571 Lecture 5, final viewpoint. Work with

$$E: \quad \overbrace{\underbrace{\cdots \oplus u^{-1}V_-}_{H_-} \oplus gX}^{E_-} \oplus \overbrace{\underbrace{V_+ \oplus uV_+}_{H_+}}^{E_+}$$

$H_+^\perp = E_-$

$H_-^\perp = E_+$

You get some kind of filtration inside X.

~~Inside E you~~ Recall the old viewpoint

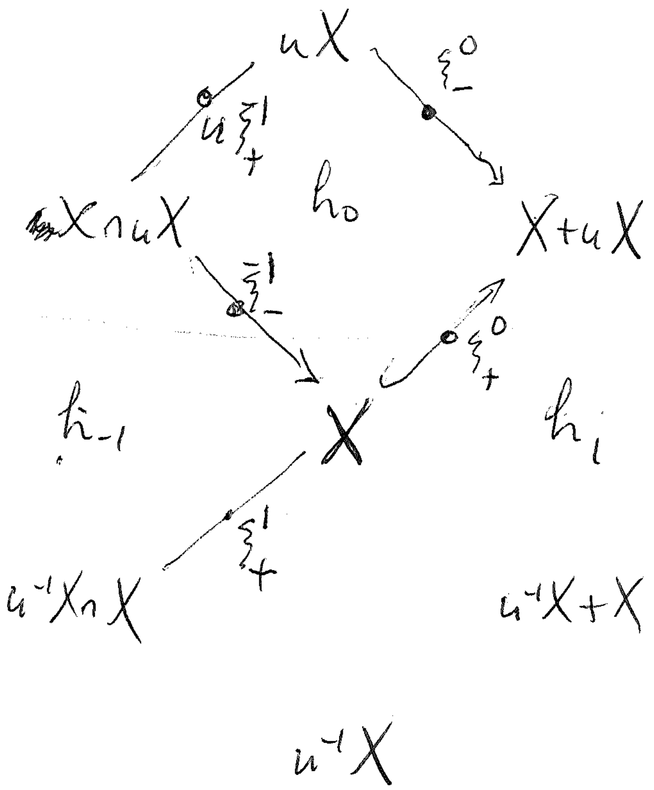
namely X, $X+uX$, $X+uX+u^2X$,

The outgoing picture

E has a natural array of subspaces F_{pq}

$= F_p^{in} \cap F_q^{out}$

Fix ξ_+, ξ_- unit v. gen. $V_\pm$.



Interested in $u^r E_- \cap u^s E_+ = \{u^r\xi_+, u^{r+1}\xi_+, \dots\}^\perp$

$\cap \{u^{s-1}\xi_-, u^{s-2}\xi_-, \dots\}^\perp$

Practice drawing pictures

 ~~Set~~ Set it up properly. ~~First~~

What's important is the bifiltration. Main index is the ~~depth~~ index inside X.

Alt. try scattering function

Start with (X, c) construct $Y = aX \oplus V_+ = bX \oplus V_-$

$$Y = \overline{jX + ujX} = jX \oplus \underbrace{\overline{(uj - jc)X}}_{V_+} = ujX \oplus \underbrace{\overline{(1 - ujc^*)X}}_{V_-}$$

~~scattering~~ outgoing rep. $x \longmapsto \nu_+\left(\frac{1}{z-c}x\right)$ defined for $|z| > 1$.

incoming rep. $x \longmapsto \nu_-\left(\frac{1}{1-zc^*}x\right)$ defined for $|z| < 1$.

$$\begin{cases} \frac{1}{z-c}x = \sum_{n \geq 0} z^{-n} \underbrace{c^n}_{j^* u^n j} x = j^* \frac{1}{z-u} jx \\ \frac{1}{1-zc^*}x = \sum_{n \geq 0} z^n \underbrace{(c^*)^n}_{j^* u^{-n} j} x = j^* \frac{1}{1-zu^{-1}} jx \end{cases}$$

Maybe I can use this to get scattering ~~under~~

$$\begin{cases} \frac{1}{z-c}x = j^* \frac{1}{z-u} jx \\ \frac{1}{1-zc^*}x = j^* \frac{1}{1-zu^{-1}} jx \end{cases}$$

Maybe you can understand this differently. You want the scattering:

$$\begin{array}{ccccccccc} & \oplus & u^{-2}V_+ & \oplus & u^{-1}V_+ & \oplus & V_+ & \oplus & uV_+ \\ & & & & & & \| & & \| \\ \cdots & \oplus & u^{-2}V_- & \oplus & u^{-1}V_- & \oplus & jX \oplus & V_+ \oplus & uV_+ \oplus \cdots \\ & & \| & & \| & & & & \\ \cdots & \oplus & u^{-2}V_- & \oplus & u^{-1}V_- & \oplus & V_- & \oplus & uV_- \oplus \end{array}$$

$S = j_-^* j_+ : L^2(S^1, V_+) \longrightarrow L^2(S^1, V_-)$ $\qquad S\cancel{z} = \cancel{z}S$

573 $\qquad v_+ \in V_+ \qquad v_+ = (uj - jc)(x)$

$j_-^* v_+ =$

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gave a lecture on relating contractions and partial unitaries. Given (X, c) have two partial unitaries inside:

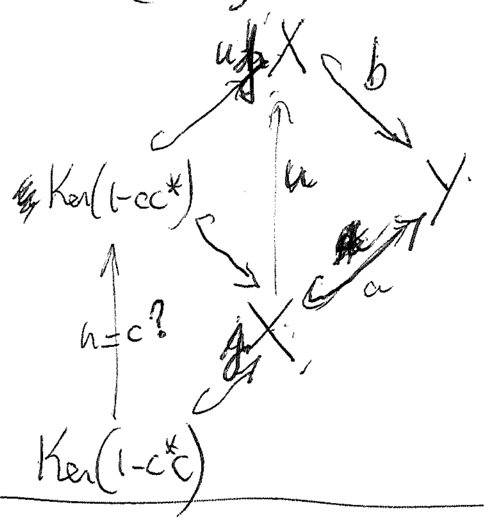
$$\begin{array}{ccc} \mathrm{Ker}(1-c^*c) & \underset{c^*}{\overset{c}{\rightleftarrows}} & \mathrm{Ker}(1-cc^*) \\ \oplus & & \oplus \\ \overline{(1-c^*c)^{1/2}X} & \rightleftarrows & \overline{(1-cc^*)^{1/2}X} \end{array}$$

$x_1 \in \mathrm{Ker}(1-c^*c) \qquad \|x\|^2 - \|cx\|^2 = (x, (1-c^*c)x) = 0$

$x_2 \in \mathrm{Ker}(1-cc^*) \qquad \|x\|^2 - \|c^*x\|^2 = 0$

$(1-c^*c)x = 0 \iff (1-c^*c)^{1/2}x = 0 \iff (x, (1-c^*c)x) = 0$

$\|x\|^2 - \|cx\|^2$



$c = j^* u j$

See if you can get this correct for the lecture. Basic idea. (X, c) gives rise to two partial unitaries, hence two scattering functions.

[redacted] The other point is the double array

Try to work it all out $\qquad a = j$ res to X

$X, c \quad$ form $\quad \overline{jX + ujX}$ with inner product

$\|ax_0 + bx_1\|^2 = \|x_0 + cx_1\|^2$

574 Given (X, c) form $\overline{jX + ujX} = X_1$

ujX

Y

jX

Notation $X_0 = jX$, $X_1 = X_0 + uX_0$, $X_2 = X_0 + uX_0 + u^2X_0$

$X_{-1} = u^{-1}X \cap X$

uX_0

η_1

uX_{-1} X_1

ξ_1

X_0

ξ_0 $u^{-1}\eta_1$

X_{-1} $u^{-1}X_1$

$u^{-1}X_0$

In terms of E and $u^k\xi_\pm$.

Try a different picture

uX_0

X_{-1} η_{-1} $u\xi_0$

uX_1

ξ_{-1} η_0 $u\xi_1$

X_0 uX_2

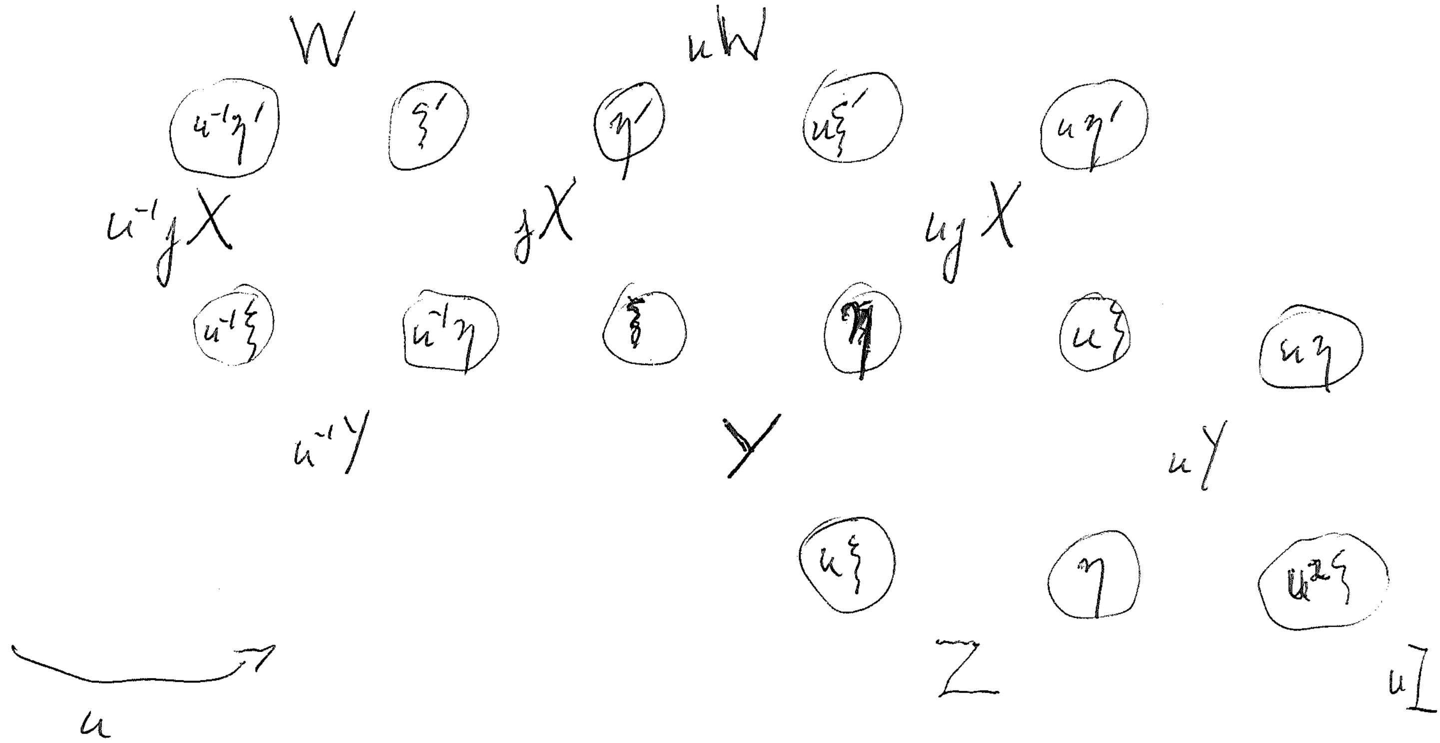
ξ_0 η_1

X_1

$u^{-1}X_0$ ξ_1 X_2

decreasing

575 ~~[illegible]~~ Start with (X, c), form two partial unitaries



Y

$jX \quad V_+ \quad uV_+$

Ψ

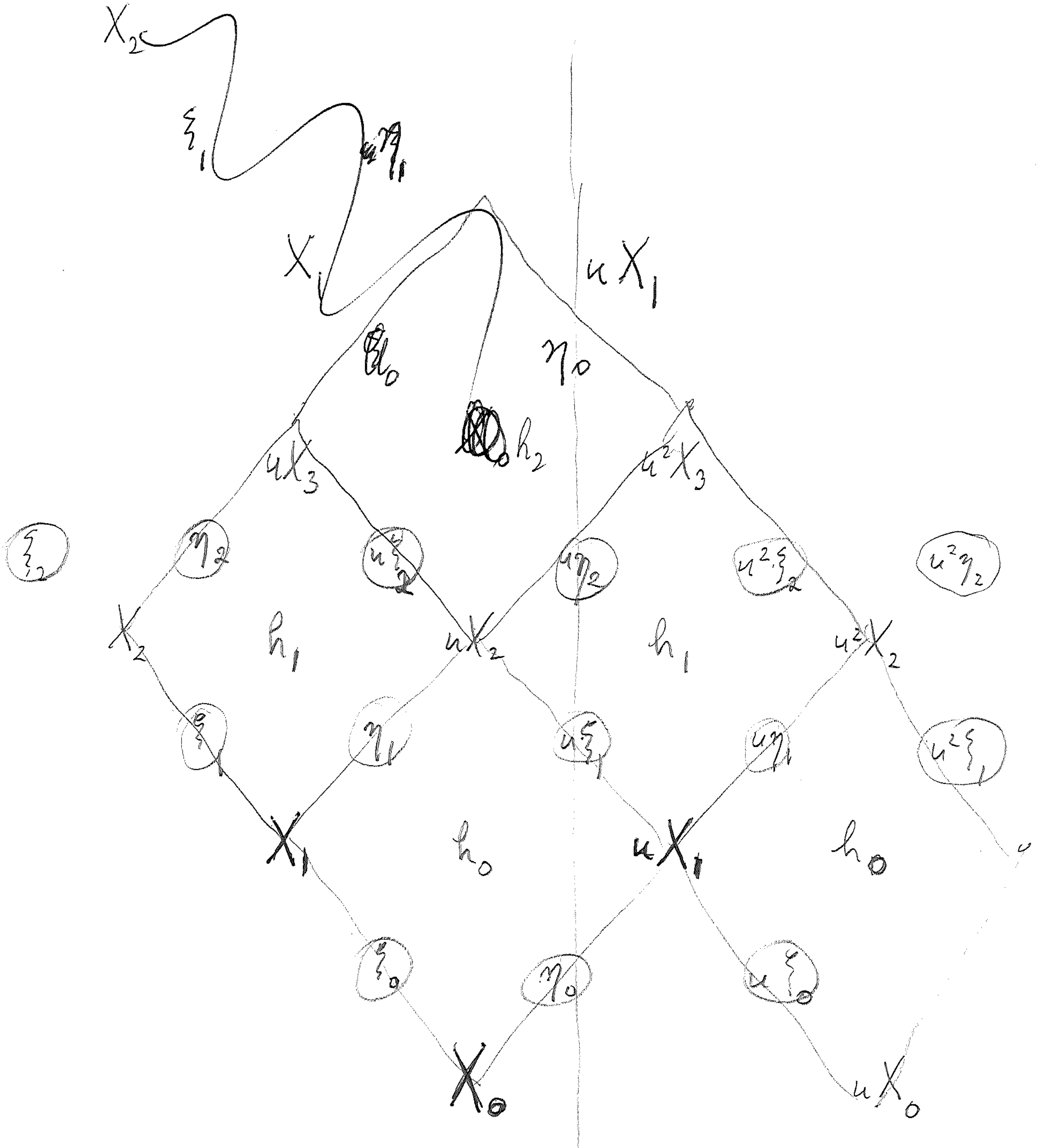
$\xi \quad u\xi \quad u^2\xi$

$\eta \quad Z$

$V_- \quad ujX$

$S_0 = \text{roughly } \dfrac{\xi}{\eta}$

576 Picture $X_1 = jX$, $X_0 = jX + ujX$



Recursion relations.

$$\begin{pmatrix} \xi_n \\ \eta_n \end{pmatrix} = \frac{1}{(1-|h_n|^2)^{1/2}} \begin{pmatrix} 1 & h_n \\ \bar{h}_n & 1 \end{pmatrix} \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \xi_{n+1} \\ \eta_{n+1} \end{pmatrix}$$

Proof.

$$\xi_0 - \eta_0 \overbrace{\eta_0^* \xi_0}^{h_0} = u\xi_1 \,\text{pos const} \qquad \text{where pos const} = (1-|h_0|^2)^{1/2}$$

$$\eta_0 - \xi_0 \underbrace{\xi_0^* \eta_0}_{\bar{h}_0} = \eta_1 \,\text{pos const} \qquad \text{pos const same.}$$

577 Clean up stuff about partial unitaries and contractions

Actually good picture is

$$\begin{tikzcd} X \arrow[r, "\sim", "s"'] \arrow[dr, "a"'] & X' \arrow[d, "a'"] \\ & Y \end{tikzcd}$$

[redacted] $b = a's$

two closed subspaces X, X' of Y and a unitary isom $s : X \xrightarrow{\sim} X'$.

~~[illegible]~~ conditions useful

~~bound states~~ $X \underset{b}{\overset{a}{\rightrightarrows}} Y$ $\qquad \overline{aX + bX} = Y.$

bound states, one def is eigenvectors eigenvalue 1.

$X \underset{b}{\overset{a}{\rightrightarrows}} Y$ Look at largest subspace $Z \subset X$ such that $aZ = bZ$. get unitary operator on Z

Another condition is that $\mathrm{spec}(c^*) \subset D$, $\mathrm{spec}(c) \subset D$

i.e. $\exists\, r < 1 \ni \dfrac{1}{z-c} = \sum_{n \geq 0} z^{-1-n} c^n$ analytic

for $|z| > r$ $\qquad x_1^* \dfrac{1}{z-c} x$ ~~analytic~~ analytic

for $|z| > r$.

~~Suppose speak you have a contr~~

Assume $\dfrac{1}{z-c}$ exists for $|z| > r$

and $\dfrac{1}{z-c^*}$ exists for $|z| > r$.

i.e. $\dfrac{1}{z^{-1}-c^*}$ exists for $0 < |z| < \frac{1}{r}$

i.e. $\dfrac{1}{1-zc^*}$ exists for $|z| < \frac{1}{r}$

So you want assume $\dfrac{1}{z-c}$ analytic $|z| > r$

$\dfrac{1}{1-zc^*}$ analytic $|z| < \frac{1}{r}$. How does this help?

578 So now your formal calculations should work.

$$\frac{z}{z-c} - 1 = \frac{z-(z-c)}{z-c} = \frac{c}{z-c}$$

$$\frac{c}{z-c} = \frac{z^{-1}c}{1-z^{-1}c} = \sum_{n\geq 1} z^{-n}c^n$$

$$\frac{1}{1-zc^*} = \sum_{n\geq 0} z^n (c^*)^n$$

$$\frac{c}{z-c} + \frac{1}{1-zc^*} = \frac{1}{z-c}\left(c(1-zc^*) + z - \cancel{c}\right)\frac{1}{1-zc^*}$$

$$= \frac{z}{z-c}\left(1-cc^*\right)\frac{1}{1-zc^*}$$

$$= \frac{1}{1-zc^*}\left((1-zc^*)c + z - \cancel{c}\right)\frac{1}{z-c}$$

$$= \frac{1}{1-zc^*}\left(1-c^*c\right)\frac{z}{z-c}$$

These identities hold provided spec c, spec $c^* \subset D$

$$j^*\frac{1}{1-zu^{-1}}j = \frac{1}{1-zc^*} \qquad \text{for } |z|<1.$$

$$j^*\frac{u}{z-u}j = \frac{c}{z-c} \qquad \text{for } |z|>1.$$

when you add ~~[illegible]~~ you get a $\mathcal{L}(X)$ valued measure on S^1.

579 ~~Digress~~ Go back to $|S|=1$ case.

Review the ~~Krei~~ philosophy

Invariant viewpoint ~~~~ T 2diml Krein space

Y Hilbert, $T\otimes Y$ ~~~~ is Krein, ~~~~

invariant form of a ~~~~ partial unitary

is an isotropic $W \subset T\otimes Y$. ~~~~

~~~~ W^0/W Krein

Start again, discuss invariant viewpoint.

T 2diml Krein space yielding frequency space $\mathbb{P}T$

Y Hilbert, then $T\otimes Y$ is naturally Krein

contraction on Y (in the disk picture) corresp. to Γ

~~~~ ≥ 0 for the Krein form. and such that

$\ell_z \otimes Y$ is ~~~~ complementary to Γ for $|z|>1$.

$$\begin{pmatrix}1\\z\end{pmatrix}Y + \begin{pmatrix}1\\c\end{pmatrix}Y = \begin{matrix}Y\\ \oplus\\ Y\end{matrix}$$

Then
$$\begin{array}{ccccc} \ell_z\otimes Y & \longrightarrow & T\otimes Y & \xrightarrow{(z\ -1)} & T/\ell_z \otimes Y \\ & & \cup & \nearrow_S & \mathcal{O}(1) \\ & & \begin{pmatrix}1\\c\end{pmatrix}Y & & \end{array}$$

~~Of try something else.~~

Point of invariant approach? (contractions / partial unitaries

have simple invariant descriptions at least for Y fin diml.

I have the idea that ~~~~ there's equiv. between $S(z)$ inner (up to S^1 constants) and (X,c) indices $1,1$, ~~no bdd states,~~ both inc. + outgo reps isometric.

Maybe you should work on fin. diml. exs.

Act Need to get started DAMN

Given (X, c) let Y = completion of (x_1, x_2) for

$$\|(x_1,x_2)\|^2 = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}^* \begin{pmatrix} 1 & c \\ c^* & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$\|x_1\|^2 + (x_1, cx_2)$
$(x_2, c^* x_1) + \|x_2\|^2$

$$\begin{pmatrix} 1 & c \\ c^* & 1 \end{pmatrix} = \begin{pmatrix} c \\ 1 \end{pmatrix}^* \begin{pmatrix} c \\ 1 \end{pmatrix} + \begin{pmatrix} 1-c^*c & 0 \\ 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ c^* \end{pmatrix}^* \begin{pmatrix} 1 \\ c^* \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1-cc^* \end{pmatrix}$$

$a : X \longrightarrow Y$ $\qquad ax = (x, 0)$
$\qquad\qquad\qquad\quad bx = (0, x)$

$$\begin{pmatrix} a^* \\ b^* \end{pmatrix} \begin{pmatrix} a & b \end{pmatrix} = \begin{pmatrix} 1 & a^*b \\ b^*a & 1 \end{pmatrix}, \qquad a^*b = c$$

$X \underset{b}{\overset{a}{\rightrightarrows}} Y$ $\qquad \|ax_1 + bx_2\|^2 = \|x_1\|^2 + (x_1, cx_2)$
$\qquad (x_2, c^* x_1) + \|x_2\|^2$

$$= \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}^* \begin{pmatrix} 1 & c \\ c^* & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Then
$$\boxed{\begin{aligned} Y &= aX \oplus \overline{(b-ac)X} \\ &= \overline{(a-bc^*)X} \oplus bX \end{aligned}}$$

$\simeq aX \oplus \overline{(1-c^*c)^{1/2}X} = aX \oplus V_+$

$\simeq \overline{(1-cc^*)^{1/2}X} \oplus bX = V_- \oplus bX.$

$c_1 = a^*b \qquad c_1^* = b^*a \qquad c_0 = ba^*$ on Y
$\qquad\qquad\qquad\qquad\qquad c_0^* = ab^*$

$$c_0^* c_0 = b a^* a b^* = b b^* = \begin{cases} 1 & \text{on } bX \\ 0 & \text{on } V_- \end{cases}$$

$$c_0 c_0^* = a a^* = \begin{cases} 1 & \text{on } aX \\ 0 & \text{on } V_+ \end{cases}$$

$$\begin{tikzcd} & X' \arrow[d, "b"] \\ X \arrow[r, "a"'] & Y \end{tikzcd}$$

$$c = a^* b : X' \longrightarrow X$$

$Y = $ completion of $X \oplus X'$

with
$$\| ax + bx' \|^2 = \|x\|^2 + (x, cx') + (cx', x) + (x', (1-c^*c)x')$$

$$= \begin{pmatrix} x \\ x' \end{pmatrix}^* \begin{pmatrix} 1 & c \\ c^* & 1 \end{pmatrix} \begin{pmatrix} x \\ x' \end{pmatrix}$$

$$= \begin{pmatrix} x \\ x' \end{pmatrix}^* \begin{pmatrix} 1 \\ c^* \end{pmatrix} \begin{pmatrix} 1 & c \end{pmatrix} \begin{pmatrix} x \\ x' \end{pmatrix} + x'^* (1 - c^* c) x'$$

$$= \| x + cx' \|^2 + \| (1-c^*c)^{1/2} x' \|^2$$

$$= \begin{pmatrix} x \\ x' \end{pmatrix}^* \begin{pmatrix} c \\ 1 \end{pmatrix} \begin{pmatrix} c^* & 1 \end{pmatrix} \begin{pmatrix} x \\ x' \end{pmatrix} + x^* (1 - cc^*) x$$

$$= \| c^* x + x' \|^2 + \| (1-cc^*)^{1/2} x \|^2$$

$$Y = aX \oplus \overline{(b-ac)X'} = bX \oplus \overline{(a-bc^*)X}$$

$$\overline{(b-ac)X'} \cong \overline{(1-c^*c)^{1/2} X'} \qquad \overline{(a-bc^*)X} \cong \overline{(1-c^*c)^{1/2} X'}$$

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$c = a^*b \qquad \|cx'\| = \|x'\| \iff bx' \in aX$

$c^* = b^*a^* \qquad \iff \exists x \quad bx' = ax$

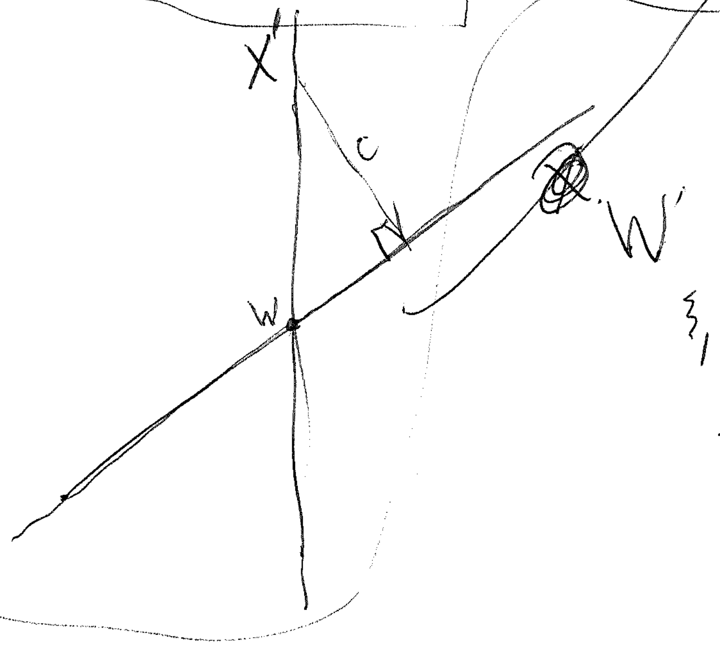
$\|c^*x\| = \|x\| \iff ax \in bX'$

$\iff \exists x' \quad ax = bx'.$

$$\boxed{\begin{tikzcd} W \arrow[r, hook] \arrow[d, hook] & X' \arrow[d, hook, "b"] \\ X \arrow[r, hook, "a"'] & Y \end{tikzcd}}$$

$W = \cancel{\ldots}\ aX \cap bX'$

$W = X \times_Y X' = \{(x, x') \mid ax = bx'\}$



~~Sut~~ Suppose $X' = uX$

$W' \qquad uW' \qquad u^2W'$

$\xi_1 \quad \eta_1 \qquad u\xi_1 \quad u\eta_1$

$X \dashrightarrow uX$

$\xi_0 \quad \eta_0$

Y

I guess you want to mention both a sequence of contractions and a sequence of partial unitaries. Look at the p. unitaries. The ~~...~~ key is maybe

$$\begin{tikzcd} X_2 \arrow[r, dash] \arrow[d, hook] & uX_2 \arrow[dl, dash] \arrow[d] \\ X_1 \arrow[r, dash] \arrow[d, hook] & uX_1 \arrow[dl, dash] \\ X_0 & \end{tikzcd}$$

Model: polys.

$F_n \overset{z}{\hookrightarrow} F_{n+1}$

583 Keep at it. Given $X \overset{a}{\underset{b}{\rightrightarrows}} Y$ $\quad a^*a = b^*b = 1_X$, $\quad \overline{aX + bX} = 1_Y$

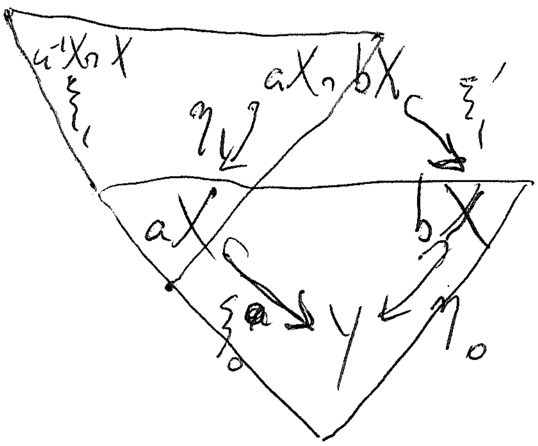
Contraction $c_1 = a^*b$ on X
$c_0 = ba^*$ in Y

$$1 - c_0^*c_0 = 1 - ab^*ba^* = 1 - aa^* = 0 \text{ on } aX$$
$$1 - c_0c_0^* = 1 - ba^*ab^* = 1 - bb^* = 0 \text{ on } bX.$$

$$c_0^*c_0 = 1 \text{ on } aX, \ 0 \text{ on } V_+$$
$$c_0c_0^* = 1 \text{ on } bX, \ 0 \text{ on } V_-$$

$\cancel{c_1^* = ba^*b}$ $\quad \|c_0 y\| = \|ba^* y\| = \|y\| \iff y \in aX$

$\|c_1^* x\| = \|a^*bx\| = \|bx\| = \|x\| \iff bx \in aX.$



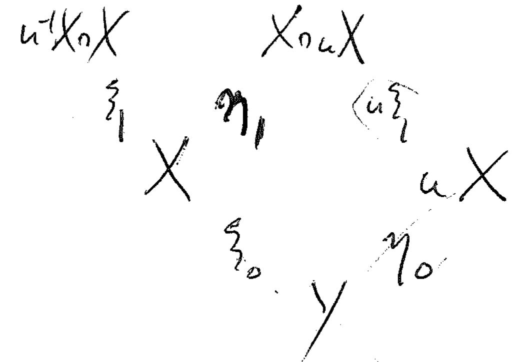
$$\xi_0 - \eta_0 \overbrace{\eta_0^*\xi_0}^{h_0} = \xi_1'(1 - |h_0|^2)^{1/2}$$
$$\eta_0 - \xi_0 \underbrace{\xi_0^*\eta_0}_{\bar h_0} = \eta_1(1 - |h_0|^2)^{1/2}$$

$$\begin{pmatrix}\xi_0 \\ \eta_0\end{pmatrix} = \frac{1}{(1 - |h_0|^2)^{1/2}} \begin{pmatrix} 1 & h_0 \\ \bar h_0 & 1 \end{pmatrix} \begin{pmatrix} \xi_1' \\ \eta_1 \end{pmatrix}$$

You [illegible] a jump in understanding. Basically given (X, c_1) you get partial unitaries.

Form $Y, a, b \ni c_1 = a^*b$. ~~Clear picture~~

Drop j.



$$\begin{pmatrix}\xi_0 \\ \eta_0\end{pmatrix} = \frac{1}{\sqrt{\ }} \begin{pmatrix} 1 & h_0 \\ \bar h_0 & 1 \end{pmatrix} \begin{pmatrix} u\xi_1 \\ \eta_1 \end{pmatrix}$$

584 You want a formula for $X_n = u^{-n+1}X \cap \cdots \cap u^{-1}X \cap X$

~~You want rather ...~~

discuss scattering

<u>Recap.</u> Given (X, c) form Y = completion of $X \oplus X$ with $\|\underbrace{\cancel{\ \ }}_{ax+bx'}\|^2 = \begin{pmatrix} x \\ x' \end{pmatrix}^* \begin{pmatrix} 1 & c \\ c^* & 1 \end{pmatrix} \begin{pmatrix} x \\ x' \end{pmatrix}$, $\quad X \underset{b}{\overset{a}{\rightrightarrows}} Y$

$$Y = aX \oplus \overline{(b-ac)X} = \overline{(a-bc^*)X} \oplus bX$$

$$\overline{(b-ac)X} \cong \overline{(1-c^*c)^{1/2}X}, \qquad \overline{(a-bc^*)X} \cong \overline{(1-cc^*)^{1/2}X}$$

$$\begin{pmatrix} a^* \\ b^* \end{pmatrix}(a\ b) = \begin{pmatrix} 1 & c \\ c^* & 1 \end{pmatrix}$$

Y char. by $a, b : X \to Y$ $\ni$ $\overline{aX + bX} = Y$ and

Then ask where ~~...~~ $c = a^*b$ preserves norm

on $\{x_1 \mid ax_1 \in bX\} \xleftarrow{\sim} \{(x_1, x_2) \mid ax_1 = bx_2\} \xrightarrow{\sim} \{x_2 \mid bx_2 \in aX\}$

$X_1 = \operatorname{Ker}(1 - c^*c) \qquad\qquad \operatorname{Ker}(1 - cc^*)$

$X_3 \quad \xi_2 \quad u\xi_3 \quad u^2X_3$

$X_2 \quad u\xi_2 \quad uX_2$

$\xi_1 \quad \eta_1 \quad u\xi_1$

$X_1 \quad uX_1$

$\xi_0 \quad Y \quad \eta_0$

Somehow this is not clear enough.

You want to put (X, c) in the middle then go down uniquely to (X_1, c_1) or up to (X_{-1}, c_{-1})

Try this again.

Given (X, c) $\qquad X = \overbrace{\operatorname{Ker}(1-c^*c)}^{W} \oplus V_+ = \overbrace{\operatorname{Ker}(1-cc^*)}^{cW} \oplus V_-$

585 $\quad c^*(W\cap cW) \underset{c^*}{\overset{c}{\rightleftarrows}} W \cap cW$

$$W \qquad\qquad cW$$

$$W \xrightarrow{a} X$$

Start with (X, c) $\quad 1-c^*c \quad 1-cc^*$ rank 1.

~~choose~~ $\cancel{1-c^*c = \xi_+ h \xi_+^*}$, $\cancel{1-cc^* = \xi_- h' \xi_-^*}$ $\cancel{\|\xi_\pm\|} = 1$

$$X = \underbrace{\operatorname{Ker}(1-c^*c)}_{aX'} \oplus \underbrace{\xi_+ \mathbb{C}}_{(1-c^*c)X} = \underset{b'X'}{\operatorname{Ker}(1-cc^*)} \oplus \underset{(1-cc^*)X}{\xi_- \mathbb{C}}$$

$$c = b'a'^* + \xi_- h \xi_+^* \qquad \overset{\text{id on } b'X'}{cc^* = b'b'^*} + \xi_- |h|^2 \xi_-^*$$

$$c^* = a'b'^* + \xi_+ \bar{h} \xi_-^* \qquad 1-cc^* = \xi_-(1-|h|^2)\xi_-^*$$

$$c^*c = a'a'^* + \xi_+ |h|^2 \xi_+^*$$

$$1-c^*c = \xi_+ (1-|h|^2) \xi_+^*$$

$$c\xi_+ = \xi_- h$$

$$\boxed{\xi_-^* c \xi_+ = h}$$

Anyway So what? $\quad c^*\xi_- = \xi_+ \bar{h}$

What's going on here? ~~[illegible]~~ You seem to have 2 constants $h = \xi_-^* c \xi_+$ and $\xi_-^* \xi_+$? Return to

$$\underset{\operatorname{Ker}(1-c^*c)}{X_2} \qquad \underset{\operatorname{Ker}(1-cc^*)}{uX_2}$$

$$X_1 \qquad uX_1$$

$$X_0$$

 Given (X, c) go after the structure

Form (E, u, j).

$$F_0 = jX + ujX = jX \oplus \xi_0 \mathbb{C}$$
$$= \eta \mathbb{C} \oplus ujX$$

$F_0 = H_-$ Go back to

W $\qquad$ uW

ξ_1 $\qquad$ η_1 $\qquad$ $u\xi_1$

$u^{-1}X$ $\qquad$ X $\qquad$ uX

$u^{-1}\xi_0$ $\qquad$ $u^{-1}\eta_0$ $\qquad$ ξ_0 $\qquad$ η_0

$u^{-1}Y$ $\qquad$ Y

Take $\underset{\xi_1}{W \subset} X \underset{\xi_0}{\subset} Y$

$$\frac{1}{k_0}\begin{pmatrix} 1 & -h_0 \\ -\bar{h}_0 & 1 \end{pmatrix}\begin{pmatrix} \xi_0 \\ \eta_0 \end{pmatrix} = \begin{pmatrix} u\xi_1 \\ \eta_1 \end{pmatrix}$$

$$E: \oplus\, u^{-2}\eta\mathbb{C} \oplus u^{-1}\eta\mathbb{C} \oplus jX \oplus \xi\mathbb{C} \oplus u\xi\mathbb{C}$$

Try a different notation where you write the orthogonal complements, increasing

Idea is that $\quad X = \xi_0^{\perp}$ in Y

$\quad W = \{\xi_0, u^{-1}\xi_0\}^{\perp}$ in Y.

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~~Given (X,c), put $Y = \operatorname{Ker}(1-c^*c)$, then~~
~~and $cY = \operatorname{Ker}(1-cc^*)$ and~~
~~the isometry $c: Y \to cY$ is unitary with~~
~~inverse c^*. Assume $1-c^*c$, $1-cc^*$ rank 1.~~
~~$X = Y \oplus \xi\mathbb{C} = cY \oplus \eta\mathbb{C}$ $\|\xi\|=\|\eta\|=1$~~

Start again with (X,c), get p.u. $\operatorname{Ker}(1-c^*c) \underset{c^*}{\overset{c}{\rightleftarrows}} \operatorname{Ker}(1-cc^*)$

$$\operatorname{Ker}(1-c^*c) = \{x\in X \mid \|cx\|=\|x\|\} \qquad \operatorname{Ker}(1-cc^*) = \{x \mid \|c^*x\|=\|x\|\}.$$

put $X_1 = \operatorname{Ker}(1-c^*c)$, $a: X_1 \to X$ inc, $b = ca : X_1 \to X$
Then $a = c^*b$, $a^*a = 1$, $b^*b = a^*c^*ca = a^*a = 1$.

because $(1-c^*c)a = 0$ ~~because $a^*a = 0$~~

Try

Repeat: Given (X,c) get p.u. $\operatorname{Ker}(1-c^*c) \underset{c^*}{\overset{c}{\rightleftarrows}} \operatorname{Ker}(1-cc^*)$
mut inverse

Put $X_1 = \operatorname{Ker}(1-c^*c) = \{x \mid \|cx\|=\|x\|\}$ $\quad a: X_1 \to X$ incl.
$cX_1 = \operatorname{Ker}(1-cc^*) = \{x \mid \|c^*x\|=\|x\|\}$ $\quad b = ca : X_1 \to X$

Then $a^*a = b^*b = 1$, $c^*ca = a$ $\quad cc^*b = b$.
$c^*b = a$ $\quad ca = b$.

Assume $1-c^*c$ $1-cc^*$ have rank 1 $\quad a^*(1-c^*c) = 0$

Pick $\xi, \eta \ni (1-c^*c)X = \xi\mathbb{C}$, $(1-cc^*)X = \eta\mathbb{C}$ $\quad \|\xi\| = \|\eta\| = 1$

~~Can write~~ $\quad c\xi\mathbb{C} = c(1-c^*c)X = (1-cc^*)cX \subset \eta\mathbb{C}$

$X = aX_1 \oplus \xi\mathbb{C}$
$= bX_1 \oplus \eta\mathbb{C}$

Let $c\xi = \eta h$
$h = \eta^* c\xi$

$1 - aa^* = \xi\xi^*$
$1 - bb^* = \eta\eta^*$

whence $c = ba^* + \eta h \xi^*$

$$c = c(aa^*) + c(\xi\xi^*) = ba^* + \eta h \xi^*$$

$$c^* = c^*bb^* + c^*\eta\eta^* = ab^* + \xi \bar{h} \eta^*$$

$$1 - c^*c = \xi(1-|h|^2)\xi^* \qquad 1 - cc^* = \eta(1-|h|^2)\eta^*$$

Next step probably to introduce c_1 on X_1 which will be either a^*b or b^*a. It should be a^*b. [redacted]

$$\begin{tikzcd} & X \\ X_1 \arrow[ur, "a"] \arrow[dr, "b"'] & \\ & X \end{tikzcd}$$

You might want $\perp$

$$X_1 \qquad\qquad uX_1$$

$$\xi_1 \qquad \eta_1 \qquad u\xi_1$$

$$u^{-1}X \qquad\qquad X \qquad\qquad uX$$

$$x_2 = \{x \in X_1 \mid cx \perp \xi_1\} \qquad ?$$

Go back to the scattering picture

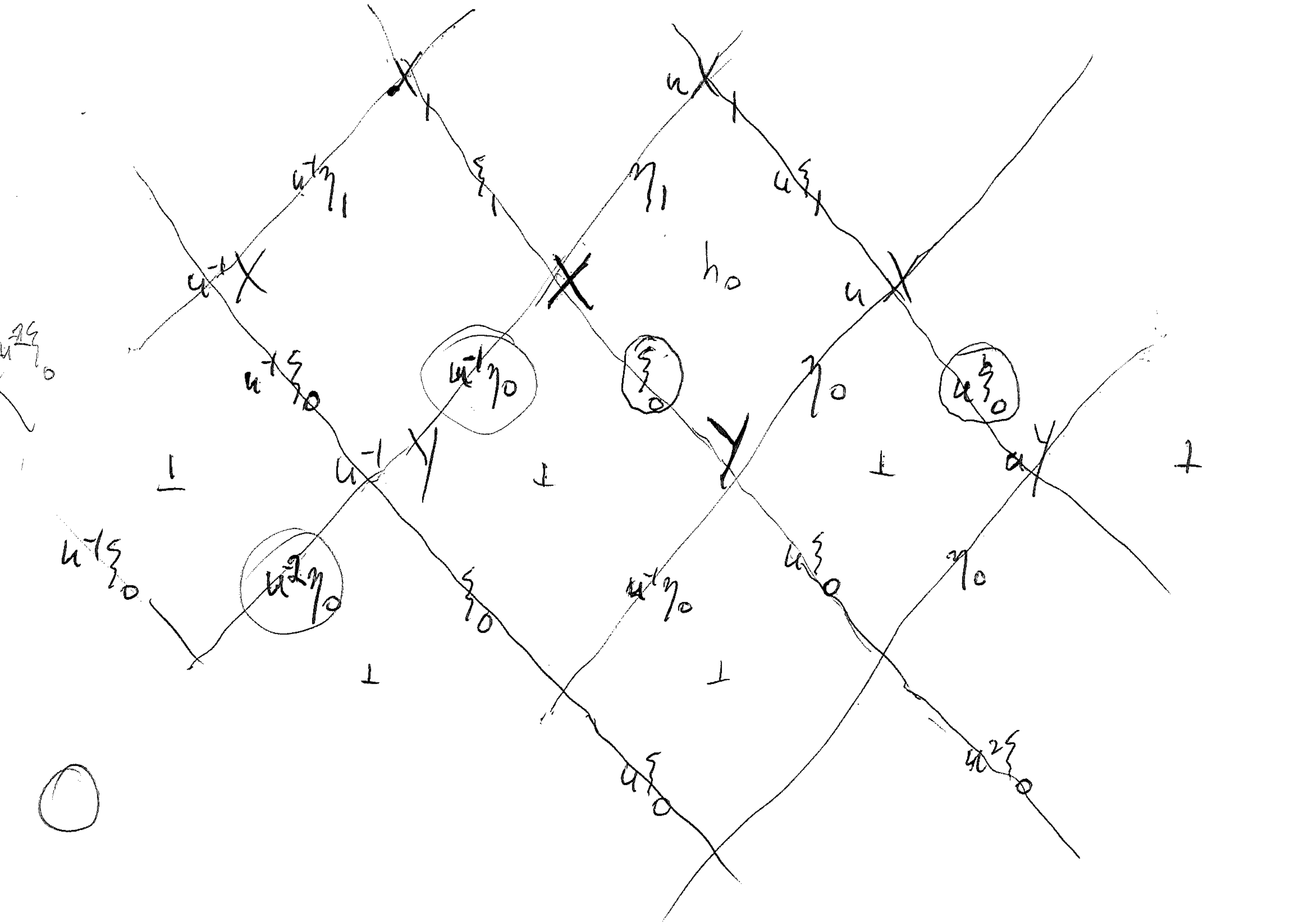
$$\begin{array}{ccccccccc}
\dots, & u^{-2}V_+ & \oplus & u^{-1}V_+ & \oplus & V_+ & \oplus & uV_+ & \cdots \\
 & \cap & & & & \| & & \| & \| \\
\oplus & u^{-1}V_- & \oplus & \mathcal{J}X & \oplus & V_+ & \oplus & uV_+ & \cdots \\
\| & \| & & & & \cup & & & \\
\cdots \oplus & u^{-1}V_- & \oplus & V_- & \oplus & uV_- & \oplus & &
\end{array}$$

automatic for $\mathcal{J}X$

F_k orth. space to $\bigoplus_{n \geq 1} u^{-n}V_- \ni \{u^{-1}\eta, u^{-2}\eta, \dots\}$

$+ \bigoplus_{n \geq k} u^n V_+ \ni \{u^k\xi, u^{k+1}\xi, \dots\}$.

589



$$\{u^{-2}\eta_0 + u^{-1}\eta_0 + X\} = \{\xi_0,\ u\xi_0,\ u^2\xi_0, \dots\}^{\perp}$$

~~Altbach~~

590 Proceed like orth polys.

$u^{-2}\eta\mathbb{C}$ $u^{-1}\eta\mathbb{C}$ $\underline{jX \quad \xi\mathbb{C}}$ $u\xi\mathbb{C}$

$\eta\mathbb{C}$ ujX

start with ξ which is ~~outside~~ $\perp$ X, apply u^{-1} to get $u^{-1}\xi$, and then restrict to the subspace X_1 of X which is $\perp$ $u^{-1}\xi$

$u^{-1}\xi = j^* u^{-1}\xi$ ~~and so forth~~

~~Let us c~~

$$X_1 = \{x \in X \mid (u^*\xi)^* jx = 0\}$$

$$\xi^* ujx = \xi^*(uj - jc)x$$

This means $\nu_+ x = 0$ i.e. $x \in (1-c^*c)X$

How to calculate?

Better approach. Use $j_+^* jx = \xi_+\xi_+^* \dfrac{1}{z-c} x$

$$j_-^* jx = \xi_-\xi_-^* \frac{1}{1-zc^*} x$$

~~Somehow you~~

Seems that you want $\xi_-^*(c^{*n}x) = 0$ $\quad n > 0$?

~~$(u^{m}\eta, jx)$~~ want $jx \perp u^k\xi_+$ for $n \geq -k$

note $jx \perp u^n\xi_+$ for $n \geq 0$.

~~$(u^{k}\xi_+, jx) = \xi_+^* u^{\ell} jx$~~ $\quad (u^{-1}\xi_+, jx) = (\xi_+, ujx)$

$= (\xi_+, (uj - jc)x) = \xi_+^* \nu_+(x)$

59 | diagram

$$X_3$$

$$\xi_3$$

$$u^{-1}X_2 \qquad\qquad X_2$$

$$\xi_2$$

$$u^{-1}X_1 \qquad\qquad X_1$$

$$\xi_1$$

$$u^{-2}X_0 \qquad\qquad u^{-1}X_0 \qquad\qquad X_0$$

$$u^{-2}\xi_0 \qquad\qquad u^{-1}\xi_0 \qquad\qquad \xi_0$$

$$u^{-2}Y \qquad\qquad u^{-1}Y \qquad\qquad Y$$

$$x \in X_0 \quad x \perp u^{-1}\xi_0 \implies x \in X_1$$

$$x \in X_1 \quad x \perp u^{-2}\xi_0 \implies x \in X_2$$

592 Start again. (Z, c_0)

$$Z = aY \oplus \xi_0 \mathbb{C} = \eta_0 \mathbb{C} \oplus bY$$

You need a good approach.

Let's take the increasing filtration viewpoint

You want

Question: Given (X, c) get decomp.

$$X = aX_1 \oplus V_+ = bX_1 \oplus V_-$$

where $aX_1 = \operatorname{Ker}(1 - c^*c)$ $\qquad$ ~~a~~

$bX_1 = \operatorname{Ker}(1 - c^*c)$ $\qquad$ $a, b : X_1 \to X$

$$a^*a = b^*b = 1 \qquad a^*b = c \qquad b^*a = c^*$$

$$V_+ = \overline{(1 - c^*c)X} \qquad V_- = \overline{(1 - cc^*)X}$$

$$V_+ \underset{c^*}{\overset{c}{\rightleftarrows}} V_-$$

strict cont.

$\|cx\| < \|x\|$ for $0 \neq x \in V_+$

$\|c^*x\| < \|x\|$ for $0 \neq x \in V_-$

Now assume $V_\pm = \xi_\pm \mathbb{C}$, $\|\xi_\pm\| = 1$

Then there are two h's, namely, what

$c : V_+ \to V_-$ is: $\quad \xi_-^* c \xi_+$

and the angle between V_+, V_- $\quad \xi_-^* \xi_+$

593 1. contraction on Y equivalent to a partial unitary on Y together with ~~strictly~~ contractive boundary condition

2. $\therefore$ partial unitary on Y equivalent to a contraction $c \ni c = cc^*c$. (i.e. c kills $(1-c^*c)X$ and $cX \subset$ Ker$(1-c^*c)$)

3. contraction c on X equivalent up to canonical isom to a p.u. $X \underset{b}{\overset{a}{\rightrightarrows}} Y \ni$ $\overline{aX + bX} = Y$
$a^*b = c$.

Begin with $\boxed{Y = aX \oplus V_+ = V_- \oplus bX}$

$d = ba^*$ $\quad c = a^*b$

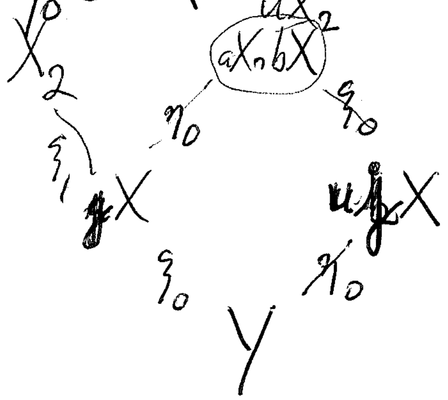
ba^*

Begin with the p. unit.

$Y = aX \oplus \xi_0 \mathbb{C} = \eta_0 \mathbb{C} \oplus bX$

set $h_0 = \eta_0^* \xi_0$. $\quad h_0 = 0$ means that $\xi_0 \in bX$ and $\eta_0 \in aX$.

$$\begin{tikzcd}
X \arrow[r, "b"] \arrow[d, "a^*b"'] & Y \arrow[r] \arrow[d, "ba^*"] \arrow[dl, "a^*"'] & V_- \\
X \arrow[r, "b"] & Y \arrow[r] & V_-
\end{tikzcd}$$



$\eta_1 = \eta_0$ $\quad ba^*$

$\xi_0 = u \xi_1$

So how to set this up.

It should be like Szego orthog polys.

You want certain equations

Start with η_0, ξ_0 $\qquad \eta_0^*\xi_0 = h_0$

$$\begin{tikzcd} jX \arrow[rr, "u=ba^*"] \arrow[dr, "\xi_0"'] & & ujX \arrow[dl, "\eta_0"] \\ & Y & \end{tikzcd} \qquad \begin{tikzcd} aX \arrow[rr, "ba^*=u"] & & bX \\ & Y & \end{tikzcd}$$

ba^*ax

Go back to $\quad Y = aX \oplus V_+ = V_- \oplus bX$

$c = ba^* \qquad d = a^*b$

$$\begin{tikzcd} aX \arrow[rr, shift left, "ba^*"] \arrow[dr] & & bX \arrow[ll, shift left, "ab^*"] \arrow[dl] \\ & Y & \end{tikzcd}$$

$aa^*ba^* : aX \longrightarrow aX$

$ax \longmapsto aa^*bx$

Keep on trying for a good notation. You want to start with (X, c), such that $(1-c^*c)$ $\langle 1-cc^*)$ have rank 1.

Set $\quad X_1 = \{x \mid \|cx\| = \|x\|\} = \ker(1-c^*c)$

ξ_0 unit vector in $(1-c^*c)X$

η ——————— $(1-cc^*)X$.

$$X = X_1 \oplus \xi_0\mathbb{C}$$
$$= \eta_0\mathbb{C} \oplus cX_1$$

$$\begin{tikzcd} X_1 \arrow[dr, "\xi_0"'] & & cX_1 \arrow[dl, "\eta_0"] \\ & X & \end{tikzcd}$$

a_1 inclusion $X_1 \hookrightarrow X \qquad c_1 = a_1^* c a_1 a_1^*$

$b_1 = ca_1$

 Where is $c_1|X_1$ unitary? This is becoming clearer

$$c_1 : \begin{array}{c} x \longmapsto cx \xrightarrow{a_1^*} a_1^* cx \\ \cap \\ X_1 \end{array}$$

Go back to c_0 on X_0, put $X_1 = \mathrm{Ker}(1 - c_0^* c_0)$

Maybe better would be to assume $(1-c_0^*c_0)$ rank 1 & choose unit v. ξ_0 in its image.

$$\xi_0^* x = 0 \iff x = c^*cx \quad \text{i.e.} \quad x \in X_1$$

You seem to be replacing $\mathrm{Ker}(1-c_0^*c_0)$ by the ~~&~~ orthogonal of V_+.

c

$$X_1 = \mathrm{Ker}(1-c_0^*c_0) = (V_+)^\perp$$

Suppose given X, c $\quad V_+ = \overline{(1-c^*c)X}$, $\quad V_- = \overline{(1-cc^*)X}$

$$V_+^\perp = \mathrm{Ker}(1-c^*c) \underset{c^*}{\overset{c}{\rightleftarrows}} V_-^\perp = \mathrm{Ker}(1-cc^*)$$

$$V_+ = \overline{(1-c^*c)X} \underset{c^*}{\overset{c}{\rightleftarrows}} V_- = \overline{(1-cc^*)X}$$

we have maps $V_+ \longrightarrow V_-$ projection also by c.

$$X \underset{b}{\overset{a}{\rightrightarrows}} Y \qquad \|cx\| = \|x\| \iff bx \in aX.$$

$$\iff bx \in V_+^\perp$$

$c = a^*b$ $\qquad aX \cap bX = V_+^\perp \cap V_-^\perp$

K_-

V_+ V_-

You want to work with orth. comps

$\overline{V_+ + V_-}$ So you have ξ_+, ξ_- with $\xi_-^* \xi_+ = h_0$. Get 2 diml subspace spanned by $\xi_\pm$. $\xi_+ - \xi_- \frac{\xi_-^* \xi_+}{h_0} = \sqrt{1-|h_0|^2}$

$$\xi_0 - \eta_0 \frac{\eta_0^* \xi_0}{h_0} = \sqrt{1-|h_0|^2} \cdot \xi_1'$$

$X_0 u X$

$\frac{\eta_0 - \xi_0 \bar{h}_0}{k_0}$	$\frac{\xi_0 - \eta_0 h_0}{k_0}$
η_0	ξ_0

Start again with (X, c) Construct dilation E. Is it possible to describe subspaces Y of E such that $i^* u^n i = (i^* u i)^n$ $n \geq 0$.

You can form $iY + uiY + u^2 iY + \cdots$

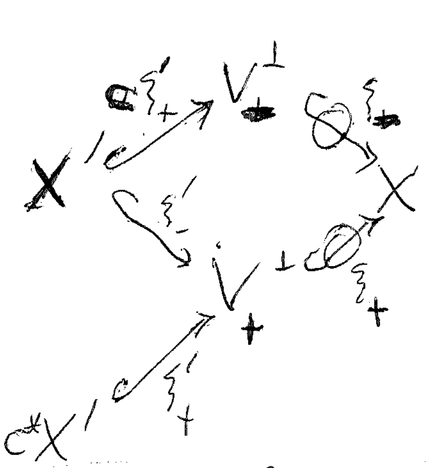
Move on to H^+/SH^+

597 Suppose given (X, c). Assume $V_\pm$ 1-dim.
$V_+ = \xi_0 \mathbb{C}$, $V_- = \eta_0 \mathbb{C}$. $h_0 = \eta_0^* \xi_0$

~~[struck-out line]~~

Form $Y \supseteq X +$

Go smaller first. Given (X, c) form $V_+ = \overline{(1-c^*c)X}$
$V_- = \overline{(1-cc^*)X}$ and $\overline{V_+ \oplus V_-}$. To simplify ass
$V_\pm = \xi_\pm \mathbb{C}$ with $\|\xi_\pm\| = 1$. Let $h_0 = \xi_-^* \xi_+$,
let $X' = X \ominus (\overline{V_+ \oplus V_-})$. We have
$V_+^\perp = \mathrm{Ker}(1-c^*c)$, $V_-^\perp = \mathrm{Ker}(1-cc^*)$. So



$$X = \mathrm{Ker}(1-c^*c) \oplus \xi_+ \mathbb{C}$$
$$= \mathrm{Ker}(1-cc^*) \oplus \xi_- \mathbb{C}$$

Start again. The go smaller step uses only the ~~angle between~~ angle $h_0 = \xi_-^* \xi_+$

(X, c) $\quad \xi_+ \mathbb{C} = (1-c^*c)X$

$\xi_- \mathbb{C} = (1-cc^*)X$

You are looking at $X' \Longrightarrow X$

Look you have $\xi_+ \;\; X \;\; \xi_-$ $\qquad \xi_-' = \frac{1}{k}\left(\xi_- - \xi_+ \overline{\xi_+^* \xi_-}\right)$ $\quad \overline{h}$

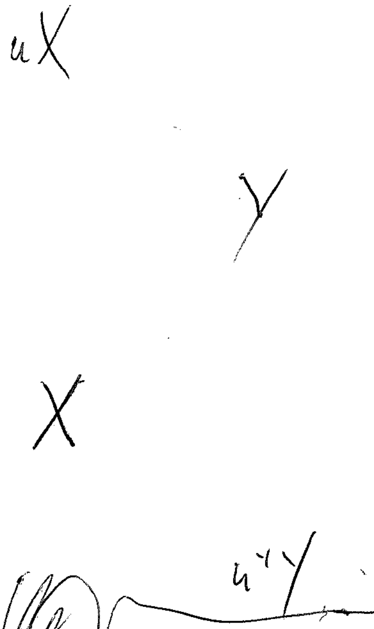
ξ_+' $\quad \mathrm{Ker}(1-c^*c)$. $\qquad \mathrm{Ker}(1-cc^*)$ $\qquad c\,\xi_+ \subset \xi_- \mathbb{C}$

ξ_-' $\qquad c\,\xi_+'$ $\qquad \xi_+' = (\xi_+ - \xi_- h_0)/k$

$c^* \mathrm{Ker}(1-c^*c) \cap \mathrm{Ker}(1-cc^*)$ $\qquad \mathrm{Ker}(1-c^*c) \cap \mathrm{Ker}(1-cc^*)$

598 Another point is that a correspondence can be iterated

$$
\begin{tikzcd}
b^{-1}(aX) \arrow[r, hook] \arrow[d, "b"] & X \arrow[r, hook, "a"] \arrow[d, "b"] & Y \\
X \arrow[r, hook, "a"] \arrow[d, "b"] & Y & \\
Y & &
\end{tikzcd}
$$

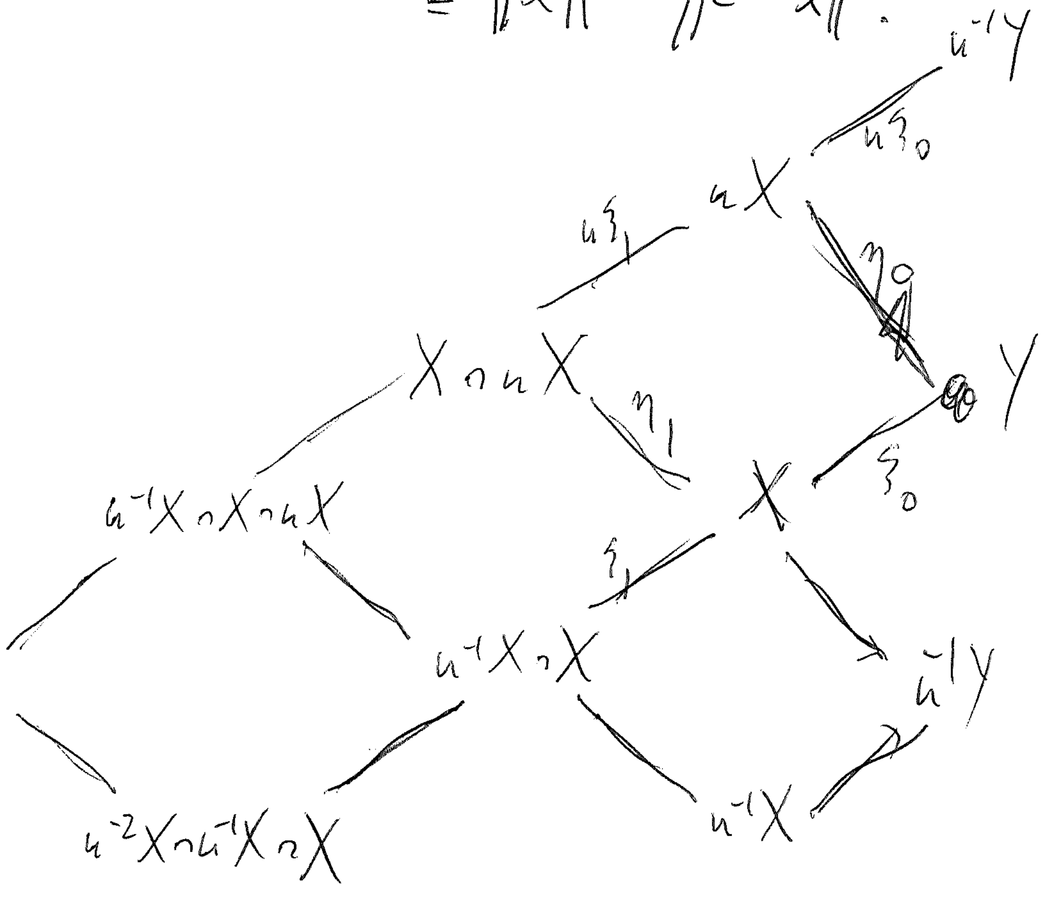


Try to get this cleaner

You would like to relate ~~uX~~

$$\sum_{k=0}^{n-1} \| y - c^{*k} x \|^2 = \sum_{k=0}^{n-1} \left(\|c^{*k} x\|^2 - \|c^{*k+1} x\|^2 \right)$$

$$= \|x\|^2 - \|c^{*n} x\|^2 .$$



599 example. orthogonal polynomials (Szego)

$$H = L^2(S^1, d\mu) \qquad F_p = \mathbb{C} + \mathbb{C}z + \dots + \mathbb{C}z^p$$

Let $\quad p_n \in (z^n + F_{n-1}) \cap F_{n-1}^{\perp}$

$\qquad q_n \in (1 + zF_{n-1}) \cap (zF_{n-1})^{\perp}$

$$F_{n-1} \underset{b = z\cdot}{\overset{a = \mathrm{inc}}{\rightrightarrows}} F_n$$

$$\begin{tikzcd}
F_{n-2} \arrow[r, "{a, \xi_{n-1}}"] \arrow[d, "{b, \eta_{n-1}}"'] & F_{n-1} \arrow[d, "{b, \eta_n}"] \\
F_{n-1} \arrow[r, "{a, \xi_n}"'] & F_n
\end{tikzcd}$$

~~[illegible]~~

$$\xi_n - \eta_n \overbrace{\eta_n^* \xi_n}^{h_n} = z\xi_{n-1}\sqrt{1-|h_n|^2}$$

$$\eta_n - \xi_n \underbrace{\xi_n^* \eta_n}_{\bar h_n} = \eta_{n-1}\sqrt{1-|h_n|^2}$$

$$\begin{pmatrix} 1 & -h_n \\ -\bar h_n & 1 \end{pmatrix}\begin{pmatrix} \xi_n \\ \eta_n \end{pmatrix} = \begin{pmatrix} z\xi_{n-1} \\ \eta_{n-1} \end{pmatrix}\sqrt{} \qquad \begin{pmatrix} \xi_n \\ \eta_n \end{pmatrix} = \frac{1}{\sqrt{}}\begin{pmatrix} 1 & h_n \\ \bar h_n & 1 \end{pmatrix}\begin{pmatrix} z\xi_{n-1} \\ \eta_{n-1} \end{pmatrix}$$

So consider $\quad Y = aX + \xi_0\mathbb{C} = bX + \eta_0\mathbb{C}$

Start with a partial unitary $X \underset{b}{\overset{a}{\rightrightarrows}} Y$

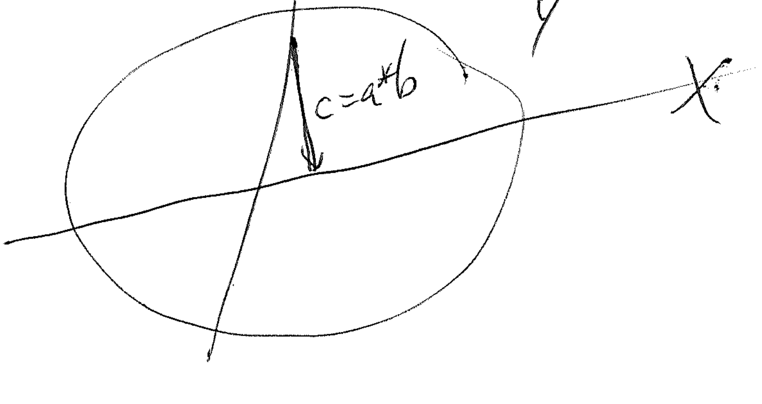
$a^*a = b^*b = 1$. Form cartesian square

$$\begin{tikzcd}
X' \arrow[r, "a'"] \arrow[d, "b'"'] & X \arrow[d, "b"] \\
X \arrow[r, "a"'] & Y
\end{tikzcd}$$

~~[illegible]~~ Maybe first you handle

$$\begin{tikzcd}
& X' \arrow[d, "b"] \\
X \arrow[r, "a"'] & Y
\end{tikzcd}$$

$c = a^*b$. X Then



600 Assume X, X' closed subspaces of the Hilbert space Y, $a : X \to Y$, $b : X' \to Y$ the inclusions. Let $c = a^*b$. Then

$$\begin{tikzcd} \mathrm{Ker}(1-c^*c) \arrow[d, phantom, "\cap"] & X \cap X' \arrow[l, "\sim"'] \arrow[r, "\sim"] & \mathrm{Ker}(1-cc^*) \arrow[d, phantom, "\cap"] \\ X' & & X \end{tikzcd}$$

b X c

~~Ker 1-~~

$$X \xrightarrow{a} Y \xleftarrow{b} X' \qquad c = a^*b : X' \to X$$

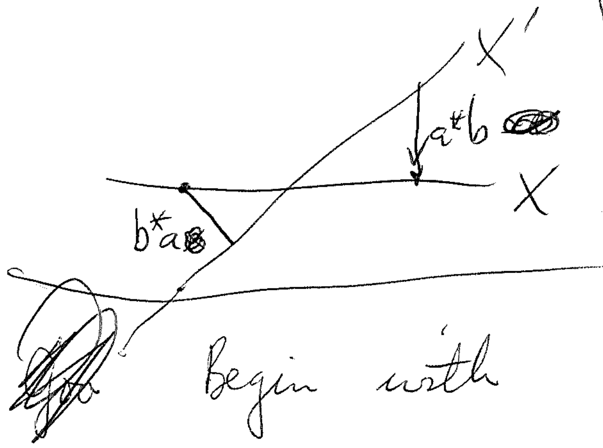
$$\begin{tikzcd} X \times_{(a,b)} X' \arrow[r, "a'"] \arrow[d, "b'"'] & X' \arrow[d, "b"] \\ X \arrow[r, "a"] & Y \end{tikzcd}$$

$$\begin{tikzcd} X \times_{(a,b)} X' \arrow[r, "\sim"] \arrow[d, "S"'] & \mathrm{Ker}(1-c^*c) \arrow[dl, shift left, "c^*"] \\ \mathrm{Ker}(1-cc^*) \arrow[ur, shift left, "c"] & \end{tikzcd}$$

$$ax = bx' \Rightarrow \left.\begin{array}{l} \overbrace{a^*b}^{c}\, x' = a^*a\, x = x \\ \underbrace{b^*a}_{c^*}\, x = b^*b\, x' = x' \end{array}\right\} \Rightarrow$$

$$\begin{tikzcd} (x,x') \in X \times_{(a,b)} X' \arrow[r, "a'"] \arrow[d, "b'"'] & \mathrm{Ker}(1-c^*c) \arrow[d, "b"] \\ \mathrm{Ker}(1-cc^*) \arrow[r, "a"] & Y \end{tikzcd}$$

601 $\qquad$ $\begin{array}{ccc} X\cap X' & \xhookrightarrow{b'} & X' \\ {\scriptstyle a'}\downarrow & & \downarrow{\scriptstyle b} \\ X & \xhookrightarrow[a]{} & Y \end{array}$ $\qquad$ $c = a^*b$, $c^* = b^*a$ $\qquad$ $\|cx'\| = \|a^*bx'\| = \|x'\|$ $\iff bx' \in aX$ $\iff x' \in X\cap X'$



Let Y, Z closed subspaces of X.

$\overline{Y+Z} = X$

Forget difficulties of notation

Begin with $\qquad X_1 \overset{a}{\underset{b}{\rightrightarrows}} X_0$

put $\quad X_2 = X_1 \times_{(a,b)} X_1$

$X_3 = X_1 \times_{(a,b)} X_1 \times_{(a,b)} X_1 = X_2 \times_{(\ )} X_2$

$$\begin{array}{ccccccc} \xrightarrow{a''} & X_2 & \xrightarrow{a'} & X_1 & \xrightarrow{a} & X_0 \\ \downarrow b'' & & \downarrow b' & & \downarrow b & & ba^* \\ X_2 & \xrightarrow{a'} & X_1 & \xrightarrow{a} & X_0 & & \\ \downarrow b' & & \downarrow b' & & & & \\ X_1 & \xrightarrow{a} & X_0 & \xleftarrow{ba^*} & & & \\ \downarrow b & & ba^* & & & & \\ X_0 & & & & & & \end{array}$$

Claim

$\|(ba^*)^n x\| = \|x\|$

$x \in X_0 \iff$

X_1

basic construction might be: $\quad X_{\#} \xrightarrow{c} X_{\#}$

~~Wh~~ What are the principles. You start with c on X and assume $1-c^*c$ $\quad 1-cc^*$ ~~have~~ small rank. Good case is $c^n x \longrightarrow 0$ all x also $c^{*n}x \longrightarrow 0$ $\forall x$.

602 Can ask about $\operatorname{Ker}\left(1-(c^*)^n c^n\right) = \{x \mid \|c^n x\| = \|x\|\}$

~~Conjectures~~ that if $c = a^*b$

$$\begin{tikzcd}
X_1 \arrow[r, "a_1"] \arrow[d, "b_1"] & X \arrow[r, "a^*"] \arrow[d, "b"] & Y \arrow[dl, dashed, "c_0 = ba^*"] \\
X \arrow[r, "a"] \arrow[d, "b"] & Y & \\
Y & &
\end{tikzcd}$$

Why not ~~try~~ start with (X, c), write $c = a^*b$.

$$\begin{tikzcd}
X_2 & & X_1 \arrow[r, "a_1"] \arrow[d, "b_1"] \arrow[dll, dashed, "c_1"'] & X \arrow[d, "b"] \arrow[dl, dashed, "c"'] \\
X_1 \arrow[rr, "a_1"] & & X \arrow[r, "a"] \arrow[d, "b"] \arrow[dll, "c"'] & Y \\
X \arrow[rr, "a"] & & Y &
\end{tikzcd}$$

$\|a^*b\,x\| = \|x\| \iff bx \in aX$ in which case

$c = a^*bx = a^*ax' = x'$

where $bx = ax'$.

The idea: $\|c^n x\| = \|x\| \iff \|c^n x\| = \|c^{n-1}x\| = \cdots = \|cx\| = \|x\|$.

Conjectures seem clear. Given (X, c) you form $X_1 \rightrightarrows X$ and then (X_1, c_1). You view X_1 as a pair ~~[illegible]~~ (x_0, x_1) such that $cx_0 = x_1$, $c^*x_1 = x_0$.

You need to reformulate in terms of ~~[illegible]~~ being orthogonal to $V_\pm$ and further images

603 10 minutes

$$X_{n-1} \overset{a}{\underset{b}{\rightrightarrows}} X_n$$

what important?
I think you want to emphasize $V_\pm$, rather than X_n

Begin with (X, c). ideas from X, c can go up or down. Characterize: Also contractions go ~~the~~ to partial unitaries either up or down.

Review the facts
objects: contraction ~~[illegible]~~ c, partial unitary

A partial unitary on X: $X' \overset{a}{\underset{b}{\rightrightarrows}} X$ is ~~a~~ equivalent to a contraction c on X such that $c = cc^*c$

$\boxed{A}$ $(X, c) \longmapsto X' = \mathrm{Ker}(1 - c^*c) \overset{a = inc}{\underset{b = ca}{\rightrightarrows}} X$

$$c^*b = c^*ca = a$$
$$cc^*b = ca = b.$$

$$b^*b = a^*c^*ca = a^*a = 1.$$

$\boxed{B}$ $(X, \underbrace{ba^*}_{c}) \longleftarrow X' \overset{a}{\underset{b}{\rightrightarrows}} X$

$$c^*c = ab^*ba^* = aa^* \implies cc^*c = caa^* = ba^*$$

$$cc^* = ba^*ab^* = bb^* \implies$$

$\boxed{A}$ is the basic map, invariant meaning, go from positive subspaces to the isotropic ones
$\boxed{B}$ is a section of $\boxed{A}$ defined the z ~~parameter~~ coordinate.

Note both constructions preserve X

 ~~Next~~ Next

$\boxed{C}$ $X' \overset{a}{\underset{b}{\rightrightarrows}} X \longmapsto c_1 = a^* b$ on X'

Anyway so now you ~~b~~ can iterate

$$(X_0, c_0) \longmapsto \left(X_1 \overset{a_0}{\underset{b_0}{\rightrightarrows}} X_0\right) \longmapsto \left(X_1, a_0^* b_0\right)$$

$$\longmapsto \left(X_2 \overset{a_1}{\underset{b_1}{\rightrightarrows}} X_1\right) \longmapsto \left(X_2, a_1^* b_1\right) \qquad \text{etc.}$$

To understand this better

$X_1 = \mathrm{Ker}(1 - c_0^* c_0)$ $\qquad$ a_0 = inclusion of X_1 in X_0
$\qquad\qquad\qquad\qquad\qquad\qquad$ $b_0 = c_0 a_0$

$b_0 = c_0 a_0 \implies c_0^* b_0 = c_0^* c_0 a_0 = a_0$, so $b_0^* b_0 = a_0^* c_0^* b_0 = a_0^* a_0 = 1$.

Note the relations. The first step: from (X_0, c_0) to $\left(X_1 \overset{a_0}{\underset{b_0}{\rightrightarrows}} X_0\right)$ forgets the boundary condition i.e. the strict contractions $c_0 : \mathrm{Ker}(a_0^*) \to \mathrm{Ker}(b_0^*)$

The second step ~~allows~~ forgets the difference between X_0 and $\overline{a_0 X_1 + b_0 X_1} \subset X_0$. The orth comp of this subspace is $\underset{V_+}{\mathrm{Ker}\, a_0^*} \cap \underset{V_-}{\mathrm{Ker}\, b_0^*}$.

Recall that there is something significant about the projection of V_+ to V_- for a partial unitary. ~~Call~~ denote this projection h_0 or $\xi_- \xi_+^*$ or $\eta_0^* \xi_0$

$h_0 : V_+ \to V_-$ is needed to describe $\overline{V_+ + V_-} \subset X_0$.

I think it's true that $(V_+ + V_-)^\perp$ in X_0

605 better $X_0 \ominus \overline{V_+ + V_-}$ is $c_0 X_2$. Digress to discuss picture.

$a_0^* b_0 = a_0^* c_0 a_0$

$$\begin{tikzcd}
 & c_0 X_2 \arrow[r, hook] \arrow[d, hook] & c_0 X_1 \arrow[d, hook] \\
X_2 \arrow[r, hook] & X_1 \arrow[r, hook] & X_0
\end{tikzcd}$$

Let $\xi \in X_1 \ni \|a_0^* b_0 \xi\| = \|\xi\|$

i.e. $c_0 \xi = b_0 \xi \in a_0 X_1 = X_1$

$$X_2 \overset{\text{def}}{=} \operatorname{Ker}\left(1 - (a_0^* b_0)^* (a_0^* b_0) \text{ on } X_1\right) \overset{\text{above}}{=} X_1 \cap c_0^{-1} X_1$$

so $c_0 X_2 = c_0 X_1 \cap X_1$. ~~Work it out differently~~

You want a picture something like

$$\begin{tikzcd}
u X_4 \arrow[r, hook] \arrow[d, hook] & u X_3 \arrow[r, hook] \arrow[d, hook] & u X_2 \arrow[r, hook] \arrow[d, hook] & u X_1 \arrow[d, hook] \\
X_3 \arrow[r, hook] & X_2 \arrow[r, hook] & X_1 \arrow[r, hook] & X_0
\end{tikzcd}$$

$X_2 = u^{-1} X_1 \cap X_1$ in X_0

$X_3 = u^{-1} X_2 \cap X_2 = u^{-2} X_1 \cap u^{-1} X_1 \cap X_1$

You ought to see if it's possible to use u.

$j_0 : X_0 \hookrightarrow E$ $\qquad$ $j_0^* u^n j_0 = \begin{cases} c_0^n & n \geq 0 \\ c_0^{*-n} & n \leq 0 \end{cases}$?

$j_1 = j_0 a_1 : X_1 \hookrightarrow X_0 \hookrightarrow E.$

What are the conjectures?

Review first:

$$(X_0, c_0) \longmapsto \left(X_1 \overset{a_0}{\underset{b_0}{\rightrightarrows}} X_0\right) \longmapsto (X_1, c_1 = a_0^* b_0)$$

$$X_1 = \{x \in X_0 \mid \|c_0 x\| = \|x\|\} = \mathrm{Ker}(1 - c_0^* c_0)$$

$$a_0 x = x \qquad b_0 x = c_0 x \quad \text{for } x \in X_1$$

Now suppose $\quad c_0 = j_0^* u j_0 \qquad j_1 = j_0 a_0$

$$j_1^* u j_1 = a_0^* j_0^* u j_0 a_0 = a_0^* c_0 a_0 = a_0^* b_0 = c_1$$

$$j_1^* u^{-1} j_1 = a_0^* j_0^* u^{-1} j_0 a_0 = a_0^* c_0^* a_0 = c_1^*$$

Review X_2

$$X_2 = \{x \in X_1 \mid \|\underbrace{c_1 x}_{a_0^* c_0 a_0 x}\| = \|x\|\}$$

$$= \left\{x \in X_0 \;\middle|\; \begin{array}{l} \|c_0 x\| = \|x\| \\ c_0 x \in a_0 X_1 = X_1 \end{array}\right\} = X_1 \cap c_0^{-1} X_1$$

It seems roughly OK, but you would like something very clean

$$\begin{tikzcd}
& uX_2 \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\text{cart}"] & uX_1 \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\text{cart}"] & uX_0 \arrow[d] \\
X_2 \arrow[r, hook] & X_1 \arrow[r, hook] & X_0 \arrow[r, hook] & X_{-1}
\end{tikzcd}$$

u

Rev. State with (X_0, c_0) form

$$X_1 = \{x \in X_0 \mid \|c_0 x\| = \|x\|\} = \mathrm{Ker}(1 - c_0^* c_0)$$

$a_0 : X_1 \to X_0$ incl. $\quad b_0 = c_0 a_0$

$$c_1 = a_0^* b_0 = a_0^* c_0 a_0 : X_1 \xhookrightarrow{a_0} X_0 \xrightarrow{c} X_0 \xrightarrow{a_0^*} X_1$$

Thus c_1 is the compression of c_0

Repeat process.

Notice that if ~~we put~~ c_0 starts out as the compression $j^* u j$. $j : X \to E$, then all c_n should be compressions of u.

You need to understand the increasing orth complements.

$$\begin{tikzcd}
 & c_0X_2 = X_1 \cap c_0X_0 \arrow[r, hook, "c_0V_1"] \arrow[d, "W_1"] & c_0X_1 \arrow[d, "W_0"] \\
X_2 \arrow[r, hook] & X_1 \arrow[r, hook, "V_0"'] & X_0
\end{tikzcd}$$

$$\mathrm{Ker}\{(1 - c_0^* c_0) \text{ on } X\}$$

Review: (X_0, c_0) given put $X_1 = \{x \in X_0 \mid \|c_0 x\| = \|x\|\}$

defining $a_0, b_0 : X_0 \to X_1$ $\quad a_0 x = x, \quad b_0 x = c_0 x$. then $a_0^* a_0 = 1$

$b_0 = c_0 a_0$ so $\quad c_0^* b_0 = c_0^* c_0 a_0 = a_0$ and

$b_0^* b_0 = a_0^* c_0^* b_0 = a_0^* a_0 = 1$. So have p.u. $(a_0, b_0 : X_1 \rightrightarrows X_0)$.

Set $c_1 = a_0^* b_0 = a_0^* c_0 a_0$ compression of c_0 to X_1.

Repeat. $X_2 = \{x \in X_1 \mid \|c_1 x\| = \|x\|\}$ $\quad c_1 x = a_0^* b_0 x$

$\|c_1 x\| = \|x\| \iff c_0 x = b_0 x \in X_1$. $\quad X_2 = X_1 \cap c_0^* X_1$

$c_0 X_2 = c_0 X_1 \cap X_1$

$$\begin{tikzcd}
 & c_0X_2 \arrow[r, hook] \arrow[dr, phantom, "\text{cart}"] & c_0X_1 \\
X_2 \arrow[r, hook, "a_2"] & X_1 \arrow[u] & X_0 \arrow[l, hook] \arrow[u]
\end{tikzcd}$$

608 Begin with (X_0, c_0), put $X_1 = \{x \in X_0 \mid \|cx\| = \|x\|\}$

$a_0 x = x$, $b_0 x = c_0 x$. You are getting

$X_2 = X_1 \cap c_0^* X_1$, $X_3 = X_2 \cap c_1^* X_2$ 2

$= X_1 \cap c_0^* X_1 \cap c_1^*(X_1 \cap c_0^* X_1)$

(X_0, c_0) $\quad X_1 = \mathrm{Ker}(1 - c_0^* c_0)$

$$\begin{tikzcd}
& c_1X_2 = X_1 \cap c_0X_1 \arrow[r, hook] \arrow[d, hook] & c_0X_1 \arrow[d, hook] \\
X_2 \arrow[r, hook, "a_1"] & X_1 \arrow[r, hook, "a_0"] & X_0
\end{tikzcd}$$

(X_0, c_0) put $X_1 = \mathrm{Ker}(1 - c_0^* c_0)$ $\quad a_0 x = x$, $b_0 x = c_0 x$ $\quad x \in X_1$

get p.u. $X_1 \overset{a_0}{\underset{b_0}{\rightrightarrows}} X_0$. put $c_1 = a_0^* b_0 : X_1 \circlearrowleft$

$X_2 = \mathrm{Ker}(1 - c_1^* c_1)$ on X_1 $= \{x \in X_1 \mid \|a_0^* b_0 x\| = \|x\|\}$

$\Updownarrow$

$c_0 x \in X_1$

$\overset{\|}{b_0 x}$

$c_1 = a_0^* b_0$

$= a_0^* c_0 a_0$ $\qquad = X_1 \cap c_0^{-1} X_1$

$$\begin{tikzcd}
& c_0X_2 \arrow[r, hook] \arrow[d, "f"] \arrow[dr, phantom, "\text{cart}"] & c_0X_1 \arrow[d, hook] \\
X_2 \arrow[r, hook, "a_1"] & X_1 \arrow[r, hook, "a_0"] & X_0
\end{tikzcd}$$

$c_0 = c_1$ on X_2

If $x \in X_2$, then $c_1 x = a_0^* c_0 a x = a_0^* \underbrace{c_0 x}_{\in X_1} = c_0 x$

$x \in X_1 \cap c_0^{-1} X_1$

Review again $(X_0, c_0) \rightsquigarrow (X_1 \overset{a_0}{\underset{b_0}{\rightrightarrows}} X_0) \longmapsto (X_1, c_1 = a_0^* b_0)$

What's to be done?

Review. (X_0, c_0) put $X_1 = \{x \in X_0 \mid \|c_0 x\| = \|x\|\} = \mathrm{Ker}(1 - c_0^* c_0)$

~~put~~ define $a_0, b_0 : X_0 \to X_1$ by $a_0 x = x$, $b_0 x = c_0 x$ $\quad \therefore b_0 = c_0 a_0$

$c_0^* b_0 = c_0^* c_0 a_0 = a_0 \quad \therefore \quad b_0^* b_0 = a_0^* c_0^* b_0 = a_0^* a_0 = 1.$

$c_0 c_0^* b_0 = c_0 a_0 = b_0$. put $c_1 = a_0^* b_0 = a_0^* c_0 a_0$

compression of c_0 on X_0 to $X_1 \subset X_0$.

~~$x \in X_1 \Rightarrow c_0 x \in X_1$ so $c_1 x = a_0^* c_0 a_0 x = a_0^* c_0 x = c_0 x$~~

From (X_1, c_1) get $X_2 = \{x \in X_1 \mid \|c_1 x\| = \|x\|\} = \{x \in X_1 \mid c_0 x \in X_1\}$

$c_1 x = a_0^* b_0 x = a_0^* c_0 x$

$X_2 = X_1 \cap c_0^{-1} X_1$, so you have

$c_0 X_2 = c_0 X_1 \cap X_1$

$$\begin{tikzcd} & c_0 X_2 \arrow[r, hook] \arrow[d, hook] \arrow[dr, phantom, "\text{cart}"] & c_0 X_1 \arrow[d, hook] \\ X_2 \arrow[r, hook, "a_2"'] \arrow[ur] & X_1 \arrow[r, hook, "a_1"'] & X_0 \end{tikzcd}$$

$x \in X_2$

$\|c_0 x\| = \|x\|$

$\|c_0^2 x\| = \|c_0 x\|$

So you should find that $X_n = \{x \mid \|c_0^n x\| = \|x\|\}$
$= \mathrm{Ker}(1 - c_0^{*n} c_0^n)$.

Good point! ~~[illegible]~~

Go back to $V_+ = \mathrm{Ker}(a_0^*)$, $V_- = \mathrm{Ker}(b_0^*)$

begin with (X, c), put $X_n = \{x \in X \mid \|c^n x\| = \|x\|\}$

$$\begin{tikzcd} c X_2 \arrow[r, hook, "c\xi_1"] \arrow[d, hook, "\eta_1"] & c X_1 \arrow[d, hook, "\eta_0"] \\ X_2 \arrow[r, hook, "\xi_1"'] & X_1 \arrow[r, hook, "\xi_0"'] & X \end{tikzcd}$$

$\xi_0 - \eta_0 h_0 = c \xi_1 \sqrt{1 - |h_0|^2}$

Your idea is that $X_1 \perp \xi_0$, $\underline{X_2 \perp \xi_0 \text{ and } c^* \xi_0}$

$X_2 = X_1 \cap c^{-1} X_1$

So $X_n = \{\xi_0, c^* \xi_0, \ldots, (c^*)^{n-1} \xi_0\}^\perp$

~~Observe~~ This should agree with $\xi_0^* \frac{1}{z - c^*} x$

$\xi_0^* \frac{1}{z - c} x$

610 Iterative formula is

$$X_1 = \operatorname{Ker}(1-c^*c) = \xi_0^{\perp}$$

$$X_2 = X_1 \cap c^{-1}X_1 = \xi_0^{\perp} \cap (c^*\xi_0)^{\perp}$$

do another way. $\nu_+(x) = (1-c^*c)^{1/2}x = \xi_0 (1-|h|^2)^{1/2} \xi_0^* x$ $\to 0$

$$\|x\|^2 - \|c^n x\|^2 = \sum_{k=0}^{n-1} \underbrace{\|(1-c^*c)^{1/2} c^k x\|^2}_{(1-|h|^2)\|\xi_0^* c^k x\|^2}$$

Start again: suppose given (X, c) form $Y = \overline{aX + bX}$

= completion $\quad \|ax + bx'\|^2 = \begin{pmatrix} x \\ x' \end{pmatrix}^* \begin{pmatrix} 1 & c \\ c^* & 1 \end{pmatrix} \begin{pmatrix} x \\ x' \end{pmatrix}$ $\qquad c = a^*b$, $c^* = b^*a$

$$= \|x + cx'\|^2 + (x', (1-c^*c)x') = \|c^*x + x'\|^2 + (x, (1-cc^*)x)$$

$$Y = aX \oplus \underbrace{\overline{(b-ac)X}}_{V_+} = \overline{(a-bc^*)X} \oplus bX$$

$c^2X_2 \quad V_-$

$$Y = jX \oplus \overline{(uj - jc)X} = \overline{(j - ujc^*)X} \oplus ujX$$

$$\begin{array}{ccccc} & & cX_2 & \hookrightarrow & cX_1 \\ & & \downarrow & & \downarrow \\ X_2 & \hookrightarrow & X_1 & \xrightarrow[\xi_0]{} & X_0 \end{array}$$

~~Better might be to form E and $Y = \overline{jX + ujX}$~~

c

$X_1 = jX$, $\quad X_0 = Y \subset E$

$c_1 = c = a_0^* b_0 \qquad c_0 = b_0 a_0^*$

$a_0 : X_1 \longrightarrow X_0$ inclusion

$b_0 = u a_0 \qquad a_0^* b_0 = a_0^* u a_0 = c$

611 Continue with weekend stuff. So what can I do? Perturbation theory? There is a puzzle about the scattering situation when ~~[illegible]~~ S is not unitary. Go over what you did yesterday.

Suppose given (X, c) you define

$$X_n = \{x \in X \mid \|c^n x\| = \|x\|\} = \mathrm{Ker}\,(1 - c^{*n}c^n)$$

$$\|x\|^2 - \|c^n x\|^2 = \sum_{k=0}^{n-1} \|c^k x\|^2 - \|c^{k+1}x\|^2$$

$$\|x\| = \|c^n x\| \implies x,\ cx \in X_{n-1}$$

$$\begin{tikzcd} x \arrow[r, bend right, "c"'] & cx \arrow[l, bend right, "c^*"'] \arrow[r, bend right, "c"'] & c^2x \arrow[l, bend right, "c^*"'] \arrow[r, bend right] & \cdots \arrow[l, bend right] \arrow[r, bend right] & c^n x \arrow[l, bend right, "c^*"'] \end{tikzcd}$$

Just what picture emerges

$$\begin{array}{cccc}
 & & & c^3X_3 \\
 & & c^2X_3 & c^2X_2 \\
 & cX_3 & cX_2 & cX_1 \\
X_3 & X_2 & X_1 & X
\end{array}$$

How does this compare with your old picture

$$\begin{aligned} Y &= aX \oplus V_+ \\ &= V_- \oplus bX \end{aligned}$$

$$\begin{aligned} c &= ba^* + \xi_- h \xi_+^* \\ c^* &= ab^* + \xi_+ \bar{h} \xi_-^* \\ c^*c &= aa^* + \xi_+ |h|^2 \xi_+^* = 1 \text{ if } |h| = 1 \end{aligned}$$

612a Anyway why not assume what you need - ~~specifically~~. Start with (X_1, c) form

$$X_0 = aX_1 \oplus V_+ = V_- \oplus bX_1 \qquad a^*b = c_1$$

$$V_+ = \overline{(b-ac)X} \text{ is } \overline{(1-c^*c)^{1/2}X}, \qquad V_- = \overline{(a-bc^*)X} \text{ is } \overline{(1-cc^*)^{1/2}X} \qquad ba^* = c_0$$

note $a^*c_0a = a^*b = c_1$

Maybe you should start with (X_0, c_0) put

$$X_1 = \{x \in X_0 \mid \|c_0 x\| = \|x\|\} = \operatorname{Ker}(1-c_0^*c_0) \overset{a}{\underset{b=c_0a}{\rightrightarrows}} X_0$$

$$V_+ = \overline{(1-c^*c)^{1/2}X_0}, \qquad V_- = \overline{(1-cc^*)^{1/2}X_0}$$

so now you have constructed the partial unitary associated to c: $X_1 \overset{a}{\underset{b=ca}{\rightrightarrows}} X_0$ where

$ax = x$, $bx = cx$ for $x \in X_1$. $\qquad ba^* = caa^*$

so $c = \underbrace{caa^*}_{ba^*} + c\xi_+\xi_+^*$ $+ \xi_- h \xi_+^*$. Do you want $1-c^*c = \xi_+(1-|h|^2)\xi_+^*$

to dilate again $\quad \overline{X_{-1}} = \overline{a_{-1}X_0 + b_{-1}X_0}$

$$X_{-1} = jX_0 + ujX_0 = jX_0 \oplus \overline{(uj-jc)X_0}$$

$$= (aX_1 \oplus V_+) \oplus \quad ?$$

Wait: go back to $X_0 = aX_1 + \xi_+\mathbb{C}$

ξ_+ unit vector in $(1-c^*c)^{1/2}X_0 = \xi_+(1-|h|^2)^{1/2}\underbrace{\xi_+^*X_0}_{\mathbb{C}}$

612b

u^2X

$uX \cap u^2X$ $\qquad uX + u^2X$

$u\xi_{-1}$

$X \cap uX \cap u^2X$ $\qquad uX$

$u\xi_0$ $\qquad \eta_{-1}$

$X \cap uX$ $\qquad X + uX$

η_0 $\qquad \xi_{-1}$

$u^{-1}X \cap X \cap uX$ $\qquad X$

ξ_0

$u^{-1}X \cap X$ $\qquad u^{-1}X + X$

$u^{-2}X \cap u^{-1}X \cap X$ $\qquad u^{-1}X$

$u^{-2}X \cap u^{-1}X$

In this picture orthonormal bases corresp to [redacted] horizontal saw ~~lines~~ paths

and probably the [redacted] path to the southwest is inadequate. You might get some experience with this from orthogonal polys. The idea would be to push a bdry condition to the circle ~~Assume~~

613 So it seems that perturbation theory tells us a lot. So what you have learned I think is that you can ~~understand~~ use orthogonal polys, Szegö theory, to understand partial unitaries.

~~Everything is mess~~

Proceed as follows. Review the Szegö theory

$L^2(S^1, d\mu)$

H, u, ξ

$$\begin{tikzcd} F_{n-1} \arrow[d, "u"] \\ F_n \end{tikzcd} \quad F_n = \mathbb{C}\xi + \mathbb{C}u\xi + \cdots + \mathbb{C}u^n\xi$$

$F_{n-1} \xhookrightarrow{p_n} F_n \quad q_n$ have a partial unitary V_n.

p_0, q_0 unit vectors in $F_0 = \mathbb{C}\xi$ $\qquad p_0 = \frac{\xi}{\|\xi\|}$

define $p_n \in F_n \ominus F_{n-1}$

$p_n = g_n u^n \xi \qquad g_n > 0.$

$q_n = g_n' \xi \qquad g_n' > 0.$

$$\begin{tikzcd} uF_{n-2} \arrow[r, hook, "up_{n-1}"] \arrow[d, "q_{n-1}"] & uF_{n-1} \arrow[d, "q_n"] \\ F_{n-1} \arrow[r, hook, "p_n"] & F_n \end{tikzcd} \qquad h_n = q_n^* p_n$$

$$\begin{pmatrix} p_n \\ q_n \end{pmatrix} = \frac{1}{k_n}\begin{pmatrix} 1 & h_n \\ \bar{h}_n & 1 \end{pmatrix}\begin{pmatrix} u\,p_{n-1} \\ q_{n-1} \end{pmatrix} \qquad \begin{aligned} h_n &= q_n^* p_n \\ &= -p_{n-1}^* q_{n-1} \end{aligned}$$

614 Anyway $H = L^2(S^1; d\mu)$

$$
\begin{tikzcd}
zF_{n-2} \arrow[r, hook, "zp_{n-1}"] \arrow[d, hook, "g_{n-1}"'] & zF_{n-1} \arrow[d, hook, "g_n"] \\
F_{n-1} \arrow[r, hook, "p_n"'] & F_n
\end{tikzcd}
$$

$$\begin{pmatrix} p_n \\ q_n \end{pmatrix} = \frac{1}{k_n}\begin{pmatrix} 1 & h_n \\ \bar h_n & 1 \end{pmatrix}\begin{pmatrix} zp_{n-1} \\ q_{n-1} \end{pmatrix}$$

$$\frac{1}{k_n}\begin{pmatrix} 1 & -h_n \\ -\bar h_n & 1 \end{pmatrix}\begin{pmatrix} p_n \\ q_n \end{pmatrix} = \begin{pmatrix} zp_{n-1} \\ q_{n-1} \end{pmatrix}$$

$$q_{n-1}^* \, z p_{n-1} =$$

$$= \frac{1}{k_n}(q_n - \bar h_n p_n)^* z p_{n-1}$$

$$= \frac{1}{k_n}(-h_n)\, p_n^* \left(p_n - h_n q_n\right)\frac{1}{k_n}$$

$$= \frac{1}{k_n^2}(-h_n)\Big(1 - h_n \underbrace{p_n^* q_n}_{\bar h_n}\Big) = -h_n$$

$$q_{n-1}^* \, z p_{n-1} = q_{n-1}^* \frac{1}{k_n}(\cancel{p_n} - h_n q_n)$$

$$q_n^* p_n = \quad = q_n^* \frac{1}{k_n}(\cancel{z p_{n-1}} + h_n q_{n-1})$$

$$
\begin{tikzcd}
\eta' \arrow[r, bend left, "\xi'"] & \eta \arrow[l, bend left, "-\xi"]
\end{tikzcd}
$$

$$\xi - \eta\overbrace{(\eta^*\xi)}^{h} = \xi' \,\overbrace{(\text{const} > 0)}^{\cancel{g}}$$

$$1 = |h|^2 + g^2 \qquad g = \sqrt{1-|h|^2}$$

$$\xi = \eta h + \xi' g \qquad \eta^*\xi = h$$

$$\begin{pmatrix} h & g \\ g_1 & h' \end{pmatrix} \quad \text{unitary} \quad g, g_1 > 0$$

$$g = g_1 \quad h' = -\bar h$$

$$\eta' = \eta g_1 + \xi' h'$$

$$\begin{matrix} & \xi' & \\ \eta' & & \eta \\ & \xi & \end{matrix}$$

$$\xi = \eta \overbrace{\eta^*\xi}^{h} + \xi' k \qquad k = \sqrt{1-|h|^2}$$

$$\eta' = \eta k' + \xi' h' \qquad k' = \sqrt{1-|h'|^2}$$

$$\begin{pmatrix} h & k \\ k' & h' \end{pmatrix} \text{ unitary} \qquad k' = k \qquad h\bar{k} + k'\overline{h'} = 0$$

$$\eta' = \eta k' + \xi' h'$$

$$\xi'^* \eta' = h' = -\bar{h} \qquad \therefore \ \eta'^* \xi' = -h.$$

$$\xi = \eta \overbrace{\eta^*\xi}^{h} + \xi' k \qquad k^2 + |h|^2 = 1. \quad k = \sqrt{1-|h|^2}$$

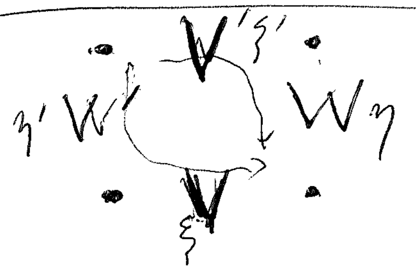
$$\eta = \xi \bar{h} + \eta' k \qquad \boxed{\eta^*\xi = h \qquad \eta'^*\xi' = -h}$$

$$k^2(\eta'^*\xi') = (\eta - \xi\bar{h})^*(\xi - \eta h)$$

$$= \underbrace{\eta^*\xi}_{h} - h\underbrace{\xi^*\xi}_{1} - \underbrace{\eta^*\eta}_{1} h + h \underbrace{\xi^*\eta}_{\bar{h}} h$$

$$= -h + |h|^2 h = (-h)k^2$$

Recap.



$$W' \oplus V \xrightarrow{\sim} V' \oplus W$$

$$\begin{matrix} V' \\ \oplus \\ W \end{matrix} \xleftarrow{\begin{pmatrix} a & b \\ c & d \end{pmatrix}} \begin{matrix} W' \\ \oplus \\ V \end{matrix}$$

more quasi-determinant stuff & but.

Go back to orth polys.

$$\frac{1}{g_n}\begin{pmatrix} 1 & h_n \\ \bar{h}_n & 1 \end{pmatrix}\begin{pmatrix} p_n \\ q_n \end{pmatrix} = \begin{pmatrix} z p_{n-1} \\ q_n \end{pmatrix}$$

$$\begin{tikzcd} zF_{n-2} \arrow[r, hook, "zp_{n-1}"] & zF_{n-1} \\ F_{n-1} \arrow[u, hook, "q_{n-1}"] \arrow[r, hook, "p_n"'] & F_n \arrow[u, hook, "q_n"'] \end{tikzcd}$$

$$h_n = q_n^* p_n = -q_{n-1}^* z p_{n-1}$$

6/6 unnormalized. $\tilde{p}_0 = \tilde{q}_0 = 1$

$$\begin{pmatrix} \tilde{p}_n \\ \tilde{q}_n \end{pmatrix} = \begin{pmatrix} 1 & h_n \\ \bar{h}_n & 1 \end{pmatrix} \begin{pmatrix} z\tilde{p}_{n-1} \\ \tilde{q}_{n-1} \end{pmatrix}$$

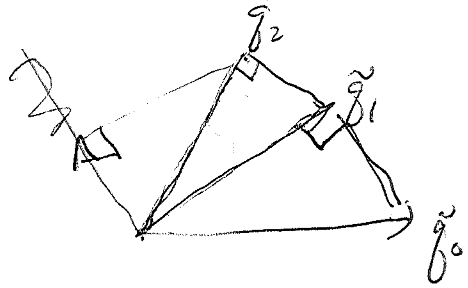
$$\tilde{p}_n = z\tilde{p}_{n-1} + h_n \tilde{q}_{n-1} \qquad \|\tilde{p}_n\|^2 = \|\tilde{p}_{n-1}\|^2 + |h_n|^2 \|\tilde{q}_{n-1}\|^2$$

$$\tilde{p}_n - h_n \tilde{q}_{n-1} = z\tilde{p}_{n-1} \qquad \|\tilde{q}_{n-1}\|^2 = \|\tilde{q}_n\|^2 + |h_n|^2 \|\tilde{p}_{n-1}\|$$

$\therefore \quad \|\tilde{q}_n\| = \|\tilde{p}_n\|$ by induction on n

$$\|\tilde{p}_n\|^2 = (1 - |h_n|^2) \|\tilde{p}_{n-1}\|^2$$

$$\|\tilde{q}_n\|^2 = \|\tilde{p}_n\|^2 = \prod_{j=1}^{n} (1 - |h_k|^2) \|\tilde{p}_0\|^2$$



$\tilde{q}_0$

$\tilde{q}_1 = \tilde{q}_0 + \bar{h}_0 z \tilde{p}_0$

$\tilde{q}_2 = \tilde{q}_0 + \bar{h}_0 z \tilde{p}_0 + \bar{h}_1 z \tilde{p}_1$

$\|\tilde{q}_2\|^2 = 1$

$\tilde{q}_\infty \exists (\neq 0) \iff (h_n)$ is ℓ^2-summable

Now go back to your ~~situation~~ basic situation where $X_n = F_n^\perp$ – Work this out.

$$H = L^2(S^1, d\mu) = Y = X \oplus \xi_0 \mathbb{C} = \eta_0 \mathbb{C} \oplus zX$$

Repeat. Start with a partial unitary?

$Y = aX \oplus \xi_+ \mathbb{C} = \xi_- \mathbb{C} \oplus bX$ but

impose a boundary condition whence $u = ba^* + \xi_- h \xi_+^*$
$|h| = 1$.

617 I'll take $h=1$. $uax = bx$ $u\xi_+ = \xi_-$.
Somehow this is not like orthogonal polynomials, but it should be, I think. Stay inside X and ignore the boundary condition.

Take $L^2(S^1, d\mu) = Y$, $\xi_+ = 1$, $X = \xi_+^\perp$
$\xi_- = u\xi_+$. You want to focus on the partial unitary. So forget $L^2(S^1, d\mu)$. Namely take $Y = aX \oplus \xi_+\mathbb{C} = \xi_-\mathbb{C} + bX$
the boundary condition only ~~[illegible] figures~~ influences what's outside X.

Let's try to formulate the situation
Begin with (X, c)

Consider (X, c)

$$\begin{array}{ccccc} cX_3 & cX_2 & cX_1 & & \\ & & \eta_0 & & \\ X_3 & X_2 & X_1 & \xi_0 & X \end{array}$$

$$X_n = \{x \in X \mid \underbrace{\|c^n x\| = \|x\|}\} = \operatorname{Ker}(1 - c^{*n}c^n)$$

$$\|c^n x\| \leq \|c^{n-1}x\| \leq \cdots \leq \|cx\| \leq \|x\|$$

$c^*c\,cx = cx$, $c^*cx = x$

$x \in X_n \iff x \in X_{n-1}$ and $cx \in X_{n-1}$

$$\begin{tikzcd} x \arrow[r, bend right, "c"'] & cx \arrow[l, bend right, "c^*"'] \arrow[r, bend right, "c"'] & {} \arrow[l, bend right] \end{tikzcd} \qquad c^n x$$

$\|c^n x\| \leq \|c^{n-1}x\| \leq \|x\|$

$$X_1 \cap cX_1 = cX_2 \qquad x \in X_2$$

$x \in X_2 \iff \|cx\| = \|x\|$ and $\|c^2 x\| = \|cx\|$
$\iff x \in X_1$ and $cx \in X_1$

618 (X,c) given form $Y = \overline{jX + ujX}$

$$\simeq jX \oplus \overline{(uj - jc)X} = \overline{(j - ujc^*)X} \oplus ujX$$

inside the standard dilation E. This gives a partial unitary $X \xrightarrow[b=uj]{a=j} Y$ yielding the contraction $c = a^*b = j^*uj$. What is

$X_1 = \{x \in X \mid \|cx\| = \|x\|\} = \ker(1 - c^*c)$? $\|cx\| = \|j^*ux\|$ equals $\|x\| = \|ux\|$ iff $ux \in X$, so $X_1 = u^{-1}X \cap X$

$X_n = \{x \in X \mid \|c^n x\| = \|x\|\}$, $\|c^n x\| = \|j^* u^n x\| = \|x\|$ iff $u^n x \in X$, so it seems that $X_n = u^{-n}X \cap X$?

$$\begin{tikzcd} X_2 \arrow[r, dash] & X_1 \arrow[r, dash] \arrow[d, dash] & X \\ u^{-1}X_1 \arrow[r, dash] \arrow[d, dash] & u^{-1}X & \\ u^{-2}X & & \end{tikzcd}$$

~~what~~

Review: (X,c) (E,u,j) $j^*u^nj = \begin{cases} c^n & n \geq 0 \\ c^{*-n} & n \leq 0 \end{cases}$

$$\rho \frac{d\theta}{2\pi} \qquad \rho = \sum_{n \geq 0} z^{-n} c^n + \sum_{n \geq 1} z^n c^{*n}$$

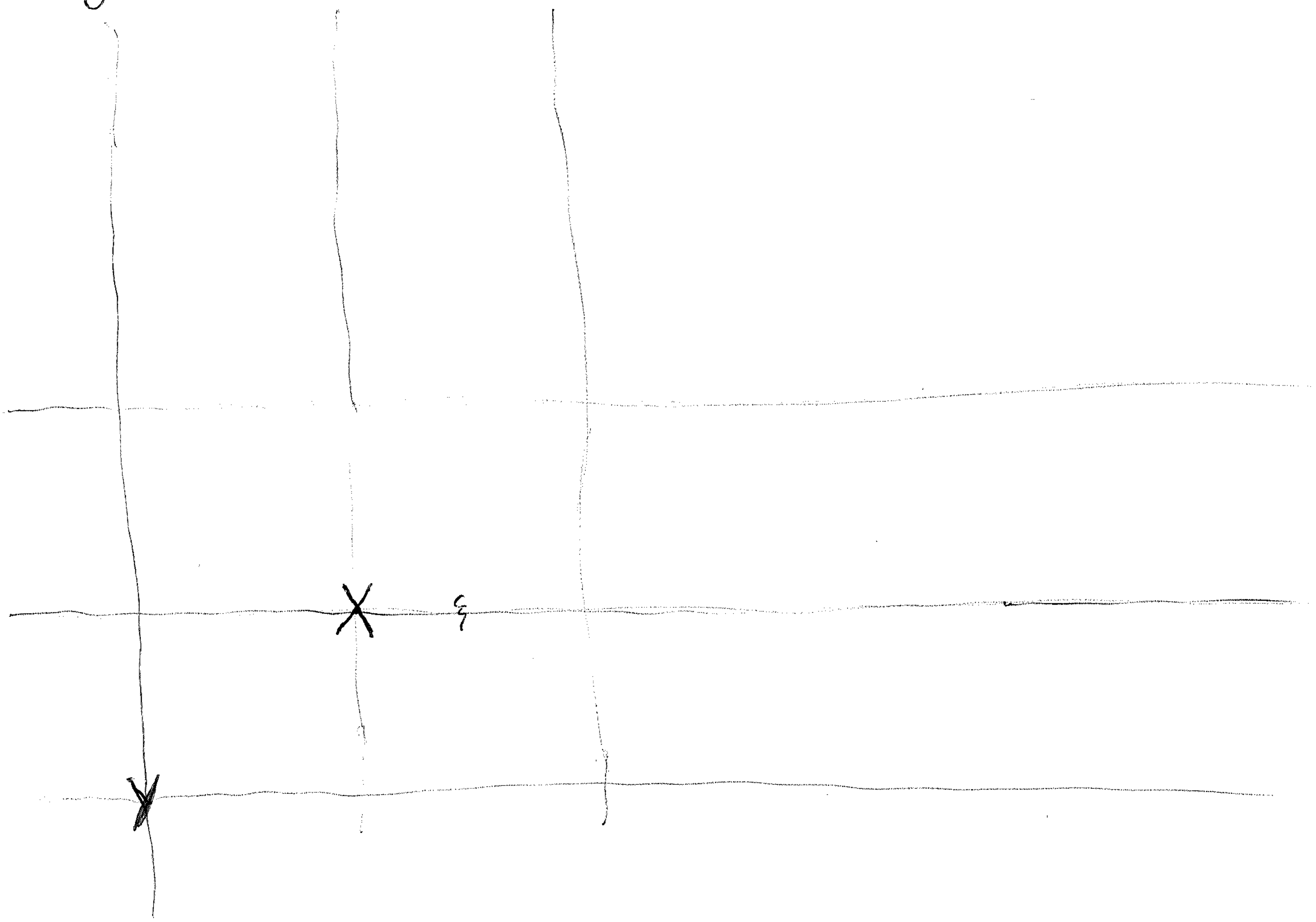
$$\int \varepsilon^n \rho \frac{d\theta}{2\pi} = j^* u^n j$$

$V_+ = \overline{(uj - jc)X}$ $\qquad v_+ x = (uj - jc)x$

$\qquad v_- x = (j - ujc^*)x$

$$E \quad \cdots \oplus u^{-1}V_- \oplus jX \oplus V_+ \oplus uV_+ \oplus \cdots$$

619 Define $F_{pq} \subset$ [scribbled out] E somehow in terms of the scattering pictures. You want $F_{pq} \subset X$ for $p,q \geq 0$. ~~F_{pq} should~~ $F_{00} = X$. Maybe ~~[scribbled out]~~ you should focus on the unit vectors. The double array consisting of the ~~[struck]~~ various orthormal bases. orthogonal polynomial picture. As p,q increase



decreasing bifiltration in directions $\uparrow \!\!\rightarrow$ if you use X, Y etc. but it's an increasing bifiltration if you emphasize V_+, V_-.

620 (X,c) form E

$$
\begin{array}{lccccccccccc}
L^2(S^1,V_+) & & & & & \oplus & u^{-1}V_+ & \oplus & V_+ & \oplus & uV_+ & \oplus \cdots \\
\downarrow j_+ & & & & & & & & \| & & \| & \\
E & \cdots \oplus & u^{-2}V_- & \oplus & u^{-1}V_- & \oplus & X & \oplus & V_+ & \oplus & uV_+ & \oplus \cdots \\
& & \| & & \| & & & & & & & \\
\uparrow j_- & \cdots \oplus & u^{-2}V_- & \oplus & u^{-1}V_- & \oplus & V_- & \oplus & uV_- & \oplus & \cdots & \\
L^2(S^1,V_-) & & & & & & & & & & &
\end{array}
$$

Two decreasing filtrations of E. Actually there are ~~[illegible]~~ probably 4, 2 for each repn, maybe related by orthogonal complements

Look at $L^2(S^1,V_+)$, better $j_+^*: E \to L^2(S^1,V_+)$.

The top of E, really, the $+$ direction of E is seen best in this repn.

obvious increasing filtration is

$$\supset u^{-2}H^2_+ \supset u^{-1}H^2_+ \supset H^2_+$$

projecting onto $u^{-n}H^2_+$ as n increases picks up more of X, the ~~orthogonal~~ kernel is then a decreasing filtration of X. You want to focus on the increasing side, building up ~~elements~~ orthonormal bases in E.

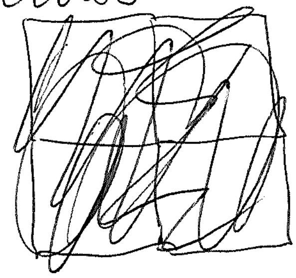
h_0	h_1	h_2	h_3	
0	h_0	h_1	h_2	
	0	h_0 (V_+)	h_1	
		0		

621 First case to consider is where

$$E = L^2(S^1, V_+) \oplus L^2(S^1, V_-)$$

$$X = H^2_-(V_+) \oplus H^2_+(V_-)$$

You don't have the right picture. Maybe you ~~need~~ want a graph. Originally you had the graph as follows



The nodes are subspaces, the edges are codim 1 inclusions, so you get

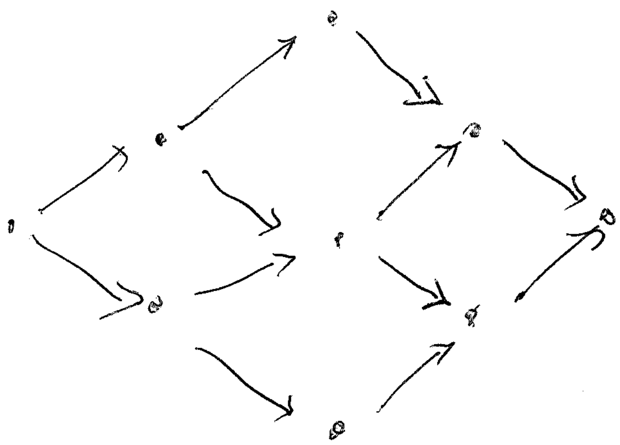
uX

$X \cap uX$

X

$u^{-1}X \cap X$

$u^{-1}X$

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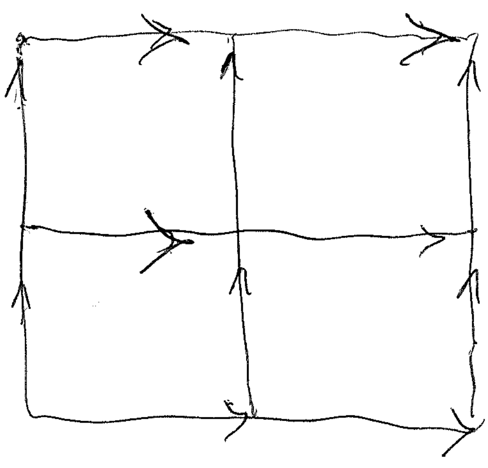
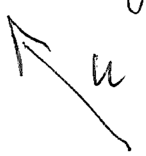
dual graph.

rotate this 45°



~~You see that~~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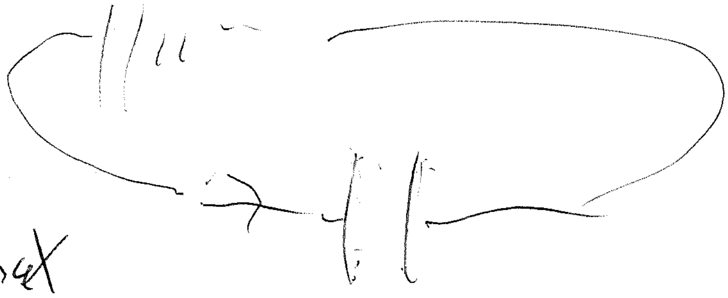
Reverse the ~~[illegible]~~ order so that you get decreasing filtration of the X-subspace and an increasing filtration for the ~~closed~~ orthogonal complements.

622 Start with



$$\begin{array}{ccc} & uX & \\ & & X\cap uX \\ & X & & u^{-1}X\cap X\cap uX \\ & & u^{-1}X\cap X & \\ & u^{-1}X & \end{array}$$

and rotate

$$\begin{array}{ccccccc}
uY & \xrightarrow{(u\xi_0)} & uX & \xrightarrow{(u\xi_1)} & X\cap uX & \xrightarrow{(u\xi_2)} & u^{-1}X\cap X\cap uX \\
\Big|(\eta_0) \ \perp & & \Big|(\eta_0) & & \Big|(\eta_1) & & \Big|(\eta_2) \\
\xrightarrow{(u\xi_0)} & & Y & \xrightarrow{(\xi_0)} & X & \xrightarrow{(\xi_1)} & u^{-1}X\cap X \xrightarrow{(\xi_2)} \\
 & & \Big| \ \perp & & \Big|(u^{-1}\eta_0) & & \Big|(u^{-1}\eta_1) \\
 & & \xrightarrow{(\xi_0)} & & u^{-1}Y & \longrightarrow & u^{-1}X \\
 & & & & \Big| & \perp & \Big|
\end{array}$$

Subspace increases $\downarrow \leftarrow$

$$\underbrace{\overset{u^{-1}\eta_0}{u^{-1}V_-} \oplus X}_{u^{-1}Y} \oplus \underbrace{V_+}_{\xi_0} \qquad \overbrace{X \oplus V_+}^{Y}$$

Set this up properly. Begin with (X, c) form E and

$$\underset{\underset{X_0}{\shortparallel}}{Y} = \overline{jX + ujX} = \underset{\underset{X_1}{\shortparallel}}{jX} \oplus \underset{\underset{V_+}{\shortparallel}}{\overline{(uj - jc)X}}$$

Also $Y = \overline{(j - ujc^*)X} \oplus ujX = V_- \oplus uX_1$

You want to arrange things in lattice form

$$\begin{tikzcd}
uX_0 \arrow[d, "(V_-)"] & uX_1 \arrow[l, "(uV_+)"'] \arrow[d, "(V_-)"] \arrow[dl, phantom, "\perp"] \arrow[dr, phantom, "h_0"] & uX_2 \arrow[d, "("] \\
X_{-1} & X_0 \arrow[l, "(uV_+)"'] \arrow[d, "(u^{-1}V_-)"] \arrow[dl, phantom, "\perp"] \arrow[dr, phantom, "\perp"] & X_1 \arrow[l, "(V_+)"'] \arrow[d, "(u^{-1}V_-)"] \\
u^{-1}X_{-2} & u^{-1}X_{-1} & u^{-1}X_0 \arrow[l, "(V_+)"']
\end{tikzcd}$$

---

check $\qquad \overline{Y + uY} = jX \oplus V_+ \oplus uV_+$

$$\overbrace{jX \oplus V_+}^{Y} \oplus uV_+$$

$$V_- \oplus \underbrace{ujX \oplus uV_+}_{uY}$$

It appears that to complete the picture you want to introduce $\underline{V_{+,n}}$

While your mind is clear, let's digress and try to understand coverings of S^1. You want to take a scattering $S(z)$ and compare $H^2(\text{double cover})/S(z)H^2(\text{double cover})$ with $H^2(S^1)/S(z)H^2(S^1)$.

624 Basic tool. $\lambda \mapsto e^{2\pi i\lambda} = z$

$$\text{UHP}/\underset{\lambda \mapsto \lambda+1}{\mathbb{Z}\text{ translation}} \xrightarrow{\sim} D - \{0\}$$

Intrinsically associated to UHP is the Hilbert space $H^2\left(\mathbb{R}, \frac{d\omega}{2\pi}\right)$.

$$L^2\left(\mathbb{R}, \frac{d\omega}{2\pi}\right) \longleftrightarrow L^2(\mathbb{R}, dt)$$

$$f(\omega) = \int e^{i\omega t}\phi(t)\,dt \longleftarrow \phi(t)$$

$$f(\omega) \longmapsto \int e^{-i\omega t} f(\omega)\frac{d\omega}{2\pi}$$

opposite signs to Harm. conventions

suppose $\phi \in L^2(\mathbb{R}_{>0}, dt)$, then $\int_0^\infty e^{i\omega t}\phi(t)\,dt$ extends analytically to the UHP.

$$f(\omega+1) = \int e^{i(\omega+1)t}\phi(t)\,dt = \int e^{i\omega t} e^{it}\phi(t)\,dt$$

So $\mathbb{Z}$-translation on UHP corresp to mult by e^{it} on $L^2(\mathbb{R}_{>0}, dt)$. I think it's true that $U(1,1)$ acts in a natural fashion?

Sections of $O(-1)$. Simplest is ~~the end~~ perhaps to use holom. For each line $\ell_z = \begin{pmatrix}1\\z\end{pmatrix}\mathbb{C}$ you want an elt. ~~So what you've~~ You are interested in sections of $O(-1)$ over $|z| = 1$, i.e.

625 Suppose $f(z)\binom{1}{z}$ is a section of $\mathcal{O}(-1)$

Fix z and an inf displacement dz. dz is a tangent vector to $\mathbb{P}^1$, equiv. a map $\mathcal{O}(-1) \longrightarrow T/\mathcal{O}(-1) \simeq \mathcal{O}(1)$

So in a natural way dz should give a quadratic form on the fibre $\mathcal{O}(-1)_z$. Put another way dz is a section of $\Omega^1 \simeq \mathcal{O}(-2)$

Fix $s = \binom{1}{z} \in \mathcal{O}(-1)_z = l_z$ ~~$l_z = f(z)\binom{1}{z}$~~

s is a holom. frame for $\mathcal{O}(1)$.

$$ds = \binom{0}{dz} \qquad \binom{1}{z}$$

$$s(z) = \binom{1}{z} \qquad ds = \binom{0}{dz} \xrightarrow{(z,\,-1)}$$

Fix a pt z of $\mathbb{P}^1$ and a tangent vector dz at that point dz should have a natural interpretation as a map $l_z \longrightarrow T/l_z \simeq l_z^\vee$ (this isom. def'd via $\Lambda^2 T = \mathbb{C}$)

$$f(z)\binom{1}{z} \longmapsto f'(z)dz\binom{1}{z} + f(z)\binom{0}{dz}$$ pair this with

$$\left(f(z)\binom{1}{z}\right)^t \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} f' \\ f'z+f \end{pmatrix} dz$$

$$= \begin{pmatrix} f & fz \end{pmatrix} \begin{pmatrix} f'z+f \\ -f' \end{pmatrix} \cancel{dz} = \left(\cancel{ff'z} + f^2 - \cancel{fzf'}\right)dz$$

$$= f(z)^2\,dz.$$

$$c\binom{1}{z} \longmapsto c\binom{0}{dz} \longmapsto \left(c\binom{1}{z}\right)^t \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} c\binom{0}{dz} = c^2\,dz$$

 So now calculate the scalar product on $l_z = \mathbb{C}\begin{pmatrix}1\\ z\end{pmatrix}$ assoc. to dz. ~~You want~~

~~to rig things~~ You want to use that ~~$c^2 dz \in \mathbb{R}_>$~~ $c^2 dz \geqslant 0$ defines a real line inside l_z. If $dz = dx$ is real then this means that $f(x)\begin{pmatrix}1\\ x\end{pmatrix}$ should yield $|f(x)|^2 dx$

For the unit circle $dz = e^{i\theta} i\, d\theta$ and the condition is $c^2 e^{i\theta} i\, d\theta \geqslant 0$ or $(c e^{i\theta/2} i^{1/2})^2 d\theta \geqslant 0$

~~so $c\begin{pmatrix}1\\ e^{i\theta}\end{pmatrix} = c e^{i\theta/2} i$~~ $\quad c e^{i\theta/2} i^{1/2} \in \mathbb{R}$

ignore $i^{1/2}$

$$c\begin{pmatrix}1\\ e^{i\theta}\end{pmatrix} = c e^{i\theta/2}\begin{pmatrix}e^{-i\theta/2}\\ e^{i\theta/2}\end{pmatrix}$$

you could require $c^2 dz$ to have a certain phase.

OKAY

$$\therefore\ f(z)\begin{pmatrix}z^{-1/2}\\ z^{1/2}\end{pmatrix} \rightsquigarrow |f(z)|^2 d\theta$$

Also the i might occur in the symplectic form, i.e. $\Lambda^2 T \simeq \mathbb{C}$.

You need to understand better the action of $SL(2,\mathbb{R})$ on H^2. Important point is that -1 is non trivial, or should be. Puzzling. Do this carefully.

$f(x)\begin{pmatrix}1\\ x\end{pmatrix}$

627 Look at $SL(2,\mathbb{R})$ acting on UHP, no you need to look at spinors. Basic idea: Look at sections over $\mathbb{R}$ of $\mathcal{O}(-1)$

$f(x)\begin{pmatrix} x \\ 1 \end{pmatrix}$, better $f(z)\begin{pmatrix} z \\ 1 \end{pmatrix}$ over the UHP.

~~Go back over So let's take $SL_2($~~

I seem to have an action of $SL_2\mathbb{R}$ on sections of $\mathcal{O}(-1)$ over $\mathbb{RP}_1$ and there's a natural scalar product.

Try again. You have the line $\ell_x = \begin{pmatrix} x \\ 1 \end{pmatrix}\mathbb{C}$

take point $c\begin{pmatrix} x \\ 1 \end{pmatrix} \in \ell_x$ and the variation ~~think of it~~

$c\begin{pmatrix} \delta x \\ 0 \end{pmatrix}$

$$c\begin{pmatrix} x \\ 1 \end{pmatrix} \longmapsto c\begin{pmatrix} \delta x \\ 0 \end{pmatrix} \in T/\ell_x \simeq \check{\ell}_x$$

$$c\begin{pmatrix} \delta x \\ 0 \end{pmatrix}^t \begin{pmatrix} 0 & +1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} x \\ 1 \end{pmatrix} c = c^2 \begin{pmatrix} \delta x & 0 \end{pmatrix}\begin{pmatrix} 1 \\ -x \end{pmatrix} = c^2 \delta x$$

So you take $f(x)\begin{pmatrix} x \\ 1 \end{pmatrix} \longmapsto \begin{pmatrix} f'(x)x\,\delta x + f(x)\,\delta x \\ f'(x)\,\delta x \end{pmatrix}$

$$\mapsto \underbrace{f(x)\begin{pmatrix} x & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}}_{\begin{pmatrix} -1 & x \end{pmatrix}} \begin{pmatrix} f'(x)x + f(x) \\ f'(x) \end{pmatrix} \delta x$$

$$= f(x)\left(-f(x)\right)\delta x = -f(x)^2\,\delta x$$

 You learn to write a section (not nec. analytic) of $\mathcal{O}(-1)$ as $f(z)\binom{z}{1} \in \mathbb{C}^2$. Then

$$f(z)\binom{z}{1} \longmapsto f\left(\frac{az+b}{cz+d}\right)\begin{pmatrix} \frac{az+b}{cz+d} \\ 1 \end{pmatrix} = \frac{1}{cz+d} f\left(\frac{az+b}{cz+d}\right)\begin{pmatrix} a & b \\ c & d \end{pmatrix}\binom{z}{1}$$

Section of $P \times^G E$ is a map $f: P \to E$ such that $\boxed{f(pg) = g^{-1} f(p)}$ $\qquad \rho(g) R_g^* f = f$

equivariance

$$f(pg_1g_2) = g_2^{-1} f(pg_1) = g_2^{-1} g_1^{-1} f(p) = (g_1g_2)^{-1} f(p).$$

$$(R^*_{g_1g_2} f)(p) = f(R_{g_1g_2} p) = f(p g_1 g_2)$$

$$(R^*_{g_1} R^*_{g_2} f)(p) = (R^*_{g_2} f)(pg_1) = f((pg_1)g_2).$$

You have $\mathcal{O}(-1) \subset \mathbb{P}^1 \times \mathbb{C}^2$

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad f($$

$$f(z)\,dz \longmapsto f\left(\frac{az+b}{cz+d}\right)\frac{dz}{(cz+d)^2}$$

~~So you~~ look at translations.

$f(z)\binom{z}{1}$ work with rational functions

rational fns. $f(z)z$

$$f(z)\binom{z}{1} \longmapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} f\left(\frac{az+b}{cz+d}\right)\begin{pmatrix} \frac{az+b}{cz+d} \\ 1 \end{pmatrix}$$

628 So now try out the UHP. Discuss what you want to do. What are you trying to do? ~~again~~ to understand $z^{1/2}$. What's the program?? basic idea is to cons

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$$L \subset \mathbb{P}_1 \times \mathbb{C}^2 \qquad L = \{(\ell, v) \mid \ell \in \mathbb{P}(\mathbb{C}^2), v \in \ell\}$$

$$L = \left\{ (z, v) \;\middle|\; z \in \mathbb{C} \cup \infty, \; v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \in \mathbb{C}^2, \frac{v_1}{v_2} = z \right\}. \quad v \in \ell_z$$

Obvious action of $SL_2(\mathbb{C})$, YES.

rational section $z \longmapsto v(z) \in \ell_z \qquad v(z) = f(z)\begin{pmatrix} z \\ 1 \end{pmatrix}$

Given $\left(z, f(z)\begin{pmatrix} z \\ 1 \end{pmatrix}\right)$ rational section

$$\left(\frac{az+b}{cz+d}, f\left(\frac{az+b}{cz+d}\right) \begin{pmatrix} \frac{az+b}{cz+d} \\ 1 \end{pmatrix} \right) = \left(\frac{az+b}{cz+d}, \frac{1}{cz+d} f\left(\frac{az+b}{cz+d}\right) \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} z \\ 1 \end{pmatrix} \right)$$

Think of a rational section as a rational map from $S \cup \infty$ to $\mathbb{C}^2$ of the form $z \mapsto f(z)\begin{pmatrix} z \\ 1 \end{pmatrix}$.

$\phi : z \longmapsto \phi(z) \in \mathbb{C}^2 \quad \ni$ the line gen. by $\phi(z)$ is $\begin{pmatrix} z \\ 1 \end{pmatrix}\mathbb{C}$. $\quad \therefore \quad \phi(z) = f(z)\begin{pmatrix} z \\ 1 \end{pmatrix}$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \phi\left(\frac{az+b}{cz+d}\right) = f\left(\frac{az+b}{cz+d}\right) \frac{1}{cz+d} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} z \\ 1 \end{pmatrix}$$

Consider $\left(z, \begin{pmatrix} z \\ 1 \end{pmatrix}\right) \xrightarrow{g} \left(\frac{az+b}{cz+d}, \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} z \\ 1 \end{pmatrix} \right)$

630 $V = \mathbb{C}^2$ $L = \{(l, v) \mid l \text{ line in } V, v \in l\} \subset \mathbb{P} \times V$

$$g(l, v) = (g(l), g(v))$$

$$l_z = \begin{pmatrix} z \\ 1 \end{pmatrix} \mathbb{C} \qquad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \Rightarrow g(l_z) = l_{\left(\frac{az+b}{cz+d}\right)}$$

section $s(l_z) = \begin{pmatrix} z \\ 1 \end{pmatrix}$. $g(s)$ is probably

$g(s) = g_V^{-1} s\, g$ right action as on fns.

$$l_z \xmapsto{g} l_{\left(\frac{az+b}{cz+d}\right)} \xmapsto{s} \begin{pmatrix} \frac{az+b}{cz+d} \\ 1 \end{pmatrix}$$

$$= \frac{1}{cz+d} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} z \\ 1 \end{pmatrix} \xmapsto{g^{-1}} \frac{1}{cz+d} \begin{pmatrix} z \\ 1 \end{pmatrix}$$

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to consider the following ~~...~~ situation. Given ~~...~~ a tangent vector δz at z there should be associated a quadratic form on $l_z = \begin{pmatrix} z \\ 1 \end{pmatrix} \mathbb{C}$. δl_z should naturally be a map $l_z \longrightarrow T/l_z$ and the latter should be isom. to $l_z^\vee$ via a symplectic form.

$$c \begin{pmatrix} z \\ 1 \end{pmatrix}^t \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \delta \begin{pmatrix} z \\ 1 \end{pmatrix} c = c (z \ \ 1) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \delta z \\ 0 \end{pmatrix} c = -c^2 \delta z$$

~~...~~

Check that $\|f\|^2 = \int_{-\infty}^{\infty} |f(x)|^2\, dx$ is preserved

under the action $f(x) \mapsto f\left(\frac{ax+b}{cx+d}\right) \frac{1}{cx+d}$. Clear

because $\int_{-\infty}^{\infty} \left| f\left(\frac{ax+b}{cx+d}\right) \right|^2 \frac{dx}{(cx+d)^2}$ YES.

63'| In particular, translation $f(x) \mapsto f(x+1)$ works.

What to do: You want to start with the link between S-functions, periodic inner functions $s(\lambda)$ on the UHP and S functions $S(z)$ on the unit disk. $S(z)$ is inner on the disk: analytic for $|z|<1$ with radial bdry values $= 1$ in abs. val. $\forall z$

Actually you start with rational S function i.e. finite Blashke product $e^{i\phi}\prod_{j=1}^{n} \frac{z-\alpha_j}{1-\bar{\alpha}_j z} = \frac{p_n(z)}{q_n(z)}$

Pull this back to the UHP via $z = e^{2\pi i\lambda}$

$$\frac{z-\alpha}{1-\bar{\alpha}z} = \frac{e^{\pi i\lambda} - \alpha e^{-\pi i\lambda}}{e^{-\pi i\lambda} - \bar{\alpha} e^{\pi i\lambda}}$$

From $S(z)$ you get a fin. dim'l Hilbert space $X = H^2_+/SH^2_+$ with a contraction having the eigenvalues $\alpha_1, \ldots, \alpha_n$. I more or less understand the structure of X, from orthogonal polys.

~~[illegible]~~

You want ~~[illegible]~~ relate

$H^2(S^1)/S(z)H^2(S^1)$ with $H^2(\mathbb{R})/S(e^{2\pi i\lambda})H^2(\mathbb{R})$

the former in some way should be the quotient of the latter by $\mathbb{Z}$ translation. Take $S(z) = z$. What is $H^2(\mathbb{R})/e^{2\pi i\lambda}H^2(\mathbb{R})$?

$$f(\lambda) = \int_0^\infty e^{2\pi i\lambda t}\,\phi(t)\,dt$$

$$e^{2\pi i\lambda} f(\lambda) = \int_0^\infty e^{2\pi i\lambda(t+1)}\,\phi(t)\,dt$$

$$= \int^\infty e^{2\pi i\lambda u}\,\phi(u-1)\,du$$

So $\quad H^2(\mathbb{R}) / e^{2\pi i\lambda} H^2(\mathbb{R}) \simeq L^2((0,1))$

$$f(\lambda) = \int_0^\infty e^{2\pi i\lambda t}\,\phi(t)\,dt$$

$$f(\lambda+1) = \int_0^\infty e^{2\pi i\lambda t}\,e^{2\pi i t}\,\phi(t)\,dt$$

~~global~~ $\quad f\left(\frac{a\lambda+b}{c\lambda+d}\right) = \frac{1}{c\lambda+d} f(\lambda)$

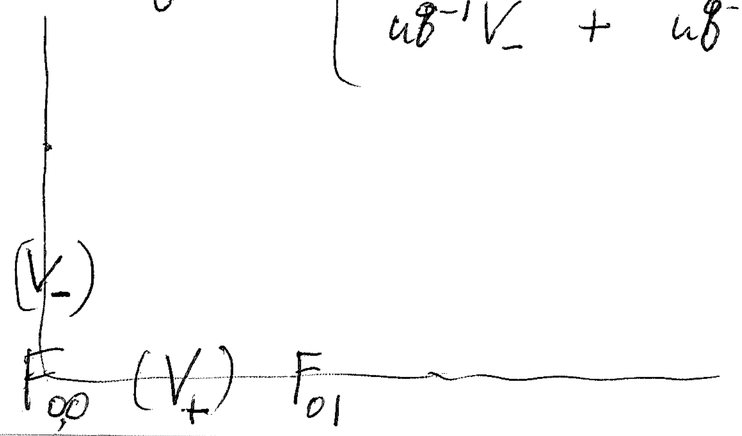
Is it possible to find an invariant function under $SL_2(\mathbb{Z})$. No ~~because~~ take $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ and you get $f(\lambda) = -f(\lambda)$. Thus k even.

Back to ~~[illegible]~~ (X, c) $\qquad E = H^2_-(V_-) \oplus X \oplus H^2_+(V_+)$

increasing family of subspaces $\qquad F_{pq} = u^q H^2_-(V_-) + u^{-p} H^2_+(V_+)$

$u F_{pq} = F_{p+1,q-1} \qquad F_{00} = X^\perp$

$$F_{pq} = \left\{ \begin{array}{l} u^{-p} V_+ + u^{-p+1} V_+ + \cdots \\ u^{q-1} V_- + u^{q-2} V_- + \cdots \end{array} \right\}$$



$F_{p0} = \cdots \oplus u^{-1}V_- \quad u^{-p}V_+ \oplus u^{-p+1}V_+ +$