^2}}{-\cancel{s^2}-s\omega^2+\cancel{s^2}+\omega^2} = \frac{s(1-s^2)}{\omega^2(1-s^2)} = \frac{s}{\omega^2}$$

$$\frac{s(1+\omega^2)}{s^2+\omega^2} = \begin{pmatrix}1 & s\\ s & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & \omega^2\end{pmatrix}\begin{pmatrix}1 & s\\ s & 1\end{pmatrix}\begin{pmatrix}0\\ 1\end{pmatrix}$$

$$\frac{s^2+\omega^2}{s(1+\omega^2)} = \begin{pmatrix}1 & s\\ s & 1\end{pmatrix}\begin{pmatrix}\omega^2 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & s\\ s & 1\end{pmatrix}\begin{pmatrix}1\\ 0\end{pmatrix}$$

You want to ~~solve~~ find a solution of

$$\psi_n = \begin{pmatrix}a_n & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & s\\ s & 1\end{pmatrix}\psi_{n+1} \qquad \psi_n = \begin{pmatrix}\psi_n^+\\ \psi_n^-\end{pmatrix}$$

where $a_n > 0$ tending to 0 such that $\psi_n \longrightarrow 0$. First discuss const coeff case. i.e. want eigenvalues of $\begin{pmatrix}a & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & s\\ s & 1\end{pmatrix} = \begin{pmatrix}a & as\\ s & 1\end{pmatrix}$

$$\lambda^2 - (a+1)\lambda + a(1-s^2) = 0$$

$$\lambda = \frac{a+1 \pm \sqrt{(a+1)^2 - 4a(1-s^2)}}{2}$$

$$(a+1)^2 - 4a(1-s^2) = a^2+2a+1-4a+4as^2$$
$$= a^2-2a+1+4as^2$$

$a>0$ small, as $a \downarrow 0$ this approaches 1.

Interested in case $s \in i\mathbb{R}$ $\quad s^2<0$

$$a^2-2a+1+\underbrace{4as^2}_{<0} < (a-1)^2$$

160 ~~So its not really clear that you~~ So are roots

$$\lambda = \frac{a+1}{2}\left(1 \pm \underbrace{\sqrt{1 - \frac{4a}{(1+a)^2}(1-s^2)}}_{1 - \frac{2a}{(1+a)^2}(1-s^2)}\right)$$

$$\lambda_1 \sim a+1 \qquad \lambda_2 \sim \frac{a}{1+a}(1-s^2) \sim a(1-s^2)$$

~~up to~~ error $O(a^2)$.

anyway $\quad w = \begin{pmatrix} a & as \\ s & 1 \end{pmatrix} w = \dfrac{aw + as}{sw+1}$

$$s w^2 + w = aw + as \qquad \text{three variables}$$

$$s w^2 + (1-a) w - as = 0$$

$$w = \frac{-(1-a) \pm \sqrt{(1-a)^2 + 4as^2}}{2s}$$

$$w = \frac{a-1}{2s} \pm \sqrt{\left(\frac{1-a}{2s}\right)^2 + a}$$

Review $\quad w = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}\begin{pmatrix} 1 & s \\ s & 1 \end{pmatrix}(w) = \dfrac{aw + as}{a^{-1}sw + a^{-1}}$

$$s w^2 + w = a^2 w + a^2 s$$

$$s w^2 + (1-a^2) w - a^2 s = 0 \qquad w = \frac{a^2-1}{2s} \pm \sqrt{\left(\frac{a^2-1}{2s}\right)^2 + a^2}$$

$$w^2 + \frac{1-a^2}{s} w - a^2 = 0$$

$$\lambda \begin{pmatrix} \psi^+ \\ \psi^- \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}\begin{pmatrix} 1 & s \\ s & 1 \end{pmatrix}\begin{pmatrix} \psi^+ \\ \psi^- \end{pmatrix} = \begin{pmatrix} a & as \\ a^{-1}s & a^{-1} \end{pmatrix}$$

$$\lambda^2 - (a + a^{-1})\lambda + (1-s^2) = 0$$

$$\lambda = \frac{a+a^{-1}}{2} \pm \sqrt{\left(\frac{a+a^{-1}}{2}\right)^2 - 1 + s^2}$$

$$\sqrt{\left(\frac{a-a^{-1}}{2}\right)^2 + s^2}$$

161 ~~Cons~~ Consider an vector space generated by elements $\psi_n^{\pm}$ $n \in \mathbb{Z}$, ~~satisfying and let define~~ s and let s be an ~~the~~ operator satisfying

$$\begin{pmatrix} \psi_n^+ \\ \psi_n^- \end{pmatrix} = \begin{pmatrix} a_n & 0 \\ 0 & a_n^{-1} \end{pmatrix} \begin{pmatrix} 1 & s \\ s & 1 \end{pmatrix} \begin{pmatrix} \psi_{n+1}^+ \\ \psi_{n+1}^- \end{pmatrix}$$

$$a_n^{-1} \psi_n^+ = \psi_{n+1}^+ + s\psi_{n+1}^-$$

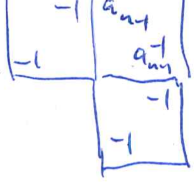
$$a_n \psi_n^- = s\psi_{n+1}^+ + \psi_{n+1}^-$$

$$s\psi_{n+1}^+ = -\psi_{n+1}^- + a_n \psi_n^-$$

$$s\psi_{n+1}^- = -\psi_{n+1}^+ + a_n^{-1} \psi_n^+$$

$$s\begin{pmatrix} \psi_{n+1}^+ \\ \psi_{n+1}^- \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \psi_{n+1}^+ \\ \psi_{n+1}^- \end{pmatrix} + \begin{pmatrix} a_n & 0 \\ 0 & a_n^{-1} \end{pmatrix} \begin{pmatrix} \psi_n^+ \\ \psi_n^- \end{pmatrix}$$

$$s\begin{pmatrix} \psi_n^+ \\ \psi_n^- \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \psi_n^+ \\ \psi_n^- \end{pmatrix} + \begin{pmatrix} a_{n-1} & 0 \\ 0 & a_{n-1}^{-1} \end{pmatrix} \begin{pmatrix} \psi_{n-1}^+ \\ \psi_{n-1}^- \end{pmatrix}$$



$$s\psi_n = -\varepsilon_x \psi_n + a\psi_{n-1} \qquad s = -\varepsilon_x + a\sigma$$

$$s^2 = (-\varepsilon_x + a\sigma)(-\varepsilon_x + a\sigma)$$

$$= \underline{1} - (\varepsilon_x a + a\varepsilon_x)\sigma + a\sigma a\sigma$$

So what does this mean?

$$\varepsilon_x a + a\varepsilon_x = \begin{pmatrix} a_n & 0 \\ 0 & a_n^{-1} \end{pmatrix} \begin{pmatrix} & -1 \\ -1 & \end{pmatrix} + \begin{pmatrix} & -1 \\ -1 & \end{pmatrix} \begin{pmatrix} a_n & \\ & a_n^{-1} \end{pmatrix}$$

$=$

162. Observe the eigenvalues λ — typically s is fixed $\in i\mathbb{R}$ and $a \searrow 0$. Then $\frac{a+a^{-1}}{2} \nearrow \infty$ while $(-1+s^2)$ is fixed and typically ≤ -1

One has $\lambda_1 + \lambda_2 = a + a^{-1}$

$$\lambda_1 \lambda_2 = 1 - s^2 \text{ fixed } \geq 1.$$

It seems as if

$$\begin{pmatrix} \psi_n^+ \\ \psi_n^- \end{pmatrix} = \begin{pmatrix} a_{n+1} & a_{n+1} s \\ a_{n+1}^{-1} s & a_{n+1}^{-1} \end{pmatrix} \begin{pmatrix} \psi_{n+1}^+ \\ \psi_{n+1}^- \end{pmatrix}$$

You want power series solutions. $\psi_n^\pm = \sum_{k=0}^{\infty} \psi_{n,k}^\pm s^k$

$$\psi_{n,k}^+ = a_{n+1}\left(\psi_{n+1,k}^+ + \psi_{n+1,k-1}^-\right)$$

$$\psi_{n,k}^- = a_{n+1}^{-1}\left(\psi_{n+1,k-1}^+ + \psi_{n+1,k}^-\right)$$

$\psi_{n,0}^+ = a_{n+1}\psi_{n+1,0}^+$ since $a_{n+1} \downarrow 0$ ~~and we want~~

this implies $\psi_{n,0}^+$ grows fast

unless $\psi_{n,0}^+ = 0$ for all n.

$\psi_{n,0}^- = a_{n+1}^{-1}\psi_{n+1,0}^-$ $\quad \therefore \quad \psi_{n,0}^- = a_1 a_2 \cdots a_n$

$$\psi_{n,1}^+ - a_{n+1}\psi_{n+1,1}^+ = a_{n+1}\psi_{n+1,0}^- = a_1 a_2 \cdots a_n a_{n+1}^2$$

$$\psi_{n,1}^- = \cancel{a_{n+1}^{-1}\psi_{n+1,0}^+} + a_{n+1}^{-1}\psi_{n+1,1}^-$$

$$\psi_{n+1,1}^+ = a_{n+1}^{-1}\psi_{n,1}^+ - a_1 \cdots a_{n+1}$$

163 Power series expansion in s

$$\psi_n(s) = \begin{pmatrix} a_{n+1} & 0 \\ 0 & a_{n+1}^{-1} \end{pmatrix} \begin{pmatrix} 1 & s \\ s & 1 \end{pmatrix} \psi_{n+1}(s) \qquad \text{here } \psi_n = \begin{pmatrix} \psi_n^+ \\ \psi_n^- \end{pmatrix}$$

$$\psi_n(0) = \begin{pmatrix} a_{n+1} & 0 \\ 0 & a_{n+1}^{-1} \end{pmatrix} \psi_{n+1}(0)$$ yields recursion relations

$$\psi_n^+(0) = a_{n+1}\,\psi_{n+1}^+(0)$$
$$\psi_n^-(0) = a_{n+1}^{-1}\,\psi_{n+1}^-(0)$$

Since $a_n \downarrow 0$ the former implies ψ^+ unbounded [redacted] unless $\psi^+ = 0$. The latter has the solution $\psi_n^-(0) = a_1 \cdots a_n$ which decays exponentially.

$$\psi_n'(s) = \begin{pmatrix} a_{n+1} & \\ & a_{n+1}^{-1} \end{pmatrix} \left[\begin{pmatrix} 1 & s \\ s & 1 \end{pmatrix} \psi_{n+1}'(s) + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \psi_{n+1}(s) \right]$$

$$\psi_n'(0) = \begin{pmatrix} a_{n+1} & \\ & a_{n+1}^{-1} \end{pmatrix} \psi_{n+1}'(0) + \begin{pmatrix} 0 & a_{n+1} \\ a_{n+1}^{-1} & 0 \end{pmatrix} \psi_{n+1}(0)$$

Let $f_n = \psi_n'(0)^+ \qquad \psi_{n+1}(0) = \begin{pmatrix} 0 \\ g_{n+1} \end{pmatrix} \qquad g_{n+1} = a_1 \cdots a_{n+1}$

Then

$$f_n = a_{n+1} f_{n+1} + a_{n+1} g_{n+1}$$

$$f_{n+1} = \frac{1}{a_{n+1}} f_n - g_{n+1}$$

$$f_1 = \frac{1}{a_1} f_0 - a_1 = \frac{1}{a_1}(f_0 - a_1^2)$$

$$f_2 = \frac{1}{a_2 a_1}(f_0 - a_1^2) - a_1 a_2 = \frac{1}{a_2 a_1}(f_0 - a_1^2 - a_1^2 a_2^2)$$

$$f_3 = \frac{1}{a_3 a_2 a_1}(f_0 - a_1^2 - a_1^2 a_2^2 - a_1^2 a_2^2 a_3^2) \qquad \text{etc.}$$

In order for this to be bounded we need $f_0 = a_1^2 + a_1^2 a_2^2 + \cdots$ which converges, as its dominated by a geometric series.

 So what am I doing? Next $\psi_n'(0)^-$ satisfies $\psi_n'(0)^- = \frac{1}{a_{n+1}} \psi_{n+1}'(0)^-$ so it can be any multiple of g: $g_n = a_1 \cdots a_n$.
The arbitrariness should be due to the fact that only a line depending analytically on s is defined by the decay condition on ψ_n as $\to \infty$.

Ask what you get from finite

$$\psi_0(s) = \underbrace{\begin{pmatrix} a_1 & 0 \\ 0 & a_1^{-1} \end{pmatrix}\begin{pmatrix} 1 & s \\ s & 1 \end{pmatrix} \cdots\cdots\cdots \begin{pmatrix} a_n & 0 \\ 0 & a_n^{-1} \end{pmatrix}\begin{pmatrix} 1 & s \\ s & 1 \end{pmatrix}} \psi_n(s)$$

n even

$$\begin{pmatrix} \deg 2n \text{ even} & \deg 2n-1 \text{ odd} \\ \deg 2n-1 \text{ odd} & \deg 2n \text{ even} \end{pmatrix}$$

$$\begin{pmatrix} 1 & s \\ s & 1 \end{pmatrix}^{2n} = \begin{pmatrix} 1+s^2 & 2s \\ 2s & 1+s^2 \end{pmatrix}^n$$

$$\parallel$$

$$\left(I + s\varepsilon\right)^{2n} = 1 + 2n s\varepsilon + \frac{2n(2n-1)}{2!} s^2 +$$

just get odd & even parts of the binomial expn.
If n odd things get reversed a bit.

Digress to ζ-function for $\mathbb{Z}[i]$, this is a UFD in fact Euclidean domain I think. Important are the primes. p odd prime in $\mathbb{Z}$, then p prime in $\mathbb{Z}[i]$ for $p \equiv 3$ (4) and p splits if $p \equiv 1$ (4).

$\pi = a + bi$ $\quad$ a, b rel. prime one even

$\pi\bar{\pi} = a^2 + b^2 \equiv 1.$ $\quad$ $\mathbb{Z}[i]/2\mathbb{Z}[i]$ 4 elt.

2 prime in $\mathbb{Z}$ $\quad$ $2 = (1+i)(1-i)$

165

$$\zeta_{\mathbb{Z}[i]}(s) = \frac{1}{(1-2^{-s})^2} \prod_{p\equiv 1(4)} \frac{1}{(1-p^{-s})^2} \prod_{p\equiv 3(4)} \frac{1}{1-p^{-2s}}$$

$$\frac{\zeta_{\mathbb{Z}[i]}(s)}{\zeta_{\mathbb{Z}}(s)} = \frac{1}{(1-2^{-s})} \prod_{p\equiv 1(4)} \frac{1}{1-p^{-s}} \prod_{p\equiv 3(4)} \frac{1}{1+p^{-s}}$$

$$= \frac{1}{1-2^{-s}} \sum_{\substack{n \text{ odd} \\ 1\le n<\infty}} \frac{\chi(n)}{n^s} \qquad \chi(p) = \begin{cases} 1 & p\equiv 1(4) \\ -1 & p\equiv 3(4) \end{cases}$$

$\chi(1) = 1$
3 = -1
5 = 1
7 -1
9 1
11 -1
13 1
15 -1

$$\chi(n) = \begin{cases} 1 & n\equiv 1\ (4) \\ -1 & n\equiv -1\ (4) \end{cases}$$

$$\Gamma(s) \sum_{\substack{n \text{ odd} \\ 1\le n<\infty}} \frac{\chi(n)}{n^s} = \int_0^\infty \underbrace{\sum_n e^{-nt}\chi(n)}\, t^s \frac{dt}{t}$$

$$\sum_{k=0}^{\infty} \left(e^{-(1+4k)t} - e^{-(3+4k)t}\right)$$

$$= (e^{-t} - e^{-3t}) \frac{1}{1-e^{-4t}}$$

$$= e^{-t}(1-e^{-2t}) \frac{1}{1-e^{-4t}}$$

$$= e^{-t} \frac{1}{1+e^{-2t}} = \frac{1}{e^t+e^{-t}}$$

back to $\psi_n(s) = \begin{pmatrix} a_{n+1} & 0 \\ 0 & a_{n+1}^{-1} \end{pmatrix}\begin{pmatrix} 1 & s \\ s & 1 \end{pmatrix} \psi_{n+1}(s)$. The attempt with power series expansion around $s=0$ looks like it won't help very much because apparently there is no way to choose initial conditions. ~~Still~~ Still domination is a good idea - examine const. coeff case.

$$\lambda\psi(s) = \underbrace{\begin{pmatrix} a & 0 \\ 0 & a^{-1}\end{pmatrix}\begin{pmatrix} 1 & s \\ s & 1\end{pmatrix}}\psi(s) = \begin{pmatrix} a & as \\ a^{-1}s & a^{-1}\end{pmatrix}\psi(s)$$

$$\lambda^2 - (a+a^{-1})\lambda + (1-s^2) = 0.$$

$$\lambda = \frac{a+a^{-1}}{2} \pm \sqrt{\left(\frac{a+a^{-1}}{2}\right)^2 - 1 + s^2}$$

Think of a as being small ~~>~~ 0 whence the large root is large like $a+a^{-1}$ and the small root is ~~…~~ like $\frac{1-s^2}{a+a^{-1}}$. Can you find $\psi(s)$

$$\underbrace{(\lambda-a)(\lambda-a^{-1})}_{\lambda^2-(a+a^{-1})\lambda+1} \overset{?}{=} s^2$$

$$\begin{pmatrix} \lambda-a & -as \\ -a^{-1}s & \lambda-a^{-1}\end{pmatrix}\psi(s) = 0$$

$$\therefore \quad \psi(s) = \begin{pmatrix} as \\ \lambda-a\end{pmatrix}\mathbb{C} = \begin{pmatrix} \lambda-a^{-1} \\ a^{-1}s\end{pmatrix}\mathbb{C}$$

which root do you want?

$$\psi_n(s) = \lambda\psi_{n+1}(s)$$

so you want λ large in order that $\psi_n(s)$ decay as $n \to \infty$.

$$\psi_0(s) = \begin{pmatrix} as \\ \lambda-a\end{pmatrix}\mathbb{C}$$

where does this work? We have s in some nbd of 0.

$$\lambda = \frac{a+a^{-1}}{2} \pm \sqrt{\left(\frac{a-a^{-1}}{2}\right)^2 + s^2}$$ is analytic in s

for $|s| < \left|\frac{a-a^{-1}}{2}\right|$ in fact sing. pts are $s = \pm i\,\frac{a-a^{-1}}{2}$

167 You want the large root

$$\lambda = \frac{a+a^{-1}}{2} + \sqrt{\left(\frac{a-a^{-1}}{2}\right)^2 + s^2}$$

Then $$\psi_0(s) = \begin{pmatrix} as \\ \lambda - a \end{pmatrix} = \begin{pmatrix} as \\ \frac{a^{-1}-a}{2} + \sqrt{\left(\frac{a^{-1}-a}{2}\right)^2 + s^2} \end{pmatrix}$$

So $$\begin{pmatrix} a & as \\ a^{-1}s & a^{-1} \end{pmatrix}\begin{pmatrix} as \\ \lambda - a \end{pmatrix} = \begin{pmatrix} a^2 s + as(\lambda - a) \\ s^2 + a^{-1}(\lambda - a) \end{pmatrix}$$

$$\lambda(\lambda - a) = (\cancel{a} + a^{-1})\lambda \mp (1 - s^2) \cancel{- \lambda a}$$
$$= a^{-1}\lambda \mp 1 + s^2$$

---

sit down and try to understand convergence.

first you need to understand the ~~range~~ singular points $s = \pm i \frac{a-a^{-1}}{2}$ In this case

$$1 - s^2 = 1 + \left(\frac{a-a^{-1}}{2}\right)^2 = \frac{a^2 - 2 + a^{-2} + 4}{4} = \left(\frac{a+a^{-1}}{2}\right)^2$$

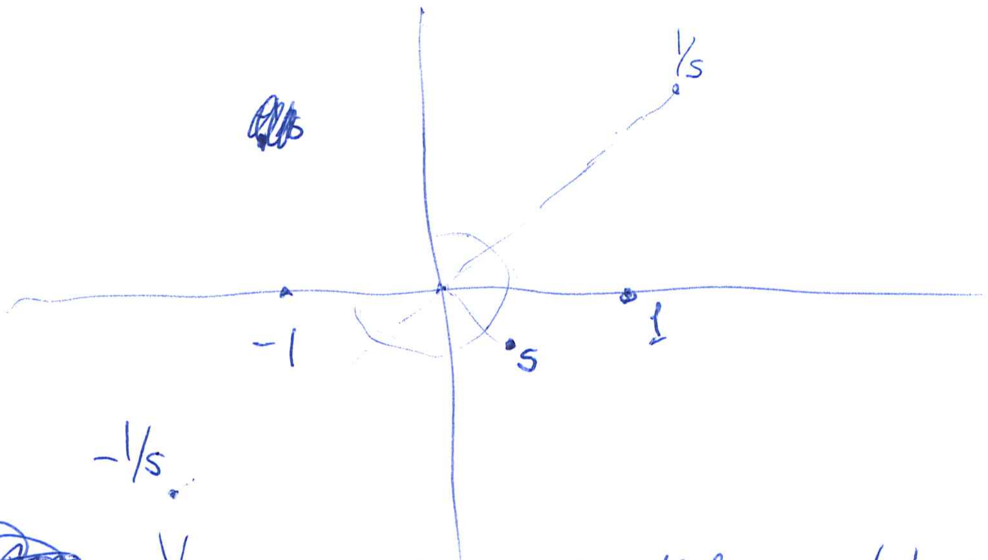
so $\lambda_1 = \lambda_2 = \frac{a+a^{-1}}{2}$ i.e. both roots are large - well? note that $\frac{a+a^{-1}}{2} \geq 1$. for $a > 0$. If we push $s = \pm i\omega$ ~~larger~~, then λ_1, λ_2 become conjugate complex with real-part $\frac{a+a^{-1}}{2}$

Do some analysis. First claim that $\forall s$ $|s| \leq M$ $\exists \varepsilon, \varepsilon'$ such that $\forall a$ $0 \leq a \leq \varepsilon'$ $\begin{pmatrix} a & as \\ a^{-1}s & a^{-1} \end{pmatrix}$ maps $D_\varepsilon(0)$ into itself.

168. Main point is that $\begin{pmatrix} 1 & s \\ s & 1 \end{pmatrix} D_\varepsilon(0)$ should not contain ∞. $\begin{pmatrix} 1 & s \\ s & 1 \end{pmatrix}(\xi) = \infty$ $\quad \xi = \begin{pmatrix} 1 & -s \\ -s & 1 \end{pmatrix}(\infty) = -\frac{1}{s}$

Thus given M bound on s want $-\frac{1}{s}$ outside $D_\varepsilon(0)$ $\quad \left|-\frac{1}{s}\right| > \varepsilon$

or $|s| < \frac{1}{\varepsilon}$. You want entire function i.e. analytic on $|s| < M$ for all M.



You need s disk $|s| \le M$ and a ξ disk $|\xi| \le \varepsilon$ and you need the maximum of $\left|\frac{\xi + s}{\xi s + 1}\right| \le \frac{|\xi| + |s|}{1 - |\xi||s|} \le \frac{\varepsilon + M}{1 - M\varepsilon}$

$a^2 \frac{\varepsilon + M}{1 - M\varepsilon} \le \varepsilon$ to enlarge M you need to decrease ε e.g. $\varepsilon = \frac{1}{2M}$

$a^2 \frac{\frac{1}{2M} + M}{1 - \frac{1}{2}} \le \frac{1}{2M}$, $\quad a^2 \underbrace{\left(\frac{1}{M} + 2M\right) 2M}_{} \le 1$

$a^2 \left(2 + 4M^2\right) \le 1.$

$$\varepsilon = \frac{k}{M} \qquad \frac{a^2\left(\frac{k}{M} + M\right)}{1-k} \leq \frac{k}{M}$$

$$a^2 (k + M^2) \leq k(1-k)$$

$$a^2 \leq \frac{1}{2+4M^2} \qquad a \sim \frac{1}{2M}$$

$$\begin{pmatrix} a & as \\ a^{-1}s & a^{-1} \end{pmatrix} \xi \qquad a^2 \frac{\xi + s}{\xi s + 1} \qquad |s| \leq M \quad |\xi| \leq \varepsilon$$

$$\left|\frac{\xi+s}{\xi s + 1}\right| \leq \frac{|\xi| + |s|}{1 - |\xi||s|} \leq \frac{\varepsilon + M}{1 - \varepsilon M}$$

$$0 \qquad a^2 \frac{\varepsilon + M}{1 - \varepsilon M} \overset{?}{\leq} \varepsilon \qquad \varepsilon = \frac{k}{M} \quad 0 < k < 1$$

$$a^2 \frac{\frac{k}{M} + M}{1-k} \overset{?}{\leq} \frac{k}{M} \qquad a^2 \leq \frac{k(1-k)}{k+M^2}$$

simplest is $k = \frac{1}{2}$ $\qquad a^2 \leq \dfrac{1}{2+4M^2}$

You should now know that once $a^2_{n+1} < \dfrac{1}{2+4M^2}$ that the disk $|\xi|_{n+1} \leq \varepsilon = \dfrac{1}{2M}$ get mapped into the disk $|\xi_n| \leq \varepsilon$, which then gives lines $l_n \subset \mathbb{C}^2$

---

digress to discuss types of functions

$$\begin{pmatrix} a_1^2 & \\ & 1 \end{pmatrix} \begin{pmatrix} 1 & s \\ s & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

 Describe

$$\begin{pmatrix} a_1^2 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & s \\ s & 1 \end{pmatrix} \cdots \begin{pmatrix} a_n^2 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & s \\ s & 1 \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

Better:

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & s \\ s & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} s \\ 1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} a_1^2 s \\ 1 \end{pmatrix}$$

$$\rightsquigarrow \begin{pmatrix} 1 & s \\ s & 1 \end{pmatrix}\begin{pmatrix} a_1^2 s \\ 1 \end{pmatrix} = \begin{pmatrix} (a_1^2+1)s \\ a_1^2 s^2 + 1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} a_2^2(a_1^2+1)s \\ a_1^2 s^2 + 1 \end{pmatrix}$$

$$\rightsquigarrow \begin{pmatrix} 1 & s \\ s & 1 \end{pmatrix}\begin{pmatrix} a_2^2(a_1^2+1)s \\ a_1^2 s^2 + 1 \end{pmatrix} = \begin{pmatrix} a_2^2(a_1^2+1)s + a_1^2 s^3 + s \\ a_2^2(a_1^2+1)s^2 + a_1^2 s^2 + 1 \end{pmatrix}$$

$$\begin{array}{c} -1 \\ \begin{pmatrix} p_0 \\ q_0 \end{pmatrix} \\ 0 \end{array} \qquad \begin{array}{c} 1 \\ \begin{pmatrix} p_1 \\ q_1 \end{pmatrix} \\ 0 \end{array} \qquad \begin{array}{c} 1 \\ \begin{pmatrix} p_2 \\ q_2 \end{pmatrix} \\ 2 \end{array} \qquad \begin{array}{c} 3 \\ \begin{pmatrix} p_3 \\ q_3 \end{pmatrix} \\ 2 \end{array}$$

$$\begin{array}{r} 2n-1 \xrightarrow{\text{odd}} \\ 2n \xrightarrow{\text{even}} \end{array} \begin{pmatrix} f(s) \\ g(s) \end{pmatrix}$$

Assume $f(1) = g(1)$

$$\begin{pmatrix} 1 & -s \\ -s & 1 \end{pmatrix}\begin{pmatrix} f(s) \\ g(s) \end{pmatrix} = \frac{f(s) - s g(s)}{-s f(s) + g(s)}$$

Idea: do this expansion

$$R(s) = \begin{pmatrix} a_1^2 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & s \\ s & 1 \end{pmatrix}\begin{pmatrix} a_2^2 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & s \\ s & 1 \end{pmatrix} \cdots$$

where $\lim a_n^2 = 0$ not only gives a discrete spectral measure ~~[illegible]~~ but it seems that $\forall n$

$$a_n^2 \leq \frac{1}{2+4M^2} \implies R(s) \text{ analytic for } |s| \leq M.$$

so that the higher coeffs might be related to the far out spectrum.

Now let's look at the linear equations.

$$\psi_n = \begin{pmatrix} a_{n+1} & \\ & a_{n+1}^{-1} \end{pmatrix} \begin{pmatrix} 1 & s \\ s & 1 \end{pmatrix} \psi_{n+1}.$$

We know that for $|s| \leq M$ and $|\xi| \leq \varepsilon = \frac{1}{2M}$

$$\left| \begin{pmatrix} 1 & s \\ s & 1 \end{pmatrix} (\xi) \right| = \left| \frac{\xi + s}{1 + s\xi} \right| \leq \frac{|\xi| + |s|}{1 - |\xi||s|} \leq \frac{\frac{1}{2M} + M}{1 - \frac{1}{2}} = \frac{1}{M} + 2M$$

so $\left| \begin{pmatrix} a & as \\ a^{-1}s & a^{-1} \end{pmatrix} (\xi) \right| \leq a^2 \left(\frac{1}{M} + 2M \right) \overset{?}{\leq} \varepsilon = \frac{1}{2M}$

when $a^2 \leq \dfrac{1}{2 + 4M^2}$

Now you want to refine this from Riemann sphere to $\mathbb{C}^2$. Assume $\begin{pmatrix} \psi_0^+ \\ \psi_0^- \end{pmatrix} = \begin{pmatrix} a & as \\ a^{-1}s & a^{-1} \end{pmatrix} \begin{pmatrix} \psi_1^+ \\ \psi_1^- \end{pmatrix}$. You want assuming $\frac{\psi_1^+}{\psi_1^-} = \xi_1$ is small, i.e. ψ_1^- large relative to ψ_1^+, you want the same for ψ_0 and you want $\|\psi_0\| \geq \frac{1}{a} \|\psi_1\|$

$$a\psi_0^- = s\psi_1^+ + \psi_1^- \implies \cancel{|\psi_0^-| \leq (s\varepsilon + 1)|\psi_1^-|}$$

$$\implies a|\psi_0^-| \geq |\psi_1^-| - |s| \left| \frac{\psi_1^+}{\psi_1^-} \right| |\psi_1^-| \geq (1 - M\varepsilon)|\psi_1^-|$$

$$\psi_0^+ = a(\psi_1^+ + s\psi_1^-)$$

$$|\psi_0^+| \leq a(|\psi_1^+| + |s||\psi_1^-|)$$

$$\leq a(\varepsilon + M)|\psi_1^-|$$

172 $\cancel{a^2 \|\psi_0\|^2 = a^2\left(|\psi_0^+|^2 + |\psi_0^-|^2\right) \leq a^2(\varepsilon^2+1)|\psi_0^-|^2}$

$$\cancel{a\|\psi_0\| \leq a(\varepsilon^2+1)^{1/2}|\psi_0^-|}$$

$$\|\psi_0\|^2 = |\psi_0^+|^2 + |\psi_0^-|^2 \qquad |\psi_0^-| \leq \|\psi_0\| \leq \sqrt{1+\varepsilon^2}\,|\psi_0^-|$$

same true for ψ_1

$$g = \begin{pmatrix} a & 0 \\ 0 & a^{-1}\end{pmatrix}\begin{pmatrix} 1 & s \\ s & 1\end{pmatrix} \qquad \begin{matrix}(1 \ \ s) = u_1^* \\ (s \ \ 1) = u_2^*\end{matrix}$$

assume $\|g\psi\|^2 = a^2|u_1^*\psi|^2 + a^{-2}|u_2^*\psi|^2 \geq a^{-2}|u_2^*\psi|^2$

$$\therefore \frac{\|g\psi\|}{\|\psi\|} \geq a^{-1}\frac{|u_2^*\psi|}{\|\psi\|}$$

$$|u_2^*\psi| = |s\psi^+ + \psi^-| \geq |\psi^-| - \overbrace{|s|}^{\leq M}\underbrace{\left|\frac{\psi^+}{\psi^-}\right|}_{\leq \varepsilon}|\psi^-|$$

$$\geq (1-M\varepsilon)|\psi^-|$$

$$\frac{\|g\psi\|}{\|\psi\|} \geq a^{-1}(1-M\varepsilon)\frac{|\psi^-|}{\|\psi\|} \qquad \begin{aligned}\|\psi\|^2 &= |\psi^+|^2 + |\psi^-|^2 \\ &\leq (\varepsilon^2+1)|\psi^-|^2\end{aligned}$$

$$\geq \frac{a^{-1}(1-M\varepsilon)}{(\varepsilon^2+1)^{1/2}} \geq \frac{a^{-1}\left(1-\frac{1}{2}\right)}{\left(\frac{1}{4M^2}+1\right)^{1/2}} \qquad ?$$

$$g\psi = \begin{pmatrix} a & 0 \\ 0 & a^{-1}\end{pmatrix}\begin{pmatrix} 1 & s \\ s & 1\end{pmatrix}\psi \qquad \varepsilon = \frac{1}{2M}$$

$$\left|\frac{(g\psi)^+}{(g\psi)^-}\right| = a^2\left|\frac{\psi^+ + s\psi^-}{s\psi^+ + \psi^-}\right| \leq a^2\frac{\left|\frac{\psi^+}{\psi^-}\right| + |s|}{1-|s|\left|\frac{\psi^+}{\psi^-}\right|} \leq a^2\frac{\varepsilon + M}{1-M\varepsilon}$$

$$\leq a^2\, 2\left(\frac{1}{2M}+M\right) \overset{?}{\leq} \varepsilon = \frac{1}{2M} \iff a^2 \leq \frac{1}{2(1+2M^2)}$$

173 But also

$$\|g\psi\|^2 = |(g\psi)^+|^2 + |(g\psi)^-|^2$$

$$|(g\psi)^-|^2 \leq \|g\psi\|^2 \leq (\varepsilon^2+1)|(g\psi)^-|^2$$

$$\|\psi\|^2 = |\psi^+|^2 + |\psi^-|^2$$

$$|\psi^-|^2 \leq \|\psi\|^2 \leq (\varepsilon^2+1)|\psi^-|^2$$

$$(g\psi)^- = a^{-1}s\psi^+ + a^{-1}\psi^-$$

$$a|(g\psi)^-| = |s\psi^+ + \psi^-| \geq (1-M\varepsilon)|\psi^-| \geq \tfrac{1}{2}|\psi^-|$$

$$2a\|g\psi\| \geq 2a|(g\psi)^-| \geq |\psi^-| \geq \frac{1}{\sqrt{\varepsilon^2+1}}\|\psi\|$$

$$2a\sqrt{\varepsilon^2+1}\,\|g\psi\| \geq \|\psi\|$$

$$4a^2(\varepsilon^2+1) \leq 4\frac{1}{2(1+2M^2)}\left(\frac{1}{4M^2}+1\right) = 4\frac{1}{2+4M^2}\,\frac{1+4M^2}{4M^2}$$

$$\leq \frac{1}{M^2}$$

What's important?

if $\left|\frac{\psi^+}{\psi^-}\right| \leq \varepsilon$ and $|s| \leq M$, then

$$\left|\frac{(g\psi)^+}{(g\psi)^-}\right| = a^2\left|\frac{\left|\frac{\psi^+}{\psi^-}\right| + s}{1 + s\left|\frac{\psi^+}{\psi^-}\right|}\right| \leq a^2\frac{\varepsilon + M}{1 - M\varepsilon} = a^2\frac{\frac{1}{2M}+M}{\frac{1}{2}}$$

$$= a^2\left(\frac{1}{M}+2M\right) \overset{?}{\leq} \varepsilon = \frac{1}{2M}$$

OK if $\quad a^2 \leq \dfrac{1}{2M\left(\frac{1}{M}+2M\right)} = \dfrac{1}{2+4M^2}$

 Also in this situation you get

$$\|\psi\| \sim |\psi^-| \quad \text{also for } g\psi.$$

Fix M and look for solutions of

$$\psi_n = \underbrace{\begin{pmatrix} a_{n+1} & 0 \\ 0 & a_{n+1}^{-1} \end{pmatrix} \begin{pmatrix} 1 & s \\ s & 1 \end{pmatrix}}_{g_{n+1}} \psi_{n+1} \qquad n \geq 0$$

which decay as $n \to \infty$. Assume that the a_n, $n \geq 1$ are sufficiently small i.e.

$$a_n^2 \leq \frac{1}{2+4M^2}$$

This ~~~~ should imply that $g_1, g_2, \ldots$ ~~~~ carry the disk $|\xi| \leq \frac{1}{2M}$ into itself i.e.

$$\left|\frac{\psi_n^+}{\psi_n^-}\right| \leq \frac{1}{2M} \implies \left|\frac{(g_n\psi_n)^+}{(g_n\psi_n)^-}\right| \leq \frac{1}{2M}$$

Use nested circles argument to obtain for any s, $|s| \leq M$ a families of lines $\ell_n^{(s)} \subset \mathbb{C}^2$ $\ni$ $\ell_n^{(s)} = g_{n+1}\ell_{n+1}$, in fact $\ell_n(s) = \begin{pmatrix} \xi \\ 1 \end{pmatrix}\mathbb{C}$ where $|\xi| \leq \frac{1}{2M}$. ~~~~ Choose $\psi_n(s) \in \ell_n(s)$ such that $\psi_0(s)^- = 1$, then $\psi_0(s)^+ = R_0(s)$ where R_0 is analytic for $|s| \leq M$. We know that $\psi_n(s)^-$ decays, and hence also $\psi_n(s)^+$ since ~~such~~ $\left|\frac{\psi_n(s)^+}{\psi_n(s)^-}\right| \leq \varepsilon = \frac{1}{2M}$

Problem of the determinant.

continuous version

$$\psi_x = \begin{pmatrix} 1 & \varepsilon h \\ \varepsilon h & 1 \end{pmatrix} \begin{pmatrix} z^{\varepsilon} & 0 \\ 0 & z^{-\varepsilon} \end{pmatrix} \psi_{x+\varepsilon} \qquad z^{\varepsilon} = e^{i\theta\varepsilon}$$

$$\psi_x = \begin{pmatrix} 1 & \varepsilon h \\ \varepsilon h & 1 \end{pmatrix} \begin{pmatrix} 1+i\theta\varepsilon & 0 \\ 0 & 1-i\theta\varepsilon \end{pmatrix} (\psi_x + \psi_x'\varepsilon)$$

175

$$\psi_x = \begin{pmatrix} 1+i\theta\varepsilon & \varepsilon h \\ \varepsilon h & 1-i\theta\varepsilon \end{pmatrix} (\underline{\psi_x} + \psi_x'\varepsilon)$$

$$\cancel{\psi_x} = \cancel{\psi_x} + \psi_x'\varepsilon + \begin{pmatrix} i\theta & h \\ h & -i\theta \end{pmatrix} \psi_x \varepsilon$$

$$-\psi_x' = \begin{pmatrix} ik & h \\ h & -ik \end{pmatrix} \psi_x$$

$$\frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} ik & h \\ h & -ik \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$\frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} ik+h & ik-h \\ h-ik & h+ik \end{pmatrix} = \begin{pmatrix} h & ik \\ ik & -h \end{pmatrix}$$

$$\partial_x \phi = \begin{pmatrix} W & S \\ S & -W \end{pmatrix} \phi$$

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \partial_x \phi = \begin{pmatrix} S & -W \\ W & S \end{pmatrix} \phi$$

$$\left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \partial_x + \begin{pmatrix} 0 & -W \\ W & 0 \end{pmatrix} \right] \phi = s\phi$$

$$\|$$

$$\begin{pmatrix} 0 & \partial_x - W \\ \partial_x + W & 0 \end{pmatrix} \phi = s\phi$$

$$(\partial_x - W)\phi_2 = s\phi_1 \qquad (\partial_x + W)\underbrace{(\partial_x - W)\phi_2}_{s\phi_1} = s^2\phi_2$$

$$(\partial_x + W)\phi_1 = s\phi_2 \qquad (\partial_x - W)(\partial_x + W)\phi_1 = s^2\phi_1$$

176 $(\partial_x - W)(\partial_x + W) = \partial_x^2 - W\partial_x + \partial_x W - W^2$

$$= \partial_x^2 - W' - W^2$$

$$\left(-\partial_x^2 + (W' + W^2)\right)\phi_1 = -s^2\phi_1$$

$$\frac{1}{s - i\omega} + \frac{1}{s + i\omega} = \frac{2s}{s^2 + \omega^2}$$

~~Start with the problem~~

Suppose you start with $R_s = \int_0^\infty \frac{s(1+\omega^2)}{s^2+\omega^2}\, d\mu(\omega) + \mu_\infty s$

can transform to S^1, ~~perform transfo~~

form z expansion, then

wait. Do circle version first, namely, take

wait - what can you do? Do circle version

Discuss aspects of the problem. First try to make clear the class of response functions.

One idea: Given unitary with cyclic vector

example $\sum_{n \geq 1} \frac{s(1+n^2)}{s^2+n^2}$

$$\sum_{n \in \mathbb{Z}} \frac{1}{s +}$$

$$\sum_{n \in \mathbb{Z}} \frac{1}{z-n} = \frac{1}{z} + \sum_{n \geq 1}\left(\frac{1}{z-n} + \frac{1}{z+n}\right)$$

$$= \frac{1}{z} + \sum_{n \geq 1} \frac{2z}{z^2 - n^2}$$

$\sin(\pi z) = 0$

177

$$\pi \frac{\cos \pi z}{\sin \pi z}$$ simple poles residue 1 ~~and~~ at $n \in \mathbb{Z}$.

$$\pi \frac{\cos \pi z}{\sin \pi z} - \sum_{n \in \mathbb{Z}} \frac{1}{z-n}$$ entire + probably bounded

$$\sum_{n=1}^{\infty} \frac{2y}{y^2+n^2} \sim \int_0^{\infty} \frac{2y}{y^2+t^2}\,dt = \int_0^{\infty} \frac{2y^2}{y^2(1+t^2)}\,dt \quad \text{fin.}$$

$\therefore$ constant + constant $=0$ as function is 0.

$$\cancel{\frac{\cos \pi i s}{\sin \pi i s} = \sum \frac{1}{is-n}}$$

$$\frac{d}{dz} \log\left(z \prod_{n \geq 1}\left(1-\frac{z^2}{n^2}\right)\right) = \frac{1}{z} + \sum_1^{\infty} \frac{1}{1-\frac{z^2}{n^2}}\left(-\frac{2z}{n^2}\right)$$

$$= \frac{1}{z} + \sum_1^{\infty} \frac{2z}{z^2-n^2} =$$

$$z \prod_{n \geq 1}\left(1-\frac{z^2}{n^2}\right) = \frac{\sin \pi z}{\pi}$$

next look at

$$\sum_{n \in \mathbb{Z}} \frac{1}{s-in} = \frac{\cos \pi i s}{\sin \pi i s}$$

$$\frac{1}{s} + \sum_{n=1}^{\infty} \frac{1}{s^2+n^2} = \frac{e^{-\pi s}+e^{\pi s}}{e^{-\pi s}-e^{\pi s}}$$

$$\pi \frac{\cos \pi z}{\sin \pi z} = \sum_{n \in \mathbb{Z}} \frac{1}{z-n}$$

178 Let $z = -is$

$$i\pi \frac{e^{i\pi(-is)} + e^{-i\pi(-is)}}{e^{i\pi(-is)} - e^{-i\pi(-is)}} = i\pi \frac{e^{\pi s} + e^{-\pi s}}{e^{\pi s} - e^{-\pi s}}$$

$$\|$$

$$\sum_{n\in\mathbb{Z}} \frac{1}{-is-n} = \sum_{n\in\mathbb{Z}} i\left(\frac{1}{i(-is-n)}\right) = i\sum_n \frac{1}{s-in}$$

$$\therefore \quad \pi \frac{e^{\pi s} + e^{-\pi s}}{e^{\pi s} - e^{-\pi s}} = \frac{1}{s} + \sum_{n=1}^{\infty} \frac{2s}{s^2+n^2}$$

$$\frac{2\frac{s}{\pi}}{\left(\frac{s}{\pi}\right)^2+n^2}$$

$$\cancel{\pi \frac{e^{\pi s} + e^{-\pi s}}{e^{\pi s} - e^{-\pi s}} - \frac{1}{s}}$$

$$\pi \frac{e^{s} + e^{-s}}{e^{s} - e^{-s}} = \frac{\pi}{s} + \sum_{n=1}^{\infty} \frac{\pi 2s}{s^2+(n\pi)^2}$$

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}(e^{2s}) = \frac{e^{s} + e^{-s}}{e^{s} - e^{-s}} = \frac{1}{s} + \sum_{n=1}^{\infty} \frac{2s}{s^2+n^2\pi^2}$$

$$\frac{1}{2s}\left(\frac{1+\frac{s^2}{2!}+\frac{s^4}{4!}}{s+\frac{s^3}{3!}} - \frac{1}{s}\right) = \frac{1}{2s\,s}\left(\frac{1+\frac{s^2}{2}+\frac{s^4}{4!}}{1+\frac{s^2}{6}} - 1\right)$$

$$= \frac{1}{2s\,s\left(1+\frac{s^2}{6}\right)}\left(\cancel{1}+\frac{s^2}{2}+\frac{s^4}{4!} - \cancel{1} - \frac{s^2}{6}\right) = \frac{\left(\frac{1}{2}-\frac{1}{6}\right)}{2} = \frac{1}{6}$$

$$\boxed{\sum_{1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}}$$

$$1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16}$$

1.25	.04
.1111	.028
.0625	.02
1.42	

179 Discuss abstractly the system

$$\psi_n = g_{n+1}\psi_{n+1} \qquad n \geq 0$$

You want to understand when there is a decaying solution. You have some kind of nested circle argument.

$$\psi_n \in \mathbb{C}^2 \quad \forall n.$$

What sort of things do you expect?

Correspondence $\quad Y \ni y = (\psi_n)_{n\geq 0}$ $\qquad X \rightrightarrows Y$

$\qquad X \ni x = (\psi_n)_{n\geq 1}$

doesn't seem to work

No. $X \ni (\psi_n)_{n\geq 0} \qquad Y \ni (\psi_n)_{n\geq 1}$

$$\psi \longmapsto \psi$$

$$X \longrightarrow Y \quad ?$$

You want the inhomog. equation.

$$\psi_n - g_{n+1}\psi_{n+1} = \phi_n \qquad n \geq 0$$

<u>What is the inhomogeneous equation</u>

Let's discuss the system of homogeneous equations $\psi_n = g_{n+1}\psi_{n+1} \quad n \geq 0.$ Kernel of operator

$$(\psi_n)_{n\geq 0} \longmapsto (\psi_n - g_{n+1}\psi_{n+1})_{n\geq 0}$$

matrix.

$$\begin{matrix} 1 & -g_1 & & \\ & 1 & -g_2 & \\ & & 1 & \end{matrix} \qquad \psi_n$$

$$\psi \longmapsto \psi - K\psi \qquad (K\psi)_n = g_{n+1}\psi_{n+1}$$

Could this be formally invertible?

$$(1-K)^{-1} = 1 + K + K^2 + \cdots$$

Lets make an effort to convert ~~collapsing~~ nested circle argument into something more like integral equations if this is possible. Take an examples. Constant coefficient case – both discrete an continuous. ~~discrete~~

$$g = \begin{pmatrix} a & as \\ a^{-1}s & a^{-1} \end{pmatrix} \qquad s \text{ fixed}$$

problem: to understand $\psi_n = g\psi_{n+1}$ $\quad n \geq 0$

you diagonalize the matrix, reduces to ~~case 1.~~ 1 dim. $\psi_n = \lambda\psi_{n+1}$ $\quad \therefore \psi_n = \lambda^{-n}\psi_0$

decays iff $|\lambda| > 1$. ~~So it is clear! Yes.~~

Review, start again. Study $\psi_n = g_{n+1}\psi_{n+1}$ $\quad n \geq 0$. this is a system of homogeneous linear equations, but a general principle says you should also consider the inhomog. equation. The solns to homog. equations are the kernel of an operator. So you form the operator

$(\psi_n)_{n\geq 0} \longmapsto (\psi_n - g_{n+1}\psi_{n+1})_{n\geq 0}$, and you want to understand both the kernel + cokernel

Example g_n constant. ~~[illegible]~~

~~[illegible]~~ Try to abstract the argument. What is the basic fact about $\begin{pmatrix} a & as \\ a^{-1}s & a^{-1} \end{pmatrix}$ you use?

Disk $|\xi| \leq \varepsilon$ gets preserved.

$$\left\| a^2 \frac{\xi + s}{1 + s\xi} \right\| \leq a^2 \frac{|\xi| + |s|}{1 - |s||\xi|} \leq a^2 \frac{\varepsilon + |s|}{1 - \varepsilon|s|}$$

Things are even better if ~~[illegible]~~ $|\xi| \leq \varepsilon \Rightarrow |g\xi| \leq k\varepsilon$ ~~[illegible]~~ with $0<k<1$. There probably is a better estimate which improves the distance from the origin to a distance between points. Thus given ξ_1, ξ_2

$$\frac{\xi_1+s}{1+s\xi_1} - \frac{\xi_2+s}{1+s\xi_2} = \frac{(s\xi_2+1)(\xi_1+s)-(s\xi_1+1)(\xi_2+s)}{(1+s\xi_1)(1+s\xi_2)}$$

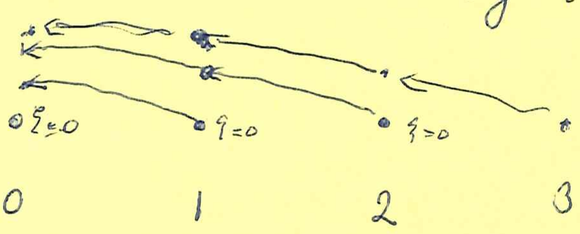
$$= \frac{s^2(\xi_2-\xi_1)+(\xi_1-\xi_2)}{} = \frac{(1-s^2)(\xi_1-\xi_2)}{(1+s\xi_1)(1+s\xi_2)}$$

$$\left\| \frac{\xi_1+s}{1+s\xi_1} - \frac{\xi_2+s}{1+s\xi_2} \right| \leq \frac{|1-s^2|\,|\xi_1-\xi_2|}{(1-|s|\xi)^2}$$

Thus you can arrange the disks to shrink to a point. Note that $1-s^2 = \det\begin{pmatrix}1 & s\\ s & 1\end{pmatrix}$

$$\cancel{\frac{az_1+b}{cz_1+d}}$$

Idea on $\mathbb{P}^1$ level. Because distances are shrunk by a factor you get a convergent sequence at position 0.



Can we proceed on the linear space level.

Look at this more generally. Suppose you have sequence g_n of FLT's. You specify a disk and shrinking no. So consider $g \in SL_2(\mathbb{C})$ carrying the unit disk into itself and shrinking distances.

need 3 parameters to describe a circle - coords of center and radius - ~~Let Look at g~~ suppose $g \in SL_2(\mathbb{C})$ carries $|z| = 1$ inside itself. Then you iterate to get fixpoints symmetrically related.

182 ~~Rotation~~ Still – how to handle ~~eigen~~ eigenvectors. Try lifting the $\mathbb{P}^1$ picture. You want some idea about $g_1 g_2 \cdots g_n \binom{0}{1}$ as $n \to \infty$. Since everything takes place for lines $\binom{\xi}{1}\mathbb{C}$ with $|\xi| \leq \varepsilon$ You must know what's going on.

$$g\begin{pmatrix}\xi\\1\end{pmatrix} = \begin{pmatrix} a & as \\ a^{-1}s & a \end{pmatrix}\begin{pmatrix}\xi\\1\end{pmatrix} = \begin{pmatrix} a(\xi+s) \\ a^{-1}(1+s\xi)\end{pmatrix} = a^{-1}(1+s\xi)\begin{pmatrix}\frac{a^2\xi+s}{1+s\xi}\\ 1\end{pmatrix}$$

so we know the 2nd component $a^{-1}(1+s\xi)$ has $|a^{-1}(1+s\xi)| \geq a^{-1}(1-|s|\varepsilon) \geq a^{-1}(\tfrac{1}{2})$.

$$\begin{pmatrix} a & a^{-1}\bar{s} \\ a\bar{s} & a^{-1}\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix} a & as \\ a^{-1}s & a\end{pmatrix} =$$

$$\begin{pmatrix}1 & \bar{s}\\ \bar{s} & 1\end{pmatrix}\begin{pmatrix} a & \\ & a^{-1}\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix} a & 0\\ 0 & a^{-1}\end{pmatrix}\begin{pmatrix}1 & s\\ s & 1\end{pmatrix}$$

$$\begin{pmatrix}1 & \bar{s}\\ \bar{s} & 1\end{pmatrix}\begin{pmatrix} a^2 & 0\\ 0 & -a^{-2}\end{pmatrix}\begin{pmatrix}1 & s\\ s & 1\end{pmatrix}$$

$$= \begin{pmatrix}1 & \bar{s}\\ \bar{s} & 1\end{pmatrix}\begin{pmatrix} a^2 & a^2 s\\ -a^{-2}s & -a^{-2}\end{pmatrix} = \begin{pmatrix} a^2 - a^{-2}|s|^2 & a^2 s - a^{-2}\bar{s}\\ a^2\bar{s} - a^{-2}s & a^2|s|^2 - a^{-2}\end{pmatrix}$$

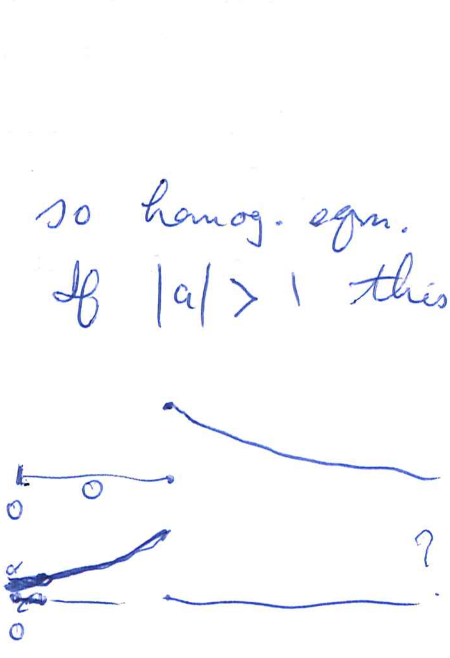
Review. Look at rank 1.

$$\psi_n = a_{n+1}\psi_{n+1} \qquad n \geq 0.$$

take const. coeff. $\psi_n = a\psi_{n+1}$ so homog. eqn. solution is $\psi_n = a^{-n}c$ c const. If $|a| > 1$ this decreases, if $|a| < 1$ this increases.

$|a| > 1$ Green's function looks like

$|a| < 1$ ______



183 Be ~~[illegible]~~ careful

$$u_0 - a u_1 = f_0$$
$$u_1 - a u_2 = f_1$$

solutions of hom. equation are $u_n = a^{-n} c$ c const.
otherwise we impose bdry condition $u_0 = 0$

$$u_0 - a u_1 = 1 \qquad u_1 = -a^{-1}$$
$$u_1 - a u_2 = 0 \qquad u_2 = a^{-1} u_1 = -a^{-2}$$
$$u_2 - a u_3 = 0 \qquad u_3 = -a^{-3}$$

this basically is what I mean by solving the Volterra equation. Note if $|a| > 1$, then the Green's function is bounded.

$$u_0 - a u_1 = 0$$
$$u_1 - a u_2 = 1$$

OK. You first look at unbounded case
No problem solving ~~unless~~ when a^{-1} $\exists$. For any f $\exists$ a unique up to a homog. soln.
Now impose boundedness $|a| < 1$ homog. soln. unbdd.
start with

---

you want to discuss a constant coeff. system, ~~[illegible]~~ of rank n, Morse-Smale diffeo, ~~geometry of~~ fixpts, incoming + outgoing submanifolds, continuous version: vector field = gradient of Morse fn.
This gives the

Tom

184 Problem: is there someway to understand the existence of a decaying eigenfunctions, e.g. compactness. how does one proceed? No way at all.

Start theory of 1 diml things.

$$\left(\partial_t - a_t\right) u = 0.$$

a_t matrix functi in general

get a path in Gl_r $\quad r =$ rank.

One dimensional determinants?

First case is $r=1$. Solution is

$$u(t) = \underbrace{e^{\int_0^t a_t \, dt}}_{\Phi(t,0)} u(0)$$

You want to understand boundary conditions

OK yesterday you had the idea that the graph of the propagator i.e. $\begin{pmatrix} 1 \\ \Phi(t,0) \end{pmatrix} V$ should have a limit as $t \to \infty$. Example.

$\Phi(t,0) = e^{tA}$. Use Jordan form to analyze. $\begin{pmatrix} 1 \\ e^{\lambda t} \end{pmatrix} \mathbb{C}$ has a clear limit for $Re(\lambda) \neq 0$. Now that you have the propagator straight to and from infinity you should be able to set boundary conditions to get a Green's function. A Boundary condition is simply a complementary subspace in $V \times V$ to $\lim_{t \to \infty} \begin{pmatrix} 1 \\ \Phi(t,0) \end{pmatrix} V$. Call this Γ_∞ let $B \subset V \times V$ sat $B \oplus \Gamma_\infty = V \times V$. Greens function

$G(t,t') =$ $H(t-t')$ + solution of homog. equation to satisfy boundary conditions

$$H(t-t') = G(t,t') \, \tfrac{+}{?} \, \Phi(t,0) \mathbb{C}$$

185 Set $t=0$.

$$0 = G(0,t') + C$$

Set $t=\infty$

$$\underbrace{1 = G(\infty,t') +}$$

$H(t-t') = G(t,t') + \Phi(t,0)C$

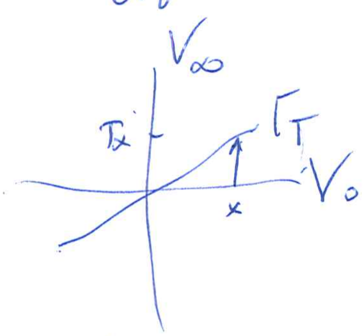
These are operators from V to V ~~except~~ but the latter two ~~have no m~~ are not defined at $t=\infty$. But the graphs are. So

$$0 = G(0,t') + C$$

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ G(t,t') \end{pmatrix} V \Big|_{t=\infty} + \begin{pmatrix} 1 \\ \Phi(t,0) \end{pmatrix} C \Big|_{t=\infty}$$

example. $\partial_t - A$ where A constant. What is $F_\infty = \lim_{t\to+\infty} \begin{pmatrix} 1 \\ e^{tA} \end{pmatrix} V$ say $V = V^+ \oplus V^-$ where spectrum of A on $\begin{vmatrix} V^+ \\ V^- \end{vmatrix}$ is in $\begin{vmatrix} RHP \\ LHP \end{vmatrix}$. Then ~~you need bdry conditions~~

$$\Gamma_\infty = \begin{pmatrix} 0 \\ 1 \end{pmatrix} V^+ \oplus \begin{pmatrix} 1 \\ 0 \end{pmatrix} V^- \subset \begin{matrix} V \\ \oplus \\ V \end{matrix}$$



$$\Gamma \subset \begin{pmatrix} T \\ 1 \end{pmatrix}$$

$$\begin{tikzcd}
 & \Gamma \arrow[d, phantom, "\cap"] & \\
V_\infty \arrow[r, "\binom{1}{0}"] & V_\infty \oplus V_0 \arrow[r, "(0\ 1)"] & V_0
\end{tikzcd}$$

somehow you find

$$\begin{tikzcd}
\Gamma \cap V_\infty \arrow[r, hook] \arrow[d] & \Gamma \arrow[r, two heads] \arrow[d, phantom, "\cap"] & \operatorname{pr}_1 \Gamma \arrow[d, hook] \\
V_\infty \arrow[r, "\binom{1}{0}"] \arrow[d] & V_\infty \oplus V_0 \arrow[r, "(0\ 1)"] \arrow[d] & V_0 \\
V/\Gamma\cap V_\infty \arrow[r, hook] & V_\infty \oplus V_0/\Gamma &
\end{tikzcd}$$

ind; domain

Example. Consider ~~a the A~~

~~$\Phi(t) = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}$ Then $\Gamma_t =$~~

$$\Phi_1(t) = e^t \qquad \Gamma_{1,\infty} = \lim_{t\to\infty} \begin{pmatrix} 1 \\ e^t \end{pmatrix} \mathbb{C} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \mathbb{C}$$

$$\Phi_2(t) = e^{-t} \qquad \Gamma_{2,\infty} = \lim_{\to} \begin{pmatrix} 1 \\ e^{-t} \end{pmatrix} \mathbb{C} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \mathbb{C}$$

You need a complement for $\Gamma_{1\infty} \oplus \Gamma_{2\infty} \subset \mathbb{C}^2 \oplus \mathbb{C}^2$

wait. There are two systems ~~these~~ added together, the prop. is

$$\begin{bmatrix} 0 \\ \\ 1 \end{bmatrix} ?$$

$$\Phi_t = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}$$

$$\Gamma_{\Phi_t} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ e^t & 0 \\ 0 & e^{-t} \end{pmatrix} \qquad \Gamma_\infty = \begin{pmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 0 \end{pmatrix} \mathbb{C}^2$$

Assume $\Gamma_\infty = \lim_{t\to\infty} \begin{pmatrix} 1 \\ \Phi(t,0) \end{pmatrix} V$ $\exists$. Assume it has the form $\Gamma_\infty \subset V \times V$ where $pr_1(\Gamma_\infty)$ is a line $l_0 \subset V$ and $\Gamma_\infty \cap 0 \times V = l_\infty$ indeterminacy, and there's a map $l_0 \longrightarrow V/l_\infty$. What sort of boundary conditions are appropriate? $l_0 \subset V_0$ is the decaying eigenline. $B \subset V_0 \times V_\infty$ complementary to Γ_∞.

What do you know?

$g_1 \cdots g_{n+1} - g_1 \cdots g_n$

$g_1\cdots g_n - g_1\cdots g_{n+1} = g_1\cdots g_n(1-g_{n+1})$

You believe that $g_1\cdots g_n$ at last partial converges to a degenerate "map" from V_0 to V_∞ which kills a decaying line.

$\mathrm{Re}(\lambda)>0$ $\quad$ $\mathrm{Re}(\lambda)<0$

Look at constant coeff case. Then $V = V^+ \oplus V^-$ eigenlines. In this case things split so Γ_∞ should consist of the zero map from the subspace V^- to the quotient space V/V^+.

You have some hopes that the boundary condition of Green's function will shed some light on determinants. Lets discuss the angles.

What should you have? You have a system

$\psi_n = g_{n+1}\psi_{n+1}$ $\quad n \geq 0$ $\quad$ or maybe $-\partial_t \psi = A_t \psi$ $\quad t \geq 0$

which leads to a unique (up to scalar factors) decaying solution. You use this decaying solution to construct Green's fn. Basically your bdry condition does not connect V_0) V_∞, i.e. the bound. condition space B is $B_0 \oplus B_\infty$, i.e. G satisfies separately a condition at 0 and The decaying one at ∞. No periodic boundary condition

---

Consider $S(z)$ analytic for $|z|<1$, continuous on $|z| \leq 1$ except at $z=-1$, and $|S(z)|=1$ for $|z|=1$, $z \neq -1$. Then $S(z)$ has a meromorphic extension

$$S(z) = S(z^*)^* \qquad z^* = +\frac{1}{z}$$

$S(s)$ — analytic in ~~RHP~~ $\mathrm{Re}(s)>0$
continuous in $\mathrm{Re}(s) \geq 0$ $\quad |S| \leq 1$.
$|S|=1$ for $\mathrm{Re}(s)=0$.

Then extend $S(s) = \left(\overline{S(\bar{s})}\right)^{-1}$ to the LHP.

Look at zeroes. Assume $S(s) \neq 0$ $\mathrm{Re}(s) > 0$, then S should be an entire function non vanishing, $\log S$ defined up to $2\pi i \mathbb{Z}$

$\mathrm{Re} \log S = \log |S|$ is a ~~bounded~~ harm function in RHP, 0 on $\mathrm{Re}(s) = 0$, ~~as~~ ~~$\mathrm{Re}\log S$~~

~~$= as$ $a \in \mathbb{R}$~~ $\mathrm{Re} \log S \leq 0$ for $\mathrm{Re}(s) \geq 0$

Only possibility $\mathrm{Re} \log S = -a \mathrm{Re}(s)$ $a \geq 0$

$\log S = -as + ir$ $S = e^{-as} S(0)$.

This means S should be determined by its zeroes. In fact look at ~~$\mathrm{Re}\log S = $~~ $\log |S|$.

$$e^{-bs} \qquad e^{-bs} e^{b} = e^{b(1-s)}$$

$$\begin{pmatrix} 1 & -s \\ -s & 1 \end{pmatrix} e^{b(1-s)} = \frac{e^{b(1-s)} - s}{1 - s e^{b(1-s)}}$$

$$\left. \frac{e^{b(1-s)}(-b) - 1}{1 - e^{b(1-s)} - s e^{b(1-s)}(-b)} \right|_{s=1} = \frac{-b-1}{1-1-(-b)} = \frac{-b-1}{b}$$

In any case go back to the harmonic function $\log|S|$ in RHP.

$$S = c \prod \frac{-s+\alpha_i}{s+\alpha_i} = \cancel{\prod \left(1 - \frac{\alpha_i}{s}\right)\left(1 + \frac{\alpha_i}{s}\right)^{-1}}$$

$$= \prod \left(1 - \frac{s}{\alpha_i}\right)\left(1 + \frac{s}{\alpha_i}\right)^{-1}$$

189 ~~Take convergence What~~

Assume $F(s)$ continuous for $Re(s) \geq 0$

$|F(s)| \leq 1$

$|F(s)| = 1$ for $Re(s) = 0$

$F(s)$ analytic for $Re(s) > 0$.

Then $F(s) = \overline{F(-\bar{s})}^{-1}$ should be a meromorphic continuation of F, where the poles of F in the LHP are reflection of the zeroes of F in the RHP. e.g.

$$F(s) = \frac{s-\alpha}{s+\bar{\alpha}}$$

$$\overline{F(-\bar{s})}^{-1} = \left(\frac{-s-\bar{\alpha}}{-s+\alpha}\right)^{-1} = \frac{s-\alpha}{s+\bar{\alpha}}$$

What can be said about zeroes of $F(s)$.

positive divisor in RHP ~~of~~ with limit ∞. ~~So you have~~

What can you construct by product

$$\prod_{k=1}^{\infty} \frac{s-\alpha_k}{s+\bar{\alpha}_k}$$

Suppose $\alpha_k \in \mathbb{R}_{>0}$

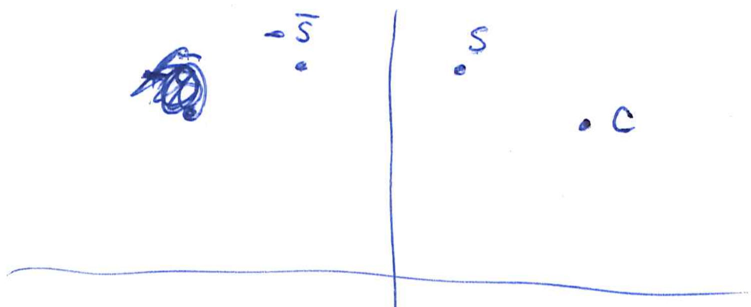
replace by

$$\prod_{k=1}^{\infty} \frac{1-\frac{s}{\alpha_k}}{1+\frac{s}{\alpha_k}}$$

$E(s)$ be a deBrange fn:

$$|E(-\bar{s})| > |E(s)| \qquad Re(s) > 0$$

e.g. $\quad s - c \qquad Re(c) > 0$



$$|s-c| < |-\bar{s}-c|$$

Ignore zeroes on $Re(s)=0$.

So given such an $E(s)$ put $F(s) = \dfrac{E(s)}{\overline{E(-\bar{s})}}$

This is analytic in RHP $\quad |\;| = 1$ on $Re(s) = 0$

So if $E(s)$ has no zeroes in RHP, then what.

$F(s)$ non vanishing $\quad \log F(s)$ defined up to $2\pi i \mathbb{Z}$

$$u(s) = Re \log F(s) = \underbrace{\log |F(s)|}_{\text{harmonic}} \begin{array}{ll} < 0 & \text{in RHP} \\ = 0 & Re(s) = 0 \end{array}$$

Poisson fmla says $\quad u(s) = -a\, Re(s) \qquad a > 0$

$$\log F(s) = -as + (i\mathbb{R}) \qquad F(s) = (S^1)\, e^{-as}$$

So we have two classes.

Suppose we start with $\quad Q(s) = \displaystyle\sum_{k=1}^{\infty} \frac{s(1+\omega_k^2)}{s^2+\omega_k^2}\, a_k$

$$\frac{1}{s-i\omega} + \frac{1}{s+i\omega} = \frac{2s}{s^2+\omega^2}$$

191 You want a measure on the circle maybe.

~~[illegible]~~

Perhaps you need to recall Poisson kernel for the circle.

$$f(z,\bar z) = \sum_{n\in\mathbb{Z}}' a_n(r) e^{in\theta}$$

Let $f(z)$ be analytic on $|z| \leq 1$. Laurent

$$f(z) = \sum_{n=0}^{\infty}{}' a_n z^n \qquad \cancel{a_n = \frac{1}{2\pi}\int_0^{2\pi} f(re^{i\theta}) e^{-}}$$

$$a_n = \frac{1}{2\pi}\int_0^{\infty} f(e^{i\theta}) e^{-in\theta} d\theta$$

$$\operatorname{Im} f(z) = \frac{f(z) - \overline{f(z)}}{2i} = \sum_{n=0}^{\infty} \frac{a_n z^n - \bar a_n \bar z^n}{2i}$$

Suppose $\operatorname{Im} f(e^{i\theta}) = \sum_{n\in\mathbb{Z}} b_n e^{in\theta}$ call this $h(e^{i\theta})$

then
$$b_n = \frac{1}{2i} a_n \qquad n \geq 1$$
$$b_n = -\frac{1}{2i}\bar a_{-n} \qquad n \leq -1$$
$$b_0 = \frac{a_0 - \bar a_0}{2i} = \operatorname{Im}(a_0).$$

$$f(e^{i\theta}) = g(e^{i\theta}) + i\,h(e^{i\theta})$$
$$b_{-n} = -\frac{1}{2i}\bar a_{+n} \qquad n \geq 1.$$

~~[illegible]~~

$$f(z) = \sum_{n\geq 0}' a_n z^n = a_0 + \sum_{n\geq 1} 2i \overbrace{b_n}^{\frac{1}{2\pi}\int h(e^{i\theta}) e^{-in\theta} d\theta} z^n$$

192 ~~Find~~ Given $f(z)$ analytic on $|z| \leqslant 1$

$$f(z) = \frac{1}{2\pi i} \oint \frac{1}{\lambda - z} f(\lambda)\, d\lambda \qquad |\lambda| = 1$$

$$= \int_0^{2\pi} \frac{1}{e^{i\theta} - z} f(e^{i\theta}) \frac{i e^{i\theta} d\theta}{2\pi i}$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \frac{1}{1 - e^{-i\theta} z} f(e^{i\theta})\, d\theta$$

OK. $f(z)$ analytic for $|z| \leqslant 1$. to express f in terms of ~~its real part~~ $Re(f)$ on the circle.

$$f(z) = \oint \frac{1}{\lambda - z} f(\lambda) \frac{d\lambda}{2\pi i} = \int_0^{2\pi} \frac{1}{e^{i\theta} - z} f(e^{i\theta}) \frac{e^{i\theta} i\, d\theta}{2\pi i}$$

$$f(z) = \int_0^{2\pi} \frac{f(e^{i\theta})}{1 - z e^{-i\theta}} \frac{d\theta}{2\pi} \qquad \text{if } |z| < 1.$$

$$\cancel{f^*(z) = \overline{f(\bar z)} = \int_0^{2\pi} \frac{\overline{f(e^{-i\theta})}}{1 - z e^{i\theta}} \frac{d\theta}{2\pi}}$$

$$f(z) = \sum_{n \geqslant 0} z^n \underbrace{\int_0^{2\pi} f(e^{i\theta}) e^{-in\theta} \frac{d\theta}{2\pi}}_{a_n}$$

$$\int_0^{2\pi} f(e^{i\theta}) e^{+in\theta} \frac{d\theta}{2\pi} = \begin{cases} 0 & n > 0. \\ \bar a_0 & n = 0 \end{cases}$$

$$\underset{+\bar a_0}{f(z)} = + \sum_{n \geqslant 0} z^n\, 2 \int_0^{2\pi} Re\left(f(e^{i\theta})\right) e^{-in\theta} \frac{d\theta}{2\pi}$$

$$= \int_0^{2\pi} \frac{2\, Re(f(e^{i\theta}))}{1 - z e^{-in\theta}} \frac{d\theta}{2\pi}$$

193

$$f(z) = \sum_{n\geq 0} z^n a_n \qquad a_n = \int f(e^{i\theta}) e^{-in\theta} \frac{d\theta}{2\pi} \quad n\geq 0$$

$$0 = \text{———} \quad n<0$$

$$0 = \int \overline{f(e^{i\theta})} e^{-in\theta} \frac{d\theta}{2\pi} \quad n\gtrless 0$$

$$\overline{a}_0 \quad n=0.$$

$$f(z)+\overline{a}_0 = \sum_{n\geq 0} z^n \int 2\mathrm{Re} f(e^{i\theta}) e^{-in\theta} \frac{d\theta}{2\pi}$$

$$= \int_0^{2\pi} \frac{1}{1-ze^{-i\theta}} \mathrm{Re} f(e^{i\theta}) \frac{d\theta}{\pi}$$

$$\mathrm{Re} f(z) + \overline{a}_0 = \int_0^{2\pi} \mathrm{Re}\left(\frac{1}{1-ze^{-i\theta}}\right) \mathrm{Re} f(e^{i\theta}) \frac{d\theta}{\pi}$$

$$\frac{1-\overline{z}e^{i\theta} + 1 - ze^{-i\theta}}{(1-ze^{-i\theta})(1-\overline{z}e^{i\theta})} \mathrm{Re}\, f(e^{i\theta}) \frac{d\theta}{2\pi}$$

OKAY go back to the determinant idea.

You are given a spectrum $\pm i\omega_n$ $n \geq 1$.

where $\omega_n \sim \frac{n}{\log n}$

get a convergent $\prod_{n=1}^{\infty}\left(1+\frac{s^2}{\omega_n^2}\right)$ determinant

$$d\log\left(1+\frac{s}{+i\omega_n}\right) = \frac{1}{1+\frac{s}{i\omega_n}} \frac{1}{i\omega_n} ds = \frac{1}{s+i\omega_n} ds$$

$$d\log\left(1+\frac{s^2}{\omega_n^2}\right) = \frac{1}{1+\frac{s^2}{\omega_n^2}} \frac{2s\,ds}{\omega_n^2} = \frac{2s\,ds}{s^2+\omega_n^2}$$

$$d\log\det = \sum_{n=1}^{\infty} \frac{2s}{s^2+\omega_n^2} ds$$

discuss char poly $\det(\lambda - A)$.

$\det(\lambda - A)$ vanishes when λ is an eigenvalue of A i.e. $\exists\, v \neq 0$ i.e. a line $\ni$ $\lambda v = Av$. Means $\binom{1}{A}V \cap \binom{1}{\lambda}V \neq 0$. $\det(\lambda - A)$ measures non transversality. $\det(1-\lambda A)$. So what can I do?

Discuss characteristic polys. You have Discuss char polys.

You are given a specific $\zeta(s)$ which you would like to interpret as "the" char. poly. of some operator. You hope to obtain this operator from a geometric 1-dim'l situation. You have some feeling for the operators you can construct, in particular, you get analytic functions of s which vanish at the points of the spectrum of the operator.

Be more specific. Take a partial unitary $X \overset{a}{\underset{b}{\rightrightarrows}} Y$ with V^+, V^- dim = 1. Assume $S(z)$ meromorphic on $\mathbb{C} \cup \infty$ except for $z = -1$, ~~You seem to be able to construct $S(z)$ analytic for $|z|<1$~~ such that $|S(z)| = 1$ for $|z| = 1$, $z \neq -1$. Actually I can construct such S as a convergent Blashke product. $\prod_{k=1}^{\infty} \frac{z - \alpha_k}{1 - \bar{\alpha}_k z}$

$$\frac{z - \alpha}{1 - \bar{\alpha} z} = \frac{\frac{1-s}{1+s} - \alpha}{1 - \bar{\alpha}\frac{1-s}{1+s}} = \frac{1 - s - \alpha(1+s)}{1+s - \bar{\alpha}(1-s)} = \frac{1-\alpha - (1+\alpha)s}{1-\bar{\alpha} + (1+\bar{\alpha})s}$$

$$= \frac{\frac{1-\alpha}{1+\alpha} - s}{\frac{1-\bar{\alpha}}{1+\bar{\alpha}} + s} \quad \frac{1+\alpha}{1+\bar{\alpha}}$$

195

$$\prod_{k=1}^{\infty} \frac{-s+\beta_k}{s+\overline{\beta}_k}$$

$-\overline{\beta}_k \cdot \quad \Big| \quad \cdot \beta_k$

$$\| \quad \prod_{k=1}^{\infty} \frac{\left(1-\frac{s}{\beta_k}\right)}{\left(1+\frac{s}{\overline{\beta}_k}\right)} \cdot \frac{\beta_k}{\overline{\beta}_k}$$

when does this converge?

$$\sum \frac{1}{|\beta_k|} < \infty$$

$$\frac{z-\alpha}{1-\overline{\alpha}z} \xrightarrow{\alpha \to -1} \frac{z+1}{1+z} = 1.$$

$$\left\| 1 - \frac{z-\alpha}{1-\overline{\alpha}z} \right\| = \left\| \frac{1-\overline{\alpha}z - z + \alpha}{1-\overline{\alpha}z} \right\| =$$

$$= \left\| \frac{(1+\alpha) - z(1+\overline{\alpha})}{1-\overline{\alpha}z} \right\|$$

$$\leq \frac{|1+\alpha| + |z||1+\alpha|}{1-|\alpha||z|} \leq |1+\alpha| \frac{1+|z|}{1-|z|}$$

So provided ~~$|1+\alpha_k| < \infty$~~. $\sum_{k=1}^{\infty} |1+\alpha_k| < \infty$

the Blashke product should converge to give an analytic function on the disk with the desired zeroes.

$$\frac{-s+\beta}{s+\overline{\beta}} \to 1 \quad \text{as } |\beta| \to \infty. \text{ NO}$$

$$\prod \frac{-s+\beta_k}{s+\overline{\beta}_k}$$

$$1 - \frac{-s+\beta}{s+\overline{\beta}} = \frac{s+\overline{\beta}+s-\beta}{s+\overline{\beta}}$$

$$\frac{-s+\beta}{s+\overline{\beta}} \frac{\overline{\beta}}{\beta} = \frac{1-\frac{s}{\beta}}{1+\frac{s}{\overline{\beta}}}$$

$$= \frac{2s}{s+|\overline{\beta}|} + \frac{\overline{\beta}-\beta}{s+\overline{\beta}}$$

196

$$-\frac{1-\frac{s}{\beta}}{1+\frac{s}{\beta}}+1 = \frac{-1+\frac{s}{\beta}+1+\frac{s}{\bar\beta}}{1+\frac{s}{\beta}} = \frac{s\left(\frac{1}{\beta}+\frac{1}{\bar\beta}\right)}{1+\frac{s}{\beta}}$$

$$\| \qquad \| \leq \frac{|s|\frac{2}{|\beta|}}{1-\frac{|s|}{|\beta|}}$$

So you get convergence if $\sum \frac{1}{|\beta_k|} < \infty$

but also if $\beta+\bar\beta$

When does $\prod_{k=1}^{\infty} \frac{z-\alpha_k}{1-z\bar\alpha_k}$ conv. abs.

assuming $\alpha_k \to -1$.

$$\left|1-\frac{z-\alpha_k}{1-z\bar\alpha_k}\right| = \left|\frac{1-z\bar\alpha-z+\alpha}{1-z\bar\alpha}\right| = \frac{|1+\alpha-z(1+\bar\alpha)|}{|1-z\bar\alpha|}$$

$$\leq \frac{|1+\alpha|+|z||1+\bar\alpha|}{1-|z|} = \frac{1+|z|}{1-|z|}|1+\alpha|$$

want $\sum |1+\alpha_k| < \infty$

What about $\prod\left(\frac{s-\beta_k}{s+\bar\beta_k}\right)$ Want terms $\to 1$.

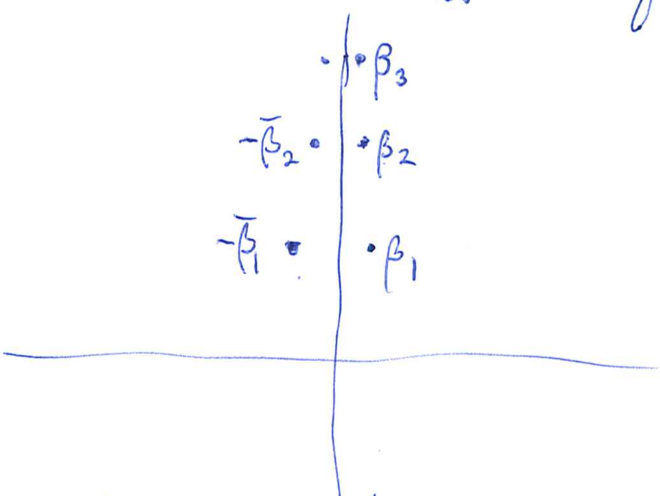
$Re(\beta_k) > 0 \qquad |\beta_k| \to \infty$

$$-\frac{s+\beta}{s+\bar\beta}\frac{\bar\beta}{\beta} = \frac{1-\frac{s}{\beta}}{1+\frac{s}{\bar\beta}} \qquad 1-\frac{1-\frac{s}{\beta}}{1+\frac{s}{\bar\beta}} = \frac{1+\frac{s}{\bar\beta}-1+\frac{s}{\beta}}{1+\frac{s}{\bar\beta}}$$

$$= \frac{s\left(\frac{1}{\beta}+\frac{1}{\bar\beta}\right)}{1+\frac{s}{\bar\beta}}$$

it seems that one can have $\beta \to \infty$ close to the imaginary axis, and in this way get interesting deB functions

197 Let's do this carefully



e.g. take

$$\beta_n = 1 + in$$

$$\frac{1}{\bar\beta_n} + \frac{1}{\beta_n} = \frac{1}{1+in} + \frac{1}{1-in} = \frac{2}{1+n^2}$$

This should lead to an interesting de B function.
Also you can proceed symmetrically i.e. combine
NO

$$\frac{1-\frac{s}{\beta}}{1+\frac{s}{\beta}} \cdot \frac{1-\frac{s}{\bar\beta}}{1+\frac{s}{\bar\beta}} = \frac{1-\frac{s}{\beta}}{1+\frac{s}{\beta}} \quad \frac{1-\frac{s}{\bar\beta}}{1+\frac{s}{\bar\beta}}$$

This doesn't help.

$$s = \frac{1-z}{1+z}$$

$$\frac{1 - \frac{1}{\beta}\frac{1-z}{1+z}}{1 + \frac{1}{\beta}\frac{1-z}{1+z}} = \frac{1+z-\beta^{-1}(1-z)}{1+z+\beta^{-1}(1-z)} = \frac{1-\beta^{-1} + z(1+\beta^{-1})}{1+\beta^{-1} + z(1-\beta^{-1})}$$

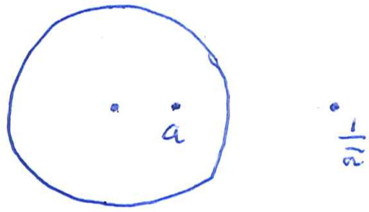
$$= \frac{(1+\beta^{-1})\left(z + \frac{1-\beta^{-1}}{1+\beta^{-1}}\right)}{(1+\beta^{-1})\left(1 + z\frac{1-\beta^{-1}}{1+\beta^{-1}}\right)} \qquad \alpha = \frac{1-\beta}{1+\beta}$$

$$\overline{f(\bar z^{-1})} = f(z)^{-1}$$

$$\overline{f(\bar z^{-1})} = \overline{f\left(e^{2\pi i\bar\omega}\right)} = \overline{F(\bar\omega)} = F(\omega)^{-1} = f(\underbrace{e^{2\pi i\omega}}_{z})^{-1}$$

198

$\cancel{1 + 2\pi} i n$



$$\left| \frac{z-a}{z-\frac{1}{\bar a}} \right| = \left| \frac{z-a}{1-\bar a z} (-\bar a) \right| \le |a| < 1. \quad ?$$

$$\left| \frac{z-a}{\frac{1}{\bar z} - a} \right| = \left| \bar z \, \frac{z-a}{1-a\bar z} \right| \le |z| \cdot 1 < 1,$$

What you want to compare are

$\cancel{|f(z)| \text{ and } |f(\tfrac{1}{\bar z})|}$

$f(z) \qquad \overline{f(z^*)}$

$$z-a, \quad \overline{z^*-a} = \frac{1}{z} - \bar a$$

$$\frac{z-a}{\frac{1}{z}-\bar a} = z \frac{z-a}{1-\bar a z}$$

$$\overline{F(\bar\omega)} = \overline{E(\bar\omega)} / E(\omega)$$

---

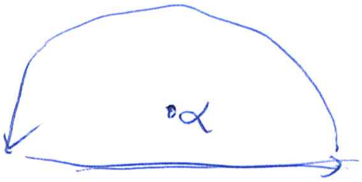
de Branges space $\left\{ f \text{ entire} \;\middle|\; \frac{f(\omega)}{E(\bar\omega)} \in H^2 \right\}$.

to get $E(\omega)$ into this space you enlarge to $f \ni \frac{f(\omega)}{\overline{E(\bar\omega)}\,(\omega+i)} \in H^2$

I need to review some Hardy space stuff.

$$\frac{1}{\lambda - \bar\alpha} \in H^2 \qquad \alpha \in UHP$$

$$\int_{-\infty}^{\infty} \overline{\frac{1}{\lambda-\bar\alpha}} f(\lambda) \frac{d\lambda}{\pi} = \int_{-\infty}^{\infty} \frac{1}{\lambda-\alpha} f(\lambda) \frac{d\lambda}{\pi}$$



f analytic in UHP
L^2 bdry values.

$$\int_0^\pi \frac{1}{Re^{i\theta}-\alpha} f(Re^{i\theta}) R\frac{e^{i\theta}\, i\,d\theta}{\pi}$$

$$= \int_0^\infty \underbrace{\frac{iRe^{i\theta}}{Re^{i\theta}-\alpha}} f(Re^{i\theta}) \frac{d\theta}{\pi}$$

$$\frac{i}{1-\frac{1}{Re^{i\theta}}\alpha}$$ bdd. $\to i$ as $R\to\infty$.

so provided $\int_0^{2\pi} |f(Re^{i\theta})|\,d\theta$ goes to zero

we have

$$\int_{-\infty}^\infty \overline{\frac{i}{(\lambda-\bar\alpha)}}\, f(\lambda)\frac{d\lambda}{2\pi} = \int_{-\infty}^\infty \frac{-i}{(\lambda-\alpha)} f(\lambda)\frac{d\lambda}{2\pi} =$$

$$= \oint \frac{f(\lambda)}{\lambda-\alpha}\frac{d\lambda}{2\pi i} = f(\alpha).$$

Conversely.

$$f(\alpha) = \int_{-\infty}^\infty \overline{\frac{i}{\lambda-\bar\alpha}}\, f(\lambda)\frac{d\lambda}{2\pi}$$

$$|f(\alpha)|^2 \le \underbrace{\left\|\frac{i}{\lambda-\bar\alpha}\right\|^2} \|f\|^2$$

$$\frac{i}{\alpha-\bar\alpha} = \frac{i}{2i\,\mathrm{Im}(\alpha)} = \frac{1}{2\,\mathrm{Im}(\alpha)}.$$

200

$$|f(\alpha)| \leq \left(\frac{1}{2\,\mathrm{Im}(\alpha)}\right)^{1/2} \|f\|$$

$$\int_{-\infty}^{\infty} \frac{1}{|\lambda-\bar\alpha|^2} \frac{d\lambda}{2\pi} = \int_{-\infty}^{\infty} \frac{1}{\lambda^2+\mathrm{Im}\,\alpha^2} \frac{d\lambda}{2\pi} = \int_{-\infty}^{\infty} \frac{1}{\lambda^2+1} \frac{d\lambda}{2\pi} \frac{1}{\mathrm{Im}\,\alpha}$$

$$\leq \frac{1}{2\,\mathrm{Im}\,\alpha}$$

$$|f(\overset{Re^{i\theta}}{\lambda})| \leq \left(\frac{1}{2R\sin\theta}\right)^{1/2} \|f\|$$

Check the 2.

$$\int_{-\infty}^{\infty} \overline{\frac{i}{2(\lambda-\bar\alpha)}} f(\lambda) \frac{d\lambda}{\pi} = \int \frac{1}{(\lambda-\alpha)} f(\lambda) \frac{d\lambda}{2\pi i} = f(\alpha)$$

$$|f(\alpha)| \leq \left\| \frac{i}{2(\lambda-\bar\alpha)} \right\| \cdot \|f\|$$

$$\left\| \frac{i}{2(\lambda-\bar\alpha)} \right\|^2 = \frac{i}{2(\alpha-\bar\alpha)} = \frac{1}{4\,\mathrm{Im}\,\alpha}$$

$$\left\| \frac{i}{2(\lambda-\bar\alpha)} \right\| = \frac{1}{2\sqrt{\mathrm{Im}\,\alpha}}$$ not sign.

$E(\omega)$ de B function, $E^\#(\omega) = \overline{E(\bar\omega)}$

$$\left\{ f \;\middle|\; \frac{f(\omega)}{E^\#(\omega)} \in H^2 \right\}.$$ e.g. $f(\omega) = \dfrac{E(\omega)}{\omega-\bar\alpha}$

$$\frac{1}{\omega-\bar\alpha} \frac{E(\omega)}{E^\#(\omega)}$$ — scattering function

analytic in the UHP <u>bdd</u>.

201 Avoid deB functions, first look at scattering functions. To construct some interesting ones. You combine zero $\lambda - \alpha$ with pole $(\lambda - \bar{\alpha})^{-1}$ to get

$\frac{\lambda-\alpha}{\lambda-\bar{\alpha}}$ exc. for $e^{i\theta}$ want $\alpha_n \to \infty$, take product

$e^{i\theta}\frac{\alpha}{\bar{\alpha}} = 1.$

$$e^{i\theta}\frac{\lambda-\alpha}{\lambda-\bar{\alpha}} = e^{i\theta}\frac{\alpha}{\bar{\alpha}}\frac{\frac{\lambda}{\alpha}-1}{\frac{\lambda}{\bar{\alpha}}-1} = \frac{1-\frac{\lambda}{\alpha}}{1-\frac{\lambda}{\bar{\alpha}}}$$ close to 1 ?

$$1 - \frac{1-\frac{\lambda}{\alpha}}{1-\frac{\lambda}{\bar{\alpha}}} = \frac{1-\frac{\lambda}{\bar{\alpha}}-1+\frac{\lambda}{\alpha}}{1-\frac{\lambda}{\bar{\alpha}}} = \frac{\left(\frac{1}{\alpha}-\frac{1}{\bar{\alpha}}\right)\lambda}{1-\frac{\lambda}{\bar{\alpha}}}$$

$$\frac{\lambda}{1-\frac{\lambda}{\bar{\alpha}}} \xrightarrow[|\alpha|\to\infty]{\text{as}} \lambda$$ $\frac{1}{\alpha_n} - \frac{1}{\bar{\alpha}_n}$ need to be l^1 seq.

what sort of possibilities arise?

before $\alpha_n = n + i$, $\frac{1}{n+i} - \frac{1}{n-i} = \frac{n-i-(n+i)}{n^2+1} = \frac{-2i}{n^2+1}$

What's the best you might do in going toward ∞.

$$\alpha = x + iy \qquad -\frac{1}{x+iy} + \frac{1}{x-iy} = \frac{2iy}{x^2+y^2}$$

try $y = x^a$

$$\frac{y}{x^2+y^2} = \frac{x^a}{x^2+x^{2a}} = \frac{1}{x^{2-a}+x^a}$$ $a=1$ no good.

observe that if $a = 1+\varepsilon$ $$\frac{1}{x^{1-\varepsilon}+x^{1+\varepsilon}} = \frac{1}{x^{1+\varepsilon}}\frac{1}{1+\frac{x^{1-\varepsilon}}{x^{1+\varepsilon}}}$$

$$= \frac{1}{x^{1+\varepsilon}}\left(\frac{1}{1+x^{-2\varepsilon}}\right) \sim \frac{1}{x^{1+\varepsilon}}$$ so $y = cx$ is critical no matter what c is.