Questions. Hodge signature then on H^{11} is a consequence of RR for surfaces. at least for classes of Line bundles.

$$h^0(D) - h^1(D) + h^2(D) \sim \cdot D^2 \cancel{+} D\cdot K$$

Example of elliptic curve × elliptic curve should have $K=0$.

$$D(D \cancel{+} K)$$

$$\to (K-D)(K-D-\cancel{K}) = D(D-K)$$

You have a real quadratic space with ~~canonical elements~~ given element H such that $h^0(aH+bD)$?

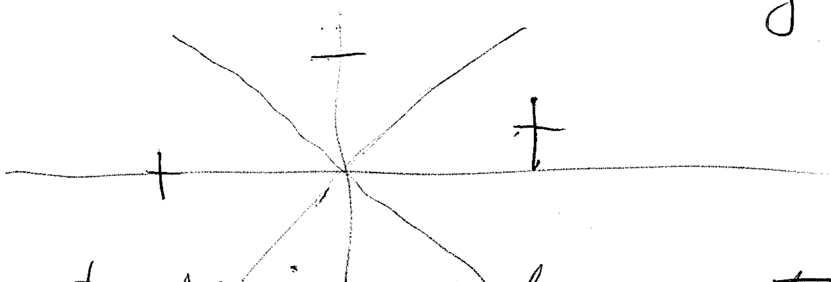
$$h^0(D) - h^1(D) + h^2(D) = D^2$$

$$h^1(D) = h^1(-D) \qquad h^2(D) = h^0(-D) \qquad \text{Serre duality}$$

$$\therefore \quad h^0(D) + h^0(-D) - h^1(D) = D^2$$

$$h^0(D+tH) = (D+tH)^2 \qquad t \gg 0.$$

real quadratic form (non-deg) has Sylvester form $\|x\|^2 - \|y\|^2$ $x \in \mathbb{R}^p$, $y \in \mathbb{R}^q$ divides $\mathbb{P}(\mathbb{R}^{p+q})$ into >0, $=0$, <0. pieces. Point is that the ~~Lorentz~~ case $p=1$



then the + region not connected

$$h^0(D) > 0 \implies D\cdot H > 0$$

$$D^2 > 0 \implies h^0(D) + h^0(-D) > 0$$

$$\implies \text{either } h^0(D) > 0 \text{ or } h^0(-D) > 0$$

$$\implies \text{either } D\cdot H > 0 \text{ or } -D\cdot H > 0$$

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ring of self correspondences on X

~~Hodge decomp thm!~~ Think carefully! Is there a link between $h^0(L)$ with L rational, i.e. in Pic of the curve over $\mathbb{F}_q$, and the quadratic space of ~~classes~~ curves on $X \times X$.

You have in mind the transition from the "additive form" ζ: $\sum_{D \geq 0} z^{\deg D}$ to the Euler product form $\prod \frac{1}{1 - Np^{-s}}$, taking log and relating to points over finite fields.

If X is an elliptic curve

$$\sum \frac{q^{h^0(L)} - 1}{q - 1} z^{\deg L}$$

$h^0(L) = \deg L$ if $\deg L > 0$

$c = X(\mathbb{F}_q)$.

$$\cancel{= 1 \cdot z^0 + z^1(c) + z^2(c) + \cdots}$$

$$= z^0 \cdot 1 + c\left(\frac{q-1}{q-1} z + \frac{q^2-1}{q-1} z^2 + \cdots\right.$$

$$= 1 + \frac{c}{q-1}\left(\underbrace{\frac{1}{1-qz} - \frac{1}{1-z}}_{\frac{1-z-1+qz}{(1-qz)(1-z)} = \frac{(q-1)z}{(1-qz)(1-z)}}\right) = 1 + c\,\frac{z}{(1-qz)(1-z)}$$

$$\frac{1-(q+1)z + qz^2 + cz}{(1-qz)(1-z)}$$

c

$$\zeta = \frac{1+(c-q-1)z+qz^2}{(1-qz)(1-z)} = \frac{(1-\alpha_1 z)(1-\alpha_2 z)}{(1-qz)(1-z)}$$

$$(c-q-1)^2 - 4q \leq 0$$

$$\left|\frac{c-q-1}{2\sqrt{q}}\right| \leq 1. \qquad c-q-1 = -\alpha_1-\alpha_2$$

$$\#X(\mathbb{F}_{q^n}) = q^n+1-\alpha_1^n-\alpha_2^n$$

$$-2q^{1/2} \leq c-q-1 \leq 2q^{1/2}$$

$$(q^{1/2}-1)^2 \leq c \leq (q^{1/2}+1)^2$$

$g=0.$
$$\sum_{n\geq 0} \frac{q^{n+1}-1}{q-1} z^n = \frac{1}{q-1}\left(\frac{q}{1-qz} - \frac{1}{1-z}\right)$$

$$= \frac{1}{q-1}\,\frac{q-\cancel{qz}-1+\cancel{qz}}{(1-qz)(1-z)} = \frac{1}{(1-qz)(1-z)}$$

The important problem here is to find an analog of the tower $\mathbb{F}_{q^n}$.

$$\zeta_{X/\mathbb{F}_q}(s) = \sum_{D\geq 0} (ND)^{-s} = \sum_{D\geq 0} z^{\deg(D)} \qquad z=q^{-s}$$

$$= \sum_{L} \frac{q^{h^0(D)}-1}{q-1} z^{\deg(L)}$$

$$\zeta_{\mathbb{Z}\cup\infty}(s) = \sum_{n=1}^{\infty} n^{-s}\, \pi^{-s/2}\Gamma(s/2) = \sum_{n=1}^{\infty} \cancel{n^{-s}} \int_0^\infty e^{-\pi n^2 t^2} t^s\, 2\frac{dt}{t}$$

$$= \int_0^\infty (\theta(t^2)-1)\, t^s \frac{dt}{t} \qquad \theta(t^2) = \sum_{n\in\mathbb{Z}} e^{-\pi n^2 t^2} = \frac{1}{t}\theta(t^{-2})$$

d

s $\quad \zeta_{\mathbb{Z}\cup\infty}(s) = \int_0^\infty \left(\theta(t^{-2})-1\right) t^{-s}\frac{dt}{t} \qquad s(1-s) = \left(\frac{1}{2}+\tau\right)\left(\frac{1}{2}-\tau\right) = \frac{1}{4}-\tau^2$

$$\zeta_{\mathbb{Z}\cup\infty}(s) = \int_0^\infty \left(t\theta(t^2)-1\right) t^{-s}\frac{dt}{t}$$

$$s(1-s)\zeta_{\mathbb{Z}\cup\infty}(s) = \int_0^\infty \left(t\theta(t^2)-1\right) \quad s(1-s)t^{-s+1}\frac{dt}{\cancel{t}}$$

$$\partial_t^2\left(t^{-s+1}\right) = \partial_t(-s+1)t^{-s} = \underbrace{(-s+1)(-s)}_{-s(1-s)}t^{-s-1}$$

$$s(1-s)\zeta_{\mathbb{Z}-\infty}(s) = \int_0^\infty \left(t\theta(t^2)-1\right)\left(-\partial_t^2\right)t^{-s+1}\,dt$$

$$= -\int_0^\infty \underbrace{\partial_t^2\left(t\theta(t^2)-1\right)}\, t^{-s}\,\frac{dt}{t}$$

decays rapidly as $t \to 0$ or ∞.

You should center at $s = \frac{1}{2}$.

$$\zeta_{\mathbb{Z}\cup\infty}(s) = \int_0^\infty \left(\theta(t^2)-1\right) t^{s}\frac{dt}{t} \qquad \begin{array}{l} \frac{1}{t} \sim \theta(t^2) \sim 1 \\ 1 \sim t\theta(t^2) \sim t \end{array}$$

$$\zeta\left(\tfrac{1}{2}+\tau\right) = \int_0^\infty \left(\theta(t^2)-1\right) t^{1/2+\tau}\frac{dt}{t}$$

$$\left(\tau+\tfrac{1}{2}\right)\left(\tau-\tfrac{1}{2}\right)\zeta\left(\tau+\tfrac{1}{2}\right) = \int_0^\infty \left(\theta(t^2)-1\right) t^{\tau-\frac{1}{2}} \left(\tau+\tfrac{1}{2}\right)\left(\tau-\tfrac{1}{2}\right)\,dt$$

$$t\frac{d}{dt}\frac{d}{dt}\left(t^{\tau+\frac{1}{2}}\right)$$

$$= \int_0^\infty \frac{d^2}{dt^2}\left(t\theta(t^2)-t\right) t^{\tau+\frac{1}{2}}\,dt$$

$$= \int_0^\infty t^{3/2}\frac{d^2}{dt^2}\left(t\theta(t^2)\right) t^{\tau}\frac{dt}{t}$$

e

$$\underset{2\upsilon\infty}{\zeta(s)} = \int_0^\infty \underbrace{(\theta(t^2)-1)\, t^{\frac{1}{2}+\tau}}_{(\theta(t^2)-1)t^{1/2}\quad t^{\tau}} \frac{dt}{t}$$

What do you know about $\theta(t^2)t^{1/2}$?

$$\theta(t^{-2})\, t^{-1/2} = t\,\theta(t^2)\, t^{-1/2} = t^{1/2}\theta(t^2)$$

$$t^{1/2}\theta(t^2) \qquad t^{-1/2}\theta(t^{-2}) = t^{-1/2}t\,\theta(t^2) = t^{1/2}\theta(t^2)$$

invariant under $t \mapsto t^{-1}$. As $t \to +\infty$

$t^{1/2}\theta(t^2) \sim t^{1/2}$

$$t^{-1/2} \overset{t\to 0}{\sim} \underbrace{t^{1/2}\theta(t^2)}_{\phi(t)} \overset{t\to+\infty}{\sim} t^{1/2}$$

$$\zeta(\tfrac{1}{2}+\tau) = \int_0^\infty \left(\phi(t) - t^{1/2}\right) t^{\tau} \frac{dt}{t}$$

$$(\tfrac{1}{2}+\tau)(\tfrac{1}{2}-\tau)\zeta(\tfrac{1}{2}+\tau) = \int_0^\infty (\phi(t)-t^{1/2})\, (\tfrac{1}{2}+\tau)(\tfrac{1}{2}-\tau)\, t^{\tau} \frac{dt}{t}$$

$$\|$$

$$\left(-t\partial_t+\tfrac{1}{2}\right)\left(t\partial_t+\tfrac{1}{2}\right) t^{\tau} \qquad \int_0^\infty (t\partial_t+\tfrac{1}{2})^*(-t\partial_t+\tfrac{1}{2})^*\left(\frac{\phi(t)-t^{1/2}}{t}\right) t^{\tau} dt$$

$$(\partial_t t + \tfrac{1}{2})\underbrace{(\partial_t t + \tfrac{1}{2})}(\tfrac{1}{t})(\quad)$$

$$\cancel{(\partial_t t + \tfrac{1}{2})}(\partial_t + \tfrac{1}{2t})\,(\phi(t)-t^{1/2})\; t^{\tau}\, dt$$

$$\cancel{-(t\partial_t+\tfrac{1}{2})\tfrac{1}{t}(\partial_t+\tfrac{1}{2t})}$$

$$\left(-\partial_t t + \tfrac{1}{2}\right)\frac{1}{t}\left(t\partial_t+\tfrac{1}{2}\right) = \frac{1}{t}\left(-t\partial_t+\tfrac{1}{2}\right)\left(t\partial_t+\tfrac{1}{2}\right)$$

~~F~~ good results. $\theta(t^2) = \sum_{n\in\mathbb{Z}} e^{-\pi n^2 t^2}$ satisfies

$$\theta(t^{-2}) = t\,\theta(t^2)$$

Thus $t^{-1/2}\theta(t^{-2}) = t^{-1/2} t\,\theta(t^2) = t^{1/2}\theta(t^2)$

so that $\phi(t) = t^{+1/2}\theta(t^2) \Rightarrow \phi(t^{-1}) = \phi(t)$.

Also $t^{-1/2} \overset{t\to 0}{\sim} \phi(t) \overset{t\to\infty}{\sim} t^{1/2}$

$$\zeta_{\mathbb{Z}\cup\infty}\left(\tfrac{1}{2}+\tau\right) = \int_0^\infty \left(\theta(t^2)-1\right) t^{\frac{1}{2}+\tau}\,\frac{dt}{t} \qquad Re(\tau) > \tfrac{1}{2}$$

$$= \int_0^\infty \left(\phi(t) - t^{1/2}\right) t^{\tau}\,\frac{dt}{t}$$

$$= \int_{-\infty}^{\infty} \left(\phi(e^x) - e^{\frac{1}{2}x}\right) e^{x\tau}\,dx$$

$$\int_{-\infty}^{\infty} \left(\partial_x^2 - \tfrac{1}{4}\right)\left(\phi(e^x) - \cancel{e^{-1/2 x}}\right) e^{x\tau}\,dx$$

$$= \int_{-\infty}^{\infty} \left(\phi(e^x) - e^{-1/2 x}\right) \underbrace{\left(\partial_x^2 - \tfrac{1}{4}\right) e^{x\tau}}_{\tau^2 - \frac{1}{4}}\,dx = \left(\tau^2 - \tfrac{1}{4}\right)\zeta_{\mathbb{Z}\cup\infty}\left(\tfrac{1}{2}+\tau\right)$$

so

$$\boxed{\begin{aligned}\left(\tau^2 - \tfrac{1}{4}\right)\zeta_{\mathbb{Z}\cup\infty}\left(\tfrac{1}{2}+\tau\right) &= \int_{-\infty}^{\infty}\left(\partial_x^2 - \tfrac{1}{4}\right)\phi(e^x)\, e^{x\tau}\,dx \\ &= \int_0^\infty \left((t\partial_t)^2 - \tfrac{1}{4}\right)\phi(t)\, t^{\tau}\,\frac{dt}{t}\end{aligned}}$$

9

You try to organize ~~...~~ curve over finite field case, emphasize the tower of finite fields which yields perhaps ~~...~~ a family of ζ functions, also maybe L functions.

So what happens? Take $k_0 = \mathbb{F}_q$ corresp $\zeta(s)$ is $\frac{1}{1-q^{-s}}$ ~~...~~. For $X/\mathbb{F}_q$ smooth + proper

$$\zeta(s) = \sum_L{}' \frac{q^{h^0(L)}-1}{q-1} q^{-s\deg(L)}$$

Ignore $\deg(L) \leq 2g$, then have

$$h\sum \frac{q^{n+1-g}-1}{q-1} q^{-sn} = \frac{h}{q-1}\left(\frac{q^{1-g}}{1-qq^{-s}} ? - \frac{1}{1-q^{-s}}\right)$$

$$= \frac{h}{q-1}\left(\frac{\cancel{q^{1-g}}\,?}{1-qz} - \frac{1}{1-z}\right)$$

Look carefully at the elliptic curve case.

$$\zeta(s) = \sum_L \frac{q^{h^0(L)}-1}{q-1} z^{\deg L}$$

sum over ~~...~~ divisor classes

$$= \sum_{\deg(L)=0} 1 + \sum_{n=1}^{\infty} \sum_{\deg(L)=n} h\frac{q^{n}-1}{q-1} z^n$$

h is class no. in this case the number of pts rational over $\mathbb{F}_q$

$$= 1 + \frac{h}{q-1}\left(\frac{1}{1-qz} - \frac{1}{1-z}\right)$$

$$= 1 + \frac{h}{q-1}\frac{\cancel{1}-z-\cancel{1}+qz}{(1-qz)(1-z)} = 1 + \frac{hz}{(1-qz)(1-z)}$$

$$= \frac{(1-qz)(1-z)+hz}{(1-qz)(1-z)} = \frac{1-(1+q-h)z+qz^2}{(1-z)(1-qz)}$$

h

$$1-(1+q-h)z+qz^2=0$$

$$\text{roots} = \frac{1+q-h \pm \sqrt{(1+q-h)^2-4q}}{2q}$$

want conjugate complex roots, since product $= q^{-1}$, this means (inverse) roots have $|\alpha_i^{-1}| = q^{-1/2}$, so this $|q^{-s}| = q^{-\mathrm{Re}(s)} = q^{-1/2} \implies \mathrm{Re}(s) = \frac{1}{2}$.

$$(1+q-h)^2 \leq 4q$$

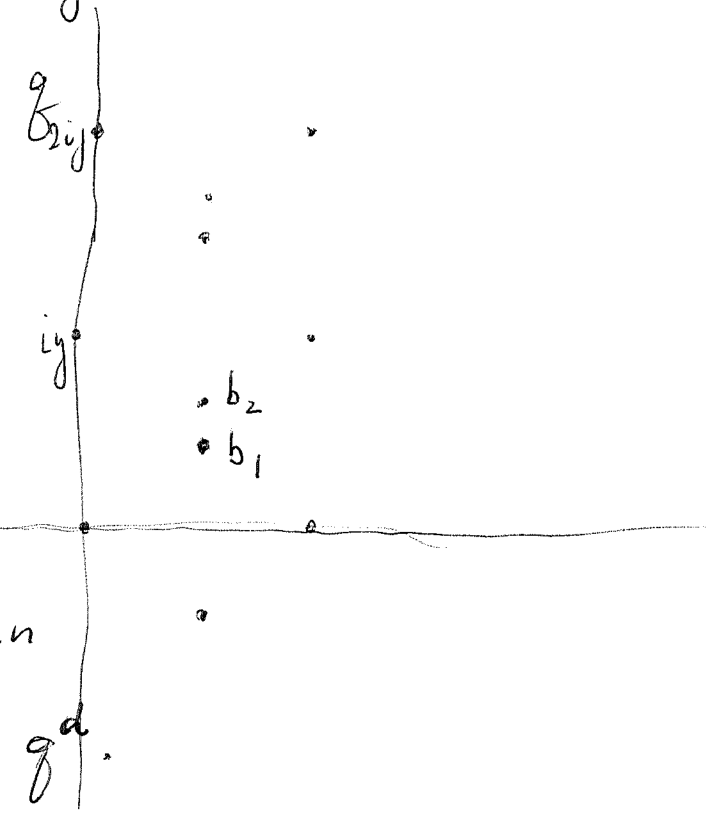
$-2q^{1/2} \leq 1+q-h \leq 2q^{1/2}$. So ~~RH~~ the RH amounts to something simple like ~~[illegible]~~

$$|h-(q+1)| \leq 2q^{1/2}$$

Now the Lefschetz formula gives What

$$h = 1-(\alpha_1+\alpha_2)+q$$

$$\zeta(s) = \frac{(1-\alpha_1 q^{-s})(1-\alpha_2 q^{-s})}{(1-q^{-s})(1-q^{1-s})}$$



$q^{-s}=1 \quad -s\log q \in 2\pi i\mathbb{Z}$

$$y = \frac{2\pi}{\log q}$$

$q^{s} = \alpha_1 \quad s\log q = \log\alpha_1 + 2\pi i n$

suppose you replace q by q^d.

$$s\log q^d = \log\alpha_1^d + 2\pi i n$$

$$s\log q = \log\alpha_1 + \frac{2\pi i n}{d}$$

i

Simple question. Suppose you try to reverse the ~~previous~~ expression of $s(s-1)\,\zeta_{\mathbb{Z}\cup\infty}$ as a Fourier transform. Do it again.

$$\zeta_{\mathbb{Z}\cup\infty} = \sum_{n=1}^{\infty} \frac{1}{n^s} \underbrace{\pi^{-s/2}\,\Gamma(s/2)}_{\int_0^\infty e^{-\pi t^2} t^s\, 2\frac{dt}{t}}$$

$$= \sum_{n=1}^{\infty} \int_0^\infty 2e^{-\pi n^2 t^2}\, t^s \frac{dt}{t} = \int_0^\infty \left(\theta(t^2)-1\right) t^s \frac{dt}{t}$$

$$\zeta_{\mathbb{Z}\cup\infty}\left(\tfrac{1}{2}+s\right) = \int_0^\infty \Big(\underbrace{t^{1/2}\theta(t^2)}_{\phi(t)} - t^{1/2}\Big)\, t^{s} \frac{dt}{t} \qquad \operatorname{Re} s > \tfrac{1}{2}$$

$$\phi(t^{-1}) = t^{-1/2}\,\theta(t^{-2}) = t^{-1/2}\, t\, \theta(t^2) = t^{1/2}\theta(t^2) = \phi(t)$$

$$\zeta_{\mathbb{Z}\cup\infty}\left(\tfrac{1}{2}+s\right) = \int_{-\infty}^{\infty} \left(\phi(e^x) - e^{1/2 x}\right) e^{xs}\, dx$$

$$\int_{-\infty}^{\infty} \left(\frac{d^2}{dx^2}-\frac{1}{4}\right)\left(\phi(e^x) \cancel{-e^{1/2x}}\right) e^{xs}\, dx$$

$$\|$$

$$\int_{-\infty}^{\infty} \left(\frac{d^2}{dx^2}-\frac{1}{4}\right)\left(\phi(e^x) - e^{1/2 x}\right) e^{xs}\, dx$$

$$\|$$

$$\int_{-\infty}^{\infty} \left(\phi(e^x) - e^{1/2 x}\right)\left(s^2-\frac{1}{4}\right) e^{xs}\, dx = \left(s^2-\frac{1}{4}\right) \zeta_{\mathbb{Z}\cup\infty}\left(\tfrac{1}{2}+s\right)$$

$$\boxed{\left(s^2-\frac{1}{4}\right) \zeta_{\mathbb{Z}\cup\infty}\left(\tfrac{1}{2}+s\right) = \int_{-\infty}^{\infty} \left(\left(\frac{d}{dx}\right)^2-\frac{1}{4}\right)\phi(e^x)\; e^{xs}\, dx}$$

Your idea will be to go backwards, namely ~~to~~ take an entire function of s like $\left(s^2-\frac{1}{4}\right)\xi_{Z\cup\infty}(s)$ is supposed to be (assuming RH) and study the inverse F.T. Shift from s to $-i\lambda$

$$Z(\lambda) = \prod_{n=1}^{\infty}\left(1-\frac{\lambda^2}{\omega_n^2}\right),$$

even real valued oscillatory along the real axis. Try a simple example like $f(\lambda)=\cos(\lambda)$ or $\frac{\sin \pi\lambda}{\pi\lambda} = \prod_{n=1}^{\infty}\left(1-\frac{\lambda^2}{n^2}\right)$.

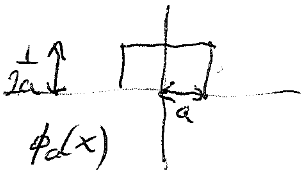
$$f(\lambda) = \int_{-\infty}^{\infty} \psi(x)\, e^{i\lambda x}\, dx$$

$$\frac{\sin \pi\lambda}{\pi\lambda} = \int_{-\pi}^{\pi} \frac{e^{i\lambda t}}{2\pi}\, dt = \left[\frac{e^{i\lambda t}}{2\pi i\lambda}\right]_{t=-\pi}^{t=\pi}$$

$$\prod_{n=1}^{\infty} \cos\left(\frac{z}{a_n}\right)$$

$$-\log\cos\theta = -\log\left(1-(1-\cos\theta)\right)$$
$$= (1-\cos\theta) + \frac{(1-\cos\theta)^2}{2} + $$
$$= \left(\frac{\theta^2}{2!}+\frac{\theta^4}{4!}\right) + \frac{1}{2}\left(\frac{\theta^2}{2!}+\frac{\theta^4}{4!}\right)^2$$
$$= \frac{\theta^2}{2} + \left(\frac{1}{4!}+\frac{1}{2\cdot 4}\right)\theta^4$$
$$\frac{1}{24}+\frac{3}{24} = \frac{4}{24} = \frac{1}{6}$$

There should be no convergence problem when $\sum \frac{1}{a_n^2} < \infty$. Actually ~~you want to use~~ Hörmander uses $\sin$? He considers a convolution of step functions



$$\int_{-a}^{a} \frac{1}{2a} e^{i\lambda x}\, dx = \left[\frac{e^{i\lambda x}}{2ai\lambda}\right]_{-a}^{a} = \frac{e^{i\lambda a}-e^{-i\lambda a}}{2i\lambda a} = \frac{\sin\lambda a}{\lambda a}$$

k

To get $\phi_{a_1} * \phi_{a_2} * \cdots\cdots$ to have compact support you want $\sum a_i < \infty$

Actually this conv. supported in $[-R, R]$ $\quad R = \sum a_i$

To get ~~conve~~ convergence you probably need

$$\frac{\sin \lambda a_n}{\lambda a_n} = 1 - \frac{(\lambda a)^2}{3!} + $$

so $\prod \frac{\sin \lambda a_n}{\lambda a_n}$ ~~will~~ should converge to an entire function of λ when $\sum a_n^2 < \infty$.

infinitely divisible measure

l

$$(\cos z)\left(\cos\frac{z}{3}\right)\cdots\cdots = \prod_{n\geq 0}\cos\left(\frac{z}{3^n}\right)$$

$$f(z) = (\cos z)\, f\left(\frac{z}{3}\right)$$

$$\log f(z) = (\log\cos z) + \log f\left(\frac{z}{3}\right)$$

$$\sum a_n z^n = \sum b_n z^n + \sum a_n \frac{z^n}{3^n}$$

$$f(z) = (1-z)\, f\left(\frac{z}{3}\right)$$

$$f(z) = (1-z)\, f(qz)$$

$$= (1-z)(1-qz)\, f(q^2 z)$$

$$\therefore f(z) = \prod_{n=1}^{\infty}(1-q^n z)$$

$$\sum_{n\geq 1} a_n z^n = \sum_{n\geq 1}\frac{z^n}{n} + \sum_{n\geq 1} a_n q^n z^n$$

$$a_n(1-q^n) = \frac{1}{n} \qquad\qquad a_n = \frac{1}{n(1-q^n)}$$

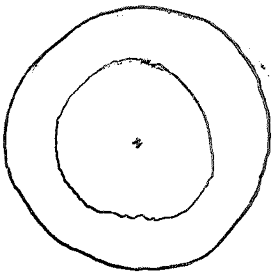
So there's an entire function of λ here with real zeroes $\lambda = q^{-n}$, $n \geq 1$, which you might study. How? You need the ~~basic~~ deformation q-theory. $\quad q = e^{2\pi i \tau}$

$\tau \in$ the UHP. Then you look at the R surface $\mathbb{C}^\times / q^{\mathbb{Z}} = \mathbb{C}/2\pi i \mathbb{Z} + 2\pi i \tau \mathbb{Z}$

$$= \mathbb{C}/2\pi i(\mathbb{Z} + \tau \mathbb{Z}).$$

line bundles, divisors.

$$e^{2\pi i \lambda} = z$$



UHP/$\mathbb{Z}$ becomes $|z| < 1$

Weierstrass σ functions $\sigma(\lambda) = \prod (\lambda - m - n\tau)$

$m, n \in \mathbb{Z}$ suitably regularized

$$\frac{\sigma'(\lambda)}{\sigma(\lambda)} = \sum \frac{1}{\lambda - m - n\tau} + ?$$

$$-\frac{d^2}{d\lambda^2} \log \sigma(\lambda) = \sum' \frac{1}{(\lambda - m - n\tau)^2} + ?$$

But $\displaystyle \sum_m \frac{1}{(\lambda - m)} = \frac{1}{\pi} \frac{\cos \pi\lambda}{\sin \pi\lambda} = \frac{i}{\pi} \frac{e^{i\pi\lambda} + e^{-i\pi\lambda}}{e^{i\pi\lambda} - e^{-i\pi\lambda}}$

$$\cancel{\text{So } -\frac{d^2}{d\lambda^2}\log\sigma(\lambda) = \sum_n \frac{1}{\pi}\cot(\pi(\lambda - n\tau))}$$

$$= \frac{i}{\pi}\left(\frac{e^{2\pi i\lambda} + 1}{e^{2\pi i\lambda} - 1}\right) = \frac{i}{\pi}\left(\frac{2}{e^{2\pi i\lambda} - 1} + 1\right)$$

635 Review - scattering

Given (X, c), get $(E, u, j: X \to E)$ $\qquad j^* u^n j = \begin{cases} c^n & n \geq 0 \\ (c^*)^{-n} & n \leq 0 \end{cases}$

$E = L^2(S^1, d\mu)$ $\quad d\mu$ measure on S^1 values in $\mathcal{L}(X)$ corresp to positive def. function

$$n \longmapsto \begin{cases} c^n & n \geq 0 \\ (c^*)^{-n} & n \leq 0 \end{cases}$$

on $\mathbb{Z}$. Proof of positivity -

Can assume $\|c\| < 1$ by replacing c by rc and letting $r \uparrow 1$. Then $d\mu = \rho \frac{d\theta}{2\pi}$ where

$$\rho(z) = \sum_{n \geq 0} z^{-n} c^n + \sum_{n > 0} z^n c^{*n} = \frac{1}{1 - z^{-1}c}(1 - cc^*)\frac{1}{1 - zc^*}$$
$$= \frac{1}{1 - zc^*}(1 - c^*c)\frac{1}{1 - z^{-1}c}$$

~~splitting~~ of E: have Hilb. space direct sum.

$$E = \cdots \oplus u^{-1}V_- \oplus jX \oplus V_+ \oplus uV_+ \oplus \cdots$$

$V_+ = \overline{\nu_+ X}$ $\qquad \nu_+ = uj - jc$ $\qquad \|\nu_+ x\|^2 = (x, (1 - c^*c)x)$

$V_- = \overline{\nu_- X}$ $\qquad \nu_- = j - ujc^*,$ $\qquad \|\nu_- x\|^2 = (x, (1 - cc^*)x).$

$j^* u^n \nu_+ = j^*(u^{n+1}j - u^n jc) = c^{n+1} - c^n c = 0$ if $n \geq 0$.

$\nu_+^* u^n \nu_+ = (j^* u^{-1} - c^* j^*) u^n \nu_+ = 0$ if $n > 0$.

$j^*(u^{-n}\nu_-) = j^*(u^{-n}j - u^{-n+1}jc^*)$ $\quad \therefore$ if $n \neq 0$.

$= (c^*)^n - (c^*)^{n-1}c^* = 0$ if $n \geq 1$.

$\nu_+^* u^{-n} \nu_- = (j^* u^{-1} - c^* j^*) u^{-n} \nu_- = 0$ if $n \geq 1$.

 Computation of ~~std. lower~~ orthog. inc. repns.

$$\begin{array}{ccccccccc}
L^2(S^1, V_+) & & \cdots\cdots \oplus u^{-1}V_+ & \oplus & V_+ & \oplus & uV_+ & & \\
\Big\downarrow j_+ & & \cap & & \cap & & \| & & \\
E & \cdots \oplus u^{-1}V_- & \oplus\, jX & \oplus & V_+ & \oplus & uV_+ & \oplus & \cdots \\
\Big\uparrow j_- & \| & \| & & \cup & & & & \\
L^2(S^1, V_-) & \oplus\, u^{-1}V_- & \oplus\, V_- & \oplus & uV_+ & \oplus & \cdots & &
\end{array}$$

$$j_+^* j x = \sum_{n \geq 1} u^{-n} \nu_+ x_n$$

$$\Big(u^{-k}\nu_+ x', \overbrace{j_+^* j x}^{\sum u^{-n}\nu_+ x_n}\Big) = (\nu_+ x', \nu_+ x_k)$$

$$= \big(j_+(u^{-k}\nu_+ x'), jx\big) = (u^{-k}\nu_+ x', jx)$$

$$= (j^* u^{-k}\nu_+ x', x) = (j^* u^{-k}(uj - jc)x', x)$$

$$= \Big(\big((c^*)^{k-1} - (c^*)^k c\big)x', x\Big)$$

$$= \big((1 - c^*c)x', c^{k-1}x\big) \qquad \therefore \quad x_k = c^{k-1}x.$$

$$\therefore \quad j_+^* j x = \sum_{n \geq 1} u^{-n} \nu_+ c^{n-1} x \longmapsto \nu_+\Big(\frac{z^{-1}}{1 - z^{-1}c} x\Big)$$

$$\| $$

$$\nu_+\Big(\frac{1}{z - c} x\Big).$$

637 ~~Let's go on to~~

Correspondences on a curve = line bundles on $X \times X$ modulo image pr_1^*, pr_2^*. To prove what? Δ, F actually the endom. F^n for $n \geq 0$, $F^0 = I$ corresp to Δ. Then

$$\text{card } X(\mathbb{F}_{q^n}) = 1 - \underbrace{(\Delta, F^n)}_{tr(F^n)} + q^n$$

There is the algebra of correspondences represented on $H^1(X, \mathbb{Z}) = \mathbb{Z}^{2g}$, perhaps a kind of symplectic representation. * algebra with trace over $\mathbb{Z}$ or $\mathbb{Q}$. The point is maybe positivity of the trace. This means you go over what you learned about sep. algs, ~~$\Omega^1 A = 0$~~. all derivations inner, $0 \to \Omega^1 A \to A \otimes A \to A \to 0$ has bimodule splitting. Ex: Clifford algebra of nondeg. quadratic forms. ~~Separable~~ In principle you should be able to work this all out for a Riemann surface using Hodge theory

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$H^2(X \times X) \simeq \text{Hom}^0(H^* X, H^* X)$ so there's an algebra structure. Composition of correspondences.

$X \times Y \times Z$

$X \times Y \qquad Y \times Z \qquad \int dy\, j(x,y) \int dz\, k(y,z)\, f(z)$

$X \qquad Y \qquad Z$

$\int dy\, j(x,y)\, k(y,z)$ push forward of $j(x,y_z)\, k(y_z, z)$ for $X \times Y \times Z \to X \times Z$

So a natural question would be to take the complex analytic case and to understand the signs. The problem is to go from the Hodge signature thm. which gives the signature of the intersection pairing on $H^2(X \times X)$ to what you need for the ring of self correspondences.

Do stuff for ~~course~~ lecture

Riemann sphere $= \mathbb{C} \cup \infty = \{$ $\mathbb{C}$-lines l in $\mathbb{C}^2 \}$

$$z \longleftrightarrow l_z = \begin{pmatrix} z \\ 1 \end{pmatrix} \mathbb{C}$$

Canonical holom. line bundle $\mathcal{O}(-1)$ over RS fibre l_z at z.

merom. section: $s(z) = \begin{pmatrix} z \\ 1 \end{pmatrix} f(z)$ $\quad f \in \mathbb{C}(z)$.

Action of $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{C})$ $\qquad s \longmapsto g^{-1} s g$

$$\underbrace{\begin{pmatrix} z \\ 1 \end{pmatrix} f(z)}_{s} \longmapsto \underbrace{\begin{pmatrix} \frac{az+b}{cz+d} \\ 1 \end{pmatrix} f\left(\frac{az+b}{cz+d}\right)}_{sg} = \underbrace{\begin{pmatrix} az+b \\ cz+d \end{pmatrix}}_{\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} z \\ 1 \end{pmatrix}} \frac{1}{cz+d} f\left(\frac{az+b}{cz+d}\right)$$

$$\therefore\ g^{-1} s g = \begin{pmatrix} z \\ 1 \end{pmatrix} \frac{1}{cz+d} f\left(\frac{az+b}{cz+d}\right)$$

differentials $\quad f(z)\, dz \longmapsto g^*(f(z)\, dz)$

$$g^*(f(z)\, dz) = \frac{1}{(cz+d)^2} f\left(\frac{az+b}{cz+d}\right)$$

$$f(t) = (\psi, e^{-iHt}\psi) = \int_0^\infty e^{-i\omega t} \underbrace{\rho(\omega)d\omega}_{\text{measure}}$$

defined for $\mathrm{Im}(t) \leq 0$

$$\rho(\omega) = \frac{1}{2\pi}\int_{-\infty - i\varepsilon}^{+\infty - i\varepsilon} e^{it\omega} f(t)\,dt$$

Fourier inversion

$$e^{i\omega(x - i\varepsilon)} = e^{i\omega x + \omega\varepsilon}$$

better seems to be this.

$$\rho(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega t} f(t)\,dt$$

assume $|f(t)| \leq C e^{-\varepsilon|t|}$. $\varepsilon > 0$.

Then $\rho(\omega)$ is analytic for $|\mathrm{Im}(\omega)| < \varepsilon$.

Say that $\binom{x}{1} f(x)$ is real section of $\mathcal{O}(-1)$ over $\mathbb{R}\cup\infty$ when $f(x)^2\,dx \geq 0$. i.e. $f(x)$

Wait: you want $\binom{z}{1} f(z)$ to be real on an oriented curve when $f(z)^2\,dz \geq 0$ along the curve.

$|z| = 1$.

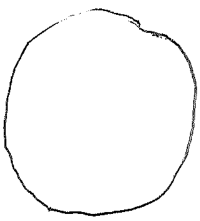
$$\underline{f(e^{i\theta})^2 e^{i\theta}\, i\, d\theta} = \underbrace{\rho(e^{i\theta})}_{\geq 0}\, d\theta$$

$$\sqrt{i}\; e^{i\theta/2} f(e^{i\theta}) \in \mathbb{R}.$$

$$dz = z\, i\, d\theta$$

$$z^{1/2} f(z)$$

up to a scalar the condition is $e^{i\theta/2} f(e^{i\theta}) \in \mathbb{R}\sqrt{-i}$. But what about L^2.



$$z = \frac{1-(-i\lambda)}{1+(-i\lambda)} = \frac{1+i\lambda}{1-i\lambda} = \frac{-\lambda+i}{\lambda+i} = \begin{pmatrix} -1 & i \\ 1 & i \end{pmatrix} \lambda$$

dz

$$dz = \frac{(-2i)\,d\lambda}{(\lambda+i)^2}$$

$$\frac{dz}{z} = \frac{(2i)\,d\lambda}{\lambda^2+1} \qquad \frac{dz}{2iz} = \frac{d\lambda}{\lambda^2+1}$$

$$\int |f(\lambda)|^2 \frac{d\lambda}{(\lambda^2+1)\pi} = \int \left|f\left(i\frac{1-z}{1+z}\right)\right|^2 \frac{dz}{2iz\,\pi}$$

$$\int \left|g\left(\frac{-\lambda+i}{\lambda+i}\right)\right|^2 \frac{d\lambda}{(\lambda^2+1)\pi} = \int |g(z)|^2 \frac{d\theta}{2\pi}$$

too confusing.

64 Lecture $RS = \mathbb{C} \cup \infty = \{\text{complex lines in } \mathbb{C}^2\}$

$$z \longmapsto l_z = \begin{pmatrix} z \\ 1 \end{pmatrix} \mathbb{C}$$

$$z \mapsto g(z) = \left(\frac{az+b}{cz+d}\right) \qquad f(z) \xmapsto{g^*} f(gz)$$

$$f(z)\,dz \longmapsto f\left(\frac{az+b}{cz+d}\right)\frac{1}{(cz+d)^2}\,ds$$

$$\begin{pmatrix} z \\ 1 \end{pmatrix} f(z) \longmapsto \underbrace{\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} \begin{pmatrix} \frac{az+b}{cz+d} \\ 1 \end{pmatrix}}_{} f\left(\frac{az+b}{cz+d}\right)$$

$$\begin{pmatrix} z \\ 1 \end{pmatrix} \frac{1}{cz+d} f\left(\frac{az+b}{cz+d}\right).$$

tangent vector $z \to z+\delta z$

Given δz you want a quadratic function on l_z

$$\delta\left[\begin{pmatrix} z \\ 1 \end{pmatrix} f(z)\right] = \begin{pmatrix} \delta z \\ 0 \end{pmatrix} f(z) + \begin{pmatrix} z \\ 1 \end{pmatrix} f'(z)\,\delta z$$

$$\begin{pmatrix} z \\ 1 \end{pmatrix} f(z) \wedge \delta\left(\begin{pmatrix} z \\ 1 \end{pmatrix} f(z)\right) = \begin{pmatrix} z \\ 1 \end{pmatrix} \wedge \begin{pmatrix} \delta z \\ 0 \end{pmatrix} f(z)^2 = f(z)^2\,\delta z$$

$s(z) \in l_z \qquad s(z+\delta z) \in l_{z+\delta z}$

$$l_z \wedge l_{z+\delta z} \subset \Lambda^2(\mathbb{C}^2) \qquad l_z$$

$$\left|\begin{pmatrix} z \\ 1 \end{pmatrix} \wedge \begin{pmatrix} z+\delta z \\ 1 \end{pmatrix}\right| = -\,\delta z$$

$$\begin{pmatrix} z \\ 1 \end{pmatrix} f(z) \longmapsto f(z)^2\,dz$$

$$-\begin{pmatrix} z \\ 1 \end{pmatrix} f(z) \wedge d\begin{pmatrix} z \\ 1 \end{pmatrix} f(z) = -\begin{vmatrix} zf & dz\,f + z\,df \\ f & df \end{vmatrix} = \underline{f^2\,dz}$$

Now suppose say $\begin{pmatrix} z \\ 1 \end{pmatrix} f(z)$ is real along a curve when $f(z)^2\,dz$

642 to make an effort to straighten this out.

$$T = \mathbb{C}^2 \qquad \Lambda^2 T \longrightarrow \mathbb{C} \qquad \begin{pmatrix}1\\0\end{pmatrix} \wedge \begin{pmatrix}0\\1\end{pmatrix} \longmapsto 1$$

$$l_z = \begin{pmatrix}z\\1\end{pmatrix}\mathbb{C} \subset T \qquad \begin{pmatrix}z\\1\end{pmatrix}\mathbb{C} = \begin{pmatrix}1\\z^{-1}\end{pmatrix}\mathbb{C}$$

How can you clarify.

$$L^2\left(\mathbb{R}, \frac{d\omega}{2\pi}\right) \ni f(\omega) \qquad g = \begin{pmatrix}a & b\\ c & d\end{pmatrix} \in SL(2,\mathbb{R})$$

$$f \longmapsto f\left(\frac{a\omega+b}{c\omega+d}\right)\frac{1}{c\omega+d}$$

$$|f(\omega)|^2\, d\omega \longmapsto \left|f\left(\frac{a\omega+b}{c\omega+d}\right)\right|^2 \frac{d\omega}{(c\omega+d)^2} = \left|f\left(\frac{a\omega+b}{c\omega+d}\right)\frac{1}{c\omega+d}\right|^2 d\omega$$

$$z = \frac{1+i\omega}{1-i\omega} = \begin{pmatrix}i & 1\\ -1 & 1\end{pmatrix}(\omega) \qquad \omega = i\,\frac{1-z}{1+z} = \begin{pmatrix}-i & i\\ 1 & 1\end{pmatrix}$$

$$1+z = \frac{2}{1-i\omega}$$

$$d\omega = i\,\frac{(1+z)(-dz) - (1-z)\,dz}{(1+z)^2} = \frac{-2i\,dz}{(1+z)^2}$$

Let $$f(\omega) = g\left(\frac{1+i\omega}{1-i\omega}\right)\frac{1}{1-i\omega}$$

Then $$\int |f(\omega)|^2\, d\omega = \int \left|g\left(\frac{1+i\omega}{1-i\omega}\right)\frac{1}{1-i\omega}\right|^2 \frac{-2i\,dz}{(1+z)^2}$$

$$= \int |g(z)|^2 \underbrace{\left|\frac{1+z}{2}\right|^2 \frac{-2i\,dz}{(1+z)^2}} = \int |g(z)|^2 \frac{dz}{2z}$$

$$\frac{1+z^{-1}}{2}\,\frac{(-i\,dz)}{1+z} = \frac{dz}{2iz} = \frac{d\theta}{2}$$

$$|f(z)|^2 \frac{dz}{iz} \longmapsto \left| f\left(\frac{az+b}{\bar b z+\bar a}\right) \frac{1}{\bar b z+\bar a}\right|^2 \frac{dz}{(\bar b z+\bar a)^2}\, i\, \frac{\bar b z+\bar a}{az+b}$$

$$= \left| f\left(\frac{az+b}{\bar b z+\bar a}\right)\right|^2 \frac{1}{(bz^{-1}+a)(az+b)(\bar b z+\bar a)^2} \qquad ?$$

$$\frac{dz}{iz} \longmapsto \frac{\bar b z+\bar a}{i(az+b)} \frac{dz}{(\bar b z+\bar a)^2} = \frac{dz}{i(}$$

$$(az+b)(\bar b z+\bar a) = (az+b)(\bar a z^{-1}+\bar b) z = |az+b|^2 z$$

~~complete confusion~~.

$$z \mapsto \frac{az+b}{\bar b z+\bar a}$$

You want $SU(1,1) = \left\{ \begin{pmatrix} a & b \\ \bar b & \bar a \end{pmatrix} \middle| \ |a|^2 - |b|^2 = 1 \right\}$.

$$\frac{dz}{iz} = \frac{1}{i}\left(\frac{a\,dz}{az+b} - \frac{\bar b\, dz}{\bar b z+\bar a}\right) = \frac{1}{i} \frac{dz}{|az+b|^2 z}$$

$$|\bar b z+\bar a|^2 = (\bar b z+\bar a)(bz^{-1}+a)$$
$$= (\bar b + \bar a z^{-1})(b+az) = |az+b|^2$$

~~$(az+b)(\bar a z^{-1}+\bar b)$~~

$$|f(z)|^2 \frac{dz}{iz} \longmapsto \underbrace{\left| f\left(\frac{az+b}{\bar b z+\bar a}\right)\right|^2 \frac{1}{|\bar b z+\bar a|^2}}_{\left| f\left(\frac{az+b}{\bar b z+\bar a}\right) \frac{1}{\bar b z+\bar a}\right|^2} \frac{dz}{iz}$$

$$|f(z)|^2 \frac{dz}{iz}$$

substitute ~~replace~~ $z = \begin{pmatrix} a & b \\ \bar b & \bar a \end{pmatrix} \omega$

$$\frac{1}{i}\frac{dz}{z} = \frac{1}{i}\left\{\frac{a\,d\omega}{a\omega+b} - \frac{\bar b\, d\omega}{\bar b\omega+\bar a}\right\} = \frac{1}{i} \frac{a(\bar b\omega+\bar a) - (a\omega+b)\bar b}{(a+b\bar\omega)(\bar b\omega+\bar a)} \frac{d\omega}{\omega}$$

$$= \frac{1}{|\bar b\omega+\bar a|^2} \frac{d\omega}{i\omega}$$

644 Thus $f(z) \longmapsto f\left(\frac{az+b}{\bar b z+\bar a}\right)\frac{1}{\bar b z+\bar a}$

gives an action of $SU(1,1)$ on functions defined on S^1 ~~compatible~~ which preserves L^2 norm. | unitary action of $SU(1,1)$ on $L^2(S^1, \frac{d\theta}{2\pi})$

$$\int |f(z)|^2 \underbrace{\frac{dz}{iz}}_{d\theta} = \int \left| f\left(\frac{az+b}{\bar b z+a}\right)\frac{1}{\bar b z+a}\right|^2 d\theta$$

$$\lambda = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\mu \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{R})$$

$$|f(\lambda)|^2 d\lambda = \left| f\left(\frac{a\mu+b}{c\mu+d}\right)\right|^2 \frac{d\mu}{(c\mu+d)^2}$$

$$\therefore \int |f(\lambda)|^2 d\lambda = \int \left| f\left(\frac{a\lambda+b}{c\lambda+d}\right)\frac{1}{c\lambda+d}\right|^2 d\mu$$

so get unitary action of $SL(2,\mathbb{R})$ on $L^2(\mathbb{R}, \frac{d\omega}{2\pi})$

Let $z = \frac{1+i\lambda}{1-i\lambda} = \begin{pmatrix} i & 1 \\ -i & 1 \end{pmatrix}\lambda = \begin{pmatrix} -1 & i \\ 1 & i \end{pmatrix}\lambda$

$$|f(z)|^2 \frac{dz}{iz} = \left| f\left(\frac{1+i\lambda}{1-i\lambda}\right)\right|^2 \frac{1}{i}\left(\frac{i\,d\lambda}{1+i\lambda} - \frac{-i\,d\lambda}{1-i\lambda}\right)$$

$$= \left| f\left(\frac{1+i\lambda}{1-i\lambda}\right)\right|^2 \frac{2\,d\lambda}{1+\lambda^2} = \left| f\left(\frac{1+i\lambda}{1-i\lambda}\right)\frac{1}{1-i\lambda}\right|^2 2\,d\lambda$$

$$s(z) = \begin{pmatrix} z \\ 1 \end{pmatrix} f(z) \qquad g^{-1} s(gz) = \begin{pmatrix} z \\ 1 \end{pmatrix}\cancel{\begin{pmatrix} az+b \\ cz+d \end{pmatrix}} \frac{1}{cz+d} f\left(\frac{az+b}{cz+d}\right)$$

$$f(z) \overset{g}{\longmapsto} f\left(\frac{az+b}{cz+d}\right)\frac{1}{cz+d}(ad-bc)^{n} \qquad \text{action of } GL(2,\mathbb{C})$$

645 Can you link the RH situation where you have $X \times X$ Lorentzian signature quadratic form and the positive * algebra of correspondences on X. This might be formally similar to the path integral versus the operator picture? This isn't clear.

You ought to be able to see easily how Lorentz signature for a subspace of $H^2(X \times X)$ leads to positivity of the trace on the corresponding algebra of operators. This should be a straightforward argument using P. Duality, Kunneth, etc.

$$(a\Delta + \Gamma) \cdot H = 0 \implies (a\Delta + \Gamma)(a\Delta + \Gamma) \leq 0$$

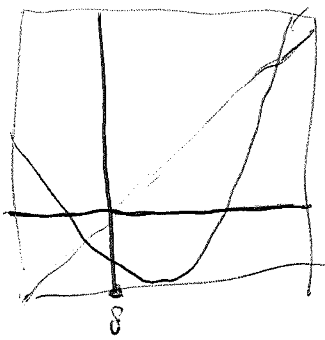
$$a^2(\Delta \cdot \Delta) + 2a(\Delta \cdot \Gamma) + \Gamma \cdot \Gamma \leq 0$$

$$H = (pr_1^* + pr_2^*)(\delta)$$

$$= X \otimes \delta + \delta \otimes X$$

$n\delta$ divisor $>> 0$
concent. at one pt.

$\Delta \cdot H$



$$\Gamma \cdot H = \underset{\overset{?}{q}n}{\Gamma \cdot (X \otimes n\delta)} + \underbrace{\Gamma \cdot (n\delta \otimes X)}_{n}$$

$$\Delta \cdot H = \Delta \cdot (X \otimes n\delta) + \Delta \cdot (n\delta \otimes X)$$

$$= 2n$$

$$\Gamma\cdot H = g+1 \qquad \left(\Gamma - \frac{g+1}{2}\Delta\right)\cdot\left(\Gamma - \frac{g+1}{2}\Delta\right) \leq 0.$$
$$\Delta\cdot H = 2$$

$$(\Gamma\cdot\Gamma) - (g+1)\underbrace{(\Gamma\cdot\Delta)}_{\# X(\mathbb{F}_q)} + \left(\frac{g+1}{2}\right)^2\underbrace{(\Delta\cdot\Delta)}_{2-2g} \leq 0$$

$$\# X(\mathbb{F}_q) \geq \left(\frac{g+1}{4}\right)(2-2g) + \frac{1}{g+1}(\Gamma\cdot\Gamma)$$

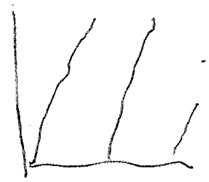
$$\# X(\mathbb{F}_q) = 1 - \sum_{i=1}^{2g}\alpha_i + q$$

Suppose $g=1$, so that $\Delta\cdot\Delta = 0$.

Take H to be $X\otimes\delta + t\,\delta\otimes X$

$$(\Gamma - a\Delta)\cdot(X\otimes\delta + t\,\delta\otimes X)$$

$$= q - a + t(1-a) = 0 \qquad \frac{q+t}{1+t} = 1 + \frac{q-1}{1+t}$$



$$q + t - (1+t)a = 0 \qquad a = \frac{q+t}{1+t} \qquad t \geq 0$$

$$1 < a < q$$

$$\Gamma\cdot\Gamma - 2a\,\# X(\mathbb{F}_q) + a^2\,\Delta\cdot\Delta \leq 0$$

$$(\Gamma - a\Delta)\cdot(\Gamma - a\Delta) = \Gamma\cdot\Gamma - 2a\overbrace{\Gamma\cdot\Delta}^{\# X(\mathbb{F}_q)} + a^2\overbrace{\Delta\cdot\Delta}^{2(1-g)} \leq 0.$$

$$(\Gamma\cdot\Delta)^2 \leq (\Gamma\cdot\Gamma)(\Delta\cdot\Delta) \qquad \text{Schwartz ineq?}$$

647 Here's the argument - you have the curves $X \otimes \delta$, $\delta \otimes X$ δ a point in X which generate a hyperbolic plane, so the intersection form is ≤ 0 on the orth. complement. So project $\Gamma + a\Delta$ onto this complement:

$$\left(\Gamma + a\Delta - bX\otimes\delta - c\,\delta\otimes X\right)\cdot X\otimes\delta = q + a - c = 0$$
$$\cdot\,\delta\otimes X = 1 + a - b = 0$$

Then $\left(\Gamma + a\Delta - bX\otimes\delta - c\,\delta\otimes X\right)^2 \leq 0$

$$\|$$

$$\left(\Gamma + a\Delta - bX\otimes\delta - c\,\delta\otimes X\right)\cdot\left(\Gamma + a\Delta\right)$$

$$\|$$

$$\left(\Gamma + a\Delta\right)^2 - \underbrace{b}_{1+a}(q+a) - \underbrace{c}_{q+1}(1+a) = \left(\Gamma + a\Delta\right)^2 - 2(1+a)(q+a)$$

so $2\left(q + (1+q)a + a^2\right) \geq \Gamma^2 + 2a\Gamma\cdot\Delta + a^2\Delta^2$

$$\underbrace{2q - \Gamma^2} + 2a\left(1 + q - \Gamma\cdot\Delta\right) + a^2\underbrace{\left(2 - \Delta^2\right)} \geq 0$$

guess $\Gamma^2 = q(2-2g)$ $\qquad \underbrace{2 - (2-2g)}_{\| \atop 2g}$

$2q - q(2-2g) = 2gq$

$$\therefore \quad 2gq + 2a\left(1 + q - \Gamma\cdot\Delta\right) + a^2 2g \geq 0 \quad \text{for all } a$$

$$\Longleftrightarrow \quad \left(1 + q - \Gamma\cdot\Delta\right)^2 \leq 2gq\,2g$$

$$\Longleftrightarrow \quad \boxed{\left|1 + q - \Gamma\cdot\Delta\right| \leq 2g\,q^{1/2}}$$

where $\Gamma\cdot\Delta = \#X(\mathbb{F}_q)$

But what might be more interesting might be the algebra of correspondences on the curve X. You should get a ~~[illegible]~~ $*$ algebra with a positive trace. Perhaps you get a unitary operator.

$$\omega = i\,\frac{1-z}{1+z} \qquad d\omega = i\,\frac{(1+z)(-dz)-(1-z)dz}{(1+z)^2}$$

$$|f(\omega)|^2 d\omega \qquad z = \frac{1+i\omega}{1-i\omega} = \begin{pmatrix} i & 1 \\ -i & 1\end{pmatrix} \qquad = (-2i)\frac{dz}{(1+z)^2}$$

$$= \left|f\left(i\,\frac{1-z}{1+z}\right)\right|^2 \frac{2^2}{(z^{1/2}+z^{-1/2})^2}\,\frac{dz}{2iz}$$

$$d\omega = -2i\,\frac{dz}{(z+1)^2}$$

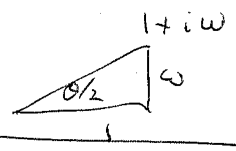
$$= \left| f\left(i\,\frac{1-z}{1+z}\right)\frac{1}{\cos\frac{\theta}{2}}\right|^2 \frac{d\theta}{2} \qquad dz = 2i\,\frac{d\omega}{(1-i\omega)^2}$$

$$z+1 = \frac{1+i\omega}{1-i\omega}+1 = \frac{2}{1-i\omega} \qquad \frac{dz}{iz} = \frac{1}{i}\left(\frac{i\,d\omega}{1+i\omega}+\frac{+i\,d\omega}{1-i\omega}\right)$$

$$d\theta = \frac{2\,d\omega}{1+\omega^2}$$

$$\frac{\theta}{2} = \arctan\omega$$

$$|f_1(z)|^2\,d\theta = |f(\omega)|^2\,\frac{2\,d\omega}{1+\omega^2}$$



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for the Nth time, try again $\quad \omega = i\,\frac{1-z}{1+z} = \begin{pmatrix} -i & i \\ 1 & 1\end{pmatrix} z$

$$\begin{pmatrix}\omega\\1\end{pmatrix} f(\omega) \longmapsto \begin{pmatrix} -i & i \\ 1 & 1\end{pmatrix}^{-1}\begin{pmatrix} i\frac{1-z}{1+z} \\ 1\end{pmatrix} f\left(i\,\frac{1-z}{1+z}\right)$$

$$= \begin{pmatrix} -i & i \\ 1 & 1\end{pmatrix}^{-1}\underbrace{\begin{pmatrix} i-iz \\ 1+z\end{pmatrix}}_{\begin{pmatrix} -i & i \\ 1 & 1\end{pmatrix}\begin{pmatrix} z \\ 1\end{pmatrix}} f\left(i\,\frac{1-z}{1+z}\right)\frac{1}{1+z} = \begin{pmatrix} z \\ 1\end{pmatrix} f\left(i\,\frac{1-z}{1+z}\right)\frac{1}{z+1}$$

So this is how to go from sections of $\mathcal{O}(-1)$ over $\mathbb{R}\cup\infty$ to sections of $\mathcal{O}(-1)$ over $|z|=1$.

649 Action of $SL(2,\mathbb{R})$ on $L^2(\mathbb{R})$, $H^2(\mathbb{R})$.

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{R})$$

$$\omega = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \lambda = \frac{a\lambda+b}{c\lambda+d}$$

$$|f(\omega)|^2\, d\omega$$

$$\|$$

$$\left|f\left(\frac{a\lambda+b}{c\lambda+d}\right)\right|^2 \frac{d\lambda}{(c\lambda+d)^2} = \left|f\left(\frac{a\lambda+b}{c\lambda+d}\right)\frac{1}{c\lambda+d}\right|^2 d\lambda$$

$$\int |f(\omega)|^2\, d\omega = \int \left|f\left(\frac{a\omega+b}{c\omega+d}\right)\frac{1}{c\omega+d}\right|^2 d\omega$$

$\therefore$ get an unitary action of $SL(2,\mathbb{R})$ on $L^2(\mathbb{R})$ by

$$f(\omega) \longmapsto f\left(\frac{a\omega+b}{c\omega+d}\right)\frac{1}{c\omega+d}$$

Action of $SU(1,1)$ on $L^2(S^1)$

$$g = \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix} \qquad |a|^2 - |b|^2 = 1.$$

$$|f_1(z)|^2 \frac{dz}{iz} \longmapsto \left|f_1\left(\frac{az+b}{\bar{b}z+\bar{a}}\right)\right|^2 \underbrace{\frac{1}{i}\left\{\frac{a\,dz}{az+b} - \frac{\bar{b}\,dz}{\bar{b}z+\bar{a}}\right\}}_{\frac{a(\bar{b}z+\bar{a}) - \bar{b}(az+b)}{\underbrace{(a+bz^{-1})(\bar{b}z+\bar{a})}_{|\bar{b}z+\bar{a}|^2}} \frac{dz}{iz}}$$

$$= \left|f_1\left(\frac{az+b}{\bar{b}z+\bar{a}}\right)\frac{1}{\bar{b}z+\bar{a}}\right|^2 \frac{dz}{iz}$$

$$\omega = i\,\frac{1-z}{1+z} = \begin{pmatrix} -i & i \\ 1 & 1 \end{pmatrix} z$$

$$|f(\omega)|^2 d\omega = \left|f\left(i\,\frac{1-z}{1+z}\right)\right|^2 \frac{(-2i)\,dz}{(z+1)^2} = |f_1(z)|^2 \frac{2}{(z^{1/2}+z^{-1/2})^2}\,\frac{dz}{iz}$$

$$= \int \left| f\left(i\,\frac{1-z}{1+z}\right)\frac{1}{z^{1/2}+z^{-1/2}}\right|^2 2\,\frac{dz}{iz}\, d\theta$$

Go back to why sections of $\mathcal{O}(-1)$ over an oriented curve have an L^2 scalar product.

$$dz \longmapsto d\left(\frac{az+b}{cz+d}\right) = \frac{(ad-bc)\,dz}{(cz+d)^2}$$

$\begin{pmatrix} z \\ 1 \end{pmatrix} f(z)$ local section of $\mathcal{O}(-1)$

Consider a t. v. δz at z.

$$\begin{pmatrix} z \\ 1 \end{pmatrix} f(z) \wedge \delta\begin{pmatrix} z \\ 1 \end{pmatrix} f(z) \in \Lambda^2 \mathbb{C}^2 \sim \mathbb{C}$$

$$\begin{vmatrix} zf & \delta z f \\ f & 0 \end{vmatrix} = -f^2 \delta z$$

~~Angles out~~ STOP.

So ask that $-f^2 \delta z \in \mathbb{R} e^{i\varphi}$ determines a real line inside L_z

---

$\begin{pmatrix} \omega \\ 1 \end{pmatrix} f(\omega)$ $\qquad \omega = i\,\dfrac{1-z}{1+z} = \begin{pmatrix} -i & i \\ 1 & 1 \end{pmatrix} z \qquad \begin{vmatrix} -i & i \\ 1 & 1 \end{vmatrix} = -2i$

maybe $\sqrt{-2i}$ to compensate for

$\downarrow \qquad \begin{pmatrix} -i & i \\ 1 & 1 \end{pmatrix}\begin{pmatrix} z \\ 1 \end{pmatrix}$

$$\begin{pmatrix} -i & i \\ 1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} i\,\dfrac{1-z}{} \\ 1+z \end{pmatrix} \frac{1}{1+z} f\left(i\,\frac{1-z}{1+z}\right) = \begin{pmatrix} z \\ 1 \end{pmatrix} f\left(i\,\frac{1-z}{1+z}\right) \frac{1}{z+1}$$

$$= \begin{pmatrix} z^{1/2} \\ z^{-1/2} \end{pmatrix} f\left(i\,\frac{1-z}{1+z}\right) \frac{1}{z^{1/2}+z^{-1/2}}$$

Map $\quad f(\omega) \longmapsto f\left(i\,\dfrac{1-z}{1+z}\right) \dfrac{\alpha}{z+1}$ $\qquad \alpha = \sqrt{-2i}$

$$\int |f(\omega)|^2 d\omega = \int \left| f\left(i\,\frac{1-z}{1+z}\right) \frac{\alpha}{z+1} \right|^2 d\theta$$

$f(\omega) = \dfrac{1}{1-i\omega} = \dfrac{z+1}{2} \qquad z = \dfrac{1+i\omega}{1-i\omega} \qquad z+1 = \dfrac{2}{1-i\omega}$

$\dfrac{1}{1-i\omega} \longrightarrow \dfrac{\alpha}{2} \qquad \dfrac{1}{1-i\omega} = \dfrac{1}{1+\frac{1-z}{1+z}} = \dfrac{1+z}{2}$

$$\omega = i\,\frac{1-z}{1+z} = \begin{pmatrix} -i & i \\ 1 & 1 \end{pmatrix} z$$

$$\begin{pmatrix} \omega \\ 1 \end{pmatrix} f(\omega) \longmapsto \begin{pmatrix} -i & i \\ 1 & 1 \end{pmatrix}^{-1} \underbrace{\begin{pmatrix} i - iz \\ 1+z \end{pmatrix}}_{\begin{pmatrix} -i & i \\ 1 & 1 \end{pmatrix}\begin{pmatrix} z \\ 1 \end{pmatrix}} f\left(i\,\frac{1-z}{1+z}\right) \frac{1}{z+1} = \begin{pmatrix} z \\ 1 \end{pmatrix} f\left(i\,\frac{1-z}{1+z}\right) \frac{1}{z+1}$$

This should give corresp. between sections of $\mathcal{O}(-1)$ over $\mathbb{R} \cup \infty$ and sections of $\mathcal{O}(-1)$ over $|z| = 1$.
Probably unique up to a scalar e.g. $\sqrt{-2i}$
Take $f(\omega) = \dfrac{1}{1 - i\omega}$. Then

$$\begin{pmatrix} z \\ 1 \end{pmatrix} \frac{1}{1 - i\,i\,\frac{1-z}{1+z}} \,\frac{1}{z+1} = \begin{pmatrix} z \\ 1 \end{pmatrix} \frac{1}{2}$$

$$\begin{pmatrix} \omega \\ 1 \end{pmatrix} \frac{1}{1 - i\omega} \longrightarrow \begin{pmatrix} i \\ 0 \end{pmatrix} \qquad \text{as } |\omega| \to \infty.$$

$$\begin{pmatrix} \omega \\ 1 \end{pmatrix} \frac{1}{\sqrt{1+\omega^2}} \longrightarrow \begin{cases} \begin{pmatrix} 1 \\ 0 \end{pmatrix} & \omega \to +\infty \\ \begin{pmatrix} -1 \\ 0 \end{pmatrix} & \omega \to -\infty \end{cases}$$

So what? What are you trying to do?
Justify the intrinsic picture. $\Gamma(C, \mathcal{O}(-1))$
intrinsically defined. $\Gamma(C, \mathcal{O}(-1))$. How to understand this thing again. ~~Completely routine~~
How can you understand.

$$\ell_z = \begin{pmatrix} z \\ 1 \end{pmatrix} \mathbb{C} \subset \mathbb{C}^2$$

$$0 \to \ell_z \longrightarrow \mathbb{C}^2 \longrightarrow \ell_z'' \to 0$$

$$0 \to \mathbb{C} \xrightarrow{\begin{pmatrix} z \\ 1 \end{pmatrix}} \mathbb{C}^2 \xrightarrow{(1\ \ -z)} \mathbb{C}$$

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so discuss example

$\mathcal{X} = \mathbb{C} \qquad x = 1 \qquad \gamma \in UHP$

$$x^* u^t x = \begin{cases} e^{i\gamma t} & t \geq 0 \\ e^{i\gamma^* t} & t \leq 0 \end{cases} = \int e^{i\omega t} \rho(\omega) \frac{d\omega}{2\pi}$$

$$\rho(\omega) = \int_{-\infty}^{0} e^{-i\omega t} e^{i\gamma^* t}\, dt + \int_{0}^{\infty} e^{-i\omega t} e^{i\gamma t}\, dt$$

$$= \frac{1}{-i\omega + i\gamma^*} + \frac{1}{i\omega - i\gamma} = \frac{i}{\omega - \gamma^*} - \frac{i}{\omega - \gamma} = \frac{(-i)(\gamma - \gamma^*)}{|\omega - \gamma|^2}$$

$$= \frac{k^2}{|\omega - \gamma|^2} \qquad k^2 = \frac{\gamma - \gamma^*}{i}$$

so

$$L^2(\mathbb{R}, \rho \tfrac{d\omega}{2\pi}) \qquad\qquad L^2(\mathbb{R})$$

$$\frac{1}{k}\, \omega - \begin{cases}\gamma \\ \gamma^*\end{cases} \longleftarrow 1$$

$$\cancel{x} \longmapsto \underbrace{\frac{k}{\omega - \gamma}}_{\in H^2_-} \text{ or } \underbrace{\frac{k}{\omega - \gamma^*}}_{H^2_+}$$

$$H^+ / (\omega - \gamma) H^+$$

$$\left(\frac{k}{\omega - \gamma}, \left(\frac{\omega - \gamma}{\omega - \gamma^*}\right) H^+\right) \qquad \overline{\left(\frac{k}{\omega - \gamma^*}, \left(\frac{\omega - \gamma}{\omega - \gamma^*}\right) H^+\right)}$$

$$\|$$

OKAY $\underbrace{\frac{1}{\omega - \gamma^*}}_{\text{pole in LHP}} \in H^+$ $\qquad \left(H^+, \underbrace{\frac{k}{\omega - \gamma}}_{\in H^-}\right) = 0.$

again

$$\left(\underbrace{\frac{k}{\omega - \bar{\gamma}^*}}_{\in H^+}, \left(\frac{\omega - \gamma}{\omega - \bar{\gamma}^*}\right) H^+\right) = \left(\underbrace{\frac{k}{\omega - \gamma}}_{\in H^-}, H^+\right) = 0$$

so what do you learn

$$\int \overline{\frac{1}{\omega - \bar{\gamma}}} f(\omega) \frac{d\omega}{2\pi} = \int \frac{1}{\omega - \gamma} f(\omega) \frac{d\omega}{2\pi i} = f(\gamma) \qquad k = \sqrt{2 \operatorname{Im} \gamma}$$

$$\left\| \frac{i}{\omega - \bar{\gamma}} \right\|^2 = \frac{i}{\gamma - \bar{\gamma}} = \frac{1}{k^2} \quad \therefore \left\| \frac{ik}{\omega - \gamma} \right\| = 1.$$

Consider curve X alg of self-correspondences of X. Basically you look at $H^2(X \times X)$ as a space of operator kernels. Focus on the problem. You will have Kunneth formula

$$H^2(X \times X) = H^0(X) \otimes H^2(X) \oplus H^1(X) \otimes H^1(X) \oplus H^2(X) \otimes H^0(X).$$

$H^2(X \times X)$ is the space of degree 0 operators on $H^*(X)$. What sort of results. One point is that $H^2(X \times X)$ is a ring sim to $\mathrm{End}^{(0)}(H^*(X))$. ~~The other point~~ ~~not clear. What's~~ Use Kunneth.

$$H^*(X \times X) \xleftarrow{\sim} H^*(X) \otimes H^*(X)$$

What do you need? use?

$H^1(X)$ is a symplectic vector space. So you have $H^*(X) = \mathbb{C} \oplus V \oplus \mathbb{C}$ with a simple multiplication. Then you

other problem. $X = \mathbb{P}_1 V$

$$T_X = \mathrm{Hom}\left(\mathcal{O}(-1), \mathcal{O} \otimes V / \mathcal{O}(-1)\right) = \mathcal{O}(1) \otimes V / \mathcal{O}$$

$$\Omega^1_X = \mathcal{O}(-2) \otimes \Lambda^2 V^*$$

$$0 \longrightarrow \mathcal{O}(-1) \longrightarrow \mathcal{O} \otimes V \longrightarrow Q \longrightarrow 0$$

$$\mathcal{O}(-1) \otimes Q \simeq \mathcal{O} \otimes \Lambda^2 V$$

$$0 \longrightarrow \underset{\mathcal{O}(-2)\otimes}{\Omega^1_X} \longrightarrow J_1(\mathcal{O}) \longrightarrow \mathcal{O} \longrightarrow 0$$

but this splits since $\mathcal{O}$ has a conn.

Use $\mathcal{O}(1)$ as the quotient

$$0 \to \mathcal{O}(-1)\otimes\Lambda^2V^* \to \mathcal{O}\otimes V^* \to \mathcal{O}(1) \to 0$$

$$0 \to \mathcal{O}(-2)\otimes\Lambda^2V^* \to \mathcal{O}(-1)\otimes V^* \to \mathcal{O} \to 0$$

$\mathcal{O}(-2)\otimes\Lambda^2V^* = \Omega^1_X$ this should represent the canonical elt of $H^1(X, \Omega^1_X)$

~~You want to ... ~~

L can. subbundle of $\mathcal{O}\otimes V$

You want a canon. isom $L^{\otimes 2} \simeq \Omega^1_X$

$$0 \to L \to \mathcal{O}\otimes V \to M \to 0$$

$$T_X = (\Omega^1_X)^\vee = \mathrm{Hom}(L, M) = L^\vee \otimes M$$

$$\Omega^1_X = L \otimes M^\vee$$

$$\boxed{\Omega^1_X = L\otimes L\otimes\Lambda^2V^*}$$

$$L\otimes M \simeq \mathcal{O}\otimes\Lambda^2V$$
$$M \simeq L^\vee\otimes\Lambda^2V$$
$$M^\vee = L\otimes\Lambda^2V^*$$

V symplectic $\quad H^*(X) = \mathbb{C}\oplus V\oplus\mathbb{C} \qquad v_1v_2 = A(v_1,v_2)\,\mathbb{C}$

satisfies P.D. $\quad H^*(X\times X) \simeq H^*(X)\otimes H^*(X)$

tensor product in super sense

$$H^*(X\times X) \xrightarrow{\sim} \mathcal{L}(H^*(X))$$
$$K \longmapsto pr_{1*}\, K\, pr_2^*$$

~~The problem is to define the ... products on~~ Problem is to link composition product on $H^2(X\times X)$ with the intersection pairing c

Look at general situation without signs, Frobenius ring ~~structure~~ structure?

$$H^2(X\times X) \simeq \mathrm{End}^{(0)}(H^*(X)) = \prod_i \mathrm{End}^{?}(H^i(X))$$

$$H^2(X\times X) = \mathbb{R}(1\otimes\mu) \oplus V\otimes V \oplus \mathbb{R}(\mu\otimes 1)$$

bases for $H^*(X\times X)$ involves $\begin{Bmatrix} 1 \\ v_1, \cdots, v_{2g} \\ \mu \end{Bmatrix} \otimes \begin{Bmatrix} \text{same} \end{Bmatrix}$

Observation: You are given a symplectic vector space V over $\mathbb{Q}$ say. Then you get certain operators on it

---

Picture: kernel picture

$$f(x) \mapsto \int dy\, K(x,y)\, f(y) = pr_{1*}\, K\, pr_2^* f$$

Then have $K^t(x,y) = K(y,x)$. and

$$tr\,(K^t L) = \cancel{\int dy\, K(\ldots)\, L(x,y)}$$

$$= \int dy \int dx\, K(y,x)\, L(x,y)$$

I guess it's clear that $tr\,(K^t L) = \int_{X\times X} KL$ is the intersection pairing.

Also $tr\,(K) = \int dx\, K(x,x) = \Delta\cdot K$

So what? Basically you seem to have identified the intersection pairing with $(K,L) = tr(K^t L)$. So you have $H^2(X \times X)$ interpreted as operators on $H^*(X)$ ~~and the intersection tr and~~ intersection pairing is $tr(K^t L)$, in particular $tr(K) = \Delta \cdot K$. Now this trace is alternating sum $\sum (-1)^i tr(K \text{ on } H^i)$. ~~So you look at~~

Look at $H^2(X \times X)$ with intersection pairing quadratic form. This is direct sum of hyperbolic 2 plane $H^2 \otimes H^0 \oplus H^0 \otimes H^2$ and $H^1 \otimes H^1$. So if you know the int. pairing is Lorentzian, you get positive trace on subspace $H^1 \otimes H^1$ of operators on H^1.

You really want to exploit this extra hyperbolic 2 plane

Review - have $H^2(X \times X)$ acting as degree 0 transf of $H^* X = H^0 \oplus H^1 \oplus H^2$ via $K \longrightarrow pr_{1*} K \, pr_2^*$

$$(Kf)(x) = \int dy \; K(x,y) f(y)$$

have transpose $K \longmapsto K^*$ $\quad K^*(x,y) = K(y,x)$ and trace $tr(K) = \int dx \; K(x,x)$ think of K as a diff'l form on $X \times X$

~~Thus Markov~~ ~~Actually~~

You need to work out the signs. Take $X = S^2$

first. $H^*(X) = \mathbb{R}1 \oplus 0 \oplus \mathbb{R}\mu$

$H^2(X \times X)$ basis: $1 \otimes \mu,\ \mu \otimes 1$

$$pr_{1*}(\mu \otimes 1)\, pr_2^* \xi = pr_{1*}\, \mu \otimes \xi = \mu \int \xi$$

$\therefore$ $\mu \otimes 1$ gives $\begin{array}{c} H^2 \longrightarrow H^2 \\ \mu \longmapsto \mu \end{array}$ and 0 on H^0

$$pr_{1*}(1 \otimes \mu)\, pr_2^* \xi = pr_{1*}(1 \otimes \mu\xi) = \int \mu \xi$$

$1 \otimes \mu$ gives $\begin{array}{c} H^0 \longrightarrow H^0 \\ 1 \longrightarrow 1 \end{array}$ and 0 on H^2

So
$$\left(\begin{array}{llll} 1 \otimes \mu & \rightsquigarrow & \text{id on } H^0, & 0 \text{ on } H^2 \\ \mu \otimes 1 & \rightsquigarrow & 0 \text{ on } H^0, & \text{id on } H^2 \end{array} \right.$$

~~So look at the intersection~~

How does this ~~~~ relate to the intersection pairing? Not clear! Your idea was that

that ~~~~ $tr(K) = \Delta \cdot K$

$K = 1 \otimes \mu$ $\qquad tr(K) = \int_X \Delta^*(1 \otimes \mu) = \int_X \mu = 1$

$K^t = \mu \otimes 1$ $\qquad tr(K^t) = \int_X \Delta^*(\mu \otimes 1) = \int_X \mu = 1.$

$$K^t K = 1 \otimes \textstyle\int_X \otimes 1\, (\mu \otimes 1 \otimes \mu) = 0.$$

$$K K^t = 1 \otimes \textstyle\int \otimes 1\, \underbrace{(1 \otimes \mu \otimes 1)}_{K}\underbrace{(1 \otimes \mu \otimes 1)}_{K^t} = 0$$

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$$H^2(X\times X) \xrightarrow{\sim} \mathrm{Hom}^{\otimes}(H^*X, H^*X)$$

$$K \qquad pr_{1*}\,K\,pr_2^* \ : \ \omega \mapsto \int^{2} dx\, K(x,y)\,\omega(y)$$

$$\mathrm{tr}(K) = \int_X \Delta^* K = \int_{X\times X} (\Delta_* 1)\, K = \Delta_* 1 \cdot K$$

$$(K^t K)(x,z) = \int dy\, K^t(x,y)\,K(y,z)$$

$$= \int dy\, K(y,x)\,K(y,z)$$

$$\mathrm{tr}(K^t K) = \int dz \int dy\ K(y,z)K(y,z)$$

$$= \iint \underbrace{dz\,dy}_{dy\,dz}\ K(y,z)^2 = K\cdot K$$

since $dx = d^2x$.

Now go back to RH calculation. You have two elements $\Delta, F \in H^2(X\times X)$, as well as $\mu\otimes 1$, $1\otimes\mu$. You formed $F - a\Delta$ made it orth to $\mu\otimes 1$, $1\otimes\mu$

$$\big(F - a\Delta - b(\mu\otimes 1) - c(1\otimes\mu)\big)\cdot(\mu\otimes 1)$$

$$= F\cdot(\mu\otimes 1) - a\,\Delta\cdot(\mu\otimes 1) - c = 0$$

$$\big(F - a\Delta - b(\mu\otimes 1) - c(1\otimes\mu)\big)\cdot(1\otimes\mu)$$

$$= F\cdot(1\otimes\mu) - a\,\Delta\cdot(1\otimes\mu) - b = 0$$

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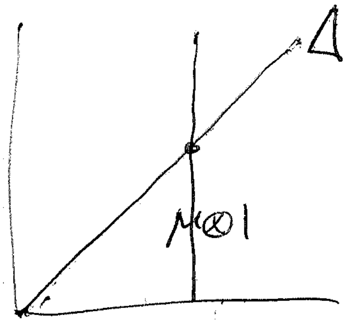
$$\Delta \cdot (\mu \otimes 1) = 1$$

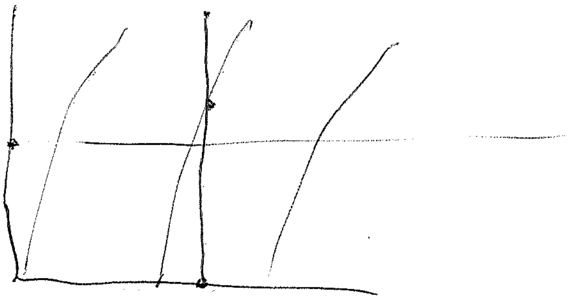
$$\Delta \cdot (1 \otimes \mu) = 1$$



$$c = 1 - a \qquad c = q - a$$

$$F \cdot (\mu \otimes 1) = 1 \qquad q$$

$$F \cdot (1 \otimes \mu) = q \qquad 1$$



$$b = q - a \qquad b = 1 - a$$

$$\left[F - a\Delta - b(\mu\otimes 1) - c(1\otimes\mu)\right] \cdot \left[F - a\Delta - b(\mu\otimes 1) - c(1\otimes \mu)\right]$$

$$= (F - a\Delta)^2 - (F - a\Delta)\cdot\left(b(\mu\otimes 1) + c(1\otimes\mu)\right)$$

$$\begin{pmatrix} 1 & -a \end{pmatrix} \begin{pmatrix} F \\ \Delta \end{pmatrix} \begin{pmatrix} \mu\otimes 1 & 1\otimes\mu \end{pmatrix} \begin{pmatrix} b \\ +c \end{pmatrix}$$

$$\begin{pmatrix} 1 & -a \end{pmatrix} \begin{pmatrix} 1 & q \\ 1 & 1 \end{pmatrix} \begin{pmatrix} q-a \\ 1-a \end{pmatrix}$$

$$\begin{pmatrix} 1-a & q-a \end{pmatrix} \begin{pmatrix} q-a \\ 1-a \end{pmatrix} = 2(1-a)(q-a)$$

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$$\Delta\cdot\Delta = 2(1-g) \qquad F\cdot F = q\,2(1-g)$$

$$\Delta\cdot F = \#\cancel{F}(F_q) = 1-\sum_{i=1}^{2g}\alpha_i + q$$

$$q(2-2g) - 2a(\Delta\cdot F) + (2-2g)a^2$$

$$-2(1-a)(q-a) \quad — \quad 2(q-a-qa+a^2)$$

$$= \left[q(2-2g)-2q\right] + 2a\left(1+q-\Delta\cdot F\right) + a^2(2-2g-2)$$

$$= -(2g)q + a\,2(1+q-\Delta\cdot F) + a^2(-2g)$$

$$a^2 - \frac{1+q-\Delta\cdot F}{g}\,a + q \geq 0 \quad \text{all } a\in\mathbb{Q}.$$

Can you ~~[illegible]~~ find a unitary operator here? How?

problem: You want a unitary operator T, perhaps something ~~better~~ slightly more general, namely, ~~in the polar decomposition,~~ to have ~~the~~ $T^*T = TT^* = q$, so that $q^{-1/2}T$ is unitary. ~~[illegible]~~ ~~Anyway what next.~~

~~[illegible]~~ Review: $H^2(X\times X) \xrightarrow{\sim} \mathrm{Hom}^0(H^*X, H^*X)$

$K(x,y)$ diff'l form $\longmapsto pr_{1*}\,K\,pr_2^*(f) = \int dy\, K(x,y)f(y)$

$$\mathrm{tr}\,K = \int_X \Delta^* K = \int_{X\times X}(\Delta_* 1)K = \Delta\cdot K = \sum_{i=0}^{2}(-1)^i\,\mathrm{tr}\,K \text{ on } H^i$$

$$\mathrm{tr}(K^t L) = K\cdot L$$

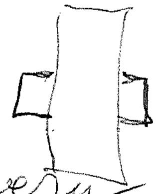
So $H^2(X,X)$ becomes an alg with an involution K^* and a trace.

660 Apply to Frobenius F ~~[illegible]~~ where $F^*F = FF^* = q$. If $tr(K^*K) > 0$ for $K \neq 0$, i.e. trace is positive, then $q^{-1/2}F$ is unitary. ~~Also~~

~~[illegible]~~ Is there a connection with $e^{-\beta H}$ and KMS.

~~[illegible]~~

$$tr(F^n) = \#X(\mathbb{F}_{q^n}) = 1 + q^n - \sum_{i=1}^{2g} \alpha_i^n$$

So you have to remove the wings. 

In the present situation things are very clear ~~because~~ from ~~you compose~~ $F^*F = FF^* = q$. ~~This should tell you that~~ What happens is that the star alg $H^2(X \times X)$ splits as the product of the star algs. $(H^0X \otimes H^2X \oplus H^2X \otimes H^0X)$ and $H^1X \otimes H^1X$ where the intersection form $tr(K^tL) = K \cdot L$ is hyperbolic on the former & negative definite on the latter. So have the * homom. $H^2(X \times X) \longrightarrow H^1(X) \otimes H^1(X)$ and by $F^*F = FF^* = q^{1/2}$ & positivity you should win.

Now let's try to construct a scenario for $\xi(s)$.

continuous version: Instead $(F)^n, (F^*)^n$ $n \geq 0$

(opposite of contractions - expansions) you want

$e^{t\alpha}$, $e^{t\alpha^*}$ $t \geq 0$.

$(1 \otimes \mu) \cdot \Delta = (\mu \otimes 1) \cdot \Delta$
$= 1$

$(1 \otimes \mu)\, (\mu \otimes 1)$ Δ F

$(\Delta - 1 \otimes \mu - \mu \otimes 1)$ $\begin{cases} \mu \otimes 1 \\ 1 \otimes \mu \end{cases}$

$F \cdot (\mu \otimes 1) = 1$

$F \cdot (1 \otimes \mu) = q$

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$$F - (1\otimes\mu) - g(\mu\otimes 1)$$

$$\bar{\Delta} = \Delta - (1\otimes\mu) - (\mu\otimes 1)$$

$$\bar{F} = F - (1\otimes\mu) - g(\mu\otimes 1)$$

$$-a = \bar{\Delta}\cdot\bar{\Delta} = \big(\Delta - (1\otimes\mu) - (\mu\otimes 1)\big)\cdot\Delta = 2-2g-2 = -2g$$

$$-b = \bar{\Delta}\cdot\bar{F} = \Delta\cdot\big(F - (1\otimes\mu) - g(\mu\otimes 1)\big) = \Delta\cdot F - 1 - g$$

$$-c = \bar{F}\cdot\bar{F} = F\cdot\big(F - (1\otimes\mu) - g(\mu\otimes 1)\big) = g(2-2g) - g - g = g(-2g)$$

$a = 2g$ $\qquad b = (1+g-\Delta\cdot F)$ $\qquad c = g(2g)$.

definite $\qquad ac - b^2 \geqslant 0$ $\qquad \Delta\cdot F = 1 - \sum \alpha_i + g$

$$g^{1/2}(2g) \geqslant \big|\underbrace{1+g-\Delta\cdot F}_{\sum_{i=1}^{2g}\alpha_i}\big|$$

Then doing this for F^n $\forall n\geqslant 0$ gives

$$(g^{1/2})^n \geqslant \left|\frac{1}{2g}\sum_{i=1}^{2g}\alpha_i^n\right| \qquad \text{this}$$

suppose $\left|\frac{1}{k}\sum_{i=1}^{k}\beta_i^n\right| \leqslant 1$ all $n\geqslant 0$.

Then $\frac{1}{k}\sum_{k=1}^{k}\sum_{n=0}^{\infty}(z\beta_i)^n$ converges for $|z|<1$.

$\frac{1}{k}\sum_{k=1}^{k}\frac{1}{1-z\beta_i}$ analytic for $|z|<1$.

$\iff |\beta_i| \leqslant 1$.

Now try a continuous version. ~~[illegible]~~ k

~~[illegible]~~ This should involve

Here is maybe a relevant point. You have F and F^* $FF^* = F^*F = q$

so $[F, F^*] = 0$. $F^n (F^*)^n = (FF^*)^n = q^n$.

Still you want $F^t (F^*)^t = q^t$

which means that $q^{-t/2} F^t = U^t$ should be unitary

for $t \geq 0$ $(U^t)^* = q^{-t/2}(F^*)^t = (U^t)^{-1}$. So

$t \mapsto U^t$ for $t \geq 0$ extends to a 1-parameter group since these U^t's are invertible - group completion of $(\mathbb{R}_{\geq 0}, +)$ ~~[illegible]~~ is $\mathbb{R}$.

---

propose to write up details of the intrinsic character of Hardy space. So begin with a review of the following two pages.

Let V 2 dim'l over $\mathbb{C}$, $\mathcal{L} = \mathcal{O}(-1) \subset \mathcal{O} \otimes V$

over $\mathbb{P}V$. ~~Given s_1, s_2 (local) sections of $\mathcal{L}$ you have~~ Given (local) sections s_1, s_2 of $\mathcal{O}\otimes V$ you ~~have~~ get ~~form~~ sections $s_1 \wedge s_2$ of $\mathcal{O} \otimes \Lambda^2 V$, $s_1 \wedge ds_2$ of $\Omega^1 \otimes \Lambda^2 V$

(if $V = \mathbb{C}^2$ $s_1 = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$ $s_2 = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix}$

$$s_1 \wedge s_2 = \begin{vmatrix} f_1 & g_1 \\ f_2 & g_2 \end{vmatrix} \qquad s_1 \wedge ds_2 = \begin{vmatrix} f_1 & dg_1 \\ f_2 & dg_2 \end{vmatrix}.$$

~~Then~~ If s_1, s_2 are local sections of the subbundle $\mathcal{L}$ ~~get~~ have

$$s_1 \wedge s_2 = 0 \qquad ds_1 \wedge s_2 + s_1 \wedge ds_2 = 0 \qquad \text{so}$$

$$s_1 \wedge ds_2 = -ds_1 \wedge s_2 = -s_2 \wedge ds_1 \quad \text{symm.}$$

$L \subset \mathcal{O} \otimes V$ canon. sub bundle.

canonical isom

$L \otimes L \otimes \Lambda^2 V^* \xrightarrow{\sim} \Omega^1$

s section (local) of L, have $ds \in \Omega^1 \otimes V$

$\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \otimes$

can form $s\, ds \in \Omega^1 \otimes$

$ds \otimes s \in \Omega^1 \otimes V \otimes \mathcal{O} \otimes V$

$\to \Omega^1 \otimes \Lambda^2 V$

$s \otimes ds \in (\mathcal{O} \otimes V) \otimes (\Omega^1 \otimes V) \longrightarrow \Omega^1 \otimes \Lambda^2 V$

$(f_1 e_1 + f_2 e_2) \wedge (df_1\, e_1 + df_2\, e_2) = (f_1\, df_2 - f_2\, df_1)\, e_1 \wedge e_2$

So what the point is that when lin. dep. get quadratic form on fibres

review Given s. Given

$$\begin{array}{ccccc} \mathcal{L} & \hookrightarrow & \mathcal{O} \otimes V & \twoheadrightarrow & \mathcal{Q} \\ & & \downarrow d & & \\ \Omega^1 \otimes \mathcal{L} & \hookrightarrow & \Omega^1 \otimes V & \twoheadrightarrow & \Omega^1 \otimes_{\mathcal{O}} \mathcal{Q} \end{array}$$

have canon. $\mathcal{L} \longrightarrow \Omega^1 \otimes_{\mathcal{O}} \mathcal{Q}$

$\mathcal{L} \otimes \mathcal{L} \longrightarrow \Omega^1 \otimes_{\mathcal{O}} \underbrace{\mathcal{Q} \otimes_{\mathcal{O}} \mathcal{L}}$

$\qquad\qquad\qquad \parallel$

$\Omega^1 \otimes_{\mathcal{O}} \mathcal{O} \otimes \Lambda^2 V = \Omega^1 \otimes \Lambda^2 V.$

Take s_1, s_2 local sections of $\mathcal{L}$. Then you get $s_1 \wedge ds_2 \in \Omega^1 \otimes \Lambda^2 V$

$s_1 \wedge s_2 \in \mathcal{O} \otimes \Lambda^2 V$

$\quad \parallel$

$\quad 0$

$s_1 \wedge ds_2$

$\quad \parallel$

$-ds_1 \wedge s_2 = s_2 \wedge ds_1$

gives obvious isom $\mathcal{L} \otimes_{\mathcal{O}} \mathcal{L} \xrightarrow{\sim} \Omega^1 \otimes \Lambda^2 V$

$$s = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \qquad t = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix}$$

$$s \wedge t = \begin{vmatrix} f_1 & g_1 \\ f_2 & g_2 \end{vmatrix} = 0 \qquad \text{assumed}$$

then $\quad s \wedge dt = \begin{vmatrix} f_1 & dg_1 \\ f_2 & dg_2 \end{vmatrix} = -\begin{vmatrix} df_1 & g_1 \\ df_2 & g_2 \end{vmatrix} = \begin{vmatrix} g_1 & df_1 \\ g_2 & df_2 \end{vmatrix}$

$$s \wedge ds = \begin{vmatrix} f_1 & df_1 \\ f_2 & df_2 \end{vmatrix} = f_1 df_2 - f_2 df_1$$

OKAY ~~take~~ so suppose $s = \begin{pmatrix} z \\ 1 \end{pmatrix} f = \begin{pmatrix} zf \\ f \end{pmatrix}$

$$s \wedge ds = zf\, df - f\, d(zf) = -f^2 dz$$

require $\quad -f^2 dz \in \mathbb{R}_{\geq 0} e^{i\phi}, \qquad z = e^{i\theta}$

$$-f^2 e^{i\theta} i\, d\theta \in \mathbb{R}_{\geq 0} e^{i\phi} \qquad -if^2 e^{i\theta} e^{-i\phi} \geq 0$$

$$(-ie^{-i\phi})(fe^{i\theta/2})^2 \geq 0. \qquad \text{choose } e^{i\phi} = -i$$

get $\quad fe^{i\theta/2} \in \mathbb{R}.$

$s = \begin{pmatrix} z \\ 1 \end{pmatrix} f$ is "real" when $fe^{i\theta/2}$ is a real fn.

$$\|s\|^2 = \int_{S^1} \frac{i}{2\pi} s \wedge ds = \int_{S^1} \frac{i}{2\pi} -f^2 e^{i\theta} i\, d\theta = \int (fe^{i\theta/2})^2 \frac{d\theta}{2\pi}$$

So s is real $\qquad s = \begin{pmatrix} z \\ 1 \end{pmatrix} f = \begin{pmatrix} z^{1/2} \\ z^{-1/2} \end{pmatrix} (e^{i\theta/2} f)$

$$\left\| \begin{pmatrix} z^{1/2} \\ z^{-1/2} \end{pmatrix} g \right\|^2 = \int |g|^2 \frac{d\theta}{2\pi}$$

665 really confused. You have $\binom{\omega}{1} f(\omega)$ a section of $\mathcal{O}(-1)$ over $\mathbb{R}$. You let substitute

$$\omega = i\,\frac{1-z}{1+z} = \begin{pmatrix} -i & i \\ 1 & 1 \end{pmatrix} z$$ and get section

$$\begin{pmatrix} z \\ 1 \end{pmatrix} f\left(i\,\frac{1-z}{1+z}\right) \frac{\alpha}{z+1} \qquad \alpha = \sqrt{-2i}$$

of $\mathcal{O}(-1)$ over $|z| = 1$.

Review s_1, s_2 sections of $\mathcal{O} \otimes V$ get $s_1 \wedge s_2$ and $s_1 \wedge ds_2$ sections of $\mathcal{O} \otimes \Lambda^2 V$ resp $\Omega^1 \otimes \Lambda^2 V$.

$$s_1 = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \quad s_2 = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} \qquad s_1 \wedge s_2 = \begin{vmatrix} f_1 & g_1 \\ f_2 & g_2 \end{vmatrix} \quad s_1 \wedge ds_2 = \begin{vmatrix} f_1 & dg_1 \\ f_2 & dg_2 \end{vmatrix}$$

If s_1, s_2 sections of subbundle $\mathcal{L} = \mathcal{O}(-1) \subset \mathcal{O} \otimes V$, then

$$s_1 \wedge s_2 = 0 \qquad ds_1 \wedge s_2 + s_1 \wedge ds_2 = 0 \qquad \text{so}$$

$s_1 \wedge ds_2 = -ds_2 \wedge s_1 = s_2 \wedge ds_1$ is a symm. pairing, $\mathcal{O}$-linear, from $\mathcal{L}$ to $\Omega^1 \otimes \Lambda^2 V$, e.g.

$$s_1 = \begin{pmatrix} zf \\ f \end{pmatrix}, \quad s_2 = \begin{pmatrix} zg \\ g \end{pmatrix} \quad \text{then}$$

$$s_1 \wedge ds_2 = \begin{vmatrix} zf & dz\,g + z\,dg \\ f & dg \end{vmatrix} = -fg\,dz$$

get canonical isom. $\mathcal{L} \otimes_{\mathcal{O}} \mathcal{L} \xrightarrow{\sim} \Omega^1 \otimes \Lambda^2 V$

Recall $GL(2,\mathbb{R})$ acts on V, $\mathbb{P}_1 V$, $\mathcal{L}$ naturally

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^* (f(z)dz) = f\left(\frac{az+b}{cz+d}\right) \frac{ad-bc}{(cz+d)^2}\, dz$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^* \begin{pmatrix} zf \\ f \end{pmatrix} = \begin{pmatrix} z \\ 1 \end{pmatrix} \frac{1}{cz+d} f\left(\frac{az+b}{cz+d}\right)$$

666 improvement

$$\begin{pmatrix} a & b \\ c & d\end{pmatrix}^* \begin{pmatrix} zf \\ f\end{pmatrix} = \underbrace{\begin{pmatrix} a & b \\ c & d\end{pmatrix}^* \begin{pmatrix} z \\ 1\end{pmatrix}} \cdot \begin{pmatrix} a & b \\ c & d\end{pmatrix}^* f$$

$$\begin{pmatrix} a & b \\ c & d\end{pmatrix}^{-1} \begin{pmatrix} \frac{az+b}{cz+d} \\ 1\end{pmatrix} = \begin{pmatrix} a & b \\ c & d\end{pmatrix}^{-1}\begin{pmatrix} az+b \\ cz+d\end{pmatrix}\frac{1}{cz+d} = \begin{pmatrix} z \\ 1\end{pmatrix}\frac{1}{cz+d}$$

pairing

$$\begin{pmatrix} zf\\ f\end{pmatrix},\begin{pmatrix} zg\\ g\end{pmatrix} \longmapsto \begin{pmatrix} zf\\ f\end{pmatrix}\wedge\begin{pmatrix} d(zg)\\ dg\end{pmatrix} = -fg\,dz$$

$$\Big\downarrow \begin{pmatrix} a & b \\ c & d\end{pmatrix}^*$$

is equivariant

$$-\begin{pmatrix} a & b \\ c & d\end{pmatrix}^* f \begin{pmatrix} a & b \\ c & d\end{pmatrix}^* g \, \frac{ad-bc}{(cz+d)^2}dz$$

Next point is intrinsic inner product

$$\begin{pmatrix} z^{1/2}f \\ z^{-1/2}f\end{pmatrix}, \text{ same } \longmapsto -\left(z^{-1/2}f\right)^2 dz = -f^2\frac{dz}{z}$$

thus the section $\begin{pmatrix} z^{\frac{1}{2}} \\ z^{-1/2}\end{pmatrix}$? not clear in your mind. So what is happening. You have have $\mathcal{L}\otimes_{\mathcal{O}}\mathcal{L} \xrightarrow{\sim} \Omega^1\otimes\Lambda^2 V$ canonical

when fixing $\Lambda^2 V \xleftarrow{\sim} \mathbb{C}$ you get a concept of real section of $\mathcal{L}$ over an oriented curve.

[redacted] Perhaps the Hilbert space itself is secondary to the splitting. Given an oriented circle you take sections of $\mathcal{O}(-1)$ over this circle, say smooth sections, or rational

667 sections (regular on the circle), the space of these sections splits into subspaces of sects holom. inside + outside the circle. Take $|z| = 1$. Look at sections of $\mathcal{O}(-1)$ over $|z| = 1$. Assume rational sections

$$s = \begin{pmatrix} z \\ 1 \end{pmatrix} f(z) \qquad f(z) \in \mathbb{C}(z).$$

Splitting

s holom. in D $\iff$ $f(z)$ holom. $f(z) = \sum_{n \geq 0} a_n z^n$

s holom outside S^1 $\iff$ $f(z) = \sum_{n < 0} a_n z^n$.

So where does the $z^{1/2}$ come from?

Look at $\mathcal{O}(-1)$ over a circle, e.g. $\mathbb{P}^1\mathbb{R}$. As a complex line bundle it is trivial, but as a real line bundle it is the Mobius bundle. Maybe you should also think of being given a connection on the real line bundle.

$$\mathbb{P}^1\mathbb{R} \subset \mathbb{P}^1\mathbb{C}$$

Try to proceed invariantly, Kreen form on V. $0 \to \mathcal{O}(-1) \hookrightarrow \mathcal{O} \otimes V \longrightarrow \mathcal{O}(1) \otimes \Lambda^2 V$

$$(\Omega^1)^\vee = \mathrm{Hom}_{\mathcal{O}}(\mathcal{O}(-1), \mathcal{O}(1) \otimes \Lambda^2 V) = \mathcal{O}(1) \otimes_{\mathcal{O}} \mathcal{O}(1) \otimes \Lambda^2 V$$

$$\Omega^1 = \mathcal{O}(-2) \otimes \Lambda^2 V^* \qquad \text{or} \qquad \mathcal{O}(-2) = \Omega^1 \otimes \Lambda^2 V$$

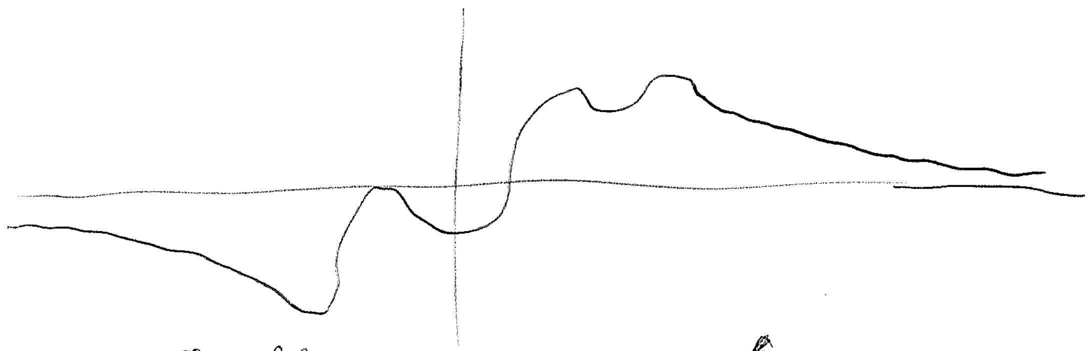
What happens should be clear for $\mathbb{P}^1\mathbb{R}$ $\mathbb{R}\mathbb{P}^1$, Section of $\mathcal{O}(-1)$ over $\mathbb{R}$ has form $\begin{pmatrix} x f(x) \\ f \end{pmatrix}$. When cont at $x = \infty$?

$$\begin{pmatrix} x f(x) \\ f(x) \end{pmatrix} = \begin{pmatrix} g(\frac{1}{x}) \\ \frac{1}{x} g(\frac{1}{x}) \end{pmatrix} ? \qquad \begin{pmatrix} x \\ 1 \end{pmatrix} f(x) = \begin{pmatrix} 1 \\ x^{-1} \end{pmatrix} g(x^{-1})$$

66 8 so $f(x) = x^{-1} g(x^{-1})$ with g cont at 0

suppose $g =$ constant c near 0. $f(x) = \frac{c}{x}$

so $\begin{pmatrix} x f(x) \\ f(x) \end{pmatrix} \approx \begin{pmatrix} c \\ \frac{c}{x} \end{pmatrix}$ as $|x| \to \infty$.



One pt. If you want a non vanishing section, then you have $\begin{pmatrix} x \\ 1 \end{pmatrix} f(x)$ with $f(x)$ non-vanishing on $\mathbb{R}$ hence $f(x) > 0$ (or < 0)

$$f(x) = \frac{1}{\sqrt{1+x^2}} \qquad \begin{pmatrix} x \\ 1 \end{pmatrix} \frac{1}{\sqrt{1+x^2}} \longrightarrow \begin{cases} 1 & x \to +\infty \\ -1 & x \to -\infty. \end{cases}$$

---

hermitian form leads to $U(1,1) \subset GL(2,\mathbb{C})$

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad g^* \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} g = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\begin{pmatrix} a^* & -c^* \\ -b^* & d^* \end{pmatrix} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \quad \text{if } \det = 1 \qquad d = a^* \quad c = b^* \qquad e^{i\phi} \begin{pmatrix} a & b \\ b^* & a^* \end{pmatrix} \quad |a|^2 - |b|^2 = 1.$$

Corresp.

669 positive energy repns of loop group.

you begin with $\partial_t^2 u = \partial_x^2 u$ wave eqn on line, say on circle with anti-periodic boundary conditions. This is a harmonic oscillator K.E. $= \int \partial_t u^2 dx$

P.E. $= \int (\partial_x u)^2 dx$, modes $u = e^{i(kx-\omega t)}$ $\omega^2 = k^2$

positive energy $\omega = |k|$.

you really want to find $e^{\tau D - tD^2}$

you have to begin with the fermion case.

Consider oscillator given by. K.E. $= \frac{1}{2}\int \dot{u}^2 dx$

P.E. $= \frac{1}{2}\int (u')^2 dx$ $\qquad \frac{d}{dt}\frac{\delta L}{\delta \dot{u}} = \frac{\delta L}{\delta u}$ $\qquad \ddot{u} = -u''$

oscillator. What are the modes? Do first on circle with anti-periodic boundary conditions. e^{ikx}

$k \in \left(\frac{1}{2} + \mathbb{Z}\right)\frac{2\pi}{L}$ if the circle is $\mathbb{R}/\mathbb{Z}L$. Basis e^{ikx} Look at simple harm. osc. $\ddot{x} + \omega^2 x = 0$

$x(t) \in \mathbb{R}$. Solution $x = \mathrm{Re}\,(Ae^{-i\omega t})$ $\qquad \omega > 0$ assumed.

Configuration space $\mathbb{R}$, Phase space $\mathbb{C}$ for a single mod. Take ~~string~~

$$u(x,t) = \sum \mathrm{Re}\left(A_k e^{-i|k|t} e^{ikx}\right)$$

It seems that you have two modes per frequency. Calculate Phase space. Complex vector space such that frequencies are positive. So it seems that left + right movers occur! Calculate phase space

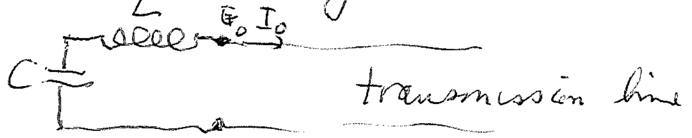
Look for $u(x,t)$ of the form $\mathrm{Re}\left(A(x)e^{-i\omega t}\right)$.

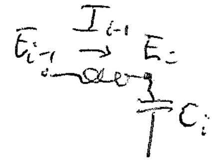
$$\mathrm{Re}\left(A''(x)e^{-i\omega t}\right) = \mathrm{Re}\left(A(x)(-\omega^2)e^{-i\omega t}\right)$$

$$\therefore \quad A'' + \omega^2 A = 0$$

 discrete version of the wave equation.
space is the line $\mathbb{Z}$. time acts as shift. The spectrum of the unitary operator is the circle – how to get positive energy?

digress – ~~classical~~ take a classical continuous situation and try to understand decay. Example





$$E_{i-1} - E_i = Ls\, I_{i-1} \qquad E = L\frac{dI}{dt}$$

$$I_{i-1} - I_i = Cs\, E_i \qquad I = C\frac{dE}{dt}$$

$$0 = \partial_x E + \partial_t I \qquad (\partial_x + \partial_t)(E+I) = 0$$

$$0 = \partial_x I + \partial_t E \qquad (\partial_x - \partial_t)(E-I) = 0$$

$$\hat{E} + \hat{I} = Ae^{-sx} \qquad E + I = Ae^{-sx+st} \quad \text{outgoing}$$

$$\hat{E} - \hat{I} = Be^{sx} \qquad E - I = Be^{sx+st} \quad \text{incoming}$$

$$\frac{E_0}{I_0} + \overbrace{Ls + \frac{1}{Cs}}^{Z} = 0 \qquad \frac{A}{B} = \frac{E_0 + I_0}{E_0 - I_0} = \begin{pmatrix} 1 & 1 \\ +1 & -1 \end{pmatrix}(-Z)$$

$$= \frac{1-Z}{1+Z}.$$

so $\displaystyle S = \frac{1-Z}{1+Z}$

Z and $s\epsilon$ RHP
so $\frac{1-Z}{1+Z}$ an

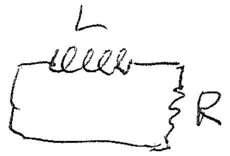
Critical are zeroes & poles i.e. where $Z = \pm 1$.

Take simple situation. Look for decay in time $Re(s) < 0$.

$$E_0 + I_0 = Ae^{st}$$

$$E_0 - I_0 = Be^{st}$$

671 Be more precise. You want a solution decaying in time.

Assume $C = \infty$. so you have  approximately

$$E = IR = -L\frac{dI}{dt}$$

$$\frac{dI}{dt} + \frac{R}{L} I = 0 \qquad I = I_0 e^{-\frac{R}{L}t}$$

$$\frac{E_0}{I_0} + Ls = 0 \qquad \frac{A}{B} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} \frac{E_0}{I_0} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} (-Ls)$$

$$= \frac{1-Ls}{1+Ls}$$

You want $B = 0$, zero incoming, then $s = -\frac{1}{L}$

$$\frac{E_0}{I_0} = -Ls = 1. \qquad \text{So for } Z = Ls + \frac{1}{Cs} \text{ you}$$

get $Z = Ls + \frac{1}{Cs} = -1$ or $LCs^2 + Cs + 1 = 0$

$$s = \frac{-C \pm \sqrt{C^2 - 4LC}}{2LC} = \frac{1}{2}\left(-\frac{1}{L} \pm \sqrt{\frac{C^2 - 4LC}{4L^2C^2}}\right) ?$$

better introduce R. $Z = Ls + \frac{1}{Cs} = -R$

$$LCs^2 + RCs + 1 = 0$$

$$s = \frac{-RC \pm \sqrt{R^2C^2 - 4LC}}{2LC} \qquad R^2C^2 < 4LC$$

$$R^2\frac{C}{L} < 4$$

$$\omega_0 \approx \frac{1}{\sqrt{LC}} \qquad \boxed{\frac{RC}{\sqrt{LC}} < 2}$$

$$s = \frac{-R \pm \sqrt{R^2 - 4\frac{L}{C}}}{2L}$$

 Your basic problem?

Basic problem – how to get positive energy when space and time are discrete. Your 1-particle space is $l^2(\mathbb{Z}) \cong L^2(S^1)$ and time evolution is translation, i.e. mult by z. ~~[illegible]~~ The spectrum is S^1, what might be "positive" energies. Possible approaches: Imaginary time, take the continuous version, pass to imaginary time, use lattice approximation.

Idea to develop – there should be a good model (Gaussian in fact) for $e^{\tau D + \cancel{t} D^2}$ in the ~~[illegible]~~ simple example of fermion – boson correspondence (Jacobi triple product)

Problem. Discrete version – You feel somehow that the S operators are the same

First work in $\mathbb{R}/2\pi L$ anti periodic bdry conditions.

e^{ikx} $\quad k \in \left(\frac{1}{2}+\mathbb{Z}\right)\frac{1}{L}$ $\qquad e^{i\left(\frac{1}{2}+\mathbb{Z}\right)\frac{1}{\cancel{L}}2\pi \cancel{L}} \overset{\text{SHIT}}{=} e^{i\pi} = -1$

1-particle space has orth basis $\frac{1}{\sqrt{2\pi L}} e^{ikx} = \varepsilon_k$ $\qquad k \in \left(\frac{1}{2}+\mathbb{Z}\right)\frac{1}{L}$

$$\int_{\mathbb{R}/2\pi L} \frac{1}{2\pi L}\, dx = 1.$$

Form fermion Fock space $\psi_k^* = \varepsilon_k \wedge \qquad \psi_k = \varepsilon_k \lrcorner$

$$\psi_k \psi_\ell^* + \psi_\ell^* \psi_k = (\varepsilon_k, \varepsilon_\ell) = \delta_{k\ell}$$

$$\psi_k \psi_\ell + \psi_\ell \psi_k = 0 \quad \text{and } *.$$

673 Form number operator $N = "\sum \psi_k^* \psi_k,"$ to be renormalized as $\sum_{k>0} \psi_k^* \psi_k - \sum_{k<0} \psi_k \psi_k^*$

also there's the energy operator roughly $H = "\sum k\psi_k^*\psi_k"$. Current operators

$J_m = "\sum' \psi_{k+m}^* \psi_k"$. ~~[illegible]~~ for $m \neq 0$ this is well-defined.

$$J_m^* = \sum' \psi_k^* \psi_{k+m} = \sum \psi_{k-m}^* \psi_k = J_{-m}$$

$$[J_m, J_n] =$$

Try invariant approach. Given ~~an~~ oriented circle, choose spin structure i.e. square root of T^* = complexified tangent bundle, there's a real structure. So you get a real vector space with scalar product. Have $u(x)$, either periodic or anti-periodic bdry. to $\int u(x)^2 dx$, get Clifford algebra, Fock space, This is the fermionic picture.

bosonic? derived from fermionic picture, ~~[illegible]~~ alt: oscillator. $\mathbb{R}$ vector space with K.E. P.E. $K.E. = \frac{1}{2}\int \partial_t u^2 dx$ $P.E. = \frac{1}{2}\int (\partial_x u)^2 dx$

need ~~[illegible]~~ connection on spinors. current operators: these occur in ferm. picture.

674 $\ddot{u} + \omega^2 u = 0$, $\omega > 0$, $u = \mathrm{Re}(Ae^{-i\omega t})$ sets up a 1-1 corres. between classical states and $A \in \mathbb{C}$.

QM $\quad p = \frac{1}{i}\partial_x \quad q = x \quad [p,q] = \frac{1}{i}$

$H = \frac{p^2}{2} + \frac{1}{2}\omega^2 q^2$ classically & QM.

$a = c(\omega q + ip) \qquad a^* = c(\omega q - ip)$

$a^*a = c^2(\omega^2 q + \underbrace{[\omega q, ip]}_{-\omega} + p^2) \qquad c^2 = \frac{1}{2}$

$a = \frac{1}{\sqrt{2}}(\omega q + ip)$
$\partial_x + \omega x$

$a^* = \frac{1}{\sqrt{2}}(\omega q - ip) = \frac{1}{\sqrt{2}}(-\partial_x + \omega x)$

$a^*a = \frac{1}{2}(-\partial_x + \omega x)(\partial_x + \omega x) = \frac{1}{2}(-\partial_x^2 + \omega^2 x^2 - \omega)$

$H = a^*a + \frac{1}{2}$

$a = c(\omega q + ip) \qquad a^* = c(\omega q - ip)$

$[a, a^*] = c^2(2\omega) \Rightarrow c = \frac{1}{\sqrt{2\omega}}$

$a^* = \frac{1}{\sqrt{2\omega}}(-ip + \omega q) = \frac{1}{\sqrt{2\omega}}(-\partial_x + \omega x)$

$a = \frac{1}{\sqrt{2\omega}}(ip + \omega q) = \frac{1}{\sqrt{2\omega}}(\partial_x + \omega x)$

$a^*a = \frac{1}{2\omega}(p^2 + \omega^2 q^2 - \omega)$

$\therefore \quad H = \frac{1}{2}(p^2 + \omega^2 q^2) = \omega\left(a^*a + \frac{1}{2}\right)$

675 $\qquad \ddot{x} + \omega^2 x = 0$

K.E. $= \frac{1}{2}\dot{x}^2 \qquad$ P.E. $= \frac{1}{2}\omega^2 x^2. \qquad L = \frac{1}{2}\dot{x}^2 - \frac{1}{2}\omega^2 x^2$

$p = \frac{\partial L}{\partial \dot{x}} = \dot{x} \qquad H = p\dot{x} - L = \frac{1}{2}p^2 + \frac{1}{2}\omega^2 x^2$

$p = \frac{1}{i}\partial_x \qquad q = x \qquad [p,q] = \frac{1}{i}$

$a = \frac{1}{\sqrt{2\omega}}(ip + \omega q) \qquad a^* = \frac{1}{\sqrt{2\omega}}(-ip + \omega q)$

$H = \frac{1}{2}p^2 + \frac{1}{2}\omega^2 q^2 = \omega(a^* a + \frac{1}{2})$ simple harm. osc.

$\partial_t^2 u - \partial_x^2 u = 0 \qquad$ K.E. $= \frac{1}{2}\int (\partial_t u)^2 dx$

P.E. $= \frac{1}{2}\int (\partial_x u)^2 dx$. Because of translation invariance phase things split ~~simply~~

Config. space consists of functions $u = u(x)$

K.E. $\frac{1}{2}\int \dot{u}^2 dx, \quad \pi(x) = \frac{\delta \mathcal{L}}{\delta \dot{u}(x)} = \dot{u}(x)$

basic 1-form $\int \dot{u}(x)\, \delta u(x)\, dx \qquad$ P.E. $= \frac{1}{2}\int (\partial_x u)^2 dx$

$= \int u(-\partial_x^2 u)\, dx$

$H = \frac{1}{2}\int \underset{\pi^2}{\dot{u}^2}\, dx + \frac{1}{2}\int (\partial_x u)^2 dx$

$\dot{q} = \frac{\partial H}{\partial p} \qquad \dot{p} = -\frac{\partial H}{\partial q}$

$\dot{u}(x) = \pi(x) \qquad \dot{\pi}(x) = +\partial_x^2 u(x)$

phase space consists of $(u(x), \dot{u}(x))$, a pair of functions, energy is $\frac{1}{2}\int (\dot{u}^2 + (\partial_x u)^2)\, dx$

duality pairing is $\int u(x)\, \dot{u}(x)\, dx$